PHY3110 FA22 HW02

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Problem 1. Let

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{bmatrix}$$

and define the following three unit vectors

$$\hat{r} = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}, \quad \hat{\phi} = \begin{bmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{bmatrix}, \quad \hat{\theta} = \begin{bmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{bmatrix}$$

easy to show that $\hat{r}$, $\hat{\phi}$ and $\hat{\theta}$ forms an orthonormal basis in $\mathbb{R}^3$.

Note that

$$\mathbf{F} = m\ddot{r} = m\frac{\mathrm{d}^2}{\mathrm{d}t^2}(r\hat{r}) = m\frac{\mathrm{d}}{\mathrm{d}t}(\dot{r}\hat{r} + r\dot{\theta}\sin\theta\hat{\phi} + \dot{\theta}r\hat{\theta})$$

Note that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\dot{r}\hat{r} &= \ddot{r}\hat{r} + \dot{r}\dot{\phi}\sin\theta\hat{\phi} + \dot{r}\dot{\theta}\hat{\theta} \\ \frac{\mathrm{d}}{\mathrm{d}t}r\dot{\phi}\sin\theta\hat{\phi} &= -r\dot{\phi}^2\sin^2\theta\hat{\theta} + (r\ddot{\phi}\sin\theta + \dot{r}\dot{\phi}\sin\theta + r\dot{\phi}\dot{\theta}\cos\theta)\hat{\phi} \\ -r\dot{\phi}^2\sin\theta\cos\theta\hat{\theta}\frac{\mathrm{d}}{\mathrm{d}t}r\dot{\theta}\hat{\theta} &= -r\dot{\theta}^2\hat{r} + r\dot{\phi}\dot{\theta}\cos\theta\hat{\phi} + (r\ddot{\theta} + \dot{r}\dot{\theta})\hat{\theta} \end{aligned}$$

Hence the Newton's 2^{nd} law becomes

$$\mathbf{F} = m[(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)\hat{r} + (r\ddot{\phi} + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\hat{\phi} + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\hat{\theta}]$$

Problem 2. Let L be the distance from the hanging point to the center of the ball. We should have $L > R$. Define the generalized coordinate θ, we can express the potential as

$$V(\theta) = -\cos\theta\, m_1 gL - [(C - \sqrt{L^2 - R^2} - (\theta_0 - \theta)R) + \cos(\theta_0 - \theta)\sqrt{L^2 - R^2} + R\sin(\theta_0 - \theta)]m_2 g$$

where $\theta_0 = \arcsin(R/L)$. Applying D'Alembert's principle, we would know that

$$\sum_i Q_i\delta q_i = 0 \Rightarrow -\frac{\partial V}{\partial\theta}\delta\theta = 0$$

Note that

$$\begin{aligned} \frac{\partial V}{\partial\theta} &= m_1 gL\sin\theta - m_2 gR + m_2 g\sqrt{L^2 - R^2}\sin(\theta - \theta_0) - m_2 gR\cos(\theta - \theta_0) \\ &= gL[(m_1 + m_2)\sin\theta - m_2\sin\theta_0] \end{aligned}$$

Then

$$gL[(m_1 + m_2)\sin\theta - m_2\sin\theta_0] = 0 \Rightarrow \theta = \arcsin\left(\frac{m_2\sin\theta_0}{m_1 + m_2}\right)$$

where $\theta \in [0, \theta_0]$.

Problem 3. Define the Lorentz factor $\gamma = 1/\sqrt{1-\dot{x}^2/c^2}$, then

$$L = -\frac{m_0c^2}{\gamma} - V$$

Since

$$\begin{aligned}
\frac{\partial L}{\partial x} &= -\frac{\partial V}{\partial x} \\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} &= \frac{\mathrm{d}}{\mathrm{d}t}m_0\dot{x}\gamma = m_0\ddot{x}\gamma + m_0\ddot{x}\gamma^3\frac{\dot{x}^2}{c^2} = m_0\ddot{x}\gamma^3
\end{aligned}$$

Thus the equation of motion writes

$$m_0\ddot{x}\gamma^3 + \frac{\partial V}{\partial x} = 0$$

Problem 4. Since for rigid body, the kinetic energy is

$$K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

where v is the velocity of center of mass and I is the rotational inertia at COM.

Define two generalized coordinate: x the horizontal position of center of mass, θ the acute angle formed between the rod and the ground. For the rod, we have $\mathbf{v} = \dot{x}\hat{x} + l\dot{\theta}\cos\theta\,\hat{y}$, $I = ml^2/3$, then we can calculate the kinetic and potential energy

$$\begin{aligned}
K &= \frac{1}{2}m[\dot{x}^2 + (l\dot{\theta}\cos\theta)^2] + \frac{1}{6}ml^2\dot{\theta}^2 \\
V &= \int_0^{2l}\rho\,\mathrm{d}s = \int_0^{2l\sin\theta}\frac{m}{2l}\frac{1}{\sin\theta}u\,\mathrm{d}u = mgl\sin\theta
\end{aligned}$$

Hence we can define the Lagrangian and get the equation of motion.

$$\begin{aligned}
L &= \frac{1}{2}m[\dot{x}^2 + (l\dot{\theta}\sin\theta)^2] + \frac{1}{6}ml^2\dot{\theta}^2 - mgl\cos\theta \\
\frac{\partial L}{\partial x} &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} &= m\ddot{x} \\
\frac{\partial L}{\partial \theta} &= ml^2\dot{\theta}^2\sin\theta\cos\theta + mgl\sin\theta \\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} &= \ddot{\theta}\left(ml^2\sin^2\theta + \frac{1}{3}ml^2\right) + 2ml^2\dot{\theta}^2\sin\theta\cos\theta \\
\Rightarrow m\ddot{x} &= 0 \\
\ddot{\theta}&\left(ml^2\sin^2\theta + \frac{1}{3}ml^2\right) + 2ml^2\dot{\theta}^2\sin\theta\cos\theta - ml^2\dot{\theta}^2\sin\theta\cos\theta - mgl\sin\theta = 0
\end{aligned}$$

Hence the invariant quantities are $\dot{x}$ and the total energy $E = K + V$.