PHY3110: Classical Mechanics

Lagrangian / Hamiltonian formulation

Homework	30%
Midterm	30%
Final	40%

Tutorial

Reference book:

H. Goldstein, C. Poole, J. Safko, Classical Mechanics, 3rd Edition, Pearson.

- ⊛ J.R. Taylor, Classical Mechanics, University Science Books.
- ⊛ T.W.B. Kibble, F.H. Berkshire, Classical Mechanics, 5th Edition, Imperial College Press.
- ⊛ 梁昆淼, 力学(下册) 理论力学, 4th Edition, 高等教育出版社.

Jianhui Zhang. Office: RA 319.

Classical mechanics describes the motion of macroscopic objects, which are not extremely massive and not extremely fast.

Speed $\xrightarrow{3\times10^8\ m/s}$ speed of light.

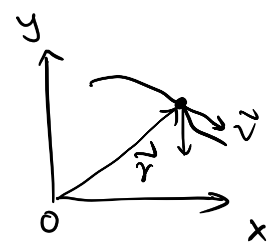
Size: $10^{-9}\,m$

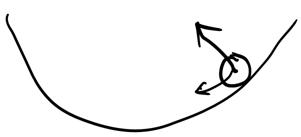
	Speed →	
Size ↑	Classical Mechanics	Relativistic Mechanics ✓
$10^{-9}\,m$	Quantum Mechanics ✗	Quantum Field Theory ✗

Review of Newtonian Mechanics

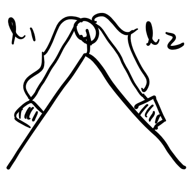
vectorial quantities of motion:

$$\vec{r},\ \vec{v},\ \vec{F},\ \vec{p}=m\vec{v},\ \vec{L}=\vec{r}\times\vec{p}=\vec{r}\times m\vec{v}$$



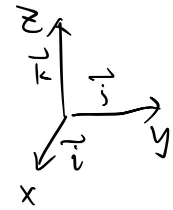


Constraint.



$l_1+l_2=\text{const.}$

Analytical mechanics:



It uses scalar quantities of motion:

kinetic energy: $T = \frac{1}{2} m \vec{v}^2$.

$\vec{v} \cdot \vec{v}$

$= (v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}) \cdot (v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k})$

$= v_1^2 + v_2^2 + v_3^2$

potential energy:

$V = V(\vec{r})$.

$\vec{i} \cdot \vec{i} = 1$,

$\vec{i} \cdot \vec{j} = \vec{i} \cdot \vec{k} = \vec{k} \cdot \vec{j} = 0$

Constraints:

They are used to reduce the number of dofs.

The formalism of analytical mechanics can also be generalized to electrodynamics, statistical / quantum mechanics, relativity & QFTs.

Newton's 2nd law:

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) = m\vec{a} \rightarrow \frac{d\vec{v}}{dt}.$$

Valid in an inertial frame.

$$\frac{d\vec{p}}{dt} = 0 \quad \text{if } \vec{F} = 0. \Rightarrow \vec{p} = \text{const}.$$

$\vec{p}$ is conserved if $\vec{F} = 0$.

Angular momentum $\vec{L}$ & torque $\vec{N}$.

$$\vec{L} = \vec{r} \times \vec{p}.$$

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r}\times\vec{p}) = \underbrace{\frac{d\vec{r}}{dt}}\times(m\vec{v}) + \vec{r}\times\frac{d\vec{p}}{dt}$$

$$= \underbrace{\vec{v}\times m\vec{v}}_{=0} + \underbrace{\vec{r}\times\vec{F}}_{=\vec{N}}$$

$(\vec{A}\times\vec{B})_i \quad (i=1,2,3)$

$=\sum_{j,k}\varepsilon_{ijk}A_jB_k$ summation over indices

$$\Rightarrow \frac{d\vec{L}}{dt} = \vec{N} = \vec{r}\times\vec{F}$$

$\vec{L}$ is conserved if $\vec{N}=0$

permutation

$\varepsilon_{123}=1,$

$\varepsilon_{132}=-1$

$\varepsilon_{312}=+1$

$\varepsilon_{iij}=0$

$\varepsilon_{122}=0$

$\varepsilon_{133}=0.$

$$(\vec{A}\times\vec{B})_1$$

$$= \varepsilon_{1jk}A_jB_k$$

$$= \varepsilon_{123}A_2B_3 + \underbrace{\varepsilon_{132}}_{=-\varepsilon_{123}}A_3B_2$$

$$= \varepsilon_{123}A_2B_3 - \varepsilon_{123}A_3B_2$$

$$= A_2B_3 - A_3B_2$$

$$V_2V_3 - V_3V_2 = 0$$

$$(\vec{A}\times\vec{B})\times\vec{c}$$

Work done by the external force:

$$W_{12} = \int^2 \vec{F}\cdot d\vec{s} = \int_1^2 \frac{d\vec{p}}{\cancel{dt}}\cdot\vec{v}\,\cancel{dt}$$

$$= \int_1^2 m\vec{v}\cdot d\vec{v}$$

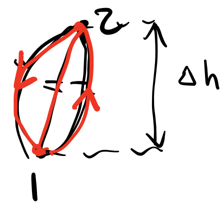
$$= \frac{1}{2}m\vec{v}_2^2 - \frac{1}{2}m\vec{v}_1^2$$

$$= T_2 - T_1$$

$$W = \oint \vec{F}\cdot d\vec{s} = 0$$

Define a scalar function $V(\vec{r})$.

$$\vec{F} = -\nabla V(\vec{r}).$$



V: potential energy,

$\vec{F}$: conservative force.

$$\underbrace{\int_1^2 \vec{F}\cdot ds + \int_2^1 \vec{F}\cdot ds}_{= \oint \vec{F}\cdot d\vec{s} = 0}$$

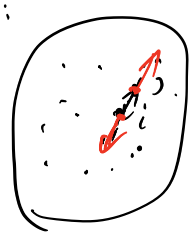
$$W_{12} = V_1 - V_2 = T_2 - T_1$$

$$\Rightarrow \underbrace{V_1 + T_1}_{E_1} = \underbrace{V_2 + T_2}_{E_2}$$

Total energy is conserved.

Consider a system of multiple particles:

$$\frac{d\vec{p}_i}{dt} = \dot{\vec{p}}_i = \vec{F}_i^{(e)} + \sum_j \vec{F}_{ji}$$



Assume Newton's 3rd law holds:

Forces that act on each other are equal & opposite.

$$\vec{F}_{ji} = -\vec{F}_{ij}$$

$$\sum_i \dot{\vec{p}}_i = \frac{d}{dt}\left(\sum_i \vec{p}_i\right) = \sum_i \vec{F}_i^{(e)} + \underbrace{\sum_{\substack{i,j\\ i\neq j}} \vec{F}_{ji}}_{=0}$$

$$= \sum_i \vec{F}_i^{(e)}$$

Center of mass of the system:

$$\vec{R} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} = \frac{\sum_i m_i \vec{r}_i}{M} \qquad M = \sum_i m_i .$$

$$\dot{\vec{P}} = \sum_i \dot{\vec{p}}_i = M\ddot{\vec{R}} = \vec{F}^{(e)} = \sum_i \vec{F}_i^{(e)} .$$

$\vec{P}$ is conserved if $\vec{F}^{(e)} = 0$

Total AM:

$$\frac{d\vec{L}}{dt} = \dot{\vec{L}} = \frac{d}{dt} \sum_i \vec{r}_i \times \underline{\vec{p}_i}$$

$$= \sum_i \vec{r}_i \times \vec{F}_i^{(e)} + \sum_{\substack{ij \\ i \neq j}} \vec{r}_i \times \vec{F}_{ji}$$

Assume the mutual force between 2 particles lie along the line between them.

$$\sum_{\substack{ij \\ i \neq j}} \vec{r}_i \times \vec{F}_{ji} \Longrightarrow \frac{1}{2} \sum_{\substack{ij \\ i \neq j}} \underbrace{\vec{r}_{ij}}_{(\vec{r}_i - \vec{r}_j)} \times \vec{F}_{ji} = 0$$

$$\boxed{\dot{\vec{L}} = \vec{N}^{(e)} = \sum_i \vec{r}_i \times \vec{F}_i^{(e)} .}$$

$\vec{L}$ is conserved if $\vec{N}^{(e)} = 0$.

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i = \sum_i (\vec{R} + \vec{r}_i') \times m_i (\vec{V} + \vec{v}_i')$$

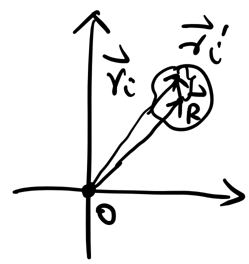
$$= \sum_i \vec{R} \times m_i \vec{V} + \sum_i \vec{r}_i' \times m_i \vec{v}_i'$$

$$+ \cancel{\underbrace{\sum_i (m_i \vec{r}_i')}_{=0} \times \vec{V}} + \cancel{\vec{R} \times \underbrace{\sum_i (m_i \vec{r}_i')}_{=0}}$$

$$= \sum_i \vec{R} \times m_i \vec{V} + \sum_i \vec{r}_i' \times \vec{p}_i'$$

$$\Downarrow$$

$$\vec{R} \times M \vec{V}$$



$$\vec{r}_i = \vec{R} + \vec{r}_i'$$

$$\vec{v}_i = \vec{V} + \vec{v}_i'$$

$$\sum_i m_i \vec{r}_i = \sum_i m_i \vec{r}_i' + \underbrace{\sum_i m_i}_{M} \vec{R}$$

Total AM = AM of COM + AM of motion around COM.

$$W_{12} = T_2 - T_1 \qquad T = \sum_i \frac{1}{2} m_i \vec{v}_i^2$$

$$T = \frac{1}{2} \sum_i m_i (\vec{V} + \vec{v}_i') \cdot (\vec{V} + \vec{v}_i')$$

$$= \frac{1}{2} \underbrace{\sum_i m_i}_{M} \vec{V}^2 + \frac{1}{2} \sum_i m_i \vec{v}_i'^2$$

$$\sum_i m_i \vec{r}_i' = 0$$

$$\Downarrow$$

$$\sum_i m_i \vec{v}_i' = 0$$

Total KE = COM KE + KE of motion around COM

If the external/internal forces are conservative,
we can define $V = \sum_i V_i + \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} V_{ij}$

$E = T + V = \text{const}.$

Constraints:

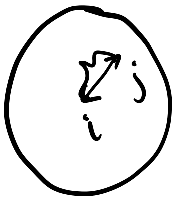
$\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N$ for N-particle system.

$$f(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_N, t) = 0$$

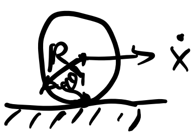
holonomic constraint.

Examples:

Rigid body



$$(\vec{r}_i - \vec{r}_j)^2 - c_{ij}^2 = 0$$



$$\dot{x} - R\dot{\theta} = 0$$

$$\Rightarrow \quad \frac{dx}{\cancel{dt}} - R\frac{d\theta}{\cancel{dt}} = 0$$

$$\Rightarrow \quad d(x - R\theta) = 0$$

$$\Rightarrow \quad x - R\theta = \text{const}$$

A constraint of the form

$$\sum_i g_i(x_1, x_2, \cdots x_n)\, dx_i = 0.$$

$\Downarrow$ multiplying some function $f(x_1, x_2, \ldots, x_n)$

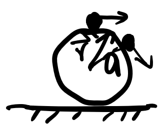
Total differential $dG(x_1, x_2, \ldots, x_n) = 0$

$$\Downarrow$$

$$G(x_1, x_2, \ldots, x_n) = \text{const}.$$

holonomic constraint

Non-holonomic constraint



$$\vec r^2 - a^2 \geq 0$$

$\boxed{\begin{array}{l} \vec r^2 - a^2 = 0, \text{ before it leaves} \\ \vec r^2 - a^2 > 0. \text{ after it leaves.} \end{array}}$

If the constraint is time-dep. $\rightarrow$ rhenomous

time-indep. $\rightarrow$ scleronomous.

With the imposed constraints,

$\vec r_i$ are no longer all indep.

it is convenient to introduce some new variables,

<u>generalized coordinates</u>.

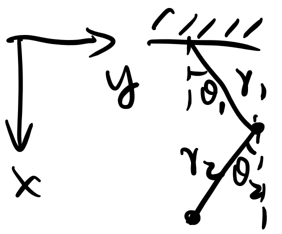
Suppose we have a N-particle system.

$1 \leq i \leq 3N, \vec r_i$ $\quad$ 3N dofs. $\quad$ k constraint Eqs.

$1 \leq j \leq 3N-k, q_j$. $\quad$ 3N-k dofs.

we can express

$$\left.\begin{array}{l} \vec r_1 = \vec r_1(q_1, q_2, \cdots, q_{3N-k}, t) \\ \vec r_2 = \vec r_2(q_1, q_2, \cdots, q_{3N-k}, t) \\ \vdots \\ \vec r_N = \vec r_N(q_1, q_2, \cdots, q_{3N-k}, t) \end{array}\right\} \Rightarrow q_j = q_j(\vec r_1, \vec r_2, \cdots \vec r_N, t).$$



(x_1, y_1) & (x_2, y_2) with constraints

$r_1 = \text{const}, \quad r_2 = \text{const}.$

$\Rightarrow$ 2 indep. dofs,

can also be chosen as θ_1, θ_2 — generalized coords.

D'Alembert's principle & Lagrange Eqs.

Formulate mechanics such that the forces of constraints do not appear.

Hint from rigid body:

Internal forces of constraints do no work.

Virtual displacement:

$\vec{r}_i \to \vec{r}_i + \delta\vec{r}_i \longrightarrow$ consistent with the constraints on the system at a given t.

$$f(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_N, t) = 0.$$

v.a.

$$f(\vec{r}_1 + \delta\vec{r}_1, \vec{r}_2 + \delta\vec{r}_2, \cdots, \vec{r}_N + \delta\vec{r}_N, t) = 0$$

real displacement:

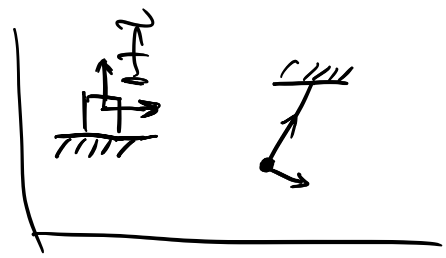
$$f(\vec{r}_1 + d\vec{r}_1, \vec{r}_2 + d\vec{r}_2, \cdots, \vec{r}_N + d\vec{r}_N, t + \delta t) = 0$$

Consider a system in equilibrium.

$$\vec{F}_i = 0 \Rightarrow \sum_i \vec{F}_i \cdot \delta\vec{r}_i = 0.$$

Separate $\vec{F}_i = \vec{F}_i^{(a)} + \vec{f}_i$ constraint force.

$$\sum_i \left(\vec{F}_i^{(a)} + \vec{f}_i\right) \cdot \delta\vec{r}_i = 0$$

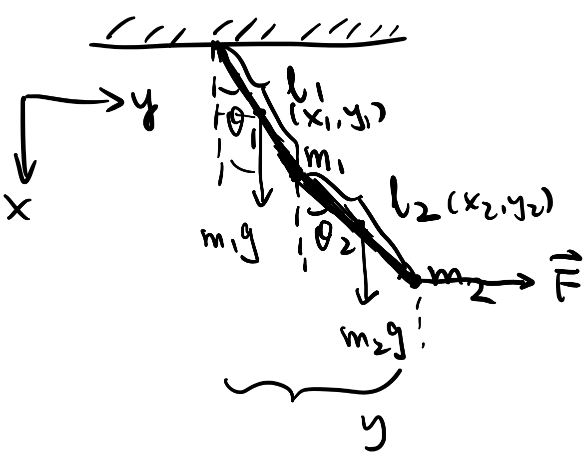


$$\sum_i \vec{f}_i \cdot \delta\vec{r}_i = 0$$

$$\Rightarrow \boxed{\sum_i \vec{F}_i^{(a)} \cdot \delta\vec{r}_i = 0.}$$

Virtual work principle.

Example:



$$m_1 g\, \delta x_1 + m_2 g\, \delta x_2 + F \delta y = 0$$

$$\begin{cases} x_1 = \dfrac{l_1}{2}\cos\theta_1 \\ x_2 = l_1 \cos\theta_1 + \dfrac{l_2}{2}\cos\theta_2 \\ y = l_1 \sin\theta_1 + l_2 \sin\theta_2 \end{cases}$$

$$\overbrace{\left(F\cos\theta_1 - \frac{m_1 g}{2}\sin\theta_1 - m_2 g \sin\theta_1\right)}^{=0} l_1\, \delta\theta_1$$

$$+ \underbrace{\left(F\cos\theta_2 - \frac{m_2 g}{2}\sin\theta_2\right)}_{=0} l_2\, \delta\theta_2 = 0.$$

$$\Rightarrow \tan\theta_1 = \frac{2F}{(m_1 + 2m_2) g}, \quad \tan\theta_2 = \frac{2F}{m_2 g}.$$

For a system moving under external forces.

$$\vec{F}_i - \dot{\vec{p}}_i = 0$$

$$\Rightarrow \quad \sum_i (\vec{F}_i - \dot{\vec{p}}_i) \cdot \delta \vec{r}_i = 0.$$

$$\Rightarrow \quad \sum_i (\vec{F}_i^{(a)} - \dot{\vec{p}}_i) \cdot \delta \vec{r}_i = 0.$$

D'Alembert's principle

$$\delta \vec{r}_i \longrightarrow \delta q_j$$

For holonomic constraints,

$n = 3N - k$. # of indep. dofs.

$$\vec{r}_i = \vec{r}_i (q_1, \cdots, q_n, t).$$

$1 \le i \le N.$

$$\vec{v}_i = \frac{d\vec{r}_i}{dt} = \frac{\partial \vec{r}_i}{\partial t} + \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j$$

Virtual displacement:

$$\delta \vec{r}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j$$

$$\sum_i \vec{F}_i^{(a)} \cdot \delta \vec{r}_i = \sum_{i,j} \vec{F}_i^{(a)} \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j = \sum_j Q_j \delta q_j$$

$$Q_j = \sum_i \vec{F}_i^{(a)} \cdot \frac{\partial \vec{r}_i}{\partial q_j}.$$

generalized force.

$$\sum_i \dot{\vec{p}}_i \cdot \delta\vec{r}_i = \sum_{ij} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\,\delta q_j$$

$$= \sum_{ij}\left[\frac{d}{dt}\left(m_i\dot{\vec{r}}_i\cdot\frac{\partial \vec{r}_i}{\partial q_j}\right) - m_i \dot{\vec{r}}_i\cdot\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)\right]\delta q_j$$

$\circledast$ $$\underline{\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)} = \frac{\partial}{\partial t}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) + \sum_k \frac{\partial}{\partial q_k}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)\dot{q}_k$$

$$= \frac{\partial}{\partial q_j}\left(\underbrace{\frac{\partial \vec{r}_i}{\partial t} + \sum_k \frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k}_{=\frac{d\vec{r}_i}{dt}}\right)$$

$$= \underline{\frac{\partial}{\partial q_j}\frac{d\vec{r}_i}{dt}}\,.$$

$\circledast$ $$\boxed{\frac{\partial \vec{r}_i}{\partial q_j} = \frac{\partial \vec{v}_i}{\partial \dot{q}_j}} = \frac{\partial}{\partial \dot{q}_j}\left(\sum_k \frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \vec{r}_i}{\partial t}\right)$$

$$\sum_{i,j}\left[\frac{d}{dt}\left(m_i\dot{\vec{r}}_i\cdot\frac{\partial \vec{v}_i}{\partial \dot{q}_j}\right) - m_i\dot{\vec{r}}_i\cdot\underline{\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)}\right]\delta q_j$$

$$= \sum_{i,j}\left[\frac{d}{dt}\left(m_i\vec{v}_i\cdot\frac{\partial \vec{v}_i}{\partial \dot{q}_j}\right) - m_i\vec{v}_i\cdot\frac{\partial \vec{v}_i}{\partial q_j}\right]\delta q_j$$

$$= \sum_j\left[\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\Big(\underbrace{\sum_i \frac{1}{2} m_i \vec{v}_i^{\,2}}_{T}\Big)\right) - \frac{\partial}{\partial q_j}\Big(\underbrace{\sum_i \frac{1}{2} m_i \vec{v}_i^{\,2}}_{T}\Big)\right]\delta q_j$$

$$= \sum_j\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right]\delta q_j$$

$$\Rightarrow \underbrace{\sum_j \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j\right]}\delta q_j = 0$$

δq_js are indep. of each other,

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j = 0.} \quad \text{for } 1 \le j \le n.$$

For conservative systems. $V = V(|\vec{r}|)$

$$\vec{F}_i = -\nabla_i \underline{V} \longrightarrow \underline{\text{potential energy}}.$$

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_i \nabla_i V \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

$$\Rightarrow \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial (T-V)}{\partial q_j} = 0.$$

If V does not depend on generalized velocities,

$$\frac{\partial V}{\partial \dot{q}_j} = 0$$

$$\Rightarrow \underline{\frac{d}{dt}\frac{\partial (T-V)}{\partial \dot{q}_j} - \frac{\partial (T-V)}{\partial q_j} = 0.}$$

Define the Lagrangian

$$L = T - V.$$

Lagrange's Eq.

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0.}$$

$E = T + V.$

The choice of the Lagrangian is not unique.

$$L' = L + \underbrace{\frac{dF(q_i,t)}{dt}}_{\Delta} \quad \& \quad L$$

give the same EOM.

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial \Delta}{\partial \dot q}\right) - \dfrac{\partial \Delta}{\partial q} = 0. \\[2ex] \Delta = \dfrac{dF}{dt} = \underline{\dfrac{\partial F}{\partial q}} \dot q + \dfrac{\partial F}{\partial t}. \Rightarrow \dfrac{\partial \Delta}{\partial \dot q} = \dfrac{\partial F}{\partial q} \\[2ex] \dfrac{d}{dt}\left(\dfrac{\partial F}{\partial q}\right) = \dfrac{\partial}{\partial q}\left(\dfrac{dF}{dt}\right) = \dfrac{\partial \Delta}{\partial q}. \end{cases}$$

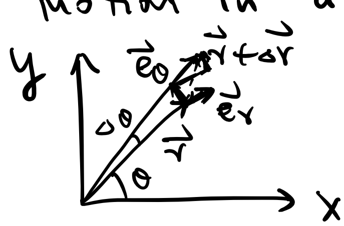
Examples of Lagrange's Eqs. (Lagrangian formalism)

1) $L = T - V$ for a single particle moving under force $\vec F$.

$$T = \frac{1}{2} m (\dot x^2 + \dot y^2 + \dot z^2).$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot x}\right) - \frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot x}\right) - \cancel{\frac{\partial V}{\partial x}} = m\ddot x = F_x$$

2) Polar coords. motion in a 2D plane.



$\Delta \vec r = (\Delta \vec x, \Delta \vec y)$.

$(x, y) \longrightarrow (r, \theta)$

$x = r\cos\theta, \quad y = r\sin\theta.$

$$\Rightarrow \left.\begin{aligned} \frac{dx}{dt} &= \dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta} \\ \frac{dy}{dt} &= \dot{y} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta} \end{aligned}\right\}$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2).$$

$$\vec{r} = x\vec{e}_x + y\vec{e}_y = r\vec{e}_r.$$

$$\lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \underbrace{\frac{\Delta r}{\Delta t}}\vec{e}_r + \underbrace{\frac{r\Delta\theta}{\Delta t}}\vec{e}_\theta.$$

radial velocity $= \dfrac{dr}{dt} = \dot{r}$.

angular velocity $= r\dfrac{d\theta}{dt} = r\dot{\theta}$.

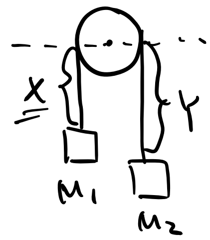
The generalized force

$$Q_r = \vec{F}\cdot\frac{\partial\vec{r}}{\partial r} = \vec{F}\cdot\vec{e}_r = F_r. \qquad Q_\theta = \vec{F}\cdot\frac{\partial\vec{r}}{\partial\theta} = \vec{F}\cdot r\vec{e}_\theta = rF_\theta.$$

Lagrange's Eqs.

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = Q_r &\Rightarrow \underline{m\ddot{r} - mr\dot{\theta}^2 = F_r}. \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = Q_\theta &\Rightarrow \underline{mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = rF_\theta}. \end{aligned}\right\}$$

Atwood's machine



$L = T - V.$

$x + y = \text{const} = l$

$\Rightarrow y = l - x.$

$$T = \frac{1}{2} M_1 \dot{x}^2 + \frac{1}{2} M_2 (\dot{\cancel{l}} - \dot{x})^2 = \frac{1}{2}(M_1 + M_2)\dot{x}^2.$$

$$V = -M_1 g x - M_2 g (l - x)$$

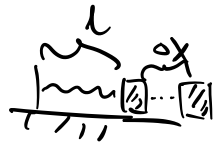
$$L = \frac{1}{2}(M_1 + M_2)\dot{x}^2 + M_1 g x + M_2 g (l - x).$$

1 EOM.

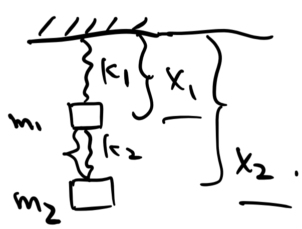
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0.$$

$$\frac{\partial L}{\partial \dot{x}} = (M_1 + M_2)\dot{x}, \quad \frac{\partial L}{\partial x} = (M_1 - M_2) g$$

$$\Rightarrow (M_1 + M_2)\ddot{x} = (M_1 - M_2) g.$$



$V = \frac{1}{2} k \Delta x^2$



T, V.

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$V = -m_1 g x_1 - m_2 g x_2 + \frac{1}{2} k_1 (x_1 - l_1)^2 + \frac{1}{2} k_2 (x_2 - x_1 - l_2)^2.$$

Assume the equilibrium lengths are l_1, l_2.

EOM:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0, \qquad i = 1, 2.$$

$$\begin{cases} m_1\ddot{x}_1 = m_1 g + k_2(x_2 - x_1 - l_2) - k_1(x_1 - l_1) \\ m_2\ddot{x}_2 = m_2 g - k_2(x_2 - x_1 - l_2) \end{cases}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j . \qquad 1 \le j \le \underline{n}$$

Suppose we have a potential (generalized potential)

$$U = U(q_j, \dot{q}_j)$$

such that $Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$

Define $L = T - U$.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 .$$

Lorentz force on a moving charge:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

we define the scalar & vector potentials $\phi(\vec{r}, t)$, $\underline{\vec{A}(\vec{r}, t)}$.

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} . \qquad \vec{B} = \nabla \times \vec{A} .$$

$$\hookrightarrow \quad \nabla \cdot \vec{B} = 0$$

$$\vec{F} = q\left[\underline{-\nabla\phi} - \underline{\frac{\partial \vec{A}}{\partial t}} + \underline{\vec{v} \times (\nabla \times \vec{A})}\right] .$$

Consider x-component of Lorentz force law.

$$F_x = q\left[-\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t} + v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - v_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right]$$

$$[\vec{v} \times (\nabla \times \vec{A})]_i = \varepsilon_{ijk} v_j (\nabla \times \vec{A})_k \qquad i=1, x$$

$$= \varepsilon_{ijk} v_j \cdot \varepsilon_{kmn} \nabla_m A_n$$

$$\boxed{\varepsilon_{kij}\varepsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}}$$

$$= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) v_j \nabla_m A_n$$

$$= v_n \nabla_i A_n - v_m \nabla_m A_i$$

$$F_x = q\left[-\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t} + v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - v_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right]$$

$$= q\left[-\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t} + \underbrace{v_x \frac{\partial A_x}{\partial x} + v_y \frac{\partial A_y}{\partial x} + v_z \frac{\partial A_z}{\partial x}}_{\vec{v} \cdot \frac{\partial \vec{A}}{\partial x}} \underbrace{- v_x \frac{\partial A_x}{\partial x} - v_y \frac{\partial A_x}{\partial y} - v_z \frac{\partial A_x}{\partial z}}_{-\vec{v} \cdot \nabla A_x}\right]$$

$$= q\left[-\frac{\partial \phi}{\partial x} - \frac{d A_x}{d t} + \vec{v} \cdot \frac{\partial \vec{A}}{\partial x}\right]$$

$$\frac{d A_x}{dt} = \frac{\partial A_x}{\partial t} + \vec{v} \cdot \nabla A_x$$

If we define

$$U = q\phi - q\vec{A} \cdot \vec{v}.$$

$$\Rightarrow F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}}\right) = \boxed{-q\frac{\partial \phi}{\partial x} + q\vec{v} \cdot \frac{\partial \vec{A}}{\partial x} - q\frac{d A_x}{dt}}$$

$$L = T - U.$$

EOM follows from L.