PHY3110 SP23 HW10

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Problem 1. The Lagrangian for a system can be written as

$$L = a\dot{x}^2 + b\frac{\dot{y}}{x} + c\dot{x}\dot{y} + fy^2\dot{x}\dot{z} + g\dot{y} - k\sqrt{x^2+y^2} \tag{1}$$

where a, b, c, f, g, k are constants. What is the Hamiltonian? What quantities are conserved?

Solution. The canonical momentum reads

$$\begin{aligned} p_x &= 2a\dot{x} + c\dot{y} + fy^2\dot{z} \\ p_y &= \frac{b}{x} + c\dot{x} + g \\ p_z &= fy^2x \end{aligned}$$

The Hamiltonian is

$$\begin{aligned} H &= \dot{x}p_x + \dot{y}p_y + \dot{z}p_z - L \\ &= a\dot{x}^2 + c\dot{x}\dot{y} + fy^2\dot{x}\dot{z} + k\sqrt{x^2+y^2} \\ &= \dot{x}p_x - a\dot{x}^2 \\ &= \frac{p_z}{fy^2}\left(p_x - a\frac{p_z}{fy^2}\right) + k\sqrt{x^2+y^2} \end{aligned}$$

H is conserved since $\partial L/\partial t = 0$. Also, p_z is conserved since z is cyclic. Moreover, y is conserved since H not depend on p_y.

Problem 2. For a given Lagrangian

$$L = \dot{q}_1^2 + \frac{\dot{q}_2^2}{a+bq_1^2} + k_1q_1^2 + k_2\dot{q}_1\dot{q}_2 \tag{2}$$

with a, b, k_1, k_2 being constants, find the equations of motion in the Hamiltonian formulation.

Solution. From the Lagrangian we can obtain that

$$\begin{aligned} p_1 &= 2\dot{q}_1 + k_2\dot{q}_2 \\ p_2 &= k_2\dot{q}_1 + \frac{2}{a+bq_1^2}\dot{q}_2 \end{aligned}$$

Hence

$$\begin{aligned} \dot{q}_1 &= \frac{2p_1 - (a+bq_1^2)k_2p_2}{4 - k_2^2(a+bq_1^2)} \\ \dot{q}_2 &= \frac{(2p_2 - k_2p_1)(a+bq_1^2)}{4 - k_2(a+bq_1^2)} \end{aligned}$$

And

$$\begin{aligned} \dot{p}_1 &= 2k_1q_1 - \frac{2bq_1}{a+bq_1^2}\dot{q}_2^2 = 2k_1q_1 - \frac{2bq_1}{a+bq_q^2}\left[\frac{(2p_2 - k_2p_1)(a+bq_1^2)}{4 - k_2(a+bq_1^2)}\right]^2 \\ \dot{p}_2 &= 0 \end{aligned}$$

Problem 3. A Hamiltonian of one degree of freedom has the form

$$H = \frac{p^2}{2a} - bqpe^{-\alpha t} + \frac{ba}{2}q^2e^{-\alpha t}(\alpha + be^{-\alpha t}) + \frac{kq^2}{2} \tag{3}$$

where a, b, α, k are constants.

a) Find the Lagrangian corresponding to this Hamiltonian.

b) Find an equivalent Lagrangian that is not explicitly depend on time.

c) What is the Hamiltonian corresponding to the second Lagrangian, and what is the relationship between the two Hamiltonians?

Solution.

a) We can find the expression of $\dot{q}$

$$\dot{q} = \frac{p}{a} - bqe^{-\alpha t},\ p = a\dot{q} + abe^{-\alpha t}$$

Hence

$$\begin{aligned} L &= p\dot{q} - H \\ &= \frac{p^2}{2a} - \frac{ab}{2}q^2e^{-\alpha t}(\alpha + be^{-\alpha t}) - \frac{kq^2}{2} \\ &= \frac{1}{2}a\dot{q}^2 - \frac{1}{2}kq^2 - \frac{\mathrm{d}}{\mathrm{d}t}abq^2e^{-\alpha t} \end{aligned}$$

b) Since $L' = L + \mathrm{d}F/\,\mathrm{d}t$ is equivalent to L, we have a L' not explicitly depend on time

$$L' = \frac{1}{2}a\dot{q}^2 - \frac{1}{2}kq^2$$

c) The Hamiltonian corresponds to L' is

$$H' = \frac{1}{2}q\dot{q}^2 - \frac{1}{2}kq^2 = \frac{p^2}{2a} + \frac{1}{2}kq^2$$

The relationship between two momentums are

$$p = p' - \frac{\partial}{\partial \dot{q}}\frac{\mathrm{d}F}{\mathrm{d}t}$$

Since $H = p\dot{q} - L$, we have

$$H = pq - L = \left(p' - \frac{\partial}{\partial \dot{q}}\frac{\mathrm{d}F}{\mathrm{d}t}\right)\dot{q} - L' + \frac{\mathrm{d}F}{\mathrm{d}t} = H' + \frac{\mathrm{d}F}{\mathrm{d}t} - \dot{q}\frac{\partial}{\partial \dot{q}}\frac{\mathrm{d}F}{\mathrm{d}t}$$

where $F = abq^2e^{-\alpha t}$ in this case.

Problem 4.

a) The Lagrangian for a system with one degree of freedom reads

$$L = \frac{m}{2}(\dot{q}^2\sin^2\omega t + \dot{q}q\omega\sin 2\omega t + q^2\omega^2) \tag{4}$$

What is the corresponding Hamilton? Is it conserved?

b) Introduce a new coordinate defined by $Q = q \sin \omega t$. Find the Lagrangian in terms of the new coordinate and the corresponding Hamiltonian. Is H conserved?

Solution.

a) The momentum and the velocity in terms of coordinate and momentum are

$$p = m \sin^2 \omega t \dot{q} + \frac{m}{2} q\omega \sin \omega t, \ \dot{q} = \frac{1}{m \sin^2 \omega t}\left(p - \frac{m}{2} q\omega \sin 2\omega t\right)$$

Hence we have

$$H = p\dot{q} - L = \frac{m}{2} \sin^2 \omega t \dot{q}^2 - \frac{m}{2} q^2 \omega^2 = \frac{1}{2m \sin^2 \omega t}\left(p - \frac{m}{2} q\omega \sin 2\omega t\right)^2 - \frac{m}{2} q^2 \omega^2$$

It is not conserved since $\partial L/\partial t \neq 0$.

b) We can derive

$$\begin{aligned} q^2 \sin^2 \omega t &= \dot{Q}^2 + \omega^2 Q^2 \cot^2 \omega t - 2Q\dot{Q}\omega \cot \omega t \\ q\dot{q}\omega \sin 2\omega t &= 2\dot{Q}Q\omega \cot \omega t - 2Q^2\omega^2 \cot^2 \omega t \\ q^2\omega^2 &= \frac{Q^2\omega^2}{\sin \omega t} \end{aligned}$$

Hence the new Lagrangian and Hamiltonian are

$$L = \frac{m}{2}(\dot{Q}^2 + Q^2\omega^2), \ H = \frac{m}{2}(P^2 - Q^2\omega^2)$$

The new Hamiltonian is conserved since the new Lagrangian does not depend on time specifically.