PHY 5410: Homework Week 4

6.3 Prove the following recursion formula for the matrix elements of r^k

$$\langle r^k \rangle = \langle nl | r^k | nl \rangle$$

in the hydrogen atom:

$$\frac{(k+1)}{n^2}\langle r^k \rangle - (2k+1)a\langle r^{k-1}\rangle + \frac{k}{4}\left[\,(2l+1)^2 - k^2\,\right] a^2 \langle r^{k-2}\rangle = 0 \quad .$$

6.13 The Hermitian vector operator $\boldsymbol{A}$ corresponding to the Lenz vector is

$$\boldsymbol{A} = \frac{1}{2m}(\boldsymbol{p}\times\boldsymbol{L} - \boldsymbol{L}\times\boldsymbol{p}) - \frac{Ze^2}{r}\boldsymbol{r} \quad .$$

Show that $\boldsymbol{A}^\dagger = \boldsymbol{A}$ and that $\boldsymbol{A}$, as in classical theory, is a constant of motion and is normal to $\boldsymbol{L}$, i.e.,

$$[\boldsymbol{A}, H] = 0 \quad , \quad \boldsymbol{A}\cdot\boldsymbol{L} = \boldsymbol{L}\cdot\boldsymbol{A} = 0 \quad ,$$

where H is the Hamiltonian of the hydrogen atom.

7.1 Let the Hamiltonian be

$$H = \frac{1}{2m}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}(\boldsymbol{x},t)\right)^2 + e\Phi(\boldsymbol{x},t) \quad .$$

Prove the continuity equation $(\partial/\partial t)\,\psi^*\psi + \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0$, with

$$\begin{aligned}\boldsymbol{j} &\equiv \frac{\hbar}{2m\mathrm{i}}\left[\psi^*\boldsymbol{\nabla}\psi - (\boldsymbol{\nabla}\psi^*)\psi - \frac{2\mathrm{i}e}{\hbar c}\boldsymbol{A}(\boldsymbol{x},t)\psi^*\psi\right] \\ &\equiv \frac{1}{2m}\left(\psi^*\left(\frac{\hbar}{\mathrm{i}}\boldsymbol{\nabla} - \frac{e}{c}\boldsymbol{A}(\boldsymbol{x},t)\right)\psi + \text{c.c.}\right) \quad .\end{aligned}$$

7.3 Consider a particle with mass m and charge e in a homogeneous electromagnetic field $\boldsymbol{B} = (0,0,B), \boldsymbol{E} = (E,0,0)$ with $|E| < |B|$. Take the gauge $\boldsymbol{A} = (0, Bx, 0)$. Determine the eigenfunctions and eigenvalues for the Hamiltonian

$$H = \frac{1}{2m}(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A})^2 - eEx \quad .$$

In the case $E = 0$, discuss also the degeneracy of the energy levels.