Name: ____________________ SID: ____________________

MTH4220(2021 Fall): Quiz 2

- This quiz will last for 1 hour.
- No phones or discussion, book and notes closed, no calculators.
- (**Notation**)In the whole quiz, we may use t, x, y as independent variable. We usually use u as dependent variable unless addressed.
- The fundamental solution of the heat equation on the real line is
$$\Phi(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{|x|^2}{4kt}}.$$
- Some useful identities
$$\int_{-\infty}^{\infty} e^{-p^2}\, dp = \sqrt{\pi}.$$

give a counterexample if it's false.

PART I. True or False problems: ~~explain briefly why if it's false~~, you'll only get at most 2 points if you don't. (7 points each, 35 points in total)

1. If $\phi(x)$ is an even function on $[-l, l]$, then its full Fourier series on $[-l, l]$ equals the its ~~sine~~ cosine series on $[0, l]$.

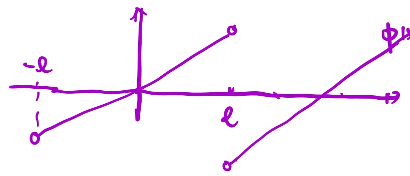
   A. True B False ✓

   $\phi(x) = 1$, even.

   Full series = Cosine series = 1.

2. If $\phi(x)$ is continuous on $(0, l)$, then its odd periodic extension of $\phi_e(x)$ is also continuous on $(-\infty, \infty)$.

   A. True B False ✓

   $\phi(x) = x$ on $(0,l)$

3. For any $f \in L^2[a, b]$, the finite sum of a classical Fourier series of f, denoted by $S_N f$ converges to f in $L^2[a, b]$ as N tends to infinity. This is the completeness of Fourier series in L^2.

   A. True ✓ B False

4. Uniform convergence ~~always~~ implies L^2-convergence and L^2-convergence ~~always~~ implies pointwise convergence.

   A. True B False ✓

   ①. $f_n(x) = \begin{cases} \frac{1}{\sqrt{n}} & x \in (-\frac{1}{n}, \frac{1}{n}) \\ 0, & x \notin (-\frac{1}{n}, \frac{1}{n}) \end{cases}$ $f(x) = 0$

   / $f_n(x) \to f(x)$ uniformly, not in L^2.

5. Let u be a smooth harmonic function on a open domain Ω, then the Poisson's formula tells that the harmonic function inside any disk $D \subset \Omega$ can be represented in terms of its value on the disk boundary.

   A. True ✓ B False

Part II. State clearly the theorem(7pts)

What is the strong and weak Maximum principle for harmonic functions, respectively?

Ω is a bdd, connected, open set on $\mathbb{R}^n$.

Weak M.P. Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is a harmonic fct ($\Delta u = 0$), then

$$\max_{\partial\Omega} u = \max_{\Omega} u.$$

Strong M.P. Suppose. $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is a harmonic fct, then

$$u(x) \leq \max_{\partial\Omega} u. \quad \forall x \in \Omega.$$

The equality holds only if $u \equiv \max_{\partial\Omega} u$. on Ω.

Part III. Short answer problems: Be sure to justify your write-up (58 points in total)

1. (18 pts) Consider the initial value problem of the heat equation on the real line:

$$\begin{cases} \partial_t u = \partial_x^2 u, & -\infty < x < \infty, t > 0 \\ u(x,0) = \phi(x). \end{cases} \tag{1}$$

(12pts) (a) If $\phi(x) = e^x$, find the solution u to the initial value problem above.

(3pts) (b) Suppose $\phi \geqslant 0$ and $\phi(x) \not\equiv 0$, show that $u(x,t) > 0$ for any $x \in (-\infty, \infty)$ and $t > 0$. This implies the heat equation has infinite propagation rate.

(3pts) (c) Show that the system is stable with respect to initial data under max-norm.

Sol. (a) $\Phi(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$ is the F.S.

then $u(x,t) = \Phi * \phi = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} \cdot e^y \, dy.$

$$= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{1}{4t}(x^2 - 2xy + y^2 - 4ty)} dy$$

$$= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{1}{4t}(x^2 + y^2 - 2(x+2t)y)} dy$$

$$= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{1}{4t}(x^2 + (y-(x+2t))^2 - (x+2t)^2)} dy$$

$$= \frac{1}{\sqrt{4\pi t}} e^{-\frac{1}{4t}(x^2 - (x+2t)^2)} \int_{\mathbb{R}} e^{-\frac{1}{4t}(y-(x+2t))^2} dy$$

$$= \frac{1}{\sqrt{4\pi t}} e^{x+t} \sqrt{2t} \int_{\mathbb{R}} e^{-z^2} dz \quad \left(z = \frac{y-(x+2t)}{\sqrt{2t}} \right)$$

$$= e^{x+t}.$$

(b) $u(x,t) = \int_{\mathbb{R}} \Phi(x-y,t)\phi(y)\, dy > 0$ since $\Phi > 0$. ($\forall t > 0$)
$\phi \not\equiv 0$.

(c) $u_1 - u_2 = \int_{\mathbb{R}} \Phi(x-y,t)(\phi_1(y) - \phi_2(y))\, dy$

$\Rightarrow \max |u_1 - u_2| \leq \int_{\mathbb{R}} \Phi(x-y,t)\, dy \cdot \max |\phi_1 - \phi_2| \leq \max |\phi_1 - \phi_2|.$

Some students use M.P. to derive it which is not correct since M.P. is not valid on a inifite domain in general

You'll only get 1 pt in this case.

2. (20pts) Consider the initial value problem of the heat equation on a finite interval

$$\begin{cases} \partial_t u = \partial_x^2 u, & 0 < x < l; \\ u(x,0) = \phi(x), & 0 < x < l; \\ \partial_x u(0,t) = 0 = \partial_x u(l,t) \end{cases} \tag{2}$$

(a) Write down the series solution. (There is no need to write out the steps.)

(b). Find $u(x,t)$ if $\phi(x) = x$

~~(b) Show that the series solution is a classical solution to the heat equation for $t > 0$.~~

(c) Show that for $\phi(x) = x$, the initial data is satisfied in the sense $\lim_{t\to 0^+} u(x,t) = \phi(x)$ for any $x \in (0,l)$. & Show $\lim_{t\to\infty} \max_x |u(x,t)| = 0$.

Sol. (a) Neumann B.C.

$u(x,t) = X(x)T(t)$, $\quad XT' = X''T. \Rightarrow \frac{X''}{X} = \frac{T'}{T} = -\lambda$

$$\Rightarrow \begin{cases} X_n'' = -\lambda_n X_n \\ T_n' = -\lambda_n T. \end{cases} \quad \text{eigenfcts: } \cos\left(\frac{n\pi}{l}x\right), \quad \lambda_n = \frac{n^2\pi^2}{l^2}, \; n = 0,1,2,\dots$$

$$\boxed{u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{l}\right)^2 t} \cos\left(\frac{n\pi}{l}x\right).}$$

(b). $\phi(x) = x$, then.

$$x = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{l}x\right)$$

$$\Rightarrow \quad A_n = \frac{2}{l}\int_0^l x\cos\left(\frac{n\pi}{l}x\right)dx$$

$$= \frac{2}{l}\int_0^l x \, d\left(\frac{l}{n\pi}\sin\left(\frac{n\pi}{l}x\right)\right) \qquad (n\neq 0)$$

$$= \frac{2}{n\pi} x \sin\left(\frac{n\pi}{l}x\right)\Big|_0^l - \frac{2}{l}\int_0^l \frac{l}{n\pi}\sin\left(\frac{n\pi}{l}x\right)dx$$

$$= -\frac{2}{n\pi}\int_0^l \sin\left(\frac{n\pi}{l}x\right)dx$$

$$= \frac{2l}{(n\pi)^2}\cos\left(\frac{n\pi}{l}x\right)\Big|_0^l$$

$$= \frac{2l}{(n\pi)^2}\left((-1)^n - 1\right) \qquad (n\neq 0)$$

10pts

$$A_0 = \frac{2}{l}\int_0^l x\,dx = l$$

$$\boxed{\phi(x) = \frac{l}{2} + \sum_{n=1}^{\infty} \frac{2l}{(n\pi)^2}\left((-1)^n - 1\right) e^{-\left(\frac{n\pi}{l}\right)^2 t}\cos\left(\frac{n\pi}{l}x\right)}$$

5 ① $\phi(x)$, $\phi'(x)$ are continuous on $(0,\ell)$, hence

$$S_N\phi \to \phi \quad \text{pointwisely}$$

5 ② $\left|A_n\cos\left(\frac{n\pi}{\ell}x\right)\right| \leq \frac{4\ell}{(n\pi)^2}$ & $\sum_{n=1}^{\infty}\frac{4\ell}{(n\pi)^2} < \infty$

we have $S_N\phi$ uniformly converges, hence pointwise & L^2 converges.

Or

② $\phi(x) \in L^2$. hence

$$S_N\phi \to \phi \quad \text{in } L^2$$

(3 pts only)

Note. the uniform convergence can't be derived directly from the theorems in the book since ϕ does not s.t. the B.C. (Neumann for Cosine series)

(c). $$u(x,t) = \sum_{n=1}^{\infty}\underbrace{\frac{2\ell}{(n\pi)^2}\left((-1)^n-1\right)e^{-\left(\frac{n\pi}{\ell}\right)^2 t}\cos\left(\frac{n\pi}{\ell}x\right)}_{u_n}.$$

Since

$$|u_n| \leq \frac{2\ell}{n^2\pi^2}\cdot 2, \ \forall (x,t). \quad \& \quad \sum_{n=1}^{\infty}\frac{4\ell}{n^2\pi^2} < \infty.$$

$$\lim_{t\to 0^+} u(x,t) = \lim_{t\to 0^+}\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty}\lim_{t\to 0^+} u_n = \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi}{\ell}x\right) = \phi(x)$$

Similarly.

$$\lim_{t\to\infty} u(x,t) = \frac{\ell}{2} + \sum_{n=1}^{\infty}\lim_{t\to\infty}\frac{2\ell}{(n\pi)^2}\left((-1)^n-1\right)e^{-\left(\frac{n\pi}{\ell}\right)^2 t}\cos\left(\frac{n\pi}{\ell}x\right)$$

$$= \frac{\ell}{2}.$$

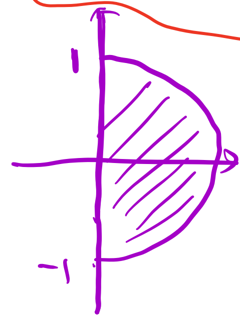
3. (20 pts) Considering the ~~Laplacian~~ equation on a given domain Ω,

$$\Delta u(x) = u(x)/|x|^2 \quad x \in \Omega. \tag{3}$$

Let Ω be the the half disk domain $\Omega := \{x \in \mathbb{R}^2 : |x| < 1, x_1 > 0\}$ find a series solution(under polar coordinate) to the equation subject to boundary condition

$$u(x) = \begin{cases} 2x_2^2, & |x| = 1, x_1 > 0; \\ 0, & x_1 = 0, x_2 \in (-1, 1). \end{cases}$$

10pts as long as the separation is correct.



Sol. • $\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2$, $u = R(r)\Theta(\theta)$

$\Rightarrow R''(r)\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = \frac{R\Theta}{r^2}$

$\Rightarrow \frac{r^2R''}{R} + \frac{rR'}{R} + 1 = -\frac{\Theta''}{\Theta} = \lambda$

$\Rightarrow \begin{cases} \Theta'' = -\lambda\Theta \\ r^2R'' + rR' - R = \lambda R. \end{cases}$

• B.C.

On $x_1 = 0$, $x_2 \in (-1,1)$, $u = 0$, that is, $\Theta(0) = \Theta(\pi) = 0$.

• $\Theta_n = \sin(n\theta)$. $\lambda_n = n^2$, $n \in \mathbb{N}\setminus\{0\}$

You'll get full pts until this step.

• $r^2R_n'' + rR_n' = (1+\lambda_n)R_n = (n^2+1)R_n$.

$R_n = r^\alpha$, $(\alpha(\alpha-1) + \alpha)r^\alpha = (n^2+1)r^\alpha$

$\Rightarrow \alpha^2 = n^2+1 \Rightarrow \alpha = \pm\sqrt{n^2+1}$.

$\Rightarrow R_n = r^{\sqrt{n^2+1}}$ or $r^{-\sqrt{n^2+1}}$. $n = 1, 2, \cdots$.

Hence

$$u(r,\theta) = \sum_{n=1}^{\infty}\left(A_n r^{\sqrt{n^2+1}} + B_n r^{-\sqrt{n^2+1}}\right)\sin(n\theta).$$

Reject the case. r has negative exponents, (singular), we have

$$\boxed{u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{\sqrt{n^2+1}} \sin(n\theta)}$$

From B.C. $u(x) = 2x_2^2$. on $|x|=1$, $x_1>0$.

under polar coordinates, $r=1$, $x_2 = \sin\theta$

$$u(1,\theta) = 2\sin^2\theta = 1-\cos(2\theta).$$

$$\Rightarrow \quad 1-\cos(2\theta) = \sum_{n=1}^{\infty} A_n \sin(n\theta), \qquad \theta\in(0,\pi).$$

$$A_n = \frac{2}{\pi}\int_0^{\pi} (1-\cos(2\theta))\sin(n\theta)\, d\theta.$$

$$= -\frac{2}{\pi}\int_0^{\pi}\sin(n\theta)\,d\theta + \frac{2}{\pi}\int_0^{\pi}\cos(2\theta)\sin(n\theta)\,d\theta.$$

$$= +\frac{2}{\pi}\frac{1}{n}\cos(n\theta)\Big|_0^{\pi} + \frac{2}{\pi}\int_0^{\pi}\cos(2\theta)\sin(n\theta)\,d\theta$$

$$= \frac{2}{n\pi}\left((-1)^n-1\right) + \frac{1}{\pi}\int_0^{\pi}\left[\sin((2+n)\theta) - \sin((2-n)\theta)\right]d\theta$$

$$\boxed{\cos\alpha\sin\beta = \frac{1}{2}(\sin(\alpha+\beta)-\sin(\alpha-\beta))}$$

① $n\neq 2$

$$A_n = \frac{2}{n\pi}\left((-1)^n-1\right) + \frac{1}{\pi}\left[-\frac{1}{2+n}\cos((2+n)\theta) + \frac{1}{2-n}\cos((2-n)\theta)\right]\Big|_0^{\pi}$$

$$= \frac{2}{n\pi}\left((-1)^n-1\right) + \frac{1}{\pi}\left[\frac{-(-1)^{2+n}+1}{n+2} + \frac{(-1)^{2-n}-1}{2-n}\right]$$

$$= \frac{2}{n\pi}\left((-1)^n-1\right) + \frac{(-1)^n-1}{\pi}\left(-\frac{1}{n+2}+\frac{1}{2-n}\right)$$

② $n=2$.

$$A_2 = \frac{1}{\pi}\int_0^{\pi}\sin((2+n)\theta)\,d\theta$$

$$= -\frac{(-1)^n-1}{n+2}.$$

4. (Bonus–2pts) State and prove the mean value property for harmonic functions in $\mathbb{R}^n$.

Check the notes