PHY 5410: Homework Week 13

1.10 (a) Show, by verifying the relation

$$n(\mathbf{x})\,|\phi\rangle = \delta(\mathbf{x}-\mathbf{x}')\,|\phi\rangle \ ,$$

that the state

$$|\phi\rangle = \psi^\dagger(\mathbf{x}')\,|0\rangle$$

($|0\rangle$ = vacuum state) describes a particle with the position $\mathbf{x}'$.
(b) The operator for the total particle number reads:

$$\hat{N} = \int d^3x\, n(\mathbf{x}) \ .$$

Show that for spinless particles

$$[\psi(\mathbf{x}), \hat{N}] = \psi(\mathbf{x}) \ .$$

2.1 Calculate the static structure function for noninteracting fermions

$$S^0(\mathbf{q}) \equiv \frac{1}{N}\langle \phi_0 \mid \hat{n}_{\mathbf{q}}\hat{n}_{-\mathbf{q}} \mid \phi_0\rangle \ ,$$

where $\hat{n}_{\mathbf{q}} = \sum_{\mathbf{k},\,\sigma} a^\dagger_{\mathbf{k}\,\sigma} a_{\mathbf{k}+\mathbf{q}\,\sigma}$ is the particle density operator in the momentum representation and $|\phi_0\rangle$ is the ground state. Take the continuum limit $\sum_{\mathbf{k}\,,\sigma} = 2V\int d^3k/(2\pi)^3$ and calculate $S^0(\mathbf{q})$ explicitly.
Hint: Consider the cases $\mathbf{q}=0$ and $\mathbf{q}\neq 0$ separately.

2.6 Derive the following relations for Fermi operators:

a)

$$\mathrm{e}^{-\alpha a^\dagger} a \mathrm{e}^{\alpha a^\dagger} = a - \alpha^2 a^\dagger + \alpha(aa^\dagger - a^\dagger a)$$

$$\mathrm{e}^{-\alpha a} a^\dagger \mathrm{e}^{\alpha a} = a^\dagger - \alpha^2 a - \alpha(aa^\dagger - a^\dagger a)$$

b)

$$\mathrm{e}^{\alpha a^\dagger a} a \mathrm{e}^{-\alpha a^\dagger a} = \mathrm{e}^{-\alpha} a$$

$$\mathrm{e}^{\alpha a^\dagger a} a^\dagger \mathrm{e}^{-\alpha a^\dagger a} = \mathrm{e}^{-\alpha} a^\dagger \ .$$