PHY5410 FA22 HW04

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Problem 1 (6.3)**.** Using the relation that $u(r) = rR(r)$, $\varrho = \kappa r$ (more strictly, we should use the notation $v(\varrho) = u(r) = u(\varrho/\kappa)$ rather than $u(\varrho)$ to denote $u(\varrho/\kappa)$), define the following notations

$$\langle r^k \rangle = 4\pi \int r^2 \, \mathrm{d}r \; r^k R(r)^2$$

$$\langle \varrho^k \rangle_u = 4\pi \int \mathrm{d}\varrho \; \varrho^k u(\varrho)^2$$

one can verify

$$\langle \varrho^k \rangle_u = 4\pi \int \mathrm{d}\kappa r \; (\kappa r)^k r^2 R(r)^2 = 4\pi\kappa^{k+1} \int r^2 \, \mathrm{d}ri \; r^k R^r(r) = \kappa^{k+1} \langle r^k \rangle$$

Using the differential equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\varrho^2} - \frac{l(l+1)}{\varrho^2} + \frac{2n}{\varrho} - 1 \right] u(\varrho) = 0 \tag{1}$$

Multiply $\varrho^{k+1} u'_{nl}(\varrho)$ on the left, we get

$$\varrho^{k+1} u' u'' - \varrho^{k-1} l(l+1) u u' + 2n \varrho^k u u' - \varrho^{k+1} u u' = 0$$

Using the fact that

$$4\pi \int \mathrm{d}\varrho \; \varrho^m u u' = -\frac{m}{2} 4\pi \int \mathrm{d}\varrho \; \varrho^{m-1} u^2 = -\frac{m}{2} \langle \varrho^{m-1} \rangle_u$$

$$4\pi \int \mathrm{d}\varrho \; \varrho^m u u' = -2\pi(m-1) \int \mathrm{d}\varrho \; \varrho^{m-1} (u')^2$$

$$4\pi \int \mathrm{d}\varrho \; \varrho^m u u'' = -4\pi m \int \mathrm{d}\varrho \; \varrho^{m-1} u u' - 4\pi \int \mathrm{d}\varrho \; \varrho^m (u')^2$$

we can multiply the expression by 4π and integrate it

$$-2\pi(k+1) \int \mathrm{d}\varrho \; \varrho^k (u')^2 + \frac{1}{2}(k-1) l(l+1) \langle \varrho^{k-2} \rangle_u - nk \langle \varrho^{k-1} \rangle_u + \frac{1}{2}(k+1) \langle \varrho^k \rangle_u = 0 \tag{2}$$

we can also multiply the equation by $\varrho^k u_{nl}(\varrho)$

$$\varrho^k u u'' - l(l+1) \varrho^{k-2} u^2 + 2n \varrho^{k-1} u^2 - \varrho^k u^2 = 0$$

multiply this equation by 4π and integrate it

$$\frac{1}{2} k(k-1) \langle \varrho^{k-2} \rangle_u - 4\pi \int \mathrm{d}\varrho \; \varrho^k (u')^2 - l(l+1) \langle \varrho^{k-2} \rangle_u + 2n \langle \varrho^{k-1} \rangle_u - \langle \varrho^k \rangle_u = 0 \tag{3}$$

Combing equation 2 and 3 to eliminate $(u')^2$ term, we can get

$$[\frac{1}{2}(k-1) l(l+1) - \frac{1}{4} k(k^2-1) + \frac{1}{2} l(l+1)(k+1)] \langle \varrho^{k-2} \rangle_u - [nk + n(k+1)] \langle \varrho^{k-1} \rangle_u + (k+1) \langle \varrho^k \rangle_u = 0$$

$$\frac{k}{4} [4l(l+1) - (k^2-1)] \langle \varrho^{k-2} \rangle_u - n(2k+1) \langle \varrho^{k-1} \rangle_u + (k+1) \langle \varrho^k \rangle_u = 0$$

$$\frac{k}{4}[(2l+1)^2-k^2]\langle\varrho^{k-2}\rangle_u - n(2k+1)\langle\varrho^{k-1}\rangle_u + (k+1)\langle\varrho^k\rangle_u = 0$$

Plugin $\kappa = 1/na$ and $\langle\varrho^k\rangle_u = \kappa^{k+1}\langle r^k\rangle$, we get

$$\begin{aligned}
&\frac{k}{4}[(2l+1)^2-k^2]\langle r^{k-2}\rangle - n(2k+1)\kappa\langle r^{k-1}\rangle + (k+1)\kappa^2\langle r^k\rangle = 0\\
\Rightarrow\ &\frac{k+1}{n^2}\langle r^k\rangle - (2k+1)a\langle r^{k-1}\rangle + \frac{k}{4}[(2l+1)^2-k^2]a^2\langle r^{k-2}\rangle = 0
\end{aligned}$$

Problem 2 (6.13). Let A_i denotes the ith component of $\mathbf{A}$, then

$$A_i = \frac{1}{2m}(\epsilon_{ijk}p_jL_k - \epsilon_{ijk}L_jp_k) - \frac{Ze^2}{r}x_i = \frac{1}{2m}\epsilon_{ijk}(p_jL_k + L_kp_j) - \frac{Ze^2}{r}x_i$$

where the summation would be taken in place of j and k. Since $p_j^\dagger = p_j$, $L_k^\dagger = L_k$, then we can prove that A_i is hermitian

$$A_i^\dagger = \frac{1}{2m}\epsilon_{ijk}(L_k^\dagger p_j^\dagger + p_j^\dagger L_k^\dagger) - \frac{Ze^2}{r}x_i^\dagger = \frac{1}{2m}\epsilon_{ijk}(L_kp_j + p_jL_k) - \frac{Ze^2}{r}x_i = A_i$$

Therefore $\mathbf{A}$ is hermitian.

To prove $[\mathbf{A}, H] = 0$, we can show that $[A_i, H] = 0$. A_i could be simplified to the following expression by direct summation over j and k.

$$\begin{aligned}
A_i &= \frac{1}{2m}\epsilon_{ijk}\epsilon_{kmn}(p_jx_mp_n + x_mp_np_j) - \frac{Ze^2}{r}x_i\\
&= \frac{1}{2m}\epsilon_{ijk}\epsilon_{kmn}[(x_mp_j - i\hbar\delta_{jm}p_n + x_mp_np_j)] - \frac{Ze^2}{r}x_i\\
&= \frac{1}{2m}\epsilon_{kij}\epsilon_{kmn}(2x_mp_jp_n - i\hbar\delta_{jm}p_n) - \frac{Ze^2}{r}x_i\\
&= \frac{1}{2m}(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})(2x_mp_jp_n - i\hbar\delta_{jm}p_n)
\end{aligned}$$

Since

$$\begin{aligned}
\delta_{im}\delta_{jn}x_mp_jp_n &= \delta_{jn}x_ip_jp_n = x_ip^2\\
\delta_{in}\delta_{jm}x_mp_jp_n &= \delta_{jm}x_mp_ip_j = (x\cdot p)p_i\\
\delta_{im}\delta_{jm}\delta_{jn}i\hbar p_n &= \delta_{ij}\delta_{jn}i\hbar p_n = i\hbar p_i\\
\delta_{in}\delta_{jm}^2p_n &= \delta_{jm}^2 i\hbar p_i = 3i\hbar p_i
\end{aligned}$$

Then we have

$$A_i = \frac{1}{m}x_ip^2 - \frac{1}{m}(x\cdot p)p_i + \frac{1}{m}i\hbar p_i - \frac{Ze^2}{r}x_i$$

Using the fact that $H = p^2/2m - Ze^2/r$, we have

$$\begin{aligned}
A_iH &= \frac{1}{2m^2}[x_i(p^2)^2 - (x\cdot p)p^2p_i + i\hbar p^2p_i] - \frac{Ze^2}{2m}\frac{1}{r}x_ip^2 - \frac{Ze^2}{m}[x_ip^2\frac{1}{r} - (x\cdot p)p_i\frac{1}{r} + i\hbar p_i\frac{1}{r}] + \frac{Ze^4}{r^2}x_i\\
HA_i &= \frac{1}{2m^2}[p^2x_ip^2 - p^2(x\cdot p)p_i + i\hbar p^2p_i] - \frac{Ze^2}{2m}p^2\frac{1}{r}x_i - \frac{Ze^2}{m}[\frac{1}{r}x_ip^2 - \frac{1}{r}(x\cdot p)p_i + \frac{1}{r}i\hbar p_i] + \frac{Ze^4}{r^2}x_i
\end{aligned}$$

Using the commutation relationship

$$p^2 x_i = x_i p^2 - 2i\hbar p_i$$
$$p^2(x\cdot p) = \sum_j p^2 x_j p_j = \sum_j x_j p^2 p_j - 2i\hbar p_j^2 = (x\cdot p)p^2 - 2i\hbar p^2$$

Then it is easy to show that

$$p^2 x_i p^2 - p^2(x\cdot p)p_i = x_i(p^2)^2 - 2i\hbar p^2 p_i - (x\cdot p)p^2 p_i + 2i\hbar p^2 p_i = x_i(p^2)^2 - (x\cdot p)p^2 p_i$$

Moreover, using the fact that

$$\begin{aligned}
p_i\frac{1}{r} &= i\hbar\frac{1}{r^3}x_i + \frac{1}{r}p_i \\
p_i p_j\frac{1}{r} &= i\hbar\frac{1}{r^3}x_i p_j + i\hbar\frac{1}{r^3}x_j p_i + \frac{1}{r}p_i p_j + \hbar^2\frac{\delta_{ij}r^2 - 3x_i x_j}{r^5} \\
p^2\frac{1}{r} &= 2i\hbar\frac{1}{r^3}(x\cdot p) + \frac{1}{r}p^2 \\
p^2 x_i\frac{1}{r} &= x_i p^2\frac{1}{r} - 2i\hbar p_i\frac{1}{r} = 2i\hbar\frac{1}{r^3}x_i(x\cdot p) + \frac{1}{r}x_i p^2 + 2\hbar^2\frac{1}{r^3}x_i - 2i\hbar\frac{1}{r}p_i
\end{aligned}$$

we have

$$\begin{aligned}
-\frac{Ze^2}{2m}\frac{1}{r}x_i p^2 - \frac{Ze^2}{m}[x_i p^2\frac{1}{r} - (x\cdot p)p_i\frac{1}{r} + i\hbar p_i\frac{1}{r}] &= -\frac{Ze^2}{2m}\frac{1}{r}x_i p^2 - \frac{Ze^2}{m}\Big[2i\hbar\frac{1}{r^3}x_i(x\cdot p) + \frac{1}{r}x_i p^2 \\
&\quad - i\hbar\frac{1}{r^3}x_i(x\cdot p) - i\hbar\frac{1}{r}p_i - \frac{1}{r}(x\cdot p)p_i + 2\hbar^2\frac{1}{r^3}x_i \\
&\quad - \hbar^2\frac{1}{r^3}x_i + i\hbar\frac{1}{r}p_i\Big] \\
&= -\frac{Ze^2}{2m}\frac{1}{r}x_i p_i - \frac{Ze^2}{m}\Big[\frac{1}{r}x_i p^2 - \frac{1}{r}(x\cdot p)p_i \\
&\quad + i\hbar\frac{1}{r^3}x_i(x\cdot p) + \hbar^2\frac{1}{r^3}x_i\Big] \\
-\frac{Ze^2}{2m}p^2\frac{1}{r}x_i - \frac{Ze^2}{m}[\frac{1}{r}x_i p^2 - \frac{1}{r}(x\cdot p)p_i + \frac{1}{r}i\hbar p_i] &= -\frac{Ze^2}{2m}\frac{1}{r}x_i p^2 - \frac{Ze^2}{m}i\hbar\frac{1}{r^3}x_i(x\cdot p) - \frac{Ze^2}{m}\hbar^2\frac{1}{r^3}x_i \\
&\quad + \frac{Ze^2}{m}i\hbar\frac{1}{r}p_i - \frac{Ze^2}{m}\Big[\frac{1}{r}x_i p^2 - \frac{1}{r}(x\cdot p)p_i + i\hbar\frac{1}{r}p_i\Big] \\
&= -\frac{Ze^2}{2m}\frac{1}{r}x_i p^2 - \frac{Ze^2}{m}\Big[\frac{1}{r}x_i p^2 - \frac{1}{r}(x\cdot p)p_i \\
&\quad + i\hbar\frac{1}{r^3}x_i(x\cdot p) + \hbar^2\frac{1}{r^3}x_i\Big]
\end{aligned}$$

Therefore $A_i H = HA_i$, hence $[A_i, H] = 0$, we have $[\mathbf{A}, H] = 0$.

The proof of $\mathbf{A}\cdot\mathbf{L} = \mathbf{L}\cdot\mathbf{A} = 0$ begins from the following claims.

Claim. $\mathbf{L}\cdot x = x\cdot\mathbf{L} = 0$.

Proof. Note that

$$x\cdot\mathbf{L} = \sum_{ijk}\epsilon_{ijk}x_i x_j p_k = \sum_k\left(\sum_{ij}\epsilon_{ijk}x_i x_j\right)p_k = \sum_k 0 p_k = 0$$
$$\mathbf{L}\cdot x = \sum_{ijk}\epsilon_{ijk}x_j p_k x_i = \sum_{ijk}\epsilon_{ijk}x_j(x_i p_k - i\hbar\delta_{ik}) = \sum_{ijk}\epsilon_{ijk}x_i x_j p_k - i\hbar\epsilon_{ijk}\delta_{ik}x_j = 0$$

□

Claim. $\mathbf{L}\cdot(p\times\mathbf{L}-\mathbf{L}\times p)=(p\times\mathbf{L}-\mathbf{L}\times p)\cdot\mathbf{L}=0$.

Proof. Note that

$$\begin{aligned}
\mathbf{L}\cdot(p\times\mathbf{L}-\mathbf{L}\times p) &= \sum_{ijk}\epsilon_{ijk}L_ip_jL_k-\epsilon_{ijk}L_iL_jp_k=\sum_{ijk}\epsilon_{ijk}L_i(p_jL_k+L_kp_j)\\
&= \sum_{ijk}\epsilon_{ijk}(2L_iL_kp_j+i\hbar\sum_l\epsilon_{kjl}p_l)\\
&= \sum_{ijk}2\epsilon_{ijk}L_iL_kp_j-i\hbar\epsilon_{ijk}\epsilon_{kjl}L_ip_l\\
&= -\sum_j 2i\hbar L_jp_j-\sum_k i\hbar L_kp_k+\sum_i 3i\hbar L_ip_i\\
&= 0\\
(p\times\mathbf{L}-\mathbf{L}\times p)\cdot\mathbf{L} &= \sum_{ijk}\epsilon_{ijk}(p_jL_k+L_kp_j)L_i\\
&= \sum_{ijk}\epsilon_{ijk}(2p_jL_k+i\hbar\sum_l\epsilon_{kjl}p_l)L_i\\
&= \sum_j 2i\hbar p_jL_j+\sum_k i\hbar p_kL_k-3\sum_i i\hbar p_iL_i\\
&= 0
\end{aligned}$$

□

Therefore, it is easy to show that

$$\begin{aligned}
\mathbf{L}\cdot\mathbf{A} &= \frac{1}{2m}\mathbf{L}\cdot(p\times\mathbf{L}-\mathbf{L}\times p)-(\mathbf{L}\times x)\frac{Ze^2}{r}=0\\
\mathbf{A}\cdot\mathbf{L} &= \frac{1}{2m}(p\times\mathbf{L}-\mathbf{L}\times p)\dot{\mathbf{L}}-\frac{Ze^2}{r}(x\times\mathbf{L})=0
\end{aligned}$$

Problem 3 (7.1)**.** Since

$$\begin{aligned}
\nabla\cdot\mathbf{j} &= \frac{1}{2m}\left[\psi^*(-i\hbar\nabla^2\psi-\frac{e}{c}A\cdot\nabla\psi-\frac{e}{c}(\nabla\cdot\mathbf{A})\psi)+(\nabla\psi^*)\cdot(-i\hbar\nabla\psi-\frac{e}{c}\mathbf{A}\psi)\right]+\text{c.c.}\\
&= \frac{1}{2m}\left(-i\hbar\psi^*\nabla^2\psi-\frac{e}{c}\psi^*\mathbf{A}\cdot\nabla\psi-\frac{e}{c}(\nabla\cdot\mathbf{A})\psi^*\psi-i\hbar\nabla\psi^*\cdot\nabla\psi-\frac{e}{c}\nabla\psi^*\cdot\mathbf{A}\psi\right)+\text{c.c.}\\
&= \frac{1}{2m}\left(-i\hbar\psi^*\nabla^2\psi-\frac{e}{c}\psi^*\mathbf{A}\cdot\nabla\psi-\frac{e}{c}(\nabla\cdot\mathbf{A})\psi^*\psi-\frac{e}{c}\nabla\psi^*\cdot\mathbf{A}\psi\right)+\text{c.c.}\\
&= \frac{1}{2m}\left(-i\hbar\psi^*\nabla^2\psi-\frac{2e}{c}\psi^*\mathbf{A}\cdot\nabla\psi-\frac{e}{c}(\nabla\cdot\mathbf{A})\psi^*\psi\right)+\text{c.c.}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial t}\psi^*\psi &= \psi^*\frac{\partial}{\partial t}\psi+\text{c.c.}\\
&= \frac{1}{i\hbar}\psi^*H\psi+\text{c.c.}\\
&= \frac{1}{i\hbar}\psi^*\left(-\frac{\hbar^2}{2m}\nabla^2\psi+\frac{i\hbar e}{mc}\mathbf{A}\cdot\nabla\psi+\frac{i\hbar e}{2mc}(\nabla\cdot\mathbf{A})\psi+\frac{e^2}{2mc}\mathbf{A}^2\psi+e\Phi\psi\right)+\text{c.c.}\\
&= \frac{1}{2m}\left(i\hbar\psi^*\nabla^2\psi+\frac{2e}{c}\psi^*\mathbf{A}\cdot\nabla\psi+\frac{e}{c}(\nabla\cdot\mathbf{A})\psi^*\psi\right)+\text{c.c.}
\end{aligned}$$

Hence $\partial_t\psi^*\psi+\nabla\cdot\mathbf{j}=0$.

Problem 4 (7.3). Let the equation be $H\psi = \lambda\psi$, where the hamiltonian H equals to

$$H = \frac{1}{2m}\left[p_x^2 + \left(p_y - \frac{e}{c}Bx\right)^2 + p_z^2\right] - eEx$$

Suppose the solution in the form of $\psi(x, y, z) = e^{ik_2 y}e^{ik_3 z}\psi(x)$, then we can get a hamiltonian H_x only related to x

$$\begin{aligned}
H_x &= \frac{1}{2m}\left[p_x^2 + \left(\hbar k_2 - \frac{e}{c}Bx\right)^2 + \hbar^2 k_3^2\right] - eEx \\
&= \frac{1}{2m}\left[p_x^2 + \left(-\hbar k_2 + \frac{e}{c}Bx\right)^2 - 2meEx + \hbar^2 k_3^2\right] \\
&= \frac{1}{2m}\left[p_x^2 + \left(\frac{e}{c}Bx + (-\hbar k_2 - \frac{mcE}{B})\right)^2 - (-\hbar k_2 - \frac{mcE}{B})^2 + \hbar^2(k_2^2 + k_3^3)\right]
\end{aligned}$$

Borrow the idea from a one-dimensional harmonic oscillator where

$$\begin{aligned}
H &= \frac{1}{2m}[p^2 + (m\omega x)^2] \\
a_\pm &= \frac{1}{\sqrt{2\hbar m\omega}}(m\omega x \mp ip) \\
\psi_0(x) &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} \\
\psi_n(x) &= A_n(a_+)^n\psi_0(x) \text{ with } E_n = \left(n + \frac{1}{2}\right)\hbar\omega
\end{aligned}$$

then we can define ω_B, x' and some operators

$$\begin{aligned}
\omega_B &= \frac{eB}{mc} \\
x' &= x + \frac{c}{eB}(-\hbar k_2 - \frac{mcE}{B}) \\
p' &= p_{x'} = p_x \\
a_\pm &= \frac{1}{\sqrt{2\hbar m\omega_B}}(m\omega_B x' \mp ip')
\end{aligned}$$

Then we have the solution to $\psi(x')$ and $\psi(x, y, z)$

$$\begin{aligned}
\psi_0(x') &= \left(\frac{m\omega_B}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega_B}{2\hbar}x'^2} = \left(\frac{m\omega_B}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega_B}{2\hbar}(x+\frac{c}{eB}(-\hbar k_2 - mcE))^2} \\
\psi_n(x') &= A_n(a_+)^n\psi_0(x') \text{ with } E_n = \left(n + \frac{1}{2}\right)\hbar\omega_B + C \\
\psi(x, y, z) &= e^{ik_2 y}e^{ik_3 z}\left(\frac{m\omega_B}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega_B}{2\hbar}(x+\frac{c}{eB}(-\hbar k_2 - mcE))^2} \\
&= e^{ik_2 y}e^{ik_3 z}\left(\frac{m\omega_B}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega_B}{2\hbar}(x - x_0)^2}
\end{aligned}$$

where A_n is the normalization constant and

$$C = \frac{1}{2m}\left[-(-\hbar k_2 - \frac{mcE}{B})^2 + \hbar^2(k_2^2 + k_3^3)\right]$$

Let L_x denote the restriction on x and L_y denote the periodic condition on y. Then $e^{ik_2 y}$ should have period L_y, which means

$$k_2 = \frac{2n\pi}{L_y},\ n \in \mathbb{N}$$

Also, the center x_0 satisfies $0 \le x_0 = \hbar c k_2 / eB \le L_x$, then we have

$$\frac{\hbar c}{eB}\frac{2n\pi}{L_y} \le L_x \Rightarrow n \le \frac{L_x L_y B e}{2\pi\hbar c} = N$$

then N is the degeneracy.