Brief Papers

Exponential Stability of Globally Projected Dynamic Systems

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***Abstract*—In this paper, we further analyze and prove the stability and convergence of the dynamic system proposed by Friesz *et al.*, whose equilibria solve the associated variational inequality problems. Two sufficient conditions are provided to ensure the asymptotic stability of this system with a monotone and asymmetric mapping by means of an energy function. Meanwhile this system with a monotone and gradient mapping is also proved to be asymptotically stable using another energy function. Furthermore, the exponential stability of this system is also shown under strongly monotone condition. Some obtained results improve the existing ones and the given conditions can be easily checked in practice. Since this dynamic system has wide applications, the obtained results are significant in both theory and application.**

***Index Terms*—Energy function, exponential stability, projected dynamic systems, variational inequality.**

I. Introduction

Consider the following dynamic system proposed by Friesz *et al.* [3]:

$$\frac{dx}{dt} = \lambda \{P_\Omega[x - \alpha F(x)] - x\} \tag{1}$$

where λ and α are positive constants, Ω is a nonempty closed convex subset of R^n, F is a mapping from Ω to R^n, and P_Ω: $R^n \longrightarrow \Omega$ is a projection operator defined by

$$P_\Omega(x) = \arg\min_{u \in \Omega} \|x - u\|$$

where $\|\cdot\|$ denotes the Euclidean norm.

It is well known that $x^* \in \Omega$ is an equilibrium point of system (1) if and only if x^* is a solution of the following variational inequality VI(F, Ω): find $x^* \in \Omega$ such that

$$(x - x^*)^T F(x^*) \geq 0, \qquad \forall\, x \in \Omega. \tag{2}$$

Thus, (1) can be used to solve the equilibrium models arising in various fields of economics and transportation science [3], fixed point problems, optimization problems such as linear and quadratic programming, nonlinear programming, linear and nonlinear complementary problems, etc. Moreover, system (1) can be realized easily by a circuit [2], [12], then it is very amenable to parallel implementation and can be applied easily to neural networks for optimization problems. Therefore, it is very important to study the stability and convergence of system (1).

There are several papers in the literature dealing with this system [3], [9], [11]–[13]; its stability analysis can be found in [3], [11]–[13]. Especially, Smith *et al.* [9] and Xia *et al.* [11] have studied the basic properties of (1). As mentioned in [11], the given conditions by Freisz *et al.* [3] for the global asymptotic stability of (1) do not seem to be guaranteed in practice. Xia *et al.* [11] further analyzed the stability of system (1) with a monotone and symmetric mapping, however, the global asymptotic stability requires the boundedness of the solution set of (2), which ensures the boundedness of the level set of the given Lyapunov functions. Moreover, the conditions for the exponential convergence in [11] are too strong. For an affine and strongly monotone mapping, Xia *et al.* [13] proved that system (1) with initial point $x^0 \in \Omega$ is globally asymptotically stable, and globally exponentially stable under the condition $F(x^*) = 0$ (x^* is a solution of (2)) by using a Lyapunov function. But this function can not be used to prove the exponential convergence of system (1) when $F(x^*) \neq 0$.

Besides the above consideration, motivated by the importance of the asymptotic stability and exponential stability of the equilibrium point for networks in many continuous optimization problems [2], we further analyze and prove the asymptotic stability and exponential stability of system (1) in this paper. For system (1) with an asymmetric mapping, based on the functions in [4], [13], we: 1) define a new energy function by introducing a new parameter; 2) provide two sufficient conditions to ensure that every solution of system (1) converges to an exact solution of problem (2) for any starting point $x^0 \in \Omega$; and 3) prove the exponential stability under strongly monotone condition. For system (1) with a gradient mapping, we prove the asymptotic stability and exponential stability under the mild conditions, respectively, by defining another new energy function. Moreover, system (1) with a Lipschitz continuous and strongly monotone mapping is also shown to be globally exponentially stable. Since these new energy functions overcome the drawbacks of those in [11], [13], some obtained results improve the corresponding results in [11], [13].

This paper is organized as follows. Several basic definitions and lemmas are given in Section II. In Section III, the asymptotic stability and exponential stability of dynamic system (1) are analyzed and proved. The illustrative examples are provided in Section IV. The conclusion is given in Section V.

Manuscript received May 30, 2001; revised March 4, 2002 and September 25, 2002.

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Digital Object Identifier 10.1109/TNN.2003.809409

II. Preliminaries

In this section, several basic definitions and lemmas are given, which are useful for our later discussions.

Definition 1 [6]: A mapping F is said to be monotone on $\Omega \subseteq R^n$ if, for each pair of points $x,\, y \in \Omega$, there is

$$(x-y)^T\left[F(x)-F(y)\right] \geq 0$$

F is said to be strictly monotone on Ω if strict inequality holds whenever $x \neq y$; F is said to be strongly monotone on Ω if, for each pair of points $x,\, y \in \Omega$, there exists a positive constant μ such that

$$(x-y)^T\left[F(x)-F(y)\right] \geq \mu \|x-y\|^2.$$

Lemma 1 [10]: If a mapping F is continuously differentiable on an open convex set D including Ω, then F is monotone (strongly monotone) on Ω if and only if its Jacobian matrix $\nabla F(x)$ is positive semidefinite (uniformly positive definite) for all $x \in \Omega$.

Lemma 2 [6]: 1) If F is strictly monotone on Ω, then problem (2) has at most one solution on Ω. 2) If F is strongly monotone on Ω, then problem (2) has a unique solution on Ω.

The following Lemma ensures that the solution $x(t)$ of (1) if it exists will eventually stay in Ω as t becomes large enough.

Lemma 3 [11]: Assume that F is locally Lipschitz continuous in a domain R^n. Then the solution $x(t)$ of (1) will approach exponentially the feasible set Ω when the initial point $x^0 \notin \Omega$. Moreover $x(t) \subset \Omega$ when $x^0 \in \Omega$.

A basic property of a projection on a closed convex set is [7]

$$[u-P_\Omega(u)]^T\left[P_\Omega(u)-v\right] \geq 0, \qquad \forall\, u \in R^n,\, v \in \Omega. \quad (3)$$

Throughout this paper, we assume that $\Omega^* = \{x \in \Omega | x \text{ solves (2)}\} \neq \emptyset$ and there exists a finite $x^* \in \Omega^*$. A mapping F is said to be a gradient mapping on Ω, if there exists a differentiable function $f\colon \Omega \longrightarrow R^1$ such that $\nabla f(x) = F(x)$ for all $x \in \Omega$; otherwise, F is said to be an asymmetric mapping on Ω.

The following notation will be used in the later discussion. We denote $\tilde{x} = P_\Omega(x-\alpha F(x))$. $\nabla F(x)$ denotes the Jacobian matrix of the differentiable mapping F evaluated at x. I denotes the identity matrix of order n. For any vector $u \in R^n$, u^T denotes its transpose.

III. Stability Analysis

In order to analyze the stability and convergence of (1) in detail, we divide this section into three subsections.

A. Stability Analysis for An Asymmetric Mapping

Let $F(x)$ be an asymmetric mapping on Ω, then based on the regularized gap function introduced by Fukushima [4], we define the following function:

$$\boxed{\begin{aligned} V_1(x) = {} & \alpha(x-\tilde{x})^T F(x) - \frac{1}{2}\|x-\tilde{x}\|^2 + \frac{\beta}{2}\|x-x^*\|^2 \\ & \forall x \in \Omega \qquad (4) \end{aligned}}$$

where $x^* \in \Omega^*$ is finite, α is given in (1), and β is a nonnegative constant. Obviously, $V_1(x)$ with $\beta = 1$ is just the Lyapunov function introduced by Xia *et al.* [13] for an affine mapping. However, as we will show later [see Lemma 4–3) below], it cannot be used to prove the exponential convergence for (1) with $x^0 \in \Omega$ when $F(x^*) \neq 0$. Now, we explore some properties for $V_1(x)$.

Lemma 4: Let $V_1(x)$ be the function defined in (4), then the followings are true.

1) $V_1(x) \geq (1/2)\left(\|x-\tilde{x}\|^2 + \beta\|x-x^*\|^2\right)$ for all $x \in \Omega$.
2) Assume that $F(x)$ is monotone on Ω and continuously differentiable on an open convex set D including Ω. Then for $0 \leq \beta \leq 1$, we have

$$(x-\tilde{x})^T \nabla V_1(x) \geq \alpha\beta(x-x^*)^T F(x), \qquad \forall\, x \in \Omega. \quad (5)$$

3) Assume that $F(x)$ is strongly monotone on Ω with modulus $\mu > 0$ and continuously differentiable on an open convex set D including Ω. Then for $0 \leq \beta \leq 1$, we have

$$(x-\tilde{x})^T \nabla V_1(x) \geq \min\{1-\beta,\, 2\alpha\mu\} V_1(x), \qquad \forall\, x \in \Omega. \quad (6)$$

Proof: 1) Setting $u = x - \alpha F(x)$ and $v = x \in \Omega$ in (3), there is

$$\alpha(x-\tilde{x})^T F(x) \geq \|x-\tilde{x}\|^2. \quad (7)$$

Inequality (7) implies 1).

2) By setting $u = x - \alpha F(x)$ and $v = x^*$ in (3), we get

$$[x-x^*+\alpha F(x)]^T(x-\tilde{x}) \geq \|x-\tilde{x}\|^2 + \alpha(x-x^*)^T F(x). \quad (8)$$

Since $F(x)$ is continuously differentiable on D, according to [6], $V_1(x)$ is also continuously differentiable on D and

$$\boxed{\nabla V_1(x) = \alpha F(x) + (\alpha\nabla F(x) - I)(x-\tilde{x}) + \beta(x-x^*).}$$

Then by (8), $\forall\, x \in \Omega$, we have

$$\boxed{\begin{aligned} &(x-\tilde{x})^T \nabla V_1(x) \\ &\quad = \alpha(x-\tilde{x})^T \nabla F(x)(x-\tilde{x}) - \|x-\tilde{x}\|^2 \\ &\qquad + \alpha(1-\beta)(x-\tilde{x})^T F(x) + \beta\left[x-x^*+\alpha F(x)\right]^T(x-\tilde{x}) \\ &\quad \geq (1-\beta)\left[\alpha(x-\tilde{x})^T F(x) - \|x-\tilde{x}\|^2\right] \\ &\qquad + \alpha(x-\tilde{x})^T \nabla F(x)(x-\tilde{x}) + \alpha\beta(x-x^*)^T F(x). \quad (9) \end{aligned}}$$

From the monotonicity of $F(x)$ on Ω and Lemma 1, we know that $\nabla F(x)$ is positive semidefinite on Ω. Hence, 2) holds by (7) and (9).

3) Since $F(x)$ is strongly monotone on Ω with modulus $\mu > 0$, $\forall\, x \in \Omega$, we have

$$\begin{cases} (x-x^*)^T\left[F(x)-F(x^*)\right] \geq \mu\|x-x^*\|^2 \\ (x-\tilde{x})^T \nabla F(x)(x-\tilde{x}) \geq \mu\|x-\tilde{x}\|^2. \end{cases} \quad (10)$$

On the other hand, we have

$$(x-x^*)^T F(x) = (x-x^*)^T\left[F(x)-F(x^*)\right] + (x-x^*)^T F(x^*). \quad (11)$$

From (2), (7) and (9)–(11), $\forall x \in \Omega$, we have

$$\boxed{\begin{aligned}&(x-\tilde{x})^T \nabla V_1(x)\\ &\quad \geq (1-\beta)\left[\alpha(x-\tilde{x})^T F(x) - \tfrac{1}{2}\|x-\tilde{x}\|^2\right]\\ &\qquad + \left(\alpha\mu - \frac{1-\beta}{2}\right)\|x-\tilde{x}\|^2 + \alpha\beta\mu\|x-x^*\|^2\\ &\quad \geq \min\{1-\beta,\, 2\alpha\mu\} V_1(x).\end{aligned}}$$

This completes the proof. ∎

The results in Lemma 4 pave a way for the following stability results of system (1).

Theorem 1: Assume that $F(x)$ is monotone on Ω and continuously differentiable on an open convex set D including Ω. In addition, if one of the following conditions is satisfied:

1) let $x^* \in \Omega^*$, $\forall x \in \Omega$, there exists a $k(x, x^*) > 0$ such that

$$(x-x^*)^T F(x) = 0 \quad \text{implies} \quad F(x) = k(x, x^*) F(x^*);$$

2) let $x^* \in \Omega^*$, $\forall x \in \Omega$

$$(x-x^*)^T F(x) = 0 \quad \text{implies} \quad x \text{ is a solution of (2).}$$

Then $\forall \alpha > 0$, system (1) is Lyapunov stable, and for any initial point $x^0 \in \Omega$, the trajectory corresponding to system (1) will converge to an exact solution of (2). In particular, $\forall \alpha > 0$, system (1) with $x^0 \in \Omega$ is globally asymptotically stable when $\Omega^* = \{x^*\}$.

Proof: Since $F(x)$ is continuously differentiable on D, and the mapping $P_\Omega(\cdot)$ is nonexpansive, it is easy to verify that $\tilde{x} - x$ is locally Lipschitz continuous on D. Thus, for each $x^0 \in \Omega$, there exists a unique continuous solution $x(t)$ for system (1) with $x(0) = x^0$. Let $[0, T)$ be its maximal interval of existence, then $x(t) \in \Omega$ for all $t \in [0, T)$ from Lemma 3.

Now, we fix β with $0 < \beta \leq 1$ in (4), then it is easy to see that all level sets of $V_1(x)$ are bounded [10] by Lemma 4-1). From Lemma 4-2), (11), the monotonicity of $F(x)$ on Ω and (2), $\forall t \in [0, T)$, we have

$$\begin{aligned}\frac{d}{dt} V_1(x(t)) &= -\lambda\,[x(t)-\tilde{x}(t)]^T \,\nabla V_1(x(t))\\ &\leq -\alpha\beta\lambda\,[x(t)-x^*]^T\, F(x(t)) \leq 0. \qquad (12)\end{aligned}$$

(12) indicates that $\{x(t)|0 \leq t < T\} \subset L(x^0) = \{x \in \Omega | V_1(x) \leq V_1(x^0)\}$. Thus, $T = +\infty$ from the boundedness of $L(x^0)$.

On the other hand, Lemma 4–1) and inequality (12) indicate that function $V_1(x)$ is nonnegative for all $x \in \Omega$ and nonincreasing along the trajectory $x(t)$ of (1). Thus, $V_1(x)$ is a Lyapunov function of (1). From Theorem 3.2 in [8], system (1) is Lyapunov stable.

From the boundedness of $L(x^0)$, $\{x(t)|t \geq 0\}$ is bounded. Thus, there exist a limit point $\overline{x}$ and a sequence $\{t_n\}$ with $0 < t_1 < t_2 < \cdots < t_n < t_{n+1} < \cdots$ and $t_n \longrightarrow +\infty$ as $n \longrightarrow +\infty$ such that

$$\lim_{n\longrightarrow+\infty} x(t_n) = \overline{x}. \qquad (13)$$

(13) indicates that $\overline{x} \in \Omega$ is an ω-limit point of $\{x(t)\,|\,t \geq 0\}$.

From LaSalle invariant set theorem (Theorem 3.4 in [8]), we know that the solution $x(t)$ of (1) converges to M as $t \to \infty$, where M is the largest invariant set in

$$R = \left\{x \in L(x^0) \,|\, \frac{dV_1(x)}{dt} = 0\right\}.$$

1) If $dV_1(x)/dt = 0$, then $(x-x^*)^T F(x) = 0$ by (12). Thus, $F(x) = k(x, x^*)F(x^*)$ $(k(x, x^*) > 0)$ from the hypothesis. Hence, for all $y \in \Omega$, there is

$$\begin{aligned}(y-x)^T F(x) &= (y-x^*)^T F(x) + (x^*-x)^T F(x)\\ &= k(x, x^*)(y-x^*)^T F(x^*) \geq 0\end{aligned}$$

that is to say $x \in \Omega^*$. Therefore, $dx/dt = 0$.

Conversely, if $dx/dt = 0$, then $dV_1(x)/dt = 0$. Therefor, $dx/dt = 0 \Leftrightarrow dV_1(x)/dt = 0$. From the above proof, $\overline{x} \in \Omega$ is an ω-limit point of $\{x(t)\,|\,t \geq 0\}$. Thus, $\overline{x} \in M \subseteq R \subseteq \Omega^*$.

Similar to the definition of $V_1(x)$, we define

$$\begin{aligned}\overline{V_1}(x) &= \alpha(x-\tilde{x})^T F(x) - \frac{1}{2}\|x-\tilde{x}\|^2 + \frac{\beta}{2}\|x-\overline{x}\|^2\\ &\qquad x \in \Omega\end{aligned}$$

where β is fixed with $0 < \beta \leq 1$. Then similar to the proofs of Lemma 4–1) and 2), we can conclude that $\overline{V}_1(x) \geq (\beta/2)\|x-\overline{x}\|^2$ for all $x \in \Omega$ and $\overline{V}_1(x(t))$ is monotonically nonincreasing on $[0, +\infty)$. From the continuity of function $\overline{V}_1(x)$, it follows that $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$\overline{V}_1(x) < \frac{\beta\varepsilon^2}{2}, \qquad \text{if } \|x-\overline{x}\| \leq \delta. \qquad (14)$$

From (13), (14), and the monotonicity of $\overline{V}_1(x(t))$, there exists a natural number N such that

$$\|x(t)-\overline{x}\|^2 \leq \frac{2}{\beta}\overline{V}_1(x(t)) \leq \frac{2}{\beta}\overline{V}_1(x(t_N)) < \varepsilon^2, \quad \text{when } t \geq t_N.$$

That is, $\lim_{t\longrightarrow+\infty} x(t) = \overline{x}$. This indicates that the solution $x(t)$ of (1) converges to a point in Ω^*, i.e., the solution $x(t)$ of (1) converges to a solution of (2).

2) Similar to the proof for 1), we can also prove for each $x^0 \in \Omega$, the solution $x(t)$ of (1) converges to a solution of (2).

In particular, if $\Omega^* = \{x^*\}$, then for each $x^0 \in \Omega$, the trajectory $x(t) \in \Omega$ will converge to x^* by the above analysis. Thus, system (1) with $x^0 \in \Omega$ is globally asymptotically stable. This completes the proof. ∎

Remark: If $F\colon \Omega \longrightarrow R^n$ is a G-map [1], that is, there exists a positive function $k(x, y)$ on $\Omega \times \Omega$ such that, for all $x, y \in \Omega$

$$(y-x)^T F(x) = 0 \quad \text{implies} \quad F(x) = k(x, y)F(y).$$

Then Theorem 1-1) is satisfied.

When $F(x)$ is strictly monotone on Ω, from Lemma 2–1) and Theorem 1, we have the following results.

Corollary 1: Assume that $F(x)$ is strictly monotone on Ω and continuously differentiable on an open convex set D including Ω. Then $\forall \alpha > 0$, system (1) with $x^0 \in \Omega$ is globally asymptotically stable at the unique solution x^* of (2).

Theorem 2: Assume that $F(x)$ is strongly monotone on Ω with modulus $\mu > 0$ and continuously differentiable on an open convex set D including Ω. Then system (1) with $x^0 \in \Omega$ converges to the unique solution x^* of (2) exponentially.

Proof: Since $F(x)$ is strongly monotone on Ω, $\Omega^* = \{x^*\}$ by Lemma 2-2). From the proof of Theorem 1, for each $x^0 \in \Omega$, $x(t) \in \Omega$ is the unique solution of system (1) with $x(0) = x^0 \in \Omega$ for all $t \geq 0$.

Now, we fix β in (4) such that $0 < \beta < 1$, then $\min\{1 - \beta, 2\alpha\mu\} > 0$. From (6), $\forall\, t \geq 0$, we have

$$\begin{aligned}\frac{d}{dt} V_1(x(t)) &= -\lambda \left[x(t) - \tilde{x}(t)\right]^T \nabla V_1(x(t))\\ &\leq -\lambda \min\{1-\beta,\, 2\alpha\mu\} V_1(x(t)).\end{aligned}$$

Thus

$$V_1(x(t)) \leq V_1(x^0) e^{-\lambda \min\{1-\beta,\, 2\alpha\mu\} t}, \qquad \forall\, t \geq 0$$

or

$$\|x(t) - x^*\| \leq \sqrt{\frac{2}{\beta} V_1(x^0)}\, \exp\left(-\frac{\lambda \min\{1-\beta, 2\alpha\mu\}}{2}\, t\right) \qquad \forall\, t \geq 0.$$

This completes the proof. ■

Theorem 2 shows that $\forall\, \alpha > 0$, system (1) with $x^0 \in \Omega$ is exponentially stable. From Theorem 2, we have the following useful result.

Corollary 2: Assume that $\nabla F(x)$ is positive definite and continuous on Ω, which is bounded. Then system (1) with $x^0 \in \Omega$ converges to the unique solution x^* of (2) exponentially.

B. Stability Analysis for a Gradient Mapping

When $F(x)$ is a gradient mapping on an open convex set D including Ω, we have the following stability result for (1).

Theorem 3: Assume that F is a gradient mapping on an open convex set D including Ω and locally Lipschitz continuous on D. In addition, if $F(x)$ is monotone on Ω, then $\forall\, \alpha > 0$, system (1) is Lyapunov stable, and for any initial point $x^0 \in \Omega$, the trajectory corresponding to system (1) will converge to an exact solution of (2). Furthermore, if $F(x)$ is strongly monotone on Ω with modulus $\mu > 0$, then system (1) with $x^0 \in \Omega$ converges to the unique solution x^* of (2) exponentially.

Proof: The assumptions imply that for each $x^0 \in \Omega$, there exists a unique continuous solution $x(t)$ of (1) with $x(0) = x^0$. Let $[0, T)$ be its maximal interval of existence, then $x(t) \in \Omega$ for all $t \in [0, T)$ from Lemma 3.

Since $F(x)$ is a gradient mapping on D, there exists a differentiable function $f\colon D \longrightarrow R^1$ such that $\nabla f(x) = F(x)$ for all $x \in D$. From the monotonicity of $F(x)$ on Ω, we know that function $f(x)$ is differentiable convex on Ω [10], and the set of global minimizers of $f(x)$ with respective to Ω coincides with Ω^*. Thus $f(x) \geq f(x^*)$ for all $x \in \Omega$. Let

$$V_2(x) = \alpha\left[f(x) - f(x^*)\right] + \tfrac{1}{2}\|x - x^*\|^2, \qquad \forall\, x \in \Omega \tag{15}$$

where $x^* \in \Omega^*$ is finite and α is given in (1), then $V_2(x)$ is also differentiable convex on Ω and $V_2(x) \geq (1/2)\|x - x^*\|^2$ for all $x \in \Omega$. Thus, all level sets of $V_2(x)$ are bounded [10].

Again, from the monotonicity of $F(x)$ on Ω, (2), (8), and (11), $\forall\, t \geq 0$, we have

$$\begin{aligned}\frac{d}{dt} V_2(x(t)) &= \nabla V_2(x(t))^T \frac{dx(t)}{dt}\\ &= -\lambda \left[x(t) - x^* + \alpha F(x(t))\right]^T \left[x(t) - \tilde{x}(t)\right]\\ &\leq -\lambda\|x(t) - \tilde{x}(t)\|^2 - \alpha\lambda \left[x(t) - x^*\right]^T F(x(t))\\ &\leq -\lambda\|x(t) - \tilde{x}(t)\|^2 \leq 0.\end{aligned} \tag{16}$$

Following the same arguments as Theorem 1, we can conclude that $T = +\infty$, system (1) is Lyapunov stable, and the trajectory $x(t)$ of system (1) will approach to an exact solution of problem (2).

Next, we prove the exponential convergence of (1).

Sine $F(x)$ is strongly monotone on Ω with modulus $\mu > 0$, then $\Omega^* = \{x^*\}$ by Lemma 2-2), and function $f(x)$ is uniformly convex on Ω. Thus

$$(x - x^*)^T F(x) \geq f(x) - f(x^*) + \frac{\mu}{2}\|x - x^*\|^2, \qquad \forall\, x \in \Omega.$$

From the proof of (16) and the above inequality, $\forall\, t \geq 0$, it follows that

$$\begin{aligned}\frac{d}{dt} V_2(x(t)) &\leq -\lambda\|x(t) - \tilde{x}(t)\|^2 - \alpha\lambda \left[x(t) - x^*\right]^T F(x(t))\\ &\leq -\alpha\lambda \left[f(x(t)) - f(x^*) + \frac{\mu}{2}\|x(t) - x^*\|^2\right]\\ &\leq -\lambda \min\{1,\, \alpha\mu\} V_2(x(t)).\end{aligned}$$

Therefore

$$V_2(x(t)) \leq V_2(x^0) e^{-\lambda \min\{1,\, \alpha\mu\} t}, \qquad \forall\, t \geq 0$$

or

$$\|x(t) - x^*\| \leq \sqrt{2V_2(x^0)}\, \exp\left(-\frac{\lambda \min\{1,\, \alpha\mu\}}{2}\, t\right), \qquad \forall\, t \geq 0.$$

This completes the proof. ■

Obviously, Theorem 3 does not require the boundedness of the set Ω^* and the differentiability of the mapping $F(x)$.

C. Stability Analysis for a Lipschitz Continuous Mapping

When $F(x)$ is Lipschitz continuous on Ω, we have the following exponential stability result for (1).

Theorem 4: Assume that the mapping F is strongly monotone on Ω with modulus $\mu > 0$, and that there exists a constant $L > 0$ such that

$$\|F(u) - F(v)\| \leq L\|u - v\|, \qquad \forall\, u, v \in \Omega.$$

Then system (1) with $\alpha < 4\mu/L^2$ is globally exponentially stable.

Proof: The assumptions imply $\Omega^* = \{x^*\}$. From Theorem 1 in [5] and Lemma 3, for each $x^0 \in \Omega$, there exists a unique solution $x(t) \in \Omega$ of (1) with $x(0) = x^0$ for all $t \geq 0$.

From (8), the assumptions of this theorem, $\tilde{x} \in \Omega$ and (2), $\forall x \in \Omega$, we have

$$\begin{aligned}
&(x - x^*)^T(x - \tilde{x}) \\
&\quad \geq \|x - \tilde{x}\|^2 + \alpha(\tilde{x} - x^*)^T F(x) \\
&\quad \geq \|x - \tilde{x}\|^2 + \alpha\left[(x - x^*) - (x - \tilde{x})\right]^T \left[F(x) - F(x^*)\right] \\
&\qquad + \alpha(\tilde{x} - x^*)^T F(x^*) \\
&\quad \geq \|x - \tilde{x}\|^2 + \alpha(x - x^*)^T \left[F(x) - F(x^*)\right] \\
&\qquad - \alpha(x - \tilde{x})^T \left[F(x) - F(x^*)\right] \\
&\quad \geq \|x - \tilde{x}\|^2 + \alpha\mu\|x - x^*\|^2 - \alpha L\|x - \tilde{x}\|\,\|x - x^*\| \\
&\quad \geq \alpha\left(\mu - \frac{\alpha L^2}{4}\right)\|x - x^*\|^2.
\end{aligned}$$

It follows that

$$\frac{d}{dt}\|x(t) - x^*\|^2 \leq -\alpha\lambda\left(\mu - \frac{\alpha L^2}{4}\right)\|x(t) - x^*\|^2, \qquad \forall t \geq 0.$$

This implies

$$\|x(t) - x^*\| \leq \|x^0 - x^*\| \exp\left(-\frac{\alpha\lambda(4\mu - \alpha L^2)}{8}\, t\right), \qquad \forall t \geq 0.$$

This completes the proof. ∎

IV. Illustrative Examples

Example 1: Consider a class of nonlinear complementary problems below. Find a vector $x \in R^n$ such that

$$\begin{cases} x_i F_i(x) = 0, \quad F_i(x) \geq 0, \quad x_i \geq 0, \quad \forall\, i \in L, \\ F_i(x) = 0, \quad \forall i \in N \backslash L \end{cases} \tag{17}$$

where F is a continuously differentiable mapping from $\Omega = \{x \in R^n \mid x_i \geq 0, i \in L\}$ to R^n, $N = \{1, 2, \ldots, n\}$, $L \subseteq N$, $x = (x_1, x_2, \ldots, x_n)^T \in \Omega$.

Since problem (17) is a special case of problem (2), the dynamic system (1) can be used to solve it.

Now, we let $L = N$ and $F(x) = C(x) + Ax + q$ where

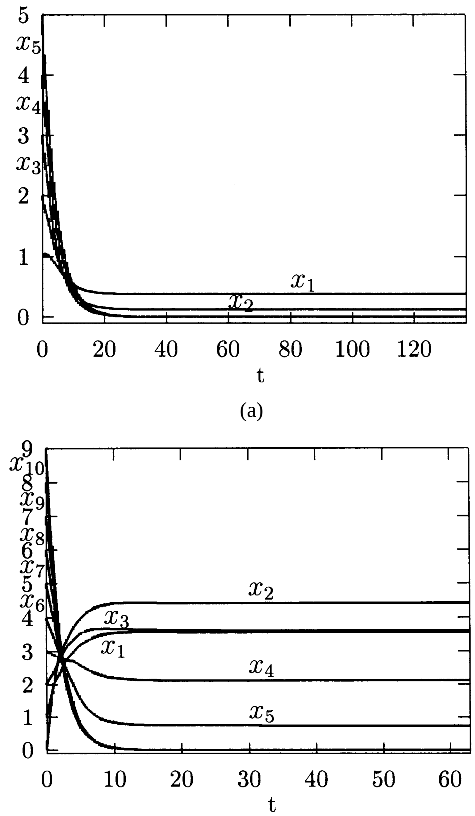
$$C(x) = [\arctan(x_1), \arctan(x_2), \ldots, \arctan(x_n)]^T$$

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$$

and

$$\left(q = \left(-\frac{n}{2} + 1, -\frac{n}{2} + 2, \cdots, \frac{n}{2} - 1, \frac{n}{2}\right)^T.\right)$$

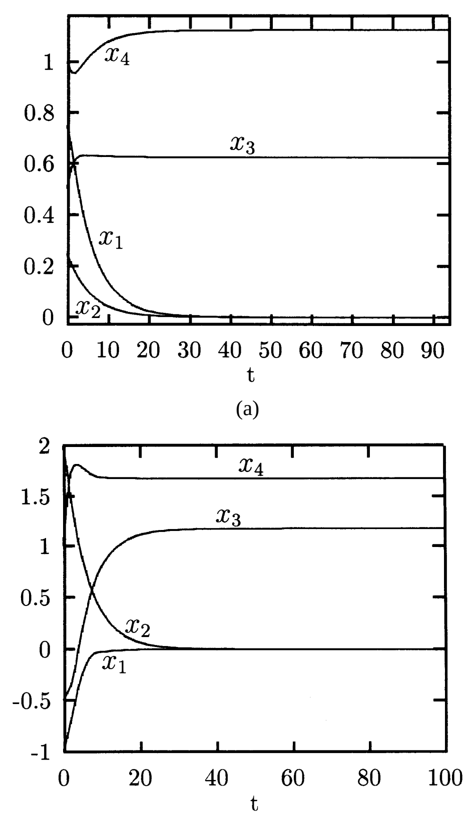
Then it is easy to verify that the mapping F is strongly monotone on R^n, and when $n = 5$ and 10, its solutions are



(a)

(b)

Fig. 1. Transient behavior of (1) in Example 1. (a) First initial point. (b) Second initial point.



(a)

(b)

Fig. 2. Transient behavior of (1) in Example 2. (a) $x^0 = (0.75, 0.25, 0.5, 1)^T$. (b) $x^0 = (-1, 2, -0.5, 1)^T$.

$x^* = (0.381\,475, 0.127\,386, 0, 0, 0)^T$ and (3.555 657, 4.407 95, 3.607 953, 2.108 374, 0.736 715, 0, 0, 0, 0, 0)^T, respectively.

Let $\alpha = \lambda = 1$, Fig. 1(a) and 1(b) show the trajectories of system (1) with different initial points (1, 2, 3, 4, 5)^T and (0, 1, 2, 3, 4, 5, 6, 7, 8, 9)^T, respectively.

Example 2: We let $L = N$ and $F(x) = Mx + q$ in (17), where

$$M = \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 2 \\ -1 & 1 & 2 & -2 \\ 1 & -2 & -2 & 2 \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}.$$

Then this problem has infinitely many solutions $x^* = (0, 0, \xi, 0.5 + \xi)^T$ $(\xi \geq 0)$, and it is easy to prove that this problem satisfies Theorem 1 ii).

Let $\alpha = \lambda = 1$, Fig. 2(a) and 2(b) show that the trajectories of system (1) with different initial points (0.75, 0.25, 0.5, 1)^T, and (-1, 2, -0.5, 1)^T converge to the exact solutions $x^* = (0, 0, 0.625, 1.125)^T$ and (0, 0, 1.1776, 1.6776)^T, respectively.

V. Conclusion

In this paper, we further analyzed the stability of the globally projected dynamic system proposed by Friesz *et al.* in detail. Two sufficient conditions are provided to ensure that every solution of system (1) with a monotone and asymmetric mapping converges to an exact solution of problem (2) for any starting point $x^0 \in \Omega$ by introducing a new energy function. Meanwhile, system (1) with a monotone gradient mapping has been shown to be asymptotically stable using another new energy function. Furthermore, the exponential stability of system (1) has been also proved under certain conditions. Some obtained results improve the corresponding ones in [11]–[13]. Theoretical analysis and illustrative examples show that the given conditions are weaker than existing ones and can be easily checked in practice. Since system (1) can be applied to solve the equilibrium models arising in various fields of economics and transportation science, and optimization problems, etc., the results obtained in this paper are significant in both theory and application.

Acknowledgment

The author would like to thank the Associate Editor and three anonymous reviewers for many valuable comments.

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