Exam Exercises

1. There are three renewal events at the end of a renewal cycle.

① failure renewal before the next inspection comes.

$$\Pr(X+H<\tau) = \Pr(X<\tau-0.5)$$
$$= F_X(\tau-0.5)$$
$$= 1-\exp(-\lambda(\tau-0.5))$$
$$= 1-\exp(-\tau+0.5)$$

② inspection renewal without replacement.

$$\Pr(X>\tau) = 1-F_X(\tau)$$
$$= 1-(1-\exp(-\lambda\tau))$$
$$= \exp(-\tau)$$

③ inspection renewal with replacement

$$\Pr(X<\tau \cap X+H>\tau) = \Pr(X<\tau \cap X>\tau-0.5)$$
$$= \int_{\tau-0.5}^{\tau} f_X(u)\,du$$
$$= \int_{\tau-0.5}^{\tau} \lambda\exp(-\lambda u)\,du$$
$$= \int_{\tau-0.5}^{\tau} \exp(-u)\,du$$

The expected cycle cost $E(CC)$

$$= C_{cm} \underset{①}{Pr(X+H<\tau)} + C_i \underset{②}{Pr(X>\tau)} + (C_i + C_{pm}) \underset{③}{Pr(X<\tau \cap X+H>\tau)}$$

The expected cycle length $E(CL)$

① failure renewal

$\underline{X+H}$ random $\qquad \sum_{t\in(0,\tau)} t\,Pr(X+H=t)$ (expected value discrete version)

$$= \sum_{t\in(0,\tau)} t\,Pr(X=t-0.5)$$

$$= \int_0^\tau t\, f_X(t-0.5)\, dt$$

$$= \int_0^\tau t\, \lambda \exp(-\lambda(t-0.5))\, dt$$

$$= \int_0^\tau t \exp(-(t-0.5))\, dt$$

② inspection renewal without replacement

$\tau \cdot Pr(X>\tau)$ ------ implies $X+H>\tau$

③ inspection renewal with replacement

$\tau \cdot Pr(X<\tau \cap X+H>\tau)$

$$E(CL) = \int_0^\tau t \exp(-(t-0.5))\,dt + \tau\Big(\underbrace{Pr(X>\tau) + Pr(X<\tau \cap X+H>\tau)}_{\text{covers the entire sample space}}\Big)$$

$$= \int_0^\tau t\exp(-(t-0.5))\,dt + \tau\, Pr(X+H>\tau)$$

$$= \int_0^\tau t\exp(-(t-0.5))\,dt + \tau\, Pr(X>\tau-0.5)$$

$$= \int_0^\tau t\exp(-(t-0.5))\,dt + \tau(1-F_X(\tau-0.5))$$

$$= \int_0^\tau t\exp(-(t-0.5))\,dt + \tau(1-(1-\exp(-\lambda(\tau-0.5))))$$

$$= \int_0^\tau t\exp(-(t-0.5))\,dt + \tau\,\cancel{1}\exp(-\cancel{\lambda}(\tau-0.5))$$

$CR(\tau) = \dfrac{E(CC)}{E(CL)}$ --- Expected cost rate

Exam Exercise

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The cdf of first passage time T_c

$$\Pr\{T_c \leq t\} = \Pr\{\theta t \geq c\}$$
$$= \Pr\{\theta \geq \frac{c}{t}\}$$
$$= 1 - F_\theta(\frac{c}{t})$$

Since, $\theta \sim$ Exponential $(\lambda = 1)$

$$\Pr\{T_c \leq t\} = 1 - (1 - \exp(-\lambda \frac{c}{t}))$$
$$= \cancel{1}\exp(-\frac{4000}{t})$$

$$f_{T_c}(t) = \frac{d F_{T_c}(t)}{dt}$$
$$= \exp(-\frac{4000}{t})(+\frac{4000}{t^2})$$

The conditional first passage time T_H, given that. ~~$T_{3500} = 4000$~~ $T_{4000} = 3500$

~~$\theta 3500$~~

$\theta 3500 = 4000$

$\therefore \theta = \frac{4000}{3500}$ this condition specifies θ.

$H = 5000 = \theta T_H = \frac{4000}{3500} T_H$

$\therefore T_H = 5000 \cdot \frac{3500}{4000}$ this condition thus specifies the T_H

$\therefore$ given the condition $T_{4000} = 3500$ revolutions

T_H is a constant number

$\therefore \Pr\{T_H = 5000 \cdot \frac{3500}{4000} \mid T_{4000} = 3500\} = 1$

$$\underbrace{\Pr\{T_H < t \mid T_c = 3500\}}_{\text{condition}} = \begin{cases} 1 & \text{if } 5000 \cdot \frac{3500}{4000} < t \\ 0 & \text{otherwise} \end{cases}$$

There are two possible renewal evenes.

① failure renewal in a certain inspection interval $((i-1)\tau, i\tau)$

$$Pr(T_H < i\tau \cap (i-1)\tau < T_c < i\tau)$$

$$= \int_{(i-1)\tau}^{i\tau} f_{T_c}(u)\, Pr(T_H < i\tau \mid T_c = u)\, du$$

$$= \int_{(i-1)\tau}^{i\tau} \exp\left(-\frac{4000}{\cancel{t}u}\right) \cdot 4000 \cdot \cancel{t}u^{-2}\, Pr(T_H < i\tau \mid T_c = u)\, du$$

$$Pr(T_H < i\tau \mid T_c = u) = \begin{cases} 1 & \text{if } 5000 \cdot \frac{u}{4000} < i\tau \\ 0 & \text{otherwise} \end{cases}$$

② inspection renewal at time point $i\tau$.

$$Pr(T_H > i\tau \cap (i-1)\tau < T_c < i\tau)$$

$$= \int_{(i-1)\tau}^{i\tau} f_{T_c}(u)\, Pr(T_H > i\tau \mid T_c = u)\, du$$

$$= \int_{(i-1)\tau}^{i\tau} \exp\left(-\frac{4000}{u}\right) \cdot 4000 \cdot u^{-2} \cdot Pr(T_H > i\tau \mid T_c = u)\, du$$

$$Pr(T_H > i\tau \mid T_c = u) = \begin{cases} 1 & \text{if } 5000 \cdot \frac{u}{4000} > i\tau \\ 0 & \text{otherwise.} \end{cases}$$

The expected cycle length $E(CL)$

① failure renewal in a certain inspection interval $((i-1)\tau, i\tau)$

T_H random

$$\sum_{t \in ((i-1)\tau, i\tau)} t \cdot Pr(T_H = t \cap (i-1)\tau < T_C < i\tau)$$

$$= \sum_{t \in ((i-1)\tau, i\tau)} t \sum_{u \in ((i-1)\tau, i\tau)} Pr(T_C = u) \cdot Pr(T_H = t \mid T_C = u)$$

$$= \sum_{t \in ((i-1)\tau, i\tau)} t \int_{(i-1)\tau}^{i\tau} f_{T_C}(u) . Pr(T_H = t \mid T_C = u) \, du$$

where $Pr(T_H = t \mid T_C = u) = \begin{cases} 1 & \text{if } 5000 \frac{u}{4000} = t \\ 0 & \text{otherwise} \end{cases}$

$$\boxed{1 \text{ Or}} = \int_{(i-1)\tau}^{i\tau} f_{T_C}(u) \frac{5000 u}{4000} Pr(T_H < i\tau \mid T_C = u) \, du$$

another expression

where $Pr(T_H < i\tau \mid T_C = u) = \begin{cases} 1 & \text{if } 5000 \frac{u}{4000} < i\tau . \\ 0 & \text{otherwise.} \end{cases}$

② inspection renewal at time point $i\tau$

$$i\tau \cdot Pr(T_H > i\tau \cap (i-1)\tau < T_C < i\tau)$$

$$= i\tau \sum_{u \in ((i-1)\tau, i\tau)} Pr(T_C = u) \, Pr(T_H > i\tau \mid T_C = u)$$

$$= i\tau \int_{(i-1)\tau}^{i\tau} f_{T_C}(u) \, Pr(T_H > i\tau \mid T_C = u) \, du$$

where $Pr(T_H > i\tau \mid T_C = u) = \begin{cases} 1 & \text{if } 5000 \frac{u}{4000} > i\tau . \\ 0 & \text{otherwise} \end{cases}$

The expected cycle cost $E(CC)$

① failure renewal

$$((i-1)C_i + C_{cm}) \cdot Pr(T_H < i\tau \cap (i-1)\tau < T_c < i\tau)$$

② inspection renewal

$$(iC_i + C_{pm}) \cdot Pr(T_H > i\tau \cap (i-1)\tau < T_c < i\tau)$$

Considering all inspection intervals.

$$E(CL) = \sum_{i=1}^{\infty} \Big(\int_{(i-1)\tau}^{i\tau} f_{T_c}(u) \frac{5000u}{4000} Pr(T_H < i\tau \mid T_c = u)\, du$$

$$+ i\tau \int_{(i-1)\tau}^{i\tau} f_{T_c}(u) Pr(T_H > i\tau \mid T_c = u)\, du \Big)$$

$$E(CC) = \sum_{i=1}^{\infty} \Big(((i-1)C_i + C_{cm}) \cdot Pr(T_H < i\tau \cap (i-1)\tau < T_c < i\tau)$$

$$+ (iC_i + C_{pm}) \cdot Pr(T_H > i\tau \cap (i-1)\tau < T_c < i\tau) \Big)$$

$$CR(\tau, C) = \frac{E(CC)}{E(CL)}$$