PHY5410 FA22 HW#02

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Problem 1 (3.10).

Problem 2 (3.15). The energy of the system is

$$E = T + V = \frac{p^2}{2m} + cx^4$$

Substitute p by $px = \hbar/2$

$$E = \frac{\hbar^2}{8mx^2} + cx^4$$

Find the minimum of E by letting $\mathrm{d}E/\,\mathrm{d}x = 0$

$$\frac{\mathrm{d}E}{\mathrm{d}x} = 4cx^3 - \frac{\hbar^2}{4mx^3} = 0 \Rightarrow x_0 = \left(\frac{\hbar^2}{16mc}\right)^{1/6}$$

Then the ground state energy approximately equals to

$$E_0 = \frac{\hbar^2}{8mx_0^2} + cx_0^4 = \frac{\hbar^2}{8m}\left(\frac{16mc}{\hbar^2}\right) + c\left(\frac{\hbar^2}{16mc}\right)^{2/3} = 3c\left(\frac{\hbar^2}{16mc}\right)^{2/3}$$

Problem 3 (4.2). Using the fact that the energy uncertainty of a free wave packet is $\Delta E = p_0 \Delta p/m$ and the time uncertainty is $\Delta t = m\Delta x/p_0$. Then

$$\Delta E = \frac{\hbar}{2dm} p_0$$
$$\Delta E \Delta t = \sqrt{1 + \left(\frac{t\hbar}{2md^2}\right)^2} \frac{\hbar}{2}$$

Problem 4 (5.3).

Claim. $L_i = L_i^\dagger$

Proof. Using the fact $[x_i, p_j] = i\hbar\delta_{ij}$ (x_i, p_j commute when $i \neq j$)

$$\begin{aligned} L_i^\dagger &= [\epsilon_{ijk}(x_j p_k - x_k p_j)]^\dagger \\ &= \epsilon_{ijk}(p_k^\dagger x_j^\dagger - p_j^\dagger x_k^\dagger) \\ &= \epsilon_{ijk}(p_k x_j - p_j x_k) \\ &= \epsilon_{ijk}(x_j p_k - x_k p_j) = L_i \end{aligned}$$ □

Claim. $\langle\psi|L_i^2\psi\rangle \geq 0$.

Proof. Since $L_i = L_i^\dagger$, we have

$$\begin{aligned} \langle\psi|L_i^2\psi\rangle &= \langle L_i^\dagger\psi|L_i\psi\rangle \\ &= \langle L_i\psi|L_i\psi\rangle \geq 0 \end{aligned}$$ □

By the claims above, we can derive that

$$\langle\psi|\mathbf{L}^2\psi\rangle = 0 \Leftrightarrow \sum_i \langle\psi|L_i^2\psi\rangle = 0 \Leftrightarrow \langle\psi|L_i^2\psi\rangle = \langle L_i\psi|L_i\psi\rangle = 0 \Leftrightarrow |L_i\psi\rangle = |0\rangle \Rightarrow \langle\psi|L_i\psi\rangle = 0$$

note that we can derive $|L_i\psi\rangle = |0\rangle$ if $|L_i\psi\rangle$ is indeed a continuous function under any representations.