PHY5410 FA22 HW06

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Problem 1 (9.5).

(a) Since

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}S_x(t) &= \frac{i}{\hbar}[H, S_x]_H = \frac{i}{\hbar}\frac{eB}{mc}[S_z, S_x]_H = -\frac{eB}{mc}S_y(t) \\
\frac{\mathrm{d}}{\mathrm{d}t}S_y(t) &= \frac{i}{\hbar}[H, S_y]_H = \frac{i}{\hbar}\frac{eB}{mc}[S_z, S_y]_H = \frac{eB}{mc}S_x(t) \\
\frac{\mathrm{d}}{\mathrm{d}t}S_z(t) &= \frac{i}{\hbar}[H, S_z]_H = \frac{i}{\hbar}\frac{eB}{mc}[S_z, S_z]_H = 0
\end{aligned}$$

Thus we have a set of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} S_x(t) \\ S_y(t) \end{bmatrix} = \begin{bmatrix} 0 & -eB/mc \\ eB/mc & 0 \end{bmatrix}\begin{bmatrix} S_x(t) \\ S_y(t) \end{bmatrix}$$

The solution is

$$\begin{bmatrix} S_x(t) \\ S_y(t) \end{bmatrix} = \exp\left(\begin{bmatrix} 0 & -\omega t \\ \omega t & 0 \end{bmatrix}\right)\begin{bmatrix} S_x(0) \\ S_y(0) \end{bmatrix} = \begin{bmatrix} \cos\omega t\ S_x(0) - \sin\omega t\ S_y(0) \\ \sin\omega t\ S_x(0) + \cos\omega t\ S_y(0) \end{bmatrix}$$

where $\omega = eB/mc$.

(b) Note that S_z could be written in $\mathbb{R}^{2\times 2}$ under the basis representation

$$S_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Then

$$\Psi(t) = \exp(-i\omega S_z t/\hbar)\Psi(0) = \exp\left(-i\frac{\omega}{\hbar}\frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}t\right)\Psi(0) = \begin{bmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ae^{-i\omega t/2} \\ be^{i\omega t/2} \end{bmatrix}$$

(c) It should be clear that the probability of getting $|\uparrow\rangle$ would be $|a|^2$ and $|\downarrow\rangle$ should be $|b|^2$. If the spin is oriented in the x direction when $t = 0$, then $[a \quad b]^T$ should be an eigenvector of S_x. Let $a = b$ and $|a| = |b| = 1/\sqrt{2}$, then we have the probability of getting $|\uparrow\rangle$ equals to

$$P(|\uparrow\rangle) = |\langle\uparrow|\psi\rangle|^2 = |ae^{-i\omega t/2}|^2 = |a|^2 = \frac{1}{2}$$

(d) Pick eigenfunction of S_x where $S_x|\uparrow_x\rangle = \hbar/2\,|\uparrow_x\rangle$. Let

$$|\uparrow_x\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle$$

Then the probability of getting $|\uparrow_x\rangle$ is

$$\begin{aligned}
P(|\uparrow_x\rangle) &= |\langle\uparrow_x|\psi\rangle^2| \\
&= \left[\frac{1}{\sqrt{2}}(ae^{-i\omega t/2} + be^{i\omega t/2})\right]^2 \\
&= \frac{1}{2}(|a|^2 + |b|^2 + a^*be^{i\omega t} + ab^*e^{-i\omega t}) \\
&= \frac{1}{2} + \frac{1}{2}\cos\omega t = \cos^2(\omega t/2)
\end{aligned}$$

using the fact that $a = b$.

(e) Using the equation

$$i\hbar\frac{\partial}{\partial t}\Psi = \frac{e_0}{mc}\frac{\hbar}{2}BS_z\Psi \Rightarrow i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}a(t)\\b(t)\end{bmatrix} = \frac{e_0}{mc}\frac{\hbar}{2}B\sigma_z\begin{bmatrix}a(t)\\b(t)\end{bmatrix}$$

then it can be solved refer to (b).

Problem 2 (10.2)**.** Note that

$$\mathbf{S}_1\mathbf{S}_2 = S_{1z}S_{2z} + \frac{1}{2}S_{1+}S_{2-} + \frac{1}{2}S_{1-}S_{2+}$$

Let $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ be a set of orthonormal basis. Then we can derive that

$$\begin{aligned}
H|\uparrow\uparrow\rangle &= \left(-\frac{a+b}{2}B\hbar + \frac{1}{4}J\hbar^2\right)|\uparrow\uparrow\rangle \\
H|\downarrow\downarrow\rangle &= \left(\frac{a+b}{2}B\hbar + \frac{1}{4}J\hbar^2\right)|\downarrow\downarrow\rangle \\
H|\uparrow\downarrow\rangle &= \left(-\frac{a-b}{2}B\hbar - \frac{1}{4}J\hbar^2\right)|\uparrow\downarrow\rangle + \frac{1}{2}J\hbar^2|\downarrow\uparrow\rangle \\
H|\downarrow\uparrow\rangle &= \left(\frac{a-b}{2}B\hbar - \frac{1}{4}J\hbar^2\right)|\downarrow\uparrow\rangle + \frac{1}{2}J\hbar^2|\uparrow\downarrow\rangle
\end{aligned}$$

To diagonalize the last two terms, we can consider the eigenvalues and eigenvectors of the following matrix

$$\begin{bmatrix} -\left(\frac{a-b}{2}B\hbar + \frac{1}{4}J\hbar^2\right) & \frac{1}{2}J\hbar^2 \\ \frac{1}{2}J\hbar^2 & \left(\frac{a-b}{2}B\hbar - \frac{1}{4}J\hbar^2\right)\end{bmatrix} \Rightarrow \lambda = -\frac{1}{4}J\hbar^2 \pm \frac{1}{2}\sqrt{(a-b)^2B^2\hbar^2 + J^2\hbar^4}$$

Therefore we have four eigenvalues corresponding to four eigenstates.

$$\begin{aligned}
\lambda_1 &= -\frac{a+b}{2}B\hbar + \frac{1}{4}J\hbar^2 \\
\lambda_2 &= -\frac{1}{4}J\hbar^2 + \frac{1}{2}\sqrt{(a-b)^2B^2\hbar^2 + J^2\hbar^4} \\
\lambda_3 &= -\frac{1}{4}J\hbar^2 - \frac{1}{2}\sqrt{(a-b)^2B^2\hbar^2 + J^2\hbar^4} \\
\lambda_4 &= \frac{a+b}{2}B\hbar + \frac{1}{4}J\hbar^2
\end{aligned}$$

Problem 3 (11.4)**.** Define $|n_0\rangle$ as the nth eigenvector of H_0, $\psi_0 = |0_0\rangle$, $H_1 = x$, $\lambda = -eE$. Using the fact that

$$\begin{aligned}
a_+ &= \frac{1}{\sqrt{2m\omega\hbar}}(m\omega x - ip) \\
a_- &= \frac{1}{\sqrt{2m\omega\hbar}}(m\omega x + ip) \\
[a_-, x] &= \frac{1}{\sqrt{2m\omega\hbar}}i[p, x] = \sqrt{\frac{\hbar}{2m\omega}} \\
[a_-^n, x] &= na_-^{n-1}[a_-, x] = n\sqrt{\frac{\hbar}{2m\omega}}a_-^{n-1}
\end{aligned}$$

The we can conclude that

$$\langle n_0|x|n_0\rangle = 0$$

$$\begin{aligned}
\langle q_0|x|n_0\rangle &= \frac{1}{\sqrt{q!n!}}\langle a_+^q\psi_0|x|a_+^n\psi_0\rangle = \frac{1}{\sqrt{q!n!}}\langle\psi_0|a_-^q x a_+^n\psi_0\rangle = \frac{1}{\sqrt{q!n!}}\langle\psi_0|\left(xa_-^q + q\sqrt{\frac{\hbar}{2m\omega}}a_-^{q-1}\right)a_+^n\psi_0\rangle \\
&= \frac{1}{\sqrt{q!n!}}q\sqrt{\frac{\hbar}{2m\omega}}\langle\psi_0|a_-^{q-1}a_+^n\psi_0\rangle = \frac{1}{\sqrt{q!n!}}q\sqrt{\frac{\hbar}{2m\omega}}\langle a_+^{q-1}\psi_0|a_+^n\psi_0\rangle
\end{aligned}$$

Let $q > n$, $\langle q_0|x|n_0\rangle$ would vanish if $q \neq n+1$ due to the orthogonality of eigenfunctions. Hence

$$\langle q_0|x|n_0\rangle = \begin{cases}\sqrt{\frac{(n+1)\hbar}{2m\omega}} & q = n+1 \\ 0 & \text{otherwise}\end{cases}$$

We can further generalize the conclusion to

$$\langle q_0|x|n_0\rangle = \begin{cases}\sqrt{\frac{(n+1)\hbar}{2m\omega}} & q = n+1 \\ \sqrt{\frac{n\hbar}{2m\omega}} & q = n-1 \\ 0 & \text{otherwise}\end{cases}$$

Then, we have

$$\begin{aligned}
E_n^0 &= \left(n+\frac{1}{2}\right)\hbar\omega \\
E_n^1 &= \langle n_0|H_1|n_0\rangle = 0 \\
E_n^2 &= -\frac{(n+1)\hbar}{2m\omega}\frac{1}{\hbar\omega} + \frac{n\hbar}{2m\omega}\frac{1}{\hbar\omega} = -\frac{1}{2m\omega^2} \\
|n_1\rangle &= -\frac{1}{\hbar\omega}\sqrt{\frac{(n+1)\hbar}{2m\omega}}|(n+1)_0\rangle + \frac{1}{\hbar\omega}\sqrt{\frac{n\hbar}{2m\omega}}|(n-1)_0\rangle
\end{aligned}$$

The energy be expanded to second-order equals to

$$E = E_0 + \lambda E_1 + \lambda^2 E_2 = \left(n+\frac{1}{2}\right)\hbar\omega - \frac{e^2E^2}{2m\omega^2}$$

By some simple algebras, we can show the exact result is

$$H = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{m\omega^2}{2}\left(x - \frac{eE}{m\omega^2}\right)^2 - \frac{e^2E^2}{2m\omega^2} \Rightarrow E_n = \left(n+\frac{1}{2}\right)\hbar\omega - \frac{e^2E^2}{2m\omega^2}$$

Then we can see that it is the second-order approximation is the same as the exact result.