

PHY5410 FA22 HW10

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Problem 1 (18.2). Let the solution be in the form

$$\psi_l(\mathbf{x}) = R_l(r)Y_{lm}(\theta, \phi)$$

In this case, we know that

$$R_l(r) = \frac{u_l(r)}{r} = \begin{cases} A j_l(kr) & r < a \\ B j_l(kr + \delta_l) & r \geq a \end{cases}$$

where $k = \sqrt{2mE/\hbar^2}$.

(a) By the continuity wrt r and $p_r = -i\hbar(\partial_r + 1/r)$ of $R(r)$ at $r = a$, we have

$$\begin{aligned} A j_l(ka) &= B j_l(ka + \delta_l) \\ \lim_{\epsilon \rightarrow 0} \int_{a-\epsilon}^{a+\epsilon} dr H\psi(\mathbf{x}) &= \lim_{\epsilon \rightarrow 0} \int_{a-\epsilon}^{a+\epsilon} dr E\psi(\mathbf{x}) \\ \Rightarrow \lambda A j_l(ka) &= A k j'_l(ka) - B k j'_l(ka + \delta_l) \end{aligned}$$

note that j'_l does not denote the derivative but denotes the radial derivative

$$j'_l(\varrho) = \left[\frac{1}{\varrho} \frac{d}{d\varrho} \varrho \right] j_l(\varrho)$$

Substituting $A/B = j_l(ka + \delta_l)/j_l(ka)$, we get an equation

$$\begin{aligned} \lambda j_k(ka + \delta_l) &= k j_l(ka + \delta_l) \frac{j'_l(ka)}{j_l(ka)} - k j'_l(ka + \delta_l) \\ \Rightarrow \frac{j'_l(ka + \delta_l)}{j_l(ka + \delta_l)} &= \frac{j'_l(ka)}{j_k(ka)} - \frac{\lambda}{k} \end{aligned}$$

Assume the asymptotic solution applies here, where

$$j_l(\varrho) = \frac{1}{\varrho} \sin\left(\varrho - \frac{l\pi}{2}\right) \quad j'_l(\varrho) = \frac{1}{\varrho} \cos\left(\varrho - \frac{l\pi}{2}\right)$$

Hence

$$\begin{aligned} \cot(ka + \delta_l - l\pi/2) &= \cot(ka - l\pi/2) - \frac{\lambda}{k} \\ \Rightarrow \delta_l &= \arctan\left[\frac{k}{k \cot(ka - l\pi/2) - \lambda}\right] - ka + \frac{l\pi}{2} \end{aligned}$$

if we define $\xi = ka$, $g = \lambda a$, then

$$\delta_l = \arctan\left[\frac{\xi}{\xi \cot(\xi - l\pi/2) - g}\right] - \xi + \frac{l\pi}{2}$$

(b) For $l = 0$ we have

$$\sigma_0 = \frac{4\pi}{k^2} \sin^2 \delta_l$$

- (c) The condition for δ_0 to take its maximum is the same as for $|\sin \delta_l/k|$ to take its maximum.
 (d) Given $g = \lambda a$ large enough, the arctan term in the δ_l will vanish. Hence

$$\delta_l(k) = -ka + \frac{l\pi}{2}$$

Hence

$$\sigma_0(k) = \frac{4\pi}{k^2} \sin^2(ka)$$

Then easy to verify that

$$\sigma_0(k) \leq \lim_{k' \rightarrow 0} \sigma_0(k')\sigma_0(k') = 4\pi a^2$$

then σ_0 approximates its maximum $4\pi a^2$ when k is sufficiently small.

Problem 2 (18.4). The radial part of the wave function is

$$R_l(a) = \begin{cases} 0 & r < a \\ A j_l(kr) + B n_l(kr) & r \geq a \end{cases}$$

To satisfy the continuity

$$\begin{aligned} R_l(a) &= A j_l(ka) + B n_l(ka) = C j_l(ka + \delta_l) = 0 \\ \Rightarrow \delta_l &= \arctan \frac{j_l(ka)}{n_l(ka)} \approx ka - \frac{l\pi}{2} \end{aligned}$$

For $l = 0$, we have $\delta_0 = ka$. Then

$$\sigma_0 = \frac{4\pi}{k^2} \sin^2 ka$$

$\sigma_0 \approx 4\pi a^2$ when ka small.

Problem 3 (18.5). Let the potential be

$$V(r) = \begin{cases} V & r < a \\ 0 & r \geq a \end{cases}$$

Hence the solution could be written as

$$\psi(\mathbf{x}) = R_l(r) Y_{lm}(\theta, \phi) \quad R_l = \begin{cases} A j_l(qr) & r < a \\ B j_l(kr + \delta_l) & r \geq a \end{cases}$$

where $q = \sqrt{2m(E - V)/\hbar^2}$, $k = \sqrt{2mE/\hbar^2}$. According to the continuity of $R(r)$ wrt r and p_r at $r = a$, we have

$$A j_l(qa) = B j_l(ka + \delta_l) \quad A q j'_l(qa) = B k j'_l(kr + \delta_l) \Rightarrow \frac{j'_l(ka + \delta_l)}{j_l(ka + \delta_l)} = \frac{q j'_l(qa)}{k j_l(qa)}$$

Plugin $l = 0$, we have

$$\begin{aligned} \tan(ka + \delta_l) &= \frac{k}{q} \tan qa \\ \Rightarrow \delta_l &= \arctan \left[\frac{k}{q} \tan qa \right] - ka \end{aligned}$$