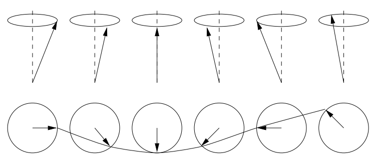
Answers: Problem set IV

1. (a) From Ehrenfest's theorem, the equation of motion for the spin is given by $-i\hbar\frac{d\hat{\mathbf{S}}_m}{dt} = [\hat{H}, \hat{\mathbf{S}}_m]$. Making use of the spin commutation relation, we have (summation on repeated spin indicies assumed)

$$-i\hbar\frac{d\hat{S}^\beta_m}{dt} = -J[\hat{S}^\alpha_m, \hat{S}^\beta_m](\hat{S}^\alpha_{m+1} + \hat{S}^\alpha_{m-1}) = -i\hbar J\epsilon^{\alpha\beta\gamma}\hat{S}^\gamma_m(\hat{S}^\alpha_{m+1} + \hat{S}^\alpha_{m-1}) .$$

We thus obtain the required equation of motion.

(b) Since $\mathbf{S}_{m+1}+\mathbf{S}_{m-1} \simeq 2\mathbf{S}|_{x=m}+\partial^2\mathbf{S}|_{x=m}$ and, for classical vectors, $\mathbf{S}\times\mathbf{S} = 0$, we obtain the required equation of motion.



(c) Substituting the expression for $\mathbf{S}(x, t)$, we find that the equation is solved with $\omega(k) = Jk^2\sqrt{S^2 - c^2}$. The corresponding spin configuration is shown right.

(d) Substituting for the spin raising and lowering operators, the identity is clear. Expanding to the spin raising and lowering operators to leading order in $\frac{a^\dagger a}{2S}$ about the ferromagnetic ground state (in which all spins are aligned along $\hat{e}_z$, we obtain

$$\hat{H} = -JNS^2 + JS\sum_m \left\{a^\dagger_m a_m + a^\dagger_{m+1}a_{m+1} - \left(a^\dagger_m a_{m+1} + \text{h.c.}\right)\right\} + O(S^0) ,$$

where h.c. denotes the Hermitian conjugate. Rearranging, we obtain the required expression for the Hamiltonian.

(e) With the definitions given in the problem,

$$[a_k, a^\dagger_{k'}] = \frac{1}{N}\sum_{m,n} e^{-ikm+ik'n}\underbrace{[a_m, a^\dagger_n]}_{\delta_{mn}} = \frac{1}{N}\sum_m e^{-i(k-k')m} = \delta_{kk'} .$$

Then subtituted into the Hamiltonian,

$$\begin{aligned}\hat{H} &= -JNS^2 + S\sum_{kk'}\underbrace{\frac{1}{N}\sum_m e^{i(k-k')m}}_{\delta_{kk'}}(e^{ik} - 1)(e^{-ik'} - 1)a^\dagger_k a_{k'} \\ &= -JNS^2 + S\sum_k |e^{ik} - 1|^2 a^\dagger_k a_k .\end{aligned}$$

From this result we obtain the required dispersion relation.

2. Standard bookwork allows a derivation of the amplitude $c_n(t)$. In the present case, with $V(t) = e\mathcal{E}_0 z e^{-t/\tau}$, the matrix element $\langle\psi_{2s}|z|\psi_{1s}\rangle = 0$ since the $1s$ and $2s$ wavefunctions both have even parity while z has odd parity. Therefore the probability of finding the atom in the $2s$ state is identically zero.

The matrix elements $\langle\psi_{2p\pm1}|z|\psi_{1s}\rangle = 0$ since the ϕ part of the integral will vanishes,

$$\langle\psi_{2p\pm1}|z|\psi_{1s}\rangle \sim \int_0^{2\pi} d\phi e^{\pm i\phi} = 0 .$$

The only non-zero matrix element is:

$$\begin{aligned}\langle\psi_{2p_0}|z|\psi_{1s}\rangle &= \left(\frac{1}{32\pi a_0^5}\right)^{1/2}\left(\frac{1}{\pi a_0^3}\right)^{1/2}\int r^2 dr\, r^2 e^{-r/a_0}e^{-r/2a_0}\int 2\pi\sin\theta d\theta\cos^2\theta \\ &= \frac{1}{4\sqrt{2}\pi a_0^4}\cdot\frac{4!}{(3/2a_0)^5}\cdot\frac{4\pi}{3} = \frac{256a_0}{243\sqrt{2}} .\end{aligned}$$

Taking the limit as $t \to \infty$, the t' integral is given by,

$$\int_0^\infty dt' e^{-t'/\tau}e^{i(E_{2p}-E_{1s})t'/\hbar} = \frac{1}{1/\tau - i\Delta E/\hbar} ,$$

where $\Delta E = E_{2p} - E_{1s} = 3R_\infty/4$. Putting all this together we obtain the probability of being in the $2p_0$ state after a long time as

$$|c_{2p_0}(\infty)|^2 = \frac{e^2 \mathcal{E}_0^2 a_0^2 2^{15}}{3^{10}} \cdot \frac{1}{\Delta E^2 + \hbar^2/\tau^2}\,.$$

3. From the lecture notes, the decay rate for unpolarized light is given by,

$$A = \frac{\omega^3 |\mathbf{d}_{kj}|^2}{3\pi\epsilon_0 c^3 \hbar}\,,$$

and the lifetime is thus $\tau = 1/A$. Take for example the $2p_0$ state of Hydrogen decaying to 1s (the other 2p states must have the same lifetime, but this one depends on the same matrix elements that we computed in in the previous question. Only the z-component of $\mathbf{d}$ is non-zero for this transition, (the ϕ integral yields zero if you compute the matrix elements of x or y) giving,

$$\langle 2p_0 |\, ez\, |1s\rangle = \frac{256 e a_0}{243\sqrt{2}} = 6.31 \times 10^{-30}\,\mathrm{Cm}\,.$$

The energy of the emitted photon is

$$\hbar\omega = \frac{3}{4}R_\infty = \frac{3}{4} \cdot \frac{me^4}{2(4\pi\epsilon_0)^2\hbar^2} \qquad \Rightarrow \qquad \omega = 1.56 \times 10^{16}\,\mathrm{Hz}\,.$$

Hence, the lifetime of the state is $\tau = 1.56 \times 10^{-9}$ s.

The only lower lying state to which 3s can decay is 2p according to the selection rules. We can expect the matrix element $\langle 3s|\, ez\, |2p\rangle \sim e a_0$ on dimensional grounds, and thus not very different from $\langle 2p|\, ez\, |1s\rangle$. The main difference between the lifetimes of the 3s and 2p levels will arise from the difference in ω^3. For the 3s→2p transition,

$$\hbar\omega = (\frac{1}{4} - \frac{1}{9})R_\infty = \frac{5}{36}R_\infty\,.$$

The ratio of the lifetimes is therefore approximately

$$\frac{\tau(3s)}{\tau(2p)} \sim \left(\frac{3}{4} \cdot \frac{36}{5}\right)^3 \sim 150\,.$$

The only state lying below 2s is 1s, but the decay 2s→1s is not allowed by the electric dipole selection rules. The 2s state is "metastable". The dominant decay is actually via two-photon emission, a process which can arise through second order perturbation theory, and occurs very slowly. In practice, atoms may well make transitions from 2s to 2p (for example) before decay takes place as a result of collision processes. Alternatively, decay of the 2s state may be induced by the application of an external electric field, which mixes 2s and 2p through the Stark effect.

4. From the lecture notes, the Born Approximation gives,

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 \left| \int V(\mathbf{r}) e^{i\boldsymbol{\Delta}\cdot\mathbf{r}} d^3r \right|^2\,,$$

where $\boldsymbol{\Delta}$ is the difference beweeen incoming and outgoing wave vectors, of magnitude $2k\sin^2(\theta/2)$. In the case where $V(\mathbf{r}) = V(r)$, i.e. where the potential is centrally symmetric, it is convenient to take $\boldsymbol{\Delta}$ as the axis of polar coordinates for the purpose of integration, so that $\boldsymbol{\Delta}\cdot\mathbf{r} = |\boldsymbol{\Delta}| r\cos\theta'$. The integral thus becomes

$$\int V(\mathbf{r}) e^{i\boldsymbol{\Delta}\cdot\mathbf{r}} d^3r = \int V(r) e^{i\Delta r\cos\theta'} 2\pi \sin\theta' d\theta' r^2 dr$$
$$= 2\pi \int V(r) r^2 dr \left[\frac{e^{i\Delta r\cos\theta'}}{i\Delta r}\right]_0^\pi = \frac{4\pi}{\Delta} \int V(r) r dr \sin(\Delta r)\,,$$

and hence

$$\frac{d\sigma}{d\Omega} = \left(\frac{2m}{\Delta\hbar^2}\right)^2 \left|\int V(r) r dr \sin(\Delta r)\right|^2 .$$

Taking $V(r) = -V_0$ for $r \leq a$, and $V(r) = 0$ otherwise, the integral becomes (integrating by parts),

$$\begin{aligned} -V_0 \int_0^a r\sin(\Delta r)dr &= -V_0 \left\{ \left[-r\frac{\cos(\Delta r)}{\Delta}\right]_0^a + \int_0^a \frac{\cos(\Delta r)}{\Delta} dr \right\} \\ &= -\frac{V_0}{\Delta^2}(\sin(\Delta a) - \Delta a \cos(\Delta a)) , \end{aligned}$$

and thus

$$\frac{d\sigma}{d\Omega} = \left[\frac{2mV_0}{\hbar^2\Delta^3}(\sin(\Delta a) - \Delta a\cos(\Delta a))\right]^2 .$$

In the low energy limit, $\Delta \to 0$,

$$\sin(\Delta a) - \Delta a \cos(\Delta a) \approx \Delta a - \frac{1}{3!}(\Delta a)^3 - \Delta a(1 - (\Delta a)^2/2) = (\Delta a)^3/3 ,$$

and hence

$$\frac{d\sigma}{d\Omega} = \left(\frac{2mV_0 a^3}{3\hbar^2}\right)^2 .$$

This is independent of Δ and hence independent of θ, so isotropic, as required. The total cross-section is obtained by integrating over solid angles, which simply involves multiplying by 4π in this case

$$\sigma_{\text{tot}} = 4\pi \left(\frac{2mV_0 a^3}{3\hbar^2}\right)^2 .$$

5. (a) When $kR \ll 1$, s-wave scattering dominates. In this case, the problem is equivalent to a one-dimensional scattering problem with an infinite wall at the origin and a δ-function repulsive potential at $r = R$.

   The wavefunction has the solution,

$$u(r) = \begin{cases} C\sin kr & r < R \\ \sin(kr + \delta_0) & r > R \end{cases}$$

   From the continuity condition on the wavefunction and the derivative, we obtain

$$\begin{aligned} &A\sin(kR) = \sin(KR + \delta_0) \\ &kA\cos(kR) - k\cos(kR + \delta_0) = U_0 \sin(kR + \delta_0) . \end{aligned}$$

   From the first equation, we obtain $A = \frac{\sin(kR+\delta_0)}{\sin(kR)}$ which substituted into the second equation, leads to the relation

$$\delta_0 = \tan^{-1}\left[\frac{k\tan(kR)}{k - U_0\tan(kR)}\right] - kR .$$

   The structure is similar to that obtained for the spherical square potential but with different resonant behaviour.

   (b) With $U_0 \gg 1/R, k$, and $U_0 \tan(kR) \gg k$, we obtain the resonance condition

$$\frac{k\tan(kR)}{k - U_0\tan(kR)} \simeq \frac{k}{-U_0\tan(kR)} \simeq 0 ,$$

   i.e. $\delta_0 \simeq -kR$, the value that it would have for a hard sphere.

(c) Now supose that $\tan(kR)$ is small. In this case, we have a resonance when $k - U_0 \tan(kR) = 0$, i.e. $\tan(kR) = \frac{k}{U_0} \ll 1$, and

$$\delta_0 = \frac{\pi}{2} - kR \simeq \frac{\pi}{2}\,.$$

The cross-section $\sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_0 \simeq \frac{4\pi}{k^2}$. The resonance is near $\tan(kR) = 0$, which implies that $kR = (2n+1)\pi/2$, the quasi-bound state of the well.

6. Substituting the definition of $S(\Lambda)$ into the defining condition we obtain

$$\begin{aligned}\left(1 - \frac{i}{4}\Sigma_{\alpha\beta}\omega^{\alpha\beta}\right)\gamma^\mu\left(1 + \frac{i}{4}\Sigma_{\gamma\delta}\omega^{\gamma\delta}\right) &= \gamma^\mu + \frac{i}{4}\left[\gamma^\mu, \Sigma_{\alpha\beta}\right]\omega^{\alpha\beta} + \cdots \\ &= \left(g^\mu{}_\nu - \omega^\mu{}_\nu\right)\gamma^\nu.\end{aligned}$$

Rearranging the left and right hand sides, we obtain

$$\frac{i}{4}\left[\gamma^\mu, \Sigma_{\alpha\beta}\right]\omega^{\alpha\beta} = -\omega^{\beta\alpha}g^\mu{}_\beta\gamma_\alpha \equiv \omega^{\alpha\beta}g^\mu{}_\beta\gamma_\alpha,$$

from which we obtain the required identity. The latter equation is shown to be consistent with the solution $\Sigma_{\alpha\beta} = (i/2)[\gamma_\alpha, \gamma_\beta]$ by making use of the anticommutation relation of the γ matrices.

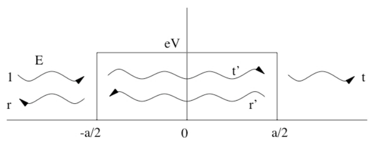
7. Using the identity

$$[\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}, \mathbf{S}\cdot\hat{\mathbf{p}}] = \hat{p}_i\hat{p}_j\begin{pmatrix} 0 & [\sigma_i,\sigma_j] \\ [\sigma_i,\sigma_j] & 0\end{pmatrix} = 2i\epsilon^{ijk}\hat{p}_i\hat{p}_j\begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0\end{pmatrix}.$$

Therefore, since $\hat{\mathbf{p}}\times\hat{\mathbf{p}} = 0$, we find that the Hamiltonian commutes with the Helicity operator.
Turning to the angular momentum, taking each term separately,

$$\begin{aligned}&[\hat{H}, \hat{L}_i] = \epsilon_{ijk}[\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}, \hat{x}_j\hat{p}_k] = \epsilon_{ijk}\left(\alpha_l\hat{p}_l\hat{x}_j\hat{p}_k - \hat{x}_j\hat{p}_k\alpha_l\hat{p}_l\right) \\ &\quad = \epsilon_{ijk}\left(-i\alpha_l\delta_{lj}\hat{p}_k\right) = -i\boldsymbol{\alpha}\times\hat{\mathbf{p}}\,. \\ &[\hat{H}, \mathbf{S}] = [\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}, \mathbf{S}] = \frac{1}{2}(\alpha_i\hat{p}_i\sigma_j - \sigma_j\alpha_i\hat{p}_i) \\ &\quad = \frac{1}{2}\left[\begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} \\ \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} & 0\end{pmatrix}, \begin{pmatrix}\boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma}\end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}[\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}, \boldsymbol{\sigma}] \\ &\quad = -i\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\hat{\mathbf{p}}\times\boldsymbol{\sigma} = -i\hat{\mathbf{p}}\times\boldsymbol{\alpha}.\end{aligned}$$

Putting these terms together we find $[\hat{H}, \hat{\mathbf{J}}] = 0$.



8. Applying the plane wave solution of the Dirac equation $\psi(p) = e^{-p\cdot x}u(p)$ (defined in this form for positive and negative energy states) to the two edges of the potential step, we obtain the boundary conditions

$$\begin{aligned}\begin{pmatrix} 1 \\ 0 \\ \frac{p}{E+m} \\ 0\end{pmatrix} e^{-ipa/2} \quad &+ r\begin{pmatrix} 1 \\ 0 \\ -\frac{p}{E+m} \\ 0\end{pmatrix} e^{ipa/2} \\ &= t'\begin{pmatrix} 1 \\ 0 \\ \frac{p'}{E'+m} \\ 0\end{pmatrix} e^{-ip'a/2} + r'\begin{pmatrix} 1 \\ 0 \\ -\frac{p'}{E'+m} \\ 0\end{pmatrix} e^{ip'a/2}\end{aligned}$$

$$t'\begin{pmatrix} 1 \\ 0 \\ \frac{p'}{E'+m} \\ 0\end{pmatrix} e^{ip'a/2} \quad + \quad r'\begin{pmatrix} 1 \\ 0 \\ -\frac{p'}{E'+m} \\ 0\end{pmatrix} e^{-ip'a/2} = t\begin{pmatrix} 1 \\ 0 \\ \frac{p}{E+m} \\ 0\end{pmatrix} e^{ipa/2},$$

where the reflection and transmission coefficients are defined in the figure.

From these equations we obtain

$$\begin{aligned}
2e^{-ipa/2} &= t'(1+\zeta)e^{-ip'a/2} + r'(1-\zeta)e^{ip'a/2} \\
2re^{ipa/2} &= t'(1-\zeta)e^{-ip'a/2} + r'(1+\zeta)e^{ip'a/2} \\
te^{ipa/2} &= e^{ip'a/2}t' + e^{-ip'a/2}r' \\
te^{ipa/2} &= \zeta\left(e^{ip'a/2}t' - e^{-ip'a/2}r'\right).
\end{aligned}$$

Rearranging these equations we obtain

$$r' = \frac{2}{1+\zeta}\frac{1}{\mu e^{-ip'a} - \mu^{-1}e^{-ip'a}}e^{-i(p-p')a/2},$$

where $\mu = (1-\zeta)/(1+\zeta)$. Finally, with this result, we obtain

$$t = e^{-ipa}\frac{1}{\cos(p'a) - i\sin(p'a)(1+\zeta^2)/2\zeta}$$

From this result, we obtain the expression for the transmitted current shown in the question.

For energies $E' > m$, the particles traverse the barrier as a plane wave. In particular, when $p'a = n\pi$ there is perfect transmission. For $m > E' > -m$, p' is imaginary and exchange of particles occurs by resonant tunnelling across the barrier. For energies $E' < -m$, the Klein paradox regime, p' is real and positive, and there is again perfect transmission when $p'a = n\pi$. Here the transmission is mediated by negative energy states under the barrier.