MAT4220 FA22 HW04

Haoran Sun (haoransun@link.cuhk.edu.cn)

Problem 1 (P89 Q3). Easy to obtain

$$\frac{1}{i}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

and $\lambda = \beta^2 \geq 0$. Then we have

$$\begin{aligned} T(t) &= e^{-i\lambda t} \\ X(x) &= A\cos\beta x + B\sin\beta x \end{aligned}$$

Applying $X(0) = X(l) = 0$, we have $B = 0$ and $\beta_n = n\pi/l$, $n \in \mathbb{N}$. Hence

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} e^{-i(n\pi)^2 t/l^2}$$

Problem 2 (P92 Q4).

(a) Easy to obtain

$$\frac{1}{k}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

and we can show that $\lambda \geq 0$ given the periodic boundary condition. Hence, we have $\lambda = \beta^2$

$$X(x) = A\cos\beta x + B\sin\beta x,\ X'(x) = -A\beta\sin\beta x + B\beta\cos\beta x$$

Plug the P.B.C into it, and we get

$$\begin{aligned} X(l) &= A\cos\beta l + B\sin\beta l = X(-l) = A\cos\beta l - B\sin\beta l \\ X'(l) &= -A\beta\sin\beta l + B\beta\cos\beta l = X'(-l) = A\beta\sin\beta l + B\beta\cos\beta l \\ \Rightarrow B\sin\beta l &= A\beta\sin\beta l = 0 \end{aligned}$$

Hence we have $\beta_n = n\pi/l$ and $\lambda_n = (n\pi/l)^2$.

(b) Using the result from (a), it should be obvious that

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi x}{l} + B_n\sin\frac{n\pi x}{l}\right)e^{-(n\pi/l)^2 t}$$

Problem 3 (P101 Q9).

(a) Let $\lambda = 0$, $X(x) = ax + b$, $X'(x) = a$. Plug in the boundary condition, we have

$$X_0(x) = x - 1$$

(b) Let $\lambda = \beta^2$, then

$$\begin{aligned} X(x) &= A\cos\beta x + B\sin\beta x \\ X'(x) &= A(-\beta\sin\beta x) + B\beta\cos\beta x \end{aligned}$$

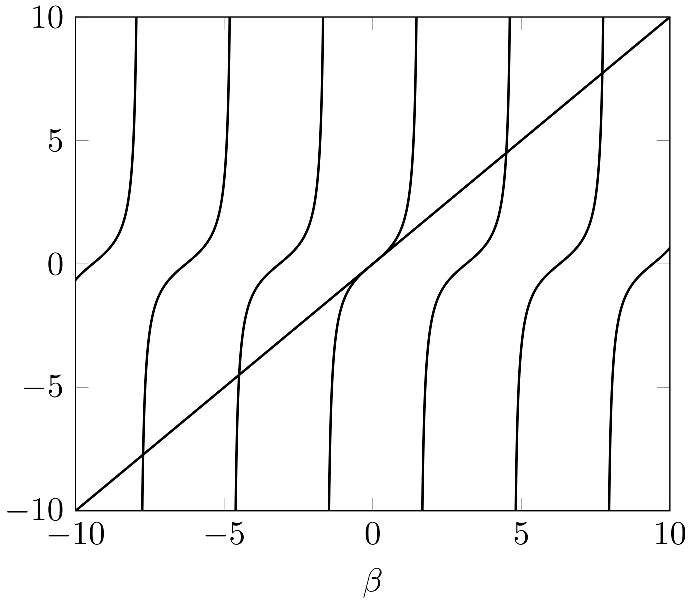
Plug in the boundary condition, we have

$$\begin{bmatrix} 1 & \beta \\ \cos\beta & \sin\beta \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

The determinant should be zero to get non-trivial solutions

$$\sin\beta - \beta\cos\beta = 0 \Rightarrow \tan\beta = \beta$$

(c) Sketch:



Thus there are infinitely many solutions.

(d) Suppose we have negative eigenvalues where $\lambda = (i\beta)^2$, then

$$\begin{aligned} X(x) &= A\cosh\beta x + B\sinh\beta x \\ X'(x) &= A\beta\sinh\beta x + B\beta\cosh\beta x \end{aligned}$$

Since

$$\begin{aligned} X(0) &= X'(x) \Rightarrow A = B \\ X(1) &= A\frac{e+e^{-\beta}}{2} + B\frac{e-e^{-\beta}}{2} = Ae^{-\beta} > 0 \end{aligned}$$

thus we cannot have negative eigenvalues.

Problem 4 (P111 Q2)**.**

(a) For sine series

$$\begin{aligned} A_n &= \frac{2}{l}\int_0^l x^2 \sin\frac{n\pi}{l}\,\mathrm{d}x \\ &= \frac{2}{n\pi}\int_0^l \left(-x^2\cos\frac{n\pi}{l}x\Big|_0^l + 2\int_0^l x\cos\frac{n\pi}{l}x\,\mathrm{d}x\right) \end{aligned}$$

$$
\begin{aligned}
&= \cdots \\
&= \left(\frac{2}{n\pi} + \frac{8}{(n\pi)^3}\right) l^2
\end{aligned}
$$

Then

$$
x^2 = \sum_{n=1}^{\infty} A_n n\pi x \quad \text{with} \quad A_n = \frac{2}{n\pi} + \frac{8}{(n\pi)^3}
$$

(b) For cosine series

$$
\begin{aligned}
A_0 &= \frac{l}{2}\int_0^l x^2 \,\mathrm{d}x = \frac{2}{3}l^2 \\
A_n &= \frac{l}{2}\int_0^l x^2 \cos\frac{n\pi}{l}x \,\mathrm{d}x \\
&= \frac{2}{n\pi}\left(x^2 \sin\frac{n\pi}{l}x\Big|_0^l - 2\int_0^l x\sin\frac{n\pi}{l}x \,\mathrm{d}x\right) \\
&= \cdots \\
&= \frac{4l^2}{(n\pi)^2}
\end{aligned}
$$

Then

$$
x^2 = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} \cos\frac{n\pi}{l}x
$$

Problem 5 (P112 Q9)**.** Due to the Neumann boundary condition, using the fact that

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}A_0 + \frac{1}{2}B_0 t + \sum_{n=1}^{n}\left(A_n\cos\frac{n\pi}{l}t + B_n\sin\frac{n\pi c}{l}t\right)\cos\frac{n\pi}{l}x \\
u(x,0) &= \frac{1}{2}A_0 + \sum_{n=1}^{n} A_n\cos\frac{n\pi}{l}x = 0 \\
u_t(x,0) &= \frac{1}{2}B_0 + \sum_{n=1}^{\infty} B_n\frac{n\pi c}{l}\cos\frac{n\pi}{l}x = \cos^2 x
\end{aligned}
$$

We can conclude that $A_n = 0$ and

$$
\begin{aligned}
B_0 &= \frac{2}{\pi}\int_0^{\pi} \cos^2 x \,\mathrm{d}x = 1 \\
B_n nc &= \frac{2}{\pi}\int_0^{\pi} \cos^2 x \cos nx \,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}(\cos 2x + 1)\cos nx \,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} \cos 2x\cos nx + \frac{1}{\pi}\int_0^{\pi}\cos nx \,\mathrm{d}x \\
&= \frac{1}{2}\delta_{2n}
\end{aligned}
$$

which means the only two non-zero terms are B_0 and B_2, therefore

$$
u(x,t) = \frac{1}{2}t + \frac{1}{4c}\sin 2ct\cos 2x
$$