PHY5410 FA22 HW03

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Problem 1 (5.1).

(a) We can show that $\langle L_x \rangle = 0$ algebraically. Similarly, we can also show that $\langle L_y \rangle = 0$. Then, $L_\pm$ vanish automatically since $L_\pm = L_x \pm iL_y$.

$$
\begin{aligned}
i\hbar \langle Y_{l,l}|L_x Y_{l,l}\rangle &= \langle Y_{l,l}|[L_y, L_z]Y_{l,l}\rangle = \langle Y_{l,l}|(L_yL_z - L_zL_y)Y_{l,l}\rangle \\
&= l\langle Y_{l,l}|L_y Y_{l,l}\rangle - \langle Y_{l,l}|L_zL_y Y_{l,l}\rangle = l\langle Y_{l,l}|L_y Y_{l,l}\rangle - \langle Y_{l,l}|L_zL_y Y_{l,l}\rangle \\
&= l\langle Y_{l,l}|L_y Y_{l,l}\rangle - \langle L_yL_z Y_{l,l}|Y_{l,l}\rangle = l\langle Y_{l,l}|L_y Y_{l,l}\rangle - l\langle Y_{l,l}|L_y Y_{l,l}\rangle \\
&= 0 \Rightarrow \langle L_x\rangle = 0
\end{aligned}
$$

(b) By (a) we know that $\langle L_x \rangle = \langle L_y \rangle = 0$, then $\Delta L_x^2 = \langle L_x^2 \rangle$ and $\Delta L_y^2 = \langle L_y^2 \rangle$. Using the fact that $\langle L_z^2 \rangle = l^2\hbar^2$, we can derive that $\Delta L_x^2 + \Delta L_y^2 = l\hbar^2$.

$$
\begin{aligned}
&\mathbf{L}^2 = L_x^2 + L_y^2 + L_z^2 \\
\Rightarrow &\langle \mathbf{L}^2\rangle = \langle L_x^2\rangle + \langle L_y^2\rangle + \langle L_z^2\rangle = l(l+1)\hbar^2 \\
\Rightarrow &l\hbar^2 = \Delta L_x^2 + \Delta L_y^2
\end{aligned}
$$

Also, according to the uncertainty relationship $\Delta A \Delta B \geq |\langle [A, B]\rangle|/2$, we have

$$
\Delta L_x \Delta L_y \geq \frac{1}{2}\langle i\hbar L_z\rangle = \frac{1}{2}\hbar^2 l
$$

Note that the following set of equations could only have a unique solution $\Delta L_x = \Delta L_y = \hbar\sqrt{l/2}$. Apparently, in the state $Y_{l,l}$, we have $\Delta L_z = 0$.

$$
\begin{cases}
\Delta L_x^2 + \Delta L_y^2 = l\hbar^2 \\
\Delta L_x \Delta L_y \geq \hbar^2 l/2
\end{cases}
$$

(c) From (b) we know that $\Delta L_x^2 + \Delta L_y^2 = \hbar^2 l(l-1) - \hbar^2 m^2$. Hence the expression $\Delta L_x^2 + \Delta L_y^2$ takes its minimum when m^2 takes its maximum, i.e., $m = \pm l$.

Problem 2 (5.6).

(a) If we write $Y_{1,1}$ under the cartesian coordinate, we would get

$$
Y_{1,1} = Y_{1,1}(x, y, z),\ L_z Y_{1,1}(x, y, z) = \hbar^2 Y_{1,1}(x, y, z)
$$

Suppose we perform a rotational transformation $(x, y, z) \to (-z, y, x)$, then $Y_{1,1}(-z, y, x)$ would directly be the eigenfunction of L_z with eigenvalue $\hbar^2 l$. Since $\mathbf{L}^2$ is rotational invariant, we still have $\mathbf{L}^2 Y_{11}(-z, y, x) = \hbar^2 l(l+1)Y_{11}(-z, y, x)$.

$$
\begin{aligned}
L_x Y_{1,1}(-z, y, x) &= (yp_z - zp_y)Y_{1,1}(-z, y, x) = [(-z)p_y - yp_{(-z)}]Y_{1,1}(-z, y, x) \\
&= [x'p_y' - y'p_{x'}]Y_{1,1}(x', y', x) = \hbar^2 l Y_{1,1}(x', y', x) \\
&= \hbar^2 l Y_{1,1}(-z, y, x)
\end{aligned}
$$

(b) Using the algebraic properties of angular momentum to solve this problem.

Remark. For convenience, I would like to drop all $\hbar$ terms in the equation, i.e. I would written $L_z Y_{1,1} = Y_{1,1}$ instead of $L_z Y_{1,1} = \hbar^2 Y_{1,1}$. For notations, let $X_{1,1}$ denote the eigenfunction of L_x with $L_x X_{1,1} = X_{1,1}$ and $\mathbf{L}^2 X_{1,1} = 2X_{1,1}$.

Represent the angular momentum operator under the matrix algebra: let $Y \in \mathbb{R}^n$ and $L_x, L_y, L_z \in \mathbb{R}^{n\times n}$.

$$aY_{1,-1} + bY_{1,0} + cY_{0,1} \mapsto \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

According to the expression of ladder operator $L_+ = L_x + iL_y$ and $L_- = L_x - iL_y$, we have

$$L_+ = L_x + iL_y = \sqrt{2}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$
$$L_- = L_x - iL_y = \sqrt{2}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

where the $\sqrt{2}$ term comes from the normalization rule. Therefore we can solve L_x and L_y accordingly.

$$L_x = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, L_y = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & i & 0 \\ -i & 0 & i \\ 0 & -i & 0 \end{bmatrix}$$

By some calculations, we can show that the characteristic equation of L_x is $\lambda^3 - \lambda = 0$, which means L_x have three distinct eigenvalues -1, 0, and 1. Then, the eigenvector corresponding to $\lambda = 1$ could be explicitly determined by finding $\ker(L_x - I)$. Due to the normalization condition, the vector would be $v_1 = \begin{bmatrix} 1/2 & 1/\sqrt{2} & 1/2 \end{bmatrix}^T$.

$$v_1 \in \ker(L_x - I) = \ker \begin{bmatrix} -1 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1 \end{bmatrix} \text{ and } \|v_1\| = 1 \Rightarrow v_1 = \begin{bmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{bmatrix}$$

Therefore $X_{1,1} = Y_{1,-1}/2 + Y_{1,0}/\sqrt{2} + Y_{1,1}/2$ is a normalized eigenfunction of L_x with $L_x X_{1,1} = \hbar^2 X_{1,1}$.

Problem 3 (6.2)**.**

(a)

(b)