MAT4220 FA22 HW07

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Problem 1 (P160 Q7). The solution is in the form of

$$u(r,\theta,\phi) = \frac{c_1}{r} + c_2 + \frac{1}{6}r^2$$

Applying the boundary condition where $u(a,\theta,\phi) = u(b,\theta,\phi) = 0$, we can solve c_1 and c_2

$$c_1 = \frac{ab}{6}\frac{b^2-a^2}{b-a} = \frac{1}{6}ab(a+b) \quad c_2 = \frac{1}{6}\frac{b^3-a^3}{b-a} = \frac{1}{6}(a^2+ab+b^2)$$

Problem 2 (P165 Q6). Let $X(x)Y(y)Z(z)$ be a solution, then

$$\Delta u = 0 \Rightarrow \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

set boundary conditions

$$u_x(0,y,z) = u_x(1,y,z) = u_y(x,0,z) = u_y(x,1,z) = u_z(x,y,0) = 0$$

We have

$$X_m(x) = \cos m\pi x,\ Y_n(y) = \cos n\pi y$$

Hence

$$\frac{Z''}{Z} = (m^2+n^2)\pi^2 \Rightarrow Z(z) = A\cosh\sqrt{m^2+n^2}\pi z$$

Let the solution be in the form

$$\begin{aligned} u(x,y,z) = \frac{1}{4}A_{00} + \frac{1}{2}\sum_{m=1}^{\infty} A_{m0}\cos m\pi x\cosh m\pi z + \frac{1}{2}\sum_{n=1}^{\infty} A_{0n}\cos n\pi y\cosh m\pi z \\ + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\cos m\pi\cos n\pi y\cosh\sqrt{m^2+n^2}\pi z \end{aligned}$$

Hence

$$\begin{aligned} u(x,y,1) = g(x,y) = \sum_{mn} A_{mn}\cos m\pi\cos n\pi y\cosh\sqrt{m^2+n^2}\pi \\ \Rightarrow A_{mn} = \frac{4}{\cosh\sqrt{m^2+n^2}\pi}\int_0^1\int_0^1 \mathrm{d}x\,\mathrm{d}y\cos m\pi x\cos n\pi y g(x,y) \end{aligned}$$

Problem 3 (P172 Q2). Since

$$u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n(A_n\cos n\theta + B_n\sin n\theta)$$

we have

$$\begin{aligned} u(a,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} a^n(A_n\cos n\theta + B_n\sin n\theta) = 1 + 3\sin\theta \\ \Rightarrow A_0 = 2,\ A_n = 0,\ B_1 = \frac{3}{a},\ B_2 = \cdots = 0 \end{aligned}$$

which means $u(r,\theta) = 1 + 3r\sin\theta/a$.

Problem 4 (P176 Q4)**.** Let

$$\begin{aligned} u(r,\theta) &= \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^{-n}(A_n \cos n\phi + B_n \sin n\phi) \\ A_n &= \frac{a^n}{\pi}\int_0^{2\pi} \mathrm{d}\phi h(\phi)\cos n\phi \\ B_n &= \frac{a^n}{\pi}\int_0^{2\pi} \mathrm{d}\phi h(\phi)\sin n\phi \end{aligned}$$

Therefore

$$\begin{aligned} u(r,\theta) &= \frac{1}{2}\int_0^{2\pi} \mathrm{d}\phi\ h(\phi) + \sum_{n=1}^{\infty}(a/r)^{-n}\int_0^{2\pi}[h(\phi)\cos n\phi\cos n\theta + h(\phi)\sin n\phi \sin n\theta] \\ &= \frac{1}{\pi}\int_0^{2\pi} \mathrm{d}\phi\ h(\phi)\left[\frac{1}{2} + \sum_{n=1}^{\infty}(a/r)^{-n}(\cos n\phi\cos n\theta + \sin n\phi\sin n\theta)\right] \\ &= \frac{1}{\pi}\int_0^{2\pi} \mathrm{d}\phi\ h(\phi)\left[\frac{1}{2} + \sum_{n=1}^{\infty}(a/r)^{-n}\cos n(\phi-\theta)\right] \\ &= \frac{1}{\pi}\int_0^{2\pi} \mathrm{d}\phi\ h(\phi)\left[\frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty}(a/r)^{-n}e^{in\varphi} + (a/r)^{-n}e^{-in\varphi}\right] \\ &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi\ h(\phi)\left[1 + \frac{ae^{i\varphi}}{r-ae^{i\varphi}} + \frac{ae^{-i\varphi}}{r-ae^{-i\varphi}}\right] \\ &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi\ h(\phi)\frac{r^2-a^2}{r^2+a^2-2ar\cos(\theta-\phi)} \end{aligned}$$

Problem 5 (P176 Q13)**.** In this case, the eigenfunction $\Theta(\theta)$ and $R(r)$ have the form

$$\Theta_n(\theta) = \sin\frac{n\pi(\theta-\alpha)}{\beta-\alpha} \quad R_n(r) = A_n r^{\frac{n\pi}{\beta-\alpha}} + B_n r^{-\frac{n\pi}{\beta-\alpha}}$$

Therefore

$$\begin{aligned} u(r,\theta) &= \sum_{n=1}^{\infty}\left(A_n r^{\frac{n\pi}{\beta-\alpha}} + B_n r^{-\frac{n\pi}{\beta-\alpha}}\right)\sin\frac{n\pi(\theta-\alpha)}{\beta-\alpha} \\ A_n &= \frac{2}{\beta-\alpha}\frac{1}{a^{\frac{2n\pi}{\beta-\alpha}} - b^{\frac{2n\pi}{\beta-\alpha}}}\int_\alpha^\beta \mathrm{d}\theta \sin\frac{n\pi(\theta-\alpha)}{\beta-\alpha}[a^{\frac{n\pi(\theta-\alpha)}{\beta-\alpha}}g(\theta) - b^{\frac{n\pi(\theta-\alpha)}{\beta-\alpha}}h(\theta)] \\ B_n &= \frac{2}{\beta-\alpha}\frac{1}{a^{-\frac{2n\pi}{\beta-\alpha}} - b^{-\frac{2n\pi}{\beta-\alpha}}}\int_\alpha^\beta \mathrm{d}\theta \sin\frac{n\pi(\theta-\alpha)}{\beta-\alpha}[a^{-\frac{n\pi(\theta-\alpha)}{\beta-\alpha}}g(\theta) - b^{-\frac{n\pi(\theta-\alpha)}{\beta-\alpha}}h(\theta)] \end{aligned}$$