PHY5410 FA22 HW08

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Problem 1 (11.6).

(a) Let $\mathbf{B} = \nabla \times \mathbf{A}$ where

$$\mathbf{B} = \nabla \times \begin{bmatrix} 0 \\ Bx \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ B \end{bmatrix}$$

Then we have

$$H = \frac{1}{2m}p^2 + \frac{e^2 B^2}{2mc}x^2 \Rightarrow \frac{1}{2mE}p^2 + \frac{e^2 B^2}{2mc^2 E}x^2 = 1$$

Consider the classical case: pick particular p s.t. The equation intersects the p_x-x plane and obtains ellipse. Let $p_y = p_z = 0$ to simplify calculation. Hence the area enclosed by p and x equals to

$$\pi ab = \frac{2mcE}{eB}\pi$$

Using the Bohr-Sommerfeld quantization condition, we have

$$\oint d\mathbf{x}\, \mathbf{p} = \left(n + \frac{1}{2}\right)h$$

$$\oint d\mathbf{x}\, \mathbf{p} = \oint dx\, p_x = \pi ab$$

$$\Rightarrow E = \left(n + \frac{1}{2}\right)\hbar \frac{eB}{mc} = \left(n + \frac{1}{2}\right)\hbar\omega$$

(b) Solve the limiting case on xy plane by setting $H = 0$.

$$H = \frac{1}{2m}\left(p - \frac{e}{c}\mathbf{A}\right)^2 = 0$$

Hence we have $p_x = p_z = 0$, $p_y = eBx/c$. Thus

$$\oint d\mathbf{x}\, \mathbf{p} = \oint dy\, \frac{eB}{c}x = \frac{e}{c}\Psi = \left(n + \frac{1}{2}\right)\hbar \Rightarrow \Phi = \left(n + \frac{1}{2}\right)\frac{hc}{e}$$

Problem 2 (12.2). Define the Hamiltonian of the system as the following expression

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 r^2$$

And pick eigenfunctions $\phi_{nlm} = R_{nl}(r)Y_{lm}$, where m_l denotes the orbital quantum number and m_s denotes the spin quantum number. Define $N = l + 2n$, then for each N, we have

$$N = 2k \Rightarrow l = 0, 2, \ldots, N; \quad N = 2k + 1 \Rightarrow l = 1, 3, \ldots, N$$

For each l, we have $m_l = -l, -l+1, \ldots, l$, total $2l + 1$ degeneracy. Hence, for both even and odd N, we have overall degeneracy equals to

$$\sum_{l=0,2,\ldots,N} (2l + 1) = \frac{1}{2}(N + 1)(N + 2)$$

$$\sum_{l=1,3,\ldots,N} (2l+1) = \frac{1}{2}(N+1)(N+2)$$

It the particle has spin s, the degeneracy becomes to $(N+1)(N+2)(2s+1)/2$.

In this case, we have H_2 equals to

$$H_2 = \frac{\omega^2}{2mc^2}\mathbf{S}\cdot\mathbf{L}$$

Let $\mathbf{J} = \mathbf{L} + \mathbf{S}$, and pick eigenvector $|j, j_z, l\rangle$, then

$$H_2 |j, j_z, l\rangle = \frac{\hbar^2}{2}[j(j+1) - s(s+1) - l(l+1)]|j, j_z, l\rangle$$

Consider the case of spin-$1/2$ particles, if $l > 0$

$$j(j+1) - s(s+1) - l(l+1) = \begin{cases} l & j = l+1/2 \\ -l-1 & j = l-1/2 \end{cases}$$

For a fixed N, each l will be split into 2 lines with $j = l + 1/2$ and $j = l - 1/2$. Hence, the spectrum will have the form of

$$E_{N,j,l} = \left(N + \frac{3}{2}\right)\hbar\omega + \frac{\hbar^2\omega^2}{2mc^2}[j(j+1) - s(s+1) - l(l+1)]$$

(a) $l = 0$, $E_{N,j,0}$ has degeneracy equals to 2 (note that $j = 1/2$ when $l = 0$).

(b) $l > 0$, $E_{N,j,l}$ has degeneracy $2j + 1$ corresponds to different j_z.

Therefore we have energy described by N, j, l and the degeneracy equals to $2j + 1$.

Problem 3 (16.2). Let

$$H_0 = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x, \quad H_1 = -eEx$$

Let $|n\rangle$ denote the original solutions and $|n'\rangle$ denote the perturbed solutions. Then we can verify that

$$\langle x - \frac{eE}{m\omega^2}|n\rangle = \langle x|n'\rangle$$

According to the sudden approximation, the transition probability of $\psi \to m'$ would be $|\langle m'|\psi\rangle|^2$. Note that

$$\langle n'|0\rangle = \frac{1}{\sqrt{n!}}\langle (a'_+)^n 0'|0\rangle$$

and

$$a'_+ = \frac{1}{\sqrt{2m\omega\hbar}}\left[m\omega\left(x - \frac{eE}{m\omega^2}\right) - ip\right] = a_+ - \frac{1}{\sqrt{2m\omega\hbar}}\frac{eE}{\omega} = a_+ - b$$

Hence

$$\langle n'|0\rangle = \frac{1}{\sqrt{n!}}\langle 0'|(a'_-)^n 0\rangle = \frac{1}{\sqrt{n!}}\langle 0'|(-b)^n 0\rangle \frac{(-b)^n}{\sqrt{n!}}\langle 0'|0\rangle$$

$$= \frac{(-b)^n}{\sqrt{n!}} A^2 \int_{-\infty}^{\infty} \mathrm{d}x \ \exp\left(-\frac{m\omega}{2\hbar}x^2\right) \exp\left[-\frac{m\omega}{2\hbar}(x - eE/m\omega^2)^2\right]$$

$$= \frac{(-b)^n}{\sqrt{n!}} A^2 \int_{-\infty}^{\infty} \mathrm{d}x \ \exp\left[-\left(x - \frac{eE}{2m\omega^2}\right)\right] \exp\left(-\frac{e^2E^2}{2m\omega^3\hbar}\right)$$

$$= \frac{(-b)^n}{\sqrt{n!}} \exp\left(-\frac{e^2E^2}{2m\omega^3\hbar}\right)$$

Then $P_{0\to n'}$ would be

$$P_{0\to n'} = |\langle n'|0\rangle|^2 = \frac{b^{2n}}{n!} \exp\frac{2m\omega^3\hbar}{e^2E^2} = \frac{1}{n!}\left(\frac{e^2E^2}{2m\omega^3\hbar^2}\right)^n \exp\frac{2m\omega^3\hbar}{e^2E^2}$$

Easy to verify that $\sum_{n'} P_{0\to n'} = 1$.