Pauli algebra

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Let $\{\mathbb{1}, \sigma_1, \sigma_2, \sigma_3\}$ be the basis of the Pauli algebra. Let

$$\sigma_\mu = \begin{pmatrix} \mathbb{1} \\ \sigma \end{pmatrix} = \begin{pmatrix} \mathbb{1} \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}, \quad A^\mu = \begin{pmatrix} a_0 \\ \mathbf{a} \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad B^\mu = \begin{pmatrix} b_0 \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix}. \tag{1}$$

Let $a = |\mathbf{a}|$, $b = |\mathbf{b}|$. We consider Euclidean metric.

Here are some useful relations.

1. Multiplication.

$$(\mathbf{a} \cdot \sigma)(\mathbf{b} \cdot \sigma) = \mathbf{a} \cdot \mathbf{b}\mathbb{1} + i(\mathbf{a} \times \mathbf{b}) \cdot \sigma. \tag{2}$$

Some further results from Eq. (2).

(a) Commutators and anticommutators.

$$[\mathbf{a} \cdot \sigma, \mathbf{b} \cdot \sigma] = 2i(\mathbf{a} \times \mathbf{b}) \cdot \sigma, \tag{3}$$
$$\{\mathbf{a} \cdot \sigma, \mathbf{b} \cdot \sigma\} = 2\mathbf{a} \cdot \mathbf{b}\mathbb{1}. \tag{4}$$

(b) $[\mathbf{a} \cdot \sigma, \mathbf{b} \cdot \sigma] = 0$ if $\mathbf{a} \parallel \mathbf{b}$.

(c) $\{\mathbf{a} \cdot \sigma, \mathbf{b} \cdot \sigma\} = 0$ if $\mathbf{a} \perp \mathbf{b}$.

In 4D,

$$(A^\mu \sigma_\mu)(B^\mu \sigma_\mu) = A^\mu B_\mu \mathbb{1} + [b_0 \mathbf{a} + a_0 \mathbf{b} + i(\mathbf{a} \times \mathbf{b})] \cdot \sigma. \tag{5}$$

2. Adjoint action.

$$\begin{aligned} (\mathbf{a} \cdot \sigma)(\mathbf{b} \cdot \sigma)(\mathbf{a} \cdot \sigma) &= (\mathbf{a} \cdot \sigma)\left[\left(\mathbf{b}_\parallel + \mathbf{b}_\perp\right) \cdot \sigma\right](\mathbf{a} \cdot \sigma) \\ &= a^2\left(\mathbf{b}_\parallel - \mathbf{b}_\perp\right) \cdot \sigma &(6) \\ &= \left[(2\mathbf{a} \cdot \mathbf{b})\mathbf{a} - a^2\mathbf{b}\right] \cdot \sigma. &(7) \end{aligned}$$

In 4D, let $\tilde{A}^\mu = \begin{pmatrix} a_0 \\ -\mathbf{a} \end{pmatrix}$. Then

$$\begin{aligned} (A^\mu \sigma_\mu)(B^\mu \sigma_\mu)\left(\tilde{A}^\mu \sigma_\mu\right) &= B_0 A^\mu \tilde{A}_\mu \mathbb{1} + \left[A^\mu \tilde{A}_\mu \mathbf{b}_\parallel + A^\mu A_\mu \mathbf{b}_\perp + 2ia_0(\mathbf{a} \times \mathbf{b})\right] \cdot \sigma &(8) \\ &= B_0 A^\mu \tilde{A}_\mu \mathbb{1} + [-2(\mathbf{a} \cdot \mathbf{b})\mathbf{a} + A^\mu A_\mu \mathbf{b} + 2ia_0(\mathbf{a} \times \mathbf{b})] \cdot \sigma. &(9) \end{aligned}$$

3. Exponential maps.

$$e^{\mathbf{a}\cdot\sigma} = \cosh{(a)}\mathbb{1} + \frac{\sinh{(a)}}{a}\mathbf{a}\cdot\sigma \tag{10}$$

$$e^{i\mathbf{a}\cdot\sigma} = \cos{(a)}\mathbb{1} + i\frac{\sin{(a)}}{a}\mathbf{a}\cdot\sigma. \tag{11}$$

Adjoint action with exponential maps.

$$e^{\mathbf{a}\cdot\sigma}(\mathbf{b}\cdot\sigma)e^{-\mathbf{a}\cdot\sigma} = \left[\mathbf{b}_{\parallel} + \cosh{(2a)}\mathbf{b}_{\perp} + i\sinh{(2a)}\left(\frac{\mathbf{a}}{a}\times\mathbf{b}_{\perp}\right)\right]\cdot\sigma, \tag{12}$$

$$e^{i\mathbf{a}\cdot\sigma}(\mathbf{b}\cdot\sigma)e^{-i\mathbf{a}\cdot\sigma} = \left[\mathbf{b}_{\parallel} + \cos{(2a)}\mathbf{b}_{\perp} - \sin{(2a)}\left(\frac{\mathbf{a}}{a}\times\mathbf{b}_{\perp}\right)\right]\cdot\sigma. \tag{13}$$