PHY3110 SP23 HW05

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Problem 1. Find the orbits of a point mass moving in a central force field $F = -kr$, where k is a positive constant. What if k is a negative constant?

Solution. Instead of using polar coordinate (r, θ), use the (x, y) Cartesian coordinate system. Then we have equations

$$m\ddot{x} = -kx,\ m\ddot{y} = -ky$$

This leads to the solution

$$x = A\sin[\omega(t - t_1)],\ y = B\cos[\omega(t - t_2)]$$

which is a ellipse (where $\omega = \sqrt{\frac{k}{m}}$).
Suppose k negative, we have a diffent solution

$$x = A\sinh[\omega(t - t_1)],\ y = B\cosh[\omega(t - t_2)]$$

which is a hyperbola (where $\omega = \sqrt{\frac{-k}{m}}$).

Problem 2. A point mass m moves in a central force field with $F = -\frac{\alpha}{r^2}$. If its orbit is an ellipse with the semi-major axis α, derive the following relation between its velocity and r, a

$$v^2 = \alpha\left(\frac{2}{r} - \frac{1}{\alpha}\right) \tag{1}$$

Solution. For ellipse orbits, $E < 0$ and we have such relation

$$a = -\frac{k}{2E}$$

In this case we have $k = \alpha m$, hence we have

$$E = -\frac{\alpha m}{2a} = \frac{1}{2}mv^2 - \frac{\alpha m}{r} \Rightarrow v^2 = \alpha\left(\frac{2}{r} - \frac{1}{a}\right)$$

For parabola orbits, $E = 0$ and hence

$$E = 0 = \frac{1}{2}mv^2 - \frac{\alpha m}{r} \Rightarrow v^2 = \frac{2\alpha}{r}$$

For hyperbola orbits, $E > 0$ and we have such relation

$$E = \frac{\alpha m}{2a} = \frac{1}{2}mv^2 - \frac{\alpha m}{r} \Rightarrow v^2 = \alpha\left(\frac{2}{r} + \frac{1}{a}\right)$$

Problem 3. Consider the scattering produced by a repulsive force $F = \frac{k}{r^3}$, show that the cross section takes the form

$$\sigma(\theta) = \frac{k\pi^2}{2E}\frac{(\pi - \theta)}{\theta^2(2\pi - \theta)^2\sin\theta} \tag{2}$$

Solution. The potential energy takes the form $V = \frac{k}{r^2}$. Using the relation

$$E = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + \frac{k}{2r^2}$$

We know that r takes its minimum r_m when $\dot{r} = 0$, therefore

$$\left.\begin{aligned} l^2 &= 2mEs^2 \\ E &= \frac{l^2}{2mr_m^2} + \frac{k}{2r_m^2} \end{aligned}\right\} \Rightarrow r_m = \left(S^2 + \frac{k}{2E}\right)^{1/2} = \frac{1}{u_m}$$

Using the formula

$$\begin{aligned} \psi &= \int_0^{u_m} \left(1 - \frac{V}{E} - S^2u^2\right)^{-1/2} S\,\mathrm{d}u \\ &= \int_0^{u_m} \left(1 - \frac{ku^2}{2E} - S^2u^2\right)^{-1/2} S\,\mathrm{d}u \\ &= \int_0^{u_m} \left[1 - \left(\frac{k}{2E} + S^2\right)u^2\right]^{-1/2} S\,\mathrm{d}u \\ &= S\left(\frac{k}{2E} + S^2\right)^{-1/2} \arcsin\left(\frac{k}{2E} + S^2\right)^{1/2} u\Bigg|_0^{u_m} \\ &= \frac{\pi}{2}S\left(\frac{k}{2E} + S^2\right)^{-1/2} \end{aligned}$$

Hence we have θ equals to

$$\theta = \pi - 2\psi = \pi\left[1 - S\left(\frac{k}{2E} + S^2\right)^{-1/2}\right]$$

Therefore

$$S = \left[\frac{k}{2E}\frac{(\pi - \theta)^2}{\theta(2\pi - \theta)}\right]^{1/2}$$

Hence

$$\frac{\mathrm{d}S}{\mathrm{d}\theta} = -S^{-1}\frac{k\pi^2}{2E}\frac{(\pi - \theta)}{\theta^2(2\pi - \theta)^2}$$

Therefore we have the differential cross-section equals to

$$\sigma(\theta) = \frac{S}{\sin\theta}\left|\frac{\mathrm{d}S}{\mathrm{d}\theta}\right| = \frac{k\pi^2}{2E}\frac{\pi - \theta}{\theta^2(2\pi - \theta)^2\sin\theta}$$

Problem 4. Show that for an antisymmetric 3×3 matrix $\mathbf{A}$, the matrix $\mathbf{B} = (\mathbf{1}+\mathbf{A})(\mathbf{1}-\mathbf{A})^{-1}$ is orthogonal, where $\mathbf{1}$ is the identity matrix.

Solution. Note that

$$\begin{aligned} \mathbf{B}^T &= [(\mathbf{1} - \mathbf{A})^{-1}]^T(\mathbf{1} + \mathbf{A})^T = (\mathbf{1} + \mathbf{A})^{-1}(\mathbf{1} - \mathbf{A}) \\ \mathbf{B}^T\mathbf{B} &= (\mathbf{1} + \mathbf{A})^{-1}(\mathbf{1} - \mathbf{A})(\mathbf{1} + \mathbf{A})(\mathbf{1} - \mathbf{A})^{-1} \end{aligned}$$

Since $(\mathbf{1} + \mathbf{A})(\mathbf{1} - \mathbf{A}) = (\mathbf{1} - \mathbf{A})(\mathbf{1} + \mathbf{A})$, we have

$$\mathbf{B}^T\mathbf{B} = (\mathbf{1} + \mathbf{A})^{-1}(\mathbf{1} + \mathbf{A})(\mathbf{1} - \mathbf{A})(\mathbf{1} - \mathbf{A})^{-1} = \mathbf{1}$$

which means that $\mathbf{B}$ is an orthogonal matrix.