MAT4220 FA22 HW02

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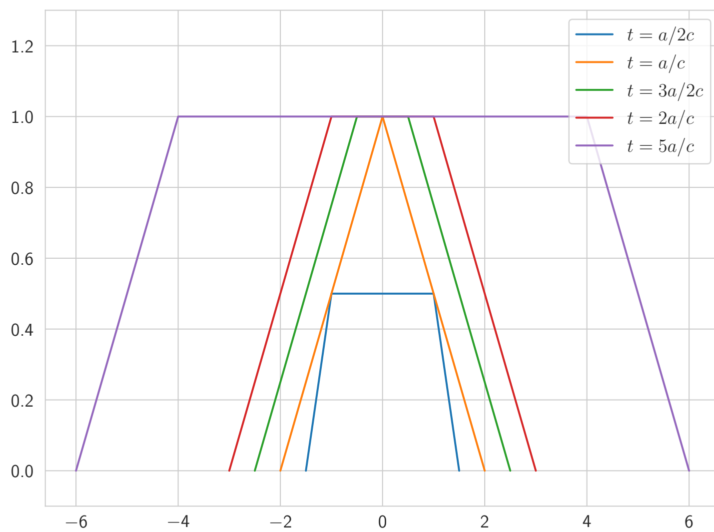
Problem 1 (P38 Q2). Using the formula for the wave equation

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}(\phi(ct+x)+\phi(-ct+x)) + \frac{1}{2c}\int_{-ct+x}^{ct+x}\phi(s)\,\mathrm{d}s \\
&= \frac{1}{2}[\log(1+(ct+x)^2)+\log(1+(-ct+x)^2)] + (4s+\frac{1}{2}s^2)\Big|_{-ct+x}^{ct+x} \\
&= \frac{1}{2}[\log(1+(ct+x)^2)+\log(1+(-ct+x)^2)] + 8ct + 2xct
\end{aligned}
$$

Problem 2 (P38 Q5). Sketch:



Problem 3 (P38 Q7). Since ϕ and ψ are odd functions, then

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}(\phi(ct+x)+\phi(-ct+x)) + \frac{1}{2c}\int_{-ct+x}^{ct+x}\psi(s)\,\mathrm{d}s \\
\Rightarrow u(-x,t) &= \frac{1}{2}(\phi(ct-x)+\phi(-ct-x)) + \frac{1}{2c}\int_{-ct-x}^{ct-x}\psi(s)\,\mathrm{d}s \\
&= \frac{1}{2}(-\phi(-ct+x)-\phi(ct+x)) + \frac{1}{2c}\int_{ct+x}^{-ct+x}\psi(-u)\,\mathrm{d}(-u) \\
&= \frac{1}{2}(-\phi(-ct+x)-\phi(ct+x)) + \frac{1}{2c}\int_{ct+x}^{-ct+x}\psi(u)\,\mathrm{d}u \\
&= -\frac{1}{2}(\phi(-ct+x)+\phi(ct+x)) - \frac{1}{2c}\int_{-ct+x}^{ct+x}\psi(u)\,\mathrm{d}u = -u(x,t)
\end{aligned}
$$

Problem 4 (P38 Q9). Setting

$$
\begin{aligned}
\xi &= x - t \\
\eta &= 4x + t
\end{aligned}
$$

we have

$$\begin{aligned}\partial_x &= \partial_\xi + 4\partial_\eta \\ \partial_t &= -\partial_\xi + \partial_\eta\end{aligned}$$

Then

$$\begin{aligned}\partial_{xx} - 3\partial_{xt} - 4\partial_{tt} &= (\partial_x - 4\partial_t)(\partial_x + \partial_t) \\ &= 25\partial_\xi\partial_\eta\end{aligned}$$

Then the general solution of the equation $\partial_\xi\partial_\eta u = 0$ would be

$$u(x,t) = F(\xi) + G(\eta) = F(x-t) + G(4x+t)$$

Then

$$\begin{aligned}&\phi(x) = F(x) + G(4x) \\ &\psi(x) = -F'(x) + G'(4x) \Rightarrow \Psi(x) = -F(x) + \frac{1}{4}G(4x)\end{aligned}$$

where $\Psi(x)$ is any function that $\Psi'(x) = \psi(x)$, then

$$\begin{aligned}F(x) &= \frac{1}{5}(\phi(x) - 4\Psi(x)) \\ G(4x) &= \frac{4}{5}(\phi(x) + \Psi(x)) \Rightarrow G(x) = \frac{4}{5}(\phi(x/4) + \Psi(x/4)) \\ u(x) &= \frac{1}{5}(\phi(x-t) - 4\Psi(x-t)) + \frac{4}{5}(\phi(x+t/4) + \Psi(x+t/4)) \\ &= \frac{1}{5}(\phi(x-t) + 4\phi(x+t/4)) + \frac{4}{5}\int_{x-t}^{x+t/4} \psi(s)\,\mathrm{d}s\end{aligned}$$

According to the boundary condition $u(x,0) = x^2$, $u_t(x,0) = e^x$, we have

$$u(x) = \frac{1}{5}[(x-t)^2 + 4(x+t/4)^2] + \frac{4}{5}(e^{x+t/4} - e^{x-t})$$

Problem 5 (P41 Q1)**.** Using the conservation law, we know that

$$E(t) = E(0) = \frac{1}{2}\int_\Omega \psi^2(x) + c^2\phi'^2(x) = \frac{1}{2}\int_\Omega 0\,\mathrm{d}x = 0$$

Then

$$E(t) = \frac{1}{2}\int_\Omega u_t^2(x,t) + c^2 u_x^2(x,t)\,\mathrm{d}x = 0 \Rightarrow u_t = 0, u_x = 0$$

by the first vanish theorem. Then we can solve that $u = k$ for some constant k. Since $\phi(x) = u(x,0) = 0$, we have $u(x,t) = u(x,0) = 0$.