PHY5410 Homework #1

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2.1

(a) $\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \left(\int_{-\infty}^{\infty} e^{-\alpha x^2} dx \cdot \int_{-\infty}^{\infty} e^{-\alpha y^2} dy\right)^{\frac{1}{2}}$

$= \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\alpha(x^2+y^2)} dx\,dy\right)^{\frac{1}{2}}$

$= \left(\int_0^{2\pi}\int_0^{+\infty} e^{-\alpha r^2} r\,dr\,d\theta\right)^{\frac{1}{2}}$

$= \left(2\pi \cdot \frac{1}{2\alpha}\right)^{\frac{1}{2}} \overset{Re(\alpha)>0}{=} \sqrt{\frac{\pi}{\alpha}}$

(b) Since $\alpha^2 k_i^2 - i x_i k_i = (\alpha k_i - \frac{i x_i}{2\alpha})^2 + \frac{x_i^2}{4\alpha^2}$

$\Rightarrow \int_{-\infty}^{+\infty} d^3k\, e^{i\vec{k}\cdot\vec{x}} e^{-k^2\alpha^2}$

$= \int_{-\infty}^{\infty} dk_1 dk_2 dk_3 \prod_{j=1}^{3} e^{ik_jx_j - \alpha^2 k_j^2}$

$= \int_{-\infty}^{\infty} dk_1 dk_2 dk_3 \prod_{j=1}^{3} e^{-(\alpha k_j - \frac{ix_j}{2\alpha})^2} \cdot e^{-\frac{x_j^2}{4\alpha^2}}$

$= \prod_{j=1}^{3} \int_{-\infty}^{\infty} dk_j\, e^{(\alpha k_j - \frac{ix_j}{2\alpha})^2} \cdot e^{-\frac{x_j^2}{4\alpha^2}}$

$= \prod_{j=1}^{3} e^{-\frac{x_j^2}{4\alpha}} \sqrt{\frac{\pi}{\alpha^2}} = \frac{\pi^{\frac{3}{2}}}{\alpha^3} e^{-\frac{|x|^2}{4\alpha}}$ $(Re(\alpha) \geq 0)$

$(|x|^2 = x_i^2 + x_j^2 + x_k^2)$

2.4

(a) $\hat{p} = -i\hbar\nabla$, WTS: $\forall \phi, \psi$, we have

$\langle \phi | \hat{p}\psi \rangle = \langle \hat{p}\phi | \psi \rangle$

Proof.

$\int_\Omega d^3\vec{x}\, \phi^*(-i\hbar\nabla\psi) \quad \left(= \langle \phi | \hat{p}\psi \rangle\right)$

$= \underbrace{\int_\Omega d^3\vec{x}\, -i\hbar\nabla(\phi^*\psi)} - \int_\Omega d^3\vec{x}\, \psi(-i\hbar\nabla\phi^*)$

$= 0 + \int_\Omega d^3\vec{x}\, (i\hbar\nabla\phi^*)\psi$

↑ apply boundary condition

$= \int_\Omega d^3\vec{x}\, (-i\hbar\nabla\phi)^*\psi$

$= \langle \hat{p}\phi | \psi \rangle \Rightarrow p^\dagger = p$ □

Since $p = p^\dagger$, $\forall \phi, \psi$

$\langle \phi | p^2\psi \rangle = \langle p\phi | p\psi \rangle = \langle p^2\phi | \psi \rangle$

$\Rightarrow (p^2)^\dagger = p^2$

(b) $\langle \phi | AB\psi \rangle = \langle A^\dagger\phi | B\psi \rangle = \langle B^\dagger A^\dagger\phi | \psi \rangle$

$\Rightarrow (AB)^\dagger = B^\dagger A^\dagger$

(c) $[AB, C] = ABC - CAB$

$= ABC + ACB - ACB - CAB$

$= A[B, C] + [A, C]B$

2.3 (a)

Since $[[A,B],B] = [A,B]B - B[A,B] = 0$

$\Rightarrow B[A,B] = [A,B]B$. $B, [A,B]$ commute

Therefore by induction

$B^k[A,B] = [A,B]B^k$ ✓

Then

$[A,B^n] = [A, BB^{n-1}] = ABB^{n-1} - BAB^{n-1} + BAB^{n-1} - BB^{n-1}A$

$= B[A,B^{n-1}] + [A,B]B^{n-1}$

$= B[A,B^{n-1}] + B^{n-1}[A,B]$

$= B[A,BB^{n-2}] + B^{n-1}[A,B]$

$= B^2[A,B^{n-2}] + B[A,B]B^{n-2} + B^{n-1}[A,B]$

$= B^2[A,B^{n-2}] + 2B^{n-1}[A,B]$ $[A,B], B$ commute

induction

$= \cdots$

$= B^{n-1}[A,B] + (n-1)B^{n-1}[A,B]$

$= nB^{n-1}[A,B]$

2.7

For time-independent Sch-eq, we have time factor $\varphi(t) = e^{-iEt/\hbar}$

Therefore

$\Psi_a(x,t) = \psi_a(x)e^{-iE_at/\hbar}$

$\Psi_b(x,t) = \psi_b(x)e^{iE_bt/\hbar}$

$\Psi = \frac{1}{\sqrt{2}}(\Psi_a + \Psi_b)$

Check if Ψ is normalized:

$\int dx\, \Psi^* \Psi$

orthogonal

$= \int dx\, \frac{1}{2}\left(|\psi_a|^2(x) + |\psi_b|^2(x) + \psi_a^*(x)\psi_b(x)e^{i(E_a-E_b)t/\hbar} + \psi_b^*(x)\psi_a(x)e^{i(E_b-E_a)t/\hbar}\right)$

$= \frac{1}{2}\cdot 2 = 1$

Then the probability density at (x,t) is

$|\psi_a|^2(x) + |\psi_b(x)|^2 + \psi_a^*(x)\psi_b(x)e^{i(E_a-E_b)t/\hbar} + \psi_b^*(x)\psi_a(x)e^{i(E_b-E_a)t/\hbar}$

If $E_a = E_b$, $|\Psi(x,t)|^2$ constant at x, independent to t

Else, $|\Psi(x,t)|^2$ change periodically at x

$$T = \frac{|E_b-E_a|}{2\pi\hbar} = \frac{|E_b-E_a|}{h}$$

T is the period -1

Nevertheless,

$$\int dx\, |\Psi(x,t)|^2 = 1$$

the normalization condition is independent from time t.

2.8.

Since $\forall \psi$ we have

$$p_i^2 f(x)\psi(x) = p_i\left(\psi(x) p_i f(x) + f(x) p_i \psi(x)\right)$$

$$= (p_i\psi)(p_i f) + \psi(p_i^2 f) + (p_i f)(p_i\psi) + f(p_i^2\psi)$$

$$f(x) p_i^2 \psi(x) = f p_i^2 \psi$$

$$\Rightarrow [p_i^2, f(x)]\psi(x)$$

$$= (p_i^2 f(x))\cdot\psi(x) + 2(p_i f(x)) p_i \psi(x)$$

$$\Rightarrow [p_i^2, f(x)] = (p_i^2 f(x)) + 2(p_i f(x)) p_i$$

Therefore

$$[p_i^2, f(x)] = p_i^2 f(x) + 2(p_i f(x)) p_i$$

$$= -\hbar^2 \frac{\partial^2}{\partial x_i^2} f(x) - 2\hbar^2\left(\frac{\partial}{\partial x_i} f(x)\right)\frac{\partial}{\partial x_i}$$

Note that

$$L_i = \varepsilon_{ijk} x_j p_k + \varepsilon_{ikj} x_k p_j$$

$$= \varepsilon_{ijk}(x_j p_k - x_k p_j)$$

Is there a summation or no summation over j, k? Shall be made clear.

$$L_j = \varepsilon_{jki}(x_k p_i - x_i p_k)$$

Since $\varepsilon_{ijk} = \varepsilon_{jki}$, we have

$$L_i L_j = (x_j p_k - x_k p_j)(x_k p_i - x_i p_k)$$

$$= \underline{-i\hbar x_j p_i} + x_j x_k p_k p_i - \cancel{x_i} x_j p_k p_k - x_k \cancel{x_k} p_j p_i + x_i x_k p_j p_k$$

$$L_j L_i = (x_k p_i - x_i p_k)(x_j p_k - x_k p_j)$$

$$= x_j \cancel{x_k} p_i p_k - x_k x_k p_i p_j - x_i x_j p_k p_k + \underline{(-i\hbar) x_i p_j} + x_i x_k p_j p_k$$

$$\Rightarrow L_i L_j - L_j L_i = i\hbar x_i p_j - i\hbar x_j p_i$$

$$= i\hbar(x_i p_j - x_j p_i)$$

$$= \frac{i\hbar}{\varepsilon_{kij}} L_k$$

$$= \varepsilon_{kij}\, i\hbar L_k$$