MAT4220 FA22 HW06

Haoran Sun (haoransun@link.cuhk.edu.cn)

Problem 1 (P135 Q15)**.**

(a) Note that $\|\cos(n+1/2)x\|^2 = \pi/2$ on $(0,\pi)$, then we have

$$B_n = \frac{\pi}{2}\int_0^\pi \cos[(n+1/2)x]\,\mathrm{d}x = \frac{4}{(2n+1)\pi}(-1)^n$$

(b) The series converges for all x in $(-2\pi, 2\pi)$.

$$S(x) = \begin{cases} 1 & x \in (-\pi,\pi) \\ -1 & x \in (-2\pi,-\pi)\cup(\pi,2\pi) \\ 0 & x = \pm\pi \end{cases}$$

(c) Using Parseval's equality, we have

$$\begin{aligned} \sum_{n=0} Bn^2 \|X_n(x)\|^2 &= \|\phi(x)\|^2 \\ \Rightarrow \sum_{n=0} \frac{16}{\pi}\frac{1}{(2n+1)^2}\frac{\pi}{2} &= \pi \\ \sum_{n=0} \frac{1}{(2n+1)^2} &= \frac{\pi}{8} \end{aligned}$$

Problem 2 (P145 Q4)**.**

(a) Let $T(t)X(t)$ satisfies the boundary condition, then

$$\frac{T'(t)}{kT} = \frac{X''(x)}{X(x)} = -\lambda$$

Assume that $\lambda \geq 0$. If $\lambda = 0$, we have $X_0(x) = A + Bx$. Else, let $\beta^2 = \lambda > 0$, then we have the following form of the eigenfunctions.

$$X(x) = C\cos\beta x + D\sin\beta x$$

Applying the boundary condition, we can solve that

$$\tan\frac{\beta_n l}{2} = \frac{\beta_n l}{2}$$

Hence $u(x,t)$ could be written in the form of

$$u(x,t) = A + Bx + \sum_{n=1}^\infty e^{-\beta_n^2 kt}(c_n\cos\beta_n x + d_n\sin\beta_n x)$$

(b) Suppose we can take the limit term by term, hence

$$\begin{aligned} \lim_{t\to\infty}\sum_{n=1}^\infty e^{-\beta_n^2 kt}(c_n\cos\beta_n x + d_n\sin\beta_n x) &= \sum_{n=1}^\infty \lim_{t\to\infty} e^{-\beta_n^2 kt}(c_n\cos\beta_n x + d_n\sin\beta_n x) \\ &= \sum_{n=1}^\infty 0 = 0 \end{aligned}$$

Consequently

$$\lim_{t\to\infty} u(x,t) = A + Bx$$

(c) Suppose $\lambda = -\beta^2 < 0$. Note that

$$\int_0^l u_x(x,t)\,\mathrm{d}x = u(x,t)u_x(x,t)|_{x=0}^{x=l} - \int_0^l u(x,t)u_{xx}(x,t)\,\mathrm{d}x \geq 0 \Rightarrow \int_0^l u(x,t)u_{xx}(x,t)\,\mathrm{d}x \leq 0$$

However

$$\int_0^l u(x,t)u_{xx}(x,t)\,\mathrm{d}x = T(t)^2 \int_0^l X(x)X''(x)\,\mathrm{d}x = T(t)^2\beta^2 \int_0^l X^2(x)\,\mathrm{d}x \geq 0$$

where the contradiction occurs that $\lambda = 0$ in this case. Hence $\lambda > 0$.

(d) Since $\langle 1, 1\rangle = l$, $\langle x, x\rangle = l^3/3$, then

$$A = \frac{1}{l}\int_0^l \phi(x)\,\mathrm{d}x \quad B = \frac{3}{l^3}\int_0^l x\phi(x)\,\mathrm{d}x$$

Problem 3 (P145 Q6)**.** Suppose u is in the form of

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-\frac{n^2\pi^2}{l^2}kt}\cos\frac{n\pi x}{l}$$

where $u(x,0) = \phi(x)$ continuous on $[0,l]$.

Claim. A_n bounded.

Proof. Since ϕ is continuous on $[0,l]$, then $\|\phi\|$ bounded, hence

$$|\langle \phi, \cos\frac{n\pi}{l}x\rangle| \leq \|\phi\|\|\cos\frac{n\pi}{l}\| < \infty \Rightarrow A_n = \frac{2}{l}\langle \cos\frac{n\pi}{l}, \phi\rangle < M$$ □

Claim. The following series converges $\forall t > 0$

$$\sum_{n=1}^{\infty} A_n n^k e^{-n^2 t}$$

Proof. Note that $\forall\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall m > N$ we have $\sum_N^\infty n^k e^{-nt} < \epsilon/M$, then

$$\sum_{n=1}^{\infty} |A_n n^k e^{-n^2 t}| \leq M\sum_{n=1}^{\infty} n^k e^{-n^2 t} < M\sum_{n=1}^{m-1} n^k e^{-nt} + \epsilon$$

Hence the series converges. □

According to the claims, $\exists N \in \mathbb{N}$ s.t. $\forall x$ and $\forall m > N$ we have

$$\left|\sum_m^\infty \frac{\mathrm{d}^k}{\mathrm{d}x^k} A_n e^{-\frac{n^2\pi^2}{l^2}kt}\cos\frac{n\pi x}{l}\right| \leq M\sum_{n=m}^{\infty} e^{-\frac{n^2\pi^2}{l^2}kt} < \epsilon$$

which means the series converges uniformly with respect to x. Hence, using the theorem in the appendix, we have

$$\begin{aligned}\frac{\mathrm{d}^k}{\mathrm{d}x^k}u(x,t) &= \frac{\mathrm{d}^k}{\mathrm{d}x^k}\sum_{n=1}^{\infty} A_n e^{-\frac{n^2\pi^2}{l^2}kt}\cos\frac{n\pi x}{l} = \sum_{n=1}^{\infty}\frac{\mathrm{d}^k}{\mathrm{d}x^k} A_n e^{-\frac{n^2\pi^2}{l^2}kt}\cos\frac{n\pi x}{l} \\ &= \sum_{n=1}^{\infty} B_n n^k e^{-\frac{n^2\pi^2}{l^2}kt}\cos\frac{n\pi x}{l}\end{aligned}$$

exists $\forall k$ in $t > 0$.

Problem 4 (P145 Q11)**.** Follow the same steps when proving the uniform convergence. Since the $f'(x)$ piecewise continuous $f'(x)X(x)$ also piecewise continuous, then it is integrable and hence

$$A_n = -\frac{1}{n}B_n' \quad B_n = \frac{1}{n}A_n'$$

Applying Bessel's inequality, we have the following series which is convergent

$$\sum_{n=1}^{\infty}({A_n'}^2 + {B_n'}^2) < \infty$$

Therefore

$$\begin{aligned}
\left|\sum_{n=1}^{\infty} A_n \cos nx + B_n \sin nx\right| &\leq \sum_{n=1}^{\infty} |A_n \cos nx| + |B_n \sin nx| \\
&\leq \sum_{n-1}^{\infty} |A_n| + |B_n| \\
&= \frac{1}{n}\sum_{n-1}^{\infty} |A_n'| + |B_n'| \\
&\leq \left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{1/2}\left(\sum_{n=1}^{\infty}|A_n'|^2 + |B_n'|^2 + 2|A_n'||B_n'|\right)^{1/2} \\
&\leq \left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{1/2}\left(\sum_{n=1}^{\infty}2(|A_n'|^2 + |B_n'|^2)\right)^{1/2} < \infty
\end{aligned}$$

Hence $\forall \epsilon > 0$, we can choose $N \in \mathbb{N}$ s.t. $\forall m > N$ and $\forall x$ we have

$$\begin{aligned}
\left|f(x) - \sum_{n=1}^{m-1} A_n \cos nx + B_n \sin nx\right| &= \left|\sum_{n=m}^{\infty} A_n \cos nx + B_n \sin nx\right| \\
&\leq M\sum_{n=m}^{\infty}(|A_n'|^2 + |B_n'|^2) < \epsilon
\end{aligned}$$

Hence the Fourier series converges uniformly.

Problem 5 (P160 Q5)**.** Note that $(x^2+y^2)/4+c$ is the solution of $\Delta u = 1$. Since $(x^2+y^2)/4-1/4$ satisfies the boundary condition, according to the uniqueness, the solution is $u(x,t) = (x^2+y^2)/4-1/4$.

Problem 6 (P160 Q11)**.** Suppose there is a solution u, then

$$\iiint_D f \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \iiint_D \nabla \cdot \nabla u \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \oiint_{\partial D} \nabla u \cdot \mathbf{n} \, \mathrm{d}\sigma = \oiint_{\partial D} \frac{\partial u}{\partial \mathbf{n}} \, \mathrm{d}\sigma = \oiint_{\partial D} g \, \mathrm{d}\sigma$$

Then if equality does not hold, there will be no solutions.

Problem 7 (P160 Q13)**.** Let $v = u + \epsilon|\mathbf{x}|^2$. Suppose v obtains its maximum in the interior domain of D, then $\Delta v \leq 0$. Note that

$$\Delta v = \Delta u + 4n\epsilon > 0$$

where n is the dimension. This contradicts the assumption. Hence $\max_D v = \max_{\partial D} v$. Since D is bounded, we can also show that

$$\max_D v = \max_D u \quad \max_{\partial D} v = \max_{\partial D} u$$

Hence $\max_D u = \max_{\partial D} u$.