PHY5410 FA22 HW03

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Problem 1 (5.1)**.**

(a) We can show that $\langle L_x \rangle = 0$ algebraically. Similarly, we can also show that $\langle L_y \rangle = 0$. Then, $\langle L_\pm \rangle$ vanish automatically since $L_\pm = L_x \pm iL_y$.

$$\begin{aligned}
i\hbar \langle Y_{l,m}|L_x Y_{l,m}\rangle &= \langle Y_{l,m}|[L_y, L_z]Y_{l,m}\rangle = \langle Y_{l,m}|(L_yL_z - L_zL_y)Y_{l,m}\rangle \\
&= m\hbar \langle Y_{l,m}|L_y Y_{l,m}\rangle - \langle Y_{l,m}|L_zL_yY_{l,m}\rangle = m\hbar \langle Y_{l,m}|L_y Y_{l,m}\rangle - \langle Y_{l,m}|L_zL_yY_{l,m}\rangle \\
&= m\hbar \langle Y_{l,m}|L_y Y_{l,m}\rangle - \langle L_yL_zY_{l,m}|Y_{l,m}\rangle = m\hbar \langle Y_{l,m}|L_y Y_{l,m}\rangle - m\hbar \langle Y_{l,m}|L_y Y_{l,m}\rangle \\
&= 0 \Rightarrow \langle L_x \rangle = 0
\end{aligned}$$

(b) By (a) we know that $\langle L_x \rangle = \langle L_y \rangle = 0$, then $\Delta L_x^2 = \langle L_x^2 \rangle$ and $\Delta L_y^2 = \langle L_y^2 \rangle$. Using the fact that $\langle L_z^2 \rangle = l^2\hbar^2$ and $\langle \mathbf{L}^2 \rangle = (l^2+l)\hbar^2$, we can derive that $\Delta L_x^2 + \Delta L_y^2 = l\hbar^2$.

$$\begin{aligned}
\mathbf{L}^2 &= L_x^2 + L_y^2 + L_z^2 \\
\Rightarrow \langle \mathbf{L}^2 \rangle &= \langle L_x^2 \rangle + \langle L_y^2 \rangle + \langle L_z^2 \rangle = l(l+1)\hbar^2 \\
\Rightarrow l\hbar^2 &= \Delta L_x^2 + \Delta L_y^2
\end{aligned}$$

Also, according to the uncertainty relationship $\Delta A \Delta B \geq |\langle [A, B]\rangle|/2$, we have

$$\Delta L_x \Delta L_y \geq \frac{1}{2}|\langle i\hbar L_z \rangle| = \frac{1}{2}\hbar^2 l$$

Note that the following set of equations could only have a unique solution $\Delta L_x = \Delta L_y = \hbar\sqrt{l/2}$. Apparently, in the state $Y_{l,l}$, we have $\Delta L_z = 0$.

$$\begin{cases}
\Delta L_x^2 + \Delta L_y^2 = l\hbar^2 \\
\Delta L_x \Delta L_y \geq \hbar^2 l/2
\end{cases}$$

(c) From (b) we know that $\Delta L_x^2 + \Delta L_y^2 = \hbar^2 l(l+1) - \hbar^2 m^2$. Hence the expression $\Delta L_x^2 + \Delta L_y^2$ takes its minimum when m^2 takes its maximum, i.e., $m = \pm l$.

Problem 2 (5.6)**.**

(a) If we write $Y_{1,1}$ under the cartesian coordinate, we would get

$$Y_{1,1} = Y_{1,1}(x,y,z), \ L_z Y_{1,1}(x,y,z) = \hbar^2 Y_{1,1}(x,y,z)$$

Suppose we perform a rotational transformation $(x,y,z) \to (-z,y,x)$, then $Y_{1,1}(-z,y,x)$ would directly be the eigenfunction of L_x with eigenvalue $\hbar l$. Since $\mathbf{L}^2$ is rotational invariant, we still have $\mathbf{L}^2 Y_{1,1}(-z,y,x) = \hbar^2 l(l+1) Y_{1,1}(-z,y,x)$.

$$\begin{aligned}
L_x Y_{1,1}(-z,y,x) &= (yp_z - zp_y)Y_{1,1}(-z,y,x) = [(-z)p_y - yp_{(-z)}]Y_{1,1}(-z,y,x) \\
&= [x'p'_y - y'p_{x'}]Y_{1,1}(x',y',x) = \hbar l Y_{1,1}(x',y',x) \\
&= \hbar l Y_{1,1}(-z,y,x)
\end{aligned}$$

(b) Using the algebraic property of the angular momentum operator to solve this problem.

Remark. For convenience, I would like to drop all $\hbar$ terms in the equation, i.e., I would written $L_z Y_{1,1} = Y_{1,1}$ instead of $L_z Y_{1,1} = \hbar Y_{1,1}$. For notations, let $X_{1,1}$ denote the eigenfunction of L_x with $L_x X_{1,1} = X_{1,1}$ and $\mathbf{L}^2 X_{1,1} = 2X_{1,1}$.

Represent the angular momentum operator under the matrix algebra: let $Y \in \mathbb{R}^n$ and $L_x, L_y, L_z \in \mathbb{R}^{n\times n}$.

$$aY_{1,-1} + bY_{1,0} + cY_{0,1} \mapsto \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

According to the expression of ladder operator $L_+ = L_x + iL_y$ and $L_- = L_x - iL_y$, we have

$$L_+ = L_x + iL_y = \sqrt{2}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$L_- = L_x - iL_y = \sqrt{2}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

where the $\sqrt{2}$ term comes from the normalization rule. Therefore we can solve L_x and L_y accordingly.

$$L_x = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, L_y = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & i & 0 \\ -i & 0 & i \\ 0 & -i & 0 \end{bmatrix}$$

By some calculations, we can show that the characteristic equation of L_x is $\lambda^3 - \lambda = 0$, which means L_x have three distinct eigenvalues -1, 0, and 1. Then, the eigenvector corresponding to $\lambda = 1$ could be explicitly determined by finding $\ker(L_x - I)$. Due to the normalization condition, the vector can be $v_1 = \begin{bmatrix} 1/2 & 1/\sqrt{2} & 1/2 \end{bmatrix}^T$.

$$v_1 \in \ker(L_x - I) = \ker\begin{bmatrix} -1 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1 \end{bmatrix} \text{ and } \|v_1\| = 1 \Rightarrow v_1 = \begin{bmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{bmatrix} \text{ a solution}$$

Therefore $X_{1,1} = Y_{1,-1}/2 + Y_{1,0}/\sqrt{2} + Y_{1,1}/2$ is a normalized eigenfunction of L_x with $L_x X_{1,1} = X_{1,1}$.

Problem 3 (6.2)**.**

Claim. The operator $\mathscr{L} := e^x(\mathrm{d}/\mathrm{d}x)e^{-x} = \mathrm{d}/\mathrm{d}x - 1$, and $L_r(x) = \mathscr{L}^r x^r$.

Proof. Since

$$\mathscr{L} = e^x \frac{\mathrm{d}}{\mathrm{d}x} e^{-x} = e^x\left(e^{-x}\frac{\mathrm{d}}{\mathrm{d}x} - e^{-x}\right) = \frac{\mathrm{d}}{\mathrm{d}x} - 1$$

also

$$e^x \frac{\mathrm{d}^2}{\mathrm{d}x^2} e^{-x} = e^x \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-x}\frac{\mathrm{d}}{\mathrm{d}x} - e^{-x}\right) = e^x\left(e^{-x}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2e^{-x}\frac{\mathrm{d}}{\mathrm{d}x} + e^{-x}\right) = \left(\frac{\mathrm{d}}{\mathrm{d}x} - 1\right)^2 = \mathscr{L}^2$$

Then we can prove by induction that $\mathscr{L}^n = e^x(\mathrm{d}/\mathrm{d}x)^n e^{-x}$. Therefore $L_r(x) = \mathscr{L}^r x^r$. □

(a) Using the fact that $L_r(x) = \mathscr{L}^r x^r$

$$L_r(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x} - 1\right)^r x^r = \sum_{k=0}^{r} (-1)^k \binom{r}{k} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{r-k} x^r = \sum_{k=0}^{r} (-1)^k \binom{r}{k} \frac{r!}{k!} x^k$$

Therefore

$$\begin{aligned}
L_r^s(x) &= \frac{\mathrm{d}^s}{\mathrm{d}x^s} L_r(x) = \sum_{k=s}^{r} (-1)^k \binom{r}{k} \frac{r!}{k!} \frac{k!}{(k-s)!} x^{k-s} \\
&= \sum_{k=s}^{r} (-1)^k \frac{[r!]^2}{k!(r-k)!(k-s)!} x^{k-s} = \sum_{u=0}^{r-s} (-1)^{u+s} \frac{[r!]^2}{u!(u+s)!(r-u-s)!} x^u
\end{aligned}$$

(b) Since $[\mathrm{d}/\,\mathrm{d}x, 1] = [\mathrm{d}/\,\mathrm{d}x, \mathrm{d}/\,\mathrm{d}x] = 0$, then we have

$$L_{r+m}^m = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m \left(\frac{\mathrm{d}}{\mathrm{d}x} - 1\right)^{r+m} x^{r+m} = \left(\frac{\mathrm{d}}{\mathrm{d}x} - 1\right)^{r+m} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m x^{r+m} = \left(\frac{\mathrm{d}}{\mathrm{d}x} - 1\right)^{r+m} \frac{(r+m)!}{r!} x^r$$

Therefore

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{t^r}{(r+m)!} L_{r+m}^m(x) &= \sum_{r=0}^{\infty} \left(\frac{\mathrm{d}}{\mathrm{d}x} - 1\right)^{r+m} \frac{(r+m)!}{r!} x^r \frac{1}{(r+m)!} t^r \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}x} - 1\right)^m \sum_{r=0}^{\infty} \left(\frac{\mathrm{d}}{\mathrm{d}x} - 1\right)^r t^r x^r \frac{1}{r!} \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}x} - 1\right)^m \sum_{r=0}^{\infty} \frac{t^r}{r!} \left(\frac{\mathrm{d}}{\mathrm{d}x} - 1\right)^r x^r \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}x} - 1\right)^m \sum_{r=0}^{\infty} \frac{t^r}{r!} L_r(x) \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}x} - 1\right)^m \frac{1}{1-t} \exp\left(-x \frac{t}{1-t}\right) \\
&= \frac{(-1)^m}{(1-t)^{m+1}} \exp\left(-x \frac{t}{1-t}\right)
\end{aligned}$$