

PHY3110 SP23 Notes

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0 Introduction

Grading: 30% homework, 30% midterm, 40% final.

Textbooks:

- H. Goldstein, C. Poole, J. Safko, Classical Mechanics, 3rd Edition, Pearson.
- J.R. Taylor, Classical Mechanics, University Science Books.
- T.W.B. Kibble, F.H. Berkshire, Classical Mechanics, 5th Edition, Imperial College Press.
- 梁昆森, 力学 (下册) 理论力学, 4th Edition, 高等教育出版社.

Classical mechanics describe the motion of macroscopic objects, which are not extremely massive and not extremely fast.

1 Newtonian Mechanics

Vectorial quantities of motion: position \mathbf{r} , velocity \mathbf{v} , force \mathbf{F} , momentum $\mathbf{p} = m\mathbf{v}$, angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. Equations of motion are derived from those vector quantities.

Analytical mechanics uses scalar quantities of motion

- Kinetic energy $T = \frac{1}{2}m\mathbf{v}^2$
- Potential energy $V = V(\mathbf{r})$

Equations of motion are derived from those scalar quantities.

1.1 Newton's Laws

Theorem 1.1 (Newton's 2nd law).

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = m\mathbf{a} \quad (1)$$

The formula is valid in an inertial frame.

Angular momentum \mathbf{L} and torque \mathbf{N} are also related

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \mathbf{r} \times \mathbf{F} = \mathbf{N} \quad (2)$$

Work done by external forces (note that $d\mathbf{s} = \mathbf{v} dt$)

$$W_{12} = \int_1^2 \mathbf{F} d\mathbf{s} = \int_1^2 m \frac{d\mathbf{v}}{dt} d\mathbf{s} = \int_1^2 m\mathbf{v} d\mathbf{v} = \frac{1}{2}m\mathbf{v}^2 \Big|_1^2 \quad (3)$$

Define a scalar function $V(\mathbf{r})$, then $\mathbf{F} = -\nabla V(\mathbf{r})$ is a conservative force.

$$\oint \mathbf{F} d\mathbf{s} = 0 \quad (4)$$

Center of mass of the system

$$\mathbf{R} = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i} = \frac{\sum_i m_i \mathbf{r}_i}{M} \quad (5)$$



Total momentum

$$\mathbf{P} = \sum_i m_i \mathbf{p}_i = M \dot{\mathbf{R}} \quad (6)$$

Hence \mathbf{P} is conserved if external force $\mathbf{F}^{(e)}$ is zero.

Total angular momentum

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt} \sum_i \mathbf{r}_i \times \mathbf{p}_i = \sum_i \mathbf{r}_i \times \left(\mathbf{F}_i^{(e)} + \sum_j \mathbf{F}_{ij} \right) = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{(e)} + \sum_{ij} \mathbf{r}_i \times \mathbf{F}_{ij}$$

Since \mathbf{r}_{ij} parallel to \mathbf{F}_{ij} , then

$$\sum_{ij} \mathbf{r}_i \mathbf{F}_{ij} = \frac{1}{2} \sum_{ij} \mathbf{r}_{ij} \times \mathbf{F}_{ji} = 0 \quad (7)$$

Therefore

$$\frac{d\mathbf{L}}{dt} = \mathbf{N}^{(e)} \quad (8)$$

Decomposition of the angular momentum

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i = \sum_i (\mathbf{R} + \mathbf{r}_i) \times m_i (\mathbf{V} + \mathbf{v}'_i) = \sum_i \mathbf{R} \times m_i \mathbf{V} + \sum_i \mathbf{r}'_i \times m_i \mathbf{v}'_i \quad (9)$$

1.2 Constraints

Holonomic constraint

$$f(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N, t) = 0 \quad (10)$$

Example: rigid body

$$(\mathbf{r}_i - \mathbf{r}_j)^2 - c_{ij}^2 = 0 \quad (11)$$

Example: non-sliding cylinder

$$\dot{x} - R\dot{\theta} = 0 \Rightarrow x - R\theta = \text{const}$$

A constraint of the form

$$\sum_i g_i(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) d\mathbf{x}_i = 0 \Rightarrow dG(\mathbf{x}_1, \dots) = 0 \Rightarrow G(\mathbf{x}_1, \dots) = \text{const} \quad (12)$$

Non-holonomic constraint: cannot be written in the form of holonomic constraint.

1.3 Generalized coordinates

Suppose we have a N -particle system, we will have $3N$ DOFs. With k constraints, we will have $3N - k$ DOFs. Define q_1, \dots, q_{3N-k} generalized coordinates, we have

$$\mathbf{r}_i = \mathbf{r}_i(q_1, \dots, q_{3N-k}, t) \quad (13)$$

2 Lagrange Formalism

2.1 D'Alembert's Principle

Hint from the rigid body: internal forces of constraints do not work.

Virtual displacement: $\delta \mathbf{r}_i$ is consistent with the constraints imposed on the system at a given time

$$\mathbf{r}_i \rightarrow \mathbf{r}_i + \delta \mathbf{r}_i \quad (1)$$

Theorem 2.1 (D'Alembert's principle). Consider a system in equilibrium

$$\mathbf{F}_i = 0 \Rightarrow \sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i = 0 \quad (2)$$

Separate $\mathbf{F}_i = \mathbf{F}_i^{(a)} + \mathbf{f}_i$ where \mathbf{f}_i is the constraint force. Hence

$$\sum_i (\mathbf{F}_i^{(a)} + \mathbf{f}_i) \cdot \delta \mathbf{r}_i = 0 \Rightarrow \sum_i \mathbf{F}_i^{(a)} \cdot \delta \mathbf{r}_i = 0 \quad (3)$$

For a system moving under external forces

$$\mathbf{F}_i - \dot{\mathbf{p}}_i = 0 \Rightarrow \sum_i (\mathbf{F}_i - \dot{\mathbf{p}}_i) \delta \mathbf{r}_i = 0 \Rightarrow \sum_i (\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i) \delta \mathbf{r}_i = 0 \quad (4)$$

For holonomic constraints

$$\mathbf{r}_i = \mathbf{r}_i(q_1, \dots, q_n, t), \quad \mathbf{v}_i = \frac{d\mathbf{r}_i}{dt} = \frac{\partial \mathbf{r}_i}{\partial t} + \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \dot{q}_j, \quad \delta \mathbf{r}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \quad (5)$$

Define generalized force Q_j

$$\sum_i \mathbf{F}_i \delta \mathbf{r}_i = \sum_{ij} \mathbf{F}_i \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j = \sum_j Q_j \delta q_j \quad (6)$$

Then

$$\sum_i \dot{\mathbf{p}}_i \cdot \delta \mathbf{r}_i = \sum_{ij} m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j = \sum_{ij} \left[\frac{d}{dt} \left(m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \right) - m_i \dot{\mathbf{r}}_i \cdot \frac{d}{dt} \frac{\partial \mathbf{r}_i}{\partial q_j} \right] \delta q_j \quad (7)$$

$$= \sum_j \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] \delta q_j = \sum_j Q_j \delta q_j \quad (8)$$

$$(9)$$

where

$$\frac{\partial T}{\partial q_j} = \sum_{ik} \frac{\partial \dot{\mathbf{r}}_k}{\partial q_j} \frac{\partial}{\partial \dot{\mathbf{r}}_k} T = \sum_k m_k \dot{\mathbf{r}}_k \frac{\partial \dot{\mathbf{r}}_k}{\partial q_j} = \sum_k m_k \dot{\mathbf{r}}_k \frac{d}{dt} \frac{\partial \mathbf{r}_k}{\partial q_j} \quad (10)$$

Hence $\forall j$ we have

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j = 0 \quad (11)$$

Let the potential energy $V = V(\mathbf{r}_i, \dots) = V(q_j, \dots)$, then we have

$$Q_j = \sum_i \mathbf{F}_i \frac{\partial \mathbf{r}_i}{\partial q_j} = \sum_i -\nabla_i V \frac{\partial \mathbf{r}_i}{\partial q_j} = -\frac{\partial V}{\partial q_j} \quad (12)$$

Therefore

$$\frac{d}{dt} \left(\frac{\partial(T-V)}{\partial \dot{q}_j} \right) - \frac{\partial(T-V)}{\partial q_j} - Q_j = 0 \quad (13)$$

Theorem 2.2 (Lagrange's equation). Define $L = T - V$, then

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0 \quad (14)$$

The choice of Lagrangian is not unique, L' where

$$L' = L + \frac{dF(q, t)}{dt} \quad (15)$$

will give the same equations of motion as L .

Example 2.1 (Lagrange's formalism).

1) For a single particle moving under force \mathbf{F}

$$L(\mathbf{x}, \dot{\mathbf{x}}, t) = \frac{1}{2} m \dot{\mathbf{x}}^2 + \mathbf{F} \cdot \mathbf{x}$$

2) Motion in a 2D plane using polar coordinates

$$L(r, \theta, \dot{r}, \dot{\theta}, t) = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + \mathbf{F} \cdot \mathbf{r}$$

Generalized forces

$$Q_r = \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial r} = \mathbf{F} \cdot \mathbf{e}_r$$

$$Q_\theta = \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial \theta} = \mathbf{F} \cdot r \mathbf{e}_\theta$$

where

$$\mathbf{e}_r = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \quad \mathbf{e}_\theta = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$$

Equations of motion

$$\begin{aligned} m\ddot{r} - mr\dot{\theta}^2 &= \mathbf{F} \cdot \mathbf{e}_r \\ mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} &= r\mathbf{F}_\theta \end{aligned}$$

3) Atwood's machine

$$L = \frac{1}{2} (M_1 + M_2) \dot{x}^2 + M_1 g x + M_2 g (l - x)$$

The equation of motion is

$$\begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} &= 0 \\ \Rightarrow (M_1 + M_2) \ddot{x} &= (M_1 - M_2) g \end{aligned}$$

Suppose we have a potential dependent on velocity (generalized potential) and the generalized force is defined as

$$U = U(q_j, \dot{q}_j), \quad Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt} \frac{\partial U}{\partial \dot{q}_j} \quad (16)$$

Define $L = T - U$, then we still have

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \quad (17)$$

Example 2.2 (Lorentz force on a moving charge). The Lorentz force

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Define the scalar and vector potentials

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

Hence

$$\mathbf{F} = q \left[-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \times (\nabla \times \mathbf{A}) \right]$$

2.2 Hamilton's principle, variational principle

Configuration space: a space formed by the set of generalized coordinates.

$$(q_1, q_2, \dots, q_n) \text{ as function of } t \quad (18)$$

Theorem 2.3 (Hamilton's principle). Define the action integral I , where $L = T - V$ or $L = T - U$ (U is the generalized potential)

$$I = \int_{t_1}^{t_2} L dt \quad (19)$$

Then the variation of the action integral equals to zero

$$\delta I = \delta \int_{t_1}^{t_2} L(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n) dt = 0 \quad (20)$$

Add small variation on the path

$$q_i(t, \alpha) = q_i(t) + \alpha \eta(t) \quad (21)$$

where $\eta(t_1) = \eta(t_2) = 0$. Then the action will be the function of α , $I = I(\alpha)$. Hence

$$\delta I = \int_{t_1}^{t_2} \left(\sum_i \frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) dt \quad (22)$$

Change the order of differentiation $\delta \dot{q}_i = d\delta q_i / dt$, then

$$\delta I(\alpha) = \int_{t_1}^{t_2} \sum_i \left[\frac{\partial L}{\partial q_i} \delta q_i - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \right] dt + \sum_i \frac{\partial L}{\partial \dot{q}_i} \delta q_i \Big|_{t_1}^{t_2} \quad (23)$$

$$= \int_{t_1}^{t_2} \sum_i \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right] \delta q_i dt = 0 \quad (24)$$

$$\Rightarrow \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q_i} = 0 \quad (25)$$

Example 2.3 (Shortest path problem). $y = y(x)$, $ds = \sqrt{dx^2 + dy^2}$, then the action integral (path) is

$$I = \int_1^2 ds = \int_1^2 \sqrt{1 + \dot{y}^2} dx \quad (26)$$

Apply the Lagrange's equation we get

$$\frac{d}{dx} \frac{d\sqrt{1 + \dot{y}^2}}{d\dot{y}} = 0 \Rightarrow \frac{d\dot{y}}{dx} = 0 \Rightarrow y = ax + b \quad (27)$$

Example 2.4 (Solid of revolution). Differential of area $2\pi x ds = 2\pi x \sqrt{1 + \dot{y}^2} dx$, then the total area is

$$\int_1^2 2\pi x \sqrt{1 + \dot{y}^2} dx \quad (28)$$

Define the Lagrangian $L(x, y, \dot{y}) = 2\pi x \sqrt{1 + \dot{y}^2}$, by Lagrange's equation we can get

$$\frac{x\dot{y}}{\sqrt{1 + \dot{y}^2}} = \text{const} \Rightarrow y = a \cosh^{-1} \frac{x}{a} + b \quad (29)$$

Example 2.5 (The curve of fastest descent). Question: along which trajectory from point 1 to point 2, the time is shortest? The total time is

$$T = \int_1^2 \frac{ds}{v} = \int_1^2 \frac{ds}{\sqrt{2gy}} \quad (30)$$

According to Newton's laws we have $y = gv^2$, then

$$T = \int_1^2 \frac{\sqrt{1 + \dot{y}^2}}{\sqrt{2gy}} dx \quad (31)$$

Then we have $L(x, y, \dot{y})$ and we get

$$2y\ddot{y} + 1 + \dot{y}^2 = 0 \quad (32)$$

$$\Rightarrow 2y\dot{y}\ddot{y} + \dot{y} + \dot{y}^3 = \dot{y}(1 + \dot{y}^2) + y(2\dot{y}\ddot{y}) = \frac{d}{dt} \dot{y}(1 + \dot{y}^2) = 0 \quad (33)$$

$$\Rightarrow y(1 + \dot{y}^2) = \text{const} \quad (34)$$

which means that $y(1 + \dot{y}^2) = \text{const}$. The solution is $x = A(\theta - \sin \theta)$, $y = A(1 - \cos \theta)$.

2.3 Constraint

Holonomic constraints

$$f(q_1, \dots, q_n, t) = 0 \quad (35)$$

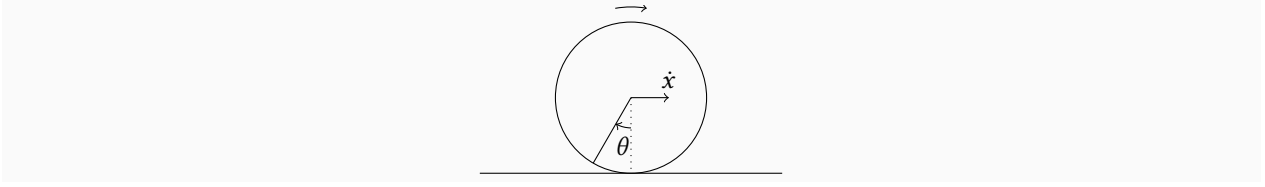
Non-holonomic constraints

$$f(q_1, \dots, q_n; \dot{q}_1, \dots, \dot{q}_n; t) = 0 \quad (36)$$

Sometimes we can convert $f(\dot{q}_i) = 0$ to $f'(q_i) = 0$.

Example 2.6 (Rolling cylinder). Rolling cylinder without sliding has the constraint

$$\dot{x} - R\dot{\theta} = 0 \Rightarrow x - R\theta = \text{const} \quad (37)$$



A commonly encountered type of non-holonomic constraint is linear constraint equations

$$\sum_{k=1}^n a_{ik} \frac{dq_k}{dt} + a_i t = 0 \quad (38)$$

For the virtual displacement δq_i

$$\sum_{k=1}^n a_{ik} \delta q_k = 0 \quad (39)$$

Suppose q_i are not independent, then the Euler-Lagrange's equation could not hold.

$$\int_{t_1}^{t_2} \sum_i \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right] \delta q_i dt = 0 \Rightarrow \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q_i} = 0$$

Include the constraint equation of q_i into the equation

$$\sum_{k=1}^n a_{ik} \delta q_k = 0 \Rightarrow \sum_{i=1}^m \lambda_i \sum_{k=1}^n a_{ik} \delta q_k = 0 \quad (40)$$

Hence

$$\int_{t_1}^{t_2} \sum_{k=1}^n \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) + \sum_{i=1}^m \lambda_i a_{ik} \right] \delta q_k dt = 0$$

Let q_1, \dots, q_{n-m} be independent generalized coordinate, q_{n-m+1}, \dots, q_n dependent generalized coordinates (i.e., they can be expressed by q_1, \dots, q_{n-m}). Choose λ_i s.t.

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) + \sum_{i=1}^m \lambda_i a_{ik} = 0 \quad (41)$$

$\forall k = n-m+1, \dots, n$. In conclusion, we have $q_1, \dots, q_n, \lambda_1, \dots, \lambda_m$ overall $n+m$ unknowns, and n Lagrange's equations and m constraint equations overall $n+m$ equations.

Remark.

- 1) It is inconvenient to reduce all q_k s to independent coordinates
- 2) If we are interested in the constraint forces

$$\sum_{k=1}^n a_{ik} dq_k + a_i t = 0$$

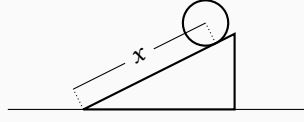
where

$$a_{ik} = \frac{\partial f_i}{\partial q_k}, \quad a_{it} = \frac{\partial f_i}{\partial t}$$

Then

$$df_i = \frac{\partial f_i}{\partial q_k} dq_k + \frac{\partial f_i}{\partial t} dt \Rightarrow df_i = 0, \quad f_i = \text{const}$$

Example 2.7 (Hoop rolling down an inclined plane).



The constraint equation writes

$$\dot{x} - r\dot{\theta} = 0 \Rightarrow x - r\theta = \text{const}$$

$a_x = 1, a_\theta = -r, a_t = 0$.

Energy terms are

$$\begin{aligned} T &= T_{\text{COM}} + T_{\text{relative}} = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}M(r\dot{\theta})^2 \\ V &= Mg(l - x) \sin \phi \\ \Rightarrow L &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}M(r\dot{\theta})^2 - Mg(l - x) \sin \phi \end{aligned}$$

Then we can write the Lagrange's equation with Lagrange's multipliers

$$\begin{aligned} \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} + a_x \lambda &= 0 \\ \Rightarrow Mg \sin \phi - M\ddot{x} + \lambda &= 0 \\ -Mr^2\ddot{\theta} - \lambda r &= 0 \\ \dot{x} = r\dot{\theta} \Rightarrow \ddot{x} &= r\ddot{\theta} \end{aligned}$$

we can get $M\ddot{x} = Mr\ddot{\theta} = -\lambda$ and $\ddot{x} = (g \sin \phi)/2$. Note that λ is the constraint force (in this case λ is the frictional force).

2.4 Lagrangian for Lorentz force

The Lorentz force is given by

$$\mathbf{F} = q[\mathbf{E} + \mathbf{v} \times \mathbf{B}] \quad (42)$$

Where

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A} \quad (43)$$

Hence, by defining the Lagrangian

$$L = \frac{1}{2}m\mathbf{v}^2 - q\phi + q\mathbf{A} \cdot \mathbf{v} \quad (44)$$

Applying the Lagrange's equation we can get the EOM (eq. 42). Take x as an example

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{d}{dt} (m\dot{x} + qA_x) = m\ddot{x} + q \frac{\partial A_x}{\partial t} + \mathbf{v} \cdot \nabla A_x \quad (45)$$

2.5 Conservation & symmetry of the system

Definition 2.1 (Cyclic coordinate). The generalized coordinate q_i is cyclic (ignorable) if

$$\frac{\partial L}{\partial q_i} = 0 \quad (46)$$

It implies the generalized momentum p_i is conserved.

Rotational symmetry. Let q_j be one of the rotational angle of spacial coordinate \mathbf{r}_i . Hence

$$d\mathbf{r}_i = \mathbf{n}_j \times \mathbf{r}_i dq_j \Rightarrow \frac{\partial \mathbf{r}_i}{\partial q_j} = \mathbf{n}_j \times \mathbf{r}_i dq_j \quad (47)$$

where \mathbf{n}_j is the normal vector of the rotation axis of q_j . Hence the generalized force of q_j writes

$$Q_{q_j} = -\frac{\partial V}{\partial q_j} = -\sum_k \frac{\partial V}{\partial \mathbf{r}_k} \frac{\partial \mathbf{r}_k}{\partial q_j} = -\frac{\partial V}{\partial \mathbf{r}_i} \frac{\partial \mathbf{r}_i}{\partial q_j} = \mathbf{F}_i \cdot (\mathbf{n}_j \times \mathbf{r}_i) = \mathbf{n}_j \cdot (\mathbf{r}_i \times \mathbf{F}_i) = \mathbf{n}_j \cdot \mathbf{N}_i \quad (48)$$

where \mathbf{N}_i stands for the torque on the i^{th} particle. The generalized momentum of q_j writes

$$p_{q_j} = \frac{\partial T}{\partial \dot{q}_j} = \sum_k \frac{\partial T}{\partial \dot{\mathbf{r}}_k} \frac{\partial \dot{\mathbf{r}}_k}{\partial \dot{q}_j} = \sum_k m_k \dot{\mathbf{r}}_k \cdot \frac{\partial \dot{\mathbf{r}}_k}{\partial \dot{q}_j} = m_i \dot{\mathbf{r}}_i \cdot (\mathbf{n}_j \times \mathbf{r}_i) = \mathbf{n}_j \cdot (\mathbf{r}_i \times m_i \dot{\mathbf{r}}_i) = \mathbf{r}_i \times \mathbf{L}_i \quad (49)$$

where \mathbf{L}_i stands for the angular momentum of the i^{th} particle. Hence, the rotational invariance implies the conservation of angular momentum.

Time translation

$$\frac{d}{dt}L = \sum_i \frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i + \frac{\partial L}{\partial t} \quad (50)$$

$$= \sum_i \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i + \frac{\partial L}{\partial t} \quad (51)$$

$$= \frac{d}{dt} \sum_i \left(\frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) + \frac{\partial L}{\partial t} \quad (52)$$

$$\Rightarrow \frac{\partial L}{\partial t} + \frac{d}{dt} \underbrace{\left(\sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right)}_H = 0 \quad (53)$$

Note that

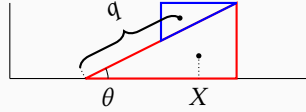
$$\sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i = 2T \quad (54)$$

Proof. Suppose r_i does not have explicit time dependence ($\partial r_i / \partial t = 0$)

$$\begin{aligned} T &= \sum_i \frac{1}{2} m_i \dot{r}_i^2 = \sum_i \frac{1}{2} m_i \left(\sum_j \frac{\partial r_i}{\partial q_j} \dot{q}_j + \frac{\partial r_i}{\partial t} \right)^2 \\ &= \sum_{ijk} \frac{1}{2} m_i \frac{\partial r_i}{\partial q_j} \frac{\partial r_i}{\partial q_k} \dot{q}_j \dot{q}_k \\ &\Rightarrow \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i = \sum_i \dot{q}_i \sum_{jk} m_k \frac{\partial r_k}{\partial q_i} \frac{\partial r_k}{\partial q_j} \dot{q}_j = \sum_{ijk} m_k \frac{\partial r_k}{\partial q_i} \frac{\partial r_k}{\partial q_j} \dot{q}_i \dot{q}_j = 2T \quad \square \end{aligned}$$

Hence we can define Hamiltonian $H = T + V$ which stands for the total energy, and H conserved if L doesn't depend on time explicitly.

Example 2.8 (Two blocks). Let M be the mass of the big block, m be the mass of the small block. Define two generalized coordinates: X stand for the position of COM of the big block, q stand for the position of COM of the small block (sloped).



Hence we can easily define the Lagrangian

$$L = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m[(\dot{X} + \dot{q} \cos \theta)^2 + \dot{q}^2 \sin^2 \theta] - mgq \sin \theta$$

There is no X dependence on the system, hence X is a cyclic coordinate.

$$\begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial \dot{X}} &= \frac{d}{dt} M\dot{X} + m(\dot{X} + \dot{q} \cos \theta) = 0 \\ \Rightarrow M\dot{X} + m(\dot{X} + \dot{q} \cos \theta) &= p_X = \text{const} \end{aligned}$$

3 The Central Force Problem

3.1 Reduction to the equivalent one-body problem

For the two-body problem, we have two choices of generalized coordinate

- (a) $\mathbf{r}_1, \mathbf{r}_2$ stand for spacial coordinates of two masses
- (b) \mathbf{R} spacial coordinate of the COM, $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

Hence, we can rewrite the kinetic energy in terms of \mathbf{R} and \mathbf{r}

$$T = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 \quad (1)$$

$$= \frac{1}{2}m_1\left(\dot{\mathbf{R}} + \frac{m_2}{m_1+m_2}\dot{\mathbf{r}}\right)^2 + \frac{1}{2}m_2\left(\dot{\mathbf{R}} - \frac{m_1}{m_1+m_2}\dot{\mathbf{r}}\right)^2 \quad (2)$$

$$= \frac{1}{2}(m_1+m_2)\dot{\mathbf{R}}^2 + \frac{m_1m_2}{m_1+m_2}\dot{\mathbf{r}}^2 = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 \quad (3)$$

Hence we can write the Lagrangian

$$L = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 - V \quad (4)$$

Suppose $V = V(\mathbf{r})$, then \mathbf{R} is a cyclic coordinate, We have $\dot{\mathbf{R}} = \text{const}$, and we can drop $\dot{\mathbf{R}}$ terms in L . Moreover, if $V = V(\|\mathbf{r}\|)$, the total angular momentum is conserved.

Use r, θ as generalized coordinates, we have

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \quad (5)$$

The Lagrange's equation wrt r writes

$$\mu\ddot{r} - \mu r\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0 \quad (6)$$

Easy to find that θ is cyclic, hence

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt}\mu r^2\dot{\theta} = 0 \Rightarrow p_\theta = \mu r^2\dot{\theta} = l = \text{const} \quad (7)$$

Theorem 3.1 (Kepler's 2nd law). Radius vector sweeps out equal areas in equal time.

Substitute $\mu r\dot{\theta}^2$ term by l , we have

$$\mu\ddot{r} - \frac{l^2}{\mu r^3} + \frac{\partial V}{\partial r} = 0 \quad (8)$$

$$\Rightarrow \mu\ddot{r} = -\frac{\partial}{\partial r}\left(V + \frac{l^2}{2\mu r^2}\right) \quad (9)$$

$$\mu\ddot{r} = -\frac{\partial}{\partial r}\dot{r}\left(V + \frac{l^2}{2\mu r^2}\right) \quad (10)$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}\mu\dot{r}^2 + V + \frac{l^2}{2\mu r^2}\right) = \frac{d}{dt}E = 0 \quad (11)$$

Hence we can get the expression of \dot{r} and a differential equation

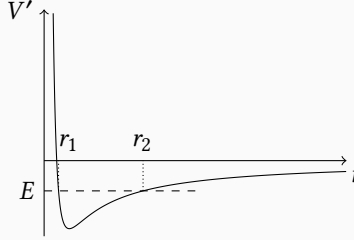
$$\dot{r} = \sqrt{\frac{2}{\mu}\left(E - V - \frac{l^2}{2\mu r^2}\right)} \quad (12)$$

Define the new effective force $f'(r)$ with an effective potential $V' = V + l^2/2\mu r^2$, we have

$$\mu\ddot{r} = -\frac{\partial V}{\partial r} + \frac{l^2}{\mu r^3} = f'(r), \quad E = \frac{1}{2}\mu\dot{r}^2 + V' = \text{const} \quad (13)$$

Example 3.1 (Gravitational force). Suppose we have $f(r) = -kr^{-2}$ and $V = -kr^{-1}$, then

$$V' = -\frac{k}{r} + \frac{l^2}{2\mu r^2}$$



- 1) $E = E_1 \geq 0$, the motion has a lower bound, $r \geq r_1$.
- 2) $V_{\min} < E_2 < 0$, the motion has lower and upper bound, $r_1 \leq r \leq r_2$.
- 3) $E_3 = V_{\min}$, the motion will shrink to a single circle $r_1 = r_2 = \text{const}$, hence it is a circular motion. In this case, the gravitational force is equal to the centrifugal force

$$\mu \ddot{r} = f(r) + \frac{l^2}{\mu r^3} = f(r) + \mu r \dot{\theta}^2 = 0 \Rightarrow f(r) = -\mu r \dot{\theta}^2$$

Remark. Let the potential be $V = -kr^{-\alpha}$, then the motion cannot have periodic behavior if $\alpha > 2$.

Example 3.2 (Harmonic oscillator). $V = kr^2/2$, we have

$$V' = \frac{1}{2}kr^2 + \frac{l^2}{2\mu r^2}$$

Theorem 3.2 (Conditions for closed orbitals, Bertrand's theorem). Stable orbitals require

$$\left. \frac{\partial^2 V'}{\partial r^2} \right|_{r=r_0} = -\frac{\partial f}{\partial r} + \frac{3l^2}{mr^4} \Big|_{r=r_0} > 0 \Rightarrow \left. \frac{\partial f}{\partial r} \right|_{r=r_0} < -\frac{3f(r_0)}{r_0} \quad (14)$$

Let $f = -kr^n$, then we have

$$-knr^{n-1} < 3kr^{n-1} \Rightarrow n > -3 \quad (15)$$

For a small perturbation from the minimum, we can write the effective potential as Taylor expansion

$$V'(r) = V'(r_0) + \left. \frac{\partial V'}{\partial r} \right|_{r=r_0} (r - r_0) + \frac{1}{2} \left. \frac{\partial^2 V'}{\partial r^2} \right|_{r=r_0} (r - r_0)^2 + \dots \quad (16)$$

$$= V'(r_0) + \frac{1}{2} \left. \frac{\partial^2 V'}{\partial r^2} \right|_{r=r_0} (r - r_0)^2 + O(|r - r_0|^3) \quad (17)$$

Hence the EOM becomes

$$m\ddot{r} = -\frac{\partial V'}{\partial r} = -\left. \frac{\partial^2 V'}{\partial r^2} \right|_{r=r_0} (r - r_0) \quad (18)$$

$$\dot{\theta} = \frac{l}{mr^2} \quad (19)$$

The solution takes the form

$$u = u_0 + a \cos \beta \theta, \quad u = \frac{1}{r}, \quad \beta^2 = \left. \frac{r}{f} \frac{\partial f}{\partial r} + 3 \right|_{r=r_0} \quad (20)$$

For finite perturbation, the orbit can be a closed only if $\beta^2 = 1$ or 4 . Then there are only two types of central-force scalar problem with the property that all bound orbitals are closed orbitals.

$$\left. \frac{r}{f} \frac{\partial f}{\partial r} \right|_{r=r_0} = \pm 2 \Rightarrow f = -kr^{-2} \text{ or } -kr \quad (21)$$

check the theorem on other books

3.2 Virial theorem

Consider a multi-particle system $1 \leq i \leq N$.

$$\dot{\mathbf{p}}_i = \mathbf{F}_i \quad (22)$$

Define new function

$$G = \sum_i \mathbf{p}_i \cdot \mathbf{r}_i \quad (23)$$

Then the time derivative of G writes

$$\frac{d}{dt}G = \sum_i m_i \dot{\mathbf{r}}_i^2 + \sum_i \mathbf{F}_i \cdot \mathbf{r}_i \quad (24)$$

Hence the time average of dG/dt is

$$\frac{1}{\tau} \int_0^\tau \frac{d}{dt}G dt = 2\bar{T} + \overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i} = \frac{1}{\tau} [G(\tau) - G(0)] \quad (25)$$

The time average equals to 0 if τ is the period of the periodic motion; even for non-periodic motion, if $\tau \rightarrow \infty$, the average will approach 0. Hence

$$2\bar{T} + \overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i} = 0 \Rightarrow \bar{T} = -\frac{1}{2} \underbrace{\overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i}}_{\text{virial of Clausius}} \quad (26)$$

Example 3.3 (Ideal gas). Temperature $T \propto T_i$ kinetic energy of i^{th} particle.

$$T_i = \frac{3}{2} k_B T \quad (27)$$

Hence the time average of total kinetic energy is

$$\overline{\sum_i T_i} = \frac{3}{2} N k_B T \quad (28)$$

The virial could be written as

$$\overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i} = \int d\mathbf{F} \cdot \mathbf{r} \quad (29)$$

Note that

$$d\mathbf{F} = -p d\mathbf{A}\mathbf{n}$$

Applying Gauss's law we have

$$\overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i} = - \int p d\mathbf{A} \mathbf{n} \cdot \mathbf{r} = - \int p dV (\nabla \cdot \mathbf{r}) = -3pV \quad (30)$$

Hence by the equation we get $pV = nk_B T$.

Example 3.4 (Gravational force). Since $\mathbf{F}_i = -\nabla_i V$, given $V_i = ar_i^{n+1}$, we have

$$\bar{T} = \frac{1}{2} \overline{\sum_i \nabla_i V \cdot \mathbf{r}_i} = \frac{n+1}{2} \overline{\sum_i ar_i^{n+1}} = \frac{n+1}{2} \bar{V} \quad (31)$$

This theorem is particularly important when the potential energy is a homogeneous function of the coordinates, i.e.

$$U(\alpha \mathbf{r}_1, \dots, \alpha \mathbf{r}_n) = \alpha^k U(\mathbf{r}_1, \dots, \mathbf{r}_n) \quad (32)$$

Note that

$$\frac{dU}{d\alpha} = \sum_a \frac{\partial \alpha \mathbf{r}_a}{\partial \alpha} \cdot \frac{\partial U}{\partial \alpha \mathbf{r}_a} = \sum_a \frac{\mathbf{r}_a}{\alpha} \cdot \frac{\partial U}{\partial \mathbf{r}_a} = k \alpha^{k-1} U \quad (33)$$

Then the equation 26 becomes

$$2\bar{T} = k\bar{U} \quad (34)$$

Equivalently

$$\bar{U} = \frac{2E}{k+2}, \quad \bar{T} = \frac{kE}{k+2} \quad (35)$$

3.3 Inverse-square force

Consider $f = -k/r^2$, $V = -k/r$, then we have EOM

$$\dot{r} = \left[\frac{2}{m} \left(E + \frac{k}{r} - \frac{l^2}{2mr^2} \right) \right]^{1/2} \Rightarrow \theta = \theta_0 + \int_{r_0}^{r_1} \frac{dr}{r^2 \left(\frac{2mE}{l^2} - \frac{2mV}{l^2} - \frac{1}{r^2} \right)^{1/2}} \quad (36)$$

Define $u = 1/r$, let $\theta_0 = 0$, then

$$\theta = \theta_0 - \int_{u_0}^{u_1} \frac{du}{\left(\frac{2mE}{l^2} + \frac{2mku}{l^2} - u^2 \right)^{1/2}} \quad (37)$$

$$\Rightarrow \theta = -\arccos \frac{\frac{l^2 u}{mk} - 1}{\sqrt{1 + \frac{2El^2}{mk^2}}}, \quad r = \frac{\frac{l^2}{mk}}{1 + \sqrt{1 + \frac{2El^2}{mk^2}} \cos \theta} = \frac{r_0}{1 + \epsilon \cos \theta} \quad (38)$$

ϵ is called the eccentricity.

ϵ	Energy	Orbit
$\epsilon > 1$	$E > 0$	Hyperbola
$\epsilon = 1$	$E = 0$	Parabola
$0 < \epsilon < 1$	$E < 0$	Ellipse
$\epsilon = 0$	$E < 0$	Circle

The major semi-axis a and minor semi-axis b are given by

$$a = \frac{r_0}{1 - \epsilon^2} = -\frac{k}{2E}, \quad b = a\sqrt{1 - \epsilon^2} = \frac{k}{\sqrt{-2mE}} \quad (39)$$

Since $mr^2\dot{\theta} = l$, then

$$\int_0^T dA = \frac{1}{2} \int_0^T r^2 \dot{\theta} dt = \frac{1}{2} \int_0^T \frac{l}{m} \theta dt = \frac{l}{2m} T = \pi ab \Rightarrow T = 2\pi a^{3/2} \sqrt{\frac{m}{k}} \quad (40)$$

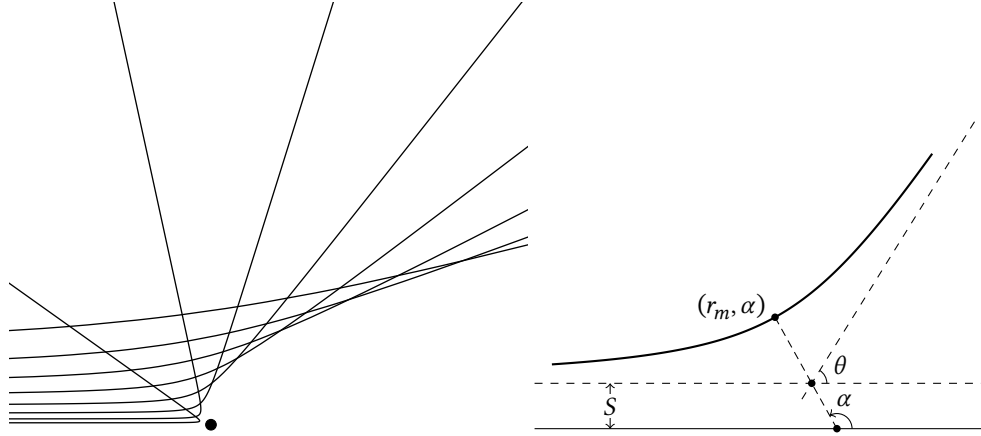
Alternatively

$$\int dt = \int \frac{dr}{\sqrt{\frac{2}{m} \left(E - V - \frac{l^2}{2mr^2} \right)}} = \sqrt{\frac{ma}{k}} \int_{r_0}^{r_1} \frac{r dr}{\sqrt{-r^2 + 2ar - b^2}} = \sqrt{\frac{ma}{k}} \int_{r_0}^{r_1} \frac{r dr}{\sqrt{-(r-a)^2 + a^2 \epsilon^2}} \quad (41)$$

Let $r - a = -a\epsilon \cos \zeta$, hence

$$t = \sqrt{\frac{ma}{k}} \int \frac{(a - a\epsilon \cos \zeta) a\epsilon \sin \zeta d\zeta}{\sqrt{a\epsilon^2(1 - \cos^2 \zeta)}} = \sqrt{\frac{ma}{k}} \int (a - a\epsilon \cos \zeta) d\zeta = \sqrt{\frac{ma^3}{k}} (\zeta - \epsilon \sin \zeta) + C \quad (42)$$

3.4 Scattering



Question: how many particles will be scattered in the given solid angle region (scattering cross section). Let $V \sim 1/r$, $f \sim 1/r^2$. Define the intensity of incident beam I

$I = \#$ of particles crossing a unit area perpendicular to the beam in unit time

The the number of particles scattered into $d\Omega$ could be expressed as

$$dN = \sigma I d\Omega, \quad \sigma = \frac{dN}{I d\Omega} \quad (43)$$

where σ is called differential cross section, which has the unit of area.

Particles in $[S, S + dS]$ would be scattered into $[\Omega, \Omega + d\Omega]$, since $d\Omega = 2\pi \sin \theta d\theta$

$$2\pi IS |dS| = \sigma I d\Omega = 2\pi \sigma I \sin \theta |d\theta| \Rightarrow \sigma = \frac{S |dS|}{\sin \theta |d\theta|} \quad (44)$$

The angular momentum of the incoming particles (wrt. to force center) is

$$l = mv_0 S = S \sqrt{2mE} \quad (45)$$

From the equation derived from central force problem

$$\alpha = \int \frac{dr}{r^2 \sqrt{\frac{2mE}{l^2} - \frac{rmV}{l^2} - \frac{1}{r^2}}} \quad (46)$$

$$= \pi + \int_{-\infty}^{r_m} \frac{dr}{r^2 \sqrt{\frac{2mE}{l^2} - \frac{rmV}{l^2} - \frac{1}{r^2}}} \quad (47)$$

Let $r_m \equiv r_{\min}$ is the closest distance. Define ψ where

$$\alpha = \pi + \int_{-\infty}^{r_m} \frac{dr}{r^2 \sqrt{\frac{2mE}{l^2} - \frac{rmV}{l^2} - \frac{1}{r^2}}} = \pi - \psi \quad (48)$$

Hence $\theta = \pi - 2\psi$. Change the variable $u = 1/r$, then

$$\psi = \int_0^{u_m=1/r_m} \frac{S du}{\sqrt{1 - \frac{V}{E} - S^2 u^2}} \quad (49)$$

Also note that

$$\sin \frac{\theta}{2} = \sin \frac{\pi - 2\psi}{2} = \cos \psi = \frac{1}{\epsilon} \quad (50)$$

Example 3.5 (Coulomb interaction). Let

$$f = \frac{ZZ'e^2}{r^2}, \quad V = \frac{ZZ'e^2}{r}, \quad r = \frac{r_0}{1 + \epsilon \cos(\alpha - \alpha')} \quad (51)$$

choose $\alpha' = \pi$. One can derive that

$$\epsilon = \sqrt{1 + \frac{2El^2}{m(ZZ'e^2)^2}} = \sqrt{1 + \left(\frac{2ES}{ZZ'e^2} \right)^2} \quad (52)$$

Using the trigonometric relationship between θ and ϵ , we have

$$S = \frac{ZZ'e^2}{2E} \cot \frac{\theta}{2}, \quad \left| \frac{dS}{d\theta} \right| = \frac{ZZ'e^2}{4E} \frac{1}{\sin^2 \frac{\theta}{2}} \quad (53)$$

$$\Rightarrow \epsilon(\theta) = \left(\frac{ZZ'e^2}{4E} \right)^2 \frac{1}{\sin^4 \frac{\theta}{2}} \quad (54)$$

check if $\alpha' = \pi$ or $\alpha' = \alpha_m$

4 Rigid Body

4.1 Coordinates of rigid body

A rigid body is a system of point masses satisfying the constraint that distance between any two points is a constant ($r_{ij} = \text{const}$ for all i, j). Let \mathbf{r}_i , \mathbf{r}_j , and \mathbf{r}_k be three points in the rigid body. Let \mathbf{r}_i has 3 DOFs, since r_{ij} is a constant, \mathbf{r}_j has two DOFs. Hence, \mathbf{r}_k only have 1 DOFs. The system has $3 + 2 + 1 = 6$ DOFs.

Or, alternatively, a rigid body has 3 coordinates for the origin of the coordinate system fixed on the rigid body, and 3 angular variables to specify the orientation of the rotated coordinate system.

4.2 Orthogonal transformations

Consider an orthogonal transformation $\mathbf{x} \mapsto \mathbf{x}'$

$$x'_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \quad (1)$$

$$x'_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \quad (2)$$

$$x'_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \quad (3)$$

Let $\mathbf{x}^2 = \mathbf{x}'^2$, we have

$$\mathbf{x}^2 = x_i x_i = x'_i x'_i = a_{ij} x_j a_{ik} x_k = \delta_{jk} x_j x_k = x_j x_j \Rightarrow a_{ij} a_{ik} = \delta_{jk} \quad (4)$$

This equation implies the orthogonal transformations have three DOFs.

Different views on rotational transformation

1. Passive view the coordinate system is transformed
2. Active view: the vector is transformed

Consider two rotational matrix A and B , let $C = AB$, then

$$C_{ij} C_{ik} = A_{im} B_{mj} A_{in} B_{nk} = A_{im} A_{in} B_{mk} B_{nk} = \delta_{nm} B_{mj} B_{nk} = B_{nj} B_{nk} = \delta_{jk} \quad (5)$$

Non-commutative $AB \neq BA$.

Define the inverse transformation

$$x_i = a'_{ij} x'_j = a'_{ij} a_{jk} x_k \Rightarrow a'_{ij} a_{jk} = \delta_{ik} \quad (6)$$

Note that

$$\left. \begin{aligned} \underbrace{a_{im} a_{ij}}_i a'_{jk} &= \delta_{mk} a'_{jk} = a'_{mk} \\ a_{im} \underbrace{a_{ij} a'_{jk}}_j &= a_{im} \delta_{ik} = a_{km} \end{aligned} \right\} \Rightarrow a_{km} = a'_{mk} \quad (7)$$

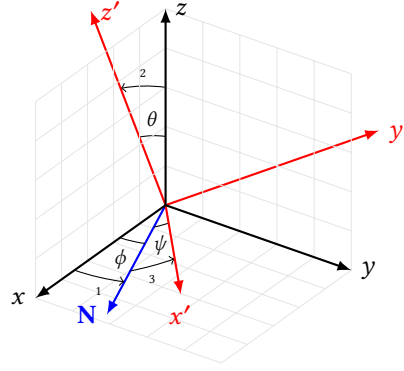
4.3 Euler angles

A convenient choice of angle variables: Euler angles. Steps:

1. Rotate around z axis by an angle ϕ , rotational matrix D
2. Rotate around ξ axis by an angle θ , rotational matrix C
3. Rotate around ξ' axis by an angle ψ , rotational matrix B

We can always write any rotational matrix A as $A = BCD$

$$A = \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ \sin \psi & -\cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & \sin \theta & -\cos \theta \end{bmatrix} \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ \sin \phi & -\cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (8)$$



Theorem 4.1 (Euler's theorem). The general displacement of a rigid body with one point fixed is a rotation around some axis \mathbf{R} . Then

$$\mathbf{R}' = A\mathbf{R} = \mathbf{R} \Rightarrow (A - I)\mathbf{R} = 0, \det(A - I) = 0 \quad (9)$$

Infinitesimal rotation transformation

$$\mathbf{x}' = A\mathbf{x} = (1 + \epsilon)\mathbf{x} \Rightarrow x'_i = x_i + \epsilon_{ij}x_j = (\delta_{ij} + \epsilon_{ij})x_j \quad (10)$$

Hence

$$(\delta_{ki} + \epsilon'_{ki})x'_i = (\delta_{ki} + \epsilon'_{ki})(\delta_{ij} + \epsilon_{ij})x_j = (\delta_{kj} + \epsilon'_{kj} + \epsilon_{kj} + \epsilon'_{ki}\epsilon_{ij})x_j = x_k \quad (11)$$

Ignoring $\epsilon'_{ki}\epsilon_{ij}$ term, we have $\epsilon'_{kj} = \epsilon_{jk}^T = -\epsilon_{kj}$, given that ϵ is an antisymmetric matrix. Let

$$\epsilon = \begin{bmatrix} 0 & -d\Omega_3 & d\Omega_2 \\ d\Omega_3 & 0 & -d\Omega_1 \\ -d\Omega_2 & d\Omega_1 & 0 \end{bmatrix}, \quad \Omega = \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix} \quad (12)$$

Then

$$d\mathbf{x} = \epsilon\mathbf{x} = d\Omega \times \mathbf{x} \quad (13)$$

4.4 Rate of change of a vector

Let \mathbf{R} be one point inside the rigid body, then the motion can be decomposed into rotational motion of the body coordinate and the translational motion of the body

$$\left(\frac{d\mathbf{R}}{dt}\right)_{\text{space}} = \left(\frac{d\mathbf{R}}{dt}\right)_{\text{body}} + \frac{d\Omega}{dt} \times \mathbf{R} \quad (14)$$

$$\Rightarrow \mathbf{V}_{\text{space}} = \mathbf{V}_{\text{body}} + \boldsymbol{\omega} \times \mathbf{R} \quad (15)$$

Decompose $\boldsymbol{\omega}$ into unit vectors corresponds to Euler angles and the body coordinate

$$\boldsymbol{\omega} = \dot{\phi}e_\phi + \dot{\theta}e_\theta + \dot{\psi}e_\psi = \omega_{x'}\mathbf{i}' + \omega_{y'}\mathbf{j}' + \omega_{z'}\mathbf{k}' \quad (16)$$

From the figure we can find that

$$e_\phi = \mathbf{k} = \sin\theta \sin\psi \mathbf{i}' + \sin\theta \cos\psi \mathbf{j}' + \cos\theta \mathbf{k}' \quad (17)$$

$$e_\theta = \cos\psi \mathbf{i}' - \sin\psi \mathbf{j}' \quad (18)$$

$$e_\psi = \mathbf{k}' \quad (19)$$

Hence

$$\omega_{x'} = \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi \quad (20)$$

$$\omega_{y'} = \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi \quad (21)$$

$$\omega_{z'} = \dot{\psi} + \dot{\phi} \cos\theta \quad (22)$$

4.5 The Coriolis effect

From equation 15 we have

$$\mathbf{a}_{\text{space}} = \mathbf{a}_{\text{body}} + \boldsymbol{\omega} \times \mathbf{V} \quad (23)$$

$$\Rightarrow m\mathbf{a}_{\text{body}} = \mathbf{F} - 2m\boldsymbol{\omega} \times \mathbf{V}_{\text{body}} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R}) \quad (24)$$

The term $-2m\boldsymbol{\omega} \times \mathbf{V}_{\text{body}}$ is the Coriolis force.

