1.

$$(x^2+y^2+z^2)dx + 2(xdx + ydy + zdz) = 0$$

$$\Rightarrow (x^2+y^2+z^2+2z)dx + 2ydy + 2zdz = 0$$

Assume that for some (x, y, z).

$f_{,x} = x^2+y^2+z^2+2x \Rightarrow f_{,xy} = 2y \; ; \; f_{,xz} = 2z$

$f_{,y} = 2y \Rightarrow f_{,yx} = f_{,yz} = 0$

$f_{,z} = 2z \Rightarrow f_{,zx} = f_{,zy} = 0$

$f_{,x}(f_{,yz} - f_{,zy}) + f_{,y}(f_{,zx} - f_{,xz}) + f_{,z}(f_{,xy} - f_{,yx})$

$= 0 + 2y(-2z) + 2z(2y) = 0$

so f is the form of holonomic constraint

2. $L = T - V = \frac{1}{2}m\dot{x}^2 + Fx$

$x(t) = A + Bt + Ct^2 \Rightarrow \dot{x}(t) = B + 2Ct$

$$\begin{cases} x(0) = 0 \Rightarrow A = 0 \\ x(t_0) = A + Bt_0 + Ct_0^2 = a \Rightarrow B = \frac{a}{t_0} - Ct_0 \end{cases}$$

to minimize the action. $I = \int_0^{t_0} L\,dt$

$$0 = \delta I = \int_0^{t_0} \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\right) dt\,\delta x$$

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)$$

$$\frac{\partial L}{\partial x} = F$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x} = \frac{d}{dt}[m(B + 2Ct)] = 2mC$$

$\Rightarrow F = 2mC \Rightarrow C = \frac{F}{2m}$. $\quad \therefore \begin{cases} A = 0 \\ B = \frac{a}{t_0} - \frac{F}{2m}t_0 \\ C = \frac{F}{2m}. \end{cases}$

3. $V = mgy$, $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$, $L = T - V$

Constraint $f(x,y) = y - Ax^2 = 0$

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} - \lambda \frac{\partial f}{\partial x} = 0 \Rightarrow m\ddot{x} + \lambda(2Ax) = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} - \lambda \frac{\partial f}{\partial y} = 0 \Rightarrow m\ddot{y} + mg - \lambda = 0 \end{cases}$$

$$\ddot{y} = \frac{d}{dt}\left(\frac{dy}{dx}\frac{dx}{dt}\right) = \frac{d}{dt}(2Ax\dot{x}) = 2A\dot{x}^2 + 2Ax\ddot{x}$$

$$2mA(\dot{x}^2 + x\ddot{x}) + mg + \frac{m\ddot{x}}{2Ax} = 0$$

$$\ddot{x} = \left(2Ax + \frac{1}{2Ax}\right)^{-1}(g + 2A\dot{x}^2)$$

$$\lambda = m\frac{g + 2A\dot{x}^2}{4A^2x^2 + 1}$$

$$F_x = -(2Ax)\lambda, \quad F_y = \lambda$$

$$\vec{F} = m\frac{g + 2A\dot{x}^2}{4A^2x^2 + 1}(-2Ax\hat{x} + \hat{y})$$

4. $L = \frac{m}{2}(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + r^2\dot{\theta}^2) - mg\rho\cos\varphi$,

constraints: $\rho = r + R$ ①

$(R+r)\varphi - r\theta = 0$ ②

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} + \sum_{i=1}^{2}\lambda_i \frac{\partial f_i}{\partial q_k} = 0$$

E-L equations:

$$\begin{cases} m\ddot{\rho} - m\rho\dot{\varphi}^2 + mg\cos\varphi = \lambda_1 \\ mr^2\ddot{\theta} = -\lambda_2 r \\ m\rho^2\ddot{\varphi} + 2m\rho\dot{\rho}\dot{\varphi} - mg\rho\sin\varphi = \lambda_2(R+r) \\ (R+r)\varphi - r\theta = 0 \\ \rho - r - R = 0 \end{cases}$$

$$\begin{cases} -m(r+R)\ddot{\varphi}^2 + mg\cos\varphi = \lambda_1 \\ m(R+r)\ddot{\varphi} = -\lambda_2 \\ m(r+R)\ddot{\varphi} - mg\sin\varphi = \lambda_2 \end{cases}$$

$$-m(r+R)\ddot{\varphi}^2 + mg\cos\varphi = \lambda_1$$

$$2m(R+r)\ddot{\varphi} = mg\sin\varphi \Rightarrow \ddot{\varphi} = \frac{1}{2}\frac{g}{R+r}\sin\varphi$$

$$\Rightarrow \dot{\varphi}\ddot{\varphi} = \frac{1}{2}\frac{g}{R+r}\sin\varphi \cdot \dot{\varphi}$$

$$= \frac{d}{dt}\dot{\varphi}^2 = -\frac{g}{R+r}\frac{d}{dt}\cos\varphi, \qquad \varphi(0) = 0$$

$$\therefore \varphi^2 = \frac{1}{R+r}(1-\cos\varphi)$$

$$\lambda_1 = -m(r+R)\varphi^2 + mg\cos\varphi$$

$$= 2mg\cos\varphi - mg$$

$$\lambda_1 = 0 \Rightarrow \varphi = 60^\circ$$