

PHY5410 FA22 HW04

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Problem 1 (6.3). Using the relation that $u(r) = rR(r)$, $\varrho = \kappa r$ (more strictly, we should use the notation $v(\varrho) = u(r) = u(\varrho/\kappa)$ rather than $u(\varrho)$ to denote $u(\varrho/\kappa)$), define the following notation

$$\langle r^k \rangle = 4\pi r^2 \int dr r^k R(r)^2$$

$$\langle \varrho^k \rangle_u = 4\pi \int d\varrho \varrho^k u(\varrho)^2$$

one can verify

$$\langle \varrho^k \rangle_u = 4\pi \int d\kappa r (\kappa r)^k r^2 R(r)^2 = 4\pi \kappa^{k+1} \int r^2 dr r^k R(r)^2 = \kappa^{k+1} \langle r^k \rangle$$

Using the differential equation

$$\left[\frac{d^2}{d\varrho^2} - \frac{l(l+1)}{\varrho^2} + \frac{2n}{\varrho} - 1 \right] u(\varrho) = 0 \quad (1)$$

Multiply $\varrho^{k+1} u'_{nl}(\varrho)$ on the left, we get

$$\varrho^{k+1} u' u'' - \varrho^{k-1} l(l+1) u u' + 2n \varrho^k u u' - \varrho^{k+1} u u' = 0$$

Using the fact that

$$4\pi \int d\varrho \varrho^m u u' = -\frac{m}{2} 4\pi \int d\varrho \varrho^{m-1} u^2 = -\frac{m}{2} \langle \varrho^{m-1} \rangle_u$$

$$4\pi \int d\varrho \varrho^m u u' = -2\pi(m-1) \int d\varrho \varrho^{m-1} (u')^2$$

$$4\pi \int d\varrho \varrho^m u u'' = -4\pi m \int d\varrho \varrho^{m-1} u u' - 4\pi \int d\varrho \varrho^m (u')^2$$

we can multiply the expression by 4π and integrate it

$$-2\pi(k+1) \int d\varrho \varrho^k (u')^2 + \frac{1}{2}(k-1)l(l+1) \langle \varrho^{k-2} \rangle_u - nk \langle \varrho^{k-1} \rangle_u + \frac{1}{2}(k+1) \langle \varrho^k \rangle_u = 0 \quad (2)$$

we can also multiply the equation by $\varrho^k u_{nl}(\varrho)$

$$\varrho^k u u'' - l(l+1) \varrho^{k-2} u^2 + 2n \varrho^{k-1} u^2 - \varrho^k u^2 = 0$$

multiply this equation by 4π and integrate it

$$\frac{1}{2}k(k-1) \langle \varrho^{k-2} \rangle_u - 4\pi \int d\varrho \varrho^k (u')^2 - l(l+1) \langle \varrho^{k-2} \rangle_u + 2n \langle \varrho^{k-1} \rangle_u - \langle \varrho^k \rangle_u = 0 \quad (3)$$

Combing equation 2 and 3 to eliminate $(u')^2$ term, we can get

$$\left[\frac{1}{2}(k-1)l(l+1) - \frac{1}{4}k(k^2-1) + \frac{1}{2}l(l+1)(k+1) \right] \langle \varrho^{k-2} \rangle_u - [nk + n(k+1)] \langle \varrho^{k-1} \rangle_u + (k+1) \langle \varrho^k \rangle_u = 0$$

$$\frac{k}{4} [4l(l+1) - (k^2-1)] \langle \varrho^{k-2} \rangle_u - n(2k+1) \langle \varrho^{k-1} \rangle_u + (k+1) \langle \varrho^k \rangle_u = 0$$

$$\frac{k}{4}[(2l+1)^2 - k^2] \langle \varrho^{k-2} \rangle_u - n(2k+1) \langle \varrho^{k-1} \rangle_u + (k+1) \langle \varrho^k \rangle_u = 0$$

Plugin $\kappa = 1/na$ and $\langle \varrho^k \rangle_u = \kappa^{k+1} \langle r^k \rangle$, we get

$$\begin{aligned} & \frac{k}{4}[(2l+1)^2 - k^2] \langle r^{k-2} \rangle - n(2k+1)\kappa \langle r^{k-1} \rangle + (k+1)\kappa^2 \langle r^k \rangle = 0 \\ \Rightarrow & \frac{k+1}{n^2} \langle r^k \rangle - (2k+1)a \langle r^{k-1} \rangle + \frac{k}{4}[(2l+1)^2 - k^2]a^2 \langle r^{k-2} \rangle = 0 \end{aligned}$$

Problem 2 (6.13). Let A_i denotes the i th component of \mathbf{A} , then

$$A_i = \frac{1}{2m}(\epsilon_{ijk} p_j L_k - \epsilon_{ijk} L_j p_k) - \frac{Ze^2}{r} x_i = \frac{1}{2m} \epsilon_{ijk} (p_j L_k + L_k p_j) - \frac{Ze^2}{r} x_i$$

where the summation would be taken in place of j and k . Since $p_j^\dagger = p_j$, $L_k^\dagger = L_k$, then we can prove that A_i is hermitian

$$A_i^\dagger = \frac{1}{2m} \epsilon_{ijk} (L_k^\dagger p_j^\dagger + p_j^\dagger L_k^\dagger) - \frac{Ze^2}{r} x_i^\dagger = \frac{1}{2m} \epsilon_{ijk} (L_k p_j + p_j L_k) - \frac{Ze^2}{r} x_i = A_i$$

Therefore \mathbf{A} is hermitian.

To prove $[\mathbf{A}, H] = 0$, we can show that $[A_i, H] = 0$. A_i could be simplified to the following expression by direct summation over j and k .

$$\begin{aligned} A_i &= \frac{1}{2m} \epsilon_{ijk} \epsilon_{kmn} (p_j x_m p_n + x_m p_n p_j) - \frac{Ze^2}{r} x_i \\ &= \frac{1}{2m} \epsilon_{ijk} \epsilon_{kmn} [(x_m p_j - i\hbar \delta_{jm} p_n + x_m p_n p_j)] - \frac{Ze^2}{r} x_i \\ &= \frac{1}{2m} \epsilon_{kij} \epsilon_{kmn} (2x_m p_j p_n - i\hbar \delta_{jm} p_n) - \frac{Ze^2}{r} x_i \\ &= \frac{1}{2m} (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) (2x_m p_j p_n - i\hbar \delta_{jm} p_n) \end{aligned}$$

Since

$$\begin{aligned} \delta_{im} \delta_{jn} x_m p_j p_n &= \delta_{jn} x_i p_j p_n = x_i p^2 \\ \delta_{in} \delta_{jm} x_m p_j p_n &= \delta_{jm} x_m p_i p_j = (x \cdot p) p_i \\ \delta_{im} \delta_{jn} \delta_{jn} i\hbar p_n &= \delta_{ij} \delta_{jn} i\hbar p_n = i\hbar p_i \\ \delta_{in} \delta_{jm}^2 p_n &= \delta_{jm}^2 i\hbar p_i = 3i\hbar p_i \end{aligned}$$

Then we have

$$A_i = \frac{1}{m} x_i p^2 - \frac{1}{m} (x \cdot p) p_i + \frac{1}{m} i\hbar p_i - \frac{Ze^2}{r} x_i$$

Using the fact that $H = p^2/2m - Ze^2/r$, we have

$$\begin{aligned} A_i H &= \frac{1}{2m^2} [x_i (p^2)^2 - (x \cdot p) p^2 p_i + i\hbar p^2 p_i] - \frac{Ze^2}{2m} \frac{1}{r} x_i p^2 - \frac{Ze^2}{m} [x_i p^2 \frac{1}{r} - (x \cdot p) p_i \frac{1}{r} + i\hbar p_i \frac{1}{r}] + \frac{Ze^4}{r^2} x_i \\ H A_i &= \frac{1}{2m^2} [p^2 x_i p^2 - p^2 (x \cdot p) p_i + i\hbar p^2 p_i] - \frac{Ze^2}{2m} p^2 \frac{1}{r} x_i - \frac{Ze^2}{m} [\frac{1}{r} x_i p^2 - \frac{1}{r} (x \cdot p) p_i + \frac{1}{r} i\hbar p_i] + \frac{Ze^4}{r^2} x_i \end{aligned}$$

Using the commutation relationship

$$p^2 x_i = x_i p^2 - 2i\hbar p_i$$

$$p^2(x \cdot p) = \sum_j p^2 x_j p_j = \sum_j x_j p^2 p_j - 2i\hbar p_j^2 = (x \cdot p)p^2 - 2i\hbar p^2$$

Then it is easy to show that the blue parts are equal

$$p^2 x_i p^2 - p^2(x \cdot p)p_i = x_i(p^2)^2 - 2i\hbar p^2 p_i - (x \cdot p)p^2 p_i + 2i\hbar p^2 p_i = x_i(p^2)^2 - (x \cdot p)p^2 p_i$$

Moreover, using the fact that

$$p_i \frac{1}{r} = i\hbar \frac{1}{r^3} x_i + \frac{1}{r} p_i$$

$$p_i p_j \frac{1}{r} = i\hbar \frac{1}{r^3} x_i p_j + i\hbar \frac{1}{r^3} x_j p_i + \frac{1}{r} p_i p_j + \hbar^2 \frac{\delta_{ij} r^2 - 3x_i x_j}{r^5}$$

$$p^2 \frac{1}{r} = 2i\hbar \frac{1}{r^3} (x \cdot p) + \frac{1}{r} p^2$$

$$p^2 x_i \frac{1}{r} = x_i p^2 \frac{1}{r} - 2i\hbar p_i \frac{1}{r} = 2i\hbar \frac{1}{r^3} x_i (x \cdot p) + \frac{1}{r} x_i p^2 + 2\hbar^2 \frac{1}{r^3} x_i - 2i\hbar \frac{1}{r} p_i$$

we have

$$\begin{aligned} -\frac{Ze^2}{2m} \frac{1}{r} x_i p^2 - \frac{Ze^2}{m} [x_i p^2 \frac{1}{r} - (x \cdot p)p_i \frac{1}{r} + i\hbar p_i \frac{1}{r}] &= -\frac{Ze^2}{2m} \frac{1}{r} x_i p^2 - \frac{Ze^2}{m} \left[2i\hbar \frac{1}{r^3} x_i (x \cdot p) + \frac{1}{r} x_i p^2 \right. \\ &\quad \left. - i\hbar \frac{1}{r^3} x_i (x \cdot p) - i\hbar \frac{1}{r} p_i - \frac{1}{r} (x \cdot p)p_i + 2\hbar^2 \frac{1}{r^3} x_i \right. \\ &\quad \left. - \hbar^2 \frac{1}{r^3} x_i + i\hbar \frac{1}{r} p_i \right] \\ &= -\frac{Ze^2}{2m} \frac{1}{r} x_i p_i - \frac{Ze^2}{m} \left[\frac{1}{r} x_i p^2 - \frac{1}{r} (x \cdot p)p_i \right. \\ &\quad \left. + i\hbar \frac{1}{r^3} x_i (x \cdot p) + \hbar^2 \frac{1}{r^3} x_i \right] \\ -\frac{Ze^2}{2m} p^2 \frac{1}{r} x_i - \frac{Ze^2}{m} \left[\frac{1}{r} x_i p^2 - \frac{1}{r} (x \cdot p)p_i + \frac{1}{r} i\hbar p_i \right] &= -\frac{Ze^2}{2m} \frac{1}{r} x_i p^2 - \frac{Ze^2}{m} i\hbar \frac{1}{r^3} x_i (x \cdot p) - \frac{Ze^2}{m} \hbar^2 \frac{1}{r^3} x_i \\ &\quad + \frac{Ze^2}{m} i\hbar \frac{1}{r} p_i - \frac{Ze^2}{m} \left[\frac{1}{r} x_i p^2 - \frac{1}{r} (x \cdot p)p_i + i\hbar \frac{1}{r} p_i \right] \\ &= -\frac{Ze^2}{2m} \frac{1}{r} x_i p^2 - \frac{Ze^2}{m} \left[\frac{1}{r} x_i p^2 - \frac{1}{r} (x \cdot p)p_i \right. \\ &\quad \left. + i\hbar \frac{1}{r^3} x_i (x \cdot p) + \hbar^2 \frac{1}{r^3} x_i \right] \end{aligned}$$

Hence the red parts are also equal. Therefore $[A_i, H] = 0$, we have $[\mathbf{A}, H] = 0$.

The proof of $\mathbf{A} \cdot \mathbf{L} = \mathbf{L} \cdot \mathbf{A} = 0$ begins from the following claims.

Claim. $\mathbf{L} \cdot \mathbf{x} = \mathbf{x} \cdot \mathbf{L} = 0$.

Proof. Note that

$$\begin{aligned} \mathbf{x} \cdot \mathbf{L} &= \sum_{ijk} \epsilon_{ijk} x_i x_j p_k = \sum_k \left(\sum_{ij} \epsilon_{ijk} x_i x_j \right) p_k = \sum_k 0 p_k = 0 \\ \mathbf{L} \cdot \mathbf{x} &= \sum_{ijk} \epsilon_{ijk} x_j p_k x_i = \sum_{ijk} \epsilon_{ijk} x_j (x_i p_k - i\hbar \delta_{ik}) = \sum_{ijk} \epsilon_{ijk} x_i x_j p_k - i\hbar \epsilon_{ijk} \delta_{ik} x_j = 0 \end{aligned}$$

□

Claim. $\mathbf{L} \cdot (\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}) = (\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}) \cdot \mathbf{L} = 0$.

Proof. Note that

$$\begin{aligned}
 \mathbf{L} \cdot (\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}) &= \sum_{ijk} \epsilon_{ijk} L_i p_j L_k - \epsilon_{ijk} L_i L_j p_k = \sum_{ijk} \epsilon_{ijk} L_i (p_j L_k + L_k p_j) \\
 &= \sum_{ijk} \epsilon_{ijk} (2L_i L_k p_j + i\hbar \sum_l \epsilon_{kjl} p_l) \\
 &= \sum_{ijk} 2\epsilon_{ijk} L_i L_k p_j - i\hbar \epsilon_{ijk} \epsilon_{kjl} L_i p_l \\
 &= - \sum_j 2i\hbar L_j p_j - \sum_k i\hbar L_k p_k + \sum_i 3i\hbar L_i p_i \\
 &= 0 \\
 (\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}) \cdot \mathbf{L} &= \sum_{ijk} \epsilon_{ijk} (p_j L_k + L_k p_j) L_i \\
 &= \sum_{ijk} \epsilon_{ijk} (2p_j L_k + i\hbar \sum_l \epsilon_{kjl} p_l) L_i \\
 &= \sum_j 2i\hbar p_j L_j + \sum_k i\hbar p_k L_k - 3 \sum_i i\hbar p_i L_i \\
 &= 0
 \end{aligned}$$

□

Therefore, it is easy to show that

$$\begin{aligned}
 \mathbf{L} \cdot \mathbf{A} &= \frac{1}{2m} \mathbf{L} \cdot (\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}) - (\mathbf{L} \times \mathbf{x}) \frac{Ze^2}{r} = 0 \\
 \mathbf{A} \cdot \mathbf{L} &= \frac{1}{2m} (\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}) \cdot \mathbf{L} - \frac{Ze^2}{r} (\mathbf{x} \times \mathbf{L}) = 0
 \end{aligned}$$

Problem 3 (7.1). Since

$$\begin{aligned}
 \nabla \cdot \mathbf{j} &= \frac{1}{2m} \left[\psi^* (-i\hbar \nabla^2 \psi - \frac{e}{c} \mathbf{A} \cdot \nabla \psi - \frac{e}{c} (\nabla \cdot \mathbf{A}) \psi) + (\nabla \psi^*) \cdot (-i\hbar \nabla \psi - \frac{e}{c} \mathbf{A} \psi) \right] + \text{c.c.} \\
 &= \frac{1}{2m} \left(-i\hbar \psi^* \nabla^2 \psi - \frac{e}{c} \psi^* \mathbf{A} \cdot \nabla \psi - \frac{e}{c} (\nabla \cdot \mathbf{A}) \psi^* \psi - i\hbar \nabla \psi^* \cdot \nabla \psi - \frac{e}{c} \nabla \psi^* \cdot \mathbf{A} \psi \right) + \text{c.c.} \\
 &= \frac{1}{2m} \left(-i\hbar \psi^* \nabla^2 \psi - \frac{e}{c} \psi^* \mathbf{A} \cdot \nabla \psi - \frac{e}{c} (\nabla \cdot \mathbf{A}) \psi^* \psi - \frac{e}{c} \nabla \psi^* \cdot \mathbf{A} \psi \right) + \text{c.c.} \\
 &= \frac{1}{2m} \left(-i\hbar \psi^* \nabla^2 \psi - \frac{2e}{c} \psi^* \mathbf{A} \cdot \nabla \psi - \frac{e}{c} (\nabla \cdot \mathbf{A}) \psi^* \psi \right) + \text{c.c.}
 \end{aligned}$$

and

$$\begin{aligned}
 \frac{\partial}{\partial t} \psi^* \psi &= \psi^* \frac{\partial}{\partial t} \psi + \text{c.c.} \\
 &= \frac{1}{i\hbar} \psi^* H \psi + \text{c.c.} \\
 &= \frac{1}{i\hbar} \psi^* \left(-\frac{\hbar^2}{2m} \nabla^2 \psi + \frac{i\hbar e}{mc} \mathbf{A} \cdot \nabla \psi + \frac{i\hbar e}{2mc} (\nabla \cdot \mathbf{A}) \psi + \frac{e^2}{2mc} \mathbf{A}^2 \psi + e\Phi \psi \right) + \text{c.c.} \\
 &= \frac{1}{2m} \left(i\hbar \psi^* \nabla^2 \psi + \frac{2e}{c} \psi^* \mathbf{A} \cdot \nabla \psi + \frac{e}{c} (\nabla \cdot \mathbf{A}) \psi^* \psi \right) + \text{c.c.}
 \end{aligned}$$

Hence $\partial_t \psi^* \psi + \nabla \cdot \mathbf{j} = 0$.

Problem 4 (7.3). Let the equation be $H\psi = \lambda\psi$, where the hamiltonian H equals to

$$H = \frac{1}{2m} \left[p_x^2 + \left(p_y - \frac{e}{c} Bx \right)^2 + p_z^2 \right] - eEx$$

Suppose the solution in the form of $\psi(x, y, z) = e^{k_2 y} e^{k_3 z} \psi(x)$, then we can get a hamiltonian H_x only related to x

$$\begin{aligned} H_x &= \frac{1}{2m} \left[p_x^2 + \left(-i\hbar k_2 - \frac{e}{c} Bx \right)^2 - \hbar^2 k_3^2 \right] - eEx \\ &= \frac{1}{2m} \left[p_x^2 + \left(i\hbar k_2 + \frac{e}{c} Bx \right)^2 - 2meEx - \hbar^2 k_3^2 \right] \\ &= \frac{1}{2m} \left[p_x^2 + \left(\frac{e}{c} Bx + (i\hbar k_2 B - mcE) \right)^2 - (i\hbar k_2 B - mcE)^2 - \hbar^2 (k_2^2 + k_3^2) \right] \end{aligned}$$

Borrow the idea from a one-dimensional harmonic oscillator where

$$\begin{aligned} H &= \frac{1}{2m} [p^2 + (m\omega x)^2] \\ a_{\pm} &= \frac{1}{\sqrt{2\hbar m\omega}} (m\omega x \mp ip) \\ \psi_0(x) &= \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} e^{-\frac{m\omega}{2\hbar} x^2} \\ \psi_n(x) &= A_n (a_+)^n \psi_0(x) \text{ with } E_n = \left(n + \frac{1}{2} \right) \hbar\omega \end{aligned}$$

then we can define ω_B , x' and some operators

$$\begin{aligned} \omega_B &= \frac{eB}{mc} \\ x' &= x + \frac{c}{eB} (i\hbar k_1 B - mcE) \\ p' &= p_{x'} = p_x \\ a_{\pm} &= \frac{1}{\sqrt{2\hbar m\omega_B}} (m\omega_B x' \mp ip') \end{aligned}$$

Then we have the solution to $\psi(x')$

$$\begin{aligned} \psi_0(x') &= \left(\frac{m\omega_B}{\pi\hbar} \right)^{1/4} e^{-\frac{m\omega_B}{2\hbar} x'^2} = \left(\frac{m\omega_B}{\pi\hbar} \right)^{1/4} e^{-\frac{m\omega_B}{2\hbar} \left(x + \frac{c}{eB} (i\hbar k_1 B - mcE) \right)^2} \\ \psi_n(x') &= A_n (a_+)^n \psi_0(x') \text{ with } E_n = \left(n + \frac{1}{2} \right) \hbar\omega_B + C \end{aligned}$$

where A_n is the normalization constant and

$$C = -\frac{1}{2m} [(i\hbar k_2 B - mcE)^2 + \hbar^2 (k_2^2 + k_3^2)]$$