Answers: Problem set I

1. **(a)** In one dimension, the current operator is specified by $\hat{j} = \frac{1}{2m}(\psi^* \hat{p}\psi + \psi(\hat{p}\psi)^*)$. Applied to the left hand side of the system (outside the region of the potential), where $\psi(x) = Ae^{ikx} + Be^{-ikx}$, we have $j_{\rm left} = \frac{\hbar k}{m}(|A|^2 - |B|^2)$; similarly $j_{\rm right} = \frac{\hbar k}{m}(|C|^2 - |D|^2)$. Current conservation demands that $j_{\rm left} = j_{\rm right}$, i.e. $|A|^2 + |D|^2 = |C|^2 + |B|^2$ – all of the incoming beam is transferred to the outgoing beam, a statement of particle conservation. This conservation of current can be written as $\langle \Psi_{\rm in}|\Psi_{\rm in}\rangle = \langle \Psi_{\rm out}|\Psi_{\rm out}\rangle$. Therefore, using the expression for $|\psi_{\rm out}\rangle$, we have

$$\langle \Psi_{\rm in}|\Psi_{\rm in}\rangle = \langle \Psi_{\rm out}|\Psi_{\rm out}\rangle = \langle \Psi_{\rm in} \underbrace{S^\dagger S}_{\stackrel{!}{=}\mathbb{I}} \Psi_{\rm in}\rangle \,,$$

leading to the unitarity condition.

(b) From the unitarity condition, it follows that

$$S^\dagger S = \mathbb{I} = \begin{pmatrix} |t|^2 + |r|^2 & rt^* + r^* t \\ rt^* + r^* t & |t|^2 + |r|^2 \end{pmatrix}$$

Comparing the matrix elements, we obtained the required result.

(c) For the δ-function scattering problem, the wavefunction is given by

$$\psi(x) = \begin{cases} e^{ikx} + re^{-ikx} & x < 0 \\ te^{ikx} & x > 0 \end{cases}$$

For the δ-function potential, the wavefunction must remain continuous as $x = 0$, and

$$\partial_x \psi|_{+\epsilon} - \partial_x \psi|_{-\epsilon} = -\frac{2maV_0}{\hbar^2}\psi(0)\,.$$

This translates to the conditions $1 + r = t$, and $ik(t - 1 + r) = -\frac{2maV_0}{\hbar^2}t$. As a result, we obtain

$$t = \frac{ik}{ik + \frac{maV_0}{\hbar^2}}\,,$$

as required. Using the result from (b), since the potential is symmetric, we have

$$r^2 = \frac{t}{t^*}(|t|^2 - 1) = -\frac{ik + \gamma}{-ik + \gamma}\left(\frac{k^2}{k^2 + \gamma^2} - 1\right) = \frac{-ik + \gamma}{ik + \gamma}\frac{\gamma^2}{k^2 + \gamma^2} = \frac{\gamma^2}{(ik + \gamma)^2}$$

As a result, we obtain the required expression for r.

2. (a) We are told that $\hat{H}|\psi_1\rangle = E_1|\psi_1\rangle$ and $\hat{H}|\psi_2\rangle = E_2|\psi_2\rangle$, where $E_1 \neq E_2$. Therefore $\langle\psi_1|\hat{H}|\psi_2\rangle = \int \psi_1^* \hat{H}\psi_2\, dx = \int \psi_1^* E_2 \psi_2\, dx = E_2\langle\psi_1|\psi_2\rangle$. Since $\hat{H}$ is Hermitian, we can write

$$\langle\psi_1|\hat{H}|\psi_2\rangle = \int (\hat{H}\psi_1)^*\psi_2\, dx = \int (E_1\psi_1)^*\psi_2\, dx = E_1^*\langle\psi_1|\psi_2\rangle = E_1\langle\psi_1|\psi_2\rangle\,,$$

since E_1 and E_2 are real. Therefore, $(E_1 - E_2)\langle\psi_1|\psi_2\rangle = 0$ and, if $E_1 \neq E_2$ then $\langle\psi_1|\psi_2\rangle = 0$, i.e. $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal.

(b) If $\hat{A}|\psi_1\rangle = |\psi_2\rangle$ and $\hat{A}|\psi_2\rangle = |\psi_1\rangle$, then adding them, $\hat{A}(|\psi_1\rangle + |\psi_2\rangle) = |\psi_1\rangle + |\psi_2\rangle$ and subtracting, $\hat{A}(|\psi_1\rangle - |\psi_2\rangle) = |\psi_2\rangle - |\psi_1\rangle = -(|\psi_1\rangle - |\psi_2\rangle)$. Hence we have an eigenvector of $a = +1$ corresponding to a normalized eigenvector $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$ and an eigenvalue $a = -1$ corresponding to eigenvector $\frac{1}{\sqrt{2}}(|\psi_1\rangle - |\psi_2\rangle)$.

(c) The time-dependent Schrödinger equation is $\hat{H}\psi = E\psi = i\hbar\partial_t\psi$, hence $\psi(t) = \psi(t=0)e^{-iEt/\hbar}$. Since $|\psi_1\rangle$ and $|\psi_2\rangle$ are eigenstates of the Hamiltonian $\hat{H}$ then we can write, $|\psi(t)\rangle = \frac{1}{\sqrt{2}}[|\psi_1\rangle e^{-iE_1t/\hbar} - |\psi_2\rangle e^{-iE_2t/\hbar}]$.

$$P = |\langle\psi(t=0)|\psi(t)\rangle|^2 = \frac{1}{4}\left|[\langle\psi_1| - \langle\psi_2|][|\psi_1\rangle e^{-iE_1t/\hbar} - |\psi_2\rangle e^{-iE_2t/\hbar}]\right|^2$$
$$= \frac{1}{2}\left[1 + \cos((E_1 - E_2)t/\hbar)\right] = \cos^2((E_1 - E_2)t/2\hbar)$$

3. **(a)** Differentiating the left hand side of the given expression with respect to β, one obtains

$$e^{-\beta a^\dagger}\underbrace{[a, a^\dagger]}_{=1}e^{\beta a^\dagger} = 1\,.$$

Integrating, we therefore have that $e^{-\beta a^\dagger}ae^{\beta a^\dagger} = \beta +$"integration constant". By setting $\beta = 0$ we can deduce that the "constant" must be a yielding the required result. Using this result, we have that

$$e^{-\beta a^\dagger}a|\beta\rangle = \underbrace{e^{-\beta a^\dagger}ae^{\beta a^\dagger}}_{=\beta+a}|0\rangle = (\beta + a)|0\rangle = \beta|0\rangle\,.$$

Lastly, to obtain the normalization, we have that

$$\langle\beta|\beta\rangle = N^2\langle 0|e^{\beta^* a}|\beta\rangle = N^2\langle 0|\sum_{n=0}^{\infty}\frac{(\beta^* a)^n}{n!}|\beta\rangle$$
$$= N^2\sum_{n=0}^{\infty}\frac{(\beta^*\beta)^n}{n!}\langle 0|\beta\rangle = N^2e^{|\beta|^2} \stackrel{!}{=} 1\,.$$

i.e. $N = e^{-|\beta|^2/2}$ as required.

(b) For the harmonic oscillator, the creation and annihilation operators are related to the phase space operators by $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)$, and $\hat{p} = -i\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger)$. Therefore, we have

$$\langle\hat{x}\rangle = \sqrt{\frac{\hbar}{2m\omega}}(\beta + \beta^*), \qquad \langle\hat{p}\rangle = -i\sqrt{\frac{\hbar m\omega}{2}}(\beta - \beta^*)\,.$$

Then, using the identity $(\Delta x)^2 = \langle(x - \langle x\rangle)^2\rangle = \langle x^2\rangle - \langle x\rangle^2$, we have

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}\langle\beta|(a^2 + aa^\dagger + a^\dagger a + (a^\dagger)^2|\beta\rangle = \frac{\hbar}{2m\omega}(1 + (\beta + \beta^*)^2),$$
$$\langle\hat{p}^2\rangle = -\frac{\hbar m\omega}{2}\langle\beta|(a^2 - aa^\dagger - a^\dagger a + (a^\dagger)^2|\beta\rangle = -\frac{\hbar}{2m\omega}(-1 + (\beta - \beta^*)^2)\,.$$

As a result, we find that $\Delta x = \frac{\hbar}{2m\omega}$ and $\Delta p = \frac{\hbar m\omega}{2}$ leading to the required expression.

(c) The equation follows simply from the definion of the operator a and the solution may be checked by substitution.

(d) Using the time-evolution of the stationary states, $|n(t)\rangle = e^{-iE_nt/\hbar}|n(0)\rangle$, where $E_n = \hbar\omega(n + 1/2)$, it follows that

$$|\beta(t)\rangle = e^{-i\omega t/2}e^{-|\beta|^2/2}\frac{\beta^n}{\sqrt{n!}}e^{-in\omega t}|n\rangle = e^{-i\omega t/2}|e^{-i\omega t}\beta\rangle\,.$$

Therefore, during the time-evolution, the coherent state form is preserved but the centre of mass and momentum follow that of the classical oscillator,

$$x_0(t) = A\cos(\varphi + \omega t), \qquad p_0(t) = m\omega A\sin(\varphi + \omega t)\,.$$

The width of the wavepacket remains constant.

4. **(a)** Starting with the electron Hamiltonian $\hat{H} = \frac{1}{2m}(\hat{\mathbf{p}} - q\mathbf{A})^2$, substitution of the expression for the vector potential leads to the required result.

   (b) Setting $\omega = m = \hbar = 1$, we immediately obtained the required dimensionless form of the Hamiltonian.

   (c) Straightforward substitution of the differential operators leads to the required identities. As a result, we can confirm that

$$[a, a^\dagger] = \frac{2}{4}\left([\partial_{\bar{z}}, \bar{z}] - [z, \partial_z]\right) = 1\,,$$

   and similarly $[b, b^\dagger] = 1$. In the complex coordinate representation,

$$\hat{H} = \frac{1}{2}\left(-i(\partial_z + \partial_{\bar{z}}) - \frac{i}{4}(z - \bar{z})\right)^2 + \frac{1}{2}\left(-(\partial_{\bar{z}} - \partial_z) + \frac{1}{4}(z + \bar{z})\right)^2$$
$$= -2\partial_{\bar{z}}\partial_z + \frac{1}{8}\bar{z}z + \frac{1}{2}(z\partial_{\bar{z}} - \bar{z}\partial_z) = 2\left(-\partial_z + \frac{\bar{z}}{4}\right)\left(\partial_{\bar{z}} + \frac{z}{4}\right) + \frac{1}{2} = a^\dagger a + \frac{1}{2}$$

   (d) The angular momentum operator is given by ($\hbar = 1$)

$$\hat{L}_z = (\mathbf{x} \times \hat{\mathbf{p}})_z = -i(x\partial_y - y\partial_x) = -\frac{i}{2}\left((z + \bar{z})(\partial_z - \partial_{\bar{z}}) - (z - \bar{z})(\partial_z + \partial_{\bar{z}})\right)$$
$$= -2\left(-\partial_z + \frac{\bar{z}}{4}\right)\left(\partial_{\bar{z}} + \frac{z}{4}\right) + 2\left(-\partial_{\bar{z}} + \frac{z}{4}\right)\left(\partial_z + \frac{\bar{z}}{4}\right) = -(a^\dagger a - b^\dagger b)\,.$$

   (e) In the coordinate representation, the condition $a|0,0\rangle = 0$ translates to the equation

$$\sqrt{2}\left(\partial_{\bar{z}} + \frac{z}{4}\right)\langle \mathbf{r}|0,0\rangle = 0$$

   We thus obtain the Gaussian expression for the wavefunction. Then, using the relation $|0,m\rangle = \frac{(b^\dagger)^m}{\sqrt{m!}}|0,0\rangle$, we have

$$\langle \mathbf{r}|0,m\rangle = \frac{1}{\sqrt{m!}}2^{m/2}\left(\partial_z + \frac{\bar{z}}{4}\right)^m \frac{1}{\sqrt{2\pi}}e^{-\bar{z}z/4} = \frac{1}{\sqrt{2\pi 2^m m!}}z^m e^{-\bar{z}z/4}\,,$$

   as required.

   (f) If we populate the states of the lowest Landau with electrons, starting from states of the lowest angular momentum m, the wavefunction is a Slater determinant involving entries $\phi_m(\mathbf{r}_j) = z_j^m e^{-|z_j|^2/4}$. Taking the determinant, and making use of the Vandemonde determinant identity, one obtains the required many-electron wavefunction.

5. The spin operator in the (θ, ϕ) direction, $\hat{\mathbf{S}}_{\theta\phi}$, can be found by forming the scalar product of the spin operator $\hat{\mathbf{S}}$ with a unit vector in the (θ, ϕ) direction, $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Therefore

$$\hat{\mathbf{S}}_{\theta\phi} = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix}.$$

   We need the eigenvalues of the matrix, i.e.

$$\frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda\begin{pmatrix} u \\ v \end{pmatrix}.$$

   Eliminating u and v, we find $\lambda^2 = 1$ and hence the eigenvalues of $\hat{\mathbf{S}}_{\theta\phi}$ are $\pm\hbar/2$, as expected. Substituting the values $\lambda = \pm 1$ back into the equations relating u and v, we can infer the ratios, $\frac{u}{v} = e^{-i\phi}\cot(\theta/2)$ and $-e^{-i\phi}\tan(\theta/2)$. So, in matrix notation, the eigenstates are

$$\begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix} \text{ and } \begin{pmatrix} \sin(\theta/2) \\ -e^{i\phi}\cos(\theta/2) \end{pmatrix},$$

for eigenvalues $+\hbar/2$ and $-\hbar/2$ respectively. The spin states in the x-direction are obtained by setting $\theta = \pi/2$, $\phi = 0$, and the spin states in the y-direction are obtained by setting $\phi = \pi/4$ in these general formulae.

6. The four states are given by $\phi_1 = \chi_+(1)\chi_+(2)$, $\phi_2 = \chi_-(1)\chi_-(2)$, $\phi_3 = \frac{1}{\sqrt{2}}[\chi_+(1)\chi_-(2)+\chi_-(1)\chi_+(2)]$, and $\phi_4 = \frac{1}{\sqrt{2}}[\chi_+(1)\chi_-(2)-\chi_-(1)\chi_+(2)]$, where ϕ_1, ϕ_2 and ϕ_3 are symmetric under particle interchange and ϕ_4 is antisymmetric.
Since $\hat{S}_+\hat{S}_- = \hat{S}_x^2 + \hat{S}_y^2 + i(\hat{S}_y\hat{S}_x - \hat{S}_x\hat{S}_y) = \hat{\mathbf{S}}^2 - \hat{S}_z^2 + \hbar\hat{S}_z$,

$$\hat{\mathbf{S}}^2 = \hat{S}_+\hat{S}_- + \hat{S}_z^2 - \hbar\hat{S}_z\,. \tag{16.1}$$

Noting that $\hat{S}_z = \hat{S}_z^{(1)} + \hat{S}_z^{(2)}$, and likewise for $\hat{S}_\pm$, and using the basic results for the operation of the spin operators on the single particle states,

$$\hat{S}_z^{(1)}\chi_\pm(1) = \pm\frac{1}{2}\hbar\chi_\pm(1), \qquad \hat{S}_\mp^{(1)}\chi_\pm(1) = \hbar\chi_\mp(1), \qquad \hat{S}_\pm^{(1)}\chi_\pm(1) = 0\,,$$

we can now calculate the effect of applying $\hat{\mathbf{S}}^2$ to each state. For example

$$\hat{\mathbf{S}}^2\phi_1 = \hbar^2[2\chi_+(1)\chi_+(2) + \chi_+(1)\chi_+(2) - \chi_+(1)\chi_+(2)] = 2\hbar^2\phi_1\,,$$

where the three terms in square brackets correspond to the three operators on the right hand side of Eq. (16.1). Likewise ϕ_2. Clearly $\hat{S}_z\phi_3 = 0 = \hat{S}_z\phi_4$, so only the $\hat{S}_+\hat{S}_-$ term need be considered. We find

$$\hat{S}_+\hat{S}_-\chi_+(1)\chi_-(2) = \hat{S}_+\hbar\chi_-(1)\chi_-(2) = \hbar^2(\chi_+(1)\chi_-(2) + \chi_-(1)\chi_+(2))\sqrt{2}\hbar^2\phi_3\,.$$

Similarly $\hat{S}_+\hat{S}_-\chi_-(1)\chi_+(2) = \sqrt{2}\hbar^2\phi_3$ so that

$$\hat{\mathbf{S}}^2\phi_3 = 2\hbar^2\phi_3, \qquad \hat{\mathbf{S}}^2\phi_4 = 0\,.$$

Thus, ϕ_1, ϕ_2, ϕ_3 all have eigenvalue $2\hbar^2$ and hence $S = 1$, while ϕ_4 has $S = 0$.
The state ψ is clearly an eigenstate of $\hat{S}_z$ with eigenvalue 0, and must therefore be a linear combination of ϕ_3 and ϕ_4. The $S = 1$ component is thus the ϕ_3 term in the wavefunction with amplitude

$$\begin{aligned} c_3 &= \langle\phi_3|\psi\rangle = \langle\sqrt{\frac{2}{3}}\chi_+(1)\chi_-(2) + \sqrt{\frac{1}{3}}\chi_-(1)\chi_+(2)|\sqrt{\frac{1}{2}}\chi_+(1)\chi_-(2) + \sqrt{\frac{1}{2}}\chi_-(1)\chi_+(2)\rangle \\ &= \sqrt{1/3} + \sqrt{1/6} \end{aligned}$$

The probability of $S = 1$ is therefore $|c_3|^2 = \frac{3+2\sqrt{2}}{6} = 0.971$.

7. Taking as a basis the states of the $\hat{S}_z$ operator, we can deduce the form of the $\hat{S}_x$ operator most easily from the action of the spin raising and lower operators. Noting that, for spin $S = 1$, $\hat{S}_\pm|S = 1, m\rangle = \hbar[2 - m(m+1)]^{1/2}|1, m \pm 1\rangle$, we can construct the matrix elements of $\hat{S}_\pm$. The latter are given by

$$\hat{S}_+ = \sqrt{2}\hbar\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \hat{S}_- = \hat{S}_+^\dagger\,.$$

Then, using the relation, $\hat{S}_x = \frac{1}{2}(\hat{S}_+ - \hat{S}_-)$, we obtained the matrix elements of the operator and corresponding eigenstates,

$$\hat{S}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \begin{matrix} m_x = 1 \\ \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \end{matrix} \quad \begin{matrix} m_x = 0 \\ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \end{matrix} \quad \begin{matrix} m_x = -1 \\ \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \end{matrix}$$

When placed in a magnetic field, B, the molecules will acquire an energy $\mu B m_z \hbar$, where μ is the magnetic moment of the molecule, which in this case equals twice the magnetic moment of the proton, and $m_z\hbar$ is the eigenvalue of $\hat{S}_z$. At $t = 0$, the molecules enter the magnetic field in the $m_x = 1$ state, after which their wavefunction evolves with time in the usual way, i.e.

$$\psi(t) = e^{-i\mu B\hat{S}_z t/\hbar}\psi(0) = \frac{1}{2}\begin{pmatrix} e^{-i\mu Bt} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e^{i\mu Bt} \end{pmatrix}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{-i\mu Bt} \\ \sqrt{2} \\ e^{i\mu Bt} \end{pmatrix}.$$

Thus, if $\mu Bt = (2n+1)\pi$, with n an integer, the molecules will be in a pure $m_x = -1$ state, and none will pass the second filter. The time is given by $t = L/v = L\sqrt{m/2E}$, where $L = 20$ mm and m and E are the mass and energy of the molecules respectively. We thus have

$$\mu = \frac{(2n+1)\pi\hbar}{BL}\left(\frac{2E}{m}\right)^{1/2} = 2.84\ 10^{-26}\ \mathrm{JT}^{-1}$$

and hence the proton magnetic moment is $1.42\ 10^{-26}\ \mathrm{JT}^{-1}$.

Note that the result can also be obtained by treating the problem as one of classical precession. The couple $= \mu B = L\Omega$, where $L = \hbar$ is the angular momentum and Ω the angular frequency of precession. If $\Omega t = (2n+1)\pi$, the molecules have precessed into the $m_x = -1$ state, and the result readily follows.

8. Substituting for the definition of the spin raising and lowering operators using the Holstein-Primakoff transformation, the commutator is obtained as

$$\frac{1}{2S\hbar^2}[\hat{S}^+, \hat{S}^-] = \left(1 - \frac{a^\dagger a}{2S}\right)^{1/2} \overbrace{aa^\dagger}^{a^\dagger a + 1} \left(1 - \frac{a^\dagger a}{2S}\right)^{1/2} - a^\dagger\left(1 - \frac{a^\dagger a}{2S}\right)a$$
$$= \left(1 - \frac{a^\dagger a}{2S}\right) + a^\dagger a\left(1 - \frac{a^\dagger a}{2S}\right) - a^\dagger a + \frac{a^\dagger a^\dagger a a}{2S} = 1 - \frac{a^\dagger a}{S}.$$

With $\hat{S}^z = \hbar(S - a^\dagger a)$, we obtain the required commutation relation $[\hat{S}^+, \hat{S}^-] = 2\hbar\hat{S}^z$.

9. For the case $\ell_1 = 1$, $\ell_2 = 2$, a table of possible ways of forming each value of $M = m_1 + m_2$ is shown right. The largest value of M is 3, so the largest value of L must be 3. There must also be a state with $M = 2$ corresponding to $L = 3$, but we have two states with $M = 2$. Therefore there must be a state with $L = 2$ as well. We need two states with $M = 1$, one for each of the $L = 3, 2$ multiplets, but we actually have three states with $M = 1$, so there must be an $L = 1$ state as well. All of the M states are now accounted for.

M	(m_1, m_2)		
3	$(1,2)$		
2	$(1,1)$	$(0,2)$	
1	$(1,0)$	$(0,1)$	$(-1,2)$
0	$(1,-1)$	$(0,0)$	$(-1,1)$
-1	$(1,-2)$	$(0,-1)$	$(-1,0)$
-2	$(0,-2)$	$(-1,1)$	
-3	$(-1,-2)$		

For the case $\ell_1 = 3$, $\ell_2 = 1$, we can again for a table (see right). Following the same logic as before, we see that states with $L = 4, 3, 2$ just account for all the states.

M	(m_1, m_2)		
4	$(3,1)$		
3	$(3,0)$	$(2,1)$	
2	$(3,-1)$	$(2,0)$	$(1,1)$
1	$(2,-1)$	$(1,0)$	$(0,1)$
0	$(1,-1)$	$(0,0)$	$(-1,1)$
-1	$(0,-1)$	$(-1,0)$	$(-2,1)$
-2	$(-1,-1)$	$(-2,0)$	$(-3,1)$
-3	$(-2,-1)$	$(-3,0)$	
-4	$(-3,-1)$		

To construct the states explicitly, we start by writing the $L = 3$ $M = 3$ state, since there is only one way of forming $M = 3$, viz. $|3,3\rangle = |1,1\rangle \otimes |2,2\rangle$. We then operate with the lowering operator $\hat{L}_-$, which is simply the sum of the lowering operators for the two separate particles. Recalling that:

$$\hat{L}_-|\ell, m\rangle = \sqrt{\ell(\ell+1) - m(m-1)}\hbar|\ell, m-1\rangle\,,$$

we obtain $\sqrt{6}\hbar|3,2\rangle = \sqrt{2}\hbar|1,0\rangle \otimes |2,2\rangle + \sqrt{4}\hbar|1,1 \otimes |2,1\rangle$, where the first term on the right hand side comes from lowering the $\ell = 1$ state and the second from lowering the $\ell = 2$ state. Hence $|3,2\rangle = \sqrt{1/3}\hbar|1,0\rangle \otimes |2,2\rangle + \sqrt{2/3}\hbar|1,1\rangle \otimes |2,1\rangle$. The state $|2,2\rangle$ must be the orthogonal linear combination, i.e. $|2,2\rangle = \sqrt{2/3}\hbar|1,0\rangle \otimes |2,2\rangle - \sqrt{1/3}\hbar|1,1\rangle \otimes |2,1\rangle$. Further states could be computed in the same way if required.

10. The commutation relations are given by $[J_i, J_j] = i\hbar\epsilon_{ijk}J_k$.

(a) First define the eigenstates ψ_m: $J_z\psi_m = m\hbar\psi_m$. To see if $J_\pm\psi_m$ is an eigenstate of J_z, we need to look at $J_zJ_\pm\psi_m$, which is equal to $J_\pm J_z\psi_m - [J_\pm, J_z]\psi_m$. The required commutator is $[J_\pm, J_z] = [J_x, J_z] \pm i[J_y, J_z]$, from the definition of $J_\pm$. From the basic commutators given at the start, this is $[J_\pm, J_z] = \hbar) - iJ_y \pm -J_x) = - \pm \hbar J_\pm$ (if treating $\pm$ like a number is confusing, do this separately for J_+ and J_-). Going back to $J_zJ_\pm\psi_m$, we can now write this as $J_\pm J_z\psi_m + \pm\hbar J_\pm\psi_m$. The first term is just $J_\pm m\hbar\psi_m$, so this is $(m \pm 1)\hbar(J_\pm\psi_m)$. Thus, $J_\pm\psi_m$ is an eigenstate of J_z, with eigenvalue $(m \pm 1)\hbar$. This establishes the raising and lowering property of $J_\pm$.

(b) Two electrons would have a total spin of $S = 1$ or 0. Adding a third spin 1/2 particle creates total spin $S = 3/2$ or 1/2 from the $S = 1$ two-particle state. The $S = 0$ two-particle state becomes $S = 1/2$ only on adding the third particle, so total $S = 3/2$ or 1/2 are the only possibilities.

(c) The states with well-defined values of m_1, m_2, and m_3 for the z spin components of all particles are the 'uncoupled basis'. Where all particles are 'spin up', this state may be written as $|\uparrow\uparrow\uparrow\rangle$. This state is also the $m_S = 3/2$ state of total $S = 3/2$ (there is no other way to get $m_1 + m_2 + m_3 = 3/2$ in the uncoupled basis). We can therefore write $|S = 3/2, m_S = 3/2\rangle = |\uparrow\uparrow\uparrow\rangle$. To get from here to $|S = 3/2, m_S = 1/2\rangle$, we need to apply $J_- = S_-^{(1)} + S_-^{(2)} + S_-^{(3)}$. In other words, the total lowering operator is the sum of the lowering operator for each separate spin (reasonably enough) this follows from the definition of J_- and $J_x = S_x^{(1)} + S_x^{(2)} + S_x^{(3)}$, etc. Now, we need to use the given normalization result. This says that

$$\begin{aligned} J_-|S = 3/2, m_S = 3/2\rangle &= \sqrt{15/4 - 3/4}\hbar|S = 3/2, m_S = 1/2\rangle \\ &= \sqrt{3}\hbar|S = 3/2, m_S = 1/2\rangle\,. \end{aligned}$$

Notice that the total quantum number, J, is the same as the overall spin quantum number, S in this case. Therefore $|S = 3/2, m_S = 1/2\rangle = (1/\sqrt{3})J_-|S = 3/2, m_S = 3/2\rangle$. Using the given normalization result again for a single state, $S_-|1/2, 1/2\rangle = \sqrt{3/4 + 1/4}\hbar|1/2, -1/2\rangle$. This establishes the required result.