PHY 5410: Homework Week 12

1.1 Show that the fully symmetrized (antisymmetrized) basis functions

$$S_{\pm}\varphi_{i_1}(x_1)\varphi_{i_2}(x_2)\ldots\varphi_{i_N}(x_N)$$

are complete in the space of the symmetric (antisymmetric) wave functions $\psi_{s/a}(x_1, x_2, \ldots, x_N)$.

Hint: Assume that the product states $\varphi_{i_1}(x_1)\ldots\varphi_{i_N}(x_N)$, composed of the single-particle wave functions $\varphi_i(x)$, form a complete basis set, and express $\psi_{s/a}$ in this basis. Show that the expansion coefficients $c^{s/a}_{i_1,\ldots,i_N}$ possess the symmetry property $\frac{1}{\sqrt{N!}}S_{\pm}c^{s/a}_{i_1,\ldots,i_N} = c^{s/a}_{i_1,\ldots,i_N}$. The above assertion then follows directly by utilizing the identity $\frac{1}{\sqrt{N!}}S_{\pm}\psi_{s/a} = \psi_{s/a}$ demonstrated in the main text.

1.3 For a simple harmonic oscillator, $[a, a^\dagger] = 1$, (or for the equivalent Bose operator) prove the following relations:

$$\begin{aligned}
[a, \mathrm{e}^{\alpha a^\dagger}] &= \alpha \mathrm{e}^{\alpha a^\dagger}, & \mathrm{e}^{-\alpha a^\dagger}\, a\, \mathrm{e}^{\alpha a^\dagger} &= a + \alpha\,, \\
\mathrm{e}^{-\alpha a^\dagger}\, \mathrm{e}^{\beta a}\, \mathrm{e}^{\alpha a^\dagger} &= \mathrm{e}^{\beta\alpha}\mathrm{e}^{\beta a}, & \mathrm{e}^{\alpha a^\dagger a}\, a\, \mathrm{e}^{-\alpha a^\dagger a} &= \mathrm{e}^{-\alpha}\, a\,,
\end{aligned}$$

where α and β are complex numbers.

1.4 For independent harmonic oscillators (or noninteracting bosons) described by the Hamiltonian

$$H = \sum_i \epsilon_i a_i^\dagger a_i$$

determine the equation of motion for the creation and annihilation operators in the Heisenberg representation,

$$a_i(t) = \mathrm{e}^{\mathrm{i}Ht/\hbar} a_i \mathrm{e}^{-\mathrm{i}Ht/\hbar}\,.$$

Give the solution of the equation of motion by (i) solving the corresponding initial value problem and (ii) by explicitly carrying out the commutator operations in the expression $a_i(t) = \mathrm{e}^{\mathrm{i}Ht/\hbar} a_i \mathrm{e}^{-\mathrm{i}Ht/\hbar}$.