PHY5410 FA22 HW12

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Problem 1 (1.10).

(a) Let $|\phi\rangle = \psi(\mathbf{x}')|0\rangle$, then we have

$$\begin{aligned}
n(\mathbf{x})|\phi\rangle &= \psi^\dagger(\mathbf{x})\psi(\mathbf{x})\psi^\dagger(\mathbf{x}')|0\rangle = \psi^\dagger(\mathbf{x})[\psi^\dagger(\mathbf{x}')\psi(\mathbf{x}) + \delta(\mathbf{x}-\mathbf{x}')]|0\rangle \\
&= \psi^\dagger(\mathbf{x})\psi^\dagger(\mathbf{x}')\psi(\mathbf{x})|0\rangle + \psi^\dagger(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}')|0\rangle \\
&= \psi^\dagger(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}')|0\rangle
\end{aligned}$$

(b) Note that

$$\hat{N} = \int \mathrm{d}\mathbf{x}\, n(\mathbf{x}) = \int \mathrm{d}\mathbf{x} \sum_{ij} a_i^\dagger a_j \langle i|x\rangle\langle x|j\rangle = \sum_{ij} a_i^\dagger a_j \int \mathrm{d}\mathbf{x}\,\langle i|x\rangle\langle x|j\rangle = \sum_{ij} a_i^\dagger a_j \delta_{ij} = \sum_i a_i^\dagger a_j$$

Using the properties where $[a_i, a_j] = 0$, $[a_i, a_j^\dagger] = \delta_{ij}$, then we have

$$\psi(\mathbf{x})\hat{N} = \sum_{ij} a_i a_j^\dagger a_j \langle x|j\rangle = \sum_{ij}(a_j^\dagger a_i + \delta_{ij})a_j\langle x|j\rangle = \hat{N}\psi(\mathbf{x}) + \psi(\mathbf{x})$$

Problem 2 (2.1).

Problem 3 (2.6).

(a)

$$\begin{aligned}
e^{-\alpha a^\dagger} a e^{\alpha a^\dagger} &= (1-\alpha a^\dagger)a(1+\alpha a^\dagger) = (a - \alpha a^\dagger a)(1+\alpha a^\dagger) = a - \alpha^2 a^\dagger + \alpha(aa^\dagger - a^\dagger a) \\
e^{-\alpha a} a^\dagger e^{\alpha a} &= (1-\alpha a)a^\dagger(1+\alpha a) = (a^\dagger - \alpha a a^\dagger)(1+\alpha a) = a^\dagger - \alpha^2 a + \alpha(a^\dagger a - aa^\dagger)
\end{aligned}$$

(b)

$$\begin{aligned}
e^{\alpha a^\dagger a} a e^{-\alpha a^\dagger a} &= (1 + \alpha a^\dagger a + \cdots)a \sum_n \frac{(-\alpha)^n}{n!}(a^\dagger a)^n \\
&= (1 + \alpha a^\dagger a + \cdots) \sum_n \frac{(-\alpha)^n}{n!} a(a^\dagger a)(a^\dagger a)^{n-1} \\
&= (1 + \alpha a^\dagger a + \cdots) \sum_n \frac{(-\alpha)^n}{n!} (1 - a^\dagger a) a (a^\dagger a)^{n-1} \\
&= (1 + \alpha a^\dagger a + \cdots) \sum_n \frac{(-\alpha)^n}{n!} a (a^\dagger a)^{n-1} \\
&= (1 + \alpha a^\dagger a + \cdots) \sum_n \frac{(-\alpha)^n}{n!} a = e^{-\alpha} a \\
e^{\alpha a^\dagger} a^\dagger e^{-\alpha a^\dagger a} &= \sum_n \frac{\alpha^n}{n!}(a^\dagger a)^n a^\dagger (1 + \alpha a^\dagger a + \cdots) \\
&= \sum_n \frac{\alpha^n}{n!}(a^\dagger a)^{n-1}(a^\dagger a) a^\dagger (1 + \alpha a^\dagger a + \cdots)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_n \frac{\alpha^n}{n!}(a^\dagger a)^{n-1} a^\dagger (1 + \alpha a^\dagger a + \cdots) \\
&= \sum_n \frac{\alpha^n}{n!} a^\dagger (1 + \alpha a^\dagger a + \cdots) \\
&= e^\alpha a^\dagger
\end{aligned}
$$