

PHY3110 FA22 HW02

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Problem 1. Let

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r \sin \theta \cos \phi \\ r \sin \theta \sin \phi \\ r \cos \theta \end{bmatrix}$$

and define the following three unit vectors

$$\hat{r} = \begin{bmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{bmatrix}, \quad \hat{\phi} = \begin{bmatrix} -\sin \phi \\ \cos \phi \\ 0 \end{bmatrix}, \quad \hat{\theta} = \begin{bmatrix} \cos \theta \cos \phi \\ \cos \theta \sin \phi \\ -\sin \theta \end{bmatrix}$$

easy to show that \hat{r} , $\hat{\phi}$ and $\hat{\theta}$ forms an orthonormal basis in \mathbb{R}^3 .

Note that

$$\mathbf{F} = m\ddot{\mathbf{r}} = m \frac{d^2}{dt^2}(r\hat{r}) = m \frac{d}{dt}(\dot{r}\hat{r} + r\dot{\theta} \sin \theta \hat{\phi} + \dot{\theta} r \hat{\theta})$$

Note that

$$\begin{aligned} \frac{d}{dt}\dot{r}\hat{r} &= \ddot{r}\hat{r} + \dot{r}\dot{\theta} \sin \theta \hat{\phi} + \dot{r}\dot{\theta} \hat{\theta} \\ \frac{d}{dt}r\dot{\theta} \sin \theta \hat{\phi} &= -r\dot{\phi}^2 \sin^2 \theta \hat{\theta} + (r\ddot{\phi} \sin \theta + \dot{r}\dot{\phi} \sin \theta + r\dot{\phi}\dot{\theta} \cos \theta)\hat{\phi} \\ -r\dot{\phi}^2 \sin \theta \cos \theta \frac{d}{dt}r\dot{\theta} \hat{\theta} &= -r\dot{\theta}^2 \hat{r} + r\dot{\phi}\dot{\theta} \cos \theta \hat{\phi} + (r\ddot{\theta} + \dot{r}\dot{\theta})\hat{\theta} \end{aligned}$$

Hence the Newton's 2nd law becomes

$$\mathbf{F} = m[(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2 \theta)\hat{r} + (r\ddot{\phi} + 2\dot{r}\dot{\phi} \sin \theta + 2r\dot{\theta}\dot{\phi} \cos \theta)\hat{\phi} + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2 \sin \theta \cos \theta)\hat{\theta}]$$

Problem 2. Let L be the distance from the hanging point to the center of the ball. We should have $L > R$. Define the generalized coordinate θ , we can express the potential as

$$V(\theta) = -\cos \theta m_1 g L - [(C - \sqrt{L^2 - R^2} - (\theta_0 - \theta)R) + \cos(\theta_0 - \theta)\sqrt{L^2 - R^2} + R \sin(\theta_0 - \theta)]m_2 g$$

where $\theta_0 = \arcsin(R/L)$. Applying D'Alembert's principle, we would know that

$$\sum_i Q_i \delta q_i = 0 \Rightarrow -\frac{\partial V}{\partial \theta} \delta \theta = 0$$

Note that

$$\begin{aligned} \frac{\partial V}{\partial \theta} &= m_1 g L \sin \theta - m_2 g R + m_2 g \sqrt{L^2 - R^2} \sin(\theta - \theta_0) - m_2 g R \cos(\theta - \theta_0) \\ &= gL[(m_1 + m_2) \sin \theta - m_2 \sin \theta_0] \end{aligned}$$

Then

$$gL[(m_1 + m_2) \sin \theta - m_2 \sin \theta_0] = 0 \Rightarrow \theta = \arcsin\left(\frac{m_2 \sin \theta_0}{m_1 + m_2}\right)$$

where $\theta \in [0, \theta_0]$.

Problem 3. Define the Lorentz factor $\gamma = \sqrt{1 - \dot{x}^2/c^2}$, then

$$L = -\frac{m_0 c^2}{\gamma} - V$$

Since

$$\begin{aligned}\frac{\partial L}{\partial x} &= -\frac{\partial V}{\partial x} \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} &= \frac{d}{dt} m_0 \dot{x} \gamma = m_0 \ddot{x} \gamma + m_0 \dot{x} \gamma^3 \frac{\dot{x}}{c^2} = m_0 \ddot{x} \gamma^3\end{aligned}$$

Thus the equation of motion writes

$$m_0 \ddot{x} \gamma^3 + \frac{\partial V}{\partial x} = 0$$

Problem 4. Since for rigid body, the kinetic energy is

$$K = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2$$

where v is the velocity of center of mass and I is the rotational inertia at COM.

Define two generalized coordinate: x the horizontal position of center of mass, θ the acute angle formed between the rod and the ground. For the rod, we have $\mathbf{v} = \dot{x}\hat{x} + l\dot{\theta} \cos \theta \hat{y}$, $I = ml^2/3$, then we can calculate the kinetic and potential energy

$$\begin{aligned}K &= \frac{1}{2} m [\dot{x}^2 + (l\dot{\theta} \cos \theta)^2] + \frac{1}{6} ml^2 \dot{\theta}^2 \\ V &= \int_0^{2l} \rho ds = \int_0^{2l \cos \theta} \frac{m}{2l} \frac{1}{\cos \theta} u du = mgl \cos \theta\end{aligned}$$

Hence we can define the Lagrangian and get the equation of motion.

$$\begin{aligned}L &= \frac{1}{2} m [\dot{x}^2 + (l\dot{\theta} \cos \theta)^2] + \frac{1}{6} ml^2 \dot{\theta}^2 - mgl \cos \theta \\ \frac{\partial L}{\partial x} &= 0 \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} &= m\ddot{x} \\ \frac{\partial L}{\partial \theta} &= -ml^2 \dot{\theta}^2 \sin \theta \cos \theta + mgl \sin \theta \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} &= \ddot{\theta} \left(ml^2 \cos^2 \theta + \frac{1}{3} ml^2 \right) - 2ml^2 \dot{\theta}^2 \sin \theta \cos \theta \\ &\Rightarrow \ddot{\theta} \left(ml^2 \cos^2 \theta + \frac{1}{3} ml^2 \right) - 2ml^2 \dot{\theta}^2 \sin \theta \cos \theta + ml^2 \dot{\theta}^2 \sin \theta \cos \theta - mgl \sin \theta = 0\end{aligned}$$

Hence the invariant quantities are \dot{x} and the total energy.