

## PHY3110 FA22 HW03

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**Problem 1.** It is non-holonomic since we cannot find a function  $f = f(x, y, z)$  and write the constraint into

$$df = f_x dx + f_y dy + f_z dz = 0$$

In this case, we have

$$(x^2 + y^2 + z^2 + 2x) dx + 2y dy + 2z dz = 0$$

Note that

$$\frac{\partial}{\partial z}(x^2 + y^2 + z^2) \neq \frac{\partial}{\partial x}(2z)$$

which means that we cannot find such  $f$ .

**Problem 2.** Minimize the action is equivalent to solve the Lagrange's equation, hence

$$\begin{aligned} L(x, \dot{x}, t) &= \frac{1}{2}m\dot{x}^2 + Fx \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} &= m\ddot{x} = 2C \Rightarrow \begin{cases} A = 0 \\ B = \frac{a}{t_0} - \frac{Ft_0}{2m} \\ C = \frac{F}{2m} \end{cases} \\ \frac{\partial L}{\partial x} &= F \end{aligned}$$

**Problem 3.** Let two generalized coordinates be  $x$  and  $y$ , then the system could be written as

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy, \quad y = Ax^2 \Rightarrow 2Ax dx - dy = 0$$

Hence we can write Lagrange's equation with constraint

$$\begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} - 2Ax\lambda &= 0 \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} + \lambda &= 0 \end{aligned} \Rightarrow \begin{cases} \ddot{x} = \frac{2Ax\lambda}{m} \\ \ddot{y} = -\frac{\lambda}{m} + g \end{cases}$$

Note that

$$y = Ax^2 \Rightarrow 2Ax\dot{x} = \dot{y} \Rightarrow \ddot{y} = 2A\dot{x}^2 + 2Ax\ddot{x}$$

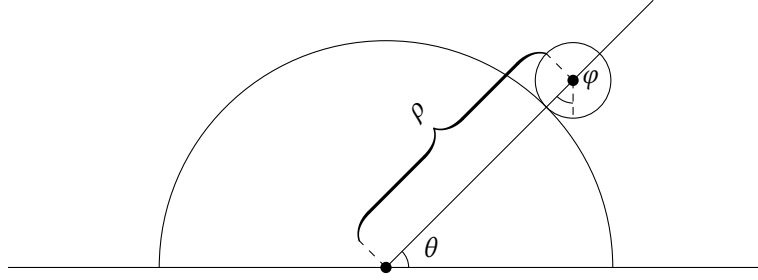
we can solve  $\lambda = \lambda(x, \dot{x}, t)$  as

$$\lambda = -\frac{2Am\dot{x}^2 + mg}{1 + 4A^2x^2}$$

Hence we have constraint force for two coordinates

$$Q_x = 2Ax\lambda = -\frac{4A^2m\dot{x}^2 + Axmg}{1 + 4A^2x^2}, \quad Q_y = -\lambda = \frac{2Am\dot{x}^2 + mg}{1 + 4A^2x^2}$$

**Problem 4.** Setup the generalized coordinate  $\rho$ ,  $\theta$  and  $\varphi$  as the following figure.



Hence, we have the system

$$L = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\theta}^2) + \frac{1}{2}mr^2\dot{\varphi}^2 - mg\rho \sin \theta$$

subject to  $\rho \geq R + r, \rho d\theta + r d\varphi = 0$

Since we only consider the case when hoop is still on the cylinder, we can generalize the constraint to  $d\rho = 0$ . Using two different Lagrange's multipliers  $\lambda$  and  $\mu$ , we can get following set of equations

$$\begin{aligned} m\ddot{\rho} - m\rho\dot{\theta}^2 + mg \sin \theta - \lambda &= 0 \\ m\rho^2\ddot{\theta} + 2m\rho\dot{\rho}\dot{\theta} + mg\rho \cos \theta - \rho\mu &= 0 \\ mr^2\ddot{\varphi} - r\mu &= 0 \end{aligned}$$

Note that  $\rho d\theta + r d\varphi = 0 \Rightarrow \dot{\rho}\dot{\theta}/r + \rho\ddot{\theta}/r + \ddot{\varphi} = 0$ , substitute  $\ddot{\varphi}$  into the third equation, and use the solved  $\mu$ , we have

$$\mu = -m\rho\dot{\theta} - m\rho\ddot{\theta}, \quad 2m\rho^2\ddot{\theta} + 3m\rho\dot{\rho}\dot{\theta} + mg\rho \cos \theta = 0$$

Applying  $\dot{\rho} = 0$  and boundary condition, we have

$$\begin{aligned} \ddot{\theta} &= -\frac{g}{2\rho} \cos \theta, \quad \dot{\theta} \frac{d\dot{\theta}}{d\theta} = -\frac{g}{2\rho} \cos \theta \\ \Rightarrow \dot{\theta}^2 &= \frac{g}{\rho}(1 - \sin \theta) \end{aligned}$$

When the hoop is about to leave the cylinder, we should have  $\lambda = 0$  (and  $\dot{\rho} = 0$ ), then

$$\begin{aligned} -m\rho\dot{\theta}^2 + mg \sin \theta &= 0 \\ 2mg \sin \theta &= mg \\ \Rightarrow \theta &= \frac{\pi}{6} \end{aligned}$$