

PHY3110 FA22 HW01

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Problem 1. Since

$$\begin{aligned}[a \times (b \times c)]_i &= \epsilon_{ijk} a_j (b \times c)_k = \epsilon_{ijk} \epsilon_{klm} a_j b_l c_m = (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) a_j b_l c_m = \delta_{il} \delta_{jm} a_j b_l c_m - \delta_{im} \delta_{jl} a_j b_l c_m \\ &= b_i (a \cdot c) - c_i (a \cdot b)\end{aligned}$$

Hence we have $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$.

Suppose $b = \nabla$, then we should consider the commutation relationship of b and c , i.e., $[b, c] = bc - cb$ might be nonzero. At that time, the formula might not hold.

Problem 2. Let

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r \sin \theta \cos \phi \\ r \sin \theta \sin \phi \\ r \cos \theta \end{bmatrix}$$

and define the following three unit vectors

$$\hat{r} = \begin{bmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{bmatrix}, \quad \hat{\phi} = \begin{bmatrix} \cos \phi \\ \sin \phi \\ 0 \end{bmatrix}, \quad \hat{\theta} = \begin{bmatrix} \cos \theta \cos \phi \\ \cos \theta \sin \phi \\ -\sin \theta \end{bmatrix}$$

easy to show that $\hat{r}, \hat{\phi}$ and $\hat{\theta}$ forms an orthonormal basis in \mathbb{R}^3 . Then under spherical coordinates we have

$$d\mathbf{r} = dr \cdot \hat{r} + r d\theta \cdot \hat{\theta} + r \sin \theta d\phi \cdot \hat{\phi}$$

square the equation, we get

$$d\mathbf{r}^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

Problem 3. Since

$$\begin{aligned}\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} &= \sum_j \frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j} \dot{q}_j + \frac{\partial^2 T}{\partial \dot{q}_i \partial q_j} \ddot{q}_j + \frac{\partial^2 T}{\partial \dot{q}_i \partial t} \\ \dot{T} &= \sum_i \frac{\partial T}{\partial q_i} \dot{q}_i + \frac{\partial T}{\partial \dot{q}_i} \ddot{q}_i + \frac{\partial T}{\partial t} \\ \frac{\partial \dot{T}}{\partial \dot{q}_i} &= \sum_j \frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j} \ddot{q}_j + \frac{\partial^2 T}{\partial \dot{q}_i \partial q_j} \dot{q}_j + \frac{\partial^2 T}{\partial \dot{q}_i \partial t}\end{aligned}$$

Then

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} + \frac{\partial T}{\partial q_i} - \frac{\partial \dot{T}}{\partial \dot{q}_i} = 0$$

Add this equation to the lagrange equation, then we have

$$\frac{\partial \dot{T}}{\partial \dot{q}_i} - 2 \frac{\partial T}{\partial q_i} = Q_i$$

Problem 4.

(a) The condition should be

$$\frac{\partial f}{\partial x_i} = Mg_i$$

(b) Define function $f(x, y, z) = x^2 + y^2 + z^2 + xy + yz + xz$, then we have

$$df = (2x + y + z) dx + (x + 2y + z) dy + (x + y + 2z) dz$$

Hence the following constraint is holonomic.

$$(2x + y + z) dx + (x + 2y + z) dy + (x + y + 2z) dz = 0$$

The second constraint is not holonomic, since it cannot be written into full differential form.

Problem 5. Use the generalized coordinate θ to describe the system, hence

$$T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2 \cos^2 \theta \dot{\theta}^2, \quad V = \frac{1}{2}kx^2 = \frac{1}{2}kl^2 \left(\frac{1}{\cos \theta} - 1 \right)^2, \quad L = T - V$$

Therefore

$$\begin{aligned} \frac{\partial L}{\partial \theta} &= -ml^2 \sin \theta \cos \theta \dot{\theta}^2 - kl^2 \left(\frac{1}{\cos \theta} - 1 \right) \frac{\sin \theta}{\cos^2 \theta} \\ \frac{\partial L}{\partial \dot{\theta}} &= ml^2 \cos^2 \theta \dot{\theta} \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} &= ml^2 \cos^2 \theta \ddot{\theta} - 2ml^2 \sin \theta \cos \theta \dot{\theta}^2 \end{aligned}$$

Hence the equation of motion becomes

$$\begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} &= 0 \\ \Rightarrow \cos^2 \theta \ddot{\theta} - \sin \theta \cos \theta \dot{\theta}^2 + \omega^2 \left(\frac{1}{\cos \theta} - 1 \right) \frac{\sin \theta}{\cos^2 \theta} &= 0 \end{aligned}$$