Fourier Everything for Physicists

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1 Fourier Series

Definition 1 [Fourier series]**:** *Consider a function in Hilbert space*

$$f : [t_0, t_0 + T] \to \mathbb{C}, \tag{1}$$

with $t_0, T \in \mathbb{R}$*,* $T \neq 0$*. We define the Fourier series (FS) of* f *and the Fourier coefficients* $\tilde{f}_n \in \mathbb{C}$ *by*

$$f(t) = \sum_{n=-\infty}^{\infty} \tilde{f}_n e^{i2\pi nt/T}, \tag{2a}$$

$$\tilde{f}_n = \frac{1}{T} \int_{t_0}^{t_0+T} dt\, f(t) e^{-i2\pi nt/T}. \tag{2b}$$

Equations (2a)–(2b) suggest that adding a periodic boundary condition to f does not change the Fourier series. Therefore we can define the Fourier series for a periodic function.

Definition 2 [Fourier series, periodic]**:** *Consider a periodic function in Hilbert space*

$$f : \mathbb{R} \to \mathbb{C}, \tag{3}$$

with period $T \in \mathbb{R}/\{0\}$*. We define the Fourier series of* f *and the Fourier coefficients* $\tilde{f}_n \in \mathbb{C}$ *by*

$$f(t) = \sum_{n=-\infty}^{\infty} \tilde{f}_n e^{i2\pi nt/T}, \tag{4a}$$

$$\tilde{f}_n = \frac{1}{T} \int_{T} dt\, f(t) e^{-i2\pi nt/T}, \tag{4b}$$

where $\int_T$ *means the integral over any interval of length* T*.*

Remark 1.1*:* Hermiticity. If $f(t)$ is real, then $\tilde{f}_n$ is Hermitian, meaning $\tilde{f}_{-n} = \tilde{f}_n^*$. Vice versa.

Theorem 1 [Parseval-Plancherel]**:**

$$\frac{1}{T} \int_T |f(t)|^2\, dt = \sum_{n=-\infty}^{\infty} \left|\tilde{f}_n\right|^2. \tag{5}$$

Proof. This can be proved by using the integral representation of the Kronecker delta function or the summation representation of the Dirac delta function. See Sec. A.2.2 for the exact forms of these delta functions. ☐

Remark 1.2*:* Angular variable. Let

$$\omega_n = \frac{2\pi}{T} n \tag{6}$$

for $n \in \mathbb{Z}$. Then we can define the angular version of Fourier series in Eqs. (4a)–(4b).

Definition 3 [Fourier series, periodic, angular]**:**

$$f(t) = \sum_{n=-\infty}^{\infty} \tilde{f}(\omega_n) e^{i\omega_n t}, \tag{7a}$$

$$\tilde{f}(\omega_n) = \frac{1}{T}\int_T dt\, f(t) e^{-i\omega_n t}, \tag{7b}$$

where we have redefined $\tilde{f}(\omega_n) = \tilde{f}_n$.

The Parseval-Plancherel theorem is

$$\frac{1}{T}\int_T |f(t)|^2\, dt = \sum_{n=-\infty}^{\infty} \left|\tilde{f}(\omega_n)\right|^2. \tag{8}$$

Remark 1.3*:* Higher dimensions. The Fourier series (7a)–(7b) and the Parseval-Plancherel theorem (8) can be easily generalized to the 3D space, i.e.,

$$f(\mathbf{r}) = \sum_{\mathbf{k}} \tilde{f}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{9a}$$

$$\tilde{f}(\mathbf{k}) = \frac{1}{V}\int_V d^3\mathbf{r} f(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}}, \tag{9b}$$

with

$$\frac{1}{V}\int_V |f(\mathbf{r})|^2\, dx = \sum_{\mathbf{k}} \left|\tilde{f}(\mathbf{k})\right|^2. \tag{10}$$

Here $\mathbf{r}$ and $\mathbf{k}$ are both 3-dimensional vectors, with $\mathbf{r}$ in the 3D box constraint by $x, y, z \in [-L/2, L/2]$ with volume $V = L^3$, and $\mathbf{k}$ in the reciprocal space with $k_\alpha = \frac{2\pi n_\alpha}{L}$, $n_\alpha \in \mathbb{Z}$ for $\alpha \in \{x, y, z\}$.

Remark 1.4*:* Rescaling of $\tilde{f}(\omega_n)$. The Fourier coefficients $\tilde{f}(\omega_n)$ defined in Eq. (7b) can be rescaled by introducing $h \in \mathbb{C}$ such that

$$f(t) = h\sum_{n=-\infty}^{\infty} \tilde{f}(\omega_n) e^{i\omega_n t}, \tag{11a}$$

$$\tilde{f}(\omega_n) = \frac{1}{hT}\int_T dt\, f(t) e^{-i\omega_n t}, \tag{11b}$$

This **will** change the prefactors in the Parseval-Plancherel theorem (8) such that

$$\frac{1}{T}\int_T |f(t)|^2\, dt = |h|^2 \sum_{n=-\infty}^{\infty} \left|\tilde{f}(\omega_n)\right|^2. \tag{12}$$

2 Fourier Transform

Definition 4 [Fourier transform]**:** *Consider a function*

$$f : \mathbb{R} \to \mathbb{C}. \tag{13}$$

We define the Fourier transform (FT) $\mathcal{F}$ *of* f *and the inverse Fourier transform* $\mathcal{F}^{-1}$ *as*

$$\mathcal{F}[f(t)] = \tilde{f}(\nu) = \int_{-\infty}^{\infty} dt\, f(t) e^{-i2\pi\nu t}, \tag{14a}$$

$$\mathcal{F}^{-1}\left[\tilde{f}(\nu)\right] = f(t) = \int_{-\infty}^{\infty} d\nu\, \tilde{f}(\nu) e^{i2\pi\nu t}, \tag{14b}$$

where

$$\tilde{f} : \mathbb{R} \to \mathbb{C} \tag{15}$$

is the Fourier conjugate of f acting on the reciprocal space.

Remark 2.1*:* Hermiticity. If $f(t)$ is real, then $\tilde{f}(\nu)$ is Hermitian, meaning $\tilde{f}(-\nu) = \tilde{f}^*(\nu)$. Vice versa.

Remark 2.2*:* We do not specify whether we define FT on L^1, L^2, or $\mathcal{S}$. However, we will talk about FT on $\mathcal{S}^*$ when we discuss the FT of Dirac delta functions.

Theorem 2 [Parseval-Plancherel]**:**

$$\int_{-\infty}^{\infty} dt \left|f(t)\right|^2 = \int_{-\infty}^{\infty} d\nu \left|\tilde{f}(\nu)\right|^2. \tag{16}$$

This theorem also means FT is unitary on L^2.

Proof. This can be proved by using the integral representation of the Dirac delta function. See Sec. A.2.2. This theorem can also be proved using the Wiener-Khinchin theorem 4 by taking the limit $t' \to 0$. □

Remark 2.3*:* Higher dimensions. The Fourier transform in 3D is straightforward. The Parseval-Plancherel theorem will look the same.

Theorem 3 [Convolution]**:** *The Fourier transform of the convolution of $f(t)$ and $g(t)$, denoted by $f(t) * g(t)$, is the product of the Fourier transforms of $f(t)$ and $g(t)$, i.e.,*

$$\mathcal{F}[f(t) * g(t)] = \mathcal{F}[f(t)]\mathcal{F}[g(t)]. \tag{17}$$

Equivalently, the convolution of $f(t)$ and $g(t)$ is the inverse Fourier transform of $\tilde{f}(\nu)\tilde{g}(\nu)$, i.e.,

$$f(t) * g(t) = \mathcal{F}^{-1}\left[\tilde{f}(\nu)\tilde{g}(\nu)\right], \tag{18}$$

or explicitly

$$\int_{-\infty}^{\infty} dt'\, f(t')g(t - t') = \int_{-\infty}^{\infty} d\nu\, \tilde{f}(\nu)\tilde{g}(\nu)e^{i2\pi\nu t}. \tag{19}$$

Theorem 4 [Wiener-Khinchin]**:** *The Fourier transform of the autocorrelation function of $f(t)$, denoted by $a_f(t)$, is the power spectrum $S(\nu) = \left|\tilde{f}(\nu)\right|^2$ of $f(t)$, i.e.,*

$$\mathcal{F}[a_f(t)] = S(\nu). \tag{20}$$

Equivalently, the autocorrelation function of $f(t)$ is the inverse Fourier transform of $S(\nu)$, i.e.,

$$a_f(t) = \mathcal{F}^{-1}[S(\nu)], \tag{21}$$

or explicitly,

$$\int_{-\infty}^{\infty} dt'\, f^*(t')f(t + t') = \int_{-\infty}^{\infty} d\nu \left|\tilde{f}(\nu)\right|^2 e^{i2\pi\nu t}, \tag{22}$$

We often normalize the power spectrum when we only care about the relative magnitudes of the spectrum. Notice that when f is even, the Wiener-Khinchin theorem can be derived from the Convolution theorem. However, in general, they treat different properties of the function f.

Remark 2.4*:* Angular variable. Let

$$\omega = 2\pi\nu. \tag{23}$$

Then we can define the angular Fourier transform and the inverse transform in Eqs. (14a)–(14b).

Definition 5 [Fourier transform, angular]**:**

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} dt\, f(t) e^{-i\omega t}, \tag{24a}$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, \tilde{f}(\omega) e^{i\omega t}. \tag{24b}$$

Remark 2.5*:* Rescaling of $\tilde{f}(\omega)$. [1] The Fourier transform $\tilde{f}(\omega)$ defined in Eq. (24a) can be rescaled by introducing $h \in \mathbb{C}$ such that

$$\tilde{f}(\omega) = h \int_{-\infty}^{\infty} dt\, f(t) e^{-i\omega t}, \tag{25a}$$

$$f(t) = \frac{1}{2\pi h} \int_{-\infty}^{\infty} d\omega\, \tilde{f}(\omega) e^{i\omega t}. \tag{25b}$$

This **will** change the prefactors in the Parseval-Plancherel theorem, the convolution theorem, and the Wiener-Khinchin theorem. The most commonly used rescaling is either $h = 1$ or $h = \dfrac{1}{\sqrt{2\pi}}$ (symmetric prefactors). To avoid the problem of 2π once and for all, stick to the non-angular Fourier transform.

3 Discrete time Fourier transform

Definition 6 [discrete time Fourier transform]**:** *Consider a continuous function $f(t)$ on $\mathbb{R}$. Suppose we sample $f(t)$ with a period of T and get a sequence of sampled values defined by $x_n = f(nT + t_0)$ for $n \in \mathbb{Z}$ and $t_0 \in \mathbb{R}$. Without loss of generality, we can let $t_0 = 0$ and have*

$$x_n = f(nT). \tag{26}$$

Then the discrete time Fourier transform (DTFT) of the sequence $\{x_n\}$ and its inverse transform are defined by

$$X(\nu) = \sum_{n=-\infty}^{\infty} x_n e^{i2\pi n\nu T}, \tag{27a}$$

$$x_n = T \int_{\frac{1}{T}} d\nu\, X(\nu) e^{-i2\pi n\nu T}. \tag{27b}$$

Here $x_n \in \mathbb{C}$, and $X : \mathbb{R} \to \mathbb{C}$.

Remark 3.1*:* Relation between DTFT and FS. Finding the DTFT of a sampled sequence is equivalent to finding the original function given its Fourier coefficients.

Comparing DTFT in Eqs. (27a)–(27b) with FS in Eqs. (4a)–(4b). we immediately see that they are same if we make the following identifications:

$$\begin{aligned} t &\leftrightarrow \nu, \\ T &\leftrightarrow \frac{1}{T}, \\ f(t) &\leftrightarrow X(\nu), \\ \tilde{f}_n &\leftrightarrow x_n. \end{aligned} \tag{28}$$

Remark 3.2*:* Relation between DTFT and FT. Finding the DTFT of a sampled sequence of $f(t)$ is equivalent to finding the FT of the sampled function $f_T(t)$ which is defined as

$$f_T(t) \equiv f(t) Ш_T(t), \tag{29}$$

[1] This rescaling is because we have homogeneous equations.

where the properties of the Dirac comb function is given in Appx. B.

To see this, using the sampling property of $Ш_T(t)$ given in Eq. (87) we find the Fourier transform of $f_T(t)$ by

$$\begin{aligned}\mathcal{F}[f_T(t)] &= \mathcal{F}[f(t)Ш_T(t)] \qquad (30)\\ &= \mathcal{F}\left[\sum_{n=-\infty}^{\infty} f(nT)\delta(t-nT)\right]\\ &= \sum_{n=-\infty}^{\infty} f(nT)\mathcal{F}[\delta(t-nT)]\\ &= \sum_{n=-\infty}^{\infty} f(nT)e^{-i2\pi n\nu T}\\ &= \sum_{n=-\infty}^{\infty} x_n e^{i2\pi n\nu T}, \qquad (31)\end{aligned}$$

which agrees with the DTFT of x_n defined in Eq. (27a).

Remark 3.3*:* Dick effect. Using Eq. (30) and the Fourier series of $Ш_T(t)$ in Eq. (90), we can also get the relation between $\mathcal{F}[f_T(t)]$ and $\mathcal{F}[f(t)]$.

$$\begin{aligned}\mathcal{F}[f_T(t)] &= \mathcal{F}[f(t)Ш_T(t)]\\ &= \mathcal{F}\left[f(t)\frac{1}{T}\sum_{n=-\infty}^{\infty} e^{i2\pi nt/T}\right]\\ &= \frac{1}{T}\sum_{n=-\infty}^{\infty}\mathcal{F}\left[f(t)e^{i2\pi nt/T}\right] \qquad (32)\end{aligned}$$

or

$$\tilde{f}_T(\nu) = \frac{1}{T}\sum_{n=-\infty}^{\infty}\left[\tilde{f}\left(\nu-\frac{n}{T}\right)\right], \qquad (33)$$

which can be interpreted as the Dick effect.

Theorem 5 [Poisson summation]**:**

$$\sum_{m=-\infty}^{\infty} f(m) = \sum_{n=-\infty}^{\infty}\tilde{f}(n). \qquad (34)$$

Proof. Using Eqs. (31) and (33), we have

$$\sum_{m=-\infty}^{\infty} f(mT)e^{i2\pi m\nu T} = \frac{1}{T}\sum_{n=-\infty}^{\infty}\left[\tilde{f}\left(\nu-\frac{n}{T}\right)\right]. \qquad (35)$$

Let $T=1$ and $\nu=0$. Then we have Eq. (34). □

Theorem 6 [Nyquist–Shannon sampling]**:** *If a function $f(t)$ contains no frequencies higher than B, it is completely determined by a sampling of period $T \leq \frac{1}{2B}$.*

Remark 3.4*:* Aliasing. When the sampling period T is chosen, by the Nyquist–Shannon sampling theorem, the function $f(t)$ can be uniquely determined if it has a maximum frequency $B_m = \frac{1}{2T}$. However, if $B > B_m$, then by Eq. (33), the RHS summations of the spectra will overlap. The frequency components of the continuous function in the overlap region will be lost. In other words, the resulting overlapped spectra becomes an *alias* of the real spectra. This is called *aliasing*.

4 Discrete Fourier transform

Definition 7 [discrete Fourier transform]**:** *Consider a continuous function $f(t)$ on $\mathbb{R}$. Suppose we sample $f(t)$ with a period of T and get a sequence of sampled values defined by $x_n = f(nT+t_0)$ for $n \in \{0, 1, 2, \cdots, N\}$ with $N \in \mathbb{Z}$ and $t_0 \in \mathbb{R}$. Without loss of generality, we can let $t_0 = 0$ and have*

$$x_n = f(nT). \tag{36}$$

Then the discrete Fourier transform (DFT) of the sequence $\{x_n\}$ and its inverse transform are defined by

$$X_m = \sum_{n=0}^{N-1} x_n e^{-i2\pi nm/N}, \tag{37a}$$

$$x_n = \frac{1}{N} \sum_{m=0}^{N-1} X_m e^{i2\pi nm/N}. \tag{37b}$$

Here $x_n, X_m \in \mathbb{C}$.

Remark 4.1*:* Hermiticity. If x_n is real, then X_m is Hermitian, meaning $X_{N-m} = X_m^*$. Vice versa.

The DFT can be taken as the discrete version of FS, FT, and DTFT in different ways. It is very important in engineering but rarely used in physics derivations. The Fast Fourier transform (FFT) algorithm is developed based on DFT. We will not discuss this unless in need.

5 Relations between FS, FT, DTFT, and DFT

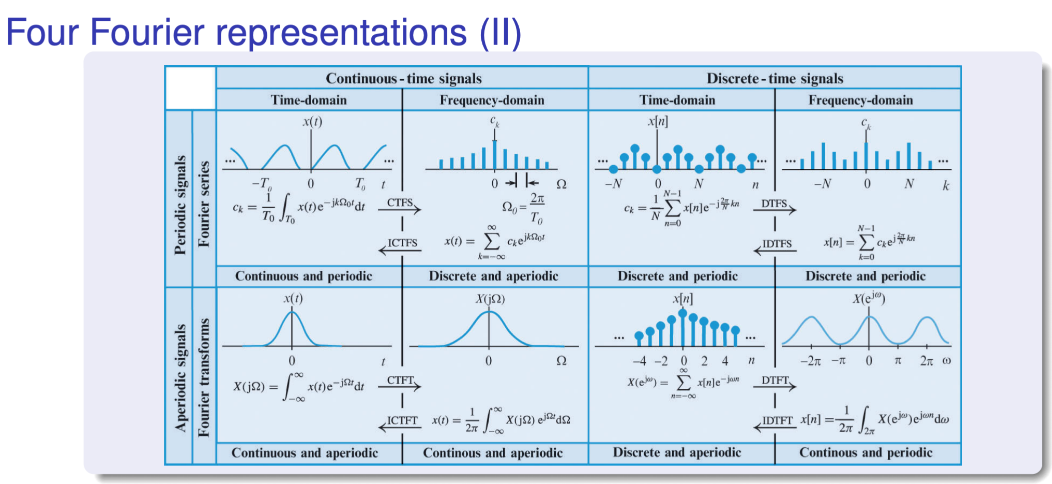

Figure 1: Relations between FS, FT, DTFT, and DFT. Source is here.

A Appendix: Dirac Delta Function

We try to define the Dirac delta function a little more rigorously than in common quantum mechanics textbooks. If one finds the math tedious, just remember that the so-called "Dirac delta function" is not really a function, but a tempered distribution, hence all its weird behaviors in Fourier analysis.

A.1 Tempered distributions

Definition 8 [tempered distribution]**:** *The topological dual space of a Schwartz space* [2] $\mathcal{S}$*, denoted by* $\mathcal{S}^*$*, is the space of continuous linear functionals* $T : \mathcal{S} \to \mathbb{C}$*. Elements of* $\mathcal{S}^*$ *are called the tempered distributions.*

Remark A.1*:* Notations. There are two other commonly used notations of $T(f)$ for $f \in \mathcal{S}$.

1. Inner product. The action of tempered distribution T on Schwartz function f can be represented by a duality pairing, which allows for defining the inner product

$$T(f) = \langle T, f \rangle \,. \tag{38}$$

2. Integral. It is computationally convenient to write tempered distribution T using an integration as if T is not a functional on f but a function on x, i.e.,

$$T(f) = \int T(x) f(x) \, dx \,. \tag{39}$$

   However, keep in mind that $T(x)$ is only meaningful when put inside the integration.

Remark A.2*:* Derivatives of tempered distributions. The nth derivative of a tempered distribution T is the tempered distribution $\partial^n T$ defined by

$$\langle \partial^n T, f \rangle = (-1)^n \langle T, \partial^n f \rangle \tag{40}$$

or

$$\int T^{(n)}(x) f(x) \, dx = (-1)^n \int T(x) f^{(n)}(x) \, dx \tag{41}$$

by repeatedly using integral by parts.

Remark A.3*:* Fourier transform of tempered distributions. Let $f \in \mathcal{S}$. We know the map $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is continuous, one-to-one, and onto, where $\mathcal{F}$ is defined in Eq. (14a).

The FT of a tempered distribution T is the tempered distribution $\mathcal{F}T$ defined by

$$\langle \mathcal{F}T, f \rangle = \langle T, \mathcal{F}f \rangle \tag{42}$$

or

$$\int \mathcal{F}[T(t)] f(\nu) \, d\nu = \int T(\nu) \mathcal{F}[f(t)] \, d\nu \,. \tag{43}$$

The inverse FT $\mathcal{F}^{-1}$ is defined similarly. The maps $\mathcal{F}, \mathcal{F}^{-1} : \mathcal{S}^* \to \mathcal{S}^*$ are both continuous, one-to-one, and onto.

[2] A Schwartz space $\mathcal{S}(\mathbb{R}^n)$ consists of all $C^\infty(\mathbb{R}^n)$ smooth functions $f(x)$ whose derivatives, including themselves, decay faster than any polynomial as $|x| \to \infty$.

A.2 Dirac delta function

A.2.1 Definition

Definition 9 [Dirac delta function]**:** *The Dirac delta function supported at* $x_0 \in \mathbb{R}^n$ *is a tempered distribution*

$$\delta_{x_0} : \mathcal{S} \to \mathbb{C} \tag{44}$$

such that

$$\langle \delta_{x_0}, f \rangle = f(x_0), \tag{45}$$

or, in the integration notation,

$$\int \delta(x - x_0) f(x)\, dx = f(x_0). \tag{46}$$

Remark A.4*:* The definition of the Dirac delta function in Eq. (46) is the familiar form in physics. Below we list some useful properties.

1. Derivatives of $\delta(x - x_0)$. By Eq. (41),

$$\int \delta^{(n)}(x - x_0) f(x)\, dx = (-1)^n f^{(n)}(x_0) \tag{47}$$

   or

$$\delta^{(n)}(x - x_0) = (-1)^n \delta(x - x_0) \frac{d^n}{dx^n}. \tag{48}$$

   Easy to see,

$$\delta^{(n)}(-x) = (-1)^n \delta^{(n)}(x). \tag{49}$$

2. Fourier transform. By Eq. (43),

$$\int \mathcal{F}[\delta(t - t_0)] f(\nu)\, d\nu = \int \delta(t - t_0) \mathcal{F}[f(\nu)]\, dt = \tilde{f}(t_0) = \int_{-\infty}^{\infty} f(\nu) e^{-i2\pi\nu t_0}\, d\nu \tag{50}$$

   or

$$\mathcal{F}[\delta(t - t_0)] = e^{-i2\pi\nu t_0}. \tag{51}$$

   The inverse transform is

$$\delta(t - t_0) = \mathcal{F}^{-1}\left[e^{-i2\pi\nu t_0}\right] = \int e^{i2\pi\nu(t - t_0)}\, d\nu\,. \tag{52}$$

   Therefore we get the Fourier representation of the Dirac delta function $\delta(t)$ as

$$\delta(t) = \int_{-\infty}^{\infty} e^{\pm i2\pi\nu t}\, d\nu\,. \tag{53}$$

   Similarly we can find the Fourier representation of the derivatives of Dirac delta function $\delta^{(n)}(t)$ as

$$\delta^{(n)}(t) = \int_{-\infty}^{\infty} (i2\pi)^n \nu^n e^{i2\pi\nu t}\, d\nu\,. \tag{54}$$

   Thus the FT of monomials can be understood as proportional to derivatives of the Dirac delta function. In summary we have for $n \in \mathbb{N}$,

$$\mathcal{F}\Big[\delta^{(n)}(t)\Big] = (i2\pi)^n \nu^n, \tag{55}$$

$$\mathcal{F}[t^n] = \left(\frac{i}{2\pi}\right)^n \delta^{(n)}(\nu), \tag{56}$$

   where we have taken the convention of FT as in Eqs. (14a)–(14b).

3. Composition.

$$\delta[f(x)] = \sum_j \frac{\delta(x - x_j)}{|f'(x_j)|} \tag{57}$$

where x_j's are the roots of $f(x)$.

A.2.2 Delta function in different settings

1. Fourier series. Let $f(t)$ be a periodic function with period T, with $\omega_n = \dfrac{2\pi n}{T}$, $n \in \mathbb{Z}$. Then we have

$$\delta_{nn'} = \frac{1}{T}\int_T dt\, e^{\pm i(\omega_n - \omega_{n'})t}, \tag{58}$$

$$\delta(t - t') = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{\pm i\omega_n (t-t')}, \tag{59}$$

2. Fourier transform. Let $f(t)$ be defined on $\mathbb{R}$, and $\omega \in \mathbb{R}$. Then we have

$$\delta(\omega - \omega') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt \;\; e^{\pm i(\omega - \omega')t}, \tag{60}$$

$$\delta(t - t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \;\; e^{\pm i\omega(t-t')}. \tag{61}$$

A.3 Other tempered distributions

Tempered distributions can be *regular* or *singular*. Regular distributions can be taken as a generalization of locally integrable functions with polynomial growth. Singular distributions can have a nonintegrable singularity or grows faster than a polynomial, such that its integral against a Schartz function need not be finite. [3]

The Dirac delta function is a singular distribution. Other examples of singular tempered distributions that are used in physics are the principal value distribution and the finite part distribution.

Definition 10 [principal value distribution]**:** *The function* $\dfrac{1}{x} : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ *has a nonintegrable singularity at* 0*. We define the principal value distribution as the singular tempered distribution*

$$\text{P.V.}\frac{1}{x} : \mathcal{S}(\mathbb{R}) \to \mathbb{C} \tag{62}$$

such that

$$\text{P.V.}\frac{1}{x}(\varphi) = \lim_{\varepsilon\to 0^+}\left[\int_{-\infty}^{-\varepsilon} dx\,\frac{\varphi(x)}{x} + \int_{\varepsilon}^{\infty} dx\,\frac{\varphi(x)}{x}\right]. \tag{63}$$

Definition 11 [finite part distribution]**:** *The function* $\dfrac{1}{|x|^2} : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}$ *has a nonintegrable singularity at* 0 *when* $n < 3$*. We define the finite part distribution as the singular tempered distribution*

$$\text{F.P.}\frac{1}{|x|^2} : \mathcal{S}(\mathbb{R}^2) \to \mathbb{C} \tag{64}$$

such that

$$\text{F.P.}\frac{1}{|x|^2}(\varphi) = \int_{|x|<1} \frac{\varphi(x) - \varphi(0)}{|x|^2}\,dx + \int_{|x|>1}\frac{\varphi(x)}{|x|^2}\,dx\,. \tag{65}$$

We can show

$$\frac{d}{dx}\text{P.V.}\frac{1}{x} = -\text{F.P.}\frac{1}{|x|^2}. \tag{66}$$

[3] Not the most rigorous definition.

A.4 Sokhotski-Plemelj theorem

Theorem 7 [Sokhotski-Plemelj, real line]**:** *The Sokhotski-Plemelj theorem along the real line is*

$$\lim_{\varepsilon\to 0}\frac{1}{x-x_0\pm i\varepsilon}=\text{P.V.}\frac{1}{x-x_0}\mp i\pi\delta(x-x_0). \tag{67}$$

We do not show the full version here.

Proof. The proof requires using the definition of the FT of tempered distributions. [4] □

The Sokhotski-Plemelj theorem is equivalent to finding the Fourier transform of the Heaviside step function, which is defined as

$$\Theta(x)=\begin{cases}1, & x>0\\ 0, & x\leq 0\end{cases}. \tag{68}$$

Taking $\Theta(x)$ as a tempered distribution, we have

$$\begin{aligned}\tilde{\Theta}(\omega)&=\int_{-\infty}^{\infty}dt\,e^{-i\omega t}\Theta(t)\\ &=\int_{0}^{\infty}dt\,e^{-i\omega t}\\ &=\lim_{\varepsilon\to 0^+}\int_{0}^{\infty}dt\,e^{-(\varepsilon+i\omega)t}\\ &=\lim_{\varepsilon\to 0^+}\frac{1}{\epsilon+i\omega}\\ &=\lim_{\varepsilon\to 0^+}-i\frac{1}{\omega-i\epsilon} \qquad (69)\\ &=\text{P.V.}\frac{1}{i\omega}+\pi\delta(\omega), \qquad (70)\end{aligned}$$

where the last step uses the the Sokhotski-Plemelj theorem in Eq. (67). The proof can go either way, both requires working out the FT of the distribution $\frac{1}{\omega-i\epsilon}$.

The relation

$$\int_{0}^{\infty}dt\,e^{-i\omega t}=\text{P.V.}\frac{1}{i\omega}+\pi\delta(\omega), \tag{71}$$

which is commonly used in quantum optics, is sometimes referred to as the *Heitler function*.

A.5 Fourier transform of $1/x$

From Eq. (70), we can also derive the Fourier transform of $\frac{1}{x}$. Noticing

$$\mathcal{F}[\Theta(t)]=\text{P.V.}\frac{1}{i\omega}+\pi\delta(\omega), \tag{72}$$

$$\mathcal{F}[\Theta(t)]-\frac{1}{2\pi}\mathcal{F}[\pi]=\text{P.V.}\frac{1}{i\omega}, \tag{73}$$

$$\mathcal{F}[2\Theta(t)-1]=-2i\text{P.V.}\frac{1}{\omega}, \tag{74}$$

[4] See here.

we have for angular Fourier transform

$$\mathcal{F}[\mathrm{sgn}(t)] = -2i\mathrm{P.V.}\frac{1}{\omega}, \tag{75}$$

$$\mathcal{F}\left[\mathrm{P.V.}\frac{1}{t}\right] = -i\pi\mathrm{sgn}(\omega), \tag{76}$$

where

$$\mathrm{sgn}(x) = \begin{cases} 1, & x > 0 \\ -1, & x \leq 0 \end{cases}. \tag{77}$$

For the nonangular Fourier transform,

$$\mathcal{F}[\mathrm{sgn}(t)] = -\frac{i}{\pi}\mathrm{P.V.}\frac{1}{\nu}, \tag{78}$$

$$\mathcal{F}\left[\mathrm{P.V.}\frac{1}{t}\right] = -i\pi\mathrm{sgn}(\nu). \tag{79}$$

Equations. (78)–(79) can be taken as the generalized version of Eqs. (14a)–(14b) for $n = -1$, where we interpret $\delta^{(-1)}$ as an integration of the δ-function, Θ. However, a constant needs to be chosen for this integration so that we can get the correct results.

A.6 Fourier transform of $1/|x|$

The key idea is to notice the distribution

$$\mathcal{F}[\ln|t|] = -\frac{1}{|\nu|} \tag{80}$$

on the Schwartz space $\{\varphi \in \mathcal{S}(\mathbb{R}) \mid \varphi(0) = 0\}$, which involves nontrivial calculations [5]. Thus

$$\mathcal{F}\left[\frac{1}{|t|}\right] = -\ln|\nu|. \tag{81}$$

B Appendix: Dirac comb function

Definition 12 [Dirac comb function]**:** *The Dirac comb function* [6] *is defined as*

$$Ш(t) = \sum_{n=-\infty}^{\infty} \delta(t-n). \tag{82}$$

The Dirac comb function is important in Fourier analysis. Below we list its properties.

1. Periodicity. Obviously, $Ш(t)$ has unit period, i.e.,

$$Ш(t) = Ш(t-n) \tag{83}$$

for $n \in \mathbb{Z}$.

One can modulate the period of $Ш(x)$ by

$$Ш\left(\frac{t}{T}\right) = T\sum_{n=-\infty}^{\infty} \delta(t-nT). \tag{84}$$

[5]See here.

[6]Other names of the Dirac comb function include the shah function, the impulse train, the sampling function, the sampling symbol, and the replicate symbol.

Sometimes we define the Dirac comb function with period T as

$$\text{Ш}_T(t) \equiv \frac{1}{T}\text{Ш}\left(\frac{t}{T}\right) = \sum_{n=-\infty}^{\infty} \delta(t - nT). \tag{85}$$

2. Parity. $\text{Ш}(t)$ is even, i.e.,

$$\text{Ш}(t) = \text{Ш}(-t). \tag{86}$$

   Obviously $\text{Ш}_T(t)$ is also even.

3. Sampling. Given a continuous function $f(t)$ on $\mathbb{R}$, its sampling of period T is given by

$$\text{Ш}_T(t)f(t) = \sum_{n=-\infty}^{\infty} f(nT)\delta(t - nT). \tag{87}$$

   Proof. By definition,

$$\text{Ш}_T(t)f(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - nT). \tag{88}$$

   Equation (88) is meaningful only when put inside an integral [7] Then Eq. (87) is obvious. □

4. Replicating. Given a continuous function $f(t)$ on $\mathbb{R}$, we can find the convolution

$$\text{Ш}_T(t) * f(t) = \sum_{n=-\infty}^{\infty} f(t - nT). \tag{89}$$

5. Fourier series. Since $\text{Ш}_T(t)$ is periodic in T, its Fourier series is well defined. By Eq. (4a), we have

$$\text{Ш}_T(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{i2\pi nt/T}. \tag{90}$$

   For $T = 1$,

$$\text{Ш}(t) = \sum_{n=-\infty}^{\infty} e^{i2\pi nt}. \tag{91}$$

6. Fourier transform. The Fourier transform of $\text{Ш}_T(t)$ is found by Eq. (14a), i.e.,

$$\mathcal{F}[\text{Ш}_T(t)] = \frac{1}{T}\text{Ш}_{\frac{1}{T}}(\nu), \tag{92}$$

   where ν is the frequency if t represents time. For $T = 1$,

$$\mathcal{F}[\text{Ш}(t)] = \text{Ш}(\nu), \tag{93}$$

   i.e., the Fourier transform of the Dirac comb function is itself.

[7] I learned this in Freshman year. I guess we can use the tempered distribution language to formalize this but I am lazy.

C Appendix: Uncertainty principle

C.1 Uncertainty principle in c-number

Consider a function $f(x)$ and its Fourier transform $\tilde{f}(k)$ defined by Eq. (24a). Suppose that we can interpret $f(x)$ and $\tilde{f}(k)$ as amplitudes distributions for x and k respectively. Then we can define the variances σ_x^2 and σ_k^2 such that

$$\sigma_x^2 = \int dx |f(x)|^2 x^2, \tag{94}$$

$$\sigma_k^2 = \int dk \big|\tilde{f}(k)\big|^2 k^2. \tag{95}$$

Here we have chosen the origin such that $\bar{x} = \bar{k} = 0$ without loss of generality. We want to find a lower bound for $\sigma_x \sigma_k$ following from the fact that f and $\tilde{f}$ are Fourier transforms of each other.

Let $A(x) = xf(x)$, $\tilde{B}(k) = k\tilde{f}(k)$. The Parseval-Plancherel theorem tells us

$$\sigma_k^2 = \int dk \big|\tilde{B}(k)\big|^2 = \int dx |B(x)|^2, \tag{96}$$

where $B(x)$ is the inverse Fourier transform of $\tilde{B}(k)$ and has the form

$$\begin{aligned}
B(x) &= \frac{1}{\sqrt{2\pi}} \int dk\ k\tilde{f}(k) e^{ikx} \\
&= \frac{1}{2\pi} \int dk\ e^{ikx} k \int dx' f(x') e^{-ikx'} \\
&= -i \frac{1}{2\pi} \int dk\ e^{ikx} \int dx' \frac{df(x')}{dx'} e^{-ikx'} \\
&= -i \int dx' \frac{df(x')}{dx'} \delta(x - x') \\
&= -i \frac{d}{dx} f(x),
\end{aligned} \tag{97}$$

where we have used the integral by parts and ignored the boundary terms.

Using the notation

$$\langle A|B\rangle = \int dx A^*(x) B(x) \tag{98}$$

for inner products defined on x-space, by Schwarz's inequality, we have

$$\begin{aligned}
\sigma_x^2 \sigma_k^2 &= \langle A|A\rangle \langle B|B\rangle \\
&\geq |\langle A|B\rangle|^2 \\
&= \left| \frac{1}{2}(\langle A|B\rangle + \langle B|A\rangle) + \frac{1}{2}(\langle A|B\rangle - \langle B|A\rangle) \right|^2
\end{aligned} \tag{99}$$

We know $\langle A|B\rangle + \langle B|A\rangle$ is always real. However, the inequality (99) cannot be further simplified without further knowledge on the commutator $\langle A|B\rangle - \langle B|A\rangle$. Luckily, for the pair of variables that are Fourier (canonical) conjugates of each other, e.g., $\{x, k\}$, the commutator $\langle A|B\rangle - \langle B|A\rangle$ is always pure imaginary.

Using Eq. (97) we can find

$$\begin{aligned}\langle A|B\rangle - \langle B|A\rangle &= -i\int dx\; xf^*(x)\frac{df(x)}{dx} - i\int dx\;\frac{df^*(x)}{dx}xf(x)\\ &= -i\int dx\; x\frac{d[f^*(x)f(x)]}{dx}\\ &= i\int dx|f(x)|^2\\ &= i. \end{aligned}\tag{100}$$

Since $\langle A|B\rangle - \langle B|A\rangle$ is pure imaginary, we can carry on with Ineq. (99) and get

$$\sigma_x^2\sigma_k^2 \geq \mathrm{Im}\{\langle A|B\rangle\}^2 = \left(\frac{\langle A|B\rangle - \langle B|A\rangle}{2i}\right)^2. \tag{101}$$

Thus we have

$$\sigma_x^2\sigma_k^2 \geq \frac{1}{4} \tag{102}$$

or

$$\sigma_x\sigma_k \geq \frac{1}{2}. \tag{103}$$

Therefore the uncertainty principle is a fundamental fact for two variables that are Fourier transforms of each other.

The equal sign in Ineq. (103) is taken iff $f(x)$ is Gaussian. The proof is shown below.

Proof. The equal sign in Ineq. (99) is taken when $A(x) \propto B(x)$, which follows from the fact that Schwarz's inequality takes equal sign iff A and B are linearly dependent of each other. Then we can get

$$f(x) = c_1 e^{ic_2x^2}, \tag{104}$$

$$A(x) = xf(x), \tag{105}$$

$$B(x) = c_2 f(x). \tag{106}$$

The equality in Ineq. (101) is taken when $\langle A|B\rangle + \langle B|A\rangle = 0$. Using Eqs. (104)–(106) we have

$$\int dx\left[x|f(x)|^2(c_2+c_2^*)\right] = 0 \tag{107}$$

$$c_2 + c_2^* = 0. \tag{108}$$

Therefore c_2 is pure imaginary. The resulting $f(x)$ is of Gaussian form. □

C.2 Uncertainty principle in quantum

The fundamental reason for the uncertainty principle in quantum mechanics is that the conjugate variables are Fourier transforms of each other, thus following the results in Sec. C. This is the general argument for, say, the energy/frequency-time uncertainty as energy eigenstates have wavefunctions $\propto e^{-iEt/\hbar}$. Below we briefly prove the uncertainty principle in quantum language. Note the similarity between Secs. C.2 and C.

Let us take the example of two arbitrary Hermitian operators $\hat{A}$ and $\hat{B}$. Given a state $|\psi\rangle$, the mean square uncertainty is defined as

$$\sigma_A^2 = \left\langle\psi\middle|\hat{A}^2\psi\right\rangle = \left\langle\hat{A}\psi\middle|\hat{A}\psi\right\rangle = \langle\psi_A|\psi_A\rangle = \int dx\,|\psi_A(x)|^2 \tag{109}$$

$$\sigma_A^2 = \left\langle\psi\middle|\hat{B}^2\psi\right\rangle = \left\langle\hat{B}\psi\middle|\hat{B}\psi\right\rangle = \langle\psi_B|\psi_B\rangle = \int dx\,|\psi_B(x)|^2. \tag{110}$$

Here we have chosen the origin such that $\left\langle \hat{A} \right\rangle = \left\langle \hat{B} \right\rangle = 0$ without loss of generality. We want to find a lower bound for $\sigma_A \sigma_B$ following from the fact that $|\psi\rangle$ may or may not simultaneously be the eigenvector of $\hat{A}$ and $\hat{B}$.

Using the Schwartz inequality, we have

$$\sigma_A^2 \sigma_B^2 = \langle \psi_A | \psi_A \rangle \langle \psi_B | \psi_B \rangle \tag{111}$$

$$\geq |\langle \psi_A | \psi_B \rangle|^2 \tag{112}$$

$$= \left| \langle \psi | \hat{A}\hat{B} | \psi \rangle \right|^2 \tag{113}$$

$$= \left| \frac{1}{2} \langle \psi | \left[\hat{A}, \hat{B} \right]_+ | \psi \rangle + \frac{1}{2} \langle \psi | \left[\hat{A}, \hat{B} \right] | \psi \rangle \right|^2 . \tag{114}$$

We know that the anticommutator $\left[\hat{A}, \hat{B} \right]_+$ is Hermitian so $\langle \psi | \left[\hat{A}, \hat{B} \right]_+ | \psi \rangle$ is always real. Suppose we assume

$$\left[\hat{A}, \hat{B} \right] = i\hat{H} \tag{115}$$

where $\hat{H}$ is a Hermitian operator, then this implies the commutator $\langle \psi | \left[\hat{A}, \hat{B} \right] | \psi \rangle$ is pure imaginary. Therefore we have

$$\sigma_A^2 \sigma_B^2 \geq \mathrm{Im} \left\{ \langle \psi | \hat{A}\hat{B} | \psi \rangle \right\}^2 = \left(\frac{1}{2i} \langle \psi | \left[\hat{A}, \hat{B} \right] | \psi \rangle \right)^2 = \frac{1}{4} \left\langle \hat{H} \right\rangle^2 , \tag{116}$$

or

$$\sigma_A \sigma_B \geq \frac{1}{2} \left\langle \hat{H} \right\rangle . \tag{117}$$

The equality is taken iff $\forall \psi$,

1. $\hat{A} |\psi\rangle = |\psi_A\rangle \propto \hat{B} |\psi\rangle = |\psi_B\rangle$.
2. $\langle \psi | \left[\hat{A}, \hat{B} \right]_+ | \psi \rangle = 0$.

For $\hat{A} = \hat{X}$ and $\hat{B} = \hat{P}$, following similar logic as in Sec. C and the commutation relation $\left[\hat{X}, \hat{P} \right] = i\hbar$, the minimum uncertainty wave function is of Gaussian form.

D Green's function

To be added. P316, 11.10, Applied Analysis, Hunter

E The Central Limit Theorem

p320, 11.12, Applied Analysis, Hunter