PHY5410 FA22 HW08

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Problem 1 (16.5). The first-order transition probability of a ground state to the nth excited state is written in the form

$$P_{0n}(t) = |\langle n,t|\psi,t\rangle|^2 = \left|\frac{1}{\hbar}\int_{t_0}^{t} \mathrm{d}t'\ e^{i\omega_{n0}t'}\langle n|V(t')|0\rangle\right|^2$$

where $\omega_{n0} = (E_n - E_0)/\hbar$. Since $t_0 < t_1$ and $t > t_2$, we have

$$\int_{t_0}^{t} \mathrm{d}t'\ e^{i\omega_{n0}t'}\langle n|V(t')|0\rangle = \int_{t_1}^{t_2} \mathrm{d}t'\ e^{i\omega_{n0}t'}\langle n|V(t')|0\rangle = \langle n|x|0\rangle\int_{t_1}^{t_2} \mathrm{d}t'\ e^{i\omega_{n0}t'}(-D\cos\Omega t)$$

Easy to prove that

$$\langle n|x|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}\delta_{n1}$$

and

$$\begin{aligned}\int_{t_1}^{t_2} \mathrm{d}t'\ e^{i\omega_{n0}t'}\cos\Omega t' = &\frac{1}{\omega_{n0}^2 - \Omega^2}(\omega_{n0}\cos\omega_{n0}t\cos\Omega - \Omega\cos\omega_{n0}t\sin\Omega t)\Big|_{t_1}^{t_2}\\ &+ i\frac{1}{\Omega^2 - \omega_{n0}^2}(\omega_{n0}\cos\omega_{n0}t\cos\Omega t + \Omega\sin\omega_{n0}t\sin\Omega t)\Big|_{t_1}^{t_2}\end{aligned}$$

Hence we can calculate the transition probability.

$$P_{0n}(t) = \frac{D^2}{2m\omega\hbar}\left|\int_{t_1}^{t_2} \mathrm{d}t'\ e^{i\omega_{n0}t'}\cos\Omega t'\right|^2\delta_{n1}$$

Note that $P_{0n}(t) = 0\ \forall n \geq 2$.

Problem 2 (16.7). The wave functions for the old ground state and the new ground state are

$$\psi(x) = \frac{\sqrt{m\lambda}}{\hbar}\exp(-m\lambda|x|/\hbar^2) \quad \psi'(x) = \frac{\sqrt{m\mu}}{\hbar}\exp(-m\mu|x|/\hbar^2)$$

Hence the transition probability is

$$\langle\psi'|\psi\rangle = 2\int_0^\infty \frac{m}{\hbar^2}\sqrt{\lambda\mu}\exp[-m(\lambda+\mu)|x|/\hbar^2] = \frac{2\sqrt{\lambda\mu}}{\lambda+\mu}$$

Problem 3 (16.10). Since

$$[\dot{x}, x] = \left[\frac{1}{i\hbar}(xH - Hx), x\right] = \frac{1}{i\hbar}(2xHx - Hx^2 - x^2H)$$

Notice that

$$\langle a|xHx|a\rangle = \sum_n \langle a|x|n\rangle\langle n|Hx|a\rangle = \sum_n E_n|\langle n|x|a\rangle|^2$$

$$\langle a|Hx^2|a\rangle = \sum_n \langle a|Hx|n\rangle \langle n|x|a\rangle = \sum_n E_a|\langle n|x|a\rangle|^2$$
$$\langle a|x^2H|a\rangle = \sum_n \langle a|x|n\rangle \langle n|xH|a\rangle = \sum_n E_a|\langle n|x|a\rangle|^2$$

Therefore

$$\langle a|[\dot{x},x]|a\rangle = \frac{2}{i\hbar}\sum_n (E_n - E_a)|\langle n|x|a\rangle|^2$$

For $H = p^2/2m + m\omega^2 x/2m$, we have

$$\dot{x} = \frac{\partial}{\partial p}H = \frac{p}{m} \Rightarrow [\dot{x},x] = -\frac{i\hbar}{m}$$

Therefore

$$\sum_n (E_n - E_a)|\langle n|x|a\rangle|^2 = \frac{\hbar^2}{2m}$$