PHY3110 SP23 HW04

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Problem 1. Use the EOM for a point mass m moving in a central potential $V(r)$, show that ($u = 1/r$, l is the angular momentum)

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u = -\frac{m}{l^2}\frac{\mathrm{d}}{\mathrm{d}u}V(1/u)$$

Solution. The EOM write

$$m\ddot{r} - \frac{l^2}{mr^3} + \frac{\mathrm{d}}{\mathrm{d}r}V = 0$$
$$\dot{\theta} = \frac{l}{mr^2}$$

Note that

$$\dot{\theta} = \frac{l}{mr^2} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t} = \frac{l}{mr^2}\frac{\mathrm{d}}{\mathrm{d}\theta}$$

Let $u = 1/r$, then

$$\dot{r} = \frac{l}{m}u^2\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{1}{u} = -\frac{l}{m}\frac{\mathrm{d}u}{\mathrm{d}\theta}, \quad \ddot{r} = -\frac{l^2}{m^2}u^2\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2}$$

Then we have

$$-\frac{l^2}{m}u^2\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} - \frac{l^2}{m}u^3 - u^2\frac{\mathrm{d}}{\mathrm{d}u}V = 0$$
$$\Rightarrow \frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u = -\frac{m}{l^2}\frac{\mathrm{d}}{\mathrm{d}u}V$$

Problem 2. If the orbit of a point mass under a central force $F(r)$ is given by $r = k\theta^2$ with k being a constant, try to derive the explicit form of $F(r)$.

Solution. Note that we have $r = k\theta^2$, and since $\dot{\theta} = l/(mr^2) = l/(mk^2r^4)$, then

$$\dot{r} = \frac{\mathrm{d}}{\mathrm{d}t}k\theta^2 = \frac{l}{mk^2}\theta^{-4}\frac{\mathrm{d}}{\mathrm{d}\theta}k\theta^2 = \frac{2l}{mk}\theta^{-3}$$
$$\ddot{r} = \cdots = -\frac{6l^2}{m^2k^3}\theta^{-8}$$

which means

$$m\ddot{r} - \frac{l^2}{mr^3} + \frac{\mathrm{d}}{\mathrm{d}r}V = 0 \Rightarrow F(r) = -\frac{\mathrm{d}}{\mathrm{d}r}V = -\frac{6kl^2}{m}r^{-4} - \frac{l^2}{m}r^{-3}$$

Problem 3. Two particles move around each other in circular orbits under gravitational forces with a period τ. If they suddenly stop at a given instant and then start to fall into each other, show that they collide after a time $\tau/(4\sqrt{2})$.

Solution. Let m denotes the reduced mass and l denotes the angular momentum. Since for the circular orbits with $r = r_0$, the energy equals to the minimum of effective potential, which means

$$\left.\frac{\partial V_{\text{eff}}}{\partial r}\right|_{r=r_0} = 0 \Rightarrow r_0 = \frac{l^2}{mk}$$

Then we can solve $\dot{\theta}$ (which is a constant) and τ

$$\dot{\theta} = \frac{l}{mr_0^2} = \frac{mk^2}{l^3}, \quad \tau = \frac{2\pi}{\dot{\theta}} = \frac{2\pi l^3}{mk^2}$$

Suppose two particles suddenly lose velocity at $r = r_0$, then the angular momentum of the system vanishes. Hence

$$E = \frac{1}{2}m\dot{r}^2 - \frac{k}{r} = -\frac{k}{r_0} \Rightarrow \dot{r} = -\left[\frac{2k}{m}\left(\frac{1}{r} - \frac{1}{r_0}\right)\right]^{1/2}$$

To calculate the time when two particles collide, simply integrate the equation

$$\begin{aligned}
\dot{r} &= -\left[\frac{2k}{m}\left(\frac{1}{r} - \frac{1}{r_0}\right)\right]^{1/2} \\
\mathrm{d}t &= -\left[\frac{2k}{m}\left(\frac{1}{r} - \frac{1}{r_0}\right)\right]^{-1/2} \mathrm{d}r \\
\int_0^{\tau'} \mathrm{d}t &= \int_{r_0}^{0} -\left[\frac{2k}{m}\left(\frac{1}{r} - \frac{1}{r_0}\right)\right]^{-1/2} \mathrm{d}r
\end{aligned}$$

Since

$$\begin{aligned}
\int_0^a \left(\frac{1}{x} - a\right)^{-1/2} \mathrm{d}x &= \int_{1/a}^{\infty} (u-a)^{-1/2}\, \mathrm{d}\frac{1}{u} \\
&= \int_{1/a}^{\infty} \frac{1}{u^2\sqrt{u-a}}\, \mathrm{d}u \\
&= \int_0^{\infty} \frac{1}{(v^2+a)^2 v}\, \mathrm{d}(v^2+a) \\
&= \int_0^{\infty} \frac{2}{(v^2+a)^2}\, \mathrm{d}v \\
&= \frac{1}{a}\frac{v}{v^2+a} + \frac{1}{a^{3/2}} \arctan \frac{v}{\sqrt{a}}\Bigg|_{v=0}^{v=\infty}
\end{aligned}$$

Hence we have the expression of τ' and we can prove its relationship with τ

$$\begin{aligned}
\tau' &= \left(\frac{2k}{m}\right)^{-1/2} \left[r_0 \frac{v}{v^2 + 1/r_0} + r_0^{3/2} \arctan \sqrt{r_0} v\right]_{v=0}^{v=\infty} \\
&= \left(\frac{2k}{m}\right)^{-1/2} r_0^{3/2} \frac{\pi}{2} \\
&= \left(\frac{m}{k}\right)^{1/2} \left(\frac{l^2}{mk^2}\right)^{3/2} \frac{\pi}{2\sqrt{2}} \\
&= \frac{l^3}{mk^2} \frac{\pi}{2\sqrt{2}} \\
&= \frac{\tau}{4\sqrt{2}}
\end{aligned}$$

Problem 4. A particle moves in a force field described by

$$V(r) = -k\frac{e^{-ar}}{r}$$

where k, a are positive constants

a) Use the effective potential to discuss the qualitative nature of the orbits for different values of energy and angular momentum.

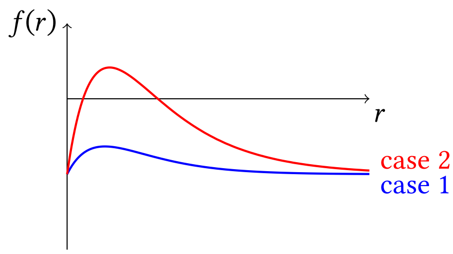
b) What is the period of the motion when the orbit is a circle?

Solution.

a) The effective potential and its first derivative writes

$$V_{\text{eff}} = -k\frac{e^{-ar}}{r} + \frac{l^2}{2m}\frac{1}{r}$$
$$\frac{\mathrm{d}}{\mathrm{d}r}V_{\text{eff}} = \frac{1}{r^3}\left(akr^2e^{-ar} + kre^{-ar} - \frac{l^2}{m}\right)$$

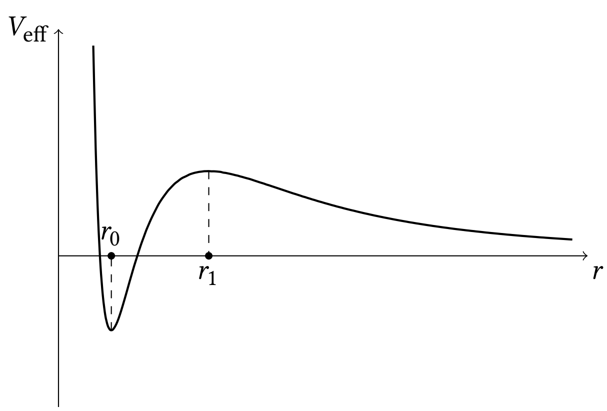
Investigating into $f(r) = akr^2e^{-ar} + kre^{-ar} - l^2/m$, we have $f'(r) = e^{-ar}(-a^2kr^2 + 2akr + k)$. This means $f(r)$ is first monotonically increasing and then decreasing, and $f(r)$ only have one extreme value. Regarding the sign of $f(r)$, there are only two cases



Since $f(r)$ shares the same sign with $\mathrm{d}V_{\text{eff}}/\,\mathrm{d}r$, we also have two cases for V_{eff}

i. V_{eff} is always decreasing, there are no bounded solutions.

ii. V_{eff} looks like the following figure



Hence, V_{eff} will have a local minimum at r_0 and a local maximum at r_1. The solution will be bounded if $E < V_{\text{eff}}(r_1)$.

b) The circular motion is only possible for the case ii. Hence we have $f(r_1) = f(r_2) = 0$, then we can solve $\dot{\theta}$ and τ (where r^* is either r_1 or r_2)

$$l^2 = mkr(1+ar)e^{-ar} \Rightarrow \dot{\theta} = \frac{l}{mr^{*2}} = \left[\frac{k(1+ar)}{mr^{*3}}\right]^{1/2} e^{-ar^*/2},\ \tau = \frac{2\pi}{\dot{\theta}} = 2\pi\left[\frac{mr^{*3}}{k(1+ar^*)}\right]^{1/2} e^{ar^*/2}$$