# MAT4220 FA22 HW03

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**Problem 1** (P52 Q3)**.** Using the formula

$$\begin{aligned}u(x,t) = \Phi \star \phi(x) &= \frac{1}{\sqrt{4\pi kt}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4kt}} e^{3y} \,\mathrm{d}y \\ &= \frac{1}{\sqrt{4\pi kt}} \int_{\mathbb{R}} e^{-\frac{(y-x-6kt)^2}{4kt}} e^{\frac{6kt(x+6kt)}{4kt}} \,\mathrm{d}y \\ &= e^{3x+9kt}\end{aligned}$$

**Problem 2** (P53 Q14)**.** Suppose $1 \neq 4kta$, we have

$$\begin{aligned}|u(x,t)| = |\Phi \star \phi(x)| &\leq |\Phi \star Ce^{ax^2}| &(1)\\ &= \frac{1}{\sqrt{4\pi kt}} \int_{\mathbb{R}} \exp\left[-\frac{(1-4kta)(y-\frac{x}{1-4kta})^2}{4kt}\right] \exp\frac{ax^2}{4kta-1} \,\mathrm{d}x &(2)\end{aligned}$$

For $t \in (0, 1/(4ak))$, we have $1 - 4kat > 0$, then

$$|u(x,t)| \leq \frac{1}{\sqrt{1-4kat}} \exp\frac{ax^2}{4kta-1} \,\mathrm{d}x$$

This upper bound makes sense $\forall x \in \Omega$, which means $u(x,t)$ is meaningful.

However, when $t \geq 1/(4kat)$, $u(x,t)$ may not make any sense. For example, let $\phi(x) = Ce^{ax^2} \leq Ce^{ax^2}$, according to equation 2, we can see that $u(x,t) = \infty$ when $t > 1/(4ak)$, a $e^{ky^2}$ $(k > 0)$ term appears in the integral term, which means the integral not exists (goes to infinity).

**Problem 3** (P60 Q3)**.** Define

$$\tilde{\phi}(x) = \phi(|x|)$$

Then

$$\tilde{w}(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{\mathbb{R}} \tilde{\phi}(y) e^{-\frac{(x-y)^2}{4kt}} = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left[e^{-\frac{(x+y)^2}{4kt}} + e^{-\frac{(x-y)^2}{4kt}}\right] \phi(y) \,\mathrm{d}y$$

**Problem 4** (P66 Q3)**.** Perform the odd extension to both $\phi(x)$ and $\psi(x)$ to $\tilde{\phi}(x)$ and $\tilde{\psi}(x)$, then we get

$$u(x,t) = \frac{1}{2}[\tilde{\phi}(x-ct) + \tilde{\phi}(x+ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \tilde{\psi}(s) \,\mathrm{d}s$$

Plugin $\phi(x) = f(x)$, $\psi(x) = cf'(x)$, then $\forall x \geq 0$, we have two cases

1. $x > ct \geq 0$, we get

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} cf'(s) \,\mathrm{d}s = f(x+ct)$$

2. $0 \le x \le ct$, we get

$$\begin{aligned} u(x,t) &= \frac{1}{2}[-f(ct-x)+f(x+ct)] + \frac{1}{2c}\left[\int_{x-ct}^{0} -cf'(-s)\,\mathrm{d}s + \int_0^{x+ct} cf'(s)\,\mathrm{d}s\right] \\ &= \frac{1}{2c}[-f(ct-x)+f(x+ct)] + \frac{1}{2}\left[-cf(ct-x)+cf(x+ct)\right] \\ &= f(x+ct)-f(ct-x) \end{aligned}$$

**Problem 5** (P67 Q10)**.** Perform odd extension of $\phi$ and $\psi$ on $(0,\pi)$ and perform even extension on whole $\mathbb{R}$, we get

$$\tilde{\phi}(x) = \cos x,\ \tilde{\psi}(x) = 0$$

Then the solution of this wave equation is

$$u(x,t) = \frac{1}{2}(\cos(x-ct)+\cos(x+ct)) = \cos x\cos ct = \cos x \cos 3t$$

using the fact that $c = 3$.

**Problem 6** (P79 Q2)**.** Using the formula that

$$u(x,t) = \frac{1}{2}[\phi(x-ct)+\phi(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x-ct}\psi(s)\,\mathrm{d}s + \int_0^t\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,\mathrm{d}y\,\mathrm{d}s$$

we have

$$\begin{aligned} u(x,t) &= \frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)} e^{ay}\,\mathrm{d}y\,\mathrm{d}s \\ &= \frac{1}{2ac}\int_0^t (e^{ax+ac(t-s)}-e^{ax-ac(t-s)})\,\mathrm{d}s \\ &= \frac{1}{2a^2c^2}(e^{ax+act}+e^{ax-act}-2e^{ax}) \\ &= \frac{1}{2a^2c^2}e^{ax}(e^{act}+e^{-act}-2) \end{aligned}$$

**Problem 7** (P79 Q14)**.** Define a new function $v(x,t) = u(x,t) - xk(t)$, then we have

$$\begin{aligned} v_{tt} - c^2 v_{xx} &= f(x,t) = -xk''(t) \\ v(x,0) &= \phi(x) = -xk(0) \\ v_t(x,0) &= \psi(x) = -xk'(0) \\ v_x(0,t) &= 0 \text{ N.B.C} \end{aligned}$$

on $x \in (0,\infty)$. Then we can perform even extension on $\phi$, $\psi$ and $f$. Applying the formula

$$\tilde{u}(x,t) = \frac{1}{2}[\tilde{\phi}(x+ct)+\tilde{\phi}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,\mathrm{d}s + \frac{1}{2c}\int_0^t \mathrm{d}s\int_{x-c(t-s)}^{x+c(t-s)}\mathrm{d}y f(y,s)$$

When $0 \le ct \le x$, the three parts red $r$, blud $b$, and orange $o$ equals to

$$\begin{aligned} r &= -xk(0) \\ b &= -xtk'(0) \end{aligned}$$

$$o = -xk(t) + xk(0) + txk'(0)$$

Then $v(x,t) = -xk(t)$, which means $u(x,t) = 0$ when $0 \le ct \le x$.

When $ct > x \ge 0$, the red part $r$ equals to

$$r = \frac{1}{2}[-(x+ct)k(0) - (ct-x)k(0)] = -ctk(0)$$

The blue part $b$ equals to

$$b = \frac{1}{2c}\int_0^{x+ct} -k'(0)s\,\mathrm{d}s + \frac{1}{2c}\int_{x-ct}^{0} k'(0)s\,\mathrm{d}s = -\frac{1}{2c}(x^2+c^2t^2)k'(0)$$

The orange part $o$ equals to

$$\begin{aligned}
o &= \left(\int_0^{t-x/c} + \int_{t-x/c}^{t}\right)\mathrm{d}s\int_{x-c(t-s)}^{x+c(t-s)} \mathrm{d}y f(y,s)\\
&= -\frac{1}{2c}\int_0^{t-x/c} \mathrm{d}s[x^2+c^2(t-s)^2]k''(s) + \int_{t-x/c}^{t} x(s-t)k''(s)\,\mathrm{d}s\\
&= -\frac{1}{2c}x^2[k'(t-x/c)-k'(0)] - \frac{c}{2}(s-t)^2k'(s)|_0^{t-x/c} + c(s-t)k(s)|_0^{t-x/c} - c\int_0^{t-x/c} k(s)\,\mathrm{d}s\\
&+\ x(s-t)k'(s)|_{t-x/c}^{t} - xk(s)|_{t-x/c}^{t}\\
&= \frac{1}{c}x^2k'(t-x/c) - xk(t) + xk(t-x/c) - \frac{1}{2c}x^2k'(t-x/c) + \frac{1}{2c}x^2k'(0)\\
&-\ \frac{1}{2c}x^2k'(t-x/c) + \frac{c}{2}t^2k'(0) - xk(t-x/c) + ctk(0) - c\int_0^{t-x/c} k(s)\,\mathrm{d}s\\
&= -xk(t) + \frac{1}{2c}x^2k'(0) + \frac{c}{2}t^2k'(0) + ctk(0) - c\int_0^{t-x/c} k(s)\,\mathrm{d}s
\end{aligned}$$

Therefore we have

$$\begin{aligned}
v(x,t) &= r+b+o\\
&= -ctk(0) - \frac{1}{2c}(x^2+c^2t^2)k'(0) - xk(t) + \frac{1}{2c}x^2k'(0) + \frac{c}{2}t^2k'(0) + ctk(0) - c\int_0^{t-x/c} k(s)\,\mathrm{d}s\\
&= -xk(t) - c\int_0^{t-x/c} k(s)\,\mathrm{d}s
\end{aligned}$$

$$\Rightarrow u(x,t) = v(x,t) + xk(t) = -c\int_0^{t-x/c} k(s)\,\mathrm{d}s$$

for $ct > x > 0$.