PHY5410 FA22 HW05

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Problem 1 (8.1). Define operator $U = e^{-iHt/\hbar}$, then we have

$$A_H B_H - B_H A_H = (U^\dagger A U)(U^\dagger B U) - (U^\dagger B U)(U^\dagger A U) = U^\dagger A B U - U^\dagger B A U = U^\dagger [A, B] U = C_H$$

Problem 2 (8.2). Using the fact that

$$\begin{aligned}
[x_H, p_H] &= [x, p]_H = i\hbar \\
[x, H] &= \frac{1}{2m}[x, p^2] = \frac{i}{m}\hbar p \\
[H, p] &= \frac{m\omega^2}{2}[x^2, p] = i\hbar m\omega^2 x
\end{aligned}$$

we have

$$\begin{aligned}
\dot{x}_H &= \frac{i}{\hbar}[H_H, x_H] = \frac{i}{\hbar}\frac{-i}{m}\hbar p_H = \frac{p_H}{m} \\
\dot{p}_H &= \frac{i}{\hbar}[H_H, p_H] = \frac{i}{\hbar} i\hbar m\omega^2 x_H = -m\omega^2 x_H
\end{aligned}$$

Then we have a system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_H \\ p_H \end{bmatrix} = \begin{bmatrix} 0 & 1/m \\ -m\omega^2 & 0 \end{bmatrix}\begin{bmatrix} x_H \\ p_H \end{bmatrix}$$

The solution is

$$\begin{bmatrix} x_H(t) \\ p_H(t) \end{bmatrix} = \exp\left(\begin{bmatrix} 0 & 1/m \\ -m\omega^2 & 0 \end{bmatrix} t\right)\begin{bmatrix} x_H(0) \\ p_H(0) \end{bmatrix}$$

Since

$$\begin{bmatrix} 0 & 1/m \\ -m\omega^2 & 0 \end{bmatrix} t = \frac{1}{2im\omega}\begin{bmatrix} 1 & -1 \\ im\omega & im\omega \end{bmatrix}\begin{bmatrix} i\omega t & \\ & -i\omega t \end{bmatrix}\begin{bmatrix} im\omega & 1 \\ -im\omega & 1 \end{bmatrix}$$

we have

$$\exp\left(\begin{bmatrix} 0 & 1/m \\ -m\omega^2 & 0 \end{bmatrix} t\right) = \frac{1}{2im\omega}\begin{bmatrix} 1 & -1 \\ im\omega & im\omega \end{bmatrix}\begin{bmatrix} e^{i\omega t} & \\ & e^{-i\omega t} \end{bmatrix}\begin{bmatrix} im\omega & 1 \\ -im\omega & 1 \end{bmatrix} = \begin{bmatrix} \cos\omega t & \frac{1}{m\omega}\sin\omega t \\ -m\omega\sin\omega t & \cos\omega t \end{bmatrix}$$

Therefore

$$\begin{aligned}
x_H(t) &= \cos\omega t x_H(0) + \frac{1}{m\omega}\sin\omega t p_H(0) \\
p_H(t) &= \cos\omega t p_H(0) - m\omega\sin\omega t x_H(0)
\end{aligned}$$

Using the fact that $a(t) = (m\omega x - ip)/\frac{1}{2m\hbar}$, we have

$$a_H(t) = \frac{1}{\sqrt{2m\hbar}}[m\omega x_H(t) - ip_H(t)] = \frac{1}{\sqrt{2m\hbar}}[m\omega e^{i\omega} x_H(0) - ie^{-i\omega} p_H(0)]$$

Problem 3 (8.4).

Case 1. $l = 1/2$, we have $m = \pm 1/2$, let

$$\alpha\,|1/2\rangle + \beta\,|-1/2\rangle \mapsto \begin{bmatrix}\alpha\\ \beta\end{bmatrix}$$

then under the matrix representation, we have

$$\begin{aligned}
L^2 &= \hbar^2\frac{3}{4}\mathbf{I}\\
L_z &= \frac{\hbar}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\\
L_x &= \frac{1}{2}(L_+ + L_-) = \frac{\hbar}{2}\left(\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\right) = \frac{\hbar}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\\
L_y &= \frac{1}{2i}(L_+ - L_-) = \frac{\hbar}{2i}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}
\end{aligned}$$

Case 2. $l = 1$, let

$$\alpha\,|1\rangle + \beta\,|0\rangle + \gamma\,|-1\rangle \mapsto \begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}$$

we have

$$\begin{aligned}
L^2 &= 2\hbar^2\mathbf{I}\\
L_z &= \frac{\hbar}{2}\begin{bmatrix}1 & & \\ & 0 & \\ & & -1\end{bmatrix}\\
L_+ &= \hbar\begin{bmatrix} & \sqrt{2} & \\ & & \sqrt{2}\\ & & \end{bmatrix}\\
L_- &= \hbar\begin{bmatrix} & & \\ \sqrt{2} & & \\ & \sqrt{2} & \end{bmatrix}\\
L_x &= \frac{\hbar}{2}\begin{bmatrix}0 & \sqrt{2} & 0\\ \sqrt{2} & 0 & \sqrt{2}\\ 0 & \sqrt{2} & 0\end{bmatrix}\\
L_y &= \frac{\hbar}{2i}\begin{bmatrix}0 & \sqrt{2} & 0\\ -\sqrt{2} & 0 & \sqrt{2}\\ 0 & -\sqrt{2} & 0\end{bmatrix}
\end{aligned}$$

Case 3. $l = 3/2$, let

$$\alpha\,|3/2\rangle + \beta\,|1/2\rangle + \gamma\,|-1/2\rangle + \delta\,|-3/2\rangle \mapsto \begin{bmatrix}\alpha\\ \beta\\ \gamma\\ \delta\end{bmatrix}$$

we have

$$L^2 = \frac{15}{4}\hbar^2\mathbf{I}$$

$$
\begin{aligned}
L_z &= \hbar \begin{bmatrix} 3/2 & & & \\ & 1/2 & & \\ & & -1/2 & \\ & & & -3/2 \end{bmatrix} \\
L_+ &= \hbar \begin{bmatrix} & \sqrt{3} & & \\ & & 2 & \\ & & & \sqrt{3} \\ & & & \end{bmatrix} \\
L_- &= \hbar \begin{bmatrix} & & & \\ \sqrt{3} & & & \\ & 2 & & \\ & & \sqrt{3} & \end{bmatrix} \\
L_x &= \frac{\hbar}{2} \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix} \\
L_y &= \frac{\hbar}{2i} \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 0 & 2 & 0 \\ 0 & -2 & 0 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{bmatrix}
\end{aligned}
$$

Case 4. $l = 2$, let

$$
\alpha\left|2\right\rangle + \beta\left|1\right\rangle + \gamma\left|0\right\rangle + \delta\left|-1\right\rangle + \epsilon\left|-2\right\rangle \mapsto \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \epsilon \end{bmatrix}
$$

we have

$$
\begin{aligned}
L^2 &= 6\hbar^2 \mathbf{I} \\
L_z &= \hbar \begin{bmatrix} 2 & & & & \\ & 1 & & & \\ & & 0 & & \\ & & & -1 & \\ & & & & -2 \end{bmatrix} \\
L_+ &= \hbar \begin{bmatrix} & 2 & & & \\ & & \sqrt{6} & & \\ & & & \sqrt{6} & \\ & & & & 2 \\ & & & & \end{bmatrix} \\
L_- &= \hbar \begin{bmatrix} & & & & \\ 2 & & & & \\ & \sqrt{6} & & & \\ & & \sqrt{6} & & \\ & & & 2 & \end{bmatrix}
\end{aligned}
$$

$$L_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$$

$$L_y = \frac{\hbar}{2i}\begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ -2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & -\sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & -\sqrt{6} & 0 & 2 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$$

Problem 4 (8.5)**.**

(a) Let $\mathbf{p}_i$ denotes the momentum operator of the ith particle

$$\mathbf{p}_i = \begin{bmatrix} p_{ix} \\ p_{iy} \\ p_{iz} \end{bmatrix}$$

Then easy to derive that

$$\begin{aligned}
[H, \mathbf{P}] &= \left(\sum_{ij} \frac{1}{2} V(|\mathbf{x}_i - \mathbf{x}_j|)\right)\left(\sum_n \mathbf{p}_n\right) - \left(\sum_n \mathbf{p}_n\right)\left(\sum_{ij} \frac{1}{2} V(|\mathbf{x}_i - \mathbf{x}_j|)\right) \\
&= \sum_{nj} V(r_{nj})\mathbf{p}_n - \sum_{nj} \mathbf{p}_n V(r_{nj})
\end{aligned}$$

By the commutation relationship where

$$p_{i\alpha}\frac{1}{r_{ij}} = i\hbar\frac{x_{i\alpha} - x_{j\alpha}}{r_{ij}^3} + \frac{1}{r_{ij}} p_{i\alpha}$$

where $\alpha = x, y, z$. Using the fact that $V(r_{ij}) = \sum_k 1/r_{ij}^k$, we have

$$\begin{aligned}
\sum_{nj} p_{n\alpha}\frac{1}{r_{nj}} &= \sum_{n<j} p_{n\alpha}\frac{1}{r_{nj}} + p_{j\alpha}\frac{1}{r_{nj}} \\
&= \sum_{n<j} i\hbar k\frac{x_{n\alpha} - x_{j\alpha}}{r_{nj}^{k+2}} + \frac{1}{r_{nj}} p_{n\alpha} + i\hbar k\frac{x_{j\alpha} - x_{n\alpha}}{r_{nj}^{k+2}} + \frac{1}{r_{nj}} p_{j\alpha} \\
&= \sum_{n<j} \frac{1}{r_{nj}} p_{n\alpha} + \frac{1}{r_{nj}} p_{j\alpha} \\
&= \sum_{nj} \frac{1}{r_{nj}} p_{n\alpha}
\end{aligned}$$

Hence $\sum_{nj} V(r_{nj})\mathbf{p}_n - \sum_{nj} \mathbf{p}_n V(r_{nj}) = 0$ and $[H, \mathbf{P}]$.

The proof of $[H, \mathbf{L}] = 0$ begins with the following claims

Claim. Let $L_{n\alpha}$ denotes the α component ($\alpha = x, y, z$) of the angular momentum operator of nth particle in the system, $L_{n\alpha} = (\mathbf{x}_n \times \mathbf{p}_n)_\alpha$. Then

$$\left[\sum_n \mathbf{p}_n^2, \sum_n L_{n\alpha}\right] = 0$$

Proof. Note that

$$\begin{aligned}\left[\sum_n \mathbf{p}_n^2, \sum_n L_{n\alpha}\right] &= \left(\sum_n \mathbf{p}_n^2\right)\left(\sum_n \epsilon_{\alpha\beta\gamma} x_{n\beta} p_{n\gamma}\right) - \left(\sum_n \epsilon_{\alpha\beta\gamma} x_{n\beta} p_{n\gamma}\right)\left(\sum_n \mathbf{p}_n^2\right) \\ &= \sum_n \epsilon_{\alpha\beta\gamma} \mathbf{p}_n^2 x_{n\beta} p_{n\gamma} - \sum_n \epsilon_{\alpha\beta\gamma} x_{n\beta} p_{n\gamma} \mathbf{p}_n^2 \\ &= \sum_n \epsilon_{\alpha\beta\gamma} (x_{n\beta}\mathbf{p}_n^2 - 2i\hbar p_{n\beta}) p_{n\gamma} - \sum_n \epsilon_{\alpha\beta\gamma} x_{n\beta} p_{n\gamma} \mathbf{p}_n^2 \\ &= -2i\hbar \sum_n \epsilon_{\alpha\beta\gamma} p_{n\beta} p_{n\gamma} = 0\end{aligned}$$

□

Claim.

$$\left[\sum_{ij} \frac{1}{r_{ij}^k}, \sum_n L_{n\alpha}\right] = 0$$

Proof. Note that

$$\begin{aligned}\left[\sum_{ij} \frac{1}{r_{ij}^k}, \sum_n L_{n\alpha}\right] &= \left(\sum_{ij} \frac{1}{r_{ij}^k}\right)\left(\sum_n \epsilon_{\alpha\beta\gamma} x_{n\beta} p_{n\gamma}\right) - \left(\sum_n \epsilon_{\alpha\beta\gamma} x_{n\beta} p_{n\gamma}\right)\left(\sum_{ij} \frac{1}{r_{ij}^k}\right) \\ &= 2\sum_{nj} \epsilon_{\alpha\beta\gamma} \frac{1}{r_{nj}^k} x_{n\beta} p_{n\gamma} - 2\sum_{nj} \epsilon_{\alpha\beta\gamma} x_{n\beta} p_{n\gamma} \frac{1}{r_{nj}^k} \\ &= 2\sum_{nj} \epsilon_{\alpha\beta\gamma} \frac{1}{r_{nj}^k} x_{n\beta} p_{n\gamma} - 2\sum_{nj} \epsilon_{\alpha\beta\gamma} x_{n\beta} \left(i\hbar k \frac{x_{n\gamma} - x_{j\gamma}}{r_{nj}^{k+2}} + \frac{1}{r_{nj}^k} p_{n\gamma}\right) \\ &= -2i\hbar k \sum_{nj} \epsilon_{\alpha\beta\gamma} x_{n\beta} \frac{x_{n\gamma} - x_{j\gamma}}{r_{nj}^{k+2}}\end{aligned}$$

Since

$$\begin{aligned}\sum_{nj} \epsilon_{\alpha\beta\gamma} x_{n\beta} x_{n\gamma} &= 0 \\ \sum_{nj} \epsilon_{\alpha\beta\gamma} x_{n\beta} x_{j\gamma} &= \sum_{n<j} \epsilon_{\alpha\beta\gamma} (x_{n\beta} x_{j\gamma} + x_{j\beta} x_{n\gamma}) \\ &= \sum_{n<j} \epsilon_{\alpha\beta\gamma} (x_{n\beta} x_{j\gamma} - x_{j\gamma} x_{n\beta}) = 0\end{aligned}$$

Therefore we have

$$\left[\sum_{nj} \frac{1}{r_{nj}^k}, \sum_n L_{n\alpha}\right] = 0$$

□

According to these two claims, we have $[H, \mathbf{L}] = 0$.

(b) Since the laplacian operator is invariant to Euclidean transformations, and $|\mathbf{x}_i - \mathbf{x}_j|$ also invariant to rotations and translations, we can verify that the following operators are invariant to rotations and translations.

$$\sum_n \mathbf{p}_n^2, \ \sum_{ij} V(|\mathbf{x}_i - \mathbf{x}_j|)$$

Then H is translationally invariant as well as rotationally invariant. Hence, $[H, \mathbf{P}] = 0$ and $[H, \mathbf{L}] = 0$