Lecture notes for 1CM30 - Service Supply Chains for Capital Goods, Spring 2014

Elementary Maintenance Models

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1. Introduction

The availability of capital goods is crucial to keep the primary processes of their owners/users up and running. Consider, aircraft, trains, wafer-steppers, and MRI scanners as examples. The inconvenience of trains and/or aircraft not running when needed (and planned) is a great inconvenience to travellers, but also a significant loss of revenue for airlines and railway operators. Wafer-steppers are used in the bottleneck production step of semi-conductor manufacturing. When a wafer-stepper is down, it causes the standstill of an entire semi-conductor factory. For the case of ASML wafer steppers, these costs are in the order of magnitude of 100000 EURO per hour. The unavailibility of MRI-scanners is perhaps the most costly as it can lead to the loss of human life under some circumstances. All these examples illustrate that keeping capital assets up and running is of critical importance. Unfortunately, keeping capital assets up and running is also a costly business.

Öner et al. (2010) estimate that the costs of maintenance and unavailibility of a capital asset over its lifetime (typically one to several decades) is typically 3 to 4 times the acquisition cost of capital assets. Maintenance (including spare parts) and (unplanned) downtime of capital goods is one of the main culprits in these cost figures. In fact, in 2003 spare part sales and services (mostly maintenance) accounted for 8% of the gross domestic product in the United States (AberdeenGroup, 2003). More recently, US bancorp estimated that the yearly expenditure in the US on spare parts amounts to 700 billion dollars which is 8% of the US gross domestic product (Jasper, 2006).

In these lecture notes, we describe several models to optimize maintenance operations, so let us first briefly consider maintenance operations. Different from regular production operations, maintenance operations are not instigated by demand from an outside customer, but by the need for maintenance of equipment. To perform maintenance, typically several resources are needed, the most important of which are:

- a specialist, mechanic, engineer or other trained professional
- tools and equipment
- spare parts.

In §1.1, we discuss different maintenance strategies and how they instigate the need for maintenance operations (and therefore also the resources mentioned above). The planning difficulties that arise in maintenance operations are discussed in §1.2.

1.1 Maintenance strategies

For the purpose of describing maintenance operations, it is convenient to think of equipment as a collection of interrelated parts. Maintenance operations consist largely in replacing parts of equipment. Maintenance strategies determine when parts or equipment need to be replaced or maintained. Throughout this subsection, we focus on the decision to maintain/replace a part, but our discussion also applies to the decision to maintain/replace equipment. Figure 1[1] gives an overview of maintenance strategies. In this subsection, we follow Figure 1 in discussing different maintenance strategies.

Modificative maintenance concerns interchanging a part with a technically more advanced part in order to make the equipment perform better[2]. This form of maintenance is usually project based and non-recurring. The maintenance strategies that occur most often are preventive and breakdown corrective maintenance. Under a breakdown corrective maintenance strategy, a part is not replaced until it has failed, while under a preventive maintenance strategy, the aim is to replace parts before failure occurs. (Off course, this aim may not always be achieved: A part can break down before its replacement occurs.) Breakdown corrective maintenance is an attractive option for parts that do not wear, such as

[1]Figure 1 was inspired by Figure 4.1 of Coetzee (1997), but has been significantly altered by the author.

[2]Sometimes maintenance is defined as any action that restores equipment to some previous state. Under this definition, modificative maintenance is an oxymoron.

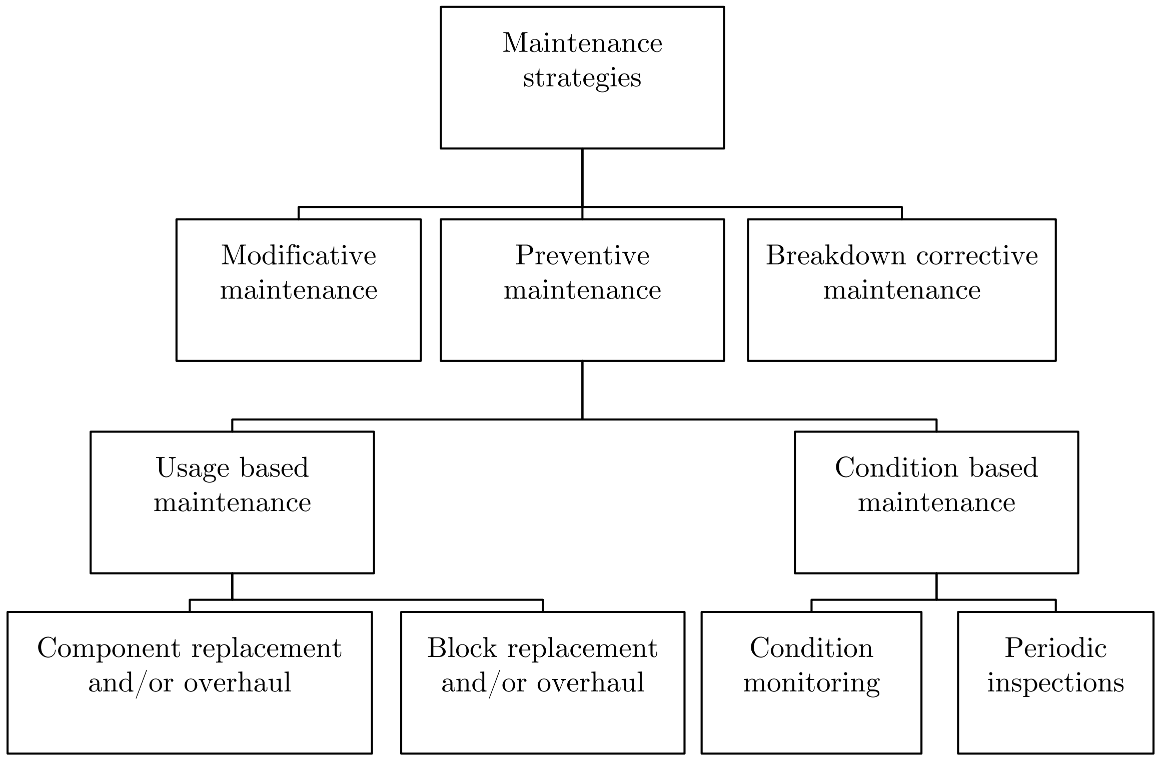

Figure 1: Maintenance strategies

electronics. For parts that do wear, it can be beneficial to follow a preventive maintenance strategy.

Preventive maintenance strategies can be further divided into usage and condition based maintenance. Under usage based maintenance, the total usage of a part is measured and maintenance is conducted when a certain threshold level has been reached. The usage of parts can be measured in many ways depending on the nature of the equipment. Time in the field is perhaps the most common mean to measure usage. For vehicles (e.g., rolling stock), mileage is a common measure of usage. The number of on-off cycles is a measure of usage for equipment that is mainly loaded at the end or beginning of on-off cycles. For example, the number of landings is a measure of usage for the landing gear of an aircraft. Since the usage of equipment is usually scheduled, the moment that maintenance is performed can also be scheduled. If there is a large set-up cost associated with maintenance, it can be beneficial to interchange several parts simultaneously (Block replacement and/or overhaul). Otherwise, maintenance can be performed on a single component (Component replacement and/or overhaul).

In condition based maintenance, the actual condition of a part is gauged and maintenance is conducted based on this. The condition of a part can be measured either periodically during inspections (Periodic inspections) or continuously through a sensor (Condition monitoring). How the condition of equipment is measured depends on the nature of equipment. Below are some examples of how the condition of equipment can be measured:

- The condition of ball-bearings can be measured via the amplitude of vibrations around the bearing (Elwany and Gebraeel, 2008).
- The condition of a metal part can be determined by visually inspecting the number and length of cracks.
- For metal systems with moving parts, the concentration of ferrous parts in the lubrication fluid is measured as an indication of the wear and need for lubrication.
- The condition of a car engine is monitored continuously while driving by the engine-oil temperature gauge.

The need for maintenance can be ascertained periodically during an inspection or at any time in case of condition monitoring.
Which types of maintenance are prevalent for a given piece of equipment depend very much on the technical nature of the equipment involved. For electronics and high-tech equipment, breakdown corrective maintenance is prevalent. For aircraft, rolling stock and other heavy machinery with moving parts, the prevalent maintenance strategies are preventive (both usage and condition based).

1.2 Uncertainty in maintenance operations

Maintenance operations are subject to considerable uncertainty. There is uncertainty both with respect to timing (When will maintenance/replacement be needed?) and content (What parts need maintenance/replacement?). The different maintenance strategies discussed in the previous subsection are organized according to these two uncertainty dimensions in Table 1[3]
Usage based and modificative maintenance can be planned for ahead of time, whereas breakdown corrective maintenance cannot be planned for at all. As a consequence of this,

[3] Table 1 has been inspired by the maintenance box of Stoneham (1998) but has been altered significantly by the author.

Table 1: Maintenance strategies organized by timing and content uncertainty.

<table>
<tr><td colspan="2" rowspan="2"></td><td colspan="2">Timing</td></tr>
<tr><td>known</td><td>unknown</td></tr>
<tr><td rowspan="2">Content</td><td>known</td><td>Usage based or modificative maintenance</td><td>Condition based maintenance (Condition monitoring)</td></tr>
<tr><td>unknown</td><td>Condition based maintenance (Periodic inspections)</td><td>Breakdown corrective maintenance</td></tr>
</table>

the resources needed for usage based and modificative maintenance can be utilized more fully than resources needed for breakdown corrective maintenance.

Condition based maintenance is a hybrid form, in which some but not all uncertainty is taken away relative to breakdown corrective maintenance. Periodic inspections can be planned, and if they lead to maintenance, you know when the maintenance needs to be conducted (right after the inspection). However, the content of the maintenance depends on what is found during the inspection. Under condition monitoring, sensors provide realtime information about the degradation of equipment. The parts that need replacement can then be inferred from the sensor signal. However, degradation usually remains an uncertain process, so that the exact time that maintenance is needed remains unknown.

Remark 1.1 Sometimes the distinction between preventive and corrective maintenance is interpreted as being synonymous to planned and unplanned maintenance. This oversimplification only captures the upper left and lower right boxes of Table 1. Condition based maintenance is a hybrid form between planned and unplanned maintenance that deserves separate attention. $\diamond$

The lecture notes are organized as follows. In §2, we review some elementary results from probability theory, reliability theory and renewal processes. With this mathematical background, we will tackle several maintenance optimization problems in the following sections.

2. Refresher in reliability theory, and renewal processes

In these lecture notes, we will mainly deal with non-negative random variables. Let the random variable T denote the time until failure of some component. If T is a continuous random variable, we denote its distribution by $F_T(t) = \mathbb{P}(T \leq t)$ and assume it has a density $f_T(t) = \frac{d}{dt}F_T(t)$. The reliability of the component at time t is the probability the components survives beyond time t and is denoted by $R(t)$.

$$R(t) = 1 - F_T(t) = 1 - \int_0^t f_T(t')dt' = \int_t^\infty f_T(t')dt'.$$

The mean time to failure ($MTTF$) is just the expectation of T:

$$\mathbb{E}[T] = \int t dF_T(t) = \int_0^\infty t f_T(t)dt = -\int_0^\infty t\frac{dR(t)}{dt}dt = [-tR(t)]_0^\infty + \int_0^\infty R(t)dt = \int_0^\infty R(t)dt. \tag{1}$$

The third equality in (1) holds because $f_T(t) = \frac{dF_T(t)}{dt} = -\frac{dR(t)}{dt}$, the fourth equality follows from integration by parts and the final equality holds because $\lim_{t\to\infty} R(t) = 0$. The variance of T is given by:

$$\begin{aligned}
\mathbf{Var}[T] &= \mathbb{E}\left[(T - \mathbb{E}[T])^2\right] = \mathbb{E}[T^2 - 2T\mathbb{E}[T] + (\mathbb{E}[T])^2] \\
&= \mathbb{E}[T^2] - 2\mathbb{E}[T]\mathbb{E}[T] - (\mathbb{E}[T])^2 = \mathbb{E}[T^2] - (\mathbb{E}[T])^2
\end{aligned} \tag{2}$$

The standard deviation of T is denoted by $\sigma_T = \sqrt{\mathbf{Var}[T]}$. Another measure that is convenient is referred to as the coefficient of variation, c_T, that satisfies:

$$c_T = \sigma_T / \mathbb{E}[T]. \tag{3}$$

Discrete random variables occur naturally in reliability engineering when a system degrades not with time but with the number of on-off cycles. An example of this was given in the introduction: Landing gear of aircraft degrade with the number of landings, not with time or mileage. For all the results in this section, there are straightforward equivalents for discrete random variables. Integrals are replaced by summations in these results

2.1 Failure rates

The distribution function and the density function of the time to failure provide only limited understanding of the physical failure mechanism that cause failures. To gain some understanding, let us consider a time t at which the component has not failed yet. We ask ourselves the following question: How likely is this component to fail in the next (small) time interval of length ε relative to the length of this interval? More formally we would like to know:

$$h(t) = \lim_{\varepsilon \downarrow 0} \mathbb{P}(T \leq t + \varepsilon | T \geq t)/\varepsilon \tag{4}$$

This function can be loosely interpreted as the instantaneous expected number of failures per time unit at time t, and is known as the failure rate, hazard rate (hence the notation $h(t)$) or mortality rate. We will refer to it as the failure rate. Now using basic probability we can write for the probability in (4)

$$\begin{aligned}
\mathbb{P}(T \leq t + \varepsilon | T \geq t) &= \frac{\mathbb{P}(T \leq t + \varepsilon \cap T \geq t)}{\mathbb{P}(T \geq t)} \\
&= \frac{\mathbb{P}(t \leq T \leq t + \varepsilon)}{\mathbb{P}(T \geq t)} \\
&= \frac{F_T(t+\varepsilon) - F_T(t)}{R(t)} \qquad (5)
\end{aligned}$$

Reinserting (5) into (4) we find

$$h(t) = \lim_{\varepsilon \downarrow 0} \frac{F_T(t+\varepsilon) - F_T(t)}{\varepsilon} \frac{1}{R(t)} = \frac{f_T(t)}{R(t)}, \tag{6}$$

where we used that the definition of a derivative and the fact that $f_T(t)$ is the derivative of $F_T(t)$. The identity in (6) is the definition of the failure rate most often used in textbooks.
The failure rate reveals some essential features about the degradation process of components.
If the failure rate is an increasing function of time $\left(\frac{dh(t)}{dt} > 0\right)$, the component degrades over time. If this is the case, we say that the time to failure is *IFR*. (IFR is the abbreviation for increasing failure rate.) Mechanical devices typically have an increasing failure rate.
If the failure rate is a decreasing function of time $\left(\frac{dh(t)}{dt} < 0\right)$, the component becomes more reliable over time (conditional on not having failed already). If this is the case, we say that the time to failure is *DFR*. (DFR, as you might have guessed, is the abbreviation for decreasing failure rate.) Electronic components often have DFR. The reason for this is that electronics are not usually subject to wear (in clean conditions at least) unless there is a manufacturing defect. However, the longer an electronic components has been functioning without problems, the more likely it is that there is no manufacturing defect.
A special case that we will consider separately in 2.2 is where the failure rate is constant (*CFR*), $\left(\frac{dh(t)}{dt} = 0\right)$. (An even more special is when the failure rate alternates from being decreasing and increasing over time like a sine function say. In this course, we will not use such "exotic" distributions.)

There is also a discrete equivalent to the failure rate. If T is a discrete random variable on the (non-negative) integers, let $p_k = \mathbb{P}(T = k)$ and $R_k = \mathbb{P}(T \geq k)$. The discrete equivalent of the failure rate, h_k, is then the probability of the component failing after k cycles conditional on it surviving at least k cycles:

$$h_k = \mathbb{P}(T = k | T \geq k) = \frac{\mathbb{P}(T = k \cap T \geq k)}{\mathbb{P}(T \geq k)} = \frac{p_k}{R_k}. \tag{7}$$

Note that $0 \leq h_k \leq 1$. The discrete hazard rate is perhaps slightly more intuitive because its definition does not involve limits and other devices from calculus.

We close this section by showing how the density and reliability function can be obtained from the failure rate. First observe that $f_T(t) = \frac{d}{dt}F_T(t) = \frac{d}{dt}(1 - R(t)) = -\frac{d}{dt}R(t)$. Inserting this into (6) yields:

$$h(t) = -\frac{1}{R(t)}\frac{dR(t)}{dt}. \tag{8}$$

Now taking (8), multiplying both sides by dt and integrating between 0 and t yields

$$\int_0^t h(u)du = -\int_0^t \frac{1}{R(t)} dR(t) = -\ln(R(t)). \tag{9}$$

Finally, solving (9) for $R(t)$ we find:

$$R(t) = \exp\left(-\int_0^t h(u)du\right). \tag{10}$$

2.2 Commonly used distributions

In this subsection, we give some results on distributions used in maintenance and reliability engineering. Before doing this, we cover some groundwork. The factorial is defined for all positive integer numbers, e.g., $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$. Sometimes it is convenient to work with an extension of the factorial so that it also applies real numbers. The Gamma-function achieves this and is defined as follows:

$$\Gamma(x) = \int_0^\infty \exp(-u)u^{x-1}du, \quad x > 0 \tag{11}$$

The gamma function has the following properties:

- $\Gamma(x + 1) = x\Gamma(x), \quad x > 0$
- $\Gamma(1) = 1$
- $\Gamma(x) = (x - 1)!, \quad x = 1, 2, \cdots$

For values between 1 and 3, the Gamma-function is evaluated in Table 2. Using the first property of the Gamma function, this table can be used to evaluate the Gamma-function for larger real numbers. For example:

$$\Gamma(5.6) = 4.6\Gamma(4.6) = 4.6 \cdot 3.6 \cdot \Gamma(3.6) = 4.6 \cdot 3.6 \cdot 2.6 \cdot \Gamma(2.6) = 4.6 \cdot 3.6 \cdot 2.6 \cdot 1.4296 = 61.55.$$

Table 2: The Gamma-function for arguments between 1 and 3.

	$\Gamma(x+y)$									
$x\backslash y$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	1.0000	0.9514	0.9182	0.8975	0.8873	0.8862	0.8935	0.9086	0.9314	0.9618
2	1.0000	1.0465	1.1018	1.1667	1.2422	1.3293	1.4296	1.5447	1.6765	1.8274

In §2.2.1-§2.2.5, we discuss continuous distribution functions and in §2.2.6, we discuss the Poisson distribution.

2.2.1 Exponential distribution

The exponential distribution is supported on $[0, \infty)$ and has the following density and distribution

$$f(x) = \lambda \exp(-\lambda x), \qquad F(x) = 1 - \exp(-\lambda x)$$

Then mean, variance and coefficient of variation of a exponential random variable X are given by:

$$\mathbb{E}[X] = 1/\lambda, \quad \mathbf{Var}[X] = 1/\lambda^2, \quad c_X = \sqrt{\mathbf{Var}[X]/\mathbb{E}^2[X]} = 1.$$

The exponential distribution is important in operations management and reliability engineering because it has the *lack of memory* property. This property means that the remaining lifetime of a part, has the same distribution as the original lifetime:

$$\begin{aligned}
\mathbb{P}(X > s+t | X > s) &= \mathbb{P}(X > s+t \cap X > s)/\mathbb{P}(X > s) \\
&= \mathbb{P}(X > s+t)/\mathbb{P}(X > s) \\
&= \exp(-\lambda(s+t))/\exp(-\lambda s) \\
&= \exp(\lambda t) \\
&= \mathbb{P}(X > t)
\end{aligned}$$

Another special property of the exponential distribution, which is closely related to the lack of memory property, is that it has a constant failure rate

$$h(t) = \lambda.$$

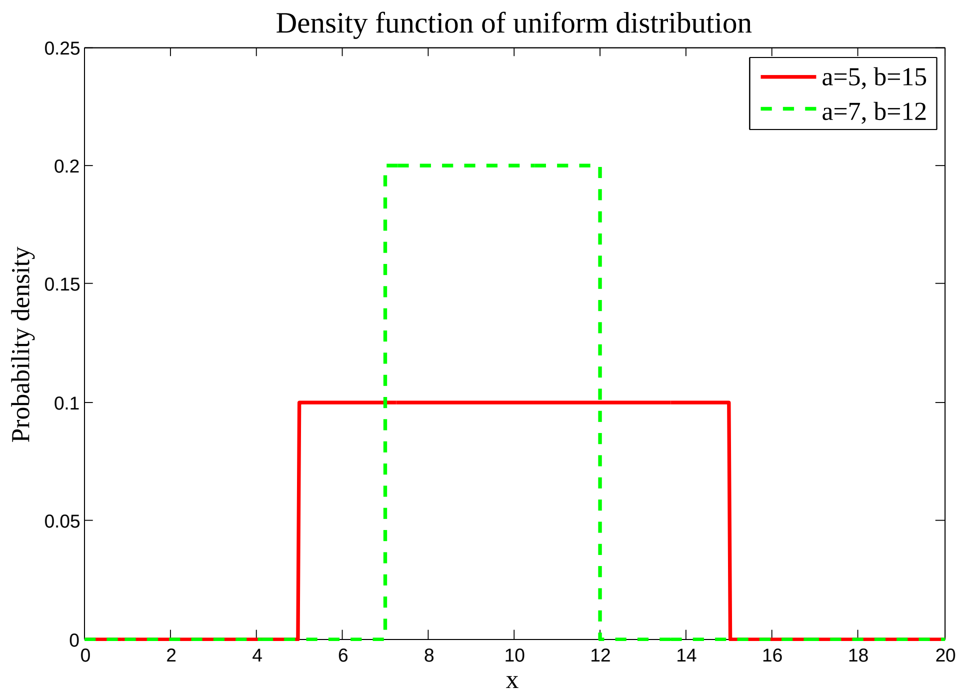

Figure 2: The probability density function of the Uniform distribution for several parameter values.

This means that the exponential distribution is the unique distribution that is both IFR and DFR.

2.2.2 Uniform distribution

The uniform distribution is supported on (a, b) and has density and distribution

$$f(x) = \begin{cases} 1/(b-a), & a \le x \le b; \\ 0, & \text{otherwise.} \end{cases} \qquad F(x) = \begin{cases} (x-a)/(b-a), & a \le x \le b; \\ 0, & x \le a; \\ 1, & b \le x. \end{cases}$$

The reason for the name uniform should be obvious from Figure 2.
If X has a uniform distribution on (a, b), then the mean, variance, and coefficient of variation are given by:

$$\mathbb{E}[X] = (a+b)/2, \quad \mathbf{Var}[X] = (b-a)^2/12, \quad c_X = \frac{b-a}{\sqrt{3}(a+b)} \tag{12}$$

The uniform distribution has an increasing failure rate:

$$h(x) = \begin{cases} 0, & x < a; \\ \frac{1}{b-x}, & a \le x \le b; \end{cases} \tag{13}$$

2.2.3 Erlang distribution

If $E_1, E_2, \cdots E_k$ are i.i.d. exponential random variables with mean λ^{-1}, then $X = \sum_{i=1}^{k} E_i$ has an Erlang[4] distribution with *shape parameter* $k \in \mathbb{N}$ and *scale parameter* $\lambda > 0$. Like the exponential distribution, the Erlang distribution is supported on $[0, \infty)$ and its density and distribution are given by:

$$f(x) = \frac{\lambda^k x^{k-1}}{(k-1)!} \exp(-\lambda x), \quad F(x) = 1 - \sum_{n=0}^{k-1} \frac{(\lambda x)^n}{n!} \exp(-\lambda x), \quad x \geq 0. \tag{14}$$

The mean, variance and coefficient of variation of X are given by:

$$\mathbb{E}[X] = k/\lambda, \quad \mathbf{Var}[X] = k/\lambda^2, \quad c_X = 1/\sqrt{k} \tag{15}$$

The failure rate of an Erlang random variable is given by:

$$h(x) = \lambda \left[\sum_{n=0}^{k-1} \frac{(k-1)!}{n!(\lambda x)^{k-1-n}} \right]^{-1}. \tag{16}$$

The failure rate of the Erlang distribution is constant for $k = 1$. (In fact, it reduces to the exponential distribution when $k = 1$.) For $k > 1$, the Erlang distribution has an increasing failure rate. Figure 3 shows the Erlang density function for several shape and scale parameters.

2.2.4 Gamma distribution

The Erlang dsitribution can be generalized by allowing k to take non-integer values. In this case, all the factorials in (14) and (16) have to be replaced by their equivalents in terms of the Gamma-function. This distribution is called the gamma distribution and is parameterized by by the *shape parameter* $\alpha > 0$ (which is equivalent to k in the Erlang distribution), and the *scale parameter* $\beta > 0$ (which is equivalent to λ in the Erlang distribution). Its density and distribution are given by:

$$f(x) = \frac{\beta^\alpha x^{\alpha-1}}{\Gamma(\alpha)} \exp(-\beta x), \quad F(x) = \int_0^x f(u)du, \quad x \geq 0. \tag{17}$$

The mean, variance and coefficient of variation of a gamma distributed random variable X are given by:

$$\mathbb{E}[X] = \alpha/\beta, \quad \mathbf{Var}[X] = \alpha/\beta^2, \quad c_X = 1/\sqrt{\alpha} \tag{18}$$

[4] The Erlang distribution has been named after Danish engineer Agner Krarup Erlang (1878-1929). Erlang is considered one of the founders of queueing theory. The Erlang distribution is often used in this field.

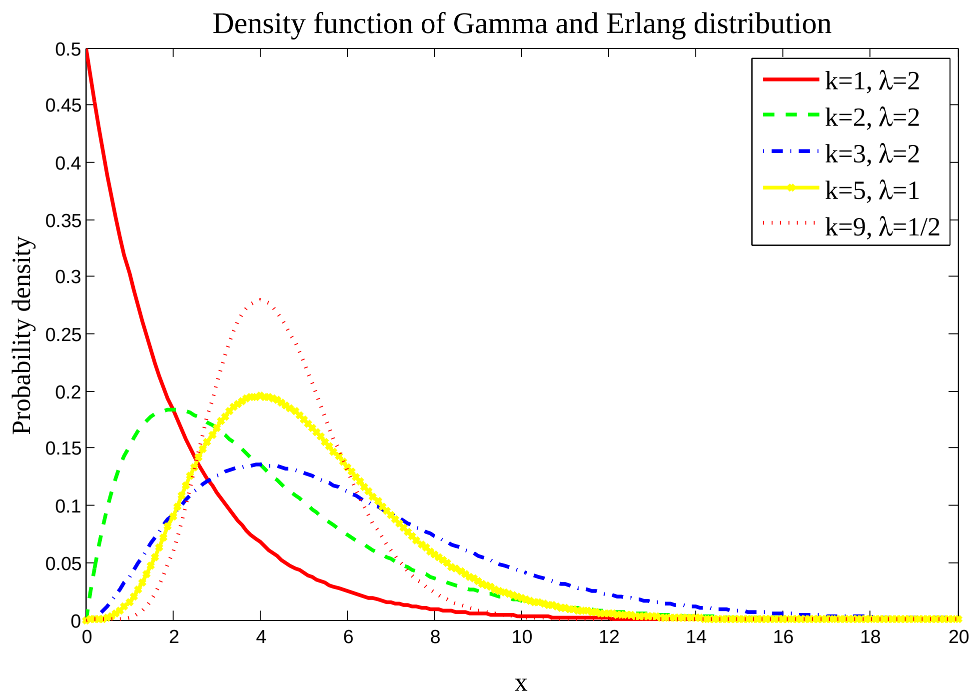

Figure 3: The Erlang (and Gamma) probability density function for several shape and scale parameters.

The failure rate of a gamma distribution is given by:

$$h(x) = \left[\int_x^\infty \left(\frac{u}{x}\right)^{\alpha-1} \exp(-\beta(u-x))du\right]^{-1} = \left[\int_0^\infty (1+v/x)^{\alpha-1} \exp(-\beta v)dv\right]^{-1}, \quad x \geq 0. \tag{19}$$

For $0 < \alpha < 1$, this failure rate is decreasing; for $\alpha > 1$ this failure rate is increasing and for $\alpha = 1$, we obtain the constant failure rate.

2.2.5 Weibul distribution

The Weibul[5] distribution is used much in reliability engineering because it provides a good fit with data in many applications and arises naturally in theory. Many components are composed of even smaller subcomponents. The time until failure of the component is therefore the shortest time until failure of any of the subcomponents. If we let $Y_1, \cdots, Y_n$

[5]The Weibul distribution has been named after Ernst Hjalmar Waloddi Weibull (1887-1979) who was a swedish engineer. He did not invent the Weibul distribution, but he made it popular among (reliability) engineers, because he showed that it arises naturally in the study of strength of materials, fatigue, rupture in solids and bearings.

denote the times until failure of the subcomponents, the time until failure of the component itself is $X = \min\{Y_1, Y_2, \cdots, Y_n\}$. If the random variables Y_1 to Y_n have finite support, then the distribution of X will approach the Weibul distribution as n approaches infinity[6]. This is the reason why the Weibul distribution arises theoretically and often provides good empirical fit with data.
The Weibul distribution has *shape* parameter β and *scale* parameter η. Its density and distribution are:

$$f(x) = \frac{\beta x^{\beta-1}}{\eta^\beta} \exp\left(-(x/\eta)^\beta\right), \quad F(x) = 1 - \exp\left(-(x/\eta)^\beta\right), \quad x \geq 0. \tag{20}$$

Note that for $\beta = 1$, the Weibul distribution reduces to the exponential distribution. The mean and variance are given by:

$$\mathbb{E}[X] = \Gamma(1 + 1/\beta)\eta, \quad \mathbf{Var}[X] = \left[\Gamma(1 + 2/\beta) - (\Gamma(1 + 1/\beta))^2\right] \eta^2. \tag{21}$$

The failure rate of the Weibul distribution has an exponential form:

$$h(x) = \frac{\beta}{\eta^\beta} x^{\beta-1}. \tag{22}$$

From this form, we immediately observe that X is IFR for $\beta > 1$, DFR for $0 < \beta < 1$, and CFR for $\beta = 1$. Figure 4 shows the Weibul probability density function for several different shape parameters.

2.2.6 Poisson distribution

The Poisson[7] distribution is a discrete distribution on the non-negative integers characterized by its mean μ only. If the random variable X has Poisson distribution with mean μ then

$$p_x = \mathbb{P}(X = x) = \exp(-\mu)\frac{\mu_x}{x!}, \quad x \in \mathbb{N}_0 = \mathbb{N} \cap \{0\} \tag{23}$$

and

$$\mathbf{Var}[X] = \mu, \quad c_X = \frac{1}{\sqrt{\mu}}. \tag{24}$$

[6]Compare this result to the central limit theorem. The central limit theorem plays an important role in statistics and says that $X = Y_1 + Y_2 + \cdots Y_n$ approaches a normal random variables as $n \to \infty$ (if the Y_i are independent and have finite first two moments). In reliability engineering, we are more often interested in $X = \min\{Y_1, Y_2, \cdots, Y_n\}$ which converges to a Weibul random variable as $n \to \infty$ (if the Y_i are non-negative and have finite support.) This result is known as the type III extreme value law.

[7]The Poisson distribution is called after the French mathematician Siméon Poisson (1781-1840).

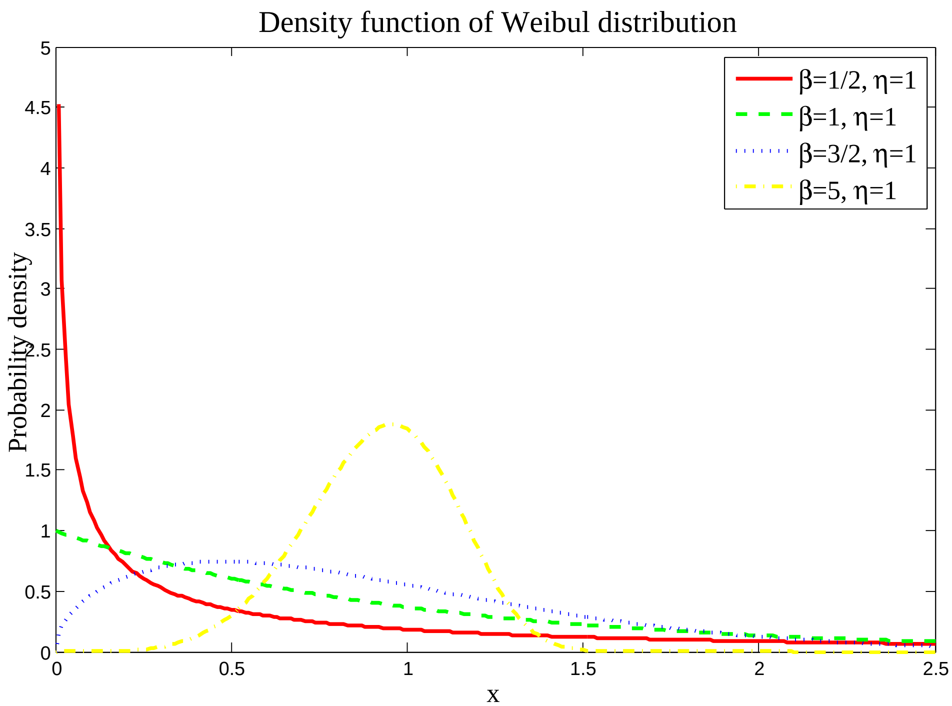

Figure 4: The Weibul probability density function for several shape parameters

Poisson probabilities satisfy the following recursive relation that is convenient in computations:

$$p_x = \frac{\mu}{x} p_{x-1}, \quad x \in \mathbb{N}. \tag{25}$$

2.3 Renewal theory

A renewal process is a counting process in with the time between events (also called renewals) are independently and identically distributed (iid). Let $X_1, X_2, \cdots$ be a sequence non-negative iid random variables with common distribution $F(x)$, density $f(x) = \frac{dF(x)}{dx}$, and mean $0 < \mathbb{E}[X_i] < \infty$. X_i is the time between the $(i-1)$-th and i-th renewal; for example X_i might represent the time to failure of a component. Now define the time the i-th renewal occurs as S_i:

$$S_i = \sum_{k=1}^{i} X_k, \qquad S_0 = 0. \tag{26}$$

If the X_i represent the time to failure of some component, then S_i represents the time until the failure of the i-th component. We may ask ourselves how many renewals have occurred up until time t. Such a process is called a renewal process and denoted by $N(t)$.

$$N(t) = \max\{i \in \mathbb{N}_0 | S_i \leq t\}, \quad t \geq 0. \tag{27}$$

Figure 5 shows an example of a renewal process with the notation that we introduced. The

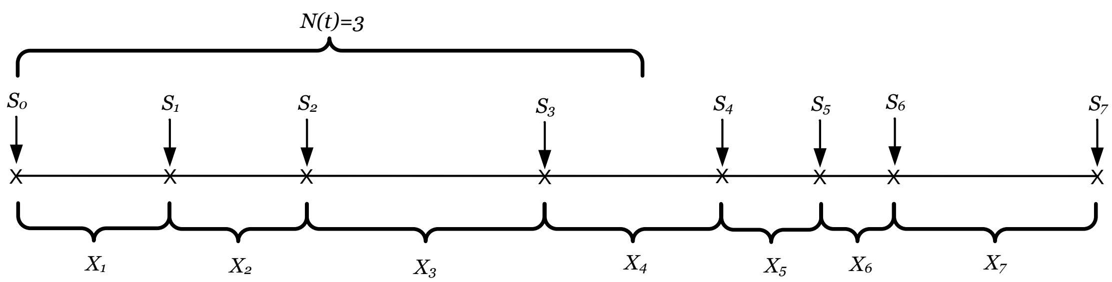

Figure 5: This Figure shows a sample path of a renewal process with the corresponding notation. Each renewal is marked by an x on the time line.

expected number of renewals up until time t, $\mathbb{E}[N(t)]$, is called the renewal function and denoted by $M(t)$. The renewal function obeys the following integral equation.

Theorem 2.1 (Renewal equation) *The renewal function $M(t) = \mathbb{E}[N(t)]$ satisfies:*

$$M(t) = F(t) + \int_0^t M(t-x)f(x)dx, \quad t \geq 0 \tag{28}$$

Proof: The proof follows from conditioning on X_1:

$$\begin{aligned}
M(t) &= \mathbb{E}[N(t)] \\
&= \int_0^\infty \mathbb{E}[N(t)|X_1 = x]f(x)dx \\
&= \int_0^t \mathbb{E}[N(t)|X_1 = x]f(x)dx \\
&= \int_0^t (1 + M(t-x))f(x)dx \\
&= F(t) + \int_0^t M(t-x)f(x)dx
\end{aligned}$$

The third equality follows because $N(t) = 0$ if $X_1 \geq t$. The fourth equality follows because if $X_1 = x \leq t$, then at least one renewal already occurred, and the expected number of renewals from x to t is just $M(t-x)$. □

Equation (28) is not immediately helpful for computing $M(t)$ because solving an integral equation (such as (28)) is, in general, more difficult than solving a differential equation. If you can guess the right form for $M(t)$ under some $f(x)$ and $F(x)$, you can plug it into (28) to check if this guess is correct. Coming up with such a guess is difficult and usually requires detailed understanding of the context.

There is also a discrete equivalent to Theorem 2.1, which is more immediately helpful in computations. Exercise 1.5 explores this further.

Example 2.1 The Poisson process is a special case of a renewal process for which $X_1, X_2, \cdots$ are exponentially distributed with mean λ^{-1}, S_k has an Erlang distribution with shape parameter k and scale parameter λ and $N(t)$ has a Poisson distribution with mean λt. From these results we immediately have that the renewal function satisfies $M(t) = \lambda t$. $\diamond$

2.4 Renewal reward theory

Now suppose that there is a reward (cost), W_i, associated with each renewal i. Furthermore, assume that $W_1, W_2, \cdots$ is an iid sequence of random variables with $|\mathbb{E}[W_i]| < \infty$. (Note that we do *not* assume that W_i and X_i are independent.) The total reward (cost) up until time t is denoted by $Y(t)$ and satisfies:

$$Y(t) = \sum_{i=1}^{N(t)} W_i. \tag{29}$$

$Y(t)$ is called a renewal reward process. A natural question that arises is: "What is the average costs per time unit for a renewal reward process?". The following theorem addresses this. The proof of this theorem is rather technical so we omit it. The interested reader is referred to Ross (1996) for a formal proof.

Theorem 2.2 (Renewal reward theorem) *The average cost per time unit of a renewal process, g, satisfies*

$$g = \lim_{t\to\infty} \frac{Y(t)}{t} = \lim_{t\to\infty} \frac{\mathbb{E}[Y(t)]}{t} = \frac{\mathbb{E}[W_i]}{\mathbb{E}[X_i]}$$

The average costs per renewal is sometimes referred to as the expected cycle cost, ECC. The expected length of a renewal is sometimes called the expected cycle length, ECL. Theorem 2.2, also known as the renewal reward theorem, says that that he expected cost per time unit is simply ECC/ECL.

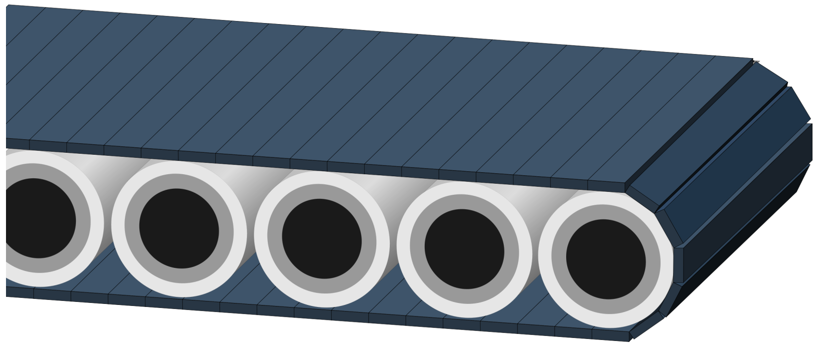

Figure 6: The actual rubber belt on a conveyor loses elasticity over time which causes slippage.

3. Elementary Maintenance Models

In this section, we are primarily concerned with the questions: "When should we conduct maintenance and what are the maintenance costs per time unit?". The answers to these questions depend very much on the setting you consider. Section 3.1 considers setting in which degradation is deterministic and Section 3.2 considers several setting where the time until failure is stochastic.

3.1 Deterministic maintenance models

The time to failure was presented as clear cut moment in time (that may be random) in the preceding discussion. Many components do not have such a clear failure moment, but their performance degrades over time. Consider for example the rubber belt of a conveyor; see Figure 6. The rubber belt loses elasticity as a result of usage in a predictable manner. This elasticity loss causes slippage of the conveyor, which causes the production to slow down. In this situation, the production loss is a reason to replace the belt before it has actually failed. Other examples of components for which this holds are fuel filters, the felt of a paper mill, and molds in glass and plastic production. The usage based maintenance policy that is explained in the introduction (see also Figure 1) is an appropriate maintenance policy in this situation. We consider the policy in which a component is replaced after τ usage. (We will assume that usage is measured in time units, but this can be changed to mileage, number of produced products etc.) The question under this policy is: "After what amount of usage should the component (belt) be replaced, i.e., what is the optimal τ?". This section provides a mathematical model to answer that question.

Suppose that the costs for replacing the belt are $C_p > 0$ euros. (The subscript p indicates the replacement is planned and the same notation will be used later.) Over usage time t, the belt slows down which causes the production rate (measured in euros per time unit) to decrease by $a > 0$ euros per time unit per time unit. (So if $a = 5$ euro/hr^2, and the production rate of a new belt/component is 200 euro/hr, then after 3 hours of usage, the remaining production rate is $200 - 3 \cdot 5 = 185$ euro/hr.) This means that after t time units, the total production loss relative to a brand new belt/component are $\int_0^t axdx = at^2/2$. The relevant costs per time unit $g(\tau)$ of replacing the the belt every τ time units are therefore:

$$g(\tau) = \frac{a\tau^2/2 + C_p}{\tau} = \frac{a\tau}{2} + \frac{C_p}{\tau}. \tag{30}$$

It is best to replace the belt after the amount of usage that minimizes $g(\tau)$. We find that

$$\frac{dg(\tau)}{d\tau} = \frac{a}{2} - \frac{C_p}{\tau^2}, \quad \frac{d^2g(\tau)}{d\tau^2} = \frac{2C_p}{\tau^3} > 0.$$

$g(\tau)$ is convex (because its second derivative is positive for all $\tau > 0$) so the optimal amount of usage (time) after which the belt should be replaced can be found by setting $dg(\tau)/d\tau = 0$ and solving for τ. Doing this yields the optimal replacement usage/time τ^*:

$$\tau^* = \sqrt{\frac{2C_p}{a}}. \tag{31}$$

Example 3.1 Consider a conveyor belt that is used 24/7 in a factory. The speed of this belt, when it is as good as new, is 5 km/hr and products on the belt are spaced 1 meter apart. The profit for each sold product is 0.5 EURO. Unfortunately, the conveyor belt speed decreases over time by 0.05 km/hr/day, and replacing the belt (C_p) costs 9600 EURO. The production rate in is $5 \cdot 1000 \cdot 0.5$ EURO/hr $= 24 \cdot 0.5 \cdot 5000 = 60,000$ EURO/day, and so $a = 0.05 \cdot 1000 \cdot 0.5 \cdot 24 = 600$ EURO/day/day. Therefore, it is best to replace the belt every $\tau^* = \sqrt{2 \cdot 9600/600} = \sqrt{32} \approx 5.66$ days. $\diamond$

3.2 Stochastic maintenance models

In traditional stochastic maintenance models, we consider the question when maintenance should be performed when the only information we have is the lifetime distribution of a component. Thoughout this section, we will use the notation in Table 3.

The models in this section are usually applied to construct maintenance programs for components. In practice, maintenance programs for components will be linked to a complete maintenance program for the equipment as a whole. This integration step is not treated here, but treated later in the course. Maintenance in this case can also be replacement. We will use these terms interchangeably.

Table 3: Notation used in traditional stochastic maintenance models

Notation		Interpretation
C_u	:	Cost of performing unplanned corrective maintenance/replacement, $C_u > 0$
C_p	:	Cost of performing planned preventive maintenance/replacement, $0 < C_p < C_u$
C_{mr}	:	Cost of performing an unplanned minimal repair. $0 < C_{mr} < C_u$
T	:	Random variable denoting the lifetime of a component.
$F_X(\cdot)$	:	Distribution function of a random variable X, $F_X(t) = \mathbb{P}(X \leq t)$
$f_X(\cdot)$	:	Densition function of a random variable X, we assume $f_X = dF_X(t)/dt$
$h_X(\cdot)$	:	Failure rate of the random variable X
$M_X(\cdot)$	:	Renewal function for a renewal process with inter-renewal times that have the distribution of X.

3.2.1 Failure based policy

The failure based policy is to replace/maintain a component every time it fails. This is the same as the breakdown corrective maintenance policy described in the Intoduction; refer also to Table 1. The costs up to time t for this policy are a renewal reward process. The expected cycle length in this case is just the mean time to failure, $\mathbb{E}[T]$ and the expected cycle costs under this policy are C_u. Therefore the expected costs per time unit, g, can be found by a straightforward application of the renewal reward theorem:

$$g = C_u/\mathbb{E}[T]. \tag{32}$$

The failure based policy is optimal when T is DFR because performing preventive maintenance does not actually improve the reliability of a component. Components with DFR are usually electronics. However, even if T is IFR, the failure based policy can be attractive if the failure rate does not grow without bound ($\lim_{x\to\infty} h_T(x) < \infty$) and $C_u - C_p$ is sufficiently small.

Example 3.2 Suppose that the lifetime of a components is uniformly distributed from 10 to 20 time units and that unplanned corrective maintenance costs 1000 EURO. If we choose to apply a failure based policy for this component, then the costs per time unit are $C_u/\mathbb{E}[T] = 1000/((20 + 10)/2) = 66.67$ EURO per time unit. (Note that the uniform distribution is IFR with $\lim_{x\to b} h_T(x) = \infty$ so it is probably a bad idea to apply failure based maintenance for this component. We consider applying preventive maintenance for this component in the next example.) $\diamond$

3.2.2 Age replacement policy

Under the age replacement policy, a component is replaced whenever it has been used for a fixed amount of time τ or if it fails before this time. This policy fits in the upper left box of Table 1 from the introduction. We let the random variable $X = \min(\tau, T)$ denote the time until a component is replaced. When a components is replaced after τ amount of time, the cost of a planned replacement C_p is incurred. The unplanned maintenance cost $C_u > C_p$ is incurred if the component is replaced due to failure before τ amount of usage. We assume the time to replace is negligible. When $\tau \to \infty$, this policy is equivalent to the failure based policy. By defining the time until replacement as a cycle, this policy can be cast as a renewal reward process. The inter-renewal time of this renewal reward process is distributed as X and the cost per cycle W is C_p with probability $1 - F_T(\tau)$ and C_u with probability $F_T(\tau)$. (Note that X and W are correlated random variables.) The ECC and ECL for the age replacement policy are:

$$ECC = F_T(\tau)C_u + (1 - F_T(\tau))C_p \tag{33}$$

$$ECL = \mathbb{E}[X] = \mathbb{E}[\min(T, \tau)] \tag{34}$$

$$= \int_0^\infty \min(x, \tau) f_T(x)dx = \int_0^\tau x f_T(x)dx + \tau R(\tau), \tag{35}$$

and the costs, which depend on τ, are $g(\tau) = ECC/ECL$. The optimal replacement time τ^* can be found by settig $dg(\tau)/d\tau = 0$ and solving for τ. If T is DFR, then $dg(\tau)/d\tau > 0$ for all τ and $\lim_{\tau\to\infty} dg(\tau)/d\tau = 0$. That is, the optimal age replacement policy reduces to a failure based policy.

Example 3.3 Reconsider the component from Example 3.2 with $C_u = 1000$, and a lifetime that is uniformly distributed between 10 and 20 time units. Suppose that planned preventive maintenance for this component costs 600 EURO. Note that under an age replacement policy only makes sense for $\tau \in (10, 20)$. For $\tau \in (10, 20)$ we have

$$ECC = F_T(\tau)C_u + (1 - F_T(\tau))C_p = \frac{\tau - 10}{10}1000 + \frac{20 - \tau}{10}600 = 40\tau + 200 \tag{36}$$

and

$$
\begin{aligned}
ECL &= \frac{1}{10}\int_{10}^{\tau} x dx + \tau\frac{20-\tau}{10} \\
&= \frac{1}{10}\left[x^2/2\right]_{x=10}^{x=\tau} + \tau\frac{20-\tau}{10} \\
&= \frac{1}{10}\frac{1}{2}\tau^2 - \frac{1}{10}\frac{1}{2}10^2 + 2\tau - \frac{1}{10}\tau^2 \\
&= -\frac{1}{20}\tau^2 + 2\tau - 5 \qquad (37)
\end{aligned}
$$

Combining (36) and (37) yields for the expected cost per time unit (after some algebra):

$$
g(\tau) = ECC/ECL = \frac{-800(\tau+5)}{\tau^2 - 40\tau + 100}, \quad \frac{dg(\tau)}{d\tau} = \frac{800(\tau^2 + 10\tau - 300)}{(\tau^2 - 40\tau + 100)^2}. \qquad (38)
$$

Now $dg(\tau)/d\tau = 0$ only if its numerator is 0 so the the optimal replacement time τ^* can be found by solving the quadratic equation $\tau^2 + 10\tau - 300 = 0$. Thus the optimal replacement time is after $\tau^* = 13.0278$ time units and the average cost per time unit under this policy is $g(\tau^*) = 57.37$ EURO per time unit. (In Example 3.2, we already noted that preventive maintenance should be useful in this case because the failure rate grows without bound. The saving compared to the failure based policy is $(66.67 - 57.37)/66.67 \approx 14\%$.) $\diamond$

Example 3.4 A machine has an Erlang distributed lifetime with shape parameter $k = 2$ and scale parameter $\lambda = 1$ (per year). So, the failure distribution is given by

$$
F_T(x) = 1 - (1+x)\,e^{-x}\ , \qquad x \geq 0 \ (x \text{ in years}).
$$

The costs of a preventive maintenance action are equal to 500 EURO. For a corrective maintenance action, the costs are 7000 EURO because unplanned maintenance interrupts the production process in which the machine functions. The maintenance manager would like to be able to evaluate the costs of different age based policies for this machine. Using the theory we built so far we find

$$
g(\tau) = \frac{7000(1-(1+\tau)e^{-\tau}) + 500(1+\tau)e^{-\tau}}{\int_0^{\tau} x^2 e^{-x} dx + (\tau^2+\tau)e^{-\tau}}. \qquad (39)
$$

Working out the integral in the denominator using integration by parts twice, we have

$$\begin{aligned}\int_0^\tau x^2 e^{-x} dx &= \left[-x^2 e^{-x}\right]_0^\tau + 2\int_0^\tau x e^{-x} dx \\ &= -\tau^2 e^{-\tau} + 2\left([-xe^{-x}]_0^\tau + \int_0^\tau e^{-x} dx\right) \\ &= -\tau^2 e^{-\tau} + 2\left(-\tau e^{-\tau} + [-e^{-x}]_0^\tau\right) \\ &= -\tau^2 e^{-\tau} - 2\tau e^{-\tau} - 2e^{-\tau} + 2 \\ &= -e^{-\tau}(\tau^2 + 2\tau + 2) + 2 \end{aligned} \tag{40}$$

Now substituting (47) back into (46), we find:

$$g(t) = \frac{7000(1-(1+\tau)e^{-\tau}) + 500(1+\tau)e^{-\tau}}{-e^{-\tau}(\tau^2+2\tau+2) + 2 + (\tau^2+\tau)e^{-\tau}} = \frac{7000 - 6500(1+\tau)e^{-\tau}}{-e^{-\tau}(\tau+2)+2}. \tag{41}$$

$\diamond$

3.2.3 Block replacement policy

Under a block replacement policy, a components is replaced at fixed times $\tau, 2\tau, 3\tau, \cdots$ and an unplanned corrective maintenance is done if the component fails between these times. This policy can be attractive when the preventive maintenance of different components can be coordinated. Consider for example a wind-turbine park at sea. It is beneficial to maintain all wind-turbines together at fixed times $\tau, 2\tau, 3\tau, \cdots$ because the cost of transporting goods and maintenance engineers out to sea only need to be incurred once for all wind-turbines together. The block replacement policy is a usage based policy; see the upper left of Table 1 in the introduction.

In the previous sections, we always defined a cycle by the time between replacements because this led to a renewal (reward) process. Under a block policy, the time between replacements is *not* a renewal process. To see why it is not, consider a component that has just been preventively replaced at time 0. The time until the next replacement is distributed as $\min(\tau, T)$. The time until the next replacement (after the first) is not distributed as $\min(T, \tau)$ unless the first replacement was a planned preventive replacement. Since the first replacement could also have been an unplanned corrective replacement, the sequence of times between replacement is not iid. Fortunately, renewal reward theory also applies to the block replacement policy by defining a cycle as the time between planned preventive replacement:

$$ECL = \tau.$$

The costs during such a cycle are C_p for the planned preventive replacement during each cycle and the costs of corrective maintenance during such a cycle. Recall that $M_T(t)$ is the expected number of renewals (failures) during an interval of length t. Therefore we have

$$ECC = C_p + C_u M_T(\tau)$$

The average costs per time unit depend on τ and are given by $g(\tau) = ECC/ECL$. Most of the time (but not always), the optimal block-replacement interval τ^* can be found by setting $dg(\tau)/d\tau = 0$ and solving for τ.

Example 3.5 Reconsider the component from Examples 3.2 and 3.3 with $C_p = 600$, $C_u = 1000$ and T has a uniform distribution from 10 to 20. Observe first that since $10 \leq T \leq 20$, $\tau^* \in (10, 20)$. (Why?) To determine the ECL, we need to find $M_T(t)$ for $t \in [10, 20]$. The number of failures in an interval of length less than 20 can be at most 1. (Why?) Therefore, $M_T(t) = F_T(t)$ for $t \in (0, 20)$; this can be verified directly using Theorem 2.1. Now it is straightforward that for $\tau \in [10, 20]$

$$ECC = C_p + C_u M_T(\tau) = 600 + 1000 F_T(\tau) = 600 + 1000\frac{\tau - 10}{10} = 100\tau - 400, \quad ECL = \tau, \tag{42}$$

so that the average costs per time unit are

$$g(\tau) = ECC/ECL = \frac{100\tau - 400}{\tau} = 100 - \frac{400}{\tau}, \quad \frac{dg(\tau)}{d\tau} = \frac{400}{\tau^2}. \tag{43}$$

Note that $g(\tau)$ is increasing because $dg(\tau)/d\tau > 0$ for $\tau \in (10, 20)$. Therefore the optimal block-replacement interval $\tau^* = 10$ time units and $g(\tau^*) = 60.00$ EURO per time unit. $\diamond$

3.2.4 Block replacement policy with minimal repair

A block replacement policy with minimal repair is identical to a regular block replacement policy with one exception: When a component fails between block replacements, it is repaired to a state that is not as good as new, but statistically identical to the state just before the failure. Such a repair is called a minimal repair and its costs are denoted C_{mr}. In practice, minimal repairs are performed by using duct-tape, tie-wraps and other ad-hoc solutions to get a component functioning again, without actually replacing it or performing thorough maintenance. A "full" repair brings the failure rate just after replacement back to $h_T(0)$. By contrast, after minimal repair, the failure rate does not change at all. Because of this, the expected number of failures between block-replacements is no longer given by $M_T(\tau)$. Recall that the failure rate can be interpreted as the expected number of failures per time unit in a very small time interval. Therefore, $\int_0^\tau h_T(x)dx$ is the expected number of failures during a block-replacement interval. Thus we have

$$ECC = C_p + C_{mr}\int_0^\tau h_T(x)dx, \quad ECL = \tau, \tag{44}$$

and the expected costs per time unit are $g(\tau) = ECC/ECL$.

Example 3.6 Reconsider Examples 3.2, 3.3, and 3.5. Suppose that the costs of a minimal repair are given by $C_{mr} = 400$ EURO. As in Example 3.5, an optimal block replacement interval must be between 10 and 20 time units long. We have for $t \in (10, 20)$:

$$\begin{aligned}
ECC &= C_p + C_{mr}\int_0^\tau h_T(x)dx \\
&= 600 + 400\int_{10}^\tau \frac{1}{20-x}dx \\
&= 600 - 400\int_{20-10}^{20-\tau} \frac{1}{u}du \\
&= 600 - 400\,[\ln(u)]_{10}^{20-\tau} \\
&= 600 - 400\,(\ln(20-\tau) - \ln(10)),
\end{aligned}$$

where the third equality follows from using the substitution $u = 20 - x$. The costs per time unit and its first derivative for $\tau \in [10, 20)$ are given by

$$g(\tau) = \frac{600 - 400\,(\ln(20-\tau) - \ln(10))}{\tau}, \quad \frac{dg(\tau)}{d\tau} = \frac{\frac{400\tau}{20-\tau} - 600 + 400(\ln(20-\tau) - \ln(10))}{\tau^2}.$$

Setting the numerator of $dg(\tau)/d\tau$ to 0 and using some algebra, we find that the optimal block replacement interval τ^* satisfies the first order condition

$$\frac{\tau^*}{20-\tau^*} + \ln(20-\tau^*) - \ln(10) - \frac{3}{2} = 0 \tag{45}$$

Unfortunately (45) cannot be solved in closed form, but a bisection search (or any other numerical root finding procedure) yields $\tau^* = 13.00$ time units so that $g(\tau^*) = 57.1$ EURO per time unit. $\diamond$

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Exercises

Exercise 1.1 *(On the age policy; level: (below) standard)*
A machine has a homogeneous (= uniform) failure distribution on the interval $[0, 10]$ (in months). So, the failure distribution is given by:

$$F(x) = \frac{x}{10} \ , \qquad 0 \leq x \leq 10 \ (x \text{ in months}).$$

For this machine the following maintenance policy is used. The machine is replaced by a new one as soon as it reaches the age of τ months, or upon failure in case the machine fails before time τ. The costs of a replacement are equal to EURO 3000. For a corrective maintenance action additional costs equal to EURO 1000 are incurred because of the disturbance of the production process that depends on the availability of the machine.
Determine the average costs of the age-based maintenance policy as a function of τ and determine the optimal policy and the corresponding average costs.

Exercise 1.2 *(On the age policy; old exam exercise; level: standard)*
A machine has an Erlang distributed lifetime with shape parameter $k = 2$ and scale parameter $\lambda = 2$ (per year). So, the failure distribution is given by:

$$F(t) \ = \ 1 - (1 + 2t)e^{-2t} \ , \qquad t \geq 0 \ (t \text{ in years}).$$

(i) Determine the failure rate function. Is it useful to apply age-based preventive maintenance for this machine?

(ii) The management decides that the machine has to be replaced by a new one as soon as the failure rate exceeds the value of $\alpha = 1$ per year, or as soon as the machine fails if this happens before the failure rate exceeds this level. Determine the time τ at which the machine has to be replaced preventively if the machine did not fail before.

(iii) The costs of a preventive maintenance action are equal to EURO 500. For a corrective maintenance action the costs are EURO 200 higher. A maintenance engineer conjectures that the rule of the management concerning preventive replacements leads to too early replacements, and hence to higher costs than needed. He suggests that preventive replacement as soon as the failure rate exceeds the value $\alpha = 4/3$ leads to lower expected costs per year. Is he right? Is there a large difference?

Exercise 1.3 *(similar to an old exam exercise; level: above standard)*
A machine has one critical component that is subject to failures. This component has a lifetime consisting of two parts: a deterministic part with length $d \geq 0$ (in months) and a Weibull distributed part with shape parameter $\beta > 0$ and scale parameter $\eta > 0$ (in months^{-1}). The failure distribution is given by:

$$F(t) \;=\; \begin{cases} 0 & \text{if } 0 \leq t < d; \\ 1 - e^{-((t-d)/\eta)^{\beta}} & \text{if } t \geq d, \end{cases} \qquad (t \text{ in months}).$$

(i) Determine the failure rate function. In which case is the function constant, increasing, and decreasing, respectively?

(ii) Is it useful to apply preventive maintenance for the component?

Assume now that $d = 0$, $\beta = 2$, and $\eta = 4$, and that the following maintenance policy is applied. If the component fails then the component itself is immediately repaired. This takes an exponential time with mean $\nu = 0.1$ months. During this time the machine is not available. After repair the component is as good as new.

(iii) Determine the expected time μ till the component fails.

(iv) Determine the long-run fraction of time that the machine is available.

Exercise 1.4 *(On the minimal repair policy; old exam exercise; level: above standard)*
A machine has a Weibull distributed lifetime with shape parameter $\beta = 2$ and scale parameter $\eta = 1$ (years), i.e., the failure distribution is given by:

$$F(t) \;=\; 1 - e^{-t^2}\,, \qquad t \geq 0 \; (t \text{ in years}).$$

If the machine fails then, by minimal repair, the machine can be brought back into the state it had just before the failure. The costs of such a minimal repair are equal to EURO 100. The alternative is to replace the machine by a new one, which costs EURO 900.

(i) The following maintenance policy is applied. The machine is replaced by a new one after τ_1 years. If the machine fails before the time τ_1, then minimal repair is applied. Determine the value of τ_1 which leads to the lowest average costs per year. How large are the average costs for this τ_1?

(ii) Suppose now that at time τ_1 as obtained under a) the machine receives a thorough revision, which is such that the failure rate is brought back to the level of time $\tau_1 - 2$. The price of this revision is EURO 300. The time needed for this revision is small and may be neglected. Next, at time $\tau_2 > \tau_1$, the machine is replaced by a new one. Determine the value of τ_2 for which the average costs are minimal, and give the corresponding average costs.

(iii) Which policy is better, the one of a) or the one of b)?

Exercise 1.5 *(On the block policy; level: above standard)*
A technical system consists of 1000 identical components. For this system, the following block maintenance policy is applied. The components are inspected at the end of each month. If a component appears to be broken, then it is immediately replaced by a new one. This costs EURO 30 per component that has to be replaced. After τ periods, $\tau \in \mathbb{N} = \{1, 2, ...\}$, all components are replaced, independently of the fact whether they are broken at that time or not. This costs EURO 10000 for the whole group of components (this is equivalent to EURO 10 per component).
Let p_i, $i \in \mathbb{N}$, denote the probability that a new components fails in the i-th month (which is observed at the end of the i-th month). These probabilities are equal to:

$$p_1 = 0,10;\ p_2 = 0,15;\ p_3 = 0,25;\ p_4 = 0,25;\ p_5 = 0,15;\ p_6 = 0,10;\ p_7 = 0 \text{ for all } i \geq 7.$$

Since the components have a maximal lifetime of 6 months, only block maintenance policies with $\tau \leq 6$ are considered.

(i) Let a cycle be defined as the time between two successive moments at which the components are replaced preventively. To determine the expected cycle costs, one has to determine the expected number of corrective maintenance actions in a cycle. For that purpose, we define M_t as the expected number of replacements after t months, $t \in \mathbb{N}$. This function may be seen as the discrete-time version of the renewal function as known from the renewal theory. For M_t, the following recursive formula holds:

$$M_t = \sum_{i=1}^{t} p_i + \sum_{i=1}^{t-1} p_i M_{t-i}\ , \qquad t \in \mathbb{N}.$$

Give an intuitive explanation for this formula.

(ii) Compute M_t for $t = 1, \ldots, 6$.

(iii) Give a formula for the average costs of the block maintenance policy with $\tau \in \{1, 2, \ldots, 6\}$.

(iv) Determine the optimal value for τ. What are the resulting average costs.

Exercise 1.6 *(On the failure-based and minimal repair policy; old exam exercise; level: standard)*
The distribution function of the lifetime of a certain machine is given by

$$F(x) = 1 - e^{-(2x)^{1.5}}, \qquad x \geq 0 \ (x \text{ in years}).$$

If the machine fails, then one can either replace the machine by a new one or apply a minimal repair, after which the machine is back in the same shape as just before its breakdown. The costs of a minimal repair are equal to 2000 EURO, while the costs of replacement by a new machine equal 5000 EURO.
For this machine two maintenance policies are considered: (i) The failure-based maintenance policy; (ii) The minimal repair policy, under which the machine is replaced by a new one after each τ time units and minimal repair is applied when a breakdown occurs.

(i) What are the average costs obtained under the failure-based maintenance policy? (Given: $\Gamma(1.667) = 0.9033$.)

(ii) Determine the failure rate function. What can be concluded from the behavior of the failure rate function for the minimal repair policy?

(iii) Determine the minimal average costs that can be obtained by the minimal repair policy. How large is the difference with respect to the average costs obtained for the failure-based maintenance policy? Are you surprised by this difference?

Exercise 1.7 *(On the age maintenance policy; old exam exercise; level: above standard)*
A dealer of an automatic orange juice press currently has about 100 customers. These customers are mainly restaurants and bars and each of them has one press.
When a customer buys a new press, he receives a (free) warranty of the manufacturer for failures within the first 3 years. If a failure occurs within the first 3 years, then the maintenance department of the manufacturer will replace the failed press by a new one for free. It is known that the presses are very reliable in the first 3 years, and hence these free replacements occur seldom.

After 3 years, a press can break down at any time. Statistics from the past suggest that the remaining lifetime after 3 years is exponential with mean 1.5 years. As a result, the lifetime distribution of a new press may be approximated by the following distribution:

$$F(t) \;=\; \begin{cases} 0 & \text{if } 0 \leq t < 3; \\ 1 - e^{-\frac{2}{3}(t-3)} & \text{if } t \geq 3, \end{cases} \qquad (t \text{ in years}).$$

If the press fails after the first 3 years, then a customer will buy a new press at the dealer, for which he will receive a warranty of 3 years again. The price of a new press is EURO 500 if it is ordered by the normal procedure. This procedure is used when the press is replaced by a new one preventively. If a press is only replaced after it has failed, then a new press is ordered and delivered by the emergency procedure, which leads to additional costs equal to C_{emer} (in EURO). These costs are subject to a change and hence C_{emer} is seen as a variable now.

(i) Determine the failure rate function for the orange juice press. Can you conclude from this function whether it is sensible, or not, to apply preventive replacements?

Suppose that at each customer, replacements of the press are applied according to an age policy with parameter τ. The corresponding average costs per year are denoted by $g(\tau)$. We also assume that the customers are rational, i.e. that the τ is chosen such that the average costs $g(\tau)$ are minimized.

(ii) Determine the average costs $g(\tau)$.

(iii) Determine the optimal age policy.

(iv) Currently a customer has to pay $C_{emer} = 200$ EURO extra when he orders a new press by the emergency procedure instead of the standard procedure. The dealer plans to charge 300 EURO extra from now on. What are the consequences of this increase for both the customers and the dealer?

Exercise 1.8 *(On the age policy; old exam exercise; level: (above) standard)*
A machine has an Erlang distributed lifetime with shape parameter $k = 2$ and scale parameter $\lambda = 1$ (per year). So, the failure distribution is given by:

$$F(t) \;=\; 1 - (1+t)e^{-t}\,, \qquad t \geq 0 \; (t \text{ in years}).$$

(i) Determine the mean and standard deviation of the lifetime of the machine.

(ii) Determine the failure rate function. Is it useful to apply preventive maintenance for this machine?

The costs for a preventive maintenance action for this machine are equal to 200 EURO. The costs for a corrective maintenance action are equal to $200 + C_{emer}$ EURO, where the factor $C_{emer} \geq 0$ represents emergency costs. The higher C_{emer} the more attractive it is to apply preventive maintenance. We now consider an age maintenance policy with parameter τ.

(iii) Determine the average costs $g(\tau)$ for the age policy with parameter τ.

(iv) The derivative of $g(\tau)$ is denoted by $g'(\tau)$. Show that: $g'(\tau) = 0$ if and only if

$$[(C_{emer} - 200)\tau - (C_{emer} + 200)] + C_{emer}e^{-\tau} \;=\; 0 \;.$$

When is $g'(\tau) > 0$? When is $g'(\tau) < 0$?

(v) Intuitively, one would think that it becomes attractive to apply a preventive maintenance action when the lifetime has become equal to the mean lifetime. What is your opinion on this intuition?

Exercise 1.9 *(On the failure-based and minimal repair policy, exercise is identical to Exercise 1.6, but with different numbers; level: standard)*
The distribution function of the lifetime of a certain machine is given by

$$F(x) \;=\; 1 - e^{-(3x)^2} \;, \qquad x \geq 0 \; (x \text{ in years}).$$

If the machine fails, then one can either replace the machine by a new one or apply a minimal repair, after which the machine is back in the same shape as just before its breakdown. The costs of a minimal repair are equal to 400 EURO, while the costs of replacement by a new machine equal 2000 EURO.
For this machine two maintenance policies are considered: (i) The failure-based maintenance policy; (ii) The minimal repair policy, under which the machine is replaced by a new one after each τ time units and minimal repair is applied when a breakdown occurs.

(i) What are the average costs obtained under the failure-based maintenance policy? (Given: $\Gamma(1.5) = 0.8862$.)

(ii) Determine the failure rate function. What can be concluded from the behavior of the failure rate function for the minimal repair policy?

(iii) Determine the minimal average costs that can be obtained by the minimal repair policy. How large is the difference with respect to the average costs obtained for the failure-based maintenance policy? Are you surprised by this difference?

Exercise 1.10 *(On the failure-based and age policy; old exam exercise; level: above standard)*
In this exercise we consider a component with a uniform (= homogeneous) failure distribution. We assume that the part may fail from the beginning. Without loss of generality, we assume that the maximum lifetime is 1. Hence the failure distribution $F(t)$ is given by $F(t) = t$, $0 \leq t \leq 1$. The component is part of an important machine. As soon as the component fails it has to be replaced by a new one.
For this component we investigate two maintenance policies, the failure-based policy and the age policy with parameter τ, $0 \leq \tau \leq 1$. The costs for a planned replacement of the component are denoted by C_p. The costs for an unplanned replacement are equal to C_u. We assume that $C_u > C_p > 0$.
The average costs under the failure-based policy are denoted by g_{fb}. The average costs under the age policy with parameter τ are denoted by $g_{age}(\tau)$. The parameter for which these costs are minimized is denoted by τ^* and $g^*_{age} = g_{age}(\tau^*)$. The ratio $(g_{fb} - g^*_{age})/g_{fb}$ denotes the relative savings that are obtained under the optimal age policy in comparison to the failure-based policy. We are interested in how large these savings are as a function of the ratio C_u/C_p.

(i) Determine g_{fb}.

(ii) Determine $g_{age}(\tau)$, $0 < \tau \leq 1$.

(iii) Determine the derivative $g'_{age}(\tau)$ of $g_{age}(\tau)$ and show that

$$g'_{age}(\tau) = 0 \quad \Leftrightarrow \quad \frac{1}{2}\left(\frac{C_u}{C_p} - 1\right)\tau^2 + \tau - 1 = 0 \ ,$$

and similarly with the "="-sign replaced by the ">"- and "<"-sign, respectively.

(iv) Show that

$$\tau^* = \frac{\sqrt{1 + 2\left(\frac{C_u}{C_p} - 1\right)} - 1}{\left(\frac{C_u}{C_p} - 1\right)}.$$

(v) Compute $(g_{fb} - g^*_{age})/g_{fb}$ for $\frac{C_u}{C_p} \downarrow 1$ and for $\frac{C_u}{C_p} = 2,\ 5,\ 13.$

(vi) Give your opinion on the relative savings that are possible by the age policy in comparison to the failure-based policy.

Exercise 1.11 *(On block replacement, above standard)*
A wind-turbine park in the north-sea consists of 10 identical turbines. Each month, a maintenance engineer visits the park and repairs turbines that have failed. Such a repair brings them to as good as new condition and costs of €500 per turbine. Once every τ months, a team of engineers visits the wind-turbine park to maintain *all* turbines and return them to as good as new condition. This preventive maintenance costs €2000 for the entire park. The lifetime of any turbine has a right-truncated Weibull distribution with $\beta = 2$ and $\eta = 5$ and truncation parameter $R = 12$, i.e., the probability a turbine fails before x months of service is given by:

$$F(x) = \begin{cases} 1 - \exp\left(-(x/\eta)^\beta\right), & \text{if } x < R; \\ 1, & \text{if } x \geq R. \end{cases}$$

This means that the lifetime of a turbine has a Weibull distribution up to time R, but is certain to fail at time R if it has not failed before time R. Let $p_i = F(i) - F(i-1)$ for $i \in \{1, \ldots, 12\}$

(i) Suppose that preventive maintenance is never conducted ($\tau = \infty$). How many visits will the maintenance engineer pay to an arbitrary turbine until he finds that it has failed? Give an expression in terms of p_i and compute the numerical answer.

(ii) Now suppose that we want to determine the monthly cost as a function of τ. To determine the expected cycle costs, one has to determine the expected number of corrective maintenance actions in a cycle. For that purpose, we define M_t as the expected number of turbine repairs after t months, $t \in \mathbb{N}$. This function may be seen as the discrete-time version of the renewal function as known from the renewal theory. For M_t, the following recursive formula holds:

$$M_t = \sum_{i=1}^{t} p_i + \sum_{i=1}^{t-1} p_i M_{t-i}\ , \qquad t \in \mathbb{N}.$$

Give an intuitive explanation for this formula.

(iii) Compute M_t for $t = 1, \ldots, 12$.

(iv) Give a formula for the average monthly cost of the block maintenance policy with $\tau \in \{1, 2, \ldots, 12\}$.

(v) Determine the optimal value for τ. What is the resulting average monthly cost?

Exercise 1.12 *(Age replacement and failure rate, standard)*
A machine has an Erlang distributed lifetime with shape parameter $k = 2$ and scale parameter $\lambda = 1$ (per year). So, the failure distribution is given by

$$F(x) = 1 - (1 + x)\, e^{-x} \ , \qquad x \geq 0 \ (x \text{ in years}).$$

We assume that the machine is needed for an infinite horizon. Corrective maintenance is executed when the machine fails, after which the machine is as good as new. Preventive maintenance is also possible. Also after preventive maintenance, the machine is as good as new. The costs of a preventive maintenance action are equal to € 500. For a corrective maintenance action, the costs are € 7000 because unplanned maintenance interrupts the production process in which the machine functions.

(i) Currently a failure-based policy is followed. Determine the yearly average costs under this policy.

(ii) Could it be useful to apply preventive maintenance for this machine? Why (not)?

(iii) We consider an age policy as an alternative for the failure-based policy. Assume that preventive maintenance is executed as soon as the failure rate (also called hazard rate) reaches the value of $\frac{1}{6}$ failures per year. If the machine fails before that time point, then corrective maintenance is executed. Determine the parameter τ_0 of the resulting age policy (i.e., the policy that executes preventive maintenance when the component age reaches τ_0 years).

(iv) Let $g(\tau)$ be the average yearly costs of the age policy with parameter T. Derive the formula for $g(\tau)$.

(v) Determine the average yearly costs $g(\tau_0)$ of the age policy as described under c). How large is the difference with respect to the average cost obtained for the failure-based policy? Do you find this difference surprising? Why (not)?

Exercise 1.13 *(All policies, Old exam, above standard)*

We consider 12 solar-energy collectors (collector for short) that are stationed in the desert. The lifetime of each collector is independently and uniformly distributed between 4 and 8 months. To perform maintenance on these collectors, an engineer needs to drive into the desert. A planned visit costs 2750 EURO and an unplanned visit costs 5000 EURO. There are material maintenance cost of 250 EURO per collector that is maintained preventively and 1000 EURO per collector that is maintained correctively. (You may assume that material needed for maintenance is always available). Failures of collectors are observed immediately because the voltage generated by the solar park drops after a failure occurs.

(a) Can it be beneficial to perform preventive maintenance on collectors? Why (not)?

(b) Suppose that a failure-based maintenance policy is used to maintain these collectors. What would be the annual total maintenance cost in this case?

(c) Suppose that an age based is used to maintain these collectors separately. What is the annual cost of an optimal age-based maintenance policy and when should collectors be replaced under such a policy?

(d) Management decides that it is silly to drive all the way into the desert (and pay either 3000 or 5000 EURO) to maintain only one collector. If an engineer can maintain multiple collectors during a visit, this might save money. Explain what a block policy is and how it can be used to avoid driving into the desert to fix only one collector.

(e) An engineer claims that under a block policy, the optimal time between block replacements, denoted by τ^*, satisfies $4 \leq \tau^* \leq 8$ (time measured in months). He further claims that any collector can fail at most once between block replacements. Is the engineer correct? Why (not)?

(f) Determine the annual cost and maintenance interval for an optimal block maintenance policy. (Hint: use the results from (e) to determine the renewal function.)

Exercise 1.14 *(Deterministic maintenance, above standard)*

Consider the air filter of a car. The price for replacing an air filter is 80 EURO. Suppose this car is driven 2000 km/month and petrol costs 1.50 EURO/l. With a new air filter, the car can drive 15 km/l, but this decreases by 1 km/l/month.

(a) Suppose you start with a new air filter. Show that up to time $0 \leq t \leq 15$ (in months) the total amount of money (in EUROS) spent on petrol is

$$\int_0^t \frac{3000}{15 - x} dx$$

when we do not replace the filter. (Hint: What are the fuel consumption rate and the money consumption rate at any time?)

(b) Use substitution to show that

$$\int_0^t \frac{3000}{15 - x} dx = 3000 \ln \left(\frac{15}{15 - t} \right)$$

(c) Consider the usage based policy in which the filter is replaced every τ time units. Use (a) and (b) to give an expression for the costs per time unit for a given τ, and evaluate this expression for $\tau = 1, 2, 3, 4, 5$.

(d) Suppose that you would like to replace your air filter after an integer number of months, when should you replace your air filter?

Answers

We provide answers for all exercises and complete solution for the last few exercises.

Exercise 1.1

Average costs: $g(\tau) = \frac{60000+2000\tau}{20\tau-\tau^2}$ EURO per month; Optimal age-based policy: $\tau^* = 8.73$ months; $g(\tau^*) = 787$ EURO per month.

Exercise 1.2

a) $h(t) = \frac{4t}{1+2t}$, $t \geq 0$.

b) $\tau = 0.5$ years.

c) $\alpha = 4/3$ is equivalent with $\tau = 1$ year.
$g(\tau) = \frac{700-200(1+2\tau)e^{-2\tau}}{1-(1+\tau)e^{-2\tau}}$, $\tau \geq 0$.
The engineer is right since $g(1) = 1234$ EURO per year $<$ 848 EURO per year $= g(0.5)$; $g(1) - g(0.5) = 386$ EURO per year.

Exercise 1.3

a)

$$h(t) = \begin{cases} 0 & \text{if } 0 \leq t < d; \\ \beta \frac{(t-d)^{\beta-1}}{\eta^\beta} & \text{if } t \geq d. \end{cases}$$

$h(t) = 0$ on $[0, d)$; on $[d, \infty)$, $h(t)$ is constant if $\beta = 1$ (equal to η^{-1}), $h(t)$ is increasing if $\beta > 1$, and $h(t)$ is decreasing if $0 < \beta < 1$.

b) Only if $d = 0$ and $0 < \beta \leq 1$, it can be said beforehand that it is not useful to apply preventive maintenance. In all other cases, i.e. if $d > 0$ or $\beta > 1$, it depends on the costs for the different maintenance actions whether preventive maintenance is useful or not.

c) $\mu = 3.54$ months.

d) 97.3 %

Exercise 1.4

a) $g(\tau_1) = \frac{900}{\tau_1} + 100\tau_1$ EURO per year, $\tau_1 \geq 0$.
$\tau_1^* = 3$ year; $g(\tau_1^*) = 600$ EURO per year.
(For comparison: the no preventive maintenance policy would lead to average costs equal to 1016 EURO per year.)

b) Average costs as a function of τ_2: $\frac{2400}{\tau^2} + 100\tau_2 - 400$, $\tau_2 \geq \tau_1$.
$\tau_2^* = 2\sqrt{6} = 4.9$ years; the corresponding average costs are equal to 580 EURO per year.

c) The policy of b) is slightly better; it is 20 EURO per year cheaper than the policy of a).

Exercise 1.5

b) $M_1 = 0.10$; $M_2 = 0.26$; $M_3 = 0.541$; $M_4 = 0.868$; $M_5 = 1.158$; $M_6 = 1.461$.

c) $g(\tau) = \frac{10000(1+3M_{\tau-1})}{\tau}$ EURO per month.

d) $\tau^* = 3$ months; $g(\tau^*) = 5933$ EURO per month.

Exercise 1.6

a) 11071 EURO per year.

b) $h(x) = 3\sqrt{2} \cdot \sqrt{x}$, $x \geq 0$. The failure rate function is increasing and goes to infinity if $x \to \infty$. Hence, we will find a finite value for the optimal τ for the minimal repair policy.

c) $\tau^* = 1.46$ years; $g(\tau^*) = 10260$ EURO per year; difference $= 811$ EURO per year.

Exercise 1.7

a)

$$h(t) = \begin{cases} 0 & \text{if } 0 \leq t < 3; \\ \frac{2}{3} & \text{if } t \geq 3; \end{cases}$$

This function is increasing, and hence preventive maintenance may be sensible.

b) Since the failure rate is 0 on $[0,3)$, only values for τ with $\tau \geq 3$ are interesting;

$$g(\tau) \;=\; \frac{500 + C_{emer}(1 - e^{-\frac{2}{3}(\tau-3)})}{3 + \frac{3}{2}(1 - e^{-\frac{2}{3}(\tau-3)})}\;, \qquad \tau \geq 3.$$

c) By analyzing $g'(\tau)$, it may be shown that $g(\tau)$ is strictly increasing on $[3,\infty)$ if $C_{emer} > 250$ EURO, constant if $C_{emer} = 250$ EURO, and strictly decreasing if $C_{emer} < 250$ EURO. Hence, $\tau^* = 3$ if $C_{emer} > 250$ EURO, $\tau^* = \infty$ if $C_{emer} < 250$ EURO, and τ^* may be taken equal to any value in $[3,\infty)$ if $C_{emer} = 250$ EURO.

d) In the current situation, $C_{emer} = 200$ EURO, and hence: (i) $\tau^* = \infty$ and $g(\tau^*) = 156$ EURO per year; (ii) Customers replace their orange juice press only correctively, on average once per 4.5 years; (iii) The dealer sells on average $\frac{100}{4.5} = 22.2$ presses per year.

If C_{emer} is increased to 300 EURO, then: (i) τ^* becomes equal to 3 years and $g(\tau^*) = 167$ EURO per year; (ii) Customers start to replace their orange juice preventively each 3 years; (iii) The dealer sells on average $\frac{100}{3} = 33.3$ presses per year.

Exercise 1.8

a) 2 years and 1.41 years, respectively.

b) $h(t) = \frac{t}{t+1}$, $t \geq 0$. The failure rate function is increasing, and thus preventive maintenance may be sensible.

c)

$$g(\tau) \;=\; \frac{200 + C_{emer}[1 - (1+\tau)e^{-\tau}]}{2 - (2+\tau)e^{-\tau}}\;, \qquad \tau \geq 0.$$

d) $g'(\tau) > 0$ if and only if $[(C_{emer} - 200)\tau - (C_{emer} + 200)] + C_{emer}e^{-\tau} > 0$;
$g'(\tau) < 0$ if and only if $[(C_{emer} - 200)\tau - (C_{emer} + 200)] + C_{emer}e^{-\tau} < 0$.

e) The following appears to hold:

- If $C_{emer} = 528$ EURO, then it is optimal to apply preventive maintenance when the lifetime has reached the mean lifetime;
- If $C_{emer} > 528$ EURO, then it is sensible to apply preventive maintenance when the lifetime has reached the mean lifetime, but earlier is even better;

- If $C_{emer} < 528$ EURO, then it is better to wait longer with applying preventive maintenance, and no preventive maintenance should be applied at all in case $C_{emer} \leq 200$.

Exercise 1.9

a) 6770 EURO per year.

b) $h(x) = 18x,\ x \geq 0$. The failure rate function is increasing and goes to infinity if $x \to \infty$. Hence, we will find a finite value for the optimal τ for the minimal repair policy.

c) $\tau^* = 0.75$ years; $g(\tau^*) = 5367$ EURO per year; difference = 1403 EURO per year.

Exercise 1.10

a) $g_{fb} = 2C_u$.

b)

$$g_{age}(\tau) = \frac{C_p + (C_u - C_p)\tau}{-\frac{1}{2}\tau^2 + \tau}, \qquad 0 < \tau \leq 1.$$

e) 0 % if $\frac{C_u}{C_p} \downarrow 1$; 6.7 % if $\frac{C_u}{C_p} = 2$; 20 % if $\frac{C_u}{C_p} = 5$; 31 % if $\frac{C_u}{C_p} = 13$.

Exercise 1.11

(i) Let N be the number of visits a maintenance engineer pays to an arbitrary turbine until he finds that it has failed. Then $\mathbb{P}\{N = i\} = p_i$ and so $\mathbb{E}[N] = \sum_{i=1}^{12} ip_i = 4.9263$.

(ii) The first term of M_t is the expected number of times the first turbine will fail in $(0, t)$. The second term described the expected number of turbine failures beyond the first failure. To find the expected number of turbine failures beyond the first, we condition on the failure time of the first turbine: If the first turbine fails in period i (with probability p_i), then the expected number of remaining failures is M_{t-i}.

(iii) Filling in the given equation, we find:

t	1	2	3	4	5	6	7	8	9	10	11	12
M_t	0.0392	0.1494	0.3124	0.5072	0.7157	0.9262	1.1338	1.3379	1.5401	1.7419	1.9440	2.1498

(iv) Note that there are 10 turbines and that turbines that fail between $\tau - 1$ and τ are maintained at a cost of 200 EURO per turbine. Let $C_u = 500$ and $C_p = 2000$ and let $g(\tau)$ be the average monthly cost for the entire park under a block replacement policy with interval τ; then we find

$$g(\tau) = \frac{10C_u M_{\tau-1} + C_p}{\tau} = \frac{5000M_{\tau-1} + 2000}{\tau}$$

(v) First observe that because a turbine always fails within 12 months the optimal $\tau \in \{1, ..., 12\}$. Thus, using the answers of (c) and (d) we find:

τ	1	2	3	4	5	6	7	8	9	10	11	12
$g(\tau)$	2000.00	1098.03	915.66	890.55	907.25	929.76	947.29	958.60	965.50	970.05	973.59	976.68

From this we conclude that the optimal τ is 4 months and that the optimal average monthly cost is 890.55 EURO.

Exercise 1.12

(i) Let X be the lifetime of a machine. For a failure based policy $ECC = 7000$ EURO and since X has an Erlang distribution with $k = 2$ and $\lambda = 1$, $ECL = k/\lambda = 2$ years. Combining we find that the average yearly cost under a failure based policy is $ECC/ECL = 7000/2 = 3500$ EURO per year.

(ii) It is only useful to apply preventive maintenance for machines that have an increasing failure rate. An Erlang distribution with $k \geq 2$ has an increasing failure rate and so it might be useful to apply preventive maintenance in this case.

(iii) The failure rate of X is given by $h(x) = f(x)/(1 - F(x)) = x/(x + 1)$. Setting $h(x) = 1/6$ and solving for x yields $\tau_0 = 1/5 = 0.2$.

(iv) Using the slides with $C_u = 7000$ and $C_p = 500$, we obtain

$$g(\tau) = \frac{7000(1 - (1 + \tau)e^{-\tau}) + 500(1 + \tau)e^{-\tau}}{\int_0^\tau x^2 e^{-x} dx + (\tau^2 + \tau)e^{-\tau}} \tag{46}$$

Working out the integral in the denominator using integration by parts twice, we have

$$\begin{aligned}
\int_0^\tau x^2 e^{-x}dx &= \left[-x^2e^{-x}\right]_0^\tau + 2\int_0^\tau xe^{-x}dx \\
&= -\tau^2 e^{-\tau} + 2\left(\left[-xe^{-x}\right]_0^\tau + \int_0^\tau e^{-x}dx\right) \\
&= -\tau^2 e^{-\tau} + 2\left(-\tau e^{-\tau} + \left[-e^{-x}\right]_0^\tau\right) \\
&= -\tau^2 e^{-\tau} - 2\tau e^{-\tau} - 2e^{-\tau} + 2 \\
&= -e^{-\tau}(\tau^2 + 2\tau + 2) + 2 \qquad (47)
\end{aligned}$$

Now substituting (47) back into (46), we find:

$$g(\tau) = \frac{7000(1-(1+\tau)e^{-\tau}) + 500(1+\tau)e^{-\tau}}{-e^{-\tau}(\tau^2+2\tau+2) + 2 + (\tau^2+\tau)e^{-\tau}} = \frac{7000 - 6500(1+\tau)e^{-\tau}}{-e^{-\tau}(\tau+2)+2} \qquad (48)$$

(v) Filling in (48), we find $g(\tau_0) = g(0.2) = 3088.15$ EURO per year. This is not surprising because C_p is much smaller than C_u and X has IFR. It is perhaps surprising that the benefit is 11.86% already, even though τ_0 is not the optimal τ for an age based policy.

Exercise 1.13

(a) Yes, because the uniform distribution has increasing failure rate and because preventive maintenance is cheaper than corrective maintenance.

(b) $ECC = C_u = 5000 + 1000 = 6000$, $ECL = \mu = (8+4)/2 = 6$, Cost per collector are 6000/6=1000 EURO per month. For the whole collector park, it is $12 \cdot 1000 = 12000$ EURO per month.

(c) Let τ denote the replacement threshold. Define

$$f(x) = \begin{cases} 1/4, & \text{if } 4 \le x \le 8; \\ 0, & \text{otherwise;} \end{cases} \qquad F(x) = \begin{cases} 0, & \text{if } x < 4; \\ (x-4)/4, & \text{if } 4 \le x \le 8; \\ 1, & \text{if } x \ge 8. \end{cases}$$

Then for $4 \le \tau \le 8$:

$$ECC = 6000F(\tau) + 3000(1 - F(\tau)) = 6000\frac{\tau-4}{4} + 3000\frac{8-\tau}{4} = 750\tau, \qquad (49)$$

$$ECL = \int_4^\tau \frac{1}{4}dt + \frac{8-\tau}{4}\tau = 2\tau - \tau^2/8 - 2, \qquad (50)$$

$$(51)$$

Setting $g'(\tau) = \frac{d}{d\tau}\frac{ECC}{ECL} = 0$ yields the following quadratic equation for τ^*:

$$300\tau - \frac{750}{8}\tau^2 - 3000 - 3000\tau + \frac{750}{4}\tau = 0$$

which yields $\tau^* = \sqrt{24} \approx 4.899$. This means the complete costs for the solar park are $12g(\sqrt{4}) = 9189.50$ EURO per month.

(d) In a block policy, all 12 solar panels are maintained preventively simultaneously every τ months. Therefore, we only need to pay the 5000 EURO to drive into the desert once for 12 collectors for each block maintenance moment.

(e) Before 4 months, collectors cannot fail and after 8 months they will all fail for sure so $4 \leq \tau^* \leq 8$. Since the time between failures is at least 4, there can happen only one failure in any interval of length less than 8.

(f) Observe (by arguments of (e)) that $M(t) = F(t)$ for $4 \leq t \leq 8$. Now we define a cycle for all 12 collectors together. We have:

$$ECC = 12M(\tau)6000 + 2750 + 12 \cdot 250 = 18000\tau - 66250, \tag{52}$$

$$ECL = \tau \tag{53}$$

$$g(\tau) = ECC/ECL = 18000 - 66250/\tau, \tag{54}$$

Observe that g is increasing in τ for $\tau > 0$, so $\tau^* = 4$. Therefore, the costs for maintaining the whole solar park will be 18000-66250/4=1437.50 EURO per month.

Exercise 1.14

(a) At time t, the car can drive $15 - t$ km/l, so the fuel consumption rate at time t is $2000/(15-t)$ l/month. Since a liter of petrol costs 1.5 EURO/l, the money consumption rate at time t is $1.5 \cdot 2000/(15-t) = 3000/(15-t)$. The total costs incurred up to time t is the integral of the money consumption rate from 0 to t.

(b) We use the substitution $u = 15 - x$ so $-du = dx$.

$$\begin{aligned}\int_0^t \frac{3000}{15-x}dx &= -3000\int_{15}^{15-t}\frac{1}{u}du \\ &= -3000\left[\ln(u)\right]_{u=15}^{u=15-t} \\ &= 3000(\ln(15) - \ln(15-t)) \\ &= 3000\ln\left(\frac{15}{15-t}\right)\end{aligned}$$

(c) Let $g(\tau)$ denote the costs per time unit.

$$g(\tau) = \frac{3000}{\tau} \ln\left(\frac{15}{15-\tau}\right) + \frac{80}{\tau}$$

Now it is straightforward to compute:

τ	1	2	3	4	5
$g(\tau)$	286.9786	254.6513	249.8102	252.6162	259.2791

(d) The filter should be replaced every three months.

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