# PHY5410 FA22 HW08

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**Problem 1** (11.6).

(a) Let $\mathbf{B} = \nabla \times \mathbf{A}$ where

$$\mathbf{B} = \nabla \times \begin{bmatrix} 0 \\ Bx \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ B \end{bmatrix}$$

Then we have

$$H = \frac{1}{2m}p^2 + \frac{e^2B^2}{2mc}x^2 \Rightarrow \frac{1}{2mE}p^2 + \frac{e^2B^2}{2mc^2E}x^2 = 1$$

Consider the classical case: pick a particular $p$ s.t. the equation intersects the $p_x$-$x$ plane and obtains an ellipse. Let $p_y = p_z = 0$ to simplify calculation. Hence the area enclosed by $p$ and $x$ equals to

$$\pi ab = \frac{2mcE}{eB}\pi$$

Using the Bohr-Sommerfeld quantization condition, we have

$$\oint \mathrm{d}\mathbf{x}\ \mathbf{p} = \left(n + \frac{1}{2}\right)h$$
$$\oint \mathrm{d}\mathbf{x}\ \mathbf{p} = \oint \mathrm{d}x\ p_x = \pi ab$$
$$\Rightarrow E = \left(n + \frac{1}{2}\right)\hbar\frac{eB}{mc} = \left(n + \frac{1}{2}\right)\hbar\omega$$

Check $H = H(p, x) = H(L, \theta)$, $H = \cdots + \mathbf{L} \cdot \mathbf{B}$?

(b) Solve the limiting case on $xy$ plane by setting $H = 0$.

$$H = \frac{1}{2m}\left(p - \frac{e}{c}\mathbf{A}\right)^2 = 0$$

Hence we have $p_x = p_z = 0$, $p_y = eBx/c$. Thus

$$\oint \mathrm{d}\mathbf{x}\ \mathbf{p} = \oint \mathrm{d}y\ \frac{eB}{c}x = \frac{e}{c}\Psi = \left(n + \frac{1}{2}\right)\hbar \Rightarrow \Phi = \left(n + \frac{1}{2}\right)\frac{hc}{e}$$

**Problem 2** (12.2). Define the Hamiltonian of the system as the following expression

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2r^2$$

And pick eigenfunctions $\phi_{nlm} = R_{nl}(r)Y_{lm_l}$, where $m_l$ denotes the orbital quantum number and $m_s$ denotes the spin quantum number. Define $N = l + 2n$, then for each $N$, we have

$$N = 2k \Rightarrow l = 0, 2, \ldots, N; \quad N = 2k + 1 \Rightarrow l = 1, 3, \ldots, N$$

For each $l$, we have $m_l = -l, -l + 1, \ldots, l$, total $2l + 1$ degeneracy. Hence, for both even and odd $N$, we have overall degeneracy equals to

$$\sum_{l=0,2,\ldots,N}(2l + 1) = \frac{1}{2}(N + 1)(N + 2)$$

$$\sum_{l=1,3,\ldots,N}(2l+1)=\frac{1}{2}(N+1)(N+2)$$

It the particle has spin $s$, the degeneracy becomes to $(N+1)(N+2)(2s+1)/2$.

In this case, we have $H_2$ equals to

$$H_2=\frac{\omega^2}{2mc^2}\mathbf{S}\cdot\mathbf{L}$$

Let $\mathbf{J}=\mathbf{L}+\mathbf{S}$, and pick eigenvector $|j,j_z,l\rangle$, then

$$H_2\,|j,j_z,l\rangle=\frac{\hbar^2}{2}[j(j+1)-s(s+1)-l(l+1)]\,|j,j_z,l\rangle$$

Hence, the spectrum will have the form of

$$E_{N,j,l}=\left(N+\frac{3}{2}\right)\hbar\omega+\frac{\hbar^2\omega^2}{2mc^2}[j(j+1)-s(s+1)-l(l+1)]$$

Consider the case of spin-1/2 particles, if $l>0$

$$j(j+1)-s(s+1)-l(l+1)=\begin{cases}l & j=l+1/2\\ -l-1 & j=l-1/2\end{cases}$$

For a fixed $N$, each $l$ will be split into 2 lines with $j=l+1/2$ and $j=l-1/2$.

(a) $l=0$, $E_{N,j,0}$ has degeneracy equals to 2 corresponds to $j_z=\pm 1/2$ (note that $j=1/2$ when $l=0$).

(b) $l>0$, $E_{N,j,l}$ has degeneracy $2j+1$ corresponds to different $j_z$.

Therefore we have energy described by $N,j,l$ and the degeneracy equals to $2j+1$.

**Problem 3** (16.2). Let

$$H_0=\frac{1}{2m}p^2+\frac{1}{2}m\omega^2x,\quad H_1=-eEx$$

Let $|n\rangle$ denote the original solutions and $|n'\rangle$ denote the perturbed solutions. Then we can verify that

$$\langle x-\frac{eE}{m\omega^2}|n\rangle=\langle x|n'\rangle$$

According to the sudden approximation, the transition probability of $\psi\to m'$ would be $|\langle m'|\psi\rangle|^2$. Note that

$$\langle n'|0\rangle=\frac{1}{\sqrt{n!}}\langle (a'_+)^n 0'|0\rangle$$

and

$$a'_+=\frac{1}{\sqrt{2m\omega\hbar}}\left[m\omega\left(x-\frac{eE}{m\omega^2}\right)-ip\right]=a_+-\frac{1}{\sqrt{2m\omega\hbar}}\frac{eE}{\omega}=a_+-b$$

Hence

$$\langle n'|0\rangle=\frac{1}{\sqrt{n!}}\langle (a'_+)^n 0'|0\rangle=\frac{1}{\sqrt{n!}}\langle 0'|(a'_-)^n 0\rangle=\frac{1}{\sqrt{n!}}\langle 0'|(-b)^n 0\rangle\frac{(-b)^n}{\sqrt{n!}}\langle 0'|0\rangle$$

$$
\begin{aligned}
&= \frac{(-b)^n}{\sqrt{n!}} A^2 \int_{-\infty}^{\infty} \mathrm{d}x \ \exp\left(-\frac{m\omega}{2\hbar}x^2\right) \exp\left[-\frac{m\omega}{2\hbar}(x - eE/m\omega^2)^2\right] \\
&= \frac{(-b)^n}{\sqrt{n!}} A^2 \int_{-\infty}^{\infty} \mathrm{d}x \ \exp\left[-\left(x - \frac{eE}{2m\omega^2}\right)\right] \exp\left(-\frac{e^2E^2}{2m\omega^3\hbar}\right) \\
&= \frac{(-b)^n}{\sqrt{n!}} \exp\left(-\frac{e^2E^2}{2m\omega^3\hbar}\right)
\end{aligned}
$$

Then $P_{0\to n'}$ would be

$$
P_{0\to n'} = |\langle n'|0\rangle|^2 = \frac{b^{2n}}{n!} \exp\frac{2m\omega^3\hbar}{e^2E^2} = \frac{1}{n!}\left(\frac{e^2E^2}{2m\omega^3\hbar}\right)^n \exp\left(-\frac{e^2E^2}{2m\omega^3\hbar}\right)
$$

Easy to verify that $\sum_{n'} P_{0\to n'} = 1$.