PHY3110 SP23 Notes

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0 Introduction

Grading: 30% homework, 30% midterm, 40% final.

Textbooks:

- H. Goldstein, C. Poole, J. Safko, Classical Mechanics, 3rd Edition, Pearson.
- J.R. Taylor, Classical Mechanics, University Science Books.
- T.W.B. Kibble, F.H. Berkshire, Classical Mechanics, 5th Edition, Imperial College Press.
- 梁昆淼，力学（下册）理论力学，4th Edition，高等教育出版社.

Classical mechanics describe the motion of macroscopic objects, which are not extremely massive and not extremely fast.

1 Newtonian Mechanics

Vectorial quantities of motion: position $\mathbf{r}$, velocity $\mathbf{v}$, force $\mathbf{F}$, momentum $\mathbf{p} = m\mathbf{v}$, angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. Equations of motion are derived from those vector quantities.

Analytical mechanics uses scalar quantities of motion

- Kinetic energy $T = \frac{1}{2}m\mathbf{v}^2$
- Potential energy $V = V(\mathbf{r})$

Equations of motion are derived from those scalar quantities.

1.1 Newton's Laws

Newton's 2$^{\text{nd}}$ law

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = m\mathbf{a} \tag{1}$$

valid in an inertial frame. Angular momentum $\mathbf{L}$ and torque $\mathbf{N}$ are also related

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \times \mathbf{p}) = \mathbf{r} \times \mathbf{F} = \mathbf{N} \tag{2}$$

Work done by external forces

$$W_{12} = \int_1^2 \mathbf{F}\,\mathrm{d}\mathbf{s} = \int_1^2 m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\,\mathrm{d}\mathbf{s} = \int_1^2 m\mathbf{v}\,\mathrm{d}\mathbf{v} = \frac{1}{2}m\mathbf{v}^2\Big|_1^2 \tag{3}$$

Define a scalar function $V(\mathbf{r})$, then $\mathbf{F} = -\nabla V(\mathbf{r})$ is a conservative force.

$$\oint \mathbf{F}\,\mathrm{d}\mathbf{s} = 0 \tag{4}$$

Center of mass of the system

$$\mathbf{R} = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i} = \frac{\sum_i m_i \mathbf{r}_i}{M} \tag{5}$$

Total momentum

$$\mathbf{P} = \sum_i m_i \mathbf{p}_i = M\dot{\mathbf{R}} \tag{6}$$

Hence $\mathbf{P}$ is conserved if external force $\mathbf{F}^{(e)}$ is zero.
Total angular momentum

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \sum_i \mathbf{r}_i \times \mathbf{p}_i = \sum_i \mathbf{r}_i \times \left(\mathbf{F}_i^{(e)} + \sum_j \mathbf{F}_{ij}\right) = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{(e)} + \sum_{ij} \mathbf{r}_i \times \mathbf{F}_{ij}$$

Since $\mathbf{r}_{ij}$ parallel to $\mathbf{F}_i j$, then

$$\sum_{ij} \mathbf{r}_i \mathbf{F}_{ij} = \frac{1}{2} \sum_{ij} \mathbf{r}_{ij} \times \mathbf{F}_{ji} = 0 \tag{7}$$

Therefore

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \mathbf{N}^{(e)} \tag{8}$$

Decomposition of the angular momentum

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i = \sum_i (\mathbf{R} + \mathbf{r}_i) \times m_i (\mathbf{V} + \mathbf{v}_i') = \sum_i \mathbf{R} \times m_i \mathbf{V} + \sum_i \mathbf{r}_i' \times m_i \mathbf{v}_i' \tag{9}$$

1.2 Constraints

Holonomic constraint

$$f(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N, t) = 0 \tag{10}$$

Example: rigid body

$$(\mathbf{r}_i - \mathbf{r}_j)^2 - c_{ij}^2 = 0 \tag{11}$$

Example: non-sliding cylinder

$$\dot{x} - R\dot{\theta} = 0 \Rightarrow x - R\theta = \text{const}$$

A constraint of the form

$$\sum_i g_i(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)\, \mathrm{d}\mathbf{x}_i = 0 \Rightarrow \mathrm{d}G(\mathbf{x}_1, \dots) = 0 \Rightarrow G(\mathbf{x}_1, \dots) = \text{const} \tag{12}$$

Non-holonomic constraint: cannot be written in the form of holonomic constraint.

1.3 Generalized coordinates

Suppose we have a N-particle system, we will have $3N$ DOFs. With k constraints, we will have $3N-k$ DOFs. Define $q_1, \dots, q_{3N-k}$ generalized coordinates, we have

$$\mathbf{r}_i = \mathbf{r}_i(q_1, \dots, q_{3N-1}, t) \tag{13}$$

2 Lagrange Formalism

2.1 D'Alembert's Principle

Hint from the rigid body: internal forces of constraints do not work.

Virtual displacement: $\delta\mathbf{r}_i$ is consistent with the constraints imposed on the system at a given time

$$\mathbf{r}_i \to \mathbf{r}_i + \delta\mathbf{r}_i \tag{14}$$

Consider a system in equilibrium

$$\mathbf{F}_i = 0 \Rightarrow \sum_i \mathbf{F}_i \cdot \delta\mathbf{r}_i = 0 \tag{15}$$

Separate $\mathbf{F}_i = \mathbf{F}_i^{(}a) + \mathbf{f}_i$ where $\mathbf{f}_i$ is the constraint force. Hence

$$\sum_i (\mathbf{F}_i^{(a)} + \mathbf{f}_i) \cdot \delta\mathbf{r}_i = 0 \Rightarrow \sum_i \mathbf{F}_i^{(a)} \cdot \delta\mathbf{r}_i = 0 \tag{16}$$

For a system moving under external forces

$$\mathbf{F}_i - \dot{\mathbf{p}}_i = 0 \Rightarrow \sum_i (\mathbf{F}_i - \dot{\mathbf{p}}_i)\delta\mathbf{r}_i = 0 \Rightarrow \sum_i (\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i)\delta\mathbf{r}_i = 0 \tag{17}$$

For holonomic constraints

$$\mathbf{r}_i = \mathbf{r}_i(q_1, \dots, q_n, t), \quad \mathbf{v}_i = \frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t} = \frac{\partial \mathbf{r}_i}{\partial t} + \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j}\dot{q}_j, \quad \delta\mathbf{r}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j}\delta q_j \tag{18}$$

Generalized force Q_j

$$\sum_i \mathbf{F}_i \delta\mathbf{r}_i = \sum_{ij} \mathbf{F}_i \frac{\partial \mathbf{r}_i}{\partial q_j}\delta q_j = \sum_j Q_j \delta q_j \tag{19}$$

Then

$$\sum_i \dot{\mathbf{p}}_i \cdot \delta\mathbf{r}_i = \sum_{ij} m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}\delta q_j = \sum_{ij} \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}\right) - m_i \dot{\mathbf{r}}_i \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathbf{r}_i}{\partial q_j}\right]\delta q_j \tag{20}$$

$$= \sum_i \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right]\delta q_j = \sum_j Q_j \delta q_j \tag{21}$$

$$\tag{22}$$

Hence $\forall j$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j = 0 \tag{23}$$

Let the potential energy $V = V(\mathbf{r}_i, \dots) = V(q_j, \dots)$, then we have

$$Q_j = \sum_i \mathbf{F}_i \frac{\partial \mathbf{r}_i}{\partial q_j} = \sum_i -\nabla_i V \frac{\partial \mathbf{r}_i}{\partial q_j} = -\frac{\partial V}{\partial q_j} \tag{24}$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} - Q_j = 0 \tag{25}$$