

# PHY5410 FA22 HW05

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**Problem 1** (8.1). Define operator  $U = e^{-iHt/\hbar}$ , then we have

$$A_H B_H - B_H A_H = (U^\dagger A U)(U^\dagger B U) - (U^\dagger B U)(U^\dagger A U) = U^\dagger A B U - U^\dagger B A U = U^\dagger [A, B] U = C_H$$

**Problem 2** (8.2). Using the fact that

$$\begin{aligned} [x_H, p_H] &= [x, p]_H = i\hbar \\ [x, H] &= \frac{1}{2m}[x, p^2] = \frac{i}{m}\hbar p \\ [H, p] &= \frac{m\omega^2}{2}[x^2, p] = i\hbar m\omega^2 x \end{aligned}$$

we have

$$\begin{aligned} \dot{x}_H &= \frac{i}{\hbar}[H_H, x_H] = \frac{i}{\hbar} \frac{-i}{m}\hbar p_H = \frac{p_H}{m} \\ \dot{p}_H &= \frac{i}{\hbar}[H_H, p_H] = \frac{i}{\hbar} i\hbar m\omega^2 x_H = -m\omega^2 x_H \end{aligned}$$

Then we have a system of differential equations

$$\frac{d}{dt} \begin{bmatrix} x_H \\ p_H \end{bmatrix} = \begin{bmatrix} 0 & 1/m \\ -m\omega^2 & 0 \end{bmatrix} \begin{bmatrix} x_H \\ p_H \end{bmatrix}$$

The solution is

$$\begin{bmatrix} x_H(t) \\ p_H(t) \end{bmatrix} = \exp \left( \begin{bmatrix} 0 & 1/m \\ -m\omega^2 & 0 \end{bmatrix} t \right) \begin{bmatrix} x_H(0) \\ p_H(0) \end{bmatrix}$$

Since

$$\begin{bmatrix} 0 & 1/m \\ -m\omega^2 & 0 \end{bmatrix} t = \frac{1}{2im\omega} \begin{bmatrix} 1 & -1 \\ im\omega & im\omega \end{bmatrix} \begin{bmatrix} i\omega t & \\ & -i\omega t \end{bmatrix} \begin{bmatrix} im\omega & 1 \\ -im\omega & 1 \end{bmatrix}$$

we have

$$\exp \left( \begin{bmatrix} 0 & 1/m \\ -m\omega^2 & 0 \end{bmatrix} t \right) = \frac{1}{2im\omega} \begin{bmatrix} 1 & -1 \\ im\omega & im\omega \end{bmatrix} \begin{bmatrix} e^{i\omega t} & \\ & e^{-i\omega t} \end{bmatrix} \begin{bmatrix} im\omega & 1 \\ -im\omega & 1 \end{bmatrix} = \begin{bmatrix} \cos \omega t & \frac{1}{m\omega} \sin \omega t \\ -m\omega \sin \omega t & \cos \omega t \end{bmatrix}$$

Therefore

$$\begin{aligned} x_H(t) &= \cos \omega t x_H(0) + \frac{1}{m\omega} \sin \omega t p_H(0) \\ p_H(t) &= \cos \omega t p_H(0) - m\omega \sin \omega t x_H(0) \end{aligned}$$

Using the fact that  $a(t) = (m\omega x - ip)/\frac{1}{2m\hbar}$ , we have

$$a_H(t) = \frac{1}{\sqrt{2m\hbar}} [m\omega x_H(t) - ip_H(t)] = \frac{1}{\sqrt{2m\hbar}} [m\omega e^{i\omega t} x_H(0) - ie^{-i\omega t} p_H(0)]$$

**Problem 3** (8.4).

Case 1.  $l = 1/2$ , we have  $m = \pm 1/2$ , let

$$\alpha |1/2\rangle + \beta |-1/2\rangle \mapsto \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

then under the matrix representation, we have

$$\begin{aligned} L^2 &= \hbar^2 \frac{3}{4} \mathbf{I} \\ L_z &= \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ L_x &= \frac{1}{2}(L_+ + L_-) = \frac{\hbar}{2} \left( \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \right) = \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ L_y &= \frac{1}{2i}(L_+ - L_-) = \frac{\hbar}{2i} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

Case 2.  $l = 1$ , let

$$\alpha |1\rangle + \beta |0\rangle + \gamma |-1\rangle \mapsto \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

we have

$$\begin{aligned} L^2 &= 2\hbar^2 \mathbf{I} \\ L_z &= \frac{\hbar}{2} \begin{bmatrix} 1 & & \\ & 0 & \\ & & -1 \end{bmatrix} \\ L_+ &= \hbar \begin{bmatrix} & \sqrt{2} & \\ & & \sqrt{2} \end{bmatrix} \\ L_- &= \hbar \begin{bmatrix} \sqrt{2} & & \\ & \sqrt{2} & \\ & & \end{bmatrix} \\ L_x &= \frac{\hbar}{2} \begin{bmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{bmatrix} \\ L_y &= \frac{\hbar}{2i} \begin{bmatrix} 0 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{bmatrix} \end{aligned}$$

Case 3.  $l = 3/2$ , let

$$\alpha |3/2\rangle + \beta |1/2\rangle + \gamma |-1/2\rangle + \delta |-3/2\rangle \mapsto \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix}$$

we have

$$L^2 = \frac{15}{4} \hbar^2 \mathbf{I}$$

$$L_z = \hbar \begin{bmatrix} 3/2 & & & \\ & 1/2 & & \\ & & -1/2 & \\ & & & -3/2 \end{bmatrix}$$

$$L_+ = \hbar \begin{bmatrix} \sqrt{3} & & & \\ & 2 & & \\ & & \sqrt{3} & \\ & & & \end{bmatrix}$$

$$L_- = \begin{bmatrix} \sqrt{3} & & & \\ & 2 & & \\ & & \sqrt{3} & \\ & & & \end{bmatrix}$$

$$L_x = \frac{\hbar}{2} \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}$$

$$L_y = \frac{\hbar}{2i} \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 0 & 2 & 0 \\ 0 & -2 & 0 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{bmatrix}$$

Case 4.  $l = 2$ , let

$$\alpha |2\rangle + \beta |1\rangle + \gamma |0\rangle + \delta |-1\rangle + \epsilon |-2\rangle \mapsto \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \epsilon \end{bmatrix}$$

we have

$$L^2 = 6\hbar^2 \mathbf{I}$$

$$L_z = \hbar \begin{bmatrix} 2 & & & & \\ & 1 & & & \\ & & 0 & & \\ & & & -1 & \\ & & & & -2 \end{bmatrix}$$

$$L_+ = \hbar \begin{bmatrix} 2 & & & & \\ & \sqrt{6} & & & \\ & & \sqrt{6} & & \\ & & & 2 & \\ & & & & \end{bmatrix}$$

$$L_- = \hbar \begin{bmatrix} 2 & & & & \\ & \sqrt{6} & & & \\ & & \sqrt{6} & & \\ & & & 2 & \\ & & & & \end{bmatrix}$$



$$L_x = \frac{\hbar}{2} \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$$

$$L_y = \frac{\hbar}{2} \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ -2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & -\sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & -\sqrt{6} & 0 & 2 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$$

**Problem 4 (8.5).**

(a) Let  $\mathbf{p}_i$  denotes the momentum operator of the  $i$ th particle

$$\mathbf{p}_i = \begin{bmatrix} p_{ix} \\ p_{iy} \\ p_{iz} \end{bmatrix}$$

Then easy to derive that

$$\begin{aligned} [H, \mathbf{P}] &= \left( \sum_{ij} \frac{1}{2} V(|\mathbf{x}_i - \mathbf{x}_j|) \right) \left( \sum_n \mathbf{p}_n \right) - \left( \sum_n \mathbf{p}_n \right) \left( \sum_{ij} \frac{1}{2} V(|\mathbf{x}_i - \mathbf{x}_j|) \right) \\ &= \sum_{nj} V(r_{nj}) \mathbf{p}_n - \sum_{nj} \mathbf{p}_n V(r_{nj}) \end{aligned}$$

By the commutation relationship where

$$p_{i\alpha} \frac{1}{r_{ij}} = i\hbar \frac{x_{i\alpha} - x_{j\alpha}}{r_{ij}^3} + \frac{1}{r_{ij}} p_{i\alpha}$$

where  $\alpha = x, y, z$ . Using the fact that  $V(r_{ij}) \propto 1/r_{ij}$ , we have

$$\begin{aligned} \sum_{nj} p_{n\alpha} \frac{1}{r_{nj}} &= \sum_{n < j} p_{n\alpha} \frac{1}{r_{nj}} + p_{j\alpha} \frac{1}{r_{nj}} \\ &= \sum_{n < j} i\hbar \frac{x_{n\alpha} - x_{j\alpha}}{r_{nj}^3} + \frac{1}{r_{nj}} p_{n\alpha} + i\hbar \frac{x_{j\alpha} - x_{n\alpha}}{r_{nj}^3} + \frac{1}{r_{nj}} p_{j\alpha} \\ &= \sum_{n < j} \frac{1}{r_{nj}} p_{n\alpha} + \frac{1}{r_{nj}} p_{j\alpha} \\ &= \sum_{nj} \frac{1}{r_{nj}} p_{n\alpha} \end{aligned}$$

Hence  $\sum_{nj} V(r_{nj}) \mathbf{p}_n - \sum_{nj} \mathbf{p}_n V(r_{nj}) = 0$  and  $[H, \mathbf{P}] = 0$ .

The proof of  $[H, \mathbf{L}] = 0$  begins with the following claims

*Claim.* Let  $L_{n\alpha}$  denotes the  $\alpha$  component ( $\alpha = x, y, z$ ) of the angular momentum operator of  $n$ th particle in the system,  $L_{n\alpha} = (\mathbf{x}_n \times \mathbf{p}_n)_\alpha$ . Then

$$\left[ \sum_n \mathbf{p}_n^2, \sum_n L_{n\alpha} \right] = 0$$

*Proof.* Note that

$$\begin{aligned}
 \left[ \sum_n \mathbf{p}_n^2, \sum_n L_{n\alpha} \right] &= \left( \sum_n \mathbf{p}_n^2 \right) \left( \sum_n \epsilon_{\alpha\beta\gamma} x_{n\beta} p_{n\gamma} \right) - \left( \sum_n \epsilon_{\alpha\beta\gamma} x_{n\beta} p_{n\gamma} \right) \left( \sum_n \mathbf{p}_n^2 \right) \\
 &= \sum_n \epsilon_{\alpha\beta\gamma} \mathbf{p}_n^2 x_{n\beta} p_{n\gamma} - \sum_n \epsilon_{\alpha\beta\gamma} x_{n\beta} p_{n\gamma} \mathbf{p}_n^2 \\
 &= \sum_n \epsilon_{\alpha\beta\gamma} (x_{n\beta} \mathbf{p}_n^2 - 2i\hbar p_{n\beta}) p_{n\gamma} - \sum_n \epsilon_{\alpha\beta\gamma} x_{n\beta} p_{n\gamma} \mathbf{p}_n^2 \\
 &= -2i\hbar \sum_n \epsilon_{\alpha\beta\gamma} p_{n\beta} p_{n\gamma} = 0
 \end{aligned}$$

*Claim.*

$$\left[ \sum_{ij} \frac{1}{r_{ij}}, \sum_n L_{n\alpha} \right] = 0$$

*Proof.* Note that

$$\begin{aligned}
 \left[ \sum_{ij} \frac{1}{r_{ij}}, \sum_n L_{n\alpha} \right] &= \left( \sum_{ij} \frac{1}{r_{ij}} \right) \left( \sum_n \epsilon_{\alpha\beta\gamma} x_{n\beta} p_{n\gamma} \right) - \left( \sum_n \epsilon_{\alpha\beta\gamma} x_{n\beta} p_{n\gamma} \right) \left( \sum_{ij} \frac{1}{r_{ij}} \right) \\
 &= 2 \sum_{nj} \epsilon_{\alpha\beta\gamma} \frac{1}{r_{nj}} x_{n\beta} p_{n\gamma} - 2 \sum_{nj} \epsilon_{\alpha\beta\gamma} x_{n\beta} p_{n\gamma} \frac{1}{r_{nj}} \\
 &= 2 \sum_{nj} \epsilon_{\alpha\beta\gamma} \frac{1}{r_{nj}} x_{n\beta} p_{n\gamma} - 2 \sum_{nj} \epsilon_{\alpha\beta\gamma} x_{n\beta} \left( i\hbar \frac{x_{n\gamma} - x_{j\gamma}}{r_{nj}^3} + \frac{1}{r_{nj}} p_{n\gamma} \right) \\
 &= -2i\hbar \sum_{nj} \epsilon_{\alpha\beta\gamma} x_{n\beta} \frac{x_{n\gamma} - x_{j\gamma}}{r_{nj}^3}
 \end{aligned}$$

Since

$$\begin{aligned}
 \sum_{nj} \epsilon_{\alpha\beta\gamma} x_{n\beta} x_{n\gamma} &= 0 \\
 \sum_{nj} \epsilon_{\alpha\beta\gamma} x_{n\beta} x_{j\gamma} &= \sum_{n<j} \epsilon_{\alpha\beta\gamma} (x_{n\beta} x_{j\gamma} + x_{j\beta} x_{n\gamma}) \\
 &= \sum_{n<j} \epsilon_{\alpha\beta\gamma} (x_{n\beta} x_{j\gamma} - x_{j\beta} x_{n\gamma}) = 0
 \end{aligned}$$

Therefore we have

$$\left[ \sum_{nj} \frac{1}{r_{nj}}, \sum_n L_{n\alpha} \right] = 0$$

According to these two claims, we have  $[H, \mathbf{L}] = 0$ .

- (b) Since the laplacian operator is invariant to Euclidean transformations, and  $|\mathbf{x}_i - \mathbf{x}_j|$  also invariant to rotations and translations, we can verify that the following operators are invariant to rotations and translations.

$$\sum_n \mathbf{p}_n^2, \sum_{ij} V(|\mathbf{x}_i - \mathbf{x}_j|)$$

Then  $H$  is translationally invariant as well as rotationally invariant. Hence,  $[H, \mathbf{P}] = 0$  and  $[H, \mathbf{L}] = 0$