PHY3110 SP23 HW07

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Problem 1. Derive Euler's equaiton from the Lagrange equations of motion.

Solution. Set up the coordinate system on the rigid body which could diagonalize the inertia tensor. Then the kinetic energy is

$$T = \frac{1}{2}\sum_i I_i \omega_i^2$$

where the angular velocity vector $\boldsymbol{\omega}$ expressed under the rotated coordinates is

$$[\boldsymbol{\omega}]_{x'y'z'} = \begin{bmatrix} \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \dot{\psi} + \dot{\phi}\cos\theta \end{bmatrix}$$

The Lagrange equation tells that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\psi}} - \frac{\partial L}{\partial \psi} &= Q_\psi \\ \frac{\mathrm{d}}{\mathrm{d}t}\sum_i \frac{\partial T}{\partial \omega_i}\frac{\partial \omega_i}{\partial \dot{\psi}} - \sum_i \frac{\partial T}{\partial \omega_i}\frac{\partial \omega_i}{\partial \psi} &= Q_\psi \\ I_{z'}\dot{\omega}_{z'} - (I_{x'} - I_{y'})\omega_{x'}\omega_{y'} &= Q_\psi \end{aligned}$$

Claim. The generalized force Q_ψ equals to the torque projected on the rotated coordinate z', namely

$$Q_\psi = N_{z'}$$

Proof. Since

$$Q_\psi = -\frac{\partial V}{\partial \psi} = -\nabla V \cdot \frac{\partial \mathbf{r}}{\partial \psi} = \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial \psi}$$

and

$$\begin{aligned} \left[\frac{\partial \mathbf{r}}{\partial \psi}\right]_{x'y'z'} &= A\left[\frac{\partial \mathbf{r}}{\partial \psi}\right]_{xyz} = A\frac{\partial}{\partial \psi}A^T[\mathbf{r}]_{x'y'z'} = BCD\frac{\partial}{\partial \psi}D^T C^T B^T [\mathbf{r}]_{x'y'z'} \\ &= B\frac{\partial}{\partial \psi}B^T[r]_{x'y'z'} = \begin{bmatrix} \cos\psi & \sin\psi & \\ -\sin\psi & \cos\psi & \\ & & 1 \end{bmatrix}\begin{bmatrix} -\sin\psi & -\cos\psi & \\ \cos\psi & -\sin\psi & \\ & & 0 \end{bmatrix}\begin{bmatrix} r_{x'} \\ r_{y'} \\ r_{z'} \end{bmatrix} = \begin{bmatrix} -r_{y'} \\ r_{x'} \\ 0 \end{bmatrix} \end{aligned}$$

Note that the inner product is invariant under orthonormal transformation

$$[\mathbf{F}]_{xyz} \cdot \left[\frac{\partial \mathbf{r}}{\partial \psi}\right]_{xyz} = [\mathbf{F}]_{x'y'z'} \cdot \left[\frac{\partial \mathbf{r}}{\partial \psi}\right]_{x'y'z'}$$

Hence

$$Q_\psi = [\mathbf{F}]_{x'y'z'} \cdot \begin{bmatrix} -r_{y'} \\ r_{x'} \\ 0 \end{bmatrix} = N_{z'}$$

□

Therefore we get one equation

$$I_{z'}\dot{\omega}_{z'} - (I_{x'} - I_{y'})\omega_{x'}\omega_{y'} = N_{z'}$$

Perform an cyclic permutation we can get other two equations

$$\begin{aligned} I_{x'}\dot{\omega}_{x'} - (I_{y'} - I_{z'})\omega_{y'}\omega_{z'} &= N_{x'} \\ I_{y'}\dot{\omega}_{y'} - (I_{z'} - I_{x'})\omega_{z'}\omega_{x'} &= N_{y'} \end{aligned}$$

Problem 2. A uniform rectangular block has mass M and sides $2a$, $2b$ and $2c$. Find the principal moments of inertia of the block

i) at its center of mass,

ii) at the center of a face of area $4ab$.

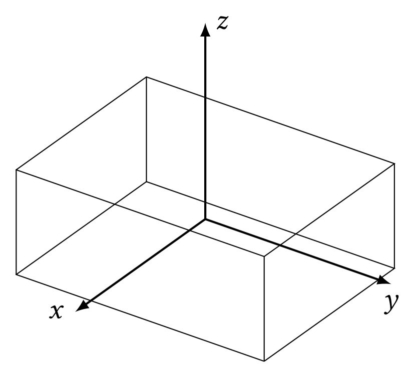
Find the moment of inertia of the block

i) About a space diagonal,

ii) about a diagonal of a face of area $4ab$.

Solution. For the principal moments of inertia

i) Set up the coordinate system similar to the following figure where the origin of the axis is coincide with the center of the rectangular block, and x, y, and z axis is parallelly aligned onto the sides with length a, b, and c, respectively.



Then I_{xx} equals to

$$I_{xx} = \int_{-a}^{a} \mathrm{d}x \int_{-b}^{b} \mathrm{d}y \int_{-c}^{c} \mathrm{d}z\ y^2 + z^2 = \frac{1}{3}M(b^2 + c^2)$$

and I_{xy} equals to

$$I_{xy} = \int_{-a}^{a} \mathrm{d}x \int_{-b}^{b} \mathrm{d}y \int_{-c}^{c} \mathrm{d}z(-xy) = 0$$

Hence we can find the inertia tensor I equals to

$$I = M\begin{pmatrix} (b^2+c^2)/3 & & \\ & (a^2+c^2)/3 & \\ & & (a^2+b^2)/3 \end{pmatrix}$$

where the diagonal terms are the principal moments of inertia.

ii) Set up a similar coordinate system as in the previous case, then I_{zz} doesn't change since the angular velocity axis doesn't change. Note that

$$I_{xx} = \int_{-a}^{a} \mathrm{d}x \int_{-b}^{b} \mathrm{d}y \int_{0}^{2c} \mathrm{d}z \; y^2 + z^2 = \frac{1}{3}M(b^2 + 4c^2)$$
$$I_{xy} = \int_{-a}^{a} \mathrm{d}x \int_{-b}^{b} \mathrm{d}y \int_{0}^{2c} \mathrm{d}z \; -xy = 0$$

Hence the tensor is

$$I = M\begin{pmatrix} (b^2+4c^2)/3 & & \\ & (a^2+4c^2)/3 & \\ & & (a^2+b^2)/3 \end{pmatrix}$$

where the diagonal terms are the principal moments of inertia.

For the moment inertia

i) To get the moment inertia along the space diagonal, we can set angular momentum vector $\boldsymbol{\omega}$ to

$$\boldsymbol{\omega} = \frac{1}{\sqrt{a^2+b^2+c^2}}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Then the moment of inertia about this axis is

$$\omega_i I_{ij} \omega_j = \frac{2M}{3} \frac{a^2b^2 + b^2c^2 + a^2c^2}{a^2+b^2+c^2}$$

ii) Let the angular momentum vector be

$$\boldsymbol{\omega} = \frac{1}{\sqrt{a^2+b^2}}\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$$

Then the moment of inertia about this axis is

$$\omega_i I_{ij} \omega_j = \frac{2M}{3} \frac{a^2b^2 + 2b^2c^2 + 2a^2c^2}{a^2+b^2}$$

Problem 3. Consider the torque-free motion of an asymmetric rigid body with one point fixed, show from Euler equations that L^2 and T (K and T are the angular momentum and kinetic energy) are conserved.

Solution. Note that $\mathbf{L}^2$ and its time derivative equals to

$$\mathbf{L}^2 = 2L_iL_i$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{L}^2 = 2L_i\dot{L}_i$$

The Euler's equation for torque-free motion is

$$\dot{L}_i + \epsilon_{ijk}\omega_j L_k = 0$$

Hence

$$L_i\dot{L}_i + \epsilon_{ijk}L_i\omega_j L_k = 0$$

$$\Rightarrow L_i\dot{L}_i = -\epsilon_{ijk}L_iL_k\omega_j = \epsilon_{ikj}L_iL_k\omega_j = 0$$

which means that $\mathbf{L}^2$ is conserved.

The kinetic energy and its time derivative writes

$$\begin{aligned} T &= \frac{1}{2}\omega_i I_{ij}\omega_j \\ \frac{\mathrm{d}}{\mathrm{d}t}T &= \frac{1}{2}\dot{\omega}_i I_{ij}\omega_j + \frac{1}{2}\omega_i I_{ij}\dot{\omega}_j = \omega_i I_{ij}\dot{\omega}_j = \omega_i\dot{L}_i \end{aligned}$$

Then from Euler's equation we know that

$$\begin{aligned} \omega_i\dot{L}_i + \epsilon_{ijk}\omega_i\omega_j L_k = 0 \\ \Rightarrow \omega_i\dot{L}_i = -\epsilon_{ijk}\omega_i\omega_j L_k = 0 \end{aligned}$$

which means that T is conserved.

Problem 4. For the axially symmetric rigid body precessing uniformly in the absence of torques, find analytical solutions for the Euler angles as a function of time.

Solution. Let $I_x = I_y \neq I_z$. Then the torque-free Euler's equation is

$$\begin{aligned} I_x\dot{\omega}_x - (I_y - I_z)\omega_y\omega_z &= 0 \\ I_y\dot{\omega}_y - (I_z - I_x)\omega_z\omega_x &= 0 \\ I_z\dot{\omega}_z &= 0 \end{aligned}$$

Hence we can solve that $\omega_z = \text{const}$. Define $a = (I_x - I_z)\omega_z/I_x$, then

$$\begin{aligned} \dot{\omega}_x &= a\omega_y \\ \dot{\omega}_y &= -a\omega_x \end{aligned} \Rightarrow \begin{aligned} \ddot{\omega}_x &= -a^2\omega_x \\ \ddot{\omega}_y &= -a^2\omega_y \end{aligned} \Rightarrow \begin{aligned} \omega_x &= A\sin(at+b) \\ \omega_y &= A\cos(at+b) \end{aligned}$$

Hence the angular momentum $\mathbf{L}$ under the body coordinates is

$$[\mathbf{L}]_b = I[\boldsymbol{\omega}]_b = \begin{bmatrix} I_xA\sin(at+b) \\ I_xA\cos(at+b) \\ I_z\omega_z \end{bmatrix}$$

Since this is a torque-free precession, $\mathbf{L}$ conserved under the space coordinate. Let $\mathbf{L}$ align onto the z axis, i.e.

$$[\mathbf{L}]_s = \begin{bmatrix} 0 \\ 0 \\ L \end{bmatrix}$$

Hence

$$[\mathbf{L}]_b = A[\mathbf{L}]_s = L\begin{bmatrix} \sin\theta\sin\psi \\ \cos\psi\sin\theta \\ \cos\theta \end{bmatrix}$$

Therefore we can establish an equality

$$A[\mathbf{L}]_s = L\begin{bmatrix} \sin\theta\sin\psi \\ \cos\psi\sin\theta \\ \cos\theta \end{bmatrix} = \begin{bmatrix} I_xA\sin(at+b) \\ I_xA\cos(at+b) \\ I_z\omega_z \end{bmatrix} \tag{1}$$

Therefore $\theta = \text{const}$. Suppose $L > 0$, we can also solve get $\psi = at + b$ (else, $\psi = at + b + \pi$). Recall from the lecture that

$$[\boldsymbol{\omega}]_b = \begin{bmatrix} \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\cos\psi \\ \dot{\psi} + \dot{\phi}\cos\theta \end{bmatrix} = \begin{bmatrix} A\sin(at+b) \\ A\cos(at+b) \\ \omega_z \end{bmatrix} \tag{2}$$

Since $\dot{\theta} = 0$ and $\psi = ax + b$, we can solve $\dot{\phi}$

$$\dot{\phi} = \frac{A}{\sin\theta}$$

Hence we have the analytical solution for Euler angles

$$\begin{aligned} \phi &= \frac{I_x A}{\sin\theta} t + C \\ \theta &= \text{const} \\ \psi &= at + b \end{aligned}$$

The solution can be further simplified since we can solve A and L wrt θ and ω_z, from Equation 1 and 2

$$L = \frac{I_z \omega_z}{\cos\theta}, \ A = \frac{L\sin\theta}{I_x} = \frac{I_z}{I_x}\tan\theta\omega_z$$

Thus

$$\begin{aligned} \phi &= \frac{I_z \omega_z}{I_x \cos\theta} \\ \theta &= \text{const} \\ \psi &= \frac{(I_x - I_z)}{I_x}\omega_z t + b \end{aligned}$$