

# PHY3110 SP23 Notes

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## 0 Introduction

**Grading:** 30% homework, 30% midterm, 40% final.

**Textbooks:**

- H. Goldstein, C. Poole, J. Safko, Classical Mechanics, 3rd Edition, Pearson.
- J.R. Taylor, Classical Mechanics, University Science Books.
- T.W.B. Kibble, F.H. Berkshire, Classical Mechanics, 5th Edition, Imperial College Press.
- 梁昆森, 力学 (下册) 理论力学, 4<sup>th</sup> Edition, 高等教育出版社.

Classical mechanics describe the motion of macroscopic objects, which are not extremely massive and not extremely fast.

## 1 Newtonian Mechanics

Vectorial quantities of motion: position  $\mathbf{r}$ , velocity  $\mathbf{v}$ , force  $\mathbf{F}$ , momentum  $\mathbf{p} = m\mathbf{v}$ , angular momentum  $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ . Equations of motion are derived from those vector quantities.

Analytical mechanics uses scalar quantities of motion

- Kinetic energy  $T = \frac{1}{2}m\mathbf{v}^2$
- Potential energy  $V = V(\mathbf{r})$

Equations of motion are derived from those scalar quantities.

### 1.1 Newton's Laws

**Theorem 1.1** (Newton's 2<sup>nd</sup> law).

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = m\mathbf{a} \quad (1)$$

The formula is valid in an inertial frame.

Angular momentum  $\mathbf{L}$  and torque  $\mathbf{N}$  are also related

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \mathbf{r} \times \mathbf{F} = \mathbf{N} \quad (2)$$

Work done by external forces

$$W_{12} = \int_1^2 \mathbf{F} d\mathbf{s} = \int_1^2 m \frac{d\mathbf{v}}{dt} d\mathbf{s} = \int_1^2 m\mathbf{v} d\mathbf{v} = \frac{1}{2}m\mathbf{v}^2 \Big|_1^2 \quad (3)$$

Define a scalar function  $V(\mathbf{r})$ , then  $\mathbf{F} = -\nabla V(\mathbf{r})$  is a conservative force.

$$\oint \mathbf{F} d\mathbf{s} = 0 \quad (4)$$

Center of mass of the system

$$\mathbf{R} = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i} = \frac{\sum_i m_i \mathbf{r}_i}{M} \quad (5)$$

Total momentum

$$\mathbf{P} = \sum_i m_i \mathbf{p}_i = M \dot{\mathbf{R}} \quad (6)$$

Hence  $\mathbf{P}$  is conserved if external force  $\mathbf{F}^{(e)}$  is zero.

Total angular momentum

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt} \sum_i \mathbf{r}_i \times \mathbf{p}_i = \sum_i \mathbf{r}_i \times \left( \mathbf{F}_i^{(e)} + \sum_j \mathbf{F}_{ij} \right) = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{(e)} + \sum_{ij} \mathbf{r}_i \times \mathbf{F}_{ij}$$

Since  $\mathbf{r}_{ij}$  parallel to  $\mathbf{F}_{ij}$ , then

$$\sum_{ij} \mathbf{r}_i \mathbf{F}_{ij} = \frac{1}{2} \sum_{ij} \mathbf{r}_{ij} \times \mathbf{F}_{ji} = 0 \quad (7)$$

Therefore

$$\frac{d\mathbf{L}}{dt} = \mathbf{N}^{(e)} \quad (8)$$

Decomposition of the angular momentum

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i = \sum_i (\mathbf{R} + \mathbf{r}_i) \times m_i (\mathbf{V} + \mathbf{v}'_i) = \sum_i \mathbf{R} \times m_i \mathbf{V} + \sum_i \mathbf{r}'_i \times m_i \mathbf{v}'_i \quad (9)$$

## 1.2 Constraints

Holonomic constraint

$$f(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N, t) = 0 \quad (10)$$

Example: rigid body

$$(\mathbf{r}_i - \mathbf{r}_j)^2 - c_{ij}^2 = 0 \quad (11)$$

Example: non-sliding cylinder

$$\dot{x} - R\dot{\theta} = 0 \Rightarrow x - R\theta = \text{const}$$

A constraint of the form

$$\sum_i g_i(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) d\mathbf{x}_i = 0 \Rightarrow dG(\mathbf{x}_1, \dots) = 0 \Rightarrow G(\mathbf{x}_1, \dots) = \text{const} \quad (12)$$

Non-holonomic constraint: cannot be written in the form of holonomic constraint.

## 1.3 Generalized coordinates

Suppose we have a  $N$ -particle system, we will have  $3N$  DOFs. With  $k$  constraints, we will have  $3N - k$  DOFs. Define  $q_1, \dots, q_{3N-k}$  generalized coordinates, we have

$$\mathbf{r}_i = \mathbf{r}_i(q_1, \dots, q_{3N-k}, t) \quad (13)$$

## 2 Lagrange Formalism

### 2.1 D'Alembert's Principle

Hint from the rigid body: internal forces of constraints do not work.

Virtual displacement:  $\delta \mathbf{r}_i$  is consistent with the constraints imposed on the system at a given time

$$\mathbf{r}_i \rightarrow \mathbf{r}_i + \delta \mathbf{r}_i \quad (1)$$

**Theorem 2.1** (D'Alembert's principle). Consider a system in equilibrium

$$\mathbf{F}_i = 0 \Rightarrow \sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i = 0 \quad (2)$$

Separate  $\mathbf{F}_i = \mathbf{F}_i^{(a)} + \mathbf{f}_i$  where  $\mathbf{f}_i$  is the constraint force. Hence

$$\sum_i (\mathbf{F}_i^{(a)} + \mathbf{f}_i) \cdot \delta \mathbf{r}_i = 0 \Rightarrow \sum_i \mathbf{F}_i^{(a)} \cdot \delta \mathbf{r}_i = 0 \quad (3)$$

For a system moving under external forces

$$\mathbf{F}_i - \dot{\mathbf{p}}_i = 0 \Rightarrow \sum_i (\mathbf{F}_i - \dot{\mathbf{p}}_i) \delta \mathbf{r}_i = 0 \Rightarrow \sum_i (\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i) \delta \mathbf{r}_i = 0 \quad (4)$$

For holonomic constraints

$$\mathbf{r}_i = \mathbf{r}_i(q_1, \dots, q_n, t), \quad \mathbf{v}_i = \frac{d\mathbf{r}_i}{dt} = \frac{\partial \mathbf{r}_i}{\partial t} + \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \dot{q}_j, \quad \delta \mathbf{r}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \quad (5)$$

Define generalized force  $Q_j$

$$\sum_i \mathbf{F}_i \delta \mathbf{r}_i = \sum_{ij} \mathbf{F}_i \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j = \sum_j Q_j \delta q_j \quad (6)$$

Then

$$\sum_i \dot{\mathbf{p}}_i \cdot \delta \mathbf{r}_i = \sum_{ij} m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j = \sum_{ij} \left[ \frac{d}{dt} \left( m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \right) - m_i \dot{\mathbf{r}}_i \cdot \frac{d}{dt} \frac{\partial \mathbf{r}_i}{\partial q_j} \right] \delta q_j \quad (7)$$

$$= \sum_i \left[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] \delta q_j = \sum_j Q_j \delta q_j \quad (8)$$

$$(9)$$

Hence  $\forall j$  we have

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j = 0 \quad (10)$$

Let the potential energy  $V = V(\mathbf{r}_i, \dots) = V(q_j, \dots)$ , then we have

$$Q_j = \sum_i \mathbf{F}_i \frac{\partial \mathbf{r}_i}{\partial q_j} = \sum_i -\nabla_i V \frac{\partial \mathbf{r}_i}{\partial q_j} = -\frac{\partial V}{\partial q_j} \quad (11)$$

Therefore

$$\frac{d}{dt} \left( \frac{\partial (T - V)}{\partial \dot{q}_j} \right) - \frac{\partial (T - V)}{\partial q_j} - Q_j = 0 \quad (12)$$

**Theorem 2.2** (Lagrange's equation). Define  $L = T - V$ , then

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0 \quad (13)$$

The choice of Lagrangian is not unique,  $L'$  where

$$L' = L + \frac{dF(q, t)}{dt} \quad (14)$$

will give the same equations of motion as  $L$ .

**Example 2.1** (Lagrange's formalism).

1) For a single particle moving under force  $\mathbf{F}$

$$L(\mathbf{x}, \dot{\mathbf{x}}, t) = \frac{1}{2} m \dot{\mathbf{x}}^2 + \mathbf{F} \cdot \mathbf{x}$$

2) Motion in a 2D plane using polar coordinates

$$L(r, \theta, \dot{r}, \dot{\theta}, t) = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + \mathbf{F} \cdot \mathbf{r}$$

Generalized forces

$$Q_r = \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial r} = \mathbf{F} \cdot \mathbf{e}_r$$

$$Q_\theta = \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial \theta} = \mathbf{F} \cdot r \mathbf{e}_\theta$$

where

$$\mathbf{e}_r = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \quad \mathbf{e}_\theta = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$$

Equations of motion

$$m\ddot{r} - mr\dot{\theta}^2 = \mathbf{F} \cdot \mathbf{e}_r$$

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = r\mathbf{F}_\theta$$

3) Atwood's machine

$$L = \frac{1}{2} (M_1 + M_2) \dot{x}^2 + M_1 g x + M_2 g (l - x)$$

The equation of motion is

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$$

$$\Rightarrow (M_1 + M_2) \ddot{x} = (M_1 - M_2) g$$

Suppose we have a potential dependent on velocity (generalized potential) and the generalized force is defined as

$$U = U(q_j, \dot{q}_j), \quad Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt} \frac{\partial U}{\partial \dot{q}_j} \quad (15)$$

Define  $L = T - U$ , then we still have

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \quad (16)$$

**Example 2.2** (Lorentz force on a moving charge). The Lorentz force

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Define the scalar and vector potentials

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

Hence

$$\mathbf{F} = q \left[ -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \times (\nabla \times \mathbf{A}) \right]$$

## 2.2 Hamilton's principle, variational principle

Configuration space: a space formed by the set of generalized coordinates.

$$(q_1, q_2, \dots, q_n) \text{ as function of } t \quad (17)$$

**Theorem 2.3** (Hamilton's principle). Define the action integral  $I$ , where  $L = T - V$  or  $L = T - U$  ( $U$  is the generalized potential)

$$I = \int_{t_1}^{t_2} L \, dt \quad (18)$$

Then the variation of the action integral equals to zero

$$\delta I = \delta \int_{t_1}^{t_2} L(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n) \, dt = 0 \quad (19)$$

Add small variation on the path

$$q_i(t, \alpha) = q_i(t) + \alpha \eta(t) \quad (20)$$

where  $\eta(t_1) = \eta(t_2) = 0$ . Then the action will be the function of  $\alpha$ ,  $I = I(\alpha)$ . Hence

$$\delta I = \int_{t_1}^{t_2} \left( \sum_i \frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) dt \quad (21)$$

Change the order of differentiation  $\delta \dot{q}_i = d\delta q_i / dt$ , then

$$\delta I(\alpha) = \int_{t_1}^{t_2} \sum_i \left[ \frac{\partial L}{\partial q_i} \delta q_i - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \right] dt + \sum_i \frac{\partial L}{\partial \dot{q}_i} \delta q_i \Big|_{t_1}^{t_2} \quad (22)$$

$$= \int_{t_1}^{t_2} \sum_i \left[ \frac{\partial L}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) \right] \delta q_i \, dt = 0 \quad (23)$$

$$\Rightarrow \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \quad (24)$$

**Example 2.3** (Shortest path problem).  $y = y(x)$ ,  $ds = \sqrt{dx^2 + dy^2}$ , then the action integral (path) is

$$I = \int_1^2 ds = \int_1^2 \sqrt{1 + \dot{y}^2} \, dx \quad (25)$$

Apply the Lagrange's equation we get

$$\frac{d}{dx} \frac{d\sqrt{1+\dot{y}^2}}{d\dot{y}} = 0 \Rightarrow \frac{d\dot{y}}{dx} = 0 \Rightarrow y = ax + b \quad (26)$$

**Example 2.4** (Solid of revolution). Differential of area  $2\pi x ds = 2\pi x \sqrt{1+\dot{y}^2} dx$ , then the total area is

$$\int_1^2 2\pi x \sqrt{1+\dot{y}^2} dx \quad (27)$$

Define the Lagrangian  $L(x, y, \dot{y}) = 2\pi x \sqrt{1+\dot{y}^2}$ , by Lagrange's equation we can get

$$\frac{x\dot{y}}{\sqrt{1+\dot{y}^2}} = \text{const} \Rightarrow y = a \cosh \frac{x}{a} + b \quad (28)$$

**Example 2.5** (The curve of fastest descent). Question: along which trajectory from point 1 to point 2, the time is shortest? The total time is

$$T = \int_1^2 \frac{ds}{v} = \int_1^2 \frac{ds}{\sqrt{2gy}} \quad (29)$$

According to Newton's laws we have  $y = gv^2$ , then

$$T = \int_1^2 \frac{\sqrt{1+\dot{y}^2}}{\sqrt{2gy}} dx \quad (30)$$

Then we have  $L(x, y, \dot{y})$  and we get **check drivation**

$$\frac{\dot{y}}{2y} + \frac{y\ddot{y}}{1+\dot{y}^2} = 0 \quad (31)$$

$$\Rightarrow \frac{d}{dx} \ln[y(1+\dot{y}^2)] = 0 \quad (32)$$

which means that  $y(1+\dot{y}^2) = \text{const}$ . The solution is  $x = a(\theta - \sin \theta)$ ,  $y = a(1 - \cos \theta)$ .

## 2.3 Constraint

Holonomic constraints

$$f(q_1, \dots, q_n, t) = 0 \quad (33)$$

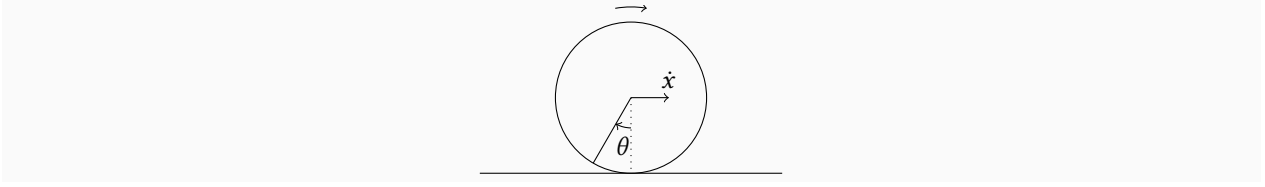
Non-holonomic constraints

$$f(q_1, \dots, q_n; \dot{q}_1, \dots, \dot{q}_n; t) = 0 \quad (34)$$

Sometimes we can convert  $f(\dot{q}_i) = 0$  to  $f'(q_i) = 0$ .

**Example 2.6** (Rolling cylinder). Rolling cylinder without sliding has the constraint

$$\dot{x} - R\dot{\theta} = 0 \Rightarrow x - R\theta = \text{const} \quad (35)$$



A commonly encountered type of non-holonomic constraint is linear constraint equations

$$\sum_{k=1}^n a_{ik} \frac{dq_k}{dt} + a_{it} = 0 \quad (36)$$

For the virtual displacement  $\delta q_i$

$$\sum_{k=1}^n a_{ik} \delta q_k = 0 \quad (37)$$

Suppose  $q_i$  are not independent, then the Euler-Lagrange's equation could not hold.

$$\int_{t_1}^{t_2} \sum_i \left[ \frac{\partial L}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) \right] \delta q_i dt = 0 \Rightarrow \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q_i} = 0$$

Include the constraint equation of  $q_i$  into the equation

$$\sum_{k=1}^n a_{ik} \delta q_k = 0 \Rightarrow \sum_{i=1}^m \lambda_i \sum_{k=1}^n a_{ik} \delta q_k = 0 \quad (38)$$

Hence

$$\int_{t_1}^{t_2} \sum_{k=1}^n \left[ \frac{\partial L}{\partial q_k} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) + \sum_{i=1}^m \lambda_i a_{ik} \right] \delta q_k dt = 0$$

Let  $q_1, \dots, q_{n-m}$  be independent generalized coordinate,  $q_{n-m+1}, \dots, q_n$  dependent generalized coordinates (i.e., they can be expressed by  $q_1, \dots, q_{n-m}$ ). Choose  $\lambda_i$  s.t.

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) + \sum_{i=1}^m \lambda_i a_{ik} = 0 \quad (39)$$

$\forall k = n-m+1, \dots, n$ . In conclusion, we have  $q_1, \dots, q_n, \lambda_1, \dots, \lambda_m$  overall  $n+m$  unknowns, and  $n$  Lagrange's equations and  $m$  constraint equations overall  $n+m$  equations.

*Remark.*

- 1) It is inconvenient to reduce all  $q_k$ s to independent coordinates
- 2) If we are interested in the constraint forces

$$\sum_{k=1}^n a_{ik} dq_k + a_{it} dt = 0$$

where

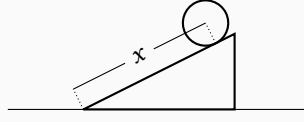
$$a_{ik} = \frac{\partial f_i}{\partial q_k}, \quad a_{it} = \frac{\partial f_i}{\partial t}$$

Then

$$df_i = \frac{\partial f_i}{\partial q_k} dq_k + \frac{\partial f_i}{\partial t} dt \Rightarrow df_i = 0, \quad f_i = \text{const}$$



**Example 2.7** (Hoop rolling down an inclined plane).



The constraint equation writes

$$\dot{x} - r\dot{\theta} = 0 \Rightarrow x - r\theta = \text{const}$$

$a_x = 1, a_\theta = -r, a_t = 0$ .

Energy terms are

$$\begin{aligned} T &= T_{\text{COM}} + T_{\text{relative}} = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}M(r\dot{\theta})^2 \\ V &= Mg(l - x) \sin \phi \\ \Rightarrow L &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}M(r\dot{\theta})^2 - Mg(l - x) \sin \phi \end{aligned}$$

Then we can write the Lagrange's equation with Lagrange's multipliers

$$\begin{aligned} \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} + a_x \lambda &= 0 \\ \Rightarrow Mg \sin \phi - M\ddot{x} + \lambda &= 0 \\ -Mr^2\ddot{\theta} - \lambda r &= 0 \\ \dot{x} = r\dot{\theta} \Rightarrow \ddot{x} &= r\ddot{\theta} \end{aligned}$$

we can get  $M\ddot{x} = Mr\ddot{\theta} = -\lambda$  and  $\ddot{x} = (g \sin \phi)/2$ . Note that  $\lambda$  is the constraint force (in this case  $\lambda$  is the frictional force). [check derivation](#)

## 2.4 Lagrangian for Lorentz force

**Definition 2.1** (Cyclic coordinate). The generalized coordinate  $q_i$  is cyclic (ignorable) if

$$\frac{\partial L}{\partial q_i} = 0 \quad (40)$$

It implies the generalized momentum  $p_i$  is conserved.

The Lorentz force is given by

$$\mathbf{F} = q[\mathbf{E} + \mathbf{v} \times \mathbf{B}] \quad (41)$$

Where

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A} \quad (42)$$

Hence, by defining the Lagrangian

$$L = \frac{1}{2}m\mathbf{v}^2 - q\phi + q\mathbf{A} \cdot \mathbf{v} \quad (43)$$

Applying the Lagrange's equation we can get the EOM (eq. 41).

## 2.5 Conservation & symmetry of the system

**Rotational symmetry.** Let  $q_j$  be one of the rotational angle of spacial coordinate  $\mathbf{r}_i$ . Hence

$$d\mathbf{r}_i = \mathbf{n}_j \times \mathbf{r}_i dq_j \Rightarrow \frac{\partial \mathbf{r}_i}{\partial q_j} = \mathbf{n}_j \times \mathbf{r}_i dq_j \quad (44)$$

where  $\mathbf{n}_j$  is the normal vector of the rotation axis of  $q_j$ . Hence the generalized force of  $q_j$  writes

$$Q_{q_j} = -\frac{\partial V}{\partial q_j} = -\sum_k \frac{\partial V}{\partial \mathbf{r}_k} \frac{\partial \mathbf{r}_k}{\partial q_j} = -\frac{\partial V}{\partial \mathbf{r}_i} \frac{\partial \mathbf{r}_i}{\partial q_j} = \mathbf{F}_i \cdot (\mathbf{n}_j \times \mathbf{r}_i) = \mathbf{n}_j \cdot (\mathbf{r}_i \times \mathbf{F}_i) = \mathbf{n}_j \cdot \mathbf{N}_i \quad (45)$$

where  $\mathbf{N}_i$  stands for the torque on the  $i^{\text{th}}$  particle. The generalized momentum of  $q_j$  writes

$$p_{q_j} = \frac{\partial T}{\partial \dot{q}_j} = \sum_k \frac{\partial T}{\partial \dot{\mathbf{r}}_k} \frac{\partial \dot{\mathbf{r}}_k}{\partial \dot{q}_j} = \sum_k m_k \dot{\mathbf{r}}_k \cdot \frac{\partial \dot{\mathbf{r}}_k}{\partial \dot{q}_j} = m_i \dot{\mathbf{r}}_i \cdot (\mathbf{n}_j \times \mathbf{r}_i) = \mathbf{n}_j \times (\mathbf{r}_i \times m_i \dot{\mathbf{r}}_i) = \mathbf{r}_j \times \mathbf{L}_i \quad (46)$$

where  $\mathbf{L}_i$  stands for the angular momentum of the  $i^{\text{th}}$  particle. Hence, the rotational invariance implies the conservation of angular momentum.

### Time translation

$$\frac{d}{dt}L = \sum_i \frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i + \frac{\partial L}{\partial t} \quad (47)$$

$$= \sum_i \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i + \frac{\partial L}{\partial t} \quad (48)$$

$$= \frac{d}{dt} \sum_i \left( \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) + \frac{\partial L}{\partial t} \quad (49)$$

$$\Rightarrow \frac{\partial L}{\partial t} + \frac{d}{dt} \underbrace{\left( \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L \right)}_H = 0 \quad (50)$$

Note that

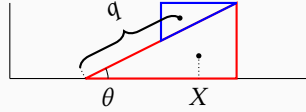
$$\sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i = 2T \quad (51)$$

*Proof.* Suppose  $r_i$  does not have explicit time dependence ( $\partial r_i / \partial t = 0$ )

$$\begin{aligned} T &= \sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 = \sum_i \frac{1}{2} m_i \left( \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \mathbf{r}_i}{\partial t} \right)^2 \\ &= \sum_{ijk} \frac{1}{2} m_i \frac{\partial \mathbf{r}_i}{\partial q_j} \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_j \dot{q}_k \\ &\Rightarrow \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i = \sum_i \dot{q}_i \sum_{jk} m_k \frac{\partial \mathbf{r}_k}{\partial q_i} \frac{\partial \mathbf{r}_k}{\partial q_j} \dot{q}_j = \sum_{ijk} m_k \frac{\partial \mathbf{r}_k}{\partial q_i} \frac{\partial \mathbf{r}_k}{\partial q_j} \dot{q}_i \dot{q}_j = 2T \quad \square \end{aligned}$$

Hence we can define Hamiltonian  $H = T + V$  which stands for the total energy, and  $H$  conserved if  $L$  doesn't depend on time explicitly.

**Example 2.8** (Two blocks). Let  $M$  be the mass of the big block,  $m$  be the mass of the small block. Define two generalized coordinates:  $X$  stand for the position of COM of the big block,  $q$  stand for the position of COM of the small block (sloped).



Hence we can easily define the Lagrangian

$$L = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m[(\dot{X} + \dot{q} \cos \theta)^2 + \dot{q}^2 \sin^2 \theta] - mgq \sin \theta$$

There is no  $X$  dependence on the system, hence  $X$  is a cyclic coordinate.

$$\begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial \dot{X}} &= \frac{d}{dt} M\dot{X} + m(\dot{X} + \dot{q} \cos \theta) = 0 \\ \Rightarrow M\dot{X}m(\dot{X} + \dot{q} \cos \theta) &= p_X = \text{const} \end{aligned}$$

### 3 The Central Force Problem

#### 3.1 Reduction to the equivalent one-body problem

For the two-body problem, we have two choices of generalized coordinate

- (a)  $\mathbf{r}_1, \mathbf{r}_2$  stand for spacial coordinates of two masses
- (b)  $\mathbf{R}$  spacial coordinate of the COM,  $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

Hence, we can rewrite the kinetic energy in terms of  $\mathbf{R}$  and  $\mathbf{r}$

$$T = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 \quad (1)$$

$$= \frac{1}{2}m_1\left(\dot{\mathbf{R}} + \frac{m_2}{m_1+m_2}\dot{\mathbf{r}}\right)^2 + \frac{1}{2}m_2\left(\dot{\mathbf{R}} - \frac{m_1}{m_1+m_2}\dot{\mathbf{r}}\right)^2 \quad (2)$$

$$= \frac{1}{2}(m_1+m_2)\dot{\mathbf{R}}^2 + \frac{m_1m_2}{m_1+m_2}\dot{\mathbf{r}}^2 = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 \quad (3)$$

Hence we can write the Lagrangian

$$L = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 - V \quad (4)$$

Suppose  $V = V(\mathbf{r})$ , then  $\mathbf{R}$  is a cyclic coordinate, We have  $\dot{\mathbf{R}} = \text{const}$ , and we can drop  $\dot{\mathbf{R}}$  terms in  $L$ . Moreover, if  $V = V(\|\mathbf{r}\|)$ , the total angular momentum is conserved.

Use  $r, \theta$  as generalized coordinates, we have

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \quad (5)$$

The Lagrange's equation wrt  $r$  writes

$$\mu\ddot{r} - \mu r\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0 \quad (6)$$

Easy to find that  $\theta$  is cyclic, hence

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt}\mu r^2\dot{\theta} = 0 \Rightarrow p_\theta = \mu r^2\dot{\theta} = l = \text{const} \quad (7)$$

**Theorem 3.1** (Kepler's 2<sup>nd</sup> law). Radius vector sweeps out equal areas in equal time.

Substitute  $\mu r\dot{\theta}^2$  term by  $l$ , we have

$$\mu\ddot{r} - \frac{l^2}{\mu r^3} + \frac{\partial V}{\partial r} = 0 \quad (8)$$

$$\Rightarrow \mu\ddot{r} = -\frac{\partial}{\partial r}\left(V + \frac{l^2}{2\mu r^2}\right) \quad (9)$$

$$\mu\ddot{r} = -\frac{\partial}{\partial r}\dot{r}\left(V + \frac{l^2}{2\mu r^2}\right) \quad (10)$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}\mu\dot{r}^2 + V + \frac{l^2}{2\mu r^2}\right) = \frac{d}{dt}E = 0 \quad (11)$$

Hence we can get the expression of  $\dot{r}$  and a differential equation

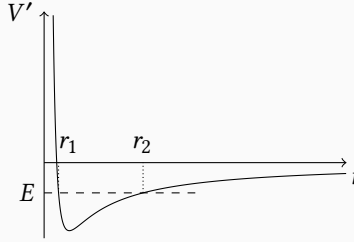
$$\dot{r} = \sqrt{\frac{2}{\mu}\left(E - V - \frac{l^2}{2\mu r^2}\right)} \quad (12)$$

Define the new effective force  $f'(r)$  with an effective potential  $V' = V + l^2/2\mu r^2$ , we have

$$\mu\ddot{r} = -\frac{\partial V}{\partial r} + \frac{l^2}{\mu r^3} = f'(r), \quad E = \frac{1}{2}\mu\dot{r}^2 + V' = \text{const} \quad (13)$$

**Example 3.1** (Gravitational force). Suppose we have  $f(r) = -kr^{-2}$  and  $V = -kr^{-1}$ , then

$$V' = -\frac{k}{r} + \frac{l^2}{2\mu r^2}$$



- 1)  $E = E_1 \geq 0$ , the motion has a lower bound,  $r \geq r_1$ .
- 2)  $V_{\min} < E_2 < 0$ , the motion has lower and upper bound,  $r_1 \leq r \leq r_2$ .
- 3)  $E_3 = V_{\min}$ , the motion will shrink to a single circle  $r_1 = r_2 = \text{const}$ , hence it is a circular motion. In this case, the gravitational force is equal to the centrifugal force

$$\mu \ddot{r} = f(r) + \frac{l^2}{\mu r^3} = f(r) + \mu r \dot{\theta}^2 = 0 \Rightarrow f(r) = -\mu r \dot{\theta}^2$$

*Remark.* Let the potential be  $V = -kr^{-\alpha}$ , then the motion cannot have periodic behavior if  $\alpha > 2$ .

**Example 3.2** (Harmonic oscillator).  $V = kr^2/2$ , we have

$$V' = \frac{1}{2}kr^2 + \frac{l^2}{2\mu r^2}$$

**Theorem 3.2** (Conditions for closed orbitals, Bertrand's theorem). Stable orbitals require

$$\left. \frac{\partial^2 V'}{\partial r^2} \right|_{r=r_0} = -\frac{\partial f}{\partial r} + \frac{3l^2}{mr^4} \Big|_{r=r_0} > 0 \Rightarrow \left. \frac{\partial f}{\partial r} \right|_{r=r_0} < -\frac{3f(r_0)}{r_0} \quad (14)$$

Let  $f = -kr^n$ , then we have

$$-knr^{n-1} < 3kr^{n-1} \Rightarrow n > -3 \quad (15)$$

For a small perturbation from the minimum, we can write the effective potential as Taylor expansion

$$V'(r) = V'(r_0) + \left. \frac{\partial V'}{\partial r} \right|_{r=r_0} (r - r_0) + \frac{1}{2} \left. \frac{\partial^2 V'}{\partial r^2} \right|_{r=r_0} (r - r_0)^2 + \dots \quad (16)$$

$$= V'(r_0) + \frac{1}{2} \left. \frac{\partial^2 V'}{\partial r^2} \right|_{r=r_0} (r - r_0)^2 + O(|r - r_0|^3) \quad (17)$$

Hence the EOM becomes

$$m\ddot{r} = -\frac{\partial V'}{\partial r} = -\left. \frac{\partial^2 V'}{\partial r^2} \right|_{r=r_0} (r - r_0) \quad (18)$$

$$\dot{\theta} = \frac{l}{mr^2} \quad (19)$$

The solution takes the form

$$u = u_0 + a \cos \beta \theta, \quad u = \frac{1}{r}, \quad \beta^2 = \left. \frac{r}{f} \frac{\partial f}{\partial r} + 3 \right|_{r=r_0} \quad (20)$$

For finite perturbation, the orbit can be a closed only if  $\beta^2 = 1$  or  $4$ . Then there are only two types of central-force scalar problem with the property that all bound orbitals are closed orbitals.

$$\left. \frac{r}{f} \frac{\partial f}{\partial r} \right|_{r=r_0} = \pm 2 \Rightarrow f = -kr^{-2} \text{ or } -kr \quad (21)$$

check the theorem on other books

### 3.2 Virial theorem

Consider a multi-particle system  $1 \leq i \leq N$ .

$$\dot{\mathbf{p}}_i = \mathbf{F}_i \quad (22)$$

Define new function

$$G = \sum_i \mathbf{p}_i \cdot \mathbf{r}_i \quad (23)$$

Then the time derivative of  $G$  writes

$$\frac{d}{dt}G = \sum_i m_i \dot{\mathbf{r}}_i^2 + \sum_i \mathbf{F}_i \cdot \mathbf{r}_i \quad (24)$$

Hence the time average of  $dG/dt$  is

$$\frac{1}{\tau} \int_0^\tau \frac{d}{dt}G dt = 2\bar{T} + \overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i} = \frac{1}{\tau} [G(\tau) - G(0)] \quad (25)$$

The time average equals to 0 if  $\tau$  is the period of the periodic motion; even for non-periodic motion, if  $\tau \rightarrow \infty$ , the average will approach 0. Hence

$$2\bar{T} + \overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i} = 0 \Rightarrow \bar{T} = -\frac{1}{2} \underbrace{\overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i}}_{\text{virial of Clausius}} \quad (26)$$

**Example 3.3** (Ideal gas). Temperature  $T \propto T_i$  kinetic energy of  $i^{\text{th}}$  particle.

$$T_i = \frac{3}{2} k_B T \quad (27)$$

Hence the time average of total kinetic energy is

$$\overline{\sum_i T_i} = \frac{3}{2} N k_B T \quad (28)$$

The virial could be written as

$$\overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i} = \int d\mathbf{F} \cdot \mathbf{r} \quad (29)$$

Note that

$$d\mathbf{F} = -p d\mathbf{A}\mathbf{n}$$

Applying Gauss's law we have

$$\overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i} = - \int p d\mathbf{A} \mathbf{n} \cdot \mathbf{r} = - \int p dV (\nabla \cdot \mathbf{r}) = -3pV \quad (30)$$

Hence by the equation we get  $pV = nk_B T$ .

**Example 3.4** (Gravational force). Since  $\mathbf{F}_i = -\nabla_i V$ , given  $V_i = ar_i^{n+1}$ , we have

$$\bar{T} = \frac{1}{2} \overline{\sum_i \nabla_i V \cdot \mathbf{r}_i} = \frac{n+1}{2} \overline{\sum_i ar_i^{n+1}} = \frac{n+1}{2} \bar{V} \quad (31)$$

This theorem is particularly important when the potential energy is a homogeneous function of the coordinates, i.e.

$$U(\alpha \mathbf{r}_1, \dots, \alpha \mathbf{r}_n) = \alpha^k U(\mathbf{r}_1, \dots, \mathbf{r}_n) \quad (32)$$

Note that

$$\frac{dU}{d\alpha} = \sum_a \frac{\partial \alpha \mathbf{r}_a}{\partial \alpha} \cdot \frac{\partial U}{\partial \alpha \mathbf{r}_a} = \sum_a \frac{\mathbf{r}_a}{\alpha} \cdot \frac{\partial U}{\partial \mathbf{r}_a} = k \alpha^{k-1} U \quad (33)$$

Then the equation 26 becomes

$$2\bar{T} = k\bar{U} \quad (34)$$

Equivalently

$$\bar{U} = \frac{2E}{k+2}, \quad \bar{T} = \frac{kE}{k+2} \quad (35)$$

### 3.3 Inverse-square force

Consider  $f = -k/r^2$ ,  $V = -k/r$ , then we have EOM

$$\dot{r} = \left[ \frac{2}{m} \left( E + \frac{k}{r} - \frac{l^2}{2mr^2} \right) \right]^{1/2} \Rightarrow \theta = \theta_0 + \int_{r_0}^{r_1} \frac{dr}{r^2 \left( \frac{2mE}{l^2} - \frac{2mV}{l^2} - \frac{1}{r^2} \right)^{1/2}} \quad (36)$$

Define  $u = 1/r$ , let  $\theta_0 = 0$ , then

$$\theta = \theta_0 - \int_{u_0}^{u_1} \frac{du}{\left( \frac{2mE}{l^2} + \frac{2mku}{l^2} - u^2 \right)^{1/2}} \quad (37)$$

$$\Rightarrow \theta = -\arccos \frac{\frac{l^2 u}{mk} - 1}{\sqrt{1 + \frac{2El^2}{mk^2}}}, \quad r = \frac{\frac{l^2}{mk}}{1 + \sqrt{1 + \frac{2El^2}{mk^2}} \cos \theta} = \frac{r_0}{1 + \epsilon \cos \theta} \quad (38)$$

$\epsilon$  is called the eccentricity.

$\epsilon$	Energy	Orbit
$\epsilon > 1$	$E > 0$	Hyperbola
$\epsilon = 1$	$E = 0$	Parabola
$0 < \epsilon < 1$	$E < 0$	Ellipse
$\epsilon = 0$	$E < 0$	Circle

The major semi-axis  $a$  and minor semi-axis  $b$  are given by

$$a = \frac{r_0}{1 - \epsilon^2} = -\frac{k}{2E}, \quad b = a\sqrt{1 - \epsilon^2} = \frac{k}{\sqrt{-2mE}} \quad (39)$$

Since  $mr^2\dot{\theta} = l$ , then

$$\int_0^T dA = \frac{1}{2} \int_0^T r^2 \dot{\theta} dt = \frac{1}{2} \int_0^T \frac{l}{m} \theta dt = \frac{l}{2m} T = \pi ab \Rightarrow T = 2\pi a^{3/2} \sqrt{\frac{m}{k}} \quad (40)$$

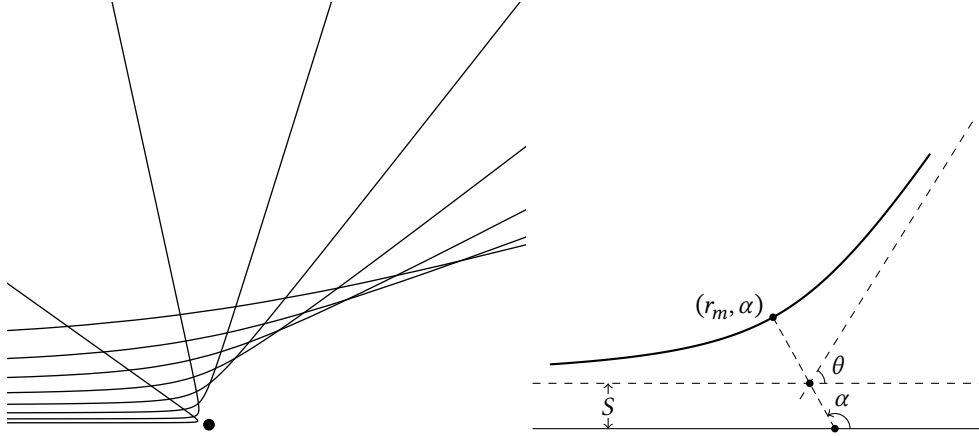
Alternatively

$$\int dt = \int \frac{dr}{\sqrt{\frac{2}{m} \left( E - V - \frac{l^2}{2mr^2} \right)}} = \sqrt{\frac{ma}{k}} \int_{r_0}^{r_1} \frac{r dr}{\sqrt{-r^2 + 2ar - b^2}} = \sqrt{\frac{ma}{k}} \int_{r_0}^{r_1} \frac{r dr}{\sqrt{-(r-a)^2 + a^2\epsilon^2}} \quad (41)$$

Let  $r - a = -a\epsilon \cos \zeta$ , hence

$$t = \sqrt{\frac{ma}{k}} \int \frac{(a - a\epsilon \cos \zeta) a\epsilon \sin \zeta d\zeta}{\sqrt{a\epsilon^2(1 - \cos^2 \zeta)}} = \sqrt{\frac{ma}{k}} \int (a - a\epsilon \cos \zeta) d\zeta = \sqrt{\frac{ma^3}{k}} (\zeta - \epsilon \sin \zeta) + C \quad (42)$$

### 3.4 Scattering



Question: how many particles will be scattered in the given solid angle region (scattering cross section). Let  $V \sim 1/r$ ,  $f \sim 1/r^2$ . Define the intensity of incident beam  $I$

$I = \#$  of particles crossing a unit area perpendicular to the beam in unit time

The the number of particles scattered into  $d\Omega$  could be expressed as

$$dN = \sigma I d\Omega, \quad \sigma = \frac{dN}{I d\Omega} \quad (43)$$

where  $\sigma$  is called differential cross section, which has the unit of area.

Particles in  $[S, S + dS]$  would be scattered into  $[\Omega, \Omega + d\Omega]$ , since  $d\Omega = 2\pi \sin \theta d\theta$

$$2\pi IS |dS| = \sigma I d\Omega = 2\pi \sigma I \sin \theta |d\theta| \Rightarrow \sigma = \frac{S |dS|}{\sin \theta |d\theta|} \quad (44)$$

The angular momentum of the incoming particles (wrt. to force center) is

$$l = mv_0 S = S \sqrt{2mE} \quad (45)$$

From the equation derived from central force problem

$$\alpha = \int \frac{dr}{r^2 \sqrt{\frac{2mE}{l^2} - \frac{rmV}{l^2} - \frac{1}{r^2}}} \quad (46)$$

$$= \pi + \int_{-\infty}^{r_m} \frac{dr}{r^2 \sqrt{\frac{2mE}{l^2} - \frac{rmV}{l^2} - \frac{1}{r^2}}} \quad (47)$$



Let  $r_m \equiv r_{\min}$  is the closest distance. Define  $\psi$  where

$$\alpha = \pi + \int_{-\infty}^{r_m} \frac{dr}{r^2 \sqrt{\frac{2mE}{l^2} - \frac{rmV}{l^2} - \frac{1}{r^2}}} = \pi - \psi \quad (48)$$

Hence  $\theta = \pi - 2\psi$ . Change the variable  $u = 1/r$ , then

$$\psi = \int_0^{u_m=1/r_m} \frac{S du}{\sqrt{1 - \frac{V}{E} - S^2 u^2}} \quad (49)$$

Also note that

$$\sin \frac{\theta}{2} = \sin \frac{\pi - 2\psi}{2} = \cos \psi = \frac{1}{\epsilon} \quad (50)$$

**Example 3.5** (Coulomb interaction). Let

$$f = \frac{ZZ'e^2}{r^2}, \quad V = \frac{ZZ'e^2}{r}, \quad r = \frac{r_0}{1 + \epsilon \cos(\alpha - \alpha')} \quad (51)$$

choose  $\alpha' = \pi$ . One can derive that

$$\epsilon = \sqrt{1 + \frac{2El^2}{m(ZZ'e^2)^2}} = \sqrt{1 + \left( \frac{2ES}{ZZ'e^2} \right)^2} \quad (52)$$

Using the trigonometric relationship between  $\theta$  and  $\epsilon$ , we have

$$S = \frac{ZZ'e^2}{2E} \cot \frac{\theta}{2}, \quad \left| \frac{dS}{d\theta} \right| = \frac{ZZ'e^2}{4E} \frac{1}{\sin^2 \frac{\theta}{2}} \quad (53)$$

$$\Rightarrow \epsilon(\theta) = \left( \frac{ZZ'e^2}{4E} \right)^2 \frac{1}{\sin^4 \frac{\theta}{2}} \quad (54)$$

check if  $\alpha' = \pi$  or  $\alpha' = \alpha_m$

## 4 Rigid Body

### 4.1 Coordinates of rigid body

A rigid body is a system of point masses satisfying the constraint that distance between any two points is a constant ( $r_{ij} = \text{const}$  for all  $i, j$ ). Let  $\mathbf{r}_i$ ,  $\mathbf{r}_j$ , and  $\mathbf{r}_k$  be three points in the rigid body. Let  $\mathbf{r}_i$  has 3 DOFs, since  $r_{ij}$  is a constant,  $\mathbf{r}_j$  has two DOFs. Hence,  $\mathbf{r}_k$  only have 1 DOFs. The system has  $3 + 2 + 1 = 6$  DOFs.

Or, alternatively, a rigid body has 3 coordinates for the origin of the coordinate system fixed on the rigid body, and 3 angular variables to specify the orientation of the rotated coordinate system.

