PHY5410 FA22 HW10

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Problem 1 (18.2). Let the solution be in the form

$$\psi_l(\mathbf{x}) = R_l(r) Y_{lm}(\theta, \phi)$$

In this case, we know that

$$R_l(r) = \frac{u_l(r)}{r} = \begin{cases} A j_l(kr) & r < a \\ B j_l(kr + \delta_l) & r \geq a \end{cases}$$ should not use the asymptotic form

where $k = \sqrt{2mE/\hbar^2}$.

(a) By the continuity wrt r and $p_r = -i\hbar(\partial_r + 1/r)$ of $R(r)$ at $r = a$, we have

$$\begin{aligned} A j_l(ka) &= B j_l(ka + \delta_l) \\ \lim_{\epsilon \to 0} \int_{a-\epsilon}^{a+\epsilon} \mathrm{d}r\ H\psi(\mathbf{x}) &= \lim_{\epsilon \to 0} \int_{a-\epsilon}^{a+\epsilon} \mathrm{d}r\ E\psi(\mathbf{x}) \\ \Rightarrow \lambda A j_l(ka) &= Akj_l'(ka) - Bkj_l'(ka + \delta_l) \end{aligned}$$

note that j_l' does not denote the derivative but denotes the radial derivative

$$j_l'(\varrho) = \left[\frac{1}{\varrho}\frac{\mathrm{d}}{\mathrm{d}\varrho}\varrho\right] j_l(\varrho)$$

Substituting $A/B = j_l(ka + \delta_l)/j_l(ka)$, we get an equation

$$\begin{aligned} \lambda j_k(ka + \delta_l) &= kj_l(ka + \delta_l)\frac{j_l'(ka)}{j_l(ka)} - kj_l'(ka + \delta_l) \\ \Rightarrow \frac{j_l'(ka + \delta_l)}{j_l(ka + \delta_l)} &= \frac{j_l'(ka)}{j_k(ka)} - \frac{\lambda}{k} \end{aligned}$$

Assume the asymptotic solution applies here, where

$$j_l(\varrho) = \frac{1}{\varrho}\sin\left(\varrho - \frac{l\pi}{2}\right) \quad j_l'(\varrho) = \frac{1}{\varrho}\cos\left(\varrho - \frac{l\pi}{2}\right)$$

Hence

$$\begin{aligned} \cot(ka + \delta_l - l\pi/2) &= \cot(ka - l\pi/2) - \frac{\lambda}{k} \\ \Rightarrow \delta_l &= \arctan\left[\frac{k}{k\cot(ka - l\pi/2) - \lambda}\right] - ka + \frac{l\pi}{2} \end{aligned}$$

if we define $\xi = ka$, $g = \lambda a$, then

$$\delta_l = \arctan\left[\frac{\xi}{\xi\cot(\xi - l\pi/2) - g}\right] - \xi + \frac{l\pi}{2}$$

(b) For $l = 0$ we have

$$\sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_l$$

(c) The condition for δ_0 to take its maximum is the same as for $|\sin\delta_l/k|$ to take its maximum.

(d) Given $g = \lambda a$ large enough, the arctan term in the δ_l will vanish. Hence

$$\delta_l(k) = -ka + \frac{l\pi}{2}$$

Hence

$$\sigma_0(k) = \frac{4\pi}{k^2}\sin^2(ka)$$

Then easy to verify that

$$\sigma_0(k) \le \lim_{k'\to 0}\sigma_0(k')\sigma_0(k') = 4\pi a^2$$

then σ_0 approximates its maximum $4\pi a^2$ when k is sufficiently small.

Problem 2 (18.4)**.** The radial part of the wave function is

$$R_l(a) = \begin{cases} 0 & r < a \\ Aj_l(kr) + Bn_l(kr) & r \ge a \end{cases}$$

To satisfy the continuity

$$R_l(a) = Aj_l(ka) + Bn_l(ka) = Cj_l(ka + \delta_l) = 0$$ should not use the asymptotic form (C term)

$$\Rightarrow \delta_l = \arctan\frac{j_l(ka)}{n_l(ka)} \approx ka - \frac{l\pi}{2}$$

For $l = 0$, we have $\delta_0 = ka$. Then

$$\sigma_0 = \frac{4\pi}{k^2}\sin^2 ka$$

$\sigma_0 \approx 4\pi a^2$ when ka small.

Problem 3 (18.5)**.** Let the potential be

$$V(r) = \begin{cases} V & r < a \\ 0 & r \ge a \end{cases}$$

Hence the solution could be written as

$$\psi(\mathbf{x}) = R_l(r)Y_{lm}(\theta,\phi) \quad R_l = \begin{cases} Aj_l(qr) & r < a \\ Bj_l(kr + \delta_l) & r \ge a \end{cases}$$

where $q = \sqrt{2m(E-V)/\hbar^2}$, $k = \sqrt{2mE/\hbar^2}$. According to the continuity of $R(r)$ wrt r and p_r at $r = a$, we have

$$Aj_l(qa) = Bj_l(ka + \delta_l) \quad Aqj_l'(qa) = Bkj_l'(kr + \delta_l) \Rightarrow \frac{j_l'(ka+\delta_l)}{j_l(ka+\delta_l)} = \frac{q}{k}\frac{j_l'(qa)}{j_l(qa)}$$

Plugin $l = 0$, we have

$$\tan(ka + \delta_l) = \frac{k}{q}\tan qa$$

$$\Rightarrow \delta_l = \arctan\left[\frac{k}{q}\tan qa\right] - ka$$