PHY3110 SP23 HW08

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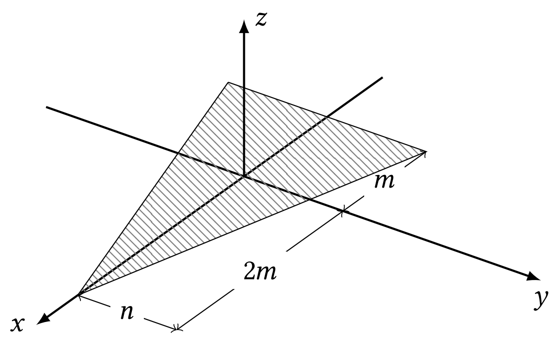
Problem 1. Find the principal moments of inertia about the center of mass of a flat rigid body in the shape of an isosceles triangle with a uniform mass density. What are the principal axes?

Solution. Easy to prove that the EOM of a uniform isosceles triangle is located at the 2/3 location of its height, hence we can set up a coordinate system as the following figure



Let the density be ρ, then the total mass $M = 3mn\rho$. Then we can compute the inertia tensor

$$
\begin{aligned}
I_{xx} &= \int_{-m}^{2m} \mathrm{d}x \int_{-n}^{n} \mathrm{d}y\, y^2\rho = \frac{2}{3}Mn^2 \\
I_{yy} &= \int_{-m}^{2m} \mathrm{d}x \int_{-n}^{n} \mathrm{d}y\, x^2\rho = 2Mm^2 \\
I_{xy} &= -\int_{-m}^{2m} \mathrm{d}x \int_{-n}^{n} \mathrm{d}y\, xy\rho = 0 \\
I_{zz} &= \int_{-m}^{2m} \mathrm{d}x \int_{-n}^{n} \mathrm{d}y\, x^2 + y^2\rho = \frac{2}{3}M(3m^2 + n^2) \\
I_{xz} &= I_{yz} = 0
\end{aligned}
$$

Hence x, y and z axes in the figure are principal axes.

Problem 2. Consider the torque-free motion of an asymmetric rigid body with one point fixed, show from Euler equations that L^2 and T (K and T are the angular momentum and kinetic energy) are conserved.

Solution. Note that $\mathbf{L}^2$ and its time derivative equals to

$$
\begin{aligned}
\mathbf{L}^2 &= 2L_iL_i \\
\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{L}^2 &= 2L_i\dot{L}_i
\end{aligned}
$$

The Euler's equation for torque-free motion is

$$\dot{L}_i + \epsilon_{ijk}\omega_j L_k = 0$$

Hence

$$
\begin{aligned}
&L_i\dot{L}_i + \epsilon_{ijk}L_i\omega_j L_k = 0 \\
&\qquad \Rightarrow L_i\dot{L}_i = -\epsilon_{ijk}L_iL_k\omega_j = \epsilon_{ikj}L_iL_k\omega_j = 0
\end{aligned}
$$

which means that $\mathbf{L}^2$ is conserved.

The kinetic energy and its time derivative writes

$$\begin{aligned} T &= \frac{1}{2}\omega_i I_{ij}\omega_j \\ \frac{\mathrm{d}}{\mathrm{d}t}T &= \frac{1}{2}\dot{\omega}_i I_{ij}\omega_j + \frac{1}{2}\omega_i I_{ij}\dot{\omega}_j = \omega_i I_{ij}\dot{\omega}_j = \omega_i \dot{L}_i \end{aligned}$$

Then from Euler's equation we know that

$$\begin{aligned} \omega_i\dot{L}_i + \epsilon_{ijk}\omega_i\omega_j L_k &= 0 \\ \Rightarrow \omega_i\dot{L}_i &= -\epsilon_{ijk}\omega_i\omega_j L_k = 0 \end{aligned}$$

which means that T is conserved.

Problem 3.

1) Express in terms of Euler's angles the constraint equations for a uniform sphere rolling without slipping on a flat horizontal surface. Show that they are nonholonomic.
2) Set up the Lagrangian equations for this problem by the method of Lagrange multipliers. Show that the translational and rotational parts of the kinetic energy are separately conserved. Are there any other constatns of motion?

Solution.

1) The constraint should be

$$\boldsymbol{\omega} \times \mathbf{R} + \mathbf{V} = 0$$

where $\boldsymbol{\omega}$ is the angular momentum vector, $\mathbf{R} = -Re_z$ is the vector pointing vertically to the plane, the end point is the contact point of the sphere and the plane, and $\mathbf{V} = \begin{bmatrix}\dot{x} & \dot{y} & 0\end{bmatrix}^T$ is the velocity of the COM of the sphere.

Express all terms wrt. spacial coordinate, then

$$\begin{aligned} [\boldsymbol{\omega}]_s &= \begin{bmatrix} \dot{\psi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \dot{\psi} + \dot{\phi}\cos\theta \end{bmatrix} \\ [\boldsymbol{\omega}\times\mathbf{R}]_s + [\mathbf{V}]_s &= \begin{bmatrix} R(\dot{\psi}\sin\theta\cos\psi - \dot{\theta}\sin\phi) + \dot{x} \\ R(\dot{\psi}\sin\theta\sin\psi + \dot{\theta}\cos\phi) + \dot{y} \\ 0 \end{bmatrix} = 0 \end{aligned}$$

The constraint is non-holonomic since we can verify it using the universal test. Consider the first equaiton

$$R\sin\theta\cos\psi\,\mathrm{d}\psi - R\sin\phi\,\mathrm{d}\theta + \mathrm{d}x = 0$$

Let ψ, ϕ, and x be three variables in the test, we have

$$1(R\cos\theta\cos\psi) + R\sin\theta\cos\psi(0) - R\sin\phi(0) \neq 0$$

which means the constraint is non-holonomic.

2) The Lagrangian is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\boldsymbol{\omega}^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I(\dot{\phi}^2 + \dot{\theta}^2 + \dot{\psi}^2 + 2\dot{\psi}\dot{\phi}\cos\theta)$$

Consider the Lagrange's equation for ϕ

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} - \mu_1\frac{\partial f_1}{\partial \dot{\phi}} - \mu_2\frac{\partial f_2}{\partial \dot{\phi}} = 0 \\ \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}I(\dot{\phi} + \dot{\psi}\cos\theta) = \dot{\omega}_z = 0 \end{aligned}$$

which means ω_z is a constant. Note that $\dot{x} = \omega_y R$, $\dot{y} = -\omega_x R$, then

$$T = \frac{1}{2}mR^2(\omega_x^2 + \omega_y^2) + \frac{1}{2}I(\omega_x^2 + \omega_y^2 + \omega_z^2) = \frac{1}{2}(mR^2 + I)(\omega_x^2 + \omega_y^2) + \frac{1}{2}I\omega_z^2$$

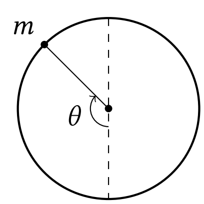
Since T constant and ω_z is also constant, we have $\omega_x^2 + \omega_y^2$ a constant. Which means that the translational kinetic energy $m(\dot{x}^2 + \dot{y}^2)/2$ and rotational kinetic energy $I\boldsymbol{\omega}^2/2$ are constant, respectively.

Problem 4. A bead of mass m is constrained to move on a hoop of radius R. The hoop rotates with constant angular velocity ω around a diameter of the hoop, which is a vertical axis (line along which gravity acts).

1) Set up the Lagrangian and obtain the equations of motion of the bead.
2) Find the critical angular velocity Ω below which the bottom of the hoop provides a stable equilibrium for the bead.
3) Find the stable equilibrium position for $\omega > \Omega$.

Solution.

1) Set up the generalized coordinate as the following figure



Then the Lagrangian and EOM are

$$\begin{aligned} L &= \frac{1}{2}mR^2(\dot{\theta}^2 + \omega^2\sin^2\theta) + mgR\cos\theta \\ mR^2\ddot{\theta} &= mR^2\omega^2 - mgR\sin\theta \end{aligned}$$

2) The effective potential is

$$V_{\text{eff}} = -\frac{1}{2}mR^2\omega^2\sin^2\theta - mgR\cos\theta$$

Its first and second-order derivative (Hessian) are

$$\begin{aligned} V'_{\text{eff}} &= -mR^2\omega^2\sin\theta\cos\theta + mgR\sin\theta \\ V''_{\text{eff}} &= -mR^2\omega^2\cos 2\theta + mgR\cos\theta \end{aligned}$$

Note that $V''_{\text{eff}}(0) = mgR - mR^2\omega^2$. If $\theta = 0$ is a equilibrium point, the sufficient condition is $V''_{\text{eff}}(0) > 0$, so the critical angular velocity Ω can be solved by $V''\text{eff}(0) = 0$. Hence

$$\Omega = \sqrt{\frac{g}{R}}$$

3) Suppose $\omega > \Omega$, and let $\sin\theta \geq 0$, then we can solve the equilibrium point by solving $V'_{\text{eff}} = 0$

$$\begin{aligned} V'_{\text{eff}} &= -mR^2\omega^2 \sin\theta\cos\theta + mgR\sin\theta = 0 \\ \Rightarrow \cos\theta &= \frac{gR}{R^2\omega^2} \end{aligned}$$

Hence we have to solutions of θ

$$\theta = \begin{cases} \arccos\frac{gR}{R^2\omega^2} \\ 2\pi - \arccos\frac{gR}{R^2\omega^2} \end{cases}$$