

PHY3110: Classical Mechanics

Lagrangian / Hamiltonian formulation

Homework	20%
Midterm	30%
Final	40%

Tutorial

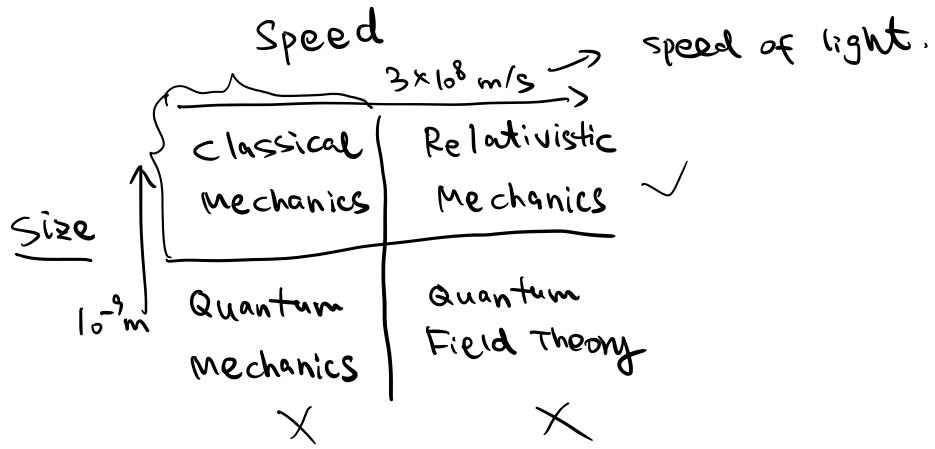
Reference book:

H. Goldstein, C. Poole, J. Safko, Classical Mechanics,
3rd Edition, Pearson.

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- ⊗ S.R. Taylor, Classical Mechanics, University Science Books.
 - ⊗ T.W.B. Kibble, F.H. Berkshire, Classical Mechanics, 5th Edition, Imperial College Press.
 - ⊗ 漆昆炽. 力学(下册) 理论力学, 4th Edition, 高等教育出版社.

Tianhui Zhang. Office: RA 319.

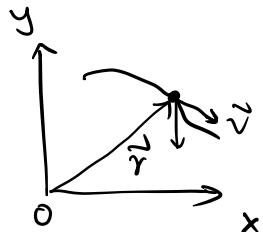
classical mechanics describes the motion of macroscopic objects, which are not extremely massive and not extremely fast.



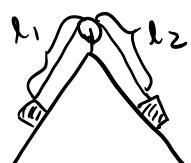
Review of Newtonian Mechanics

vectorial quantities of motion:

$$\vec{r}, \vec{v}, \vec{F}, \vec{p} = m\vec{v}, \vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$$



constraint



$$l_1 + l_2 = \text{const.}$$

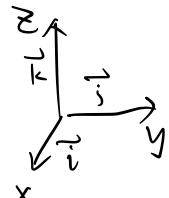
Analytical mechanics:

It uses scalar quantities of motion:

$$\text{kinetic energy: } T = \frac{1}{2} m \vec{v}^2.$$

Potential energy:

$$V = V(\vec{r}).$$



$$\vec{v} \cdot \vec{v}$$

$$= (v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}) \cdot$$

$$(v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k})$$

$$= v_1^2 + v_2^2 + v_3^2$$

$$\vec{i} \cdot \vec{i} = 1,$$

$$\vec{i} \cdot \vec{j} = \vec{i} \cdot \vec{k} = \vec{k} \cdot \vec{j} = 0$$

Constraints:

They are used to reduce the number of def's.

The formalism of analytical mechanics can also be generalized to [electrodynamics, statistical / quantum mechanics, relativity & QFTs.]

Newton's 2nd law:

$$\vec{F} = \frac{d\vec{p}}{dt} = \cancel{\frac{d}{dt}(m\vec{v})} = \vec{m}\vec{a} \Rightarrow \frac{d\vec{v}}{dt}.$$

Valid in an inertial frame.

$$\frac{d\vec{p}}{dt} = 0 \quad \text{if } \vec{F} = 0. \Rightarrow \vec{p} = \text{const.}$$

\vec{p} is conserved
if $\vec{F} = 0$.

Angular momentum \vec{L} & torque \vec{N} .

$$\vec{L} = \vec{r} \times \vec{p}.$$

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d\vec{r}}{dt} \times (m\vec{v}) + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$(\vec{A} \times \vec{B})_i \quad (i=1,2,3)$$

$$= \sum_{j,k} \epsilon_{ijk} A_j B_k \quad \text{summation over indices}$$

$$\epsilon_{123}^{\text{permutation}} = 1,$$

$$\epsilon_{132} = -1$$

$$\epsilon_{312} = +1$$

$$\epsilon_{iij} = 0$$

$$\epsilon_{122} = 0$$

$$\epsilon_{133} = 0.$$

$$(\vec{A} \times \vec{B})_i$$

$$= \epsilon_{ijk} A_j B_k$$

$$= \epsilon_{123} A_2 B_3 + \epsilon_{132} A_3 B_2$$

$$= -\epsilon_{123}$$

$$= \epsilon_{123} A_2 B_3 - \epsilon_{123} A_3 B_2$$

$$= A_2 B_3 - A_3 B_2$$

$$V_2 V_3 - V_3 V_2 = 0$$

$$(\vec{A} \times \vec{B}) \times \vec{C}$$



Work done by the external force:

$$W_{12} = \int_{1}^{2} \vec{F} \cdot d\vec{s} = \int_{1}^{2} \frac{d\vec{p}}{dt} \cdot \vec{v} dt$$

$$= \int_{1}^{2} m \vec{v} \cdot d\vec{v}$$

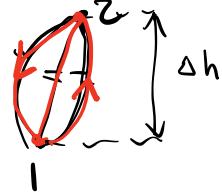
$$= \frac{1}{2} m \vec{v}_2^2 - \frac{1}{2} m \vec{v}_1^2$$

$$= T_2 - T_1$$

$$W = \oint \vec{F} \cdot d\vec{s} = 0$$

Define a scalar function $V(\vec{r})$.

$$\vec{F} = -\nabla V(\vec{r}).$$



V : potential energy

\vec{F} : conservative force.

$$\begin{aligned} & \oint S_1 \vec{F} \cdot d\vec{s} + \oint S_2 \vec{F} \cdot d\vec{s} \\ &= \oint \vec{F} \cdot d\vec{s} = 0 \end{aligned}$$

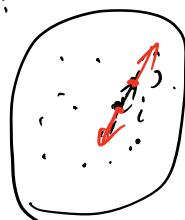
$$W_{12} = V_1 - V_2 = T_2 - T_1$$

$$\Rightarrow \underbrace{V_1 + T_1}_{E_1} = \underbrace{V_2 + T_2}_{E_2}$$

Total energy is conserved.

Consider a system of multiple particles:

$$\frac{d\vec{p}_i}{dt} = \dot{\vec{p}}_i = \vec{F}_i^{(e)} + \sum_j \vec{F}_{ji}$$



Assume Newton's 3rd law holds:

Forces that act on each other are

equal & opposite.

$$\vec{F}_{ji} = -\vec{F}_{ij}$$

$$\begin{aligned} \sum_i \dot{\vec{p}}_i &= \frac{d}{dt} \left(\sum_i \vec{p}_i \right) = \sum_i \vec{F}_i^{(e)} + \sum_{i,j} \vec{F}_{ji} = 0 \\ &= \sum_i \vec{F}_i^{(e)} \end{aligned}$$

Center of mass of the system:

$$\vec{R} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} = \frac{\sum_i m_i \vec{r}_i}{M} \quad M = \sum_i m_i.$$

$$\dot{\vec{P}} = \sum_i \dot{\vec{p}}_i = M \dot{\vec{R}} = \vec{F}^{(e)} = \sum_i \vec{F}_i^{(e)}.$$

\vec{P} is conserved if $\vec{F}^{(e)} = 0$

Total AM:

$$\begin{aligned} \frac{d\vec{L}}{dt} &= \dot{\vec{L}} = \frac{d}{dt} \sum_i \vec{r}_i \times \vec{p}_i \\ &= \sum_i \vec{r}_i \times \vec{F}_i^{(e)} + \sum_{i \neq j} \vec{r}_i \times \vec{F}_{ji} \end{aligned}$$

Assume the mutual force between 2 particles lie along the line between them.

$$\sum_{i \neq j} \vec{r}_i \times \vec{F}_{ji} \Rightarrow \frac{1}{2} \sum_{i \neq j} \vec{r}_{ij} \times \vec{F}_{ji} = 0$$

$\Downarrow (\vec{r}_i - \vec{r}_j)$

$$\boxed{\dot{\vec{L}} = \vec{N}^{(e)} = \sum_i \vec{r}_i \times \vec{F}_i^{(e)}}.$$

\vec{L} is conserved if $\vec{N}^{(e)} = 0$.

$$\begin{aligned}
 \vec{r} &= \sum_i \vec{r}_i \times \vec{p}_i = \sum_i (\vec{R} + \vec{r}'_i) \times m_i (\vec{v} + \vec{v}'_i) \\
 &= M \vec{R} \times m_i \vec{v} + \sum_i \vec{r}'_i \times m_i \vec{v}'_i \\
 &\quad + \underbrace{\sum_i (m_i \vec{r}'_i) \times \vec{v}}_{\parallel} + \vec{R} \times \underbrace{\sum_i (m_i \vec{r}'_i)}_{\parallel} \\
 &= \sum_i \vec{R} \times m_i \vec{v} + \sum_i \vec{r}'_i \times \vec{p}'_i \\
 &\quad \Downarrow \\
 &= \vec{R} \times M \vec{v}
 \end{aligned}$$

$$\begin{aligned}
 \vec{r}'_i &= \vec{R} + \vec{r}_i \\
 \vec{v}'_i &= \vec{v} + \vec{v}_i
 \end{aligned}$$

$$\sum_i m_i \vec{r}'_i = \sum_i m_i \vec{r}_i + \underbrace{\sum_i m_i \vec{R}}_{M}$$

Total AM = AM of COM. + AM of motion around COM.

$$W_{12} = T_2 - T_1, \quad T = \sum_i \frac{1}{2} m_i \vec{v}_i^2$$

$$\begin{aligned}
 T &= \frac{1}{2} \sum_i m_i (\vec{v} + \vec{v}'_i) \cdot (\vec{v} + \vec{v}'_i) \\
 &= \underbrace{\frac{1}{2} \sum_i m_i \vec{v}^2}_{M} + \frac{1}{2} \sum_i m_i \vec{v}'_i^2
 \end{aligned}$$

$$\begin{aligned}
 \sum_i m_i \vec{v}'_i &= 0 \\
 \Downarrow \\
 \sum_i m_i \vec{v}'_i &= 0
 \end{aligned}$$

Total KE = COM KE + KE of motion around COM

If the external/internal forces are conservative, we can define $V = \sum_i V_i + \frac{1}{2} \sum_{i,j} V_{ij}$
 $E = T + V = \text{const.}$

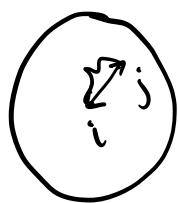
Constraints:

$\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N$ for N-particle system.

$$f(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N, t) = 0$$

holonomic constraint.

Examples:



Rigid body

$$(\vec{r}_i - \vec{r}_j)^2 - c_{ij}^2 = 0$$



$$\dot{x} - R\dot{\theta} = 0$$

$$\Rightarrow \frac{dx}{dt} - R \frac{d\theta}{dt} = 0$$

$$\Rightarrow d(x - R\theta) = 0$$

$$\Rightarrow x - R\theta = \text{const}$$

A constraint of the form

$$\sum_i g_i(x_1, x_2, \dots, x_n) dx_i = 0.$$

↓ multiplying some function $f(x_1, x_2, x_n)$

Total differential $dG(x_1, x_2, \dots, x_n) =$



$$G(x_1, x_2, \dots, x_n) = \text{const.}$$



holonomic constraint.

Non-holonomic constraint



$$\vec{r}^2 - a^2 \geq 0$$

$\vec{r}^2 - a^2 = 0$, before it leaves
 $\vec{r}^2 - a^2 > 0$, after it leaves.

If the constraint is time-dep. \rightarrow rheonomous

time-indep. \rightarrow scleronomous

With the imposed constraints,

\vec{r}_i are no longer all indep.

it is convenient to introduce some new variables,
generalized coordinates.

Suppose we have a N-particle system.

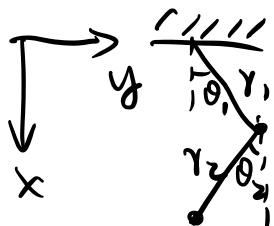
$1 \leq i \leq 3N, \vec{r}_i$ 3N dofs.

k constraint Eqs.

$1 \leq j \leq 3N-k, q_j$. $3N-k$ dofs.

We can express

$$\left. \begin{aligned} \vec{r}_1 &= \vec{r}_1(q_1, q_2, \dots, q_{3N-k}, t) \\ \vec{r}_2 &= \vec{r}_2(q_1, q_2, \dots, q_{3N-k}, t) \\ &\vdots \\ \vec{r}_N &= \vec{r}_N(q_1, q_2, \dots, q_{3N-k}, t) \end{aligned} \right\} \Rightarrow q_j = q_j(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N, t)$$



(x_1, y_1) & (x_2, y_2) with constraints

$r_1 = \text{const}$, $r_2 = \text{const}$.
 $\Rightarrow 2$ indep. dofs,

can also be chosen as θ_1, θ_2

generalized coords.

D'Alembert's principle & Lagrange Eqs.

Formulate mechanics such that the forces of constraints do not appear.

Hint from rigid body:

Internal forces of constraints do no work.

Virtual displacement:

$\vec{r}_i \rightarrow \vec{r}_i + \delta \vec{r}_i$. \rightarrow consistent with the constraints on the system at a given t.

$\boxed{\text{v.d.}}$ $f(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N, t) = 0$.

$f(\vec{r}_1 + \delta \vec{r}_1, \vec{r}_2 + \delta \vec{r}_2, \dots, \vec{r}_N + \delta \vec{r}_N, t) = 0$

real displacement:

$f(\vec{r}_1 + \underline{d\vec{r}_1}, \vec{r}_2 + \underline{d\vec{r}_2}, \dots, \vec{r}_N + \underline{d\vec{r}_N}, t + \underline{dt}) = 0$

Consider a system in equilibrium.

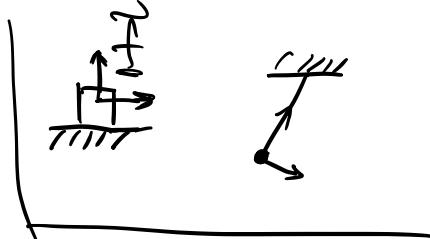
$$\vec{F}_i = 0 \rightarrow \sum_i \vec{F}_i \cdot \delta \vec{r}_i = 0.$$

Separate $\vec{F}_i = \vec{F}_i^{(a)} + \vec{f}_i$ constraint force.

$$\sum_i (\vec{F}_i^{(a)} + \vec{f}_i) \cdot \delta \vec{r}_i = 0$$

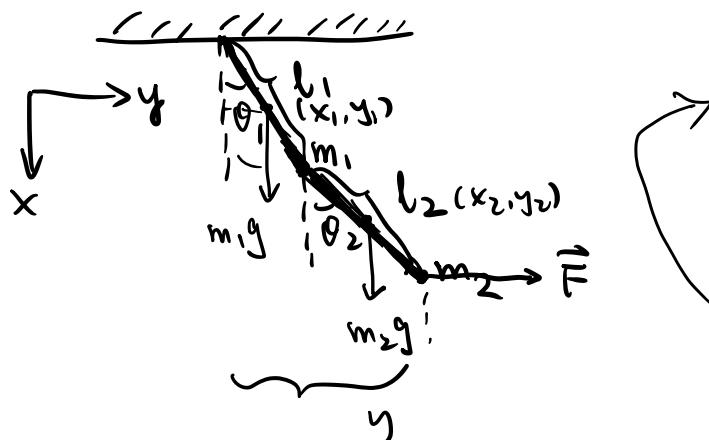
$$\sum_i \vec{f}_i \cdot \delta \vec{r}_i = 0$$

$$\Rightarrow \boxed{\sum_i \vec{F}_i^{(a)} \cdot \delta \vec{r}_i = 0.}$$



Virtual work principle.

Example:



$$m_1 g \delta x_1 + m_2 g \delta x_2 + F \delta y = 0$$

$$x_1 = \frac{l_1}{2} \cos \theta_1,$$

$$x_2 = l_1 \cos \theta_1 + \frac{l_2}{2} \cos \theta_2$$

$$y = l_1 \sin \theta_1 + l_2 \sin \theta_2$$

$$(F \cos \theta_1 - \frac{m_1 g}{2} \sin \theta_1, -m_2 g \sin \theta_1) \cdot \delta \theta_1 = 0$$

$$+ (F \cos \theta_2 - \frac{m_2 g}{2} \sin \theta_2) \cdot \delta \theta_2 = 0.$$

$$\Rightarrow \tan \theta_1 = \frac{2F}{(m_1 + 2m_2)g}, \quad \tan \theta_2 = \frac{2F}{m_2 g}.$$

For a system moving under external forces.

$$\vec{F}_i - \dot{\vec{p}}_i = 0$$

$$\Rightarrow \sum_i (\vec{F}_i - \dot{\vec{p}}_i) \cdot \delta \vec{r}_i = 0.$$

$$\Rightarrow \sum_i (\vec{F}_i^{(a)} - \dot{\vec{p}}_i) \cdot \delta \vec{r}_i = 0.$$

D'Alembert's principle

$$\delta \vec{r}_i \rightarrow \delta q_j$$

For holonomic constraints,

$$\vec{r}_i = \vec{r}_i(q_1, \dots, q_n, t)$$

$n = 3N - k$. # of
indep. d.o.f.s.

$$\vec{v}_i = \frac{d\vec{r}_i}{dt} = \frac{d\vec{r}_i}{dt} + \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j$$

Virtual displacement:

$$\delta \vec{r}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j$$

$$\sum_i \vec{F}_i^{(a)} \cdot \delta \vec{r}_i = \sum_{i,j} \vec{F}_i^{(a)} \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j = \sum_j Q_j \delta q_j$$

$$Q_j = \underbrace{\sum_i \vec{F}_i^{(a)} \cdot \frac{\partial \vec{r}_i}{\partial q_j}}_{\text{generalized force.}}$$

$$\sum_i \vec{p}_i \cdot \dot{\delta r}_i = \sum_{i,j} m_i \vec{r}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j$$

$$\rightarrow = \sum_{i,j} \left[\frac{d}{dt} \left(m_i \vec{r}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right) - m_i \vec{r}_i \cdot \frac{d}{dt} \left(\frac{\partial \vec{r}_i}{\partial q_j} \right) \right] \delta q_j$$

⊕

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial \vec{r}_i}{\partial q_j} \right) &= \frac{\partial}{\partial t} \left(\frac{\partial \vec{r}_i}{\partial q_j} \right) + \sum_k \frac{\partial}{\partial q_k} \left(\frac{\partial \vec{r}_i}{\partial q_j} \right) \dot{q}_k \\ &= \frac{\partial}{\partial q_j} \left(\frac{\partial \vec{r}_i}{\partial t} + \sum_k \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k \right) \\ &= \frac{\partial}{\partial q_j} \frac{\partial \vec{r}_i}{\partial t} \end{aligned}$$

⊕

$$\boxed{\frac{\partial \vec{r}_i}{\partial q_j} = \frac{\partial \vec{v}_i}{\partial \dot{q}_j}} = \frac{\partial}{\partial q_j} \left(\sum_k \left(\frac{\partial \vec{r}_i}{\partial q_k} \right) \dot{q}_k + \frac{\partial \vec{r}_i}{\partial t} \right)$$

$$\begin{aligned} &\sum_{i,j} \left[\frac{d}{dt} \left(m_i \vec{r}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right) - m_i \vec{r}_i \cdot \frac{d}{dt} \left(\frac{\partial \vec{r}_i}{\partial q_j} \right) \right] \delta q_j \\ &= \sum_{i,j} \left[\frac{d}{dt} \left(m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j} \right) - m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_j} \right] \delta q_j \\ &= \sum_j \left[\frac{d}{dt} \left(\frac{\partial}{\partial \dot{q}_j} \left(\sum_i m_i \vec{v}_i^2 \right) \right) - \frac{\partial}{\partial q_j} \left(\sum_i m_i \vec{v}_i^2 \right) \right] \delta q_j \\ &= \sum_j \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] \delta q_j \end{aligned}$$

$$\Rightarrow \sum_j \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j \right] \delta q_j = 0$$

δq_j s are indep. of each other,

$$\Rightarrow \boxed{\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j = 0.} \text{ for } 1 \leq j \leq n.$$

For conservative systems.

$$\vec{F}_i = -\nabla_i V \hookrightarrow \text{potential energy.}$$

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = - \sum_i \nabla_i V \cdot \frac{\partial \vec{r}_i}{\partial q_j} = - \frac{\partial V}{\partial q_j}$$

$$\Rightarrow \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial(T-V)}{\partial q_j} = 0.$$

If V does not depend on generalized velocities,

$$\frac{\partial V}{\partial \dot{q}_j} = 0$$

$$\Rightarrow \frac{d}{dt} \frac{\partial(T-V)}{\partial \dot{q}_j} - \frac{\partial(T-V)}{\partial q_j} = 0.$$

Define the Lagrangian

$$E = T - V.$$

$$L = T - V.$$

Lagrange's Eq.

$$\boxed{\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0.}$$

The choice of the Lagrangian is not unique.

$$L' = L + \underbrace{\frac{dF(q_{int})}{dt}}_{\rightarrow \Delta} \quad \& \quad L$$

give the same EOM.

$$\frac{d}{dt} \left(\frac{\partial \Delta}{\partial \dot{q}} \right) - \frac{\partial \Delta}{\partial q} = 0.$$

$$\Delta - \frac{dF}{dt} = \frac{\partial F}{\partial q} \dot{q} + \frac{\partial F}{\partial t}. \Rightarrow \frac{\partial \Delta}{\partial \dot{q}} = \frac{\partial F}{\partial q}$$

$$\frac{d}{dt} \left(\frac{\partial F}{\partial \dot{q}} \right) = \frac{\partial}{\partial \dot{q}} \left(\frac{\partial F}{\partial t} \right) = \frac{\partial \Delta}{\partial q}.$$