PHY5410 FA22 HW13

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Problem 1 (5.1).

(a) Since $\sigma^i\sigma^j = \epsilon_{ijk}\sigma^k$ $(i \neq j)$ and $(\sigma^i)^2 = \mathbb{1}$, then

$$\alpha^i\alpha^j + \alpha^j\alpha^i = \begin{bmatrix} \sigma^i\sigma^j + \sigma^j\sigma^i & 0 \\ 0 & \sigma^i\sigma^j + \sigma^j\sigma^i \end{bmatrix} = 2\delta_{ij}\mathbb{1}$$

(b)

$$\alpha^i\beta + \beta\alpha^i = \begin{bmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{bmatrix} + \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix} = 0$$

(c)

$$(\alpha^i)^2 = \begin{bmatrix} \sigma^i\sigma^i & 0 \\ 0 & \sigma^i\sigma^i \end{bmatrix} = \mathbb{1} \quad \beta^2 = \begin{bmatrix} \mathbb{1}^2 & 0 \\ 0 & (-\mathbb{1})^2 \end{bmatrix} = \mathbb{1}$$

Problem 2 (5.3). Since

$$\begin{aligned} i\hbar\partial_t\psi &= \left[c\alpha^k\left(-i\hbar\partial_k - \frac{e}{c}A_k\right) + \beta mc^2\right]\psi \\ \Rightarrow E\psi &= \left[c\alpha^k\left(-i\hbar\partial_k - \frac{e}{c}A_k\right) + \beta mc^2\right]\psi \\ E^2\psi &= \left[c\alpha^k\left(-i\hbar\partial_k - \frac{e}{c}A_k\right) + \beta mc^2\right]^2\psi \end{aligned}$$

the equation is summed over $k = 1, 2, 3$. Hence

$$\begin{aligned} E^2\psi &= \left[c\alpha^i\left(-i\hbar\partial_i - \frac{e}{c}A_i\right)c\alpha^j\left(-i\hbar\partial_j - \frac{e}{c}A_j\right) + (\alpha^k\beta + \beta\alpha^k)\cdots + \beta^2m^2c^4\right]\psi \\ &= \left[c^2\left(-i\hbar\partial_i - \frac{e}{c}A_i\right)^2 + m^2c^4\right]\psi \end{aligned}$$

Give that $A_1 = A_3 = 0$, $A_2 = Bx$, using the conclusion from HW04, where energy levels of such systems looks like

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega_B \quad \omega_B = \frac{eB}{mc}$$

Hence in this system

$$\begin{aligned} E_n^2 &= \left[(2n+1)\hbar\omega_B mc^2 + m^2c^4\right] \\ \Rightarrow E_n &= \sqrt{(2n+1)\hbar\omega_B mc^2 + m^2c^4} \end{aligned}$$

where $\omega_B = eB/mc$ (suppose $E \geq 0$).

Problem 3 (6.2). Note that

$$x^\mu = g^{\mu\nu}x_\nu \quad x_\mu = g_{\mu\nu}x^\nu$$

and

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \quad x'_{\mu} = g_{\mu\nu}\Lambda^{\nu}{}_{\sigma} g^{\sigma\rho} x_{\rho} = \Lambda_{\rho}{}^{\mu} x_{\rho}$$

Note that if we sum over b

$$\Lambda^{a}{}_{b}\Lambda_{c}{}^{b} = \Lambda^{a}{}_{b} g_{b\nu}\Lambda^{\nu}{}_{\sigma} g^{\sigma c} = g_{a\sigma} g^{\sigma c} = \delta_{ac}$$

Then $(\Lambda^{\mu}{}_{\nu})^{-1} = \Lambda_{\nu}{}^{\mu}$. Hence

$$\begin{aligned}
\partial'_{\mu} &= \frac{\partial x^{\nu}}{\partial x'^{\mu}}\partial_{\nu} = \Lambda_{\mu}{}^{\nu}\partial_{\nu} \\
\partial'^{\mu} &= \frac{\partial x_{\nu}}{\partial x'_{\mu}}\partial_{\nu} = \Lambda^{\nu}{}_{\mu}\partial^{\nu}
\end{aligned}$$

Problem 4 (6.4)**.** The Dirac equation reads

$$[-\gamma^{\mu}(i\hbar\partial_{\mu} - \frac{e}{c}A_{\mu}) + mc]\psi = 0$$

Hence

$$\begin{aligned}
[-\gamma^{\nu}(i\hbar\partial_{\nu} - \frac{e}{c}A_{\nu}) - mc][-\gamma^{\mu}(i\hbar\partial_{\mu} - \frac{e}{c}A_{\mu}) + mc]\psi &= 0 \\
[\gamma^{\nu}\gamma^{\mu}(i\hbar\partial_{\nu} - \frac{e}{c}A_{\nu})(i\hbar\partial_{\mu} - \frac{e}{c}A_{\mu}) - m^2c^2]\psi &= 0
\end{aligned}$$

Since $\gamma^{\nu}\gamma^{\mu} = g^{\nu\mu}\mathbb{1} + i\sigma^{\mu\nu}$, where $\sigma^{\mu\nu} = i[\gamma^{\mu}, \gamma^{\nu}]/2$. Then

$$\begin{aligned}
[\gamma^{\nu}\gamma^{\mu}(i\hbar\partial_{\nu} - \frac{e}{c}A_{\nu})(i\hbar\partial_{\mu} - \frac{e}{c}A_{\mu}) - m^2c^2]\psi &= 0 \\
[(g^{\nu\mu}\mathbb{1} + i\sigma^{\mu\nu})(i\hbar\partial_{\nu} - \frac{e}{c}A_{\nu})(i\hbar\partial_{\mu} - \frac{e}{c}A_{\mu}) - m^2c^2]\psi &= 0 \\
[(\partial - \frac{e}{c}A)^2 + i\sigma^{\mu\nu}(i\hbar\partial_{\nu} - \frac{e}{c}A_{\nu})(i\hbar\partial_{\mu} - \frac{e}{c}A_{\mu}) - m^2c^2]\psi &= 0 \\
[(\partial - \frac{e}{c}A)^2 - i\sigma^{\mu\nu}(i\hbar\partial_{\mu} - \frac{e}{c}A_{\mu})(i\hbar\partial_{\nu} - \frac{e}{c}A_{\nu}) - m^2c^2]\psi &= 0
\end{aligned}$$

Note that

$$\begin{aligned}
i\sigma^{\mu\nu}(i\hbar\partial_{\mu} - \frac{e}{c}A_{\mu})(i\hbar\partial_{\nu} - \frac{e}{c}A_{\nu}) &= \frac{1}{2}[i\sigma^{\mu\nu}(i\hbar\partial_{\mu} - \frac{e}{c}A_{\mu})(i\hbar\partial_{\nu} - \frac{e}{c}A_{\nu}) + i\sigma^{\nu\mu}(i\hbar\partial_{\nu} - \frac{e}{c}A_{\nu})(i\hbar\partial_{\mu} - \frac{e}{c}A_{\mu})] \\
&= \frac{1}{2}[i\sigma^{\mu\nu}(i\hbar\partial_{\mu} - \frac{e}{c}A_{\mu})(i\hbar\partial_{\nu} - \frac{e}{c}A_{\nu}) - i\sigma^{\mu\nu}(i\hbar\partial_{\nu} - \frac{e}{c}A_{\nu})(i\hbar\partial_{\mu} - \frac{e}{c}A_{\mu})] \\
&= \frac{i}{2}\sigma^{\mu\nu}[(i\hbar\partial_{\mu} - \frac{e}{c}A_{\mu}), (i\hbar\partial_{\nu} - \frac{e}{c}A_{\nu})] \\
&= \frac{\hbar}{2}\frac{e}{c}\sigma^{\mu\nu}F_{\mu\nu} \quad \text{I don't understand how this step works}
\end{aligned}$$

Therefore

$$[(\partial - \frac{e}{c}A)^2 - \frac{he}{2c}\sigma^{\mu\nu}F_{\mu\nu} - m^2c^2]\psi = 0$$

Note that

$$A = \begin{bmatrix} \Phi \\ A_1 \\ A_2 \\ A_3 \end{bmatrix} \quad E = \nabla\Phi = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \quad B = \nabla \times \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

Therefore

$$F = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{bmatrix}$$

Note that

$$\sigma^{0\nu} = \frac{i}{2}[\gamma^0, \gamma^\nu] = \frac{i}{2}[\beta, \beta\alpha^\nu] = \frac{i}{2}\beta[\beta, \alpha^\nu] = \frac{i}{2}\beta(2\beta\alpha^\nu) = i\alpha^\nu$$

Hence for $\mu = 0$ or $\nu = 0$, we have

$$\sigma^{\mu\nu}F_{\mu\nu} = 2\sigma^{0\nu}F_{0\nu} = -2i\alpha \cdot E$$

For all $\mu, \nu \geq 1$, we have

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\nu, \gamma^\nu] = \frac{i}{2}[\beta\alpha^\mu, \beta\alpha^\nu] = \frac{i}{2}[\alpha^\nu, \alpha^\mu] = -\frac{i}{2}2i\epsilon_{\nu\mu\rho}\Sigma_\rho = \epsilon_{\mu\nu\rho}\Sigma_\rho$$

Then for all $\mu, \nu \geq 1$

$$\sigma^{\mu\nu}F_{\mu\nu} = \sigma_{\mu\nu\rho}\Sigma_\rho\epsilon_{\mu\nu\sigma}B_\sigma = 2\delta_{\rho\sigma}\Sigma_\rho B_\sigma = 2\Sigma \cdot B$$

Hence

$$\sigma^{\mu\nu}F_{\mu\nu} = -2i\alpha \cdot E + 2\Sigma \cdot B = -2i(\alpha \cdot E + i\Sigma \cdot B)$$

Then the Dirac equation becomes

$$[(\partial - \frac{e}{c}A)^2 + \frac{i\hbar e}{c}(\alpha \cdot E + i\Sigma \cdot B) - m^2c^2]\psi = 0$$