PHY3110 FA22 HW03

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Problem 1. It is non-holonomic since we cannot find a function $f = f(x, y, z)$ and write the constraint into

$$\mathrm{d}f = f_x \,\mathrm{d}x + f_y \,\mathrm{d}y + f_z \,\mathrm{d}z = 0$$

In this case, we have

$$(x^2 + y^2 + z^2 + 2x)\,\mathrm{d}x + 2y\,\mathrm{d}y + 2z\,\mathrm{d}z = 0$$

Note that

$$\frac{\partial}{\partial z}(x^2 + y^2 + z^2) \neq \frac{\partial}{\partial x}(2z)$$

which means that we cannot find such f.

Problem 2. Minimize the action is equivalent to solve the Lagrange's equation, hence

$$\begin{aligned} L(x, \dot{x}, t) &= \frac{1}{2}m\dot{x}^2 + Fx \\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} &= m\ddot{x} = 2C \\ \frac{\partial L}{\partial x} &= F \end{aligned} \quad \Rightarrow \begin{cases} A = 0 \\ B = \dfrac{a}{t_0} - \dfrac{Ft_0}{2m} \\ C = \dfrac{F}{2m} \end{cases}$$

Problem 3. Let two generalized coordinates be x and y, then the system could be written as

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy, \quad y = Ax^2 \Rightarrow 2Ax\,\mathrm{d}x - \mathrm{d}y = 0$$

Hence we can write Lagrange's equation with constraint

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} - 2Ax\lambda = 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} + \lambda = 0 \end{aligned} \Rightarrow \begin{cases} \ddot{x} = \dfrac{2Ax\lambda}{m} \\ \ddot{y} = -\dfrac{\lambda}{m} + g \end{cases}$$

Note that

$$y = Ax^2 \Rightarrow 2Ax\dot{x} = \dot{y} \Rightarrow \ddot{y} = 2A\dot{x}^2 + 2Ax\ddot{x}$$

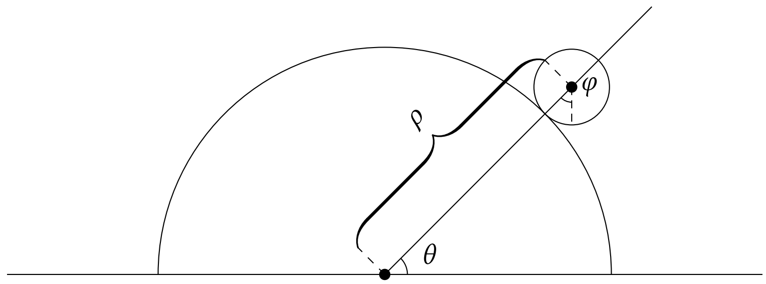
we can solve $\lambda = \lambda(x, \dot{x}, t)$ as

$$\lambda = -\frac{2Am\dot{x}^2 + mg}{1 + 4A^2x^2}$$

Hence we have constraint force for two coordinates

$$Q_x = 2Ax\lambda = -\frac{4A^2mx\dot{x}^2 + Axmg}{1 + 4A^2x^2}, \ Q_y = -\lambda = \frac{2Am\dot{x}^2 + mg}{1 + 4A^2x^2}$$

Problem 4. Set up the generalized coordinate ρ, θ and φ as the following figure.



Hence, we have the system

$$
\begin{aligned}
L &= \frac{1}{2}m(\dot\rho^2 + \rho^2\dot\theta^2) + \frac{1}{2}mr^2\dot\varphi^2 - mg\rho\sin\theta \\
&\text{subject to} \quad \rho \geq R + r,\ \rho\,\mathrm{d}\theta + r\,\mathrm{d}\varphi = 0
\end{aligned}
$$

Since we only consider the case when hoop is still on the cylinder, we can generalize the constraint to $\mathrm{d}\rho = 0$. Using two different Lagrange's multipliers λ and μ, we can get following set of equations

$$
\begin{aligned}
m\ddot\rho - m\rho\dot\theta^2 + mg\sin\theta - \lambda &= 0 \\
m\rho^2\ddot\theta + 2m\rho\dot\rho\dot\theta + mg\rho\cos\theta - \rho\mu &= 0 \\
mr^2\ddot\varphi - r\mu &= 0
\end{aligned}
$$

Note that $\rho\,\mathrm{d}\theta + r\,\mathrm{d}\varphi = 0 \Rightarrow \dot\rho\dot\theta/r + \rho\ddot\theta/r + \ddot\varphi = 0$, substitute $\ddot\varphi$ into the third equation, and use the solved μ, we have

$$
\mu = -m\dot\rho\dot\theta - m\rho\ddot\theta, \quad 2m\rho^2\ddot\theta + 3m\rho\dot\rho\dot\theta + mg\rho\cos\theta = 0
$$

Applying $\dot\rho = 0$ and boundary condition, we have

$$
\begin{aligned}
\ddot\theta = -\frac{g}{2\rho}\cos\theta, \quad \dot\theta\frac{\mathrm{d}\dot\theta}{\mathrm{d}\theta} &= -\frac{g}{2\rho}\cos\theta \\
\Rightarrow \dot\theta^2 &= \frac{g}{\rho}(1 - \sin\theta)
\end{aligned}
$$

When the hoop is about to leave the cylinder, we should have $\lambda = 0$ (and $\ddot\rho = 0$), then

$$
\begin{aligned}
-m\rho\dot\theta^2 + mg\sin\theta &= 0 \\
2mg\sin\theta &= mg \\
\Rightarrow \theta &= \frac{\pi}{6}
\end{aligned}
$$