PHY5410 FA22 HW02

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Problem 1 (3.10). The wavefunction under the momentum representation would be

$$\begin{aligned}\varphi(p,t) &= \frac{1}{\sqrt{2\pi\hbar}}\int \mathrm{d}x\ \psi(x,t)e^{-ipx/\hbar} \\ &= \frac{1}{\sqrt{2\pi\hbar}}\frac{1}{2\pi\hbar}\iint \mathrm{d}x\,\mathrm{d}p'\ g(p')e^{ix(p-p')/\hbar}e^{-iE(p')t/\hbar}e^{i\alpha(p')} \\ &= \frac{1}{\sqrt{2\pi\hbar}}\int \mathrm{d}p'\ g(p')e^{-iE(p')t/\hbar}e^{i\alpha(p')}\delta(p-p') \\ &= \frac{1}{\sqrt{2\pi\hbar}}g(p)e^{-iE(p)t/\hbar}e^{i\alpha(p)}\end{aligned}$$

and

$$|\varphi(p,t)|^2 = \frac{1}{2\pi\hbar}g^2(p)$$

Therefore

$$\begin{aligned}\langle p\rangle = \langle\varphi|p\varphi\rangle &= \frac{1}{2\pi\hbar}\int \mathrm{d}p\ pg^2(p) = p_0 \\ \langle p^2\rangle = \langle\varphi|p^2\varphi\rangle &= \frac{1}{2\pi\hbar}\int \mathrm{d}p\ p^2g^2(p)\end{aligned}$$

Using the fact that $x = ih\,\mathrm{d}/\,\mathrm{d}p$ under momentum representation, we have

$$x\varphi(p,t) = \frac{1}{\sqrt{2\pi\hbar}}g(p)(E'(p)t-\alpha'(p)\hbar)e^{-iE(t)/\hbar+i\alpha(p)} + \frac{1}{\sqrt{2\pi\hbar}}i\hbar g'(p)e^{-iE(t)/\hbar+i\alpha(p)}$$

Thus

$$\begin{aligned}\langle x\rangle &= \langle\varphi|x\varphi\rangle \\ &= \frac{1}{2\pi\hbar}\int \mathrm{d}p\ [E'(p)-\alpha'(p)\hbar]g^2(p) + i\hbar g(p)g'(p) \\ &= \frac{1}{2\pi\hbar}\int \mathrm{d}p\ [E'(p)-\alpha'(p)\hbar]g^2(p) \\ &= \langle E'(p)t-\alpha'(p)\hbar\rangle\end{aligned}$$

note that the term $g(p)g'(p)$ vanishes under integration due to the boundary condition.

$$\int \mathrm{d}p\ g(p)g'(p) = \frac{1}{2}g^2(p)\Big|_{\Omega} = 0$$

For x^2 (ignoring $g(p)g'(p)$ term)

$$\begin{aligned}\langle x^2\rangle &= \langle x\varphi|x\varphi\rangle \\ &= \frac{1}{2\pi\hbar}\int \mathrm{d}p\ \hbar^2 {g'}^2(p) + [E'(p)t-\alpha(p)\hbar]^2g^2(p) \\ &= \langle[E'(p)t-\alpha'(p)\hbar]^2\rangle + \frac{1}{2\pi\hbar}\int \mathrm{d}p\ \hbar^2{g'}^2(p)\end{aligned}$$

Plugin a gaussian wave packet with $\sigma_p = \Delta p$, i.e.

$$
\begin{aligned}
g(p) &= Ae^{\frac{-(p-p_0)^2}{4\Delta p^2}} \\
E(p) &= \frac{p^2}{2m} \\
\alpha(p) &= 0
\end{aligned}
$$

we have

$$
\begin{aligned}
\langle p \rangle &= p_0 \\
\langle p^2 \rangle &= \Delta p^2 + p_0^2 \\
\langle x \rangle &= \langle pt/m \rangle = \frac{t}{m} \langle p \rangle = \frac{p_0 t}{m} \\
\langle x^2 \rangle &= \langle p^2 t^2/m^2 \rangle + \langle \hbar^2 \left(\frac{p - p_0}{2\Delta p^2} \right)^2 \rangle \\
&= \frac{t^2}{m^2} \langle p^2 \rangle + \frac{\hbar^2}{4\Delta p^4} \langle (p - p_0)^2 \rangle \\
&= \frac{t^2}{m^2}(p_0^2 + \Delta p^2) + \frac{\hbar^2}{4\Delta p^2}
\end{aligned}
$$

In this manner, we have

$$
\begin{aligned}
\Delta x &= \sqrt{\frac{t^2}{m^2}\Delta p^2 + \frac{\hbar^2}{4\Delta p^2}} = \frac{\hbar}{2\Delta p}\sqrt{1 + \frac{4t^2\Delta p^4}{m^2\hbar^2}} \\
\Rightarrow \Delta x \Delta p &= \frac{\hbar}{2}\sqrt{1 + \frac{4t^2\Delta p^4}{m^2\hbar^2}} \geq \frac{\hbar}{2}
\end{aligned}
$$

Problem 2 (3.15). The energy of the system is

$$
E = T + V = \frac{p^2}{2m} + cx^4
$$

Substitute p by $px = \hbar/2$

$$
E = \frac{\hbar^2}{8mx^2} + cx^4
$$

Find the minimum of E by letting $\mathrm{d}E/\,\mathrm{d}x = 0$

$$
\frac{\mathrm{d}E}{\mathrm{d}x} = 4cx^3 - \frac{\hbar^2}{4mx^3} = 0 \Rightarrow x_0 = \left(\frac{\hbar^2}{16mc} \right)^{1/6}
$$

Then the ground state energy approximately equals to

$$
E_0 = \frac{\hbar^2}{8mx_0^2} + cx_0^4 = \frac{\hbar^2}{8m}\left(\frac{16mc}{\hbar^2} \right)^{1/3} + c\left(\frac{\hbar^2}{16mc} \right)^{2/3} = 3c\left(\frac{\hbar^2}{16mc} \right)^{2/3}
$$

Problem 3 (4.2)**.** Using the fact that the energy uncertainty of a free wave packet is $\Delta E = p_0 \Delta p/m$ and the time uncertainty is $\Delta t = m\Delta x/p_0$. Then

$$\Delta E = \frac{\hbar}{2dm} p_0$$
$$\Delta E \Delta t = \sqrt{1 + \left(\frac{t\hbar}{2md^2}\right)^2} \frac{\hbar}{2}$$

Problem 4 (5.3)**.**

Claim. $L_i = L_i^\dagger$

Proof. Using the fact $[x_i, p_j] = i\hbar\delta_{ij}$ (x_i, p_j commute when $i \neq j$)

$$L_i^\dagger = \epsilon_{ijk} p_k^\dagger x_j^\dagger = \epsilon_{ijk} p_k x_j = \epsilon_{ijk} x_j p_k = L_i \qquad \square$$

Claim. $\langle \psi | L_i^2 \psi \rangle \geq 0$.

Proof. Since $L_i = L_i^\dagger$, we have

$$\langle \psi | L_i^2 \psi \rangle = \langle L_i^\dagger \psi | L_i \psi \rangle = \langle L_i \psi | L_i \psi \rangle \geq 0 \qquad \square$$

By the claims above, we can derive that

$$\langle \psi | \mathbf{L}^2 \psi \rangle = 0 \Leftrightarrow \sum_i \langle \psi | L_i^2 \psi \rangle = 0 \Leftrightarrow \langle \psi | L_i^2 \psi \rangle = \langle L_i \psi | L_i \psi \rangle = 0 \Leftrightarrow |L_i \psi\rangle = |0\rangle \Rightarrow \langle \psi | L_i \psi \rangle = 0$$

note that we can derive $|L_i \psi\rangle = |0\rangle$ if $|L_i \psi\rangle$ is indeed a continuous function under any representations.