PHY5410 FA22 HW11

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Problem 1 (1.1). Denote

$$|i_1, \dots, i_N\rangle = \varphi_{i_1}(x_1) \cdots \varphi_{i_N}(x_N)$$

Assume this basis is complete

$$\sum_{i_1,\dots,i_N} |i_1, \dots, i_N\rangle \langle i_1, \dots, i_N| = 1$$

Then $\forall \psi_{s/a}$, we have

$$\sum_{i_1,\dots,i_N} |i_1, \dots, i_N\rangle \langle i_1, \dots, i_N|\psi_{s/a}\rangle = \psi_{s/a}$$

$$\sum_{i_1,\dots,i_N} \frac{1}{\sqrt{N!}} S_\pm |i_1, \dots, i_N\rangle \langle i_1, \dots, i_N|\psi_{s/a}\rangle = \frac{1}{\sqrt{N!}} S_\pm \psi_{s/a}$$

$$\sum_{i_1,\dots,i_N} \frac{1}{\sqrt{N!}} S_\pm |i_1, \dots, i_N\rangle \langle i_1, \dots, i_N|\frac{1}{N!} S_\pm^\dagger S_\pm \psi_{s/a}\rangle = \psi_{s/a}$$

$$\sum_{i_1,\dots,i_N} \frac{1}{N!} S_\pm |i_1, \dots, i_N\rangle \langle i_1, \dots, i_N|S_\pm^\dagger \psi_{s/a}\rangle = \psi_{s/a}$$

which shows that $S_\pm |i_1, \dots, i_N\rangle$ complete $\forall \psi_{s/a}$.

Problem 2 (1.3).

(a) Since $[a, (a^\dagger)^m] = m(a^\dagger)^{m-1}$

$$ae^{\alpha a^\dagger} = a \sum_n \frac{1}{n!} (\alpha a^\dagger)^n = \frac{\alpha^n}{n!} a (a^\dagger)^n = \sum_n \frac{\alpha^n}{n!} \left[n (a^\dagger)^{n-1} + (a^\dagger)^n a \right] = \alpha e^{\alpha a^\dagger} + e^{\alpha a^\dagger} a \Rightarrow [a, e^{\alpha a^\dagger}] = \alpha e^{\alpha a^\dagger}$$

(b) Note that

$$e^{-\alpha a^\dagger} a e^{\alpha a^\dagger} = e^{-\alpha a^\dagger} [e^{\alpha a^\dagger} a + \alpha e^{\alpha a^\dagger}] = a + \alpha$$

(c) Note that

$$e^{-\alpha a^\dagger} \beta a e^{\alpha a^\dagger} = \beta a + \beta \alpha \Rightarrow e^{-\alpha a^\dagger} (\beta a)^n e^{\alpha a^\dagger} = (\beta a + \beta \alpha)^n$$

Hence

$$e^{-\alpha a^\dagger} e^{\beta \alpha} e^{\alpha a^\dagger} = \sum_n \frac{1}{n!} (\beta a + \beta \alpha)^n = e^{\beta a + \beta \alpha} = e^{\beta a} e^{\beta \alpha}$$

(d) Since

$$e^{\alpha a^\dagger a} a = \sum_n \frac{1}{n!} \alpha^n (a^\dagger a)^n a = \sum_n \frac{1}{n!} \alpha^n (a^\dagger a)^{n-1} a (a^\dagger a - 1) = \sum_n \frac{1}{n!} \alpha^n a (a^\dagger a - 1)^n = a e^{\alpha (a^\dagger a - 1)}$$

Hence

$$e^{\alpha a^\dagger a} a e^{-\alpha a^\dagger a} = a e^{\alpha a^\dagger a} e^{-\alpha} a e^{-\alpha a^\dagger a} = a e^{-\alpha}$$

Problem 3 (1.4). There are two methods to solve the problem

(i) Using the differential relation

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}a_i(t) &= \frac{iHt}{\hbar}e^{iHt/\hbar}a_ie^{-iHt/\hbar} - e^{iHt/\hbar}a_i\frac{iHt}{\hbar}e^{-iHt/\hbar} \\
&= \frac{iHt}{\hbar}e^{iHt/\hbar}a_ie^{-iHt/\hbar} - \frac{iHt}{\hbar}e^{iHt/\hbar}a_ie^{-iHt/\hbar} - \frac{iHt}{\hbar}e^{iHt/\hbar}\frac{i\epsilon_it}{\hbar}a_ie^{-iHt/\hbar} \\
&= -\frac{i\epsilon_it}{\hbar}a_i \\
\Rightarrow a_i(t) &= a_i(0)e^{-i\epsilon_it/\hbar} = a_ie^{-i\epsilon_it/\hbar}
\end{aligned}$$

(ii) Using the Bose commutation relation $[a_i, a_j^\dagger] = \delta_{ij}$ and the equation from Problem 2, we have

$$a_i(t) = e^{i\epsilon_ita_i^\dagger a_i/\hbar}a_ie^{-i\epsilon_ita_i^\dagger a_i/\hbar} = a_ie^{-i\epsilon_it/\hbar}$$