MAT4220 FA22 HW08

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Problem 1 (P184 Q5)**.** Consider any solutions satisfies the boundary condition w and u where $\Delta u = 0$. Let $v = w - u$, hence $\partial v/\partial \mathbf{n} = 0$. Then

$$\begin{aligned}
E[w] &= E[u+v] \\
&= \frac{1}{2}\iiint_D \mathrm{d}\mathbf{x}\ |\nabla u + \nabla v|^2 - \oiint_{\partial D} \mathrm{d}\sigma\ (u+v)(\nabla u + \nabla v)\cdot \mathbf{n} \\
&= E[u] + E[v] + \iiint_D \mathrm{d}\mathbf{x}\ \nabla u \cdot \nabla v - \oiint_{\partial D} \mathrm{d}\sigma\ v\nabla u \cdot \mathbf{n} \\
&= E[u] + E[v] + \oiint_{\partial D} \mathrm{d}\sigma\ v\nabla u \cdot \mathbf{n} - \iiint_D \mathrm{d}\mathbf{x}\ v\Delta u - \oiint_{\partial D} \mathrm{d}\sigma\ v\nabla u \cdot \mathbf{n} \\
&= E[u] + E[v]
\end{aligned}$$

Since

$$E[v] = \frac{1}{2}\iiint_D \mathrm{d}\mathbf{x}\ |\nabla v|^2 - \oiint \mathrm{d}\sigma\ v\nabla v \cdot \mathbf{n} = \frac{1}{2}\iiint_D \mathrm{d}\mathbf{x}\ |\nabla v|^2 \geq 0$$

Hence $E[w] - E[u] = E[v] \geq 0$.

Problem 2 (P184 Q7)**.** Define the operation

$$(\nabla u_1, \nabla u_2) = \iiint_D \mathrm{d}\mathbf{x}\ \nabla u_1 \cdot \nabla u_2$$

Let

$$\tilde{w}(\mathbf{x}) = w_0 + c_1 w_1 + \cdots + c_n w_n$$

let $c_0 = 1$, hence

$$E[\tilde{w}] = (\nabla \tilde{w}, \nabla \tilde{w}) = \sum_{ij} c_i c_j (\nabla u_i, \nabla u_j)$$

To minimize the energy, we should have $\partial E[\tilde{w}]/\partial c_i = 0$, which means

$$\frac{\partial E[\tilde{w}]}{\partial c_i} = 2c_i(\nabla w_i, \nabla w_i) + 2\sum_{i\neq j} c_j (\nabla w_i, \nabla w_j) = 2\sum_j c_j(\nabla w_i, \nabla w_j) = 0$$

Hence we have a linear system with n unknowns and n equations

$$\sum_{j=1}^n c_j(\nabla w_i, \nabla w_j) = -(\nabla w_i, \nabla w_0)$$

for $i = 1, \ldots, n$.

Problem 3 (P187 Q2)**.** Since

$$-\frac{1}{4\pi}\iiint_D \mathrm{d}\mathbf{x}\frac{1}{r}\Delta\phi(\mathbf{x}) = -\frac{1}{4\pi}\oiint_{\partial D} \mathrm{d}\sigma \frac{1}{r}\nabla\phi \cdot \mathbf{n} + \frac{1}{4\pi}\iiint_D \mathrm{d}\mathbf{x}\ \nabla\frac{1}{r}\cdot\nabla\phi(\mathbf{x})$$

$$= -\frac{1}{4\pi}\oiint_{\partial D} \mathrm{d}\sigma \frac{1}{r}\nabla\phi(\mathbf{x})\cdot\mathbf{n} + \frac{1}{4\pi}\oiint_{\partial D} \mathrm{d}\sigma\phi(\mathbf{x})\nabla\frac{1}{r}\cdot\mathbf{n} - \frac{1}{4\pi}\iiint_D \mathrm{d}\mathbf{x}\ \phi(\mathbf{x})\Delta\frac{1}{r}$$

Let $D_\epsilon = B_R(\mathbf{0})\setminus B_\epsilon(\mathbf{0})$ for some $R>0$ large (ϕ vanish) and $\epsilon>0$ small, since $\Delta 1/r = 0$, then

$$\begin{aligned}
-\frac{1}{4\pi}\iiint_{D_\epsilon} \mathrm{d}\mathbf{x}\frac{1}{r}\Delta\phi(\mathbf{x}) &= -\frac{1}{4\pi}\oiint_{\partial B_\epsilon(\mathbf{0})} \mathrm{d}\sigma\frac{1}{r}\nabla\phi(\mathbf{x})\cdot\mathbf{n} + \frac{1}{4\pi}\oiint_{\partial B_\epsilon(\mathbf{0})} \mathrm{d}\sigma\phi(\mathbf{x})\nabla\frac{1}{r}\cdot\mathbf{n} \\
&= \epsilon\frac{1}{4\pi}\oiint_{\partial B_\epsilon(\mathbf{0})} \sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta\frac{\partial}{\partial r}\phi(\mathbf{x}) + \frac{1}{4\pi}\oiint_{\partial B_\epsilon(\mathbf{0})} \sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta\phi(\mathbf{x}) \\
&= \epsilon\frac{\partial}{\partial r}\phi(\mathbf{x}_1) + \phi(\mathbf{x}_2)
\end{aligned}$$

for $\mathbf{x}_1, \mathbf{x}_2 \in \partial B_\epsilon(\mathbf{0})$ according to the mean value theorem. Hence, easy to show that

$$\lim_{\epsilon\to 0} -\frac{1}{4\pi}\iiint_{D_\epsilon} \mathrm{d}\mathbf{x}\frac{1}{r}\Delta\phi(\mathbf{x}) = -\frac{1}{4\pi}\iiint_D \mathrm{d}\mathbf{x}\frac{1}{r}\Delta\phi(\mathbf{x}) = \phi(\mathbf{0})$$

Problem 4 (P196 Q1)**.** Since $G''(x)=0$, we have $G(x) = Ax+b$ except x_0. Then we have

$$G(x) = \begin{cases} Ax+B & x\in(0,x_0) \\ Cx+D & x\in(x_0,l) \end{cases}$$

Applying the boundary and continuity condition, we have

$$\begin{aligned} B &= 0 \\ Cl + D &= 0 \\ Ax_0 + B &= Cx_0 + D \end{aligned}$$

Note that $H(x) = G(x) + |x-x_0|/2$ differentiable at x_0

$$H(x) = \begin{cases} Ax+B+(x_0-x)/2 & x\in(0,x_0) \\ Cx+D+(x-x_0)/2 & x\in(x_0,l) \end{cases} \Rightarrow A - 1/2 = C + 1/2$$

Hence we have four equations, easy to solve that

$$A = \frac{l-x_0}{x_0} \quad B = 0 \quad C = -\frac{x_0}{l} \quad D = x_0$$

Hence

$$G(x,x_0) = \begin{cases} \frac{l-x_0}{l}x & x\in(0,x_0) \\ \frac{l-x}{l}x_0 & x\in(x_0,l) \end{cases}$$

Problem 5 (P196 Q6)**.**

(a) The Green's function for the half plane is

$$G(\mathbf{x},\mathbf{x}_0) = \frac{1}{2\pi}(\log|\mathbf{x}-\mathbf{x}_0| - \log|\mathbf{x}-\mathbf{x}_0^*|)$$

where $\mathbf{x}_0 = (x_0,y_0)$, $\mathbf{x}_0^* = (x_0,-y_0)$.

(b) The solution is

$$u(\mathbf{x}_0) = \int_{\partial D} \mathrm{d}\sigma\ h(\mathbf{x}) \frac{\partial}{\partial \mathbf{n}} G(\mathbf{x}, \mathbf{x}_0) = -\int_{-\infty}^{\infty} \mathrm{d}x\ h(x) \frac{\partial}{\partial y} G[(x,0),(x_0,y_0)]$$

where

$$\frac{\partial}{\partial y} G[(x,y),(x_0,y_0)] = \frac{1}{2\pi}\left[\frac{y-y_0}{(\mathbf{x}-\mathbf{x}_0)^2} - \frac{y+y_0}{(\mathbf{x}-\mathbf{x}_0^*)^2}\right]$$

(c) Plugin $h(\mathbf{x}) = 1$, we get

$$\begin{aligned} u(x_0,y_0) &= -\int_{-\infty}^{\infty} \mathrm{d}x\ \frac{1}{2\pi} \frac{0-y_0}{(x-x_0)^2+(0-y_0)^2} - \frac{0+y_0}{(x-x_0^2)+(0+y_0)^2} \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}x \frac{y_0}{(x-x_0)^2+y_0^2} \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}u \frac{1}{u^2+1} = 1 \end{aligned}$$