PHY5410 FA22 HW12

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Problem 1 (1.10).

(a) Let $|\phi\rangle = \psi(\mathbf{x}')|0\rangle$, then we have

$$\begin{aligned}
n(\mathbf{x})|\phi\rangle &= \psi^\dagger(\mathbf{x})\psi(\mathbf{x})\psi^\dagger(\mathbf{x}')|0\rangle = \psi^\dagger(\mathbf{x})[\psi^\dagger(\mathbf{x}')\psi(\mathbf{x}) + \delta(\mathbf{x}-\mathbf{x}')]|0\rangle \\
&= \psi^\dagger(\mathbf{x})\psi^\dagger(\mathbf{x}')\psi(\mathbf{x})|0\rangle + \psi^\dagger(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}')|0\rangle \\
&= \psi^\dagger(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}')|0\rangle \\
&= \psi^\dagger(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')|0\rangle
\end{aligned}$$

according to the property where $\delta(x-y)f(x) = \delta(x-y)f(y)$.

(b) Note that

$$\hat{N} = \int \mathrm{d}\mathbf{x}\, n(\mathbf{x}) = \int \mathrm{d}\mathbf{x}\, \sum_{ij} a_i^\dagger a_j \langle i|\mathbf{x}\rangle\langle\mathbf{x}|j\rangle = \sum_{ij} a_i^\dagger a_j \int \mathrm{d}\mathbf{x}\, \langle i|\mathbf{x}\rangle\langle\mathbf{x}|j\rangle = \sum_{ij} a_i^\dagger a_j \delta_{ij} = \sum_i a_i^\dagger a_j$$

Using the properties where $[a_i, a_j] = 0$, $[a_i, a_j^\dagger] = \delta_{ij}$, then we have

$$\psi(\mathbf{x})\hat{N} = \sum_{ij} a_i a_j^\dagger a_j \langle\mathbf{x}|j\rangle = \sum_{ij}(a_j^\dagger a_i + \delta_{ij})a_j\langle\mathbf{x}|j\rangle = \sum_{ij} a_j^\dagger a_j a_i \langle\mathbf{x}|j\rangle + \sum_j a_j\langle\mathbf{x}|j\rangle = \hat{N}\psi(\mathbf{x}) + \psi(\mathbf{x})$$

Problem 2 (2.1).

(a) Suppose $\mathbf{q} = 0$, then

$$S^0(0) = \frac{1}{N}\sum_{\mathbf{k}\sigma\mathbf{k}'\sigma'} \langle\phi_0|a_{\mathbf{k}\sigma}^\dagger a_{\mathbf{k}\sigma} a_{\mathbf{k}'\sigma'}^\dagger a_{\mathbf{k}'\sigma'}|\phi_0\rangle = \frac{1}{N}\sum_{\mathbf{k}\sigma\mathbf{k}'\sigma'} \langle\phi_0|n_{\mathbf{k},\sigma} n_{\mathbf{k}',\sigma'}|\phi_0\rangle = \frac{1}{N}\sum_{\mathbf{k}\sigma\mathbf{k}'\sigma'} = N$$

(b) Suppose $\mathbf{q} \neq 0$, then

$$S^0(\mathbf{q}) = \frac{1}{N}\sum_{\mathbf{k}\sigma\mathbf{k}'\sigma'} \langle\phi_0|a_{\mathbf{k}\sigma}^\dagger a_{\mathbf{k}+\mathbf{q}\sigma} a_{\mathbf{k}'\sigma'}^\dagger a_{\mathbf{k}'-\mathbf{q}\sigma'}|\phi_0\rangle$$

Since $|\phi_0\rangle$ is the ground state, we should have $|\mathbf{k}'-\mathbf{q}| < k_F$. Also, since $\mathbf{k}' \neq \mathbf{k}'-\mathbf{q}$, we should have $|\mathbf{k}'| \geq k_F$. Therefore, $a_{\mathbf{k}'\sigma'}^\dagger a_{\mathbf{k}'-\mathbf{q}\sigma'}|\phi_0\rangle$ is an excited state, which pops a ground state $\mathbf{k}'-\mathbf{q}, \sigma'$ out and add an fermion on the high-energy state $\mathbf{k}', \sigma'$. Denote this state as

$$a_{\mathbf{k}'\sigma'}^\dagger a_{\mathbf{k}'-\mathbf{q}\sigma'}|\phi_0\rangle = |\phi_{\mathbf{k}'-\mathbf{q}\sigma'}^{\mathbf{k}'\sigma'}\rangle$$

Similarly, we have

$$\langle\phi_0|a_{\mathbf{k}\sigma}^\dagger a_{\mathbf{k}+\mathbf{q}\sigma} = \langle\phi_{\mathbf{k}\sigma}^{\mathbf{k}+\mathbf{q}\sigma}|$$

where $|\mathbf{k}| < k_F$ and $|\mathbf{k}+\mathbf{q}| > k_F$. Then we have

$$S^0(\mathbf{q}) = \frac{1}{N}\sum_{\mathbf{k}\sigma\mathbf{k}'\sigma'} \langle\phi_{\mathbf{k}\sigma}^{\mathbf{k}+\mathbf{q}\sigma}|\phi_{\mathbf{k}'-\mathbf{q}\sigma'}^{\mathbf{k}'\sigma'}\rangle = \frac{1}{N}\sum_{\mathbf{k}\sigma\mathbf{k}'\sigma'} \delta_{\sigma\sigma'}\delta_{\mathbf{k},\mathbf{k}'-\mathbf{q}}\delta_{\mathbf{k}',\mathbf{k}+\mathbf{q}} = \frac{1}{N}\sum_{\mathbf{k}\sigma} 1$$

note that the sum is taken with the restraint $\mathbf{k} \in \Omega = \{\mathbf{k}||\mathbf{k}| < k_F, |\mathbf{k}+\mathbf{q}| > k_F\}$. Using the continuum limit we have

$$S^0(\mathbf{q}) = 2\frac{V}{N}\int_\Omega \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3} = \begin{cases} \frac{3}{4}|\mathbf{q}/k_F| - |\mathbf{q}/k_F|^3/16 & 0 < |\mathbf{q}| \leq 2k_F \\ 1 & |\mathbf{q}| > 2k_F \end{cases}$$

Problem 3 (2.6).

(a)

$$e^{-\alpha a^\dagger} a e^{\alpha a^\dagger} = (1-\alpha a^\dagger)a(1+\alpha a^\dagger) = (a-\alpha a^\dagger a)(1+\alpha a^\dagger) = a - \alpha^2 a^\dagger + \alpha(aa^\dagger - a^\dagger a)$$
$$e^{-\alpha a} a^\dagger e^{\alpha a} = (1-\alpha a)a^\dagger(1+\alpha a) = (a^\dagger - \alpha a a^\dagger)(1+\alpha a) = a^\dagger - \alpha^2 a + \alpha(a^\dagger a - aa^\dagger)$$

(b)

$$\begin{aligned}
e^{\alpha a^\dagger a} a e^{-\alpha a^\dagger a} &= (1+\alpha a^\dagger a+\cdots)a\sum_n \frac{(-\alpha)^n}{n!}(a^\dagger a)^n \\
&= (1+\alpha a^\dagger a+\cdots)\sum_n \frac{(-\alpha)^n}{n!}a(a^\dagger a)(a^\dagger a)^{n-1} \\
&= (1+\alpha a^\dagger a+\cdots)\sum_n \frac{(-\alpha)^n}{n!}(1-a^\dagger a)a(a^\dagger a)^{n-1} \\
&= (1+\alpha a^\dagger a+\cdots)\sum_n \frac{(-\alpha)^n}{n!}a(a^\dagger a)^{n-1} \\
&= (1+\alpha a^\dagger a+\cdots)\sum_n \frac{(-\alpha)^n}{n!}a = e^{-\alpha}a \\
e^{\alpha a^\dagger} a^\dagger e^{-\alpha a^\dagger a} &= \sum_n \frac{\alpha^n}{n!}(a^\dagger a)^n a^\dagger(1+\alpha a^\dagger a+\cdots) \\
&= \sum_n \frac{\alpha^n}{n!}(a^\dagger a)^{n-1}(a^\dagger a)a^\dagger(1+\alpha a^\dagger a+\cdots) \\
&= \sum_n \frac{\alpha^n}{n!}(a^\dagger a)^{n-1}a^\dagger(1+\alpha a^\dagger a+\cdots) \\
&= \sum_n \frac{\alpha^n}{n!}a^\dagger(1+\alpha a^\dagger a+\cdots) \\
&= e^\alpha a^\dagger
\end{aligned}$$