Solutions to the Schrödinger equation for a charged particle in a magnetic field

The Schrödinger equation for a charged particle in a magnetic field is,

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}\left[\left(-i\hbar\nabla - q\vec{A}(\vec{r},t)\right)^2 + qV(\vec{r},t)\right]\psi. \tag{1}$$

Consider the case of free particles ($V(\vec{r},t)=0$). The time independent Schrödinger equation is then,

$$\frac{1}{2m}\left[\left(-i\hbar\nabla - q\vec{A}(\vec{r},t)\right)^2\right]\psi = E\psi. \tag{2}$$

To solve this equation, it is convenient to use Landau gauge for the vector potential $\vec{A}$,

$$\vec{A} = B_z x\hat{y}. \tag{3}$$

This choice of gauge describes a uniform, magnetic field in the z-direction, $\vec{B} = B_z\hat{z}$, which can be verified by taking the curl $\vec{B} = \nabla\times\vec{A}$.

Using this gauge, the factor $\left(-i\hbar\nabla - q\vec{A}\right)^2$ can be written,

$$-\hbar^2\nabla^2 + i\hbar qB_z\nabla\cdot x\hat{y} + i\hbar qB_z x\hat{y}\cdot\nabla + q^2B_z^2x^2. \tag{4}$$

Here $\nabla\cdot x\hat{y} = x\hat{y}\cdot\nabla = x\frac{d}{dy}$ so the two middle terms can be combined and the time independent Schrödinger equation becomes,

$$\frac{1}{2m}\left(-\hbar^2\nabla^2 + i2\hbar qB_z x\frac{d}{dy} + q^2B_z^2x^2\right)\psi = E\psi. \tag{5}$$

Because y and z do not appear explicitly in this equation, it can be shown that the wave function has the form,

$$\psi(x,y,z) = e^{ik_yy}e^{ik_zz}\Phi(x), \tag{6}$$

where $\Phi(x)$ is a function of x that needs to be determined. We can verify that this form for ψ solves 5 by substituting it in,

$$\frac{1}{2m}\left(-\hbar^2\nabla^2 + i2\hbar qB_z x\frac{d}{dy} + q^2B_z^2x^2\right)e^{ik_yy}e^{ik_zz}\Phi(x) = Ee^{ik_yy}e^{ik_zz}\Phi(x). \tag{7}$$

Here $\nabla^2 e^{ik_yy}e^{ik_zz}\Phi(x) = (-k_y^2-k_z^2)e^{ik_yy}e^{ik_zz}\Phi(x) + e^{ik_yy}e^{ik_zz}\frac{d^2\Phi(x)}{dx^2}$ and $\frac{d}{dy}e^{ik_yy}e^{ik_zz}\Phi(x) = ik_ye^{ik_yy}e^{ik_zz}\Phi(x)$. The exponential factors cancel out and we are left with an equation for Φ,

$$\frac{1}{2m}\left(-\hbar^2\frac{d^2}{dx^2} + \hbar^2k_z^2 + (\hbar k_y - qB_zx)^2\right)\Phi(x) = E\Phi(x). \tag{8}$$

The term $\frac{\hbar^2k_z^2}{2m}$ is the kinetic energy in the z-direction. Motion in the z-direction is unaffected by the magnetic field. This problem decouples into motion in the plane perpendicular to the magnetic field motion parallel to the magnetic field. The total energy is the kinetic energy in the z-direction plus the energy of the particle in the $x-y$ plane, $E = \frac{\hbar^2k_z^2}{2m} + E'$, where E' is the energy of the particle in the plane. The Schrödinger equation for the motion in the plane is,

$$\frac{1}{2m}\left(-\hbar^2\frac{d^2}{dx^2} + (\hbar k_y - qB_zx)^2\right)\Phi(x) = E'\Phi(x). \tag{9}$$

This equation is mathematically equivalent to the equation for a one-dimensional harmonic oscillator. The equation for a harmonic oscillator with mass m and spring constant K that oscillates around position x_j is,

$$\left(\frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{K}{2}(x-x_j)^2\right)\Phi(x) = E'\Phi(x). \tag{10}$$

Equations 9 and 10 are equivalent if we make the associations,

$$\frac{K}{2} = \frac{q^2B_z^2}{2m}, \quad \omega_c = \frac{qB_z}{m}, \quad \text{and} \quad x_j = \frac{\hbar k_y}{qB_z} = \frac{2\pi\hbar j}{qB_zL_y}. \tag{11}$$

Here ω_c is the classical cyclotron frequency of a charged particle in a magnetic field. The last form for x_j arises because k_y can take on the values $\frac{2\pi j}{L_y}$ where j is an integer. Since we know that the energy levels of the harmonic oscillator have the form, $E = \hbar\omega(\nu+1/2)$, where

$\omega = \sqrt{K/m}$, it follows that the in-plane energy of a charged particle in a magnetic field is,

$$E' = \hbar\omega_c\left(\nu + \frac{1}{2}\right) \qquad \nu = 0, 1, 2, \cdots. \tag{12}$$

Thus the solutions to equation 2 are $\psi(x, y, z) = e^{ik_y y}e^{ik_z z}\Phi(x)$ where $\Phi(x)$ is a solution to the equation for a harmonic oscillator 9. All of the electron states with the same value of ν are said to be in the same Landau level.

Degeneracy of the Landau levels

We now consider how many states correspond to a Landau level with a single value of ν. There will be one solution for every allowed value of x_j. The allowed values of x_j are restricted by the fact that x_j must be inside the sample, $-\frac{L_x}{2} < x_j < \frac{L_x}{2}$ resulting in a restriction on k_y of $-\frac{qB_zL_x}{2\hbar} < k_y < \frac{qB_zL_x}{2\hbar}$. Furthermore, the periodic boundary conditions restrict k_y to values $k_y = \frac{2\pi j}{L_y}$ where j is an integer. Thus the allowed number of states is $\frac{qB_zL_x}{\hbar}$ divided by $\frac{2\pi}{L_y}$,

$$\text{number of states per } \nu = \frac{qB_zL_xL_y}{h}.$$

The density of states per Landau level per spin is,

$$D_0 = \frac{qB_z}{h} = \frac{m\omega_c}{h} \quad \text{J}^{-1}\text{m}^{-2}.$$

- Festkörperphyisk, Rudolf Gross and Achim Marx, 2.Auflage
- Introduction to Solid State Physics, Charles Kittel, 8th edition