PHY3110 SP23 Notes

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Contents

0 Introduction

Grading: 30% homework, 30% midterm, 40% final.

Textbooks:

- H. Goldstein, C. Poole, J. Safko, Classical Mechanics, 3rd Edition, Pearson.
- J.R. Taylor, Classical Mechanics, University Science Books.
- T.W.B. Kibble, F.H. Berkshire, Classical Mechanics, 5th Edition, Imperial College Press.
- 梁昆淼，力学（下册）理论力学，4^{th} Edition，高等教育出版社.

Classical mechanics describe the motion of macroscopic objects, which are not extremely massive and not extremely fast.

1 Newtonian Mechanics

Vectorial quantities of motion: position $\mathbf{r}$, velocity $\mathbf{v}$, force $\mathbf{F}$, momentum $\mathbf{p} = m\mathbf{v}$, angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. Equations of motion are derived from those vector quantities.

Analytical mechanics uses scalar quantities of motion

- Kinetic energy $T = \frac{1}{2}m\mathbf{v}^2$
- Potential energy $V = V(\mathbf{r})$

Equations of motion are derived from those scalar quantities.

1.1 Newton's Laws

Theorem 1.1 (Newton's 2^{nd} law).

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = m\mathbf{a} \tag{1}$$

The formula is valid in an inertial frame.

Angular momentum $\mathbf{L}$ and torque $\mathbf{N}$ are also related

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \times \mathbf{p}) = \mathbf{r} \times \mathbf{F} = \mathbf{N} \tag{2}$$

Work done by external forces (note that $\mathrm{d}\mathbf{s} = \mathbf{v}\,\mathrm{d}t$)

$$W_{12} = \int_1^2 \mathbf{F}\,\mathrm{d}\mathbf{s} = \int_1^2 m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\,\mathrm{d}\mathbf{s} = \int_1^2 m\mathbf{v}\,\mathrm{d}\mathbf{v} = \left.\frac{1}{2}m\mathbf{v}^2\right|_1^2 \tag{3}$$

Define a scalar function $V(\mathbf{r})$, then $\mathbf{F} = -\nabla V(\mathbf{r})$ is a conservative force.

$$\oint \mathbf{F}\,\mathrm{d}\mathbf{s} = 0 \tag{4}$$

Center of mass of the system

$$\mathbf{R} = \frac{\sum_i m_i\mathbf{r}_i}{\sum_i m_i} = \frac{\sum_i m_i\mathbf{r}_i}{M} \tag{5}$$

Total momentum

$$\mathbf{P} = \sum_i m_i \mathbf{p}_i = M\dot{\mathbf{R}} \tag{6}$$

Hence $\mathbf{P}$ is conserved if external force $\mathbf{F}^{(e)}$ is zero.
Total angular momentum

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\sum_i \mathbf{r}_i \times \mathbf{p}_i = \sum_i \mathbf{r}_i \times \left(\mathbf{F}_i^{(e)} + \sum_j \mathbf{F}_{ij}\right) = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{(e)} + \sum_{ij} \mathbf{r}_i \times \mathbf{F}_{ij}$$

Since $\mathbf{r}_{ij}$ parallel to $\mathbf{F}_i j$, then

$$\sum_{ij} \mathbf{r}_i \mathbf{F}_{ij} = \frac{1}{2}\sum_{ij} \mathbf{r}_{ij} \times \mathbf{F}_{ji} = 0 \tag{7}$$

Therefore

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \mathbf{N}^{(e)} \tag{8}$$

Decomposition of the angular momentum

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i = \sum_i (\mathbf{R} + \mathbf{r}_i) \times m_i(\mathbf{V} + \mathbf{v}_i') = \sum_i \mathbf{R} \times m_i \mathbf{V} + \sum_i \mathbf{r}_i' \times m_i \mathbf{v}_i' \tag{9}$$

1.2 Constraints

Holonomic constraint

$$f(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N, t) = 0 \tag{10}$$

Example: rigid body

$$(\mathbf{r}_i - \mathbf{r}_j)^2 - c_{ij}^2 = 0 \tag{11}$$

Example: non-sliding cylinder

$$\dot{x} - R\dot{\theta} = 0 \Rightarrow x - R\theta = \text{const}$$

A constraint of the form

$$\sum_i g_i(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)\, \mathrm{d}\mathbf{x}_i = 0 \Rightarrow \mathrm{d}G(\mathbf{x}_1, \dots) = 0 \Rightarrow G(\mathbf{x}_1, \dots) = \text{const} \tag{12}$$

Non-holonomic constraint: cannot be written in the form of holonomic constraint.

1.3 Generalized coordinates

Suppose we have a N-particle system, we will have $3N$ DOFs. With k constraints, we will have $3N - k$ DOFs. Define $q_1, \dots, q_{3N-k}$ generalized coordinates, we have

$$\mathbf{r}_i = \mathbf{r}_i(q_1, \dots, q_{3N-1}, t) \tag{13}$$

2 Lagrange Formalism

2.1 D'Alembert's Principle

Hint from the rigid body: internal forces of constraints do not work.

Virtual displacement: $\delta\mathbf{r}_i$ is consistent with the constraints imposed on the system at a given time

$$\mathbf{r}_i \to \mathbf{r}_i + \delta\mathbf{r}_i \tag{1}$$

Theorem 2.1 (D'Alembert's principle)**.** Consider a system in equilibrium

$$\mathbf{F}_i = 0 \Rightarrow \sum_i \mathbf{F}_i \cdot \delta\mathbf{r}_i = 0 \tag{2}$$

Separate $\mathbf{F}_i = \mathbf{F}_i^{(a)} + \mathbf{f}_i$ where $\mathbf{f}_i$ is the constraint force. Hence

$$\sum_i (\mathbf{F}_i^{(a)} + \mathbf{f}_i) \cdot \delta\mathbf{r}_i = 0 \Rightarrow \sum_i \mathbf{F}_i^{(a)} \cdot \delta\mathbf{r}_i = 0 \tag{3}$$

For a system moving under external forces

$$\mathbf{F}_i - \dot{\mathbf{p}}_i = 0 \Rightarrow \sum_i (\mathbf{F}_i - \dot{\mathbf{p}}_i)\delta\mathbf{r}_i = 0 \Rightarrow \sum_i (\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i)\delta\mathbf{r}_i = 0 \tag{4}$$

For holonomic constraints

$$\mathbf{r}_i = \mathbf{r}_i(q_1, \dots, q_n, t), \quad \mathbf{v}_i = \frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t} = \frac{\partial \mathbf{r}_i}{\partial t} + \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j}\dot{q}_j, \quad \delta\mathbf{r}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j}\delta q_j \tag{5}$$

Define generalized force Q_j

$$\sum_i \mathbf{F}_i \delta\mathbf{r}_i = \sum_{ij} \mathbf{F}_i \frac{\partial \mathbf{r}_i}{\partial q_j}\delta q_j = \sum_j Q_j \delta q_j \tag{6}$$

Then

$$\sum_i \dot{\mathbf{p}}_i \cdot \delta\mathbf{r}_i = \sum_{ij} m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}\delta q_j = \sum_{ij}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}\right) - m_i \dot{\mathbf{r}}_i \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathbf{r}_i}{\partial q_j}\right]\delta q_j \tag{7}$$

$$= \sum_j \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right]\delta q_j = \sum_j Q_j \delta q_j \tag{8}$$

$$\tag{9}$$

where

$$\frac{\partial T}{\partial q_j} = \sum_{ik} \frac{\partial \dot{\mathbf{r}}_k}{\partial q_j}\frac{\partial}{\partial \dot{\mathbf{r}}_k}T = \sum_k m_k \dot{\mathbf{r}}_k \frac{\partial \dot{\mathbf{r}}_k}{\partial q_j} = \sum_k m_k \dot{\mathbf{r}}_k \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathbf{r}_k}{\partial q_j} \tag{10}$$

Hence $\forall j$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j = 0 \tag{11}$$

Let the potential energy $V = V(\mathbf{r}_i, \dots) = V(q_j, \dots)$, then we have

$$Q_j = \sum_i \mathbf{F}_i \frac{\partial \mathbf{r}_i}{\partial q_j} = \sum_i -\nabla_i V \frac{\partial \mathbf{r}_i}{\partial q_j} = -\frac{\partial V}{\partial q_j} \tag{12}$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial(T-V)}{\partial\dot{q}_j}\right)-\frac{\partial(T-V)}{\partial q_j}-Q_j=0 \tag{13}$$

Theorem 2.2 (Langrange's equation)**.** Define $L=T-V$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}_j}\right)-\frac{\partial L}{\partial q_j}=0 \tag{14}$$

The choice of Lagrangian is not unique, L' where

$$L'=L+\frac{\mathrm{d}F(q,t)}{\mathrm{d}t} \tag{15}$$

will give the same equations of motion as L.

Example 2.1 (Lagrange's formalism)**.**

1) For a single particle moving under force $\mathbf{F}$

$$L(\mathbf{x},\dot{\mathbf{x}},t)=\frac{1}{2}m\dot{\mathbf{x}}^2+\mathbf{F}\cdot\mathbf{x}$$

2) Motion in a 2D plane using polar coordinates

$$L(r,\theta,\dot{r},\dot{\theta},t)=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)+F\cdot\mathbf{r}$$

Generalized forces

$$\begin{aligned}Q_r&=\mathbf{F}\cdot\frac{\partial\mathbf{r}}{\partial r}=\mathbf{F}\cdot\mathbf{e}_r\\ Q_\theta&=\mathbf{F}\cdot\frac{\partial\mathbf{r}}{\partial\theta}=\mathbf{F}\cdot r\mathbf{e}_\theta\end{aligned}$$

where

$$\mathbf{e}_r=\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}\quad \mathbf{e}_\theta=\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$$

Equations of motion

$$\begin{aligned}m\ddot{r}-mr\dot{\theta}^2&=\mathbf{F}\cdot\mathbf{e}_r\\ mr^2\ddot{\theta}+2mr\dot{r}\dot{\theta}&=r\mathbf{F}_\theta\end{aligned}$$

3) Atwood's machine

$$L=\frac{1}{2}(M_1+M_2)\dot{x}^2+M_1gx+M_2g(l-x)$$

The equation of motion is

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{x}}-\frac{\partial L}{\partial x}=0\\ \Rightarrow\ &(M_1+M_2)\ddot{x}=(M_1-M_2)g\end{aligned}$$

Suppose we have a potential dependent on velocity (generalized potential) and the generalized force is defined as

$$U=U(q_j,\dot{q}_j),\quad Q_j=-\frac{\partial U}{\partial q_j}+\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial U}{\partial\dot{q}_j} \tag{16}$$

Define $L = T - U$, then we still have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \tag{17}$$

Example 2.2 (Lorentz force on a moving charge)**.** The Lorentz force

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Define the scalar and vector potentials

$$E = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

Hence

$$\mathbf{F} = q\left[-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \times (\nabla \times \mathbf{A})\right]$$

2.2 Hamilton's principle, variational principle

Configuration space: a space formed by the set of generalized coordinates.

$$(q_1, q_2, \dots, q_n) \text{ as function of } t \tag{18}$$

Theorem 2.3 (Hamilton's principle)**.** Define the action integral I, where $L = T - V$ or $L = T - U$ (U is the generalized potential)

$$I = \int_{t_1}^{t_2} L \,\mathrm{d}t \tag{19}$$

Then the variation of the action integral equals to zero

$$\delta I = \delta \int_{t_1}^{t_2} L(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n)\,\mathrm{d}t = 0 \tag{20}$$

Add small variation on the path

$$q_i(t, \alpha) = q_i(t) + \alpha\eta(t) \tag{21}$$

where $\eta(t_1) = \eta(t_2) = 0$. Then the action will be the function of α, $I = I(\alpha)$. Hence

$$\delta I = \int_{t_1}^{t_2} \left(\sum_i \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i\right)\mathrm{d}t \tag{22}$$

Change the order of differentiation $\delta\dot{q}_i = \mathrm{d}\delta q_i / \mathrm{d}t$, then

$$\delta I(\alpha) = \int_{t_1}^{t_2} \sum_i \left[\frac{\partial L}{\partial q_i}\delta q_i - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i\right]\mathrm{d}t + \sum_i \frac{\partial L}{\partial \dot{q}_i}\delta q\Big|_{t_1}^{t_2} \tag{23}$$

$$= \int_{t_1}^{t_2} \sum_i \left[\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right]\delta q_i \,\mathrm{d}t = 0 \tag{24}$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{25}$$

Example 2.3 (Shortest path problem). $y = y(x)$, $\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$, then the action integral (path) is

$$I = \int_1^2 \mathrm{d}s = \int_1^2 \sqrt{1 + \dot{y}^2}\,\mathrm{d}x \tag{26}$$

Apply the Lagrange's equation we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}\sqrt{1+\dot{y}^2}}{\mathrm{d}\dot{y}} = 0 \Rightarrow \frac{\mathrm{d}\dot{y}}{\mathrm{d}x} = 0 \Rightarrow y = ax + b \tag{27}$$

Example 2.4 (Solid of revolution). Differential of area $2\pi x\,\mathrm{d}s = 2\pi x\sqrt{1+\dot{y}^2}\,\mathrm{d}x$, then the total area is

$$\int_1^2 2\pi x\sqrt{1+\dot{y}^2}\,\mathrm{d}x \tag{28}$$

Define the Lagrangian $L(x, y, \dot{y}) = 2\pi x\sqrt{1+\dot{y}^2}$, by Lagrange's equation we can get

$$\frac{x\dot{y}}{\sqrt{1+\dot{y}^2}} = \text{const} \Rightarrow y = a\cosh^{-1}\frac{x}{a} + b \tag{29}$$

Example 2.5 (The curve of fastest descent). Question: along which trajectory from point 1 to point 2, the time is shortest? The total time is

$$T = \int_1^2 \frac{\mathrm{d}s}{v} = \int_1^2 \frac{\mathrm{d}s}{\sqrt{2gy}} \tag{30}$$

According to Newton's laws we have $y = gv^2$, then

$$T = \int_1^2 \frac{\sqrt{1+\dot{y}^2}}{\sqrt{2gy}}\,\mathrm{d}x \tag{31}$$

Then we have $L(x, y, \dot{y})$ and we get

$$2y\ddot{y} + 1 + \dot{y}^2 = 0 \tag{32}$$

$$\Rightarrow 2y\dot{y}\ddot{y} + \dot{y} + \dot{y}^3 = \dot{y}(1+\dot{y}^2) + y(2\dot{y}\ddot{y}) = \frac{\mathrm{d}}{\mathrm{d}t}\dot{y}(1+\dot{y}^2) = 0 \tag{33}$$

$$\Rightarrow y(1+\dot{y}^2) = \text{const} \tag{34}$$

which means that $y(1+\dot{y}^2) = \text{const}$. The solution is $x = A(\theta - \sin\theta)$, $y = A(1 - \cos\theta)$.

2.3 Constraint

Holonomic constraints

$$f(q_1, \ldots, q_n, t) = 0 \tag{35}$$

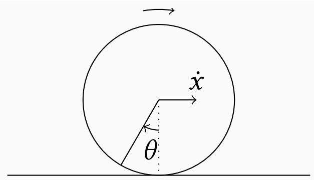
Non-holonomic constraints

$$f(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t) = 0 \tag{36}$$

Sometimes we can convert $f(\dot{q}_i) = 0$ to $f'(q_i) = 0$.

Example 2.6 (Rolling cylinder). Rolling cylinder without sliding has the constraint

$$\dot{x} - R\dot{\theta} = 0 \Rightarrow x - R\theta = \text{const} \tag{37}$$



A commonly encountered type of non-holonomic constraint is linear constraint equations

$$\sum_{k=1}^{n} a_{ik}\frac{\mathrm{d}q_k}{\mathrm{d}t} + a_i t = 0 \tag{38}$$

For the virtual displacement δq_i

$$\sum_{k=1}^{n} a_{ik}\delta q_k = 0 \tag{39}$$

Suppose q_i are not independent, then the Euler-Lagrange's equation could not hold.

$$\int_{t_1}^{t_2}\sum_i \left[\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right]\delta q_i \,\mathrm{d}t = 0 \nRightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q_i} = 0$$

Include the constraint equation of q_i into the equation

$$\sum_{k=1}^{n} a_{ik}\delta q_k = 0 \Rightarrow \sum_{i=1}^{m}\lambda_i\sum_{k=1}^{n} a_{ik}\delta q_k = 0 \tag{40}$$

Hence

$$\int_{t_1}^{t_2}\sum_{k=1}^{n}\left[\frac{\partial L}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) + \sum_{i=1}^{m}\lambda_i a_{ik}\right]\delta q_k \,\mathrm{d}t = 0$$

Let $q_1, \dots, q_{n-m}$ be independent generalized coordinate, $q_{n-m+1}, \dots, q_n$ dependent generalized coordinates (i.e., they can be expressed by $q_1, \dots, q_{n-m}$). Choose λ_i s.t.

$$\frac{\partial L}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) + \sum_{i=1}^{m}\lambda_i a_{ik} = 0 \tag{41}$$

$\forall k = n-m+1, \dots, n$. In conclusion, we have $q_1, \dots, q_n, \lambda_1, \dots, \lambda_m$ overall $n+m$ unknowns, and n Lagrange's equations and m constraint equations overall $n+m$ equations.

Remark.

1) It is inconvenient to reduce all q_ks to independent coordinates
2) If we are interested in the constraint forces

$$\sum_{k=1}^{n} a_{ik}\,\mathrm{d}q_k + a_{it}\,\mathrm{d}t = 0$$

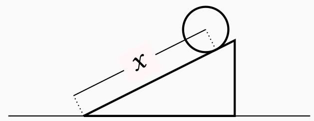
where

$$a_{ik} = \frac{\partial f_i}{\partial q_k}, \quad a_{it} = \frac{\partial f_i}{\partial t}$$

Then

$$\mathrm{d}f_i = \frac{\partial f_i}{\partial q_k}\mathrm{d}q_k + \frac{\partial f_i}{\partial t}\mathrm{d}t \Rightarrow \mathrm{d}f_i = 0,\ f_i = \text{const}$$

Example 2.7 (Hoop rooling down an inclined plane).



The constraint equation writes

$$\dot{x} - r\dot{\theta} = 0 \Rightarrow x - r\theta = \text{const}$$

$a_x = 1$, $a_\theta = -r$, $a_t = 0$.
Energy terms are

$$\begin{aligned} T &= T_{\text{COM}} + T_{\text{relative}} = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}M(r\dot{\theta})^2 \\ V &= Mg(l - x)\sin\phi \\ \Rightarrow L &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}M(r\dot{\theta})^2 - Mg(l - x)\sin\phi \end{aligned}$$

Then we can write the Lagrange's equation with Lagrange's multipliers

$$\begin{aligned} \frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} + a_x\lambda = 0 \\ \Rightarrow Mg\sin\phi - M\ddot{x} + \lambda = 0 \\ -Mr^2\ddot{\theta} - \lambda r = 0 \\ \dot{x} = r\dot{\theta} \Rightarrow \ddot{x} = r\ddot{\theta} \end{aligned}$$

we can get $M\ddot{x} = Mr\ddot{\theta} = -\lambda$ and $\ddot{x} = (g\sin\phi)/2$. Note that λ is the constraint force (in this case λ is the frictional force).

2.4 Lagrangian for Lorentz force

The Lorentz force is given by

$$\mathbf{F} = q[\mathbf{E} + \mathbf{v} \times \mathbf{B}] \tag{42}$$

Where

$$\mathbf{E} = -\nabla\phi - \frac{\partial A}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A} \tag{43}$$

Hence, by defining the Lagrangian

$$L = \frac{1}{2}m\mathbf{v}^2 - q\phi + q\mathbf{A} \cdot \mathbf{v} \tag{44}$$

Applying the Lagrange's equation we can get the EOM (eq. 42). Take x as an example

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} = \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x} + qA_x) = m\ddot{x} + q\frac{\partial A_x}{\partial t} + \mathbf{v} \cdot \nabla A_x \tag{45}$$

2.5 Conservation & symmetry of the system

Definition 2.1 (Cylic coordinate). The generalized coordinate q_i is cyclic (ignorable) if

$$\frac{\partial L}{\partial q_i} = 0 \tag{46}$$

It implies the generalized momentum p_i is conserved.

Rotational symmetry. Let q_j be one of the rotational angle of spacial coordinate $\mathbf{r}_i$. Hence

$$\mathrm{d}\mathbf{r}_i = \mathbf{n}_j \times \mathbf{r}_i \,\mathrm{d}q_j \Rightarrow \frac{\partial \mathbf{r}_i}{\partial q_j} = \mathbf{n}_j \times \mathbf{r}_i \,\mathrm{d}q_j \tag{47}$$

where $\mathbf{n}_j$ is the normal vector of the rotation axis of q_j. Hence the generalized force of q_j writes

$$Q_{q_j} = -\frac{\partial V}{\partial q_j} = -\sum_k \frac{\partial V}{\partial \mathbf{r}_k}\frac{\partial \mathbf{r}_k}{\partial q_j} = -\frac{\partial V}{\partial \mathbf{r}_i}\frac{\partial \mathbf{r}_i}{\partial q_j} = \mathbf{F}_i \cdot (\mathbf{n}_j \times \mathbf{r}_i) = \mathbf{n}_j \cdot (\mathbf{r}_i \times \mathbf{F}_i) = \mathbf{n}_j \cdot \mathbf{N}_i \tag{48}$$

where $\mathbf{N}_i$ stands for the torque on the i^{th} particle. The generalized momentum of q_j writes

$$p_{q_j} = \frac{\partial T}{\partial \dot{q}_j} = \sum_k \frac{\partial T}{\partial \dot{\mathbf{r}}_k}\frac{\partial \mathbf{r}_i}{\partial q_j} = \sum_k m_k \dot{\mathbf{r}}_k \cdot \frac{\partial \dot{\mathbf{r}}_k}{\partial \dot{q}_j} = m_i \dot{\mathbf{r}}_i \cdot (\mathbf{n}_j \times \mathbf{r}_i) = \mathbf{n}_j \times (\mathbf{r}_i \times m_i \dot{\mathbf{r}}_i) = \mathbf{r}_j \times \mathbf{L}_i \tag{49}$$

where $\mathbf{L}_i$ stands for the angular momentum of the i^{th} particle. Hence, the rotational invariance implies the conservation of angular momentum.

Time translation

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} L &= \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial L}{\partial t} &(50)\\
&= \sum_i \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial L}{\partial t} &(51)\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\sum_i \left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) + \frac{\partial L}{\partial t} &(52)\\
\Rightarrow \frac{\partial L}{\partial t} &+ \frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\left(\sum \frac{\partial L}{\partial \dot{q}_i} q_i - L\right)}_{H} = 0 &(53)
\end{aligned}$$

Note that

$$\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i = 2T \tag{54}$$

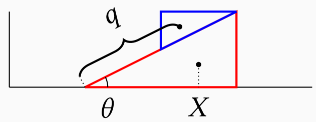
Proof. Suppose r_i does not have explicit time dependence ($\partial r_i/\partial t = 0$)

$$\begin{aligned}
T &= \sum_i \frac{1}{2} m_i \dot{r}_i^2 = \sum_i \frac{1}{2} m_i \left(\sum_j \frac{\partial r_i}{\partial q_j}\dot{q}_j + \frac{\partial r_i}{\partial t}\right)^2\\
&= \sum_{ijk} \frac{1}{2} m_i \frac{\partial r_i}{\partial q_j}\frac{\partial r_i}{\partial q_k}\dot{q}_j \dot{q}_k\\
\Rightarrow \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i &= \sum_i \dot{q}_i \sum_{jk} m_k \frac{\partial r_k}{\partial q_i}\frac{\partial r_k}{\partial q_j}\dot{q}_j = \sum_{ijk} m_k \frac{\partial r_k}{\partial q_i}\frac{\partial r_k}{\partial q_j}\dot{q}_i\dot{q}_j = 2T
\end{aligned}$$

□

Hence we can define Hamiltonian $H = T + V$ which stands for the total energy, and H conserved if L doesn't depend on time explicitly.

Example 2.8 (Two blocks). Let M be the mass of the big block, m be the mass of the small block. Define two generalized coordinates: X stand for the position of COM of the big block, q stand for the position of COM of the small block (sloped).



Hence we can easily define the Lagrangian

$$L = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m[(\dot{X} + \dot{q}\cos\theta)^2 + \dot{q}^2\sin^2\theta] - mgq\sin\theta$$

There is no X dependence on the system, hence X is a cyclic coordinate.

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{X}} = \frac{\mathrm{d}}{\mathrm{d}t}M\dot{X} + m(\dot{X} + \dot{q}\cos\theta) = 0$$
$$\Rightarrow M\dot{X}m(\dot{X} + \dot{q}\cos\theta) = p_X = \text{const}$$

3 The Central Force Problem

3.1 Reduction to the equivalent one-body problem

For the two-body problem, we have two choices of generalized coordinate

(a) $\mathbf{r}_1, \mathbf{r}_2$ stand for spacial coordinates of two masses

(b) $\mathbf{R}$ spacial coordinate of the COM, $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

Hence, we can rewrite the kinetic energy in terms of $\mathbf{R}$ and $\mathbf{r}$

$$
\begin{aligned}
T &= \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 && (1)\\
&= \frac{1}{2}m_1\left(\dot{\mathbf{R}} + \frac{m_2}{m_1+m_2}\dot{\mathbf{r}}\right)^2 + \frac{1}{2}m_2\left(\dot{\mathbf{R}} - \frac{m_1}{m_1+m_2}\dot{\mathbf{r}}\right)^2 && (2)\\
&= \frac{1}{2}(m_1+m_2)\dot{\mathbf{R}}^2 + \frac{m_1m_2}{m_1+m_2}\dot{\mathbf{r}}^2 = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 && (3)
\end{aligned}
$$

Hence we can write the Lagrangian

$$L = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 - V \tag{4}$$

Suppose $V = V(\mathbf{r})$, then $\mathbf{R}$ is a cyclic coordinate, We have $\dot{\mathbf{R}} =$ const, and we can drop $\dot{\mathbf{R}}$ terms in L. Moreover, if $V = V(\|\mathbf{r}\|)$, the total angular momentum is conserved.

Use r, θ as generalized coordinates, we have

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \tag{5}$$

The Lagrange's equation wrt r writes

$$\mu\ddot{r} - \mu r\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0 \tag{6}$$

Easy to find that θ is cyclic, hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\theta}} = \frac{\mathrm{d}}{\mathrm{d}t}\mu r^2\dot{\theta} = 0 \Rightarrow p_\theta = \mu r^2\dot{\theta} = l = \text{const} \tag{7}$$

Theorem 3.1 (Kepler's 2nd law)**.** Radius vector sweeps out equal areas in equal time.

Substitute $\mu r\dot{\theta}^2$ term by L, we have

$$
\begin{aligned}
\mu\ddot{r} - \frac{l^2}{\mu r^3} + \frac{\partial V}{\partial r} = 0 && (8)\\
\Rightarrow \mu\ddot{r} = -\frac{\partial}{\partial r}(V + \frac{l^2}{2\mu r^2}) && (9)\\
\mu\ddot{r}\dot{r} = -\frac{\partial}{\partial r}\dot{r}(V + \frac{l^2}{2\mu r^2}) && (10)\\
\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{2}\mu\dot{r}^2 + V + \frac{l^2}{2\mu r^2}) = \frac{\mathrm{d}}{\mathrm{d}t}E = 0 && (11)
\end{aligned}
$$

Hence we can get the expression of $\dot{r}$ and a differential equation

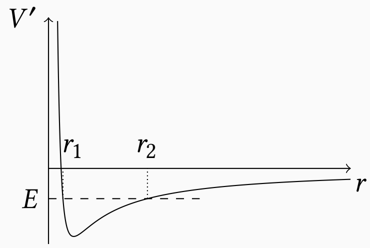
$$\dot{r} = \sqrt{\frac{2}{\mu}(E - V - \frac{l^2}{2\mu r^2})} \tag{12}$$

Define the new effective force $f'(r)$ with an effective potential $V' = V + l^2/2\mu r^2$, we have

$$\mu\ddot{r} = -\frac{\partial V}{\partial r} + \frac{l^2}{\mu r^3} = f'(r), \quad E = \frac{1}{2}\mu\dot{r}^2 + V' = \text{const} \tag{13}$$

Example 3.1 (Gravitational force)**.** Suppose we have $f(r) = -kr^{-2}$ and $V = -kr^{-1}$, then

$$V' = -\frac{k}{r} + \frac{l^2}{2\mu r^2}$$



1) $E = E_1 \geq 0$, the motion has a lower bound, $r \geq r_1$.
2) $V_{\min} < E_2 < 0$, the motion has lower and upper bound, $r_1 \leq r \leq r_2$.
3) $E_3 = V_{\min}$, the motion will shrink to a single circle $r_1 = r_2 = \text{const}$, hence it is a circular motion. In this case, the gravitational force is equal to the centrifugal force

$$\mu\ddot{r} = f(r) + \frac{l^2}{\mu r^3} = f(r) + \mu r\dot{\theta}^2 = 0 \Rightarrow f(r) = -\mu r\dot{\theta}^2$$

Remark. Let the potential be $V = -kr^{-\alpha}$, then the motion cannot have periodic behavior if $\alpha > 2$.

Example 3.2 (Harmonic oscillator)**.** $V = kr^2/2$, we have

$$V' = \frac{1}{2}kr^2 + \frac{l^2}{2\mu r^2}$$

Theorem 3.2 (Conditions for closed orbitals, Bertrand's theorem)**.** Stable orbitals require

$$\left.\frac{\partial^2 V'}{\partial r^2}\right|_{r=r_0} = -\frac{\partial f}{\partial r} + \left.\frac{3l^2}{mr^4}\right|_{r=r_0} > 0 \Rightarrow \left.\frac{\partial f}{\partial r}\right|_{r=r_0} < -\frac{3f(r_0)}{r_0} \tag{14}$$

Let $f = -kr^n$, then we have

$$-knr^{n-1} < 3kr^{n-1} \Rightarrow n > -3 \tag{15}$$

For a small perturbation from the minimum, we can write the effective potential as Taylor expansion

$$V'(r) = V'(r_0) + \left.\frac{\partial V'}{\partial r}\right|_{r=r_0}(r - r_0) + \frac{1}{2}\left.\frac{\partial^2 V'}{\partial r^2}\right|_{r=r_0}(r - r_0)^2 + \cdots \tag{16}$$

$$= V'(r_0) + \frac{1}{2}\left.\frac{\partial^2 V'}{\partial r^2}\right|_{r=r_0}(r - r_0)^2 + O(|r - r_0|^3) \tag{17}$$

Hence the EOM becomes

$$m\ddot{r} = -\frac{\partial V'}{\partial r} = -\left.\frac{\partial^2 V'}{\partial r^2}\right|_{r=r_0}(r - r_0) \tag{18}$$

$$\dot{\theta} = \frac{l}{mr^2} \tag{19}$$

The solution taks the form

$$u = u_0 + a\cos\beta\theta,\ u = \frac{1}{r},\ \beta^2 = \left.\frac{r}{f}\frac{\partial f}{\partial r} + 3\right|_{r=r_0} \tag{20}$$

For finite perturbation, the orbit can be a closed only if $\beta^2 = 1$ or 4. Then there are only two types of central-force scalar problem with the property that all bound orbitals are closed orbitals.

$$\left.\frac{r}{f}\frac{\partial f}{\partial r}\right|_{r=r_0} = \pm 2 \Rightarrow f = -kr^{-2} \text{ or } -kr \tag{21}$$

check the theorem on other books

3.2 Virial theorem

Consider a multi-particle system $1 \leq i \leq N$.

$$\dot{\mathbf{p}}_i = \mathbf{F}_i \tag{22}$$

Define new function

$$G = \sum_i \mathbf{p}_i \cdot \mathbf{r}_i \tag{23}$$

Then the time derivative of G writes

$$\frac{\mathrm{d}}{\mathrm{d}t}G = \sum_i m_i \dot{\mathbf{r}}_i^2 + \sum_i \mathbf{F}_i \cdot \mathbf{r}_i \tag{24}$$

Hence the time average of $\mathrm{d}G/\,\mathrm{d}t$ is

$$\frac{1}{\tau}\int_0^\tau \frac{\mathrm{d}}{\mathrm{d}t}G\,\mathrm{d}t = 2\overline{T} + \overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i} = \frac{1}{\tau}[G(\tau) - G(0)] \tag{25}$$

The time average equals to 0 if τ is the period of the periodic motion; even for non-periodic motion, if $\tau \to \infty$, the average will approach 0. Hence

$$2\overline{T} + \overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i} = 0 \Rightarrow \overline{T} = -\frac{1}{2}\underbrace{\overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i}}_{\text{virial of Clausius}} \tag{26}$$

Example 3.3 (Ideal gas)**.** Temperature $T \propto T_i$ kinetic energy of i^{th} particle.

$$T_i = \frac{3}{2}k_B T \tag{27}$$

Hence the time average of total kinetic energy is

$$\overline{\sum_i T_i} = \frac{3}{2}Nk_B T \tag{28}$$

The virial could be written as

$$\overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i} = \int \mathrm{d}\mathbf{F} \cdot \mathbf{r} \tag{29}$$

Note that

$$\mathrm{d}\mathbf{F} = -p\,\mathrm{d}A\mathbf{n}$$

Applying Gauss's law we have

$$\overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i} = -\int p\,\mathrm{d}A\ \mathbf{n} \cdot \mathbf{r} = -\int p\,\mathrm{d}V\ (\nabla \mathbf{r}) = -3pV \tag{30}$$

Hence by the equation we get $pV = nk_B T$.

Example 3.4 (Gravational force). Since $\mathbf{F}_i = -\nabla_i V$, given $V_i = ar_i^{n+1}$, we have

$$\overline{T} = \frac{1}{2}\overline{\sum_i \nabla_i V \cdot \mathbf{r}_i} = \frac{n+1}{2}\overline{\sum_i ar_i^{n+1}} = \frac{n+1}{2}\overline{V} \tag{31}$$

This theorem is particularly important when the potential energy is a homogeneous function of the coordinates, i.e.

$$U(\alpha\mathbf{r}_1, \dots, \alpha\mathbf{r}_n) = \alpha^k U(\mathbf{r}_1, \dots, \mathbf{r}_n) \tag{32}$$

Note that

$$\frac{\mathrm{d}U}{\mathrm{d}\alpha} = \sum_a \frac{\partial \alpha \mathbf{r}_a}{\partial \alpha} \cdot \frac{\partial U}{\partial \alpha \mathbf{r}_a} = \sum_a \frac{\mathbf{r}_a}{\alpha} \cdot \frac{\partial U}{\partial \mathbf{r}_a} = k\alpha^{k-1}U \tag{33}$$

Then the equation 26 becomes

$$2\overline{T} = k\overline{U} \tag{34}$$

Equivalently

$$\overline{U} = \frac{2E}{k+2}, \ \overline{T} = \frac{kE}{k+2} \tag{35}$$

3.3 Inverse-square force

Consider $f = -k/r^2$, $V = -k/r$, then we have EOM

$$\dot{r} = \left[\frac{2}{m}\left(E + \frac{k}{r} - \frac{l^2}{2mr^2}\right)\right]^{1/2} \Rightarrow \theta = \theta_0 + \int_{r_0}^{r_1} \frac{\mathrm{d}r}{r^2\left(\frac{2mE}{l^2} - \frac{2mV}{l^2} - \frac{1}{r^2}\right)^{1/2}} \tag{36}$$

Define $u = 1/r$, let $\theta_0 = 0$, then

$$\theta = \theta_0 - \int_{u_0}^{u_1} \frac{\mathrm{d}u}{\left(\frac{2mE}{l^2} + \frac{2mku}{l^2} - u^2\right)^{1/2}} \tag{37}$$

$$\Rightarrow \theta = -\arccos\frac{\frac{l^2u}{mk} - 1}{\sqrt{1 + \frac{2El^2}{mk^2}}}, \ r = \frac{\frac{l^2}{mk}}{1 + \sqrt{1 + \frac{2El^2}{mk^2}}\cos\theta} = \frac{r_0}{1 + \epsilon\cos\theta} \tag{38}$$

ϵ is called the eccentricity.

ϵ	Energy	Orbit
$\epsilon > 1$	$E > 0$	Hyperbola
$\epsilon = 1$	$E = 0$	Parabola
$0 < \epsilon < 1$	$E < 0$	Ellipse
$\epsilon = 0$	$E < 0$	Circle

The major semi-axis a and minor semi-axis b are given by

$$a = \frac{r_0}{1 - \epsilon^2} = -\frac{k}{2E}, \ b = a\sqrt{1 - \epsilon^2} = \frac{k}{\sqrt{-2mE}} \tag{39}$$

Since $mr^2\dot{\theta} = l$, then

$$\int_0^T \mathrm{d}A = \frac{1}{2}\int_0^T r^2\dot{\theta}\,\mathrm{d}t = \frac{1}{2}\int_0^T \frac{l}{m}\theta\,\mathrm{d}t = \frac{l}{2m}T = \pi ab \Rightarrow T = 2\pi a^{3/2}\sqrt{\frac{m}{k}} \tag{40}$$

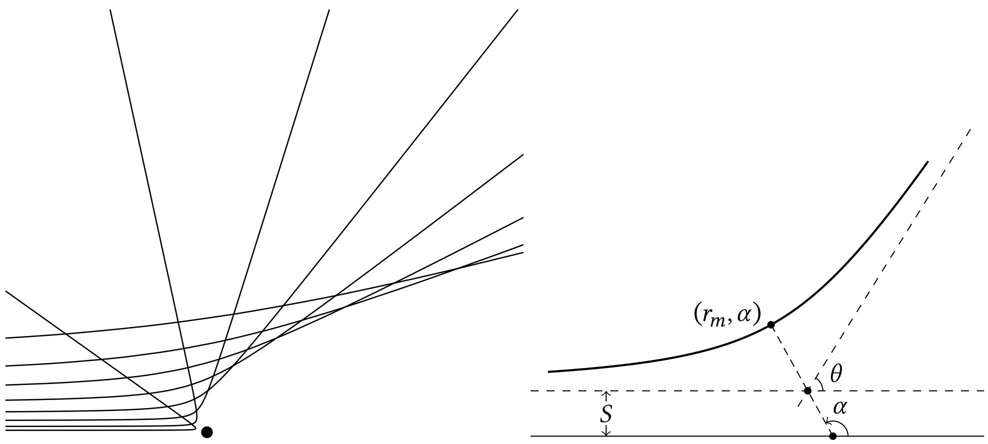
Alternatively

$$\int \mathrm{d}t = \int \frac{\mathrm{d}r}{\sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}} = \sqrt{\frac{ma}{k}} \int_{r_0}^{r_1} \frac{r\,\mathrm{d}r}{\sqrt{-r^2 + 2ar - b^2}} = \sqrt{\frac{ma}{k}} \int_{r_0}^{r_1} \frac{r\,\mathrm{d}r}{\sqrt{-(r-a)^2 + a^2\epsilon^2}} \tag{41}$$

Let $r - a = -a\epsilon\cos\zeta$, hence

$$t = \sqrt{\frac{ma}{k}} \int \frac{(a - a\epsilon\cos\zeta)a\epsilon\sin\zeta\,\mathrm{d}\zeta}{\sqrt{a\epsilon^2(1-\cos^2\zeta)}} = \sqrt{\frac{ma}{k}} \int (a - a\epsilon\cos\zeta)\,\mathrm{d}\zeta = \sqrt{\frac{ma^3}{k}}(\zeta - \epsilon\sin\zeta) + C \tag{42}$$

3.4 Scattering



Question: how many particles will be scattered in the given solid angle region (scattering cross section). Let $V \sim 1/r$, $f \sim 1/r^2$. Define the intensity of incident beam I

$$I = \#\text{ of particles crossing a unit area perpendicular to the beam in unit time}$$

The the number of particles scattered into $\mathrm{d}\Omega$ could be expressed as

$$\mathrm{d}N = \sigma I\,\mathrm{d}\Omega,\ \sigma = \frac{\mathrm{d}N}{I\,\mathrm{d}\Omega} \tag{43}$$

where σ is called differential cross section, which has the unit of area.

Particles in $[S, S + \mathrm{d}S]$ would be scattered into $[\Omega, \Omega + \mathrm{d}\Omega]$, since $\mathrm{d}\Omega = 2\pi\sin\theta\,\mathrm{d}\theta$

$$2\pi IS|\,\mathrm{d}S| = \sigma I\,\mathrm{d}\Omega = 2\pi\sigma I\sin\theta|\,\mathrm{d}\theta| \Rightarrow \sigma = \frac{S|\,\mathrm{d}S|}{\sin\theta|\,\mathrm{d}\theta|} \tag{44}$$

The angular momentum of the incoming particles (wrt. to force center) is

$$l = mv_0S = S\sqrt{2mE} \tag{45}$$

From the equation derived from central force problem

$$\alpha = \int \frac{\mathrm{d}r}{r^2\sqrt{\frac{2mE}{l^2} - \frac{rmV}{l^2} - \frac{1}{r^2}}} \tag{46}$$

$$= \pi + \int_{-\infty}^{r_m} \frac{\mathrm{d}r}{r^2\sqrt{\frac{2mE}{l^2} - \frac{rmV}{l^2} - \frac{1}{r^2}}} \tag{47}$$

Let $r_m \equiv r_{\text{min}}$ is the closest distance. Define ψ where

$$\alpha = \pi + \int_{-\infty}^{r_m} \frac{\mathrm{d}r}{r^2\sqrt{\frac{2mE}{l^2} - \frac{rmV}{l^2} - \frac{1}{r^2}}} = \pi - \psi \tag{48}$$

Hence $\theta = \pi - 2\psi$. Change the variable $u = 1/r$, then

$$\psi = \int_0^{u_m = 1/r_m} \frac{S\,\mathrm{d}u}{\sqrt{1 - \frac{V}{E} - S^2u^2}} \tag{49}$$

Also note that

$$\sin\frac{\theta}{2} = \sin\frac{\pi - 2\psi}{2} = \cos\psi = \frac{1}{\epsilon} \tag{50}$$

Example 3.5 (Coulomb interaction)**.** Let

$$f = \frac{ZZ'e^2}{r^2},\ V = \frac{ZZ'e^2}{r},\ r = \frac{r_0}{1 + \epsilon\cos(\alpha - \alpha')} \tag{51}$$

choose $\alpha' = \pi$. One can derive that

$$\epsilon = \sqrt{1 + \frac{2El^2}{m(ZZ'e^2)^2}} = \sqrt{1 + \left(\frac{2ES}{ZZ'e^2}\right)^2} \tag{52}$$

Using the trigonometric relationship between θ and ϵ, we have

$$S = \frac{ZZ'e^2}{2E}\cot\frac{\theta}{2},\ \left|\frac{\mathrm{d}S}{\mathrm{d}\theta}\right| = \frac{ZZ'e^2}{4E}\frac{1}{\sin^2\frac{\theta}{2}} \tag{53}$$

$$\Rightarrow \epsilon(\theta) = \left(\frac{ZZ'e^2}{4E}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}} \tag{54}$$

check if $\alpha' = \pi$ or $\alpha' = \alpha_m$

4 Rigid Body

4.1 Coordinates of rigid body

A rigid body is a system of point masses satisfying the constraint that distance between any two points is a constant (r_{ij} = const for all i, j). Let $\mathbf{r}_i$, $\mathbf{r}_j$, and $\mathbf{r}_k$ be three points in the rigid body. Let $\mathbf{r}_i$ has 3 DOFs, since r_ij is a constant, $\mathbf{r}_j$ has two DOFs. Hence, $\mathbf{r}_k$ only have 1 DOFs. The system has $3 + 2 + 1 = 6$ DOFs.

Or, alternatively, a rigid body has 3 coordinates for the origin of the coordinate system fixed on the rigid body, and 3 angular variables to specify the orientation of the rotated coordinate system.

4.2 Orthogonal transformations

Consider an orthogonal transformation $\mathbf{x} \mapsto \mathbf{x}'$

$$x_1' = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \tag{1}$$
$$x_2' = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \tag{2}$$
$$x_3' = a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \tag{3}$$

Let $\mathbf{x}^2 = \mathbf{x'}^2$, we have

$$\mathbf{x}^2 = x_i x_i = x_i' x_i' = a_{ij}x_j a_{ik}x_k = \delta_{jk}x_j x_k = x_j x_j \Rightarrow a_{ij}a_{ik} = \delta_{jk} \tag{4}$$

This equation implies the orthogonal transformations have three DOFs.

Different views on rotational transformation

1. Passive view the coordinate system is transformed
2. Active view: the vector is transformed

Consider two rotational matrix A and B, let $C = AB$, then

$$C_{ij}C_{ik} = A_{im}B_{mj}A_{in}B_{nk} = A_{im}A_{in}B_{mk}B_{nk} = \delta_{nm}B_{mj}B_{nk} = B_{nj}B_{nk} = \delta_{jk} \tag{5}$$

Non-commutative $AB \neq BA$.

Define the inverse transformation

$$x_i = a_{ij}' x_j' = a_{ij}' a_{jk} x_k \Rightarrow a_{ij}' a_{jk} = \delta_{ik} \tag{6}$$

Note that

$$\left.\begin{aligned} \underbrace{a_{im}a_{ij}}_{i} a_{jk}' &= \delta_{mk}a_{jk}' = a_{mk}' \\ a_{im}\,\underbrace{a_{ij}a_{jk}'}_{j} &= a_{im}\delta_{ik} = a_{km} \end{aligned}\right\} \Rightarrow a_{km} = a_{mk}' \tag{7}$$

4.3 Euler angles

A convenient choice of angle variables: Euler angles. Steps:

1. Rotate around z axis by an angle ϕ, rotational matrix D
2. Rotate around ξ axis by an angle θ, rotational matrix C
3. Rotate around ξ' axis by an angle ψ, rotational matrix B

We can always write any rotational matrix A as $A = BCD$

$$A = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ \sin\psi & -\cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ \sin\phi & -\cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8}$$

draw Euler angle figure

The general displacement of a rigid body with one point fixed is a rotation around some axis $\mathbf{R}$. Then

$$\mathbf{R}' = A\mathbf{R} = \mathbf{R} \Rightarrow (A - I)\mathbf{R} = 0,\ \ \det(A - I) = 0 \tag{9}$$