① $L = \frac{m}{2}v^2 - V(\vec{r})$ （假设处于同一势场中）

$\vec{r} = r\vec{e_r}$

$\vec{v} = \dot{\vec{r}} = \dot{r}\vec{e_r} + r\dot{\vec{e_r}}$

$= \dot{r}\vec{e_r} + r\dot{\theta}\vec{e_\theta} + r\sin\theta\dot{\varphi}\vec{e_\varphi}$

$\therefore L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2) - V(\vec{r})$

$\frac{\partial L}{\partial \dot{r}} = m\dot{r}$ $\quad \frac{\partial L}{\partial r} = mr\dot{\theta}^2 + mr\sin^2\theta\dot{\varphi}^2 - \frac{\partial V}{\partial r}$

$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$ $\quad \frac{\partial L}{\partial \theta} = mr^2\sin\theta\cos\theta\dot{\varphi}^2 - \frac{\partial V}{\partial \theta}$

$\frac{\partial L}{\partial \dot{\varphi}} = mr^2\sin^2\theta\dot{\varphi}$ $\quad \frac{\partial L}{\partial \varphi} = -\frac{\partial V}{\partial \varphi}$.

$\therefore m\ddot{r} - mr\dot{\theta}^2 - mr\sin^2\theta\dot{\varphi}^2 + \frac{\partial V}{\partial r} = 0$

$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - mr^2\sin\theta\cos\theta\dot{\varphi}^2 + \frac{\partial V}{\partial \theta} = 0$

$\frac{d}{dt}(mr^2\sin^2\theta\dot{\varphi}) + \frac{\partial V}{\partial \varphi} = mr^2\sin^2\theta\ddot{\varphi} + 2mr\sin^2\theta\dot{r}\dot{\varphi} + 2mr^2\dot{\varphi}\sin\theta\cos\theta + \frac{\partial V}{\partial r} = 0$

假设受到力 $\vec{F} = F_r\vec{e_r} + F_\theta\vec{e_\theta} + F_\varphi\vec{e_\varphi}$ 的作用

相应的广义力为 $Q_i = \vec{F}\frac{\partial \vec{r}}{\partial q_i} = \begin{cases} Q_r = F_r \\ Q_\theta = F_\theta r \\ Q_\varphi = F_\varphi r\sin\theta. \end{cases}$

② 由虚功原理.

$m_1 g\delta y_1 + m_2 g\delta y_2 = 0$

$y_1 = (l+R)\cos\theta$ $\qquad \varphi = \frac{\pi}{2} - \theta - \arccos\frac{R}{l+R}$

$y_2 = L - (\frac{\pi}{2} - \theta - \arccos\frac{R}{l+R})R - \sqrt{(l+R)^2 - R^2} + y_1$

$\therefore \delta y_1 = -(l+R)\sin\theta\,\delta\theta$

$\delta y_2 = R\delta\theta + \delta y_1 = R\delta\theta - (l+R)\sin\theta\,\delta\theta$.

$$\therefore\ -m_1 g(l+R)\sin\theta - m_2 g(l+R)\sin\theta + m_2 g R = 0$$

$$\therefore\ \sin\theta = \frac{m_2 R}{(m_1+m_2)(l+R)}$$

③ $$L = -m_0 c^2\sqrt{1-v^2/c^2} - V(r)$$

$$\frac{\partial L}{\partial v} = -m_0 c^2 \frac{1}{2}\,\frac{-\frac{2v}{c^2}}{\sqrt{1-v^2/c^2}} = \frac{m_0 v}{\sqrt{1-v^2/c^2}}$$

$$\frac{\partial L}{\partial r} = -\frac{\partial V}{\partial r}$$

$$\frac{d}{dt}\frac{\partial L}{\partial v} = \frac{m_0\dot v}{\sqrt{1-v^2/c^2}} + m_0 v\,\frac{-\frac{1}{2}\times\left(-\frac{2v}{c^2}\right)\dot v}{\left(\sqrt{1-v^2/c^2}\right)^3}$$

$$= \frac{m_0\dot v\left(1-\frac{v^2}{c^2}\right) + m_0\dot v\frac{v^2}{c^2}}{\left(\sqrt{1-v^2/c^2}\right)^3}$$

$$= \frac{m_0\dot v}{\sqrt{1-v^2/c^2}}$$

$$\therefore\ E-L = \frac{m_0\dot v}{\sqrt{1-v^2/c^2}} = -\frac{\partial V}{\partial r}$$

④ 质心：(X, Y)

$$X = x + l\cos\theta,\quad Y = l\sin\theta$$

$$L = T - V = \frac{m}{2}(\dot X^2 + \dot Y^2) + \frac{1}{2}\times\frac{1}{3}ml^2\dot\theta^2 - mgl\cos\theta.$$

$$= \frac{m}{2}\left[(\dot x - l\sin\theta\cdot\theta)^2 + l^2\cos^2\theta\cdot\dot\theta^2\right] + \frac{1}{6}ml^2\dot\theta^2 - mgl\cos\theta.$$

∵ 不包含时间 t，∴ 能量 E 守恒.

$$\frac{\partial L}{\partial x} = 0$$ ∴ 水平动量守恒.

$$\frac{\partial L}{\partial \dot x} = m\dot x \qquad \therefore\ m\dot x = 0$$

$$\frac{\partial L}{\partial \dot\theta} = ml^2\cos^2\theta\,\dot\theta + \frac{1}{3}ml^2\dot\theta$$

$$ml^2\cos^2\theta\cdot\ddot\theta - 2ml^2\cos\theta\sin\theta\,\dot\theta^2 + \frac{1}{3}ml^2\ddot\theta + mgl\cos\theta = 0$$