MAT4220 FA22 HW05

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Problem 1 (P117 Q4)**.**

(a) Note that the formula for cosine coefficients is

$$A_n = \frac{1}{l}\int_{-l}^{l} \phi(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

Since $\phi(x)$ is odd and $\cos\frac{n\pi x}{l}$ is even, we have $\phi(x)\cos\frac{n\pi x}{l}$ an odd function. Hence $A_n = 0$.

(b) Note that the formula for sine coefficients is

$$B_n = \frac{1}{l}\int_{-l}^{l} \phi(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

Since $\phi(x)$ even and $\sin\frac{n\pi x}{l}$ odd, we have $\phi(x)\sin\frac{n\pi x}{l}$ odd. Hence $B_n = 0$.

Problem 2 (P117 Q8)**.**

(a) Suppose f is even and differentiable on $(-l, l)$. Then

$$f'(x_0) = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0} -\frac{f(-x_0)-f(-x_0-h)}{h} = -f'(-x_0)$$

Then f' is odd.

Following similar steps, we can also prove that f' is even given that f is odd.

(b) Suppose f even and integrable on $(-l, l)$. Then

$$F(x) = \int_0^x f(t)\,\mathrm{d}t = \int_0^{-x} -f(-t)\,\mathrm{d}(-t) = -F(-x)$$

Then F is odd.

Following similar steps, we can also prove that F is even given that f is odd.

Problem 3 (P117 Q10)**.**

(a) $\lim_{x\to 0^+}\phi(x) = 0$

(b) $\lim_{x\to 0^+}\phi(x) = 0$ and $\lim_{x\to 0^+}\phi'(x)$ exists.

(c) $\lim_{x\to 0^+}\phi(x)$ exists.

(d) $\lim_{x\to 0^+}\phi(x)$ exists and $\lim_{x\to 0^+}\phi'(x) = 0$.

Problem 4 (P123 Q10)**.**

(a) *Proof.* Prove by induction. Easy to prove that $(Z_1, Z_2) = 0$. Suppose $Z_1, \ldots, Z_k$ are orthogonal to each other. Then

$$Y_{k+1} = X_{k+1} - \sum_{i=1}^{k}(Z_i, X_{K+1})Z_i$$

$$(Z_j, Y_{k+1}) = (Z_j, Y_{k+1}) - \sum_{k=1}^{k}(Z_i, X_{k+1})\delta_{ij} = (Z_j, Y_{k+1}) - (Z_j, y_{k+1}) = 0$$

Then Y_{k+1} is orthogonal to $Z_1, \ldots, Z_k$. Suppose $\|Y_{k+1}\| \neq 0$, then Z_{k+1} exists and $Z_1, \ldots, Z_{k+1}$ orthogonal to each other. □

(b) Let $f_1(x) = \cos x + \cos 2x$ and $f_2(x) = 3\cos x - 4\cos 2x$, then

$$\begin{aligned}
\|f_1\|^2 &= \int_0^\pi \cos^2 x + 2\cos x \cos 2x + \cos^2 2x \,\mathrm{d}x = \pi \\
\Rightarrow z_1 &= \frac{1}{\sqrt{\pi}}(\cos x + \cos 2x) \\
y_2 &= f_2 - \frac{1}{\sqrt{\pi}} \int_0^\pi (\cos x + \cos 2x)(3\cos x - 4\cos 2x)\,\mathrm{d}x = \frac{7}{2}(\cos x - \cos 2x) \\
\Rightarrow z_2 &= \frac{1}{\sqrt{\pi}}(\cos x - \cos 2x)
\end{aligned}$$

Problem 5 (P134 Q1)**.**

Remark. $\forall x \in (-1, 1)$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=0}^{n} (-1)^n x^{2n} = \frac{1}{1+x^2}$$

(a) Note that

$$|S_n - S| = \left| \frac{-(-x^2)^n}{1+x^2} \right| = \frac{x^{2n}}{1+x^2}$$

Easy to show that for each $x \in (-1, 1)$, $\forall \epsilon > 0$, we can choose $N \in \mathbb{N}$ where $x^{2N} < \epsilon$ s.t. $|S_n - S| < \epsilon\ \forall n > N$. Then S_n converges pointwisely.

(b) S_n does not converges uniformly since there always $\exists \epsilon = 1/8 > 0$, s.t. $\forall n > N \in \mathbb{N}$, we can always find $x_0 = (1/2)^{1/2n}$ where

$$|S_n(x_0) - S(x_0)| = \frac{x^{2n}}{1+x^2} > \frac{1}{2}x^{2n} = \frac{1}{4} > \epsilon = \frac{1}{8}$$

(c) S_n converges in the L^2 sense since $\forall \epsilon > 0$, we can choose $N \in \mathbb{N}$ where $N > 1/\epsilon$ s.t.

$$\left| \int_{-1}^{1} \left(\frac{x^{2n}}{1+x^2} \right)^2 \mathrm{d}x \right| \le \int_{-1}^{1} \frac{1}{4} x^{2n}\,\mathrm{d}x = \frac{1}{2}\frac{1}{2n+1} < \frac{\epsilon}{2+\epsilon} < \epsilon$$

Problem 6 (P134 Q3)**.**

(a) $\forall x \notin [1/2 - 1/n, 1/2) \cup (1/2, 1/2 + 1/n]$, we have $f_n(x) = 0$. Forall other x, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ with $1/N < |x - \frac{1}{2}|$ s.t. $\forall n > N$, we have $f_n(x) = 0$. Then $f_n \to 0$ pointwisely.

(b) $\exists \epsilon = 1$, $\exists N_1 \in \mathbb{N}$ s.t. $|\gamma_n| > \epsilon = 1\ \forall n > N_1$ since $\gamma_n \to \infty$. Then $\forall n > N_1$, $\exists x_n = x + 1/2n$ s.t. $|f_n(x)| = |\gamma_n| > \epsilon = 1$. Then $f_n \to 0$ not uniformly.

(c) Easy to show that

$$\|f_n\|^2 = \frac{2}{n} \cdot n^{2/3} = 2n^{-1/3} \to 0 \text{ as } n \to \infty$$

$f_n \to 0$ in L^2 sense.

(d) Easy to show that

$$\|f_n\|^2 = \frac{2}{n}n^2 = 2n \to \infty$$

$f_n \to 0$ not in L^2 sense.

Problem 7 (P134 Q7)**.**

(a) We can verify that

$$c_0 = \frac{1}{2}\int_{-1}^{1} \phi(x)\,\mathrm{d}x = 0$$
$$c_n = \frac{1}{2}\int_{-1}^{1} e^{-in\pi x}\phi(x) = \frac{1}{2}\left[\int_{-1}^{0} e^{-in\pi x}(-1-x)\,\mathrm{d}x + \int_{0}^{1} e^{-in\pi x}(1-x)\,\mathrm{d}x\right] = \frac{1}{in\pi}$$

(b) First non-zero term (traditional sine series) is

$$\frac{2}{\pi}\sin \pi x,\ \frac{1}{\pi}\sin 2\pi x,\ \frac{2}{3\pi}\sin 3\pi x$$

(c) Note that

$$\|\phi\|^2 = \int_{-1}^{1} \phi(x)^2\,\mathrm{d}x = \frac{2}{3}$$
$$\|S_N^2\| = \sum_{n=-N}^{N} |c_n|^2 \int_{-1}^{1} 1\,\mathrm{d}x = \frac{4}{\pi^2}\sum_{n=1}^{N}\frac{1}{n^2} \to \frac{2}{3}$$

Then $\|S_n\|$ not converge to $\|\phi\|$, then it converges in the L^2 sense.

(d) It converges pointwisely.

(e) It does not converges uniformly.