PHY 5410: Homework Week 14

5.1 Show that the matrices

$$\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

obey the algebraic relations

$$\alpha^i \alpha^j + \alpha^j \alpha^i = 2\delta^{ij}\,\mathbb{1}\,, \qquad \alpha^i \beta + \beta \alpha^i = 0\,, \qquad \alpha^{i\,2} = \beta^2 = \mathbb{1}\,.$$

5.3 Particles in a homogeneous magnetic field.
Determine the energy levels that result from the Dirac equation for a (relativistic) particle of mass m and charge e in a homogeneous magnetic field $\mathbf{B}$. Use the gauge $A^0 = A^1 = A^3 = 0,\ A^2 = Bx$.

6.2 Show, by using the transformation properties of x_μ, that $\partial^\mu \equiv \partial/\partial x_\mu$ ($\partial_\mu \equiv \partial/\partial x^\mu$) transforms as a contravariant (covariant) vector.

6.4 Derive the quadratic form of the Dirac equation

$$\left[\left(\mathrm{i}\hbar\partial - \frac{e}{c}A\right)^2 - \frac{\mathrm{i}\hbar e}{c}\left(\boldsymbol{\alpha}\boldsymbol{E} + \mathrm{i}\boldsymbol{\Sigma}\boldsymbol{B}\right) - m^2c^2\right]\psi = 0$$

for the case of external electromagnetic fields. Write the result using the electromagnetic field tensor $F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$, and also in a form explicitly dependent on $\boldsymbol{E}$ and $\boldsymbol{B}$.

Hint: Multiply the Dirac equation from the left by $\gamma^\nu\left(\mathrm{i}\hbar\partial_\nu - \frac{e}{c}A_\nu\right) + mc$ and, by using the commutation relations for the γ matrices, bring the expression obtained into quadratic form in terms of the field tensor

$$\left[\left(\mathrm{i}\hbar\partial - \frac{e}{c}A\right)^2 - \frac{\hbar e}{2c}\sigma^{\mu\nu}F_{\mu\nu} - m^2c^2\right]\psi = 0\,.$$

The assertion follows by evaluating the expression $\sigma^{\mu\nu}F_{\mu\nu}$ using the explicit form of the field tensor as a function of the fields $\boldsymbol{E}$ and $\boldsymbol{B}$.