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MTH4220(2022 Fall): Quiz 1

- This quiz will last for 50 mins.
- No phones or discussion, book and notes closed, no calculators.

83 + 2 + 10 = 95

+1

$\begin{matrix} 2 & -1 \\ 1 & 2 \end{matrix}$ $\begin{matrix} 2 & 1 \\ -1 & 2 \end{matrix}$ $x' = 2x + y$ $\partial x = 2\partial x' - \partial y'$

$y' = -x + 2y$ $\partial y = \partial x' + 2\partial y'$

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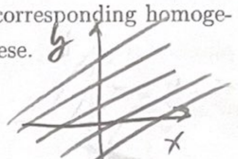
1. For a given function f, consider the first order PDE:

$$2\partial_x u + \partial_y u = f(u)$$

(a) (10pts) What kind of coordinates (x', y') you can introduce so that the terms on the left hand side of the equation can be reduced to $C\partial_{x'} u$ for some constant C.

$x' = 2x + y, \ y' = -x + 2y, \Rightarrow 2\partial_x + \partial_y = 5\partial_{x'}$ 10

(b) (10pts) What are the characteristic lines/curves to the corresponding homogeneous equation, i.e. $f(u) \equiv 0$ in the equation? Sketch these.



$y = \frac{1}{2}x + C$

$\frac{dy}{dx} = \frac{1}{2} \Rightarrow y = \frac{1}{2}x + C$ 10

(c) (5pts) A solution u to the equation is constant on any of these characteristic curves/lines. True or false? False 5

(d) (10pts) Consider $f(u) = u$. For $k, m \in \mathbb{R}$ and $g = g(x)$ given, discuss the existence and uniqueness of solutions to the equation with prescribed data: $u(x, y(x)) = g(x)$ on the line $y(x) = kx + m$.

Solve the equation:

$2\partial_x u + \partial_y u = u$

$5\partial_{x'} u = u$

$\Rightarrow \ln|u| = \frac{1}{5}x' + C(y')$

$u = F(y')e^{x'/5} = F(-x+2y)e^{(2x+y)/5}$.

For initial data $u(x, y(x))$ on $y(x) = kx + m$,

① $k = \frac{1}{2}$. solution might not exist, or only uniquely determined on $y(x) = kx + m$, Not uniquely on $\mathbb{R}^2 \setminus \{(x, kx+m), x \in \mathbb{R}\}$

② $k \neq \frac{1}{2}$. solution is uniquely determined on $\mathbb{R}^2$, $(x, y) \in \mathbb{R}^2$

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$$\begin{bmatrix} \partial\xi \\ \partial\eta \end{bmatrix} = \begin{bmatrix} 1 & -a \\ 1 & -b \end{bmatrix} \begin{bmatrix} \partial t \\ \partial x \end{bmatrix}$$

$$\underline{\begin{pmatrix} -b & a \\ -1 & 1 \end{pmatrix}} \begin{pmatrix} 1 & -a \\ 1 & -b \end{pmatrix} \qquad \begin{bmatrix} -b & -1 \\ a & 1 \end{bmatrix} \begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

$$(\partial t - a\partial x)(\partial t - b\partial x)$$

+17

2. Let $a, b \in \mathbb{R}$ be given constants, consider the second order equation

$$\partial_{tt}u - (a+b)\partial_{tx}u + ab\partial_{xx}u = 0.$$

(a) (10pts) Determine the equation type(elliptic, parabolic or hyperbolic).

$A = \begin{bmatrix} 1 & -(a+b)/2 \\ -(a+b)/2 & ab \end{bmatrix}$ $\det A = -\frac{1}{4}(a-b)^2 \Rightarrow$ parabolic if $a=b$. hyperbolic if $a \neq b$

(b) (10pts) Is it possible to introduce some suitable coordinates (ξ, η) such that the equation can be reduced to $\partial_{\xi\eta}u = 0$? If so, what are they?

$\partial x = -\partial\xi + \partial\eta$

Yes. $\xi = -x - \cancel{b}t$, $\eta = at + x$ $\Rightarrow$ $\partial t = -b\partial\xi + a\partial\eta$

-7

+5

(c) (10pts) Solve the equation for the case $a = 3, b = 4$.

Using the result from (b),

$(\partial t - a\partial x)(\partial t - b\partial x)u = 0$

$\Rightarrow (a-b)\partial\xi\, (a-b)\partial\eta\, u = 0$

plug in $a = 3, b = 4$

$\partial\xi\partial\eta u = 0.$

$\Rightarrow u = F(\xi) + G(\eta)$

$= F(-x \cancel{-} 4t) + G(3t + x)$

.5 +5

32 +2

3. Let $\Omega \subset \mathbb{R}^n$ be a smooth connected domain, $T \in (0, \infty)$ a given constant, and $\Omega_T = \Omega \times [0, T]$.

5

(a) (5pts) What is the parabolic boundary, $\partial_p \Omega_T$, of Ω_T?

$\partial_p \Omega_T = \Omega \times \{0\} \cup \partial\Omega \times [0, T]$ ✓

9

(b) (10pts) If u attains a **minimum** at the point $(\vec{x}_0, t_0) \in \Omega_T \backslash \partial_p \Omega_T$, what can you say about the signs of the first and second derivatives of u there?

$u_t(\vec{x}_0, t_0) \leq 0$, $\nabla_{\vec{x}} u(\vec{x}_0, t_0) = \vec{0} \in \mathbb{R}^n$, ✓ $\Delta_{\vec{x}} u(\vec{x}_0, t_0) \geq 0$

(c) (10pts) Does the **Minimum Principle** holds to the equation $\partial_t u - \Delta u \geqslant 0$ on Ω_T? If so, describe it.

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Yes. For the DE satisfies $\partial_t u - \Delta u \geq 0$ on Ω_T, then $\min_{\Omega_T} u = \min_{\partial_p \Omega_T} u$

(d) (10pts) Give an example of approximate function, v_ε, of u in order to show the **Minimum Principle** to the heat equation $\partial_t u - \Delta u = 0$ on Ω_T.

$V_\varepsilon = u + \varepsilon t$

8 +2

Then show that

$$\lim_{\varepsilon \to 0^+} \overset{\min}{\cancel{\max_{\Omega_T}}} v_\varepsilon = \lim_{\varepsilon \to 0^+} \overset{\min}{\cancel{\max_{\Omega_T}}} u.$$

① $\min_{\Omega_T} V_\varepsilon \geq \min_{\Omega_T} u + \min_{\Omega_T} \varepsilon t$, limit preserve inequality.

$\Rightarrow \lim_{\varepsilon \to 0^+} \min_{\Omega_T} V_\varepsilon \geq \min_{\Omega_T} u + \lim_{\varepsilon \to 0^+} \min \varepsilon t = \min_{\Omega_T} u + \lim_{\varepsilon \to 0^+} 0$

$\downarrow$ since $0 \leq t \leq T$

0

$\Rightarrow \lim_{\varepsilon \to 0^+} \min_{\Omega_T} V_\varepsilon \geq \min_{\Omega_T} u$ ✓

② $\min_{\Omega_T} u \geq \min_{\Omega_T} V_\varepsilon + \min_{\Omega_T} (-\varepsilon t)$

$\Rightarrow \min_{\Omega_T} u \geq \lim_{\varepsilon \to 0^+} \min_{\Omega_T} V_\varepsilon + \lim_{\varepsilon \to 0^+} \min_{\Omega_T} (-\varepsilon t) = \lim_{\varepsilon \to 0^+} \min_{\Omega_T} V_\varepsilon + \lim_{\varepsilon \to 0^+} -\varepsilon T$

$\downarrow$ t bdd

0

$\Rightarrow \min_{\Omega_T} u \geq \lim_{\varepsilon \to 0^+} \min_{\Omega_T} V_\varepsilon$

① + ② $\Rightarrow \min_{\Omega_T} u \leq \lim_{\varepsilon \to 0^+} \min_{\Omega_T} V_\varepsilon \leq \min_{\Omega_T} u \Rightarrow \min_{\Omega_T} u = \lim_{\varepsilon \to 0^+} \min_{\Omega_T} V_\varepsilon$