PHY3110: classical Mechanics

Lagrangian / Hamiltonian formulation

Homework	30%
Midterm	30%
Final	40%

Tutorial

Reference book:

H. Goldstein, C. Poole, J. Safko, classical Mechanics, 3rd Edition, Pearson.

⊛ J.R. Taylor, Classical Mechanics, University Science Books.

⊛ T.W.B. Kibble, F.H. Berkshire, classical Mechanics, 5th Edition, Imperial College Press.

⊛ 梁昆淼, 力学(下册) 理论力学, 4th Edition, 高等教育出版社.

Jianhui Zhang. office: RA 319.

Classical mechanics describes the motion of macroscopic objects, which are not extremely massive and not extremely fast.

Speed

3×10^{8} m/s → speed of light.

Size

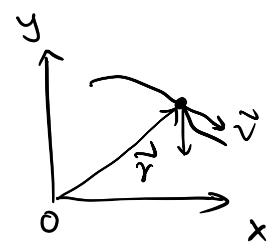
10^{-9} m

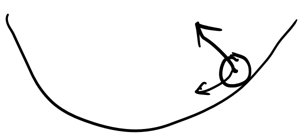
Classical mechanics	Relativistic Mechanics
Quantum mechanics	Quantum Field Theory

Review of Newtonian mechanics

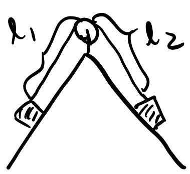
vectorial quantities of motion:

$$\vec{r},\ \vec{v},\ \vec{F},\ \vec{p}=m\vec{v},\ \vec{L}=\vec{r}\times\vec{p}=\vec{r}\times m\vec{v}$$



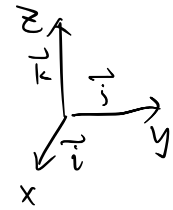


Constraint.



$l_1+l_2=\text{const.}$

Analytical mechanics:



It uses scalar quantities of motion:

kinetic energy: $T = \frac{1}{2} m \vec{v}^2$.

$\vec{v} \cdot \vec{v}$

$= (v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}) \cdot (v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k})$

$= v_1^2 + v_2^2 + v_3^2$

$\vec{i} \cdot \vec{i} = 1$,

$\vec{i} \cdot \vec{j} = \vec{i} \cdot \vec{k} = \vec{k} \cdot \vec{j} = 0$

potential energy:

$V = V(\vec{r})$.

Constraints:

They are used to reduce the number of dofs.

The formalism of analytical mechanics can also be generalized to electrodynamics, statistical/quantum mechanics, relativity & QFTs.

Newton's 2nd law:

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) = m\vec{a} \rightarrow \frac{d\vec{v}}{dt}.$$

Valid in an inertial frame.

$$\frac{d\vec{p}}{dt} = 0 \quad \text{if } \vec{F} = 0. \Rightarrow \vec{p} = \text{const}.$$

$\vec{p}$ is conserved if $\vec{F} = 0$.

Angular momentum $\vec{L}$ & torque $\vec{N}$.

$$\vec{L} = \vec{r} \times \vec{p}.$$

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r}\times\vec{p}) = \underbrace{\frac{d\vec{r}}{dt}}_{\vec{v}} \times (m\vec{v}) + \vec{r}\times\frac{d\vec{p}}{dt}$$

$$= \underbrace{\vec{v}\times m\vec{v}}_{=0} + \underbrace{\vec{r}\times\vec{F}}_{=\vec{N}}$$

$(\vec{A}\times\vec{B})_i \quad (i=1,2,3)$

$=\sum_{j,k} \varepsilon_{ijk} A_j B_k$ summation over indices

$$\Rightarrow \frac{d\vec{L}}{dt} = \vec{N} = \vec{r}\times\vec{F}$$

$\boxed{\vec{L} \text{ is conserved if } \vec{N}=0}$

permutation

$\varepsilon_{123} = 1,$

$\varepsilon_{132} = -1$

$\varepsilon_{312} = +1$

$\varepsilon_{iij} = 0$

$\varepsilon_{122} = 0$

$\varepsilon_{133} = 0.$

$$(\vec{A}\times\vec{B})_1$$

$$= \varepsilon_{1jk} A_j B_k$$

$$= \varepsilon_{123} A_2 B_3 + \underbrace{\varepsilon_{132}}_{=-\varepsilon_{123}} A_3 B_2$$

$$= \varepsilon_{123} A_2 B_3 - \varepsilon_{123} A_3 B_2$$

$$= A_2 B_3 - A_3 B_2$$

$$V_2 V_3 - V_3 V_2 = 0$$

$$(\vec{A}\times\vec{B})\times\vec{c}$$

Work done by the external force:

$$W_{12} = \int^2 \vec{F}\cdot d\vec{s} = \int_1^2 \frac{d\vec{p}}{\cancel{dt}}\cdot\vec{v}\,\cancel{dt}$$

$$= \int_1^2 m\vec{v}\cdot d\vec{v}$$

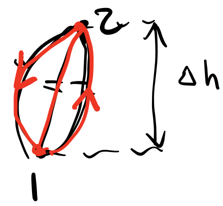
$$= \frac{1}{2} m\vec{v}_2^{\,2} - \frac{1}{2} m\vec{v}_1^{\,2}$$

$$= T_2 - T_1$$

$$W = \oint \vec{F}\cdot d\vec{s} = 0$$

Define a scalar function $V(\vec{r})$.

$$\vec{F} = -\nabla V(\vec{r}).$$



$$\underbrace{\int_1^2 \vec{F}\cdot ds + \int_2^1 \vec{F}\cdot ds}_{} = \oint \vec{F}\cdot d\vec{s} = 0$$

V: potential energy,

$\vec{F}$: conservative force.

$$W_{12} = V_1 - V_2 = T_2 - T_1$$

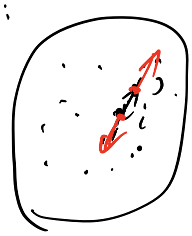
$$\Rightarrow \underbrace{V_1 + T_1}_{E_1} = \underbrace{V_2 + T_2}_{E_2}$$

$$E_1 = E_2$$

Total energy is conserved.

Consider a system of multiple particles:

$$\frac{d\vec{p}_i}{dt} = \dot{\vec{p}}_i = \vec{F}_i^{(e)} + \sum_j \vec{F}_{ji}$$



Assume Newton's 3rd law holds:

Forces that act on each other are equal & opposite.

$$\vec{F}_{ji} = -\vec{F}_{ij}$$

$$\sum_i \dot{\vec{p}}_i = \frac{d}{dt}\left(\sum_i \vec{p}_i\right) = \sum_i \vec{F}_i^{(e)} + \sum_{\substack{i,j \\ i\neq j}} \vec{F}_{ji} = 0$$

$$= \sum_i \vec{F}_i^{(e)}$$

Center of mass of the system:

$$\vec{R} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} = \frac{\sum_i m_i \vec{r}_i}{M} \qquad M = \sum_i m_i .$$

$$\dot{\vec{P}} = \sum_i \dot{\vec{P}}_i = M\ddot{\vec{R}} = \vec{F}^{(e)} = \sum_i \vec{F}_i^{(e)} .$$

$\vec{P}$ is conserved if $\vec{F}^{(e)} = 0$

Total AM:

$$\frac{d\vec{L}}{dt} = \dot{\vec{L}} = \frac{d}{dt} \sum_i \vec{r}_i \times \underline{\vec{p}_i}$$

$$= \sum_i \vec{r}_i \times \vec{F}_i^{(e)} + \sum_{\substack{ij \\ i\neq j}} \vec{r}_i \times \vec{F}_{ji}$$

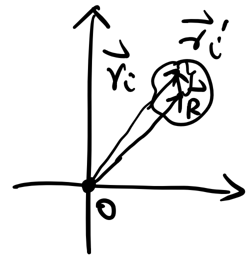
Assume the mutual force between 2 particles lie along the line between them.

$$\sum_{\substack{ij \\ i\neq j}} \vec{r}_i \times \vec{F}_{ji} \Longrightarrow \frac{1}{2} \sum_{\substack{ij \\ i\neq j}} \underset{\substack{\Downarrow \\ (\vec{r}_i - \vec{r}_j)}}{\vec{r}_{ij}} \times \vec{F}_{ji} = 0$$

$$\boxed{\dot{\vec{L}} = \vec{N}^{(e)} = \sum_i \vec{r}_i \times \vec{F}_i^{(e)} .}$$

$\vec{L}$ is conserved if $\vec{N}^{(e)} = 0$.

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i = \sum_i (\vec{R} + \vec{r}_i') \times m_i (\vec{V} + \vec{v}_i')$$



$$= \sum_i \vec{R} \times m_i \vec{V} + \sum_i \vec{r}_i' \times m_i \vec{v}_i'$$

$$+ \cancel{\underbrace{\sum_i (m_i \vec{r}_i')}_{=0} \times \vec{V}} + \cancel{\vec{R} \times \underbrace{\sum_i (m_i \vec{r}_i')}_{=0}}$$

$$\vec{r}_i = \vec{R} + \vec{r}_i'$$

$$\vec{v}_i = \vec{V} + \vec{v}_i'$$

$$= \sum_i \vec{R} \times m_i \vec{V} + \sum_i \vec{r}_i' \times \vec{p}_i'$$

$$\Downarrow$$

$$\vec{R} \times M \vec{V}$$

$$\sum_i m_i \vec{r}_i = \sum_i m_i \vec{r}_i' + \underbrace{\sum_i m_i}_{M} \vec{R}$$

Total AM = AM of COM + AM of motion around COM.

$$W_{12} = T_2 - T_1 \qquad T = \sum_i \frac{1}{2} m_i \vec{v}_i^2$$

$$\sum_i m_i \vec{r}_i' = 0$$

$$\Downarrow$$

$$\sum_i m_i \vec{v}_i' = 0$$

$$T = \frac{1}{2} \sum_i m_i (\vec{V} + \vec{v}_i') \cdot (\vec{V} + \vec{v}_i')$$

$$= \frac{1}{2} \underbrace{\sum_i m_i}_{M} \vec{V}^2 + \frac{1}{2} \sum_i m_i \vec{v}_i'^2$$

Total KE = COM KE + KE of motion around COM

If the external/internal forces are conservative, we can define $V = \sum_i V_i + \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} V_{ij}$

$E = T + V =$ const.

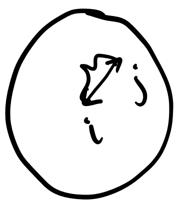
Constraints:

$\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N$ for N-particle system.

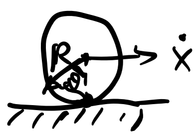
$f(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_N, t) = 0$ holonomic constraint.

Examples:

Rigid body



$$(\vec{r}_i - \vec{r}_j)^2 - c_{ij}^2 = 0$$



$$\dot{x} - R\dot{\theta} = 0$$

$$\Rightarrow \quad \frac{dx}{\cancel{dt}} - R\frac{d\theta}{\cancel{dt}} = 0$$

$$\Rightarrow \quad d(x - R\theta) = 0$$

$$\Rightarrow \quad x - R\theta = \text{const}$$

A constraint of the form

$$\sum_i g_i(x_1, x_2, \cdots x_n)\, dx_i = 0.$$

$\Downarrow$ multiplying some function $f(x_1, x_2, \dots, x_n)$

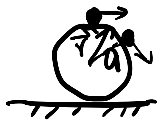
Total differential $dG(x_1, x_2, \dots, x_n) = 0$

$$\Downarrow$$

$$\underline{G(x_1, x_2, \cdots, x_n) = \text{const}.}$$

holonomic constraint

Non-holonomic constraint



$\vec{r}^2 - a^2 \geq 0$

$\vec{r}^2 - a^2 = 0$, before it leaves

$\vec{r}^2 - a^2 > 0$. after it leaves.

If the constraint is time-dep. $\rightarrow$ rhenomous

time-indep. $\rightarrow$ scleronomous.

With the imposed constraints,

$\vec{r}_i$ are no longer all indep.

it is convenient to introduce some new variables,

generalized coordinates.

Suppose we have a N-particle system.

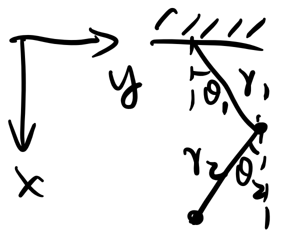
$1 \leq i \leq 3N, \vec{r}_i$ $\quad$ 3N dofs.

k constraint Eqs.

$1 \leq j \leq 3N-k, q_j$. $\quad$ 3N-k dofs.

we can express

$$\left.\begin{aligned}
\vec{r}_1 &= \vec{r}_1(q_1, q_2, \cdots, q_{3N-k}, t) \\
\vec{r}_2 &= \vec{r}_2(q_1, q_2, \cdots, q_{3N-k}, t) \\
&\vdots \\
\vec{r}_N &= \vec{r}_N(q_1, q_2, \cdots, q_{3N-k}, t)
\end{aligned}\right\} \Rightarrow q_j = q_j(\vec{r}_1, \vec{r}_2, \cdots \vec{r}_N, t).$$



(x_1, y_1) & (x_2, y_2) with constraints

$r_1 = \text{const}, \quad r_2 = \text{const}.$

$\Rightarrow$ 2 indep. dofs,

can also be chosen as θ_1, θ_2 } **generalized coords.**