MAT4220 FA22 HW09

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Problem 1 (P197 Q11)**.**

(a) Easy to verify that (18) satisfies $\Delta G = 0$ except at $\mathbf{x} = \mathbf{x}_0$, $G(\mathbf{x})|_{\partial D} = 0$, $G(\mathbf{x}) - \log|\mathbf{x}-\mathbf{x}_0|/2\pi$ finite at $\mathbf{x}_0$.

(b) Note that

$$\nabla G = \frac{1}{2\pi}\frac{1}{\rho}(\mathbf{x}-\mathbf{x}_0) - \frac{1}{2\pi}\frac{1}{\rho^*}(\mathbf{x}-\mathbf{x}_0^*)$$

Since $\mathbf{n} = \mathbf{x}/|\mathbf{x}|$, we have

$$\begin{aligned}
\nabla G \cdot \mathbf{n} &= \frac{1}{2\pi}\frac{1}{\rho}(a - r_0\cos\phi) - \frac{1}{2\pi}\frac{1}{\rho^*}(a - r_0^*\cos\phi) \\
&= \frac{1}{2\pi}\frac{a - r_0 a\cos\phi}{a^2 + r_0^2 - 2ar_0\cos\phi} - \frac{1}{2\pi}\frac{a - \frac{a^2}{r_0}\cos\phi}{a^2 + \frac{a^4}{r_0^2} - 2\frac{a^3}{r_0}\cos\phi} \\
&= \frac{1}{2\pi}\frac{1}{a}\frac{a^2 - r_0 a\cos\phi - r_0^2 + r_0 a\cos\phi}{a^2 + r_0^2 - 2ar_0\cos\phi} \\
&= \frac{a^2 - r_0^2}{2\pi a}\frac{1}{a^2 + r_0^2 - 2ar_0\cos\phi}
\end{aligned}$$

Therefore we have proved Poisson's formula since

$$u(\mathbf{x}_0) = \frac{a^2 - r_0^2}{2\pi a}\oiint_{\partial D} \mathrm{d}\sigma\ u(\mathbf{x})\frac{\partial}{\partial \mathbf{n}}G(\mathbf{x},\mathbf{x}_0) = \frac{a^2 - r_0^2}{2\pi a}\oiint_{\partial D} \mathrm{d}\sigma\ \frac{u(\mathbf{x})}{a^2 + r_0^2 - 2ar_0\cos\phi}$$

Problem 2 (P197 Q13)**.** The Green's function for the whole ball is

$$G(\mathbf{x},\mathbf{x}_0) = -\frac{1}{2\pi\rho} + \frac{a}{|\mathbf{x}_0|}\frac{1}{4\pi\rho^*}$$

Reflect the green's function wrt xy plane, we have

$$G(\mathbf{x},\mathbf{x}_0) = -\frac{1}{4\pi\rho} + \frac{a}{|\mathbf{x}_0|}\frac{1}{4\pi\rho^*} + \frac{1}{4\pi\rho_z} - \frac{a}{|\mathbf{x}_0|}\frac{1}{4\pi\rho_z^*}$$

where

$$\rho = |\mathbf{x}-\mathbf{x}_0| \quad \rho^* = |\mathbf{x}-\mathbf{x}_0^*| \quad \rho_z = |\mathbf{x}-\mathbf{x}_{0z}| \quad \rho_z^* = |\mathbf{x}-\mathbf{x}_{0z}^*|$$

where $x_0^* = a^2\mathbf{x_0}/|\mathbf{x}_0|^2$, and $\mathbf{x}_{0z}$ is the reflection of $\mathbf{x}_0$ wrt xy plane, $\mathbf{x}_{0z}^*$ is the reflection of $\mathbf{x}_0^*$ wrt xy plane.

Problem 3 (P337 Q1)**.** Easy to prove the linearity. To prove the continuity, since f integrable on Ω, then $\forall\phi_N \to \phi$, $\phi_N \in C^\infty(\Omega)$ compactly supported, let $F = |\langle |f|, 1\rangle|$ on Ω (since $|f|$ also integrable), then $\forall\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n > N$, we have $|\phi_N(x) - \phi(x)| < \epsilon/F$, and hence

$$\begin{aligned}
|\langle f, \phi_n\rangle - \langle f, \phi\rangle| &= \left|\int_\Omega \mathrm{d}x\ f(x)[\phi_n(x) - \phi(x)]\right| \\
&< \int_\Omega \mathrm{d}x\ |f(x)|\frac{\epsilon}{F} = \epsilon
\end{aligned}$$

which means the map is continuous.

Problem 4 (P337 Q2)**.** *Linearity:* direct prove by definition

$$\begin{aligned}\langle f', a\phi + b\psi\rangle &= -\langle f, a\phi' + b\psi'\rangle = -\int_\Omega \mathrm{d}x\ f(x)(a\phi'(x) + b\psi'(x)) = -a\langle f, \phi'\rangle - b\langle f, \psi'\rangle \\ &= a\langle f', \phi\rangle + b\langle f', \psi\rangle\end{aligned}$$

Continuity: since $\phi_N \to \phi$ uniformly and $\phi_N \in C^\infty(\Omega)$ compactly supported, then $\phi_N' \to \phi'$ uniformly and $\phi_N' \in C^\infty(\Omega)$ compactly supported, hence

$$\langle f, \phi_N'\rangle \to \langle f, \phi'\rangle \Rightarrow \langle f', \phi_N\rangle \to \langle f', \phi\rangle$$

Problem 5 (P337 Q5)**.**

Claim. $-c\langle H_x, \phi\rangle = \langle H_t, \phi\rangle$.

Proof. Since

$$\begin{aligned}\langle H_x, \phi\rangle &= -\iint_\Omega \mathrm{d}x\,\mathrm{d}t\ H(x-ct)\phi_x(x,t) = -\int_0^\infty \mathrm{d}t \int_{ct}^\infty \mathrm{d}x\ \phi_x(x,t) = \int_0^\infty \mathrm{d}t\ \phi(ct,t) \\ \langle H_t, \phi\rangle &= -\iint_\Omega \mathrm{d}x\,\mathrm{d}t\ H(x-ct)\phi_t(x,t) = -\int_0^\infty \mathrm{d}x \int_0^{x/c} \mathrm{d}t\ \phi_t(x,t) = -\int_0^\infty \mathrm{d}x\ \phi(x, x/c) \\ &= -c\int_0^\infty \mathrm{d}u\ \phi(cu, u)\end{aligned}$$

Hence $c\langle H_x, \phi\rangle = \langle H_t, \phi\rangle$. □

Therefore

$$\begin{aligned}c^2\langle H_{xx}, \phi\rangle &= -c^2\langle H_x, \phi_x\rangle = c\langle H_t, \phi_x\rangle = -c\langle H, \phi_{xt}\rangle \\ \langle H_{tt}, \phi\rangle &= -\langle H_t, \phi_t\rangle = c\langle H_x, \phi_t\rangle = -c\langle H, \phi_{tx}\rangle\end{aligned}$$

Therefore we have $\langle H_{tt}, \phi\rangle = c^2\langle H_{xx}, \phi\rangle$, which means that $H(x-ct)$ is a weak solution.