PHY5410 FA22 HW13

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Problem 1 (5.1).

(a) Since $\sigma^i\sigma^j = \epsilon_{ijk}\sigma^k$ $(i \neq j)$ and $(\sigma^i)^2 = \mathbb{1}$, then

$$\alpha^i\alpha^j + \alpha^j\alpha^i = \begin{bmatrix} \sigma^i\sigma^j + \sigma^j\sigma^i & 0 \\ 0 & \sigma^i\sigma^j + \sigma^j\sigma^i \end{bmatrix} = 2\delta_{ij}\mathbb{1}$$

(b)

$$\alpha^i\beta + \beta\alpha^i = \begin{bmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{bmatrix} + \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix} = 0$$

(c)

$$(\alpha^i)^2 = \begin{bmatrix} \sigma^i\sigma^i & 0 \\ 0 & \sigma^i\sigma^i \end{bmatrix} = \mathbb{1} \quad \beta^2 = \begin{bmatrix} \mathbb{1}^2 & 0 \\ 0 & (-\mathbb{1})^2 \end{bmatrix} = \mathbb{1}$$

Problem 2 (5.3). Since

$$\begin{aligned} i\hbar\partial_t\psi &= \left[c\alpha^k\left(-i\hbar\partial_k - \frac{e}{c}A_k\right) + \beta mc^2\right]\psi \\ \Rightarrow E\psi &= \left[c\alpha^k\left(-i\hbar\partial_k - \frac{e}{c}A_k\right) + \beta mc^2\right]\psi \\ E^2\psi &= \left[c\alpha^k\left(-i\hbar\partial_k - \frac{e}{c}A_k\right) + \beta mc^2\right]^2\psi \end{aligned}$$

the equation is summed over $k = 1, 2, 3$. Hence

$$\begin{aligned} E^2\psi &= \left[c\alpha^i\left(-i\hbar\partial_i - \frac{e}{c}A_i\right)c\alpha^j\left(-i\hbar\partial_j - \frac{e}{c}A_j\right) + (\alpha^k\beta + \beta\alpha^k)\cdots + \beta^2m^2c^4\right]\psi \\ &= \left[c^2\left(-i\hbar\partial_i - \frac{e}{c}A_i\right)^2 + m^2c^4\right]\psi \end{aligned}$$

Give that $A_1 = A_3 = 0$, $A_2 = Bx$, using the conclusion from HW04, where energy levels of such systems looks like

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega_B \quad \omega_B = \frac{eB}{mc}$$

Hence in this system

$$\begin{aligned} E_n^2 &= \left[(2n+1)\hbar\omega_B mc^2 + m^2c^4\right] \\ \Rightarrow E_n &= \sqrt{(2n+1)\hbar\omega_B mc^2 + m^2c^4} \end{aligned}$$

where $\omega_B = eB/mc$ (suppose $E \geq 0$).

Problem 3 (6.2). Note that

$$x_\mu = g_{\mu\nu}x^\nu \quad x^\mu = g^{\mu\nu}x_\nu$$

Then

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \frac{\partial x^\nu}{\partial x_\mu}\frac{\partial}{\partial x^\nu} = g^{\nu\mu}\partial_\nu$$

where the sum is taken on ν. Also since

$$x'^\mu = \Lambda^\mu_\nu x^\nu$$

we have

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \frac{\partial x'^\nu}{\partial x^\mu}\frac{\partial}{\partial x'^\nu} = \Lambda^\nu_\mu \partial'_\nu$$

Similarly

$$\partial_\mu = g_{\nu\mu}\partial^\nu \quad \partial$$

Problem 4 (6.4). The Dirac equation reads

$$[-\gamma^\mu(\partial_\mu - \frac{e}{c}A_\mu) + mc]\psi = 0$$

Hence

$$\begin{aligned}
[-\gamma^\nu(\partial_\nu - \frac{e}{c}A_\nu) - mc][-\gamma^\mu(\partial_\mu - \frac{e}{c}A_\mu) + mc]\psi &= 0 \\
[\gamma^\nu\gamma^\mu(\partial_\nu - \frac{e}{c}A_\nu)(\partial_\mu - \frac{e}{c}A_\mu) - m^2c^2]\psi &= 0
\end{aligned}$$

Since $\gamma^\nu\gamma^\mu = g^{\nu\mu}\mathbb{1} + i\sigma^{\mu\nu}$, where $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$. Then

$$\begin{aligned}
[\gamma^\nu\gamma^\mu(\partial_\nu - \frac{e}{c}A_\nu)(\partial_\mu - \frac{e}{c}A_\mu) - m^2c^2]\psi &= 0 \\
[(g^{\nu\mu}\mathbb{1} + i\sigma^{\mu\nu})(\partial_\nu - \frac{e}{c}A_\nu)(\partial_\mu - \frac{e}{c}A_\mu) - m^2c^2]\psi &= 0 \\
[(\partial - \frac{e}{c}A)^2 + i\sigma^{\mu\nu}(\partial_\nu - \frac{e}{c}A_\nu)(\partial_\mu - \frac{e}{c}A_\mu) - m^2c^2]\psi &= 0 \\
[(\partial - \frac{e}{c}A)^2 - i\sigma^{\mu\nu}(\partial_\mu - \frac{e}{c}A_\mu)(\partial_\nu - \frac{e}{c}A_\nu) - m^2c^2]\psi &= 0
\end{aligned}$$

Note that

$$\begin{aligned}
i\sigma^{\mu\nu}(\partial_\mu - \frac{e}{c}A_\mu)(\partial_\nu - \frac{e}{c}A_\nu) &= \frac{1}{2}[i\sigma^{\mu\nu}(\partial_\mu - \frac{e}{c}A_\mu)(\partial_\nu - \frac{e}{c}A_\nu) + i\sigma^{\nu\mu}(\partial_\nu - \frac{e}{c}A_\nu)(\partial_\mu - \frac{e}{c}A_\mu)] \\
&= \frac{1}{2}[i\sigma^{\mu\nu}(\partial_\mu - \frac{e}{c}A_\mu)(\partial_\nu - \frac{e}{c}A_\nu) - i\sigma^{\mu\nu}(\partial_\nu - \frac{e}{c}A_\nu)(\partial_\mu - \frac{e}{c}A_\mu)] \\
&= \frac{i}{2}\sigma^{\mu\nu}[(\partial_\mu - \frac{e}{c}A_\mu), (\partial_\nu - \frac{e}{c}A_\nu)] \\
&= \frac{i}{2}\sigma^{\mu\nu}F_{\mu\nu}
\end{aligned}$$