1. $(\vec{a}\times(\vec{b}\times\vec{c}))_i = \varepsilon_{ijk}\, a_j\, (\vec{b}\times\vec{c})_k$

$= \varepsilon_{ijk}\,\varepsilon_{klm}\, a_j b_l c_m$

根据 $\varepsilon_{ijk}\,\varepsilon_{klm} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})$

$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, a_j b_l c_m$

$= a_j b_i c_j - a_j b_j c_i$

$= a_j c_i b_i - a_j b_j c_i$

$= ((\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\cdot\vec{c})_i$

$\therefore\ \vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\cdot\vec{c}$

将 $\vec{b}\to\vec{\nabla}$ 后，由于导数算符与函数不能调换位置，上式不成立。

$\therefore\ (\vec{a}\times(\vec{\nabla}\times\vec{c}))_i = a_j\partial_i c_j - a_j\partial_j c_i$

$= \partial_i(a_j c_j) - a_j\partial_j c_i - c_j\partial_i a_j$

$= \nabla_i(\vec{a}\cdot\vec{c}) - (\vec{a}\cdot\vec{\nabla})c_i - (\vec{c}\times(\vec{\nabla}\times\vec{a}))_i - (\vec{c}\times\vec{\nabla})a_j$

$\therefore\ \vec{a}\times(\vec{\nabla}\times\vec{c}) = \nabla(\vec{a}\cdot\vec{c}) - (\vec{a}\cdot\vec{\nabla})\vec{c} - (\vec{c}\cdot\vec{\nabla})\vec{a} - \vec{c}\times(\vec{\nabla}\times\vec{a})$

2.

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \Rightarrow \begin{cases} dx = \sin\theta\cos\varphi\, dr + r\cos\theta\cos\varphi\, d - r\sin\theta\sin\varphi\, d\varphi \\ dy = \sin\theta\sin\varphi\, dr + r\cos\theta\sin\varphi\, d\theta + r\sin\theta\cos\varphi\, d\varphi \\ dz = \cos\theta\, dr - r\sin\theta\, d\theta. \end{cases}$$

写作矩阵形式：

$$\begin{bmatrix} dr \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} \wedge & & \\ & x & r \\ & x & x \end{bmatrix} \begin{bmatrix} dr \\ r d\theta \\ r\sin\theta\, d\varphi \end{bmatrix}$$

对于任一微分矢量 $d\vec{r}$，可用直角坐标或球坐标表示，但矢量本身不变。

$\therefore \quad d\vec{r} = dx^i \vec{e}_i = M^i_j d\bar{x}^j \vec{e}_i = d\bar{x}^j \vec{\bar{e}}_j$

由此可得球坐标与直角坐标的变换关系，$\vec{\bar{e}}_i = e_j M^j_i$

$$\therefore \begin{cases} \vec{e}_r = \sin\theta\cos\varphi\,\vec{e}_1 + \sin\theta\sin\varphi\,\vec{e}_2 + \cos\theta\,\vec{e}_3 \\ \vec{e}_\theta = \cos\theta\cos\varphi\,\vec{e}_1 + \cos\theta\sin\varphi\,\vec{e}_2 - \sin\theta\,\vec{e}_3 \\ \vec{e}_\varphi = -\sin\varphi\,\vec{e}_1 + \cos\varphi\,\vec{e}_2 \end{cases}$$

可验证：$\vec{e}_r \cdot \vec{e}_\theta = 0$，$\vec{e}_r \cdot \vec{e}_\varphi = 0$，$\vec{e}_\theta \cdot \vec{e}_\varphi = 0$

$\therefore$ 球坐标系为 <u>正交系</u>

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\varphi^2$$

3. $T = T(q, \dot{q}, t)$

$$\dot{T} = \frac{\partial T}{\partial q_j}\dot{q}_j + \frac{\partial T}{\partial \dot{q}_j}\ddot{q}_j + \frac{\partial T}{\partial t}$$

$$\frac{\partial \dot{T}}{\partial \dot{q}_i} = \frac{\partial T}{\partial q_j}\delta_{ij} + \frac{\partial T}{\partial \dot{q}_i \partial \dot{q}_j}\ddot{q}_j + \frac{\partial T}{\partial \dot{q}_i \partial t} + \frac{\partial T}{\partial \dot{q}_i \partial q_i}\dot{q}_j$$

$$= \frac{\partial T}{\partial q_i} + \frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j}\ddot{q} + \frac{\partial^2 T}{\partial \dot{q}_i \partial t} + \frac{\partial^2 T}{\partial \dot{q}_i \partial q_j}\dot{q}_j$$

$$= \frac{\partial T}{\partial q_i} + \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i}$$

$$= \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} = \frac{\partial^2 T}{\partial \dot{q}_i \partial t} + \frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j}\ddot{q}_j + \frac{\partial^2 T}{\partial \dot{q}_i \partial q_j}\dot{q}_j$$

由 $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = 0 \Rightarrow \frac{\partial \dot{T}}{\partial \dot{q}_i} - 2\frac{\partial T}{\partial q_i} = 0$

4. (a) 若(3)为完整约束，则 $\exists F: dF = \Sigma f g_i dx_i = 0$

即 $\frac{\partial F}{\partial x_i} = f g_i \quad \frac{\partial f}{\partial x_j} = f g_i \quad i \neq j$

又 $\frac{\partial^2 F}{\partial x_i \partial x_j} = \frac{\partial^2 F}{\partial x_j \partial x_i}$

$\therefore \frac{\partial f g_i}{\partial x_j} = \frac{\partial f g_i}{\partial x_i}$

(b) $(2x+y+z)dx + (x+2y+z)dy + (x+y+2z)dz = 0$

$\frac{\partial}{\partial x}(2x+y+z) = 2 = \frac{\partial}{\partial y}(x+2y+z)$

$\frac{\partial}{\partial x}(2x+y+z) = \frac{\partial}{\partial z}(x+y+2z)$

$\frac{\partial}{\partial y}(x+2y+z) = \frac{\partial}{\partial z}(x+y+2z)$

∴是完整约束.

$(x^2+y^2+z^2)dx + 2(xdx+ydy+zdz) = 0$

$\frac{\partial}{\partial y} g_1 = 2y \quad \frac{\partial}{\partial x} g_2 = 0$

设 $\exists f = \frac{\partial}{\partial y} f g_1 = f 2y + (x^2+y^2+z^2+2x)\frac{\partial f}{\partial y}$

$\frac{\partial}{\partial x} f g_2 = 2y \frac{\partial f}{\partial x}$

$2y \frac{\partial f}{\partial x} = f \cdot 2y + (x^2+y^2+z^2+2x)\frac{\partial f}{\partial y} \quad ①$

同理. $\frac{\partial}{\partial z} f g_2 = 2y \frac{\partial f}{\partial z}$

$$\frac{\partial}{\partial y} f g_3 = 2z \frac{\partial f}{\partial y} \Rightarrow 2y\frac{\partial f}{\partial z} = 2z \frac{\partial f}{\partial y} \quad ②$$

$$2z \frac{\partial f}{\partial x} = f\cdot 2z + (x^2+y^2+z^2+2x)\frac{\partial f}{\partial z} \quad ③$$

若 $f=f(x)$. ②. ③ 成立, ① $\Rightarrow f = \frac{\partial f}{\partial x} \Rightarrow f = e^x$

是完整约束.

5.

设弹簧原长为 L. 则平衡时 $mg = k(l - l_0)$

$$T = \frac{1}{2} m [(l+x)]^2 + \frac{1}{2} m [(l+x)\theta]^2$$

$$V = -mg(l+x)\cos\theta + \frac{1}{2}k(l+x-l_0)^2$$

$$L = T-V = \frac{1}{2} m x^2 + \frac{1}{2} m(l+x)^2 + mg(l+x)\cos\theta - \frac{1}{2}k\left(x+\frac{mg}{k}\right)^2$$

$\frac{\partial L}{\partial x} = mx$. $\frac{\partial L}{\partial x} = m(l+x)\theta^2 + mg\cos\theta - k\left(x+\frac{mg}{k}\right)$

$$\frac{\partial L}{\partial \theta} = m(l+x)^2\dot\theta$$

$$\frac{\partial L}{\partial \theta} = -mg(l+x)\sin\theta.$$

$$\begin{cases} m\dot{x} - m(l+x)^2\dot\theta^2 - mg\cos\theta + k\left(x+\frac{mg}{k}\right) = 0 \\ m(l+x^2)\ddot\theta + 2m(l+x)\dot\theta\dot{x} + mg\sin\theta = 0 \end{cases}$$