# PHY5410 FA22 HW10

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**Problem 1** (18.2)**.** Let the solution be in the form

$$\psi_l(\mathbf{x}) = R_l(r)Y_{lm}(\theta,\phi)$$

In this case, we know that

$$R_l(r) = \frac{u_l(r)}{r} = \begin{cases} j_l(kr) & r < a \\ Bj_l(kr) + Cn_l(kr) & r \geq a \end{cases}$$

where $k = \sqrt{2mE/\hbar^2}$.

(a) By the continuity wrt $r$ and $p_r = -i\hbar(\partial_r + 1/r)$ of $R(r)$ at $r = a$, we have

$$\begin{aligned} j_l(ka) &= Bj_l(ka) + Cn_l(ka) \\ \lim_{\epsilon\to 0}\int_{a-\epsilon}^{a+\epsilon} \mathrm{d}r\, H\psi(\mathbf{x}) &= \lim_{\epsilon\to 0}\int_{a-\epsilon}^{a+\epsilon} \mathrm{d}r\, E\psi(\mathbf{x}) \\ &\Rightarrow \lambda j_l(ka) = kj_l(ka) - Bkj_l'(ka) - Ckn_l'(ka) \end{aligned}$$

note that $j_l'$ denotes the radial derivative but does not denote the derivative

$$j_l'(\varrho) = \left[\frac{1}{\varrho}\frac{\mathrm{d}}{\mathrm{d}\varrho}\varrho\right] j_l(\varrho)$$

Hence we have the equation

$$\begin{aligned} Bj_l(ka) + Cn_l(ka) &= j_l(ka) \\ Bj_l'(ka) + Cn_l'(ka) &= j_l'(ka) - \lambda j_l(ka) \end{aligned}$$

Hence we can get

$$B =$$

(b) For $l = 0$ we have

$$\sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_l$$

(c) The condition for $\delta_0$ to take its maximum is the same as for $|\sin\delta_l/k|$ to take its maximum.

(d) Given $g = \lambda a$ large enough, the arctan term in the $\delta_l$ will vanish. Hence

$$\delta_l(k) = -ka + \frac{l\pi}{2}$$

Hence

$$\sigma_0(k) = \frac{4\pi}{k^2}\sin^2(ka)$$

Then easy to verify that

$$\sigma_0(k) \leq \lim_{k'\to 0}\sigma_0(k')\sigma_0(k') = 4\pi a^2$$

then $\sigma_0$ approximates its maximum $4\pi a^2$ when $k$ is sufficiently small.

**Problem 2** (18.4). The radial part of the wave function is

$$R_l(a) = \begin{cases} 0 & r < a \\ Aj_l(kr) + Bn_l(kr) & r \geq a \end{cases}$$

To satisfy the continuity

$$\begin{aligned} &R_l(a) = Aj_l(ka) + Bn_l(ka) = Cj_l(ka + \delta_l) = 0 \text{ should not use the asymptotic form ($C$ term)} \\ &\Rightarrow \delta_l = \arctan \frac{j_l(ka)}{n_l(ka)} \approx ka - \frac{l\pi}{2} \end{aligned}$$

For $l = 0$, we have $\delta_0 = ka$. Then

$$\sigma_0 = \frac{4\pi}{k^2} \sin^2 ka$$

$\sigma_0 \approx 4\pi a^2$ when $ka$ small.

**Problem 3** (18.5). Let the potential be

$$V(r) = \begin{cases} V & r < a \\ 0 & r \geq a \end{cases}$$

Hence the solution could be written as

$$\psi(\mathbf{x}) = R_l(r)Y_{lm}(\theta, \phi) \quad R_l = \begin{cases} Aj_l(qr) & r < a \\ Bj_l(kr + \delta_l) & r \geq a \end{cases}$$

where $q = \sqrt{2m(E - V)/\hbar^2}$, $k = \sqrt{2mE/\hbar^2}$. According to the continuity of $R(r)$ wrt $r$ and $p_r$ at $r = a$, we have

$$Aj_l(qa) = Bj_l(ka + \delta_l) \quad Aqj_l'(qa) = Bkj_l'(kr + \delta_l) \Rightarrow \frac{j_l'(ka + \delta_l)}{j_l(ka + \delta_l)} = \frac{q}{k} \frac{j_l'(qa)}{j_l(qa)}$$

Plugin $l = 0$, we have

$$\begin{aligned} \tan(ka + \delta_l) &= \frac{k}{q} \tan qa \\ \Rightarrow \delta_l &= \arctan \left[ \frac{k}{q} \tan qa \right] - ka \end{aligned}$$