

# PHY5410 FA22 HW12

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## Problem 1 (1.10).

(a) Let  $|\phi\rangle = \psi(\mathbf{x}')|0\rangle$ , then we have

$$\begin{aligned} n(\mathbf{x})|\phi\rangle &= \psi^\dagger(\mathbf{x})\psi(\mathbf{x})\psi^\dagger(\mathbf{x}')|0\rangle = \psi^\dagger(\mathbf{x})[\psi^\dagger(\mathbf{x}')\psi(\mathbf{x}) + \delta(\mathbf{x} - \mathbf{x}')]|0\rangle \\ &= \psi^\dagger(\mathbf{x})\psi^\dagger(\mathbf{x}')\psi(\mathbf{x})|0\rangle + \psi^\dagger(\mathbf{x})\delta(\mathbf{x} - \mathbf{x}')|0\rangle \\ &= \psi^\dagger(\mathbf{x})\delta(\mathbf{x} - \mathbf{x}')|0\rangle \end{aligned}$$

(b) Note that

$$\hat{N} = \int d\mathbf{x} n(\mathbf{x}) = \int d\mathbf{x} \sum_{ij} a_i^\dagger a_j \langle i|\mathbf{x}\rangle \langle \mathbf{x}|j\rangle = \sum_{ij} a_i^\dagger a_j \int d\mathbf{x} \langle i|\mathbf{x}\rangle \langle \mathbf{x}|j\rangle = \sum_{ij} a_i^\dagger a_j \delta_{ij} = \sum_i a_i^\dagger a_i$$

Using the properties where  $[a_i, a_j] = 0$ ,  $[a_i, a_j^\dagger] = \delta_{ij}$ , then we have

$$\psi(\mathbf{x})\hat{N} = \sum_{ij} a_i a_j^\dagger a_j \langle \mathbf{x}|j\rangle = \sum_{ij} (a_j^\dagger a_i + \delta_{ij}) a_j \langle \mathbf{x}|j\rangle = \sum_{ij} a_j^\dagger a_i a_j \langle \mathbf{x}|j\rangle + \sum_j a_j \langle \mathbf{x}|j\rangle = \hat{N}\psi(\mathbf{x}) + \psi(\mathbf{x})$$

## Problem 2 (2.1).

(a) Suppose  $\mathbf{q} = 0$ , then

$$S^0(0) = \frac{1}{N} \sum_{\mathbf{k}\sigma\mathbf{k}'\sigma'} \langle \phi_0 | a_{\mathbf{k}\sigma}^\dagger a_{\mathbf{k}\sigma} a_{\mathbf{k}'\sigma'}^\dagger a_{\mathbf{k}'\sigma'} | \phi_0 \rangle = \frac{1}{N} \sum_{\mathbf{k}\sigma\mathbf{k}'\sigma'} \langle \phi_0 | n_{\mathbf{k},\sigma} n_{\mathbf{k}',\sigma'} | \phi_0 \rangle = \frac{1}{N} \sum_{\mathbf{k}\sigma\mathbf{k}'\sigma'} 1 = N$$

(b) Suppose  $\mathbf{q} \neq 0$ , then

$$S^0(\mathbf{q}) = \frac{1}{N} \sum_{\mathbf{k}\sigma\mathbf{k}'\sigma'} \langle \phi_0 | a_{\mathbf{k}\sigma}^\dagger a_{\mathbf{k}+\mathbf{q}\sigma} a_{\mathbf{k}'\sigma'}^\dagger a_{\mathbf{k}'-\mathbf{q}\sigma'} | \phi_0 \rangle$$

Since  $|\phi_0\rangle$  is the ground state, we should have  $|\mathbf{k}' - \mathbf{q}| < k_F$ . Also, since  $\mathbf{k}' \neq \mathbf{k}' - \mathbf{q}$ , we should have  $|\mathbf{k}'| \geq k_F$ . Therefore,  $a_{\mathbf{k}'\sigma'}^\dagger a_{\mathbf{k}'-\mathbf{q}\sigma'} | \phi_0 \rangle$  is an excited state, which pops a ground state  $\mathbf{k}' - \mathbf{q}, \sigma'$  out and add an fermion on the high-energy state  $\mathbf{k}', \sigma'$ . Denote this state as

$$a_{\mathbf{k}'\sigma'}^\dagger a_{\mathbf{k}'-\mathbf{q}\sigma'} | \phi_0 \rangle = | \phi_{\mathbf{k}'-\mathbf{q}\sigma'}^{\mathbf{k}'\sigma'} \rangle$$

Similarly, we have

$$\langle \phi_0 | a_{\mathbf{k}\sigma}^\dagger a_{\mathbf{k}+\mathbf{q}\sigma} = \langle \phi_{\mathbf{k}\sigma}^{\mathbf{k}+\mathbf{q}\sigma} |$$

where  $|\mathbf{k}| < k_F$  and  $|\mathbf{k} + \mathbf{q}| > k_F$ . Then we have

$$S^0(\mathbf{q}) = \frac{1}{N} \sum_{\mathbf{k}\sigma\mathbf{k}'\sigma'} \langle \phi_{\mathbf{k}\sigma}^{\mathbf{k}+\mathbf{q}\sigma} | \phi_{\mathbf{k}'-\mathbf{q}\sigma'}^{\mathbf{k}'\sigma'} \rangle = \frac{1}{N} \sum_{\mathbf{k}\sigma\mathbf{k}'\sigma'} \delta_{\sigma\sigma'} \delta_{\mathbf{k},\mathbf{k}'-\mathbf{q}} \delta_{\mathbf{k}',\mathbf{k}+\mathbf{q}} = \frac{1}{N} \sum_{\mathbf{k}\sigma} 1$$

note that the sum is taken with the restraint  $\mathbf{k} \in \Omega = \{\mathbf{k} | |\mathbf{k}| < k_F, |\mathbf{k} + \mathbf{q}| > k_F\}$ . Using the continuum limit we have

$$S^0(\mathbf{q}) = 2 \frac{V}{N} \int_{\Omega} \frac{d\mathbf{k}}{(2\pi)^3} = \begin{cases} \frac{2+3|\mathbf{q}/2k_F|-|\mathbf{q}/2k_F|^3}{4} & 0 < |\mathbf{q}| \leq 2k_F \\ 1 & |\mathbf{q}| > 2k_F \end{cases}$$

**Problem 3 (2.6).**

(a)

$$\begin{aligned}
e^{-\alpha a^\dagger} a e^{\alpha a^\dagger} &= (1 - \alpha a^\dagger) a (1 + \alpha a^\dagger) = (a - \alpha a^\dagger a) (1 + \alpha a^\dagger) = a - \alpha^2 a^\dagger + \alpha (a a^\dagger - a^\dagger a) \\
e^{-\alpha a} a^\dagger e^{\alpha a} &= (1 - \alpha a) a^\dagger (1 + \alpha a) = (a^\dagger - \alpha a a^\dagger) (1 + \alpha a) = a^\dagger - \alpha^2 a + \alpha (a^\dagger a - a a^\dagger)
\end{aligned}$$

(b)

$$\begin{aligned}
e^{\alpha a^\dagger} a e^{-\alpha a^\dagger} &= (1 + \alpha a^\dagger a + \dots) a \sum_n \frac{(-\alpha)^n}{n!} (a^\dagger a)^n \\
&= (1 + \alpha a^\dagger a + \dots) \sum_n \frac{(-\alpha)^n}{n!} a (a^\dagger a) (a^\dagger a)^{n-1} \\
&= (1 + \alpha a^\dagger a + \dots) \sum_n \frac{(-\alpha)^n}{n!} (1 - a^\dagger a) a (a^\dagger a)^{n-1} \\
&= (1 + \alpha a^\dagger a + \dots) \sum_n \frac{(-\alpha)^n}{n!} a (a^\dagger a)^{n-1} \\
&= (1 + \alpha a^\dagger a + \dots) \sum_n \frac{(-\alpha)^n}{n!} a = e^{-\alpha} a \\
e^{\alpha a^\dagger} a^\dagger e^{-\alpha a^\dagger} &= \sum_n \frac{\alpha^n}{n!} (a^\dagger a)^n a^\dagger (1 + \alpha a^\dagger a + \dots) \\
&= \sum_n \frac{\alpha^n}{n!} (a^\dagger a)^{n-1} (a^\dagger a) a^\dagger (1 + \alpha a^\dagger a + \dots) \\
&= \sum_n \frac{\alpha^n}{n!} (a^\dagger a)^{n-1} a^\dagger (1 + \alpha a^\dagger a + \dots) \\
&= \sum_n \frac{\alpha^n}{n!} a^\dagger (1 + \alpha a^\dagger a + \dots) \\
&= e^{\alpha} a^\dagger
\end{aligned}$$