Suggested Solutions to Midterm Exam for MATH4220

1. (**20 points**)

   (a) (**8 points**) Solve the following problem

$$\begin{cases} \partial_t u + 4\partial_x u - 2u = 0 \\ u(x, t = 0) = x^2. \end{cases}$$

   (b) (**12 points**) Solve the problem

$$\begin{cases} 2\partial_x u + y\partial_y u = 0 \\ u(x = 0, y) = y. \end{cases}$$

   What are characteristic curves of this equation?

Solution:

(a) **Method 1:Coordinate Method**:
Use the following new coordinates

$$t' = t + 4x, \ x' = 4t - x$$

Hence $\partial_t u + 4\partial_x u = 17\partial_{t'} u = 2u$. Thus the solution is $u(t', x') = f(x')e^{\frac{2}{17}t'}$ with function f to be determined. Therefore, the general solutions are

$$u(t, x) = f(4t - x)e^{\frac{2}{17}(t+4x)}. \qquad \textbf{(5points)}$$

Moreover, the initial condition implies that

$$u(x, t = 0) = f(-x)e^{\frac{8}{17}x} = x^2,$$

or equivalently,

$$f(x) = x^2 e^{\frac{8}{17}x}.$$

Finally,

$$u(t, x) = (4t - x)^2 e^{2t}. \qquad \textbf{(3points)}$$

Method 2: Geometric Method
The corresponding characteristic curves are

$$\frac{dt}{1} = \frac{dx}{4}$$

that is, $x = 4t + C$ where C is an arbitrary constant. Then

$$\frac{d}{dt}u(t, 4t + C) = u_t(t, 4t + C) + 4u_x(t, 4t + C) = 2u(t, 4t + C).$$

Hence $u(t, 4t + C) = f(C)e^{2t}$, where f is an arbitrary function. Therefore,

$$u(t, x) = f(x - 4t)e^{2t}. \qquad \textbf{(5points)}$$

While the initial condition shows that

$$u(x, t = 0) = x^2 = f(x)$$

thus

$$u(x, t) = (x - 4t)^2 e^{2t}. \qquad \textbf{(3points)}$$

(b) The characteristic equations are

$$\frac{dx}{2} = \frac{dy}{y}$$

thus the characteristic curves are given by

$$y = Ce^{\frac{x}{2}} \qquad \textbf{(5points)}$$

where C is an arbitrary constant. Then

$$\frac{d}{dx}u(x, Ce^{\frac{x}{2}}) = u_x + \frac{C}{2}e^{\frac{x}{2}}u_y = u_x + \frac{y}{2}u_y = 0$$

Hence $u(x, Ce^{\frac{x}{2}}) = f(C)$ where f is an arbitrary function. Thus

$$u(x, y) = f(ye^{-\frac{x}{2}})$$

Besides, the auxiliary condition gives that $y = u(x = 0, y) = f(y)$. Hence, the solution is

$$u(x, y) = ye^{-\frac{x}{2}}. \qquad \textbf{(5points)}$$

2. (**20 points**)

(a) (**8 points**) Is the following initial-boundary value problem well-posed? Why?

$$\begin{cases} \partial_t u - \partial_x u = 0, & x > 0, t > 0 \\ u(x, t = 0) = \sin x, & x > 0, \\ u(x = 0, t) = 0, & t > 0. \end{cases}$$

(b) (**4 points**) For each positive integer n, is

$$u_n(x, y) = \frac{1}{n}e^{-\sqrt{n}}\sin(nx)\frac{e^{ny} - e^{-ny}}{2}$$

a solution to the following problem

$$\begin{cases} \partial_x^2 u + \partial_y^2 u = 0, & -\infty < x < +\infty, y > 0 \\ u(x, y = 0) = 0 \\ \partial_y u(x, y = 0) = \frac{1}{n}e^{-\sqrt{n}}\sin(nx). \end{cases}$$

(c) (**8 points**) Is the following Cauchy problem

$$\begin{cases} \partial_x^2 u + \partial_y^2 u = 0, & -\infty < x < +\infty, y > 0 \\ u(x, y = 0) = 0 \\ \partial_y u(x, y = 0) = 0, \end{cases}$$

well-posed? Explain in details why?

Solution:

(a) **No** (**1points**), this problem, which violates the existence, is ill-posed.
In fact, note that the characteristic lines are given by

$$\frac{dt}{1} = \frac{dx}{-1}$$

that is

$$x = -t + C$$

with arbitrary constants C. Thus the general solution to $\partial_t u - \partial_x u = 0$ is

$$u(x,t) = f(x+t)$$

with an arbitrary function f. Moreover, the initial condition shows that

$$u(x,0) = f(x) = 0$$

thus

$$u(x,t) = 0$$

which does not satisfy the boundary condition $u(x, t = 0) = \sin x$. (**7points**)

(b) It follows from simple computications (**2points**) that

$$\begin{aligned}
\partial_x u_n(x,y) &= e^{-\sqrt{n}} \cos(nx) \frac{e^{ny} - e^{-ny}}{2} \\
\partial_x^2 u_n(x,y) &= -ne^{-\sqrt{n}} \sin(nx) \frac{e^{ny} - e^{-ny}}{2} \\
\partial_y u_n(x,y) &= e^{-\sqrt{n}} \sin(nx) \frac{e^{ny} + e^{-ny}}{2} \\
\partial_y^2 u_n(x,y) &= ne^{-\sqrt{n}} \sin(nx) \frac{e^{ny} - e^{-ny}}{2}
\end{aligned}$$

and then that

$$\begin{aligned}
u(x,0) &= 0 \\
\partial_y u(x,0) &= e^{-\sqrt{n}} \sin(nx).
\end{aligned}$$

Thus $u_n(x,y)$ is indeed a solution only for $n = 1$ (**2points**).

(c) **No** (**1point**), it's ill-posed since the solution does not depends on the data continuously.
In fact (**7points**), observe that $u = 0$ is a solution to

$$\begin{cases}
\partial_x^2 u + \partial_y^2 u = 0, & -\infty < x < +\infty, y > 0 \\
u(x, y = 0) = 0 \\
\partial_y u(x, y = 0) = 0,
\end{cases}$$

and that for any positive integer k, $u_k(x,y) = \frac{1}{2k+1} e^{-\sqrt{2k+1}} \sin((2k+1)x) \frac{e^{(2k+1)y} - e^{-(2k+1)y}}{2}$ is a solution to the following problem

$$\begin{cases}
\partial_x^2 u + \partial_y^2 u = 0, & -\infty < x < +\infty, y > 0 \\
u(x, y = 0) = 0 \\
\partial_y u(x, y = 0) = e^{-\sqrt{2k+1}} \sin((2k+1)x).
\end{cases}$$

Note that

$$|\partial_y u_k(x,0) - \partial_y u(x,0)| = |e^{-\sqrt{2k+1}} \sin((2k+1)x)| \le e^{-\sqrt{2k+1}} \to 0, \text{ as } k \to +\infty.$$

However, for $x = \frac{\pi}{2}, y > 0$

$$\begin{aligned} \left| u_k(\frac{\pi}{2}, y) - u(\frac{\pi}{2}, y) \right| &= \left| \frac{1}{2k+1} e^{-\sqrt{2k+1}} \sin(\frac{(2k+1)\pi}{2}) \frac{e^{(2k+1)y} - e^{-(2k+1)y}}{2} \right| \\ &= \frac{1}{2k+1} e^{-\sqrt{2k+1}} \left| \frac{e^{(2k+1)y} - e^{-(2k+1)y}}{2} \right| \to \infty, \text{ as } k \to +\infty. \end{aligned}$$

3. (**20 points**)

(a) (**12points**) Prove the following generalized maximum principle:
if $\partial_t u - \partial_x^2 u \le 0$ on $R \equiv [0,l] \times [0,T]$, then

$$\max_R u(x,t) = \max_{\partial R} u(x,t)$$

where $\partial R = \{(x,t) \in R | \text{ either } t = 0, \text{ or } x = 0, \text{ or } x = l\}$.

(b) (**8points**) Show that if $v(x,t)$ solves the following problem

$$\begin{cases} \partial_t v = \partial_x^2 v + f(x,t), & 0 < x < l, 0 < t < T \\ v(x,0) = 0, & 0 < x < l \\ v(0,t) = 0, v(l,t) = 0, & 0 \le t \le T \end{cases}$$

with a continuous function f on $R \triangleq [0,l] \times [0,T]$. Then

$$v(x,t) \le t \max_R |f(x,t)|$$

(Hint, consider $u(x,t) = v(x,t) - t \max_R |f(x,t)|$ and apply the result in (a).)

Solution:

(a) Let $v(x,t) = u(x,t) + \epsilon x^2$ (**2points**), then v satisfies

$$\partial_t v - \partial_x^2 v = \partial_t u - \partial_x^2 u - 2\epsilon < 0 \qquad \textbf{(1point)}$$

First, **claim** that v attains its maximum on the parabolic boundary R. Let $\max_R v(x,t) = M = v(x_0, t_0)$. Suppose on the contrary, then either

i. $0 < x_0 < l, 0 < t_0 < T$.
In this case, $v_t(x_0,t_0) = v_x(x_0,t_0) = 0$ and $v_{xx}(x_0,t_0) \le 0$. Thus $\partial_t v - \partial_x^2 v\big|_{(x_0,t_0)} \ge 0$, which is impossible. (**3points**)

ii. $0 < x_0 < l, t_0 = T$.
In this case, $v_t(x_0,t_0) \ge 0, v_x(x_0,t_0) = 0$ and $v_{xx}(x_0,t_0) \le 0$. Thus $\partial_t v - \partial_x^2 v\big|_{(x_0,t_0)} \ge 0$, which is impossible. (**3points**)

Hence

$$\max_R v(x,t) = \max_{\partial R} v(x,t).$$

Then for any $(x,t) \in R$,

$$u(x,t) \le u(x,t) + \epsilon x^2 \le \max_{\partial R} v(x,t) \le \max_{\partial R} u(x,t) + \epsilon l^2 \qquad \textbf{(2points)}$$

Letting $\epsilon \to 0$ gives $u(x,t) \le \max_{\partial R} u(x,t)$ for any $(x,t) \in R$. Hence $\max_R u(x,t) = \max_{\partial R} u(x,t)$(**1point**)

(b) Let $u(x,t) = v(x,t) - t\max_R |f(x,t)|$ (**2points**), then u satisfies

$$\begin{cases} \partial_t u - \partial_x^2 u = -\max\limits_R |f(x,t)| + f(x,t) \le 0 \\ u(x,0) = 0, \\ u(0,t) = u(l,t) = -t\max\limits_R |f(x,t)| \le 0 \end{cases} \qquad \textbf{(2points)}$$

Hence the result in (a) (**2points**) implies that for any $(x,t) \in R$,

$$u(x,t) \le \max_{\partial R} u(x,t) = 0 \qquad \textbf{(2points)}$$

that is, $v(x,t) \le t\max_R |f(x,t)|$.

4. (**20 points**)

(a) (**10 points**) Consider the following problem

$$\begin{cases} \partial_t u = \partial_x^2 u + f(x,t), \quad -\infty < x < +\infty, \quad t > 0 \\ u(x, t=0) = \varphi(x). \end{cases}$$

Prove that if $\varphi(x)$ and $f(x,t)$ are even functions of x, then the solution $u(x,t)$ to above solution must be even in x.

(b) (**10 points**) Apply the result in (a) to solve the following problem

$$\begin{cases} \partial_t u = \partial_x^2 u + e^{-x^2}, \quad x > 0, t > 0 \\ u(x, t=0) = \cos x, \quad x > 0 \\ \partial_x u(x=0,t) = 0. \end{cases}$$

Solution:

(a) Suppose $u(x,t)$ is a solution to above problem, and set

$$v(x,t) = u(-x,t).$$

Then it follows from simple calculations that

$$\begin{aligned} \partial_t v(x,t) &= \partial_t u(-x,t) \\ \partial_x v(x,t) &= -\partial_x u(-x,t) \\ \partial_x^2 v(x,t) &= \partial_x^2 u(-x,t), \end{aligned}$$

then $v(x,t)$ satisfies that

$$\begin{cases} \partial_t v = \partial_x^2 v + f(-x,t), \quad x > 0, t > 0 \\ v(x,t=0) = \varphi(-x). \end{cases}$$

Note that $f(-x,t) = f(x,t)$ and $\varphi(x) = \varphi(-x)$, then $v(x,t) = u(-x,t)$ is a solution to original problem. We **claim** that the solution for this problem is unique, thus we can show that the solution is even for x, that is,

$$u(x,t) = u(-x,t). \qquad \textbf{(5points)}$$

Now we prove the claim.

Let u_1 and u_2 are two solutions to the problem. Set $w = u_1 - u_2$, then w satisfies

$$\begin{cases} \partial_t w = \partial_x^2 w, \quad -\infty < x < +\infty, \quad t > 0 \\ w(x,t=0) = 0. \end{cases}$$

Multiply $\partial_t w = \partial_x^2 w$ by w and take integral w.r.t x, then

$$\frac{d}{dt}\int_{-\infty}^{\infty}\frac{1}{2}|w|^2dx = \int_{-\infty}^{\infty}\partial_x^2 w w dx.$$

Apply the integration by parts to the RHS of above equality,

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{\infty}\frac{1}{2}|w|^2dx &= \partial_x w w\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}|\partial_x w|^2dx \\ &= -\int_{-\infty}^{\infty}|\partial_x w|^2dx,\end{aligned}$$

where the boundary terms vanish due to $w(x,0)\equiv 0$ for any $x\in\mathbb{R}$ (We assume that all functions shown in the equation is continuous). Then for any $t>0$, we have

$$\int_{-\infty}^{\infty}\frac{1}{2}|w|^2dx + \int_0^t\int_{-\infty}^{\infty}|\partial_x w|^2dx = \int_{-\infty}^{\infty}\frac{1}{2}|\varphi|^2dx = 0,$$

which implies that for any $t>0$ and x,

$$w\equiv 0,$$

equivalently

$$u_1\equiv u_2.$$

Uniqueness is proved (**5points**).

(b) First consider the following Cauchy problem:

$$\begin{cases}\partial_t u = \partial_x^2 u + e^{-x^2}, & -\infty<x<\infty, t>0\\ u(x,t=0)=\cos x.\end{cases}$$

The corresponding solution is given by

$$u(x,t) = \int_{-\infty}^{\infty}S(x-y,t)\cos(y)dy + \int_0^t\int_{-\infty}^{\infty}S(x-y,t-s)e^{-y^2}dyds, \tag{1}$$

where $S(x,t)=\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}$ is the heat kernal. (**4points**)

Note that e^{-x^2} and $\cos x$ are even functions. By the result in (a), we know that the solution to above problem is even for x, that is,

$$u(x,t)=u(-x,t), \qquad \textbf{(2points)}$$

which implies that

$$\partial_x u(0,t)=0. \qquad \textbf{(2points)}$$

Thus $u(x,t)$ given by (1) is a solution to original half-line problem, precisely

$$\begin{aligned}u(x,t) &= \int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\cos y dy + \int_0^t\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi(t-s)}}e^{-\frac{(x-y)^2}{4(t-s)}}e^{-y^2}dyds\\ &= e^{-t}\cos x + \frac{1}{2}\sqrt{4t+1}e^{-\frac{x^2}{4t+1}} - \frac{1}{2}e^{-x^2} - x\mathcal{E}rf(x) + x\mathcal{E}rf(\frac{x}{\sqrt{4t+1}}). \qquad \textbf{(2points)}\end{aligned}$$

5. (**20 points**)

(a) (**14points**) Find the general solution formula for

$$\begin{cases} \partial_t^2 u + \partial_{xt} u - 2\partial_x^2 u = 0 \\ u(x,0) = \varphi(x) \\ \partial_t u(x,0) = 0. \end{cases}$$

(b) (**6points**) In part (a), find the solution with

$$\varphi(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

and draw the graph of $u(x,1)$.

Solution:

(a) Observe that $\partial_t^2 u + \partial_{xt} u - 2\partial_x^2 u = (\partial_t - \partial_x)(\partial_t + 2\partial_x)u$, let

$$t = t' + x', \ x = -t' + 2x'$$

and $v(x', t') = u(x,t)$, then v satisfies

$$\partial_{t'x'} v = 0.$$

Thus

$$v(x', t') = f(x') + g(t')$$

with f, g being functions to be determined. Equivalently,

$$u(x,t) = f(x+t) + g(x-2t).$$

with new functions f, g to be determined(**8points**). Combining with the initial conditions, we have

$$\begin{aligned} \varphi(x) = u(x,0) \Rightarrow & \qquad f(x) + g(x) = \varphi(x) \\ 0 = \partial_t u(x,0) \Rightarrow & \qquad f'(x) - 2g'(x) = 0. \end{aligned}$$

Thus

$$f'(x) = \frac{2}{3}\varphi'(x), \quad f(x) + g(x) = \varphi(x),$$

then

$$u(x,t) = \frac{2}{3}\varphi(x+t) + \frac{1}{3}\varphi(x-2t). \qquad \textbf{(6points)}$$

(b) For initial data

$$\varphi(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1, \end{cases}$$

the solution is given by

$$\begin{aligned} u(x,t) &= \frac{2}{3}\varphi(x+t) + \frac{1}{3}\varphi(x-2t) \\ &= \begin{cases} 1, & |x+t| < 1, |x-2t| < 1 \\ \frac{2}{3}, & |x+t| < 1, |x-2t| > 1 \\ \frac{1}{3}, & |x+t| > 1, |x-2t| < 1 \\ 0, & |x+t| > 1, |x-2t| > 1. \end{cases} \end{aligned} \qquad \textbf{(4points)}$$

In particular, $t = 1$,

$$u(x,1) = \begin{cases} \dfrac{2}{3}, & -2 < x < 0 \\ \dfrac{1}{3}, & 1 < x < 3 \\ 0, & x < -2,\ x > 3,\ 0 < x < 1. \end{cases} \qquad (\mathbf{2points})$$

The graph is omitted here.