PHY3110 FA22 HW03

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Problem 1. It is non-holonomic since we cannot find a function $f = f(x, y, z)$ and write the constraint into

$$\mathrm{d}f = f_x \,\mathrm{d}x + f_y \,\mathrm{d}y + f_z \,\mathrm{d}z = 0$$

In this case, we have

$$(x^2 + y^2 + z^2 + 2x)\,\mathrm{d}x + 2y\,\mathrm{d}y + 2z\,\mathrm{d}z = 0$$

Note that

$$\frac{\partial}{\partial z}(x^2 + y^2 + z^2) \neq \frac{\partial}{\partial x}(2z)$$

which means that we cannot find such f.

Problem 2. Minimize the action is equivalent to solve the Lagrange's equation, hence

$$\begin{aligned} L(x, \dot{x}, t) &= \frac{1}{2}m\dot{x}^2 + Fx \\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} &= m\ddot{x} = 2C \\ \frac{\partial L}{\partial x} &= F \end{aligned} \quad \Rightarrow \begin{cases} A = 0 \\ B = \dfrac{a}{t_0} - \dfrac{Ft_0}{2m} \\ C = \dfrac{F}{2m} \end{cases}$$

Problem 3. Let two generalized coordinate be x and y, then the system could be written as

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy, \quad y = Ax^2 \Rightarrow 2Ax\,\mathrm{d}x - \mathrm{d}y = 0$$

Hence we can write Lagrange's equation with constraint

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} - 2Ax\lambda = 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} + \lambda = 0 \end{aligned} \Rightarrow \begin{cases} \ddot{x} = \dfrac{2Ax\lambda}{m} \\ \ddot{y} = -\dfrac{\lambda}{m} + g \end{cases}$$

Note that

$$y = Ax^2 \Rightarrow 2Ax\dot{x} = \dot{y} \Rightarrow \ddot{y} = 2A\dot{x}^2 + 2Ax\ddot{x}$$

we can solve $\lambda = \lambda(x, \dot{x}, t)$ as

$$\lambda = -\frac{2Am\dot{x}^2 + mg}{1 + 4A^2x^2}$$

Hence we have constraint force for two coordinates

$$Q_x = 2Ax\lambda = -\frac{4A^2mx\dot{x}^2 + Axmg}{1 + 4A^2x^2}, \; Q_y = -\lambda = \frac{2Am\dot{x}^2 + mg}{1 + 4A^2x^2}$$

Problem 4. Setup polar coordinates a and θ, we have constraint $a \geq R + r$. The Lagrangian could be written as

$$L = \frac{1}{2}m(\dot{a}^2 + a^2\dot{\theta}^2) + \frac{1}{2}mr^2(R\dot{\theta}/r)^2 - mga\sin\theta$$

Use the constraint $a = r + R \Rightarrow \mathrm{d}a = 0$, we can get Lagrange's equation for the system

$$m\ddot{a} - ma\dot{\theta}^2 + mg\sin\theta - \lambda = 0$$

Since $\mathrm{d}a = 0 \Rightarrow \dot{a} = 0$ and $\ddot{a} = 0$, we have

$$\lambda = mg\sin\theta - ma\dot{\theta}^2$$

Using the conservation law of energy, we have

$$mg(r+R)(1-\sin\theta) = \frac{1}{2}m[(r+R)^2 + R^2]\dot{\theta}^2$$

Substitute $\dot{\theta}^2$ into the expression of $\sin\theta$, then we have

$$\lambda = mg\left[\frac{3(r+R)^2 + R^2}{(r+R)^2 + R^2}\sin\theta - \frac{2(r+R)^2}{(r+R)^2 + R^2}\right]$$

When the hoop falls off the cylinder, we should have $\lambda = 0$, hence θ will be

$$\theta = \arcsin\frac{2(r+R)^2}{3(r+R)^2 + R^2}$$