Mid-Term Exam for MATH4220

1. (**20 points**)

   (a) (**8 points**) Solve the following problem

$$\begin{cases} \partial_t u + 4\partial_x u - 2u = 0 \\ u(x, t = 0) = x^2 \end{cases}$$

   (b) (**12 points**) Solve the problem

$$\begin{cases} 2\partial_x u + y\partial_y u = 0 \\ u(x = 0, y) = y \end{cases}$$

   What are characteristic curves of this equation?

2. (**20 points**)

   (a) (**8 points**) Is the following initial-boundary value problem well-posed? Why?

$$\begin{cases} \partial_t u - \partial_x u = 0, & x > 0, \quad t > 0 \\ u(x, t = 0) = \sin x, & t > 0 \\ u(x = 0, t) = 0, & t > 0 \end{cases}$$

   (b) (**4 points**) For each positive integer n, is

$$u_n(x, y) = \frac{1}{n} e^{-\sqrt{n}} \sin(nx) \frac{e^{ny} - e^{-ny}}{2}$$

   a solution to the following Cauchy problem

$$\begin{cases} \partial_x^2 u + \partial_y^2 u = 0, & -\infty < x < +\infty, \quad y > 0 \\ u(x, 0) = 0 \\ \partial_y u(x, y = 0) = \frac{1}{n} e^{-\sqrt{n}} \sin(nx) \end{cases}$$

   (c) (**8 points**) Is the following Cauchy problem

$$\begin{cases} \partial_x^2 u + \partial_y^2 u = 0, & -\infty < x < +\infty, \quad y > 0 \\ u(x, 0) = 0 \\ \partial_y u(x, 0) = 0 \end{cases}$$

   Well-posed? Explain in details why?

3. (**20 points**)

(a) Prove the following generalized maximum principle:
if $\partial_t u - \partial_x^2 u \leq 0$ on $R \equiv [0,l] \times [0,T]$, then

$$\max u(x,t) = \max_{\partial R} u(x,t)$$

where $\partial R = \{(x,t) \in R|$ such that either $x = 0$, or $x = l$, or $t = 0\}$.

(b) Show that if v solves the following problem

$$\begin{aligned} &\partial_t v = \partial_x^2 v + f(x,t), \quad 0 < x < l, \quad 0 < t < T \\ &v(x,0) = 0 \\ &v(0,t) = 0 = v(l,t), \quad 0 \leq x \leq T \end{aligned}$$

then

$$v(x,t) \leq t \max_R |f(x,t)|$$

(hint, applying the result in (a) to $u(x,t) = v(x,t) - t\max_R |f(x,t)|$)

4. (**20 points**)

(a) (**10 points**) Consider the following problem

$$\begin{cases} \partial_t u = \partial_x^2 u + f(x,t), & -\infty < x < +\infty, \quad t > 0 \\ u(x, t=0) = \varphi(x) \end{cases}$$

Prove that if $\varphi(x)$ and $f(x,t)$ are even functions of x, then the solution $u(x,t)$ to above solution must be even in x.

(b) (**10 points**) Apply the result in (a) to solve the following problem

$$\begin{cases} \partial_t u = \partial_x^2 u + e^{-x^2}, & x > 0, \quad t > 0 \\ u(x, t=0) = \cos x, & x > 0 \\ \partial_x u(x=0,t) = 0 \end{cases}$$

5. (**20 points**)

(a) Find the general solution formula for

$$\begin{cases} \partial_t^2 u + \partial_{xt} u - 2\partial_x^2 u = 0 \\ u(x,0) = \varphi(x) \\ \partial_t u(x,0) = 0 \end{cases}$$

(b) In part (a), find the solution with

$$\varphi(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

and draw the graph of $u(x,1)$.