Name: ____________________ SID: ____________________

MTH4220(2022 Fall): Quiz 2

- This quiz will last for 50 mins.
- No phones or discussion, book and notes closed, no calculators.

1. (a) (True or False, 5pts) Given any function set $\{X_n : [a,b] \to \mathbb{R}\}_{n=1}^{N}$, the least square approximation of f in the set is given by

$$\sum_{n=1}^{N} A_n X_n, \qquad \text{for } A_n = \frac{\langle f, X_n \rangle_{L^2}}{\langle X_n, X_n \rangle_{L^2}}. \tag{1}$$

(b) (Short answer, 10pts) What are symmetric boundary conditions on an given interval (a,b)? Examples? Why do we introduce the symmetric boundary condition?

(c) (True or False, 5pts) The Fourier sine series of $f(x) = x$ on $(0,1)$ converges pointwisely but not uniformly on $(0,1)$.

(d) (Short answer, 5pts) Construct an sequence of functions $f_n : [0,1] \to \mathbb{R}$ which is pointwisely to some function $f(x)$ but not uniformly.

2. (a) (10pts) What's the Laplacian under polar coordinates in two dimension.

   (b) (15pts) Find all the radial solutions to

$$\Delta u = 1. \tag{2}$$

   on an annulus domain $\Omega := B_2(0) \backslash B_1(0)$.

3. (15pts) Let a, b be two constants, $\Omega \subset \mathbb{R}^n$ is bounded and smooth, prove that there is a smooth solution to the boundary value problem

$$\begin{cases} \Delta u = a \dfrac{e^u}{\int_\Omega e^u \, dx} - b \dfrac{e^{-u}}{\int_\Omega e^{-u} \, dx}; \\ \left. \dfrac{\partial u}{\partial n} \right|_{\partial \Omega} = 0. \end{cases} \tag{3}$$

only if $a = b$.

4. Let u be a harmonic and smooth function in $\Omega \subset \mathbb{R}^2$.

(a) (20pts) What is mean-value property? State and prove it. Any applications you could recall?

(b) (15pts) For a ball $B_\varepsilon(x_0) \subset \Omega$ with $\varepsilon > 0$, prove that

$$\int_{\partial B_\varepsilon(x_0)} u(x) \frac{\partial}{\partial n} \log |x - x_0| dS = 2\pi u(x_0). \tag{4}$$

(c) (True or False, 5pts) Since $u, \ln |x - x_0|$ are harmonic functions in Ω, from the Green's second identity, we know that

$$\frac{1}{2\pi} \int_{\partial\Omega} \left(u \frac{\partial}{\partial n} \log |x - x_0| - \frac{\partial}{\partial n} u \log |x - x_0| \right) dS = 0. \tag{5}$$