Stochastic Differential Equations in 1 Hour

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1 Intro

Stochastic differential equations (SDE) are different from Ordinary differential equations (ODE). The reason is that integration rules for ODEs are not valid for SDEs due to the stochastic nature of the latter. Here I will go through the basics of SDEs from more of an applied perspective. Basic knowledge of probability theory is assumed.

2 Langevin equation

Definition 1 [Langevin equation]**:** *A Langevin equation is of the form*

$$\frac{dx}{dt} = a(t,x) + b(t,x)\xi(t), \tag{1}$$

where $\xi(t)$ is a rapidly fluctuating random term that satisfies

$$\langle \xi(t) \rangle = 0, \tag{2}$$
$$\langle \xi(t)\xi(t') \rangle = \delta(t - t'). \tag{3}$$

One may notice that this definition requires formal description using the modern statistical language. A theory developed by Itô describes the Langevin equation formally as a statistical differential equation.

3 Wiener process

Definition 2 [Wiener process, Brownian motion]**:** *A Wiener process (or Brownian motion) is a continuous-time* [1] *stochastic process $\{W(t_j)\}$ with the following properties:*

1. *$W(t_0) = 0$.*
2. *The process has stationary and independent increments* [2]*.*
3. *The increment $W(t_j) - W(t_k)$ has the distribution*

$$W(t_j) - W(t_k) \sim \mathcal{N}(0, t_j - t_k). \tag{4}$$

For convenience, below we also refer to ΔW or dW as the Wiener process. Notice that the dimension of $W(t)$ is *square root of time*.

[1] Lindeberg condition, the probability for the final position to be finitely different from a given position goes to zero faster than Δt.

[2] A stochastic process with stationary and independent increments is called a *Lévy process*. The Wiener process is the intersection of Gaussian processes and the Lévy processes.

4 Itô calculus

4.1 Itô integral

In general an SDE has the form

$$dx(t,\omega) = a[t, x(t,\omega)]\, dt + b[t, x(t,\omega)]\, dW(t,\omega)\,. \tag{5}$$

Here $x(t,\omega)$ is a stochastic variable with ω denoting the corresponding probability space, and $dW(t,\omega)$ is the Wiener process. For simplicity, below we ignore the ω notation. We also ignore the t dependence of x and W in our expressions as it is clear both x and dW depend on t.

Definition 3 [SDE]**:** *A stochastic differential equation(SDE) is a differential equation with the form*

$$dx = a(t,x)\, dt + b(t,x)\, dW\,, \tag{6}$$

where dW is the Wiener process.

Formally solving Eq. (6), setting $t_0 = 0$ and $x(0) = x_0$, we have

$$x(t) = x_0 + c \int_0^t ds\, a(s,x) + \int_0^t dW\, b(s,x) \tag{7}$$

Technically speaking, the integral in the third term on the RHS of Eq. (7) is undefined. Since dW is stochastic, we need to define this integral from first principles—Riemann sums. To do this, we construct the series $\{S_n\}$ where

$$S_n = \sum_{j=1}^{n} b(\tau_j)[W(t_j) - W(t_{j-1})] \tag{8}$$

with $t_{j-1} \le \tau_j \le t_j$. With normal Riemann sums the choice of τ_j does not matter. However since W is a Wiener process the integral defined as the limit of S_n depends on the choice of τ_j. There exist several methods to construct the Riemann sum (Itô , Stratonovich, etc.). Here we focus on the Itô definition by setting $\tau_j = t_{j-1}$ so that

$$S_n = \sum_{j=1}^{n} b(t_{j-1})[W(t_j) - W(t_{j-1})]. \tag{9}$$

For convenience below we write $W(t_j)$ as W_j.

Definition 4 [Itô stochastic integral]**:** *The Itô stochastic integral of function $b(t)$ is a stochastic random variable $S(t)$ defined as*

$$S = \int_0^t dW\, b(s,x) \equiv \operatorname{ms}\lim_{n\to\infty} S_n, \tag{10}$$

where ms lim *is the mean square limit* [3].

One can show that the Itô integral exists as long as $b(t)$ is continuous and nonanticipating [4] on the closed interval $[0,t]$. Below we give some examples and results relating to the Itô integral.

Example 1: Integration of polynomials.

[3] $\left\langle (S - S_n)^2 \right\rangle = 0$ in the limit $n \to \infty$. Mean square convergence implies both convergence in probability and convergence in distribution.

[4] Causality requirement.

1. Let $b(t) = c$. Then it is clear that

$$S = S_n = cW(t). \tag{11}$$

2. Let $b(t) = W(t)$. Then

$$S_n = \sum_{j=1}^{n} W_{j-1}(W_j - W_{j-1}). \tag{12}$$

   It is a common trick to reorganize terms into squares when calculating Itô stuff. Thus we have

$$\begin{aligned} S_n &= \frac{1}{2}\sum_{j=1}^{n}\left(W_j^2 - W_{j-1}^2\right) - \frac{1}{2}\sum_{j=1}^{n}\left(W_j - W_{j-1}\right)^2 \\ &= \frac{1}{2}W(t)^2 - \frac{1}{2}\sum_{j=1}^{n}\left(W_j - W_{j-1}\right)^2 \end{aligned} \tag{13}$$

   We can calculate the mean square limit of the last term. Notice

$$\left\langle \sum_{j=1}^{n}\left(W_j - W_{j-1}\right)^2 \right\rangle = \sum_{j=1}^{n}\left\langle \left(W_j - W_{j-1}\right)^2 \right\rangle = \sum_{j=1}^{n} t_j - t_{j-1} = t. \tag{14}$$

   We thus need to prove $\lim_{n\to\infty}\left\langle \left(\sum_{j=1}^{n}\left(W_j - W_{j-1}\right)^2 - t\right)^2 \right\rangle = 0$. The proof can be done by direct calculation and is not shown here. Thus we have

$$\operatorname{ms}\lim_{n\to\infty}\sum_{j=1}^{n}\left(W_j - W_{j-1}\right)^2 = t. \tag{15}$$

   Therefore

$$S = \int_0^t dW\, W = \frac{1}{2}\left[W(t)^2 - t\right], \tag{16}$$

   which is contrast to our intuition from normal integration rules.

3. In general, we can derive a general formula for integration of polynomials in W as shown in Eq. (17).

$$\int_0^t dW(t')\, W^n(t') = \frac{1}{n+1}W^{n+1}(t) - \frac{n}{2}\int_0^t dt'\, W^{n-1}(t') \tag{17}$$

   This can be proved using Eq .(18).

Theorem 1: *When put inside an Itô integral,*

$$\begin{cases} dW^2 & = dt \\ dW^{2+N} & = 0 \\ dW\, dt & = 0 \end{cases} \tag{18}$$

for $N \in \mathbb{Z}^+$.

Proof. A proof can be found in Gardiner. One intuitive understanding of this is that dW is an infinitesimal quantity of order $\frac{1}{2}$ in dt. Thus anything that is of order higher than 1 is discarded. $\square$

Example 2: Mean, variance, and covariance of S.

$$\mathrm{E}[S] = \int_0^t dW\, b = 0, \tag{19}$$

$$\mathrm{V}[S] = \mathrm{E}\left[\left(\int_0^t dW\, b\right)^2\right] = \int_0^t dt\, \mathrm{E}\big[b^2\big], \tag{20}$$

$$\mathrm{Cov}[S, S'] = \mathrm{E}\left[\int_0^t dW\, b \int_0^t dW\, b'\right] = \int_0^t dt\, \mathrm{E}[bb']. \tag{21}$$

Equations (20)–(21) are also referred to as the *Itô isometry*. These results can be calculated using Eqs. (18).

4.2 Itô 's lemma

We have already seen that the rules of classical calculus are not valid for SDEs, at least for integration. In this section we derive the chain rule for SDEs. This is actually easier than in the integration case as there is a lemma, called *Itô 's lemma*. The spirit lies in Eq. (18).

Definition 5 [Itô 's lemma]**:** *Consider a stochastic process $x(t)$. Suppose we have another stochastic process $y = y(x,t)$* [5]*. Then we have*

$$dy = b\frac{\partial y}{\partial x}\, dW + \left(\frac{\partial y}{\partial t} + a\frac{\partial y}{\partial x} + \frac{b^2}{2}\frac{\partial^2 y}{\partial x^2}\right) dt\,. \tag{22}$$

Proof. Use Taylor expansion then use Eq. (18), we have

$$\begin{aligned} dy &= \frac{\partial y}{\partial x}\, dx + \frac{1}{2}\frac{\partial^2 y}{\partial x^2}\, dx^2 + \frac{\partial y}{\partial t}\, dt + \cdots \\ &= \frac{\partial y}{\partial x}(a\, dt + b\, dW) + \frac{b^2}{2}\frac{\partial^2 y}{\partial x^2}\, dt + \frac{\partial y}{\partial t}\, dt + \cdots \\ &= \mathrm{RHS}. \end{aligned} \tag{23}$$

□

4.3 Relation with Langevin theory

Definition 6 [Langevin noise terms]**:** *We can define the Langevin noise terms by setting*

$$dW(t) = \xi(t)\, dt\,, \tag{24}$$

where $\xi(t)$ are of dimension one over square root of time and satisfy

$$\langle \xi(t)\xi(t')\rangle = \delta(t - t'). \tag{25}$$

In the Itô definition, this delta function has to obey the following conditions

$$\int_{t_1}^{t_2} dt\, f(t)\delta(t - t_1) = f(t_1), \tag{26}$$

$$\int_{t_1}^{t_2} dt\, f(t)\delta(t - t_2) = 0. \tag{27}$$

In the Stratonovich integral setting, we have

$$\int_{t_1}^{t_2} dt\, f(t)\delta(t - t_1) = \frac{1}{2} f(t_1), \tag{28}$$

$$\int_{t_1}^{t_2} dt\, f(t)\delta(t - t_2) = \frac{1}{2} f(t_2). \tag{29}$$

This is what we used in the derivation of quantum master equation.

[5] Suppose all the continuous conditions are satisfied.

4.4 Relation with Fokker-Planck equation

Consider an Itô SDE of the form Eq. (5). We want to find the probability density function for a $x(t)$. This can be done by relating the Itô SDE to its corresponding Fokker-Planck equation.

Consider an arbitrary function $y[x(t)]$ where y is not explicitly dependent on t. Using Itô 's lemma we have

$$\begin{aligned}
\left\langle \frac{dy}{dt} \right\rangle &= \frac{1}{dt}\left\langle b\frac{\partial y}{\partial x}\,dW + \left(\frac{\partial y}{\partial t} + a\frac{\partial y}{\partial x} + \frac{b^2}{2}\frac{\partial^2 y}{\partial x^2}\right)dt\right\rangle \\
&= \left\langle a\frac{\partial y}{\partial x} + \frac{b^2}{2}\frac{\partial^2 y}{\partial x^2}\right\rangle \\
&= \int dx\, p(x,t)\left[a(x,t)\frac{\partial y}{\partial x} + \frac{b(x,t)^2}{2}\frac{\partial^2 y}{\partial x^2}\right] \\
&= \int dx\, y(x)\left[-\frac{\partial[p(x,t)a(x,t)]}{\partial x} + \frac{1}{2}\frac{\partial^2\left[p(x,t)b(x,t)^2\right]}{\partial x^2}\right],
\end{aligned} \tag{30}$$

where we ignored the surface terms in the integral by parts. However, we also have

$$\left\langle \frac{dy}{dt} \right\rangle = \int dx\, y(x)p(x,t). \tag{31}$$

Since $y(t)$ is arbitrary, we have

$$p(x,t) = -\frac{\partial[p(x,t)a(x,t)]}{\partial x} + \frac{1}{2}\frac{\partial^2\left[p(x,t)b(x,t)^2\right]}{\partial x^2}, \tag{32}$$

which is the Fokker-Planck equation corresponding to the Ito SDE. One can find $p(x,t)$ by solve the Fokker-Planck equation. For simple cases, we can solve for the solutions directly. For more complicated cases, we can only solve for the probability density function from the Fokker-Planck equation and then calculate the mean, variance, etc. Below we give some examples that we can direclty solve.

5 Various processes

Example 3: The simplest case is when both the drift and diffusion are constant. Then

$$dx = \mu\,dt + \sigma\,dW\,. \tag{33}$$

We have by linearity of Itô integral

$$x(t) = x_0 + \mu t + \sigma W(t). \tag{34}$$

Thus

$$\mathrm{E}[x(t)] = x_0 + \mu t, \tag{35}$$
$$\mathrm{Var}[x(t)] = \sigma^2 t. \tag{36}$$

In the examples below we will use this first example again and again by using susbstitution of variables.

Example 4 [Ohrnstein-Uhlenbeck]**:**

$$dx = -\kappa x(t)\,dt + \sigma\,dW\,. \tag{37}$$

Consider

$$
\begin{aligned}
y(t,x) &= xe^{\kappa t} \\
dy &= \sigma e^{\kappa t}\, dW\,(t) \\
y(t,x) &= y_0 + \sigma \int_0^t e^{\kappa t}\, dW\,(t). \qquad (38)
\end{aligned}
$$

Hence

$$
x(t) = x_0 e^{-\kappa t} + \sigma \int_0^t e^{\kappa\left(t'-t\right)}\, dW\,(t'). \qquad (39)
$$

Thus

$$
\begin{aligned}
\mathrm{E}[x(t)] &= x_0 e^{-\kappa t}, \qquad (40) \\
\mathrm{Var}[x(t)] &= \left\langle \left[\sigma \int_0^t e^{\kappa\left(t'-t\right)}\, dW\,(t')\right]^2 \right\rangle \\
&= \sigma^2 \int_0^t dW\,(t') \left\langle \left[e^{\kappa\left(t'-t\right)}\right]^2 \right\rangle \\
&= \sigma^2 \int_0^t dt'\, e^{2\kappa\left(t'-t\right)} \\
&= \sigma^2 e^{-2\kappa t} \int_0^t dt'\, e^{2\kappa t'} \\
&= \frac{\sigma^2}{2\kappa}\big(1 - e^{-2\kappa t}\big). \qquad (41)
\end{aligned}
$$

Example 5 [mean-reverting]**:** The Ohrnstein-Uhlenbeck process is a special case of the mean-reverting process, given by

$$
dx = \kappa[\mu - x(t)]\, dt + \sigma\, dW\,. \qquad (42)
$$

Using the same substitution we get

$$
x(t) = x_0 e^{-\kappa t} + \mu\big(1 - e^{-\kappa t}\big) + \sigma \int_0^t e^{\kappa\left(t'-t\right)}\, dW\,(t'). \qquad (43)
$$

Thus

$$
\begin{aligned}
\mathrm{E}[x(t)] &= x_0 e^{-\kappa t} + \mu\big(1 - e^{-\kappa t}\big), \qquad (44) \\
\mathrm{Var}[x(t)] &= \frac{\sigma^2}{2\kappa}\big(1 - e^{-2\kappa t}\big). \qquad (45)
\end{aligned}
$$

Example 6 [geometric Brownian]**:**

$$
dx = \mu x(t)\, dt + \sigma x(t)\, dW\,. \qquad (46)
$$

Consider

$$
\begin{aligned}
y(x) &= \ln x \\
dy &= \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\, dW \\
y(t) &= y_0 + \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W(t). \qquad (47)
\end{aligned}
$$

Hence

$$x(t) = x_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right]. \tag{48}$$

Thus

$$\begin{aligned}
\mathrm{E}[x(t)] &= x_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t\right]\mathrm{E}\left[e^{\sigma W(t)}\right] \\
&= x_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t\right] \exp\left[\frac{\sigma^2 t}{2}\right] \\
&= x_0 e^{\mu t}, && (49) \\
\mathrm{Var}[x(t)] &= \mathrm{E}[x(t)^2] - \mathrm{E}[x(t)]^2 \\
&= x_0^2 \exp\left[(2\mu - \sigma^2)t\right]\mathrm{E}\left[e^{2\sigma W(t)}\right] - x_0^2 e^{2\mu t} \\
&= x_0^2 \exp\left[(2\mu - \sigma^2)t\right] \exp\left[2\sigma^2 t\right] - x_0^2 e^{2\mu t} \\
&= x_0^2 e^{2\mu t}\left(e^{\sigma^2 t} - 1\right), && (50)
\end{aligned}$$

where we have used the fact that for a Gaussian random variable Z

$$\mathrm{E}[e^{cZ}] = e^{\frac{c^2}{2}}. \tag{51}$$

The moments of the geometric Brownian process can also be calculated using the property of a *martingale* or using the probability distribution function that is a *lognormal distribution.*

6 References

Gardiner handbook