PHY3110 FA22 HW01

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Problem 1. Since

$$\begin{aligned}[a \times (b \times c)]_i &= \epsilon_{ijk} a_j (b \times c)_k = \epsilon_{ijk}\epsilon_{klm} a_j b_l c_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) a_j b_l c_m = \delta_{il}\delta_{jm} a_j b_l c_m - \delta_{im}\delta_{jl} a_j b_l c_m \\ &= b_i (a \cdot c) - c_i (a \cdot b)\end{aligned}$$

Hence we have $a \times (b \times c) = (a \cdot c) b - (a \cdot b) c$.

Suppose $b = \nabla$, then we should consider the commutation relationship of b and c, i.e., $[b, c] = bc - cb$ might be nonzero. At that time, the formula might not hold.

Problem 2. Let

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r \sin\theta \cos\phi \\ r \sin\theta \sin\phi \\ r \cos\theta \end{bmatrix}$$

and define the following three unit vectors

$$\hat{r} = \begin{bmatrix} \sin\theta \cos\phi \\ \sin\theta \sin\phi \\ \cos\theta \end{bmatrix}, \quad \hat{\phi} = \begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{bmatrix}, \quad \hat{\theta} = \begin{bmatrix} \cos\theta \cos\phi \\ \cos\theta \sin\phi \\ -\sin\theta \end{bmatrix}$$

easy to show that $\hat{r}$, $\hat{\phi}$ and $\hat{\theta}$ forms an orthonormal basis in $\mathbb{R}^3$. Then under spherical coordinates we have

$$\mathrm{d}\mathbf{r} = \mathrm{d}r \cdot \hat{r} + r\,\mathrm{d}\theta \cdot \hat{\theta} + r \sin\theta\,\mathrm{d}\phi \cdot \hat{\phi}$$

square the equation, we get

$$\mathrm{d}\mathbf{r}^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2 \sin^2\theta\,\mathrm{d}\phi^2$$

Problem 3. Since

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_i} &= \sum_j \frac{\partial^2 T}{\partial \dot{q}_i q_j}\dot{q}_j + \frac{\partial^2 T}{\partial \dot{q}_i \dot{q}_j}\ddot{q}_j + \frac{\partial^2 T}{\partial \dot{q}_i \partial t} \\ \dot{T} &= \sum_i \frac{\partial T}{\partial q_i}\dot{q}_i + \frac{\partial T}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial T}{\partial t} \\ \frac{\partial \dot{T}}{\partial \dot{q}_i} &= \sum_j \frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j}\ddot{q}_j + \frac{\partial^2 T}{\partial \dot{q}_i q_j}\dot{q}_j + \frac{\partial^2 T}{\partial \dot{q}_i \partial t}\end{aligned}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_i} + \frac{\partial T}{\partial q_i} - \frac{\partial \dot{T}}{\partial \dot{q}_i} = 0$$

Add this equation to the Lagrange's equation, then we have

$$\frac{\partial \dot{T}}{\partial \dot{q}_i} - 2\frac{\partial T}{\partial q_i} = Q_i$$

Problem 4.

(a) The condition should be

$$\frac{\partial f}{\partial x_i} = g_i$$

(b) Define function $f(x, y, z) = x^2 + y^2 + z^2 + xy + yz + xz$, then we have

$$\mathrm{d}f = (2x + y + z)\,\mathrm{d}x + (x + 2y + z)\,\mathrm{d}y + (x + y + 2z)\,\mathrm{d}z$$

Hence the following constraint is holonomic.

$$(2x + y + z)\,\mathrm{d}x + (x + 2y + z)\,\mathrm{d}y + (x + y + 2z)\,\mathrm{d}z = 0$$

The second constraint is not holonomic, since it cannot be written into full differential form.

Problem 5. Use the generalized coordinates θ and x to represent the system, hence we have (let $k = m\omega^2$)

$$\begin{aligned}
T &= \frac{1}{2}m[(l + x)^2\dot{\theta}^2 + \dot{x}^2] \\
V &= -mg(l + x)\cos\theta + \frac{1}{2}m\omega^2 x \\
L &= T - V
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\partial L}{\partial x} &= m(l + x)\dot{\theta}^2 + mg\cos\theta - m\omega^2 x \\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} &= m\ddot{x} \\
\frac{\partial L}{\partial \theta} &= -mg(l + x)\sin\theta \\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} &= m(l + x)^2\ddot{\theta} + 2m(l + x)\dot{\theta}\dot{x}
\end{aligned}$$

Applying the Lagrange's equation we get

$$\begin{aligned}
\ddot{x} &= (l + x)\dot{\theta}^2 + g\cos\theta - \omega^2 x \\
\ddot{\theta} &= -\frac{1}{l + x}(g\sin\theta + 2\dot{\theta}\dot{x})
\end{aligned}$$