PHY3110 SP23 HW05

Haoran Sun (haoransun@link.cuhk.edu.cn)

Problem 1. Obtain the general rotation matrix in terms of the Euler angles by performing an explicit multiplication of the tree successive rotation matrices. Verify that the matrix multiplication is associative.

Solution. The three rotation matrices are

$$D = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix},\ B = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence we have

$$A = BCD = \begin{bmatrix} \cos\phi\cos\psi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\theta\sin\psi \\ -\cos\theta\cos\psi\sin\phi - \cos\phi\sin\psi & \cos\theta\cos\phi\cos\psi - \sin\phi\sin\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\cos\phi\sin\theta & \cos\theta \end{bmatrix}$$

The matrix multiplication is associative.

Problem 2. Consider the rotation in the following order: first rotate around the x axis by an angle θ, then around z' axis by an angle ψ, and finally around the old z axis by an angle ϕ. Does this lead to the same transformation matrix as that in Problem 1? Do you have an explanation for this?

Solution. Let x denotes the original coordinate system. Let ξ, η, and ζ denote three coordinate systems after each rotation, and let $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, and $\boldsymbol{\zeta}$ denote their basis vectors ($\mathbb{R}^{3\times 3}$), respectively. Use matrices B, C, and D from Problem 1. We can immediately know the change of basis matrices are

$$\mathscr{C}_{\xi x} = C,\ \mathscr{C}_{\eta\xi} = B,\ \mathscr{C}_{\eta x} = BC$$

However, it seems not straightforward to derive change of basis matirx related to ζ. To do so, let's first derive $\boldsymbol{\zeta}$. Note that

$$\boldsymbol{\zeta} = \mathscr{C}_{x\zeta} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{\eta} = D^T \mathscr{C}_{x\eta} = D^T \mathscr{C}_{x\xi}\mathscr{C}_{\xi\eta} = D^T C^T B^T$$

Hence the change of basis matrix $\mathscr{C}_{x\zeta}$ is

$$\mathscr{C}_{\zeta x} = \mathscr{C}_{x\zeta}^T = \boldsymbol{\zeta}^T = BCD = A$$

which is the same matrix as A in the Problem 1.

Problem 3. A particle is thrown up vertically with initial speed v_0, reaches a maximum and falls back to the ground. Show that the Coriolis deflection when it again reaches the ground is opposite in direction, and four times greater in magnitude, than the Coriolis deflection when it is dropped at rest from the same maximum height.

Solution. The Coriolis force is in the form

$$-2m\boldsymbol{\omega} \times \mathbf{V}_{\text{body}}$$

Only consider the z direction of $\mathbf{V}_{\text{body}}$, since the acceleration is constant (g), we have

$$[\mathbf{V}_{\text{body}}]_z = -gt + C$$

Ignore the centrifugal term, we have $\mathbf{a} = -\mathbf{k}t + \mathbf{c}$ for some constant vector $\mathbf{k}$ and $\mathbf{c}$, hence for two cases

i. Overall time $2T$ ($v_0 = gT$)

$$\mathbf{a} = \mathbf{k}T - \mathbf{k}t,\ \mathbf{v} = \mathbf{k}Tt - \frac{1}{2}\mathbf{k}t^2 + \frac{1}{2}\mathbf{k}T^2,\ \mathbf{x} = \frac{1}{2}kTt^2 - \frac{1}{6}\mathbf{k}T^3 + \frac{1}{2}\mathbf{k}T^2 t$$
$$\Rightarrow \mathbf{x}|_0^{2T} = \frac{5}{6}\mathbf{k}T^2$$

ii. Overall time T

$$\mathbf{a} = -\mathbf{k}t,\ \mathbf{v} = \frac{1}{2}\mathbf{k}t^2,\ \mathbf{x} = \frac{1}{6}\mathbf{k}t^3$$
$$\Rightarrow \mathbf{x}|_0^{T} = -\frac{1}{6}\mathbf{k}T^2$$

Hence the first case has four times greater in magnitude.

Problem 4. Prove that for a general rigid body motion about a fixed point, the kinetic energy T satisfies

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \boldsymbol{\omega} \cdot \mathbf{N} \tag{1}$$

Solution. For any point i in the rigid body rotating around one point, one should have

$$\mathrm{d}\mathbf{r}_i = \boldsymbol{\omega} \times \mathbf{r}_i \,\mathrm{d}t$$

Hence

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sum \frac{1}{2}m_i\mathbf{v}_i^2 &= \sum m_i\mathbf{v}_i \cdot \mathbf{a}_i \\ &= \sum m_i(\boldsymbol{\omega} \times \mathbf{r}_i) \cdot \mathbf{a}_i \\ &= \sum \boldsymbol{\omega} \cdot (\mathbf{r}_i \times m_i\mathbf{a}_i) \\ &= \boldsymbol{\omega} \cdot \sum \mathbf{r}_i \times m_i\mathbf{a}_i \\ &= \boldsymbol{\omega} \cdot \mathbf{N}\end{aligned}$$

Problem 5. A uniform sphere of mass M and radius R rotates around an axis through its center of mass. How it the kinetic energy of the sphere related to the angular velocity?

Solution. Let the z axis align onto $\boldsymbol{\omega}$, then the kinetic energy is

$$\begin{aligned}T &= \iint R^2 \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi \frac{M}{4\pi R^2}\frac{1}{2}(\omega R\cos\theta)^2 \\ &= \int_0^{2\pi}\mathrm{d}\phi \int_0^{\pi}\mathrm{d}\theta \frac{MR^2\omega^2}{8\pi}\sin^3\theta \\ &= \frac{MR^2\omega^2}{3}\end{aligned}$$