Name: ____________________ SID: ____________________

MTH4220(2022 Fall): Midterm

- This exam will last for 1 hour and 45 mins.
- No phones or discussion, book and notes closed, no calculators.

1. (a) (10pts) Solve the first order PDE:

$$\partial_x u + y\partial_y u = u$$

(b) (10pts) Solve the initial value problem

$$\begin{cases} \partial_t^2 u(x,t) = c^2 \partial_x^2 u(x,t) + t, \\ u(x,0) = \sin x, \ \partial_t u(x,0) = 1. \end{cases}$$

2. (15pts) Let $\kappa > 0$ be a positive constant. Consider the following heat equation on a domain $\Omega \subset \mathbb{R}$:

$$\begin{cases} \partial_t u = \kappa \partial_x^2 u + f(x,t), & t > 0; \\ u(0,x) = \phi(x). \end{cases} \tag{1}$$

(a) If $\Omega = \mathbb{R}$ is the real line, what is the solution to the above initial value problem?

(b) If Ω is the half line $(0, \infty)$, and $\phi(x) = x$, $f(x,t) = 0$, then determine the solution subject to boundary condition $u(0,t) = 1$.

3. (25pts) Let $c > 0$ be a given constant. Consider the initial boundary value problem:

$$\begin{cases} \partial_t^2 u = c^2 \partial_x^2 u + \partial_t u, \qquad x \in (0, c). \\ \partial_x u(0,t) = \partial_x u(c,t) = 0. \\ u(x,0) = \phi(x), \quad \partial_t u(x,0) = \psi(x). \end{cases} \tag{2}$$

Use separation of variables to find a series solution.

4. (20pts) Consider the function $g(x) = x + 1$ on $(0, 2)$.

   (a) Define and sketch the even periodic extension of $g(x)$ on the real line.

   (b) Find the Fourier cosine series of the $g(x)$ on (0,2), does it converge as a series? (use knowledge from your elementary real analysis/Calculus class)

5. (20pts) Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. Consider the initial boundary value problem of the heat equation

$$\begin{cases} \partial_t u = \Delta u + f(x,t), & x \in \Omega, \quad t > 0. \\ u(x,0) = \phi(x), \quad u|_{\partial\Omega}(x,t) = \psi(x). \end{cases} \tag{3}$$

Suppose there exists a solution to the problem, answering the following questions.

(a) (3pts) Use math language to state the meaning of uniqueness of solutions to the problem.

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(b) (3pts) Use math language to state a stability of solutions subject to initial and boundary data?

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(c) (8pts) Let $\Omega = (a,b) \subset \mathbb{R}$. Prove the stability you stated in (c) and show that the stability implies uniqueness.

(d) (6pts) Is there any stability subject to initial data only under L^2-norm? Justify your answer.

(e) (Bonus 5pts) If for $p > 1$ we introduce the $L^p(\Omega)$-norm by

$$\|g\|_{L^p(\Omega)} := \left(\int_\Omega g^p(x)\, dx \right)^{1/p},$$

and $f = f(u)$, is there any stability of the system under $L^p(\Omega)$-norm? Justify your answer.