

PHY3110 SP23 Notes

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0 Introduction

Grading: 30% homework, 30% midterm, 40% final.

Textbooks:

- H. Goldstein, C. Poole, J. Safko, Classical Mechanics, 3rd Edition, Pearson.
- J.R. Taylor, Classical Mechanics, University Science Books.
- T.W.B. Kibble, F.H. Berkshire, Classical Mechanics, 5th Edition, Imperial College Press.
- 梁昆森, 力学 (下册) 理论力学, 4th Edition, 高等教育出版社.

Classical mechanics describe the motion of macroscopic objects, which are not extremely massive and not extremely fast.

1 Newtonian Mechanics

Vectorial quantities of motion: position \mathbf{r} , velocity \mathbf{v} , force \mathbf{F} , momentum $\mathbf{p} = m\mathbf{v}$, angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. Equations of motion are derived from those vector quantities.

Analytical mechanics uses scalar quantities of motion

- Kinetic energy $T = \frac{1}{2}m\mathbf{v}^2$
- Potential energy $V = V(\mathbf{r})$

Equations of motion are derived from those scalar quantities.

1.1 Newton's Laws

Newton's 2nd law

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = m\mathbf{a} \quad (1)$$

valid in an inertial frame. Angular momentum \mathbf{L} and torque \mathbf{N} are also related

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \mathbf{r} \times \mathbf{F} = \mathbf{N} \quad (2)$$

Work done by external forces

$$W_{12} = \int_1^2 \mathbf{F} \cdot d\mathbf{s} = \int_1^2 m \frac{d\mathbf{v}}{dt} \cdot d\mathbf{s} = \int_1^2 m\mathbf{v} \cdot d\mathbf{v} = \left. \frac{1}{2}m\mathbf{v}^2 \right|_1^2 \quad (3)$$

Define a scalar function $V(\mathbf{r})$, then $\mathbf{F} = -\nabla V(\mathbf{r})$ is a conservative force.

$$\oint \mathbf{F} \cdot d\mathbf{s} = 0 \quad (4)$$

Center of mass of the system

$$\mathbf{R} = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i} = \frac{\sum_i m_i \mathbf{r}_i}{M} \quad (5)$$

Total momentum

$$\mathbf{P} = \sum_i m_i \mathbf{p}_i = M \dot{\mathbf{R}} \quad (6)$$

Hence \mathbf{P} is conserved if external force $\mathbf{F}^{(e)}$ is zero.

Total angular momentum

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt} \sum_i \mathbf{r}_i \times \mathbf{p}_i = \sum_i \mathbf{r}_i \times \left(\mathbf{F}_i^{(e)} + \sum_j \mathbf{F}_{ij} \right) = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{(e)} + \sum_{ij} \mathbf{r}_i \times \mathbf{F}_{ij}$$

Since \mathbf{r}_{ij} parallel to \mathbf{F}_{ij} , then

$$\sum_{ij} \mathbf{r}_i \mathbf{F}_{ij} = \frac{1}{2} \sum_{ij} \mathbf{r}_{ij} \times \mathbf{F}_{ji} = 0 \quad (7)$$

Therefore

$$\frac{d\mathbf{L}}{dt} = \mathbf{N}^{(e)} \quad (8)$$

Decomposition of the angular momentum

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i = \sum_i (\mathbf{R} + \mathbf{r}_i) \times m_i (\mathbf{V} + \mathbf{v}'_i) = \sum_i \mathbf{R} \times m_i \mathbf{V} + \sum_i \mathbf{r}'_i \times m_i \mathbf{v}'_i \quad (9)$$

1.2 Constraints

Holonomic constraint

$$f(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N, t) = 0 \quad (10)$$

Example: rigid body

$$(\mathbf{r}_i - \mathbf{r}_j)^2 - c_{ij}^2 = 0 \quad (11)$$

Example: non-sliding cylinder

$$\dot{x} - R\dot{\theta} = 0 \Rightarrow x - R\theta = \text{const}$$

A constraint of the form

$$\sum_i g_i(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) d\mathbf{x}_i = 0 \Rightarrow dG(\mathbf{x}_1, \dots) = 0 \Rightarrow G(\mathbf{x}_1, \dots) = \text{const} \quad (12)$$

Non-holonomic constraint: cannot be written in the form of holonomic constraint.

1.3 Generalized coordinates

Suppose we have a N -particle system, we will have $3N$ DOFs. With k constraints, we will have $3N - k$ DOFs. Define q_1, \dots, q_{3N-k} generalized coordinates, we have

$$\mathbf{r}_i = \mathbf{r}_i(q_1, \dots, q_{3N-k}, t) \quad (13)$$

2 Lagrange Formalism

2.1 D'Alembert's Principle

Hint from the rigid body: internal forces of constraints do not work.

Virtual displacement: $\delta \mathbf{r}_i$ is consistent with the constraints imposed on the system at a given time

$$\mathbf{r}_i \rightarrow \mathbf{r}_i + \delta \mathbf{r}_i \quad (14)$$

Consider a system in equilibrium

$$\mathbf{F}_i = 0 \Rightarrow \sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i = 0 \quad (15)$$

Separate $\mathbf{F}_i = \mathbf{F}_i^{(a)} + \mathbf{f}_i$ where \mathbf{f}_i is the constraint force. Hence

$$\sum_i (\mathbf{F}_i^{(a)} + \mathbf{f}_i) \cdot \delta \mathbf{r}_i = 0 \Rightarrow \sum_i \mathbf{F}_i^{(a)} \cdot \delta \mathbf{r}_i = 0 \quad (16)$$

For a system moving under external forces

$$\mathbf{F}_i - \dot{\mathbf{p}}_i = 0 \Rightarrow \sum_i (\mathbf{F}_i - \dot{\mathbf{p}}_i) \delta \mathbf{r}_i = 0 \Rightarrow \sum_i (\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i) \delta \mathbf{r}_i = 0 \quad (17)$$

For holonomic constraints

$$\mathbf{r}_i = \mathbf{r}_i(q_1, \dots, q_n, t), \quad \mathbf{v}_i = \frac{d\mathbf{r}_i}{dt} = \frac{\partial \mathbf{r}_i}{\partial t} + \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \dot{q}_j, \quad \delta \mathbf{r}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \quad (18)$$

Generalized force Q_j

$$\sum_i \mathbf{F}_i \delta \mathbf{r}_i = \sum_{ij} \mathbf{F}_i \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j = \sum_j Q_j \delta q_j \quad (19)$$

Then

$$\sum_i \dot{\mathbf{p}}_i \cdot \delta \mathbf{r}_i = \sum_{ij} m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j = \sum_{ij} \left[\frac{d}{dt} \left(m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \right) - m_i \dot{\mathbf{r}}_i \cdot \frac{d}{dt} \frac{\partial \mathbf{r}_i}{\partial q_j} \right] \delta q_j \quad (20)$$

$$= \sum_i \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] \delta q_j = \sum_j Q_j \delta q_j \quad (21)$$

$$(22)$$

Hence $\forall j$ we have

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j = 0 \quad (23)$$

Let the potential energy $V = V(\mathbf{r}_i, \dots) = V(q_j, \dots)$, then we have

$$Q_j = \sum_i \mathbf{F}_i \frac{\partial \mathbf{r}_i}{\partial q_j} = \sum_i -\nabla_i V \frac{\partial \mathbf{r}_i}{\partial q_j} = -\frac{\partial V}{\partial q_j} \quad (24)$$

Therefore

$$\frac{d}{dt} \left(\frac{\partial (T - V)}{\partial \dot{q}_j} \right) - \frac{\partial (T - V)}{\partial q_j} - Q_j = 0 \quad (25)$$

Define $L = T - V$, then

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} - Q_j = 0 \quad (26)$$

The choice of Lagrangian is not unique, L' where

$$L' = L + \frac{dF(q, t)}{dt} \quad (27)$$

will give the same equations of motion as L .

Example 2.1 (Lagrange's formalism).

1) For a single particle moving under force \mathbf{F}

$$L(\mathbf{x}, \dot{\mathbf{x}}, t) = \frac{1}{2} m \dot{\mathbf{x}}^2 + \mathbf{F} \cdot \mathbf{x}$$

2) Motion in a 2D plane using polar coordinates

$$L(r, \theta, \dot{r}, \dot{\theta}, t) = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + \mathbf{F} \cdot \mathbf{r}$$

Generalized forces

$$Q_r = \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial r} = \mathbf{F} \cdot \mathbf{e}_r$$

$$Q_\theta = \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial \theta} = \mathbf{F} \cdot r \mathbf{e}_\theta$$

where

$$\mathbf{e}_r = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \quad \mathbf{e}_\theta = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$$

Equations of motion

$$m\ddot{r} - mr\dot{\theta}^2 = \mathbf{F} \cdot \mathbf{e}_r$$

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = r\mathbf{F}_\theta$$

3) Atwood's machine

$$L = \frac{1}{2}(M_1 + M_2)\dot{x}^2 + M_1gx + M_2g(l - x)$$

The equation of motion is

$$\begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} &= 0 \\ \Rightarrow (M_1 + M_2)\ddot{x} &= (M_1 - M_2)g \end{aligned}$$

Suppose we have a potential dependent on velocity (generalized potential) and the generalized force is defined as

$$U = U(q_j, \dot{q}_j), \quad Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt} \frac{\partial U}{\partial \dot{q}_j} \quad (28)$$

Define $L = T - U$, then we still have

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \quad (29)$$

Example 2.2 (Lorentz force on a moving charge). The Lorentz force

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Define the scalar and vector potentials

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

Hence

$$\mathbf{F} = q \left[-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \times (\nabla \times \mathbf{A}) \right]$$

2.2 Integral point of view of D'Alembert's principle

Configuration space: a space formed by the set of generalized coordinates.

(q_1, q_2, \dots, q_n) as function of t

Theorem 2.1 (Hamilton's principle). Define the action integral I , where $L = T - V$ or $L = T - U$ (U is the generalized potential)

$$I = \int_{t_1}^{t_2} L dt$$

Then the variation of the action integral equals to zero

$$\delta I = \delta \int_{t_1}^{t_2} L(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n) dt = 0$$

Add small variation on the path

$$q_i(t, \alpha) = q_i(t) + \alpha \eta(t)$$

where $\eta(t_1) = \eta(t_2) = 0$. Then the action will be the function of α , $I = I(\alpha)$. Hence

$$\delta I = \int_{t_1}^{t_2} \left(\sum_i \frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) dt$$

Change the order of differentiation $\delta \dot{q}_i = d\delta q_i / dt$, then

$$\begin{aligned} \delta I(\alpha) &= \int_{t_1}^{t_2} \sum_i \left[\frac{\partial L}{\partial q_i} \delta q_i - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \right] dt + \sum_i \frac{\partial L}{\partial \dot{q}_i} \delta q_i \Big|_{t_1}^{t_2} \\ &= \int_{t_1}^{t_2} \sum_i \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right] \delta q_i dt = 0 \\ &\Rightarrow \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q_i} = 0 \end{aligned}$$

Example 2.3 (Shortest path problem). $y = y(x)$, $ds = \sqrt{dx^2 + dy^2}$, then the action integral (path) is

$$I = \int_1^2 ds = \int_1^2 \sqrt{1 + \dot{y}^2} dx$$

Apply the Lagrange's equation we get

$$\frac{d}{dx} \frac{d\sqrt{1 + \dot{y}^2}}{d\dot{y}} = 0 \Rightarrow \frac{d\dot{y}}{dx} = 0 \Rightarrow y = ax + b$$

Example 2.4 (Solid of revolution). Differential of area $2\pi x ds = 2\pi x \sqrt{1 + \dot{y}^2} dx$, then the total area is

$$\int_1^2 2\pi x \sqrt{1 + \dot{y}^2} dx$$

Define the Lagrangian $L(x, y, \dot{y}) = 2\pi x \sqrt{1 + \dot{y}^2}$, by Lagrange's equation we can get

$$\frac{x\dot{y}}{\sqrt{1 + \dot{y}^2}} = \text{const} \Rightarrow y = a \cosh \frac{x}{a} + b$$

Example 2.5 (The curve of fastest descent). Question: along which trajectory from point 1 to point 2, the time is shortest? The total time is

$$T = \int_1^2 \frac{ds}{v} = \int_1^2 \frac{ds}{\sqrt{2gy}}$$

According to Newton's laws we have $y = gv^2$, then

$$T = \int_1^2 \frac{\sqrt{1 + \dot{y}^2}}{\sqrt{2gy}} dx$$

Then we have $L(x, y, \dot{y})$ and we get **check derivation**

$$\begin{aligned} \frac{\dot{y}}{2y} + \frac{y\ddot{y}}{1 + \dot{y}^2} &= 0 \\ \Rightarrow \frac{d}{dx} \ln[y(1 + \dot{y}^2)] &= 0 \end{aligned}$$

which means that $y(1 + \dot{y}^2) = \text{const}$. The solution is $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$.