PHY3110 SP23 HW11

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Problem 1. Verify the Jacobi identity for the Poisson brackets.

Solution. Let $A_q = \partial_q A$, then

$$\begin{aligned}\{A,\{B,C\}\} &= \{A, B_{q_i}C_{p_i} - B_{p_i}C_{q_i}\}\\ &= A_{q_j}(B_{q_ip_j}C_{p_i} + B_{q_i}C_{p_ip_j} - B_{p_ip_j}C_{q_i} - B_{p_i}C_{q_ip_j}) - A_{p_j}(B_{q_iq_j}C_{p_i} + B_{q_i}C_{p_iq_j} - B_{p_iq_j}C_{q_i} - B_{p_i}C_{q_iq_j})\end{aligned}$$

Hence $J = \{A,\{B,C\}\} + \{B,\{C,A\}\} + \{C,\{A,B\}\}$ is

$$\begin{aligned}J &= A_{q_j}(B_{q_ip_j}C_{p_i} + B_{q_i}C_{p_ip_j} - B_{p_ip_j}C_{q_i} - B_{p_i}C_{q_ip_j}) - A_{p_j}(B_{q_iq_j}C_{p_i} + B_{q_i}C_{p_iq_j} - B_{p_iq_j}C_{q_i} - B_{p_i}C_{q_iq_j})\\ &+ B_{q_j}(C_{q_ip_j}A_{p_i} + C_{q_i}A_{p_ip_j} - C_{p_ip_j}A_{q_i} - C_{p_i}A_{q_ip_j}) - B_{p_j}(C_{q_iq_j}A_{p_i} + C_{q_i}A_{p_iq_j} - C_{p_iq_j}A_{q_i} - C_{p_i}A_{q_iq_j})\\ &+ C_{q_j}(A_{q_ip_j}B_{p_i} + A_{q_i}B_{p_ip_j} - A_{p_ip_j}B_{q_i} - A_{p_i}B_{q_ip_j}) - C_{p_j}(A_{q_iq_j}B_{p_i} + A_{q_i}B_{p_iq_j} - A_{p_iq_j}B_{q_i} - A_{p_i}B_{q_iq_j})\\ &= 0\end{aligned}$$

Problem 2. Show by the use of Poisson brackets that for a one-dimensional harmonic oscillator there is a constant of motion u defined as

$$u(q,p,t) = \ln(p + im\omega q) - i\omega t, \quad \omega = \sqrt{\frac{k}{m}} \tag{1}$$

Solution. The Hamiltonian of a one-dimensional harmonic oscillator is

$$H(q,p,t) = \frac{1}{2m}(p^2 + m^2\omega^2 q^2)$$

Hence

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t} &= \{u, H\} + \frac{\partial u}{\partial t}\\ &= \frac{im\omega}{p + im\omega q}\frac{p}{q} - \frac{m\omega^2 q}{p + im\omega q} - i\omega\\ &= \frac{1}{p + im\omega q}(i\omega p - m\omega^2 q + m\omega^2 q - i\omega p)\\ &= 0\end{aligned}$$

Problem 3. Show that the following transformation is canonical (α is a fixed parameter):

$$\begin{aligned}x &= \frac{1}{\alpha}(\sqrt{2P_1}\sin Q_1 + P_2), \quad p_x = \frac{\alpha}{2}(\sqrt{2P_1}\cos Q_1 - Q_2)\\ y &= \frac{1}{\alpha}(\sqrt{2P_1}\cos Q_1 + Q_2), \quad p_y = -\frac{\alpha}{2}(\sqrt{2P_1}\sin Q_1 - P_2)\end{aligned} \tag{2}$$

Apply this transformation to the problem of a particle of charge q moving in a plane that is perpendicular to a constant magnetic field $\mathbf{B}$. Express the Hamiltonian fot this problem in the (Q_i, P_i) coordinates letting the parameter α take the form

$$\alpha^2 = \frac{qB}{c} \tag{3}$$

From this Hamiltonian, obtain the motion of the particle as a function of time.

Solution. To show the transformation is a canonical transformation, consider the Jacobain $\mathbf{J}$

$$
\begin{aligned}
\mathbf{J} &= \frac{\partial[\mathbf{Q}\ \ \mathbf{P}]^T}{\partial[\mathbf{q}\ \ \mathbf{p}]^T} \\
&= \begin{bmatrix} (2P_1)^{0.5}\cos Q_1/\alpha & 0 & (2P_1)^{-0.5}\sin Q_1/\alpha & 1/\alpha \\ -(2P_1)^{0.5}\sin Q_1/\alpha & 1/\alpha & (2P_1)^{-0.5}\cos Q_1/\alpha & 0 \\ -\alpha(2P_1)^{0.5}\sin Q_1 & -\alpha/2 & \alpha(2P_1)^{-0.5}\cos Q_1/2 & 0 \\ -\alpha(2P_1)^{0.5}\cos Q_1 & 0 & \alpha(2P_1)^{-0.5}\cos Q_1/2 & \alpha/2 \end{bmatrix}
\end{aligned}
$$

We can show that

$$\mathbf{J}\begin{bmatrix} & -\mathbb{1} \\ \mathbb{1} & \end{bmatrix}\mathbf{J}^T = \begin{bmatrix} & -\mathbb{1} \\ \mathbb{1} & \end{bmatrix}$$

Hence, it is a cannonical transformation.

To simplify calculation, let $c = 1$, then $\alpha^2 = qB$. Hence

$$L = \frac{1}{2}m\mathbf{v}^2 - q\phi + q\mathbf{A}\cdot\mathbf{v}, \quad H = \mathbf{v}\cdot\mathbf{p} - L = \frac{1}{2}m\mathbf{v}^2 = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2$$

Since $\mathbf{B} = \begin{bmatrix} 0 & 0 & B \end{bmatrix}^T$, there are multiple choices of $\mathbf{A}$, for example

$$\begin{bmatrix} 0 \\ Bx \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -By \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -By/2 \\ Bx/2 \\ 0 \end{bmatrix}$$

I will adopt the last one to simply the calculation. Suppose we have $A_x = -By/2$, $A_y = Bx/2$, $A_z = 0$, then

$$
\begin{aligned}
H &= \frac{1}{2m}[(p_x - qA_x)^2 + (p_y - qA_y)^2] \\
&= \frac{1}{2m}(\alpha^2 2P_1\sin^2 Q_1 + \alpha^2 2P_1\cos^2 P_1) \\
&= \frac{qB}{m}P_1
\end{aligned}
$$

In this case, the solution of EOM is

$$
\begin{aligned}
Q_1 &= \frac{qB}{m}t + C_1 \\
Q_2 &= C_2 \\
P_1 &= C_3 \\
P_2 &= C_4
\end{aligned}
$$

Problem 4. Use the method of infinitesimal canonical transformation to solve the motion of a one-dimensional harmonic oscillator as a function of time.

Solution. Let $G = H, U = q$, then

$$
\begin{aligned}
U' &= [q, H] = \frac{p}{m} \\
U'' &= [p/m, H] = -\omega^2 q \\
U''' &= -\omega^2\frac{p}{m}
\end{aligned}
$$

$$U^{(4)} = \omega^4 q$$

Therefore

$$\begin{aligned} q(t) &= q_0 + \frac{p_0}{m}t - \frac{1}{2!}\omega^2 q_0 t^2 - \frac{1}{3!}\omega^3 \frac{p_0}{m} t^3 + \frac{1}{4!}\omega^4 q_0 t^4 + \cdots \\ &= q_0 \cos \omega t + \frac{p_0}{m\omega} \sin \omega t \end{aligned}$$

Follow the same way, we can also get $p(t)$

$$p(t) = -m\omega q_0 \sin \omega t + p_0 \cos \omega t$$

where $q_0 = q(0)$ and $p_0 = p(0)$.