MAT4220 FA22 HW02

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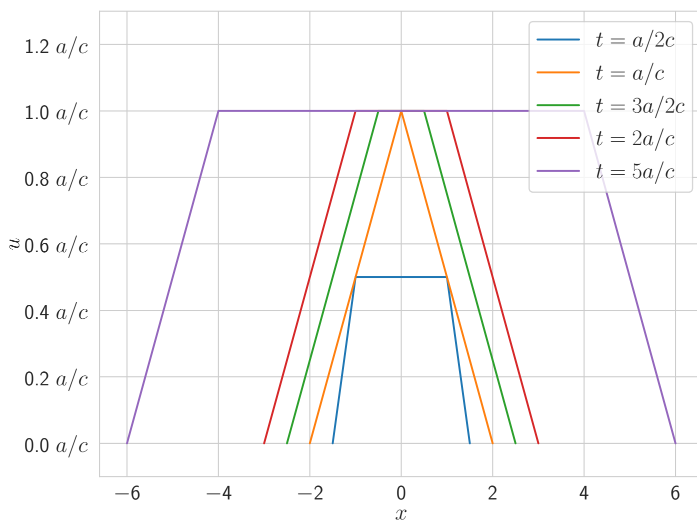
Problem 1 (P38 Q2). Using the formula for the wave equation

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}(\psi(ct+x)+\psi(-ct+x)) + \frac{1}{2c}\int_{-ct+x}^{ct+x}\phi(s)\,\mathrm{d}s \\
&= \frac{1}{2}[\log(1+(ct+x)^2)+\log(1+(-ct+x)^2)] + (4s+\frac{1}{2}s^2)\Big|_{-ct+x}^{ct+x} \\
&= \frac{1}{2}[\log(1+(ct+x)^2)+\log(1+(-ct+x)^2)] + 8ct + 2cxt
\end{aligned}
$$

Problem 2 (P38 Q5). Sketch:



Problem 3 (P38 Q7). Since ϕ and ψ are odd functions, then

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}(\phi(ct+x)+\phi(-ct+x)) + \frac{1}{2c}\int_{-ct+x}^{ct+x}\psi(s)\,\mathrm{d}s \\
\Rightarrow u(-x,t) &= \frac{1}{2}(\phi(ct-x)+\phi(-ct-x)) + \frac{1}{2c}\int_{-ct-x}^{ct-x}\psi(s)\,\mathrm{d}s \\
&= \frac{1}{2}(-\phi(-ct+x)-\phi(ct+x)) + \frac{1}{2c}\int_{ct+x}^{-ct+x}\psi(-u)\,\mathrm{d}(-u) \\
&= \frac{1}{2}(-\phi(-ct+x)-\phi(ct+x)) + \frac{1}{2c}\int_{ct+x}^{-ct+x}\psi(u)\,\mathrm{d}u \\
&= -\frac{1}{2}(\phi(-ct+x)+\phi(ct+x)) - \frac{1}{2c}\int_{-ct+x}^{ct+x}\psi(u)\,\mathrm{d}u = -u(x,t)
\end{aligned}
$$

Therefore $u(-x,t) = -u(x,t)$, u an odd function in x.

Problem 4 (P38 Q9). Setting

$$\xi = x - t$$

$$\eta = 4x + t$$

we have

$$\begin{aligned}\partial_x &= \partial_\xi + 4\partial_\eta \\ \partial_t &= -\partial_\xi + \partial_\eta\end{aligned}$$

Then

$$\begin{aligned}\partial_{xx} - 3\partial_{xt} - 4\partial_{tt} &= (\partial_x - 4\partial_t)(\partial_x + \partial_t) \\ &= 25\partial_\xi\partial_\eta\end{aligned}$$

Then the general solution of the equation $\partial_\xi\partial_\eta u = 0$ would be

$$u(x,t) = F(\xi) + G(\eta) = F(x-t) + G(4x+t)$$

Then

$$\begin{aligned}\phi(x) &= F(x) + G(4x) \\ \psi(x) &= -F'(x) + 4G'(4x) \Rightarrow \Psi(x) = -F(x) + G(4x)\end{aligned}$$

where $\Psi(x)$ is any function that $\Psi'(x) = \psi(x)$, then

$$\begin{aligned}F(x) &= \frac{1}{2}(\phi(x) - \Psi(x)) \\ G(4x) &= \frac{1}{2}(\phi(x) + \Psi(x)) \\ u(x) &= \frac{1}{2}(\phi(x-t) - \Psi(x-t)) + \frac{1}{2}(\phi(x+t/4) + \Psi(x+t/4)) \\ &= \frac{1}{2}[\phi(x-t) + \phi(x+t/4)] + \frac{1}{2}\int_{x-t}^{x+t/4} \psi(s)\,\mathrm{d}s\end{aligned}$$

According to the boundary condition $u(x,0) = x^2$, $u_t(x,0) = e^x$, we have

$$u(x) = \frac{1}{2}[(x-t)^2 + (x+t/4)^2] + \frac{1}{2}e^x(e^{t/4} - e^{-t})$$

Problem 5 (P41 Q1)**.** Using the conservation law, we know that

$$E(t) = E(0) = \frac{1}{2}\int_\Omega \psi^2(x) + c^2\phi'^2(x) = \frac{1}{2}\int_\Omega 0\,\mathrm{d}x = 0$$

Then

$$E(t) = \frac{1}{2}\int_\Omega u_t^2(x,t) + c^2u_x^2(x,t)\,\mathrm{d}x = 0 \Rightarrow u_t = 0, u_x = 0$$

by the first vanish theorem. Then we can solve that $u(x,t) = k$ for some constant k. Since $\phi(x) = u(x,0) = 0$, we have $u(x,t) = u(x,0) = 0$.

Problem 6 (P46 Q4)**.**

1. Since the maximum of u is on $\partial_p\Omega_T$, then
$$\max u(x,t) = \max_{\partial_p\Omega_T} u(x,t) = \max_{x\in[0,1]} \{0, 4x(1-x)\} = 1$$
Also, since u not a constant function, then $\forall (x,t) \in \Omega_T \setminus \partial_p\Omega_T$
$$u(x,t) < \max u(x,t) = 1$$
Similarly, according to the minimum principle, one can show that $\min u(x,t) = 0$ and thus
$$u(x,t) > 0 \ \forall (x,t) \in \Omega_T \setminus \partial_p\Omega_T$$
Finally we have $0 < u(x,t) < 1$ on $\Omega_T \setminus \partial_p\Omega_T = (0,1)\times(0,\infty)$.

2. Define $v(x,t) = u(1-x,t)$.
Claim. $v_t = v_{xx}$

Proof. Since
$$\begin{aligned} v_t(x,t) &= u_t(1-x,t) \\ v_{xx}(x,t) &= \partial_x^2 u(1-x,t) = \partial_x \frac{\mathrm{d}(1-x)}{\mathrm{d}x}\frac{\partial}{\partial(1-x)} u(1-x,t) = -\partial_x u_x(1-x,t) \\ &= -\frac{\mathrm{d}(1-x)}{\mathrm{d}x}\frac{\partial}{\partial(1-x)} u_x(1-x,t) \\ &= u_{xx}(1-x,t) \end{aligned}$$
Then $v_t(x,t) - v_{xx}(x,t) = 0$. □

We can also show that u and v have the same boundary condition where
$$v(x,0) = u(1-x,0) = 4x(1-x),\ v(0,t) = u(1,t) = u(0,t) = v(1,t) = 0$$
Then, according to the uniqueness of diffusion equation, $u(x,t) = v(x,t)$ on Ω_T.

3. The energy method shows that with $u(x,0) = 0$, $u_t(0,t) = u_t(0,l) = 0$, we have
$$\frac{\mathrm{d}}{\mathrm{d}t}E = -\kappa\int_\Omega u_x^2\,\mathrm{d}x \le 0$$
Since $0 < u < 1$ in $\Omega_T \setminus \partial_p\Omega_T$, $\forall t\ \exists x \in (0,1)$ s.t. $u_x(x,t) \ne 0$. Therefore $\forall t$
$$\frac{\mathrm{d}}{\mathrm{d}t}E = -\kappa\int_\Omega u_x^2\,\mathrm{d}x < 0$$

Problem 7 (P46 Q6)**.** To prove the comparison principle, we can prove an equivalent statement: if u is a solution to diffusion equation with $u \ge 0$ on $\partial_p\Omega_T$, then we have $u \ge 0$ on Ω_T.

Proof. According to the minimum principle
$$\min_{\Omega_T} u = \min_{\partial_p\Omega_T} u \ge 0$$
then $u \ge 0$ on Ω_T. □

Therefore $\forall u, v$ solution to the diffusion equation with same boundary condition and $u \leq v$, we know that $w = v - u$ a solution to the diffusion equation with zero boundary condition. According to our claim, on $\Omega = [0, l] \times [0, \infty)$

$$w \geq 0 \Rightarrow v - u \geq 0$$

Problem 8.

Claim. The function $v(x,t) = u(x,t) - \epsilon t$ with $\epsilon > 0$ takes its maximum on $\partial_p \Omega_T$.

Proof. Prove by contradiction. Suppose $\exists (x_\epsilon, t_\epsilon) \in \Omega_T \setminus \partial_p \Omega_T$ s.t.

$$\max_{\Omega_T} v(x,t) = v(x_\epsilon, t_\epsilon)$$

Then $v_t(x_\epsilon, t_\epsilon) = u_t(x_\epsilon, t_\epsilon) - \epsilon \geq 0$, $v_x(x_\epsilon, t_\epsilon) = 0$, $v_{xx}(x_\epsilon, t_\epsilon) = u_{xx}(x_\epsilon, t_\epsilon) \leq 0$, then we have $v_t(x_\epsilon, t_\epsilon) - \kappa v_{xx}(x_\epsilon, t_\epsilon) \geq 0$. However

$$v_t(x_\epsilon, t_\epsilon) - \kappa v_{xx}(x_\epsilon, t_\epsilon) = u_t(x_\epsilon, t_\epsilon) - \kappa u_{xx}(x_\epsilon, t_\epsilon) - \epsilon \leq -\epsilon < 0$$

Then we have contradiction. Hence v takes its maximum on some point $(x_\epsilon, t_\epsilon) \in \partial_p \Omega_T$. □

Since $\partial_p \Omega_T$ compact, we can pick a sequence of ϵ_n s.t. $(x_{\epsilon_n}, t_{\epsilon_n}) \to (x_0, t_0) \in \partial_p \Omega_T$ as $\epsilon_n \to 0^+$ (every bounded sequence has a converged subsequence). Using the fact that $\lim_{\epsilon \to 0^+} \max v = \max u$ which was proved in the class, u takes its maximum on $(x_0, y_0) \in \partial_p \Omega_T$.