

PHY5410 FA22 HW09

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Problem 1 (16.5). The first-order transition probability of a ground state to the n th excited state is written in the form

$$P_{0n}(t) = |\langle n, t | \psi, t \rangle|^2 = \left| \frac{1}{\hbar} \int_{t_0}^t dt' e^{i\omega_{n0}t'} \langle n | V(t') | 0 \rangle \right|^2$$

where $\omega_{n0} = (E_n - E_0)/\hbar$. Since $t_0 < t_1$ and $t > t_2$, we have

$$\int_{t_0}^t dt' e^{i\omega_{n0}t'} \langle n | V(t') | 0 \rangle = \int_{t_1}^{t_2} dt' e^{i\omega_{n0}t'} \langle n | V(t') | 0 \rangle = \langle n | x | 0 \rangle \int_{t_1}^{t_2} dt' e^{i\omega_{n0}t'} (-D \cos \Omega t)$$

Easy to prove that

$$\langle n | x | 0 \rangle = \sqrt{\frac{\hbar}{2m\omega}} \delta_{n1}$$

and

$$\begin{aligned} \int_{t_1}^{t_2} dt' e^{i\omega_{n0}t'} \cos \Omega t' &= \frac{1}{\omega_{n0}^2 - \Omega^2} (\omega_{n0} \cos \omega_{n0}t \cos \Omega - \Omega \cos \omega_{n0}t \sin \Omega t) \Big|_{t_1}^{t_2} \\ &+ i \frac{1}{\Omega^2 - \omega_{n0}^2} (\omega_{n0} \cos \omega_{n0}t \cos \Omega + \Omega \sin \omega_{n0}t \sin \Omega t) \Big|_{t_1}^{t_2} \end{aligned}$$

Hence we can calculate the transition probability.

$$P_{0n}(t) = \frac{D^2}{2m\omega\hbar} \left| \int_{t_1}^{t_2} dt' e^{i\omega_{n0}t'} \cos \Omega t' \right|^2 \delta_{n1}$$

Note that $P_{0n}(t) = 0 \ \forall n \geq 2$.

Problem 2 (16.7). The wave functions for the old ground state and the new ground state are

$$\psi(x) = \frac{\sqrt{m\lambda}}{\hbar} \exp(-m\lambda|x|/\hbar^2) \quad \psi'(x) = \frac{\sqrt{m\mu}}{\hbar} \exp(-m\mu|x|/\hbar^2)$$

Hence the transition probability is

$$\langle \psi' | \psi \rangle = 2 \int_0^\infty \frac{m}{\hbar^2} \sqrt{\lambda\mu} \exp[-m(\lambda + \mu)|x|/\hbar^2] = \frac{2\sqrt{\lambda\mu}}{\lambda + \mu}$$

Problem 3 (16.10). Since

$$[\dot{x}, x] = \left[\frac{1}{i\hbar} (xH - Hx), x \right] = \frac{1}{i\hbar} (2xHx - Hx^2 - x^2H)$$

Notice that

$$\langle a | xHx | a \rangle = \sum_n \langle a | x | n \rangle \langle n | Hx | a \rangle = \sum_n E_n |\langle n | x | a \rangle|^2$$

$$\begin{aligned}\langle a|Hx^2|a\rangle &= \sum_n \langle a|Hx|n\rangle \langle n|x|a\rangle = \sum_n E_n |\langle n|x|a\rangle|^2 \\ \langle a|x^2H|a\rangle &= \sum_n \langle a|x|n\rangle \langle n|xH|a\rangle = \sum_n E_n |\langle n|x|a\rangle|^2\end{aligned}$$

Therefore

$$\langle a|[\dot{x}, x]|a\rangle = \frac{2}{i\hbar} \sum_n (E_n - E_a) |\langle n|x|a\rangle|^2$$

For $H = p^2/2m + m\omega^2 x^2/2m$, we have

$$\dot{x} = \frac{\partial}{\partial p} H = \frac{p}{m} \Rightarrow [\dot{x}, x] = -\frac{i\hbar}{m}$$

Therefore

$$\sum_n (E_n - E_a) |\langle n|x|a\rangle|^2 = \frac{\hbar^2}{2m}$$