

## PHY5410 FA22 HW02

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**Problem 1** (3.10). The wavefunction under the momentum representation would be

$$\begin{aligned}
 \varphi(p, t) &= \frac{1}{\sqrt{2\pi\hbar}} \int dx \psi(x, t) e^{-ipx/\hbar} \\
 &= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{2\pi\hbar} \iint dx dp' g(p') e^{ix(p-p')/\hbar} e^{-iE(p')t/\hbar} e^{i\alpha(p')} \\
 &= \frac{1}{\sqrt{2\pi\hbar}} \int dp' g(p') e^{-iE(p')t/\hbar} e^{i\alpha(p')} \delta(p - p') \\
 &= \frac{1}{\sqrt{2\pi\hbar}} g(p) e^{-iE(p)t/\hbar} e^{i\alpha(p)}
 \end{aligned}$$

and

$$|\varphi(p, t)|^2 = \frac{1}{2\pi\hbar} g^2(p)$$

Therefore

$$\begin{aligned}
 \langle p \rangle &= \langle \varphi | p \varphi \rangle = \frac{1}{2\pi\hbar} \int dp p g^2(p) = p_0 \\
 \langle p^2 \rangle &= \langle \varphi | p^2 \varphi \rangle = \frac{1}{2\pi\hbar} \int dp p^2 g^2(p)
 \end{aligned}$$

Using the fact that  $x = i\hbar d/dp$  under momentum representation, we have

$$x\varphi(p, t) = \frac{1}{\sqrt{2\pi\hbar}} g(p) (E'(p)t - \alpha'(p)\hbar) e^{-iE(p)t/\hbar + i\alpha(p)} + \frac{1}{\sqrt{2\pi\hbar}} i\hbar g'(p) e^{-iE(p)t/\hbar + i\alpha(p)}$$

Thus

$$\begin{aligned}
 \langle x \rangle &= \langle \varphi | x \varphi \rangle \\
 &= \frac{1}{2\pi\hbar} \int dp [E'(p)t - \alpha'(p)\hbar] g^2(p) + i\hbar g(p) g'(p) \\
 &= \frac{1}{2\pi\hbar} \int dp [E'(p)t - \alpha'(p)\hbar] g^2(p) \\
 &= \langle E'(p)t - \alpha'(p)\hbar \rangle
 \end{aligned}$$

note that the term  $g(p)g'(p)$  vanishes under integration due to the boundary condition.

$$\int dp g(p)g'(p) = \frac{1}{2} g^2(p) \Big|_{\Omega} = 0$$

For  $x^2$  (ignoring  $g(p)g'(p)$  term)

$$\begin{aligned}
 \langle x^2 \rangle &= \langle \varphi | x^2 \varphi \rangle \\
 &= \frac{1}{2\pi\hbar} \int dp \hbar^2 g'^2(p) + [E'(p)t - \alpha(p)\hbar]^2 g^2(p) \\
 &= \langle [E'(p)t - \alpha'(p)\hbar]^2 \rangle + \frac{1}{2\pi\hbar} \int dp \hbar^2 g'^2(p)
 \end{aligned}$$

Plug in a gaussian wave packet with  $\sigma_p = \Delta p$ , i.e.

$$\begin{aligned} g(p) &= Ae^{\frac{-(p-p_0)^2}{4\Delta p^2}} \\ E(p) &= \frac{p^2}{2m} \\ \alpha(p) &= 0 \end{aligned}$$

we have

$$\begin{aligned} \langle p \rangle &= p_0 \\ \langle p^2 \rangle &= \Delta p^2 + p_0^2 \\ \langle x \rangle &= \langle pt/m \rangle = \frac{t}{m} \langle p \rangle = \frac{p_0 t}{m} \\ \langle x^2 \rangle &= \langle p^2 t^2 / m^2 \rangle + \langle \hbar^2 \left( \frac{p - p_0}{2\Delta p^2} \right)^2 \rangle \\ &= \frac{t^2}{m^2} \langle p^2 \rangle + \frac{\hbar^2}{4\Delta p^4} \langle (p - p_0)^2 \rangle \\ &= \frac{t^2}{m^2} (p_0^2 + \Delta p^2) + \frac{\hbar^2}{4\Delta p^2} \end{aligned}$$

In this manner, we have

$$\begin{aligned} \Delta x &= \sqrt{\frac{t^2}{m^2} \Delta p^2 + \frac{\hbar^2}{4\Delta p^2}} = \frac{\hbar}{2\Delta p} \sqrt{1 + \frac{4t^2 \Delta p^4}{m^2 \hbar^2}} \\ \Rightarrow \Delta x \Delta p &= \frac{\hbar}{2} \sqrt{1 + \frac{4t^2 \Delta p^4}{m^2 \hbar^2}} \geq \frac{\hbar}{2} \end{aligned}$$

**Problem 2** (3.15). The energy of the system is

$$E = T + V = \frac{p^2}{2m} + cx^4$$

Substitute  $p$  by  $px = \hbar/2$

$$E = \frac{\hbar^2}{8mx^2} + cx^4$$

Find the minimum of  $E$  by letting  $dE/dx = 0$

$$\frac{dE}{dx} = 4cx^3 - \frac{\hbar^2}{4mx^3} = 0 \Rightarrow x_0 = \left( \frac{\hbar^2}{16mc} \right)^{1/6}$$

Then the ground state energy approximately equals to

$$E_0 = \frac{\hbar^2}{8mx_0^2} + cx_0^4 = \frac{\hbar^2}{8m} \left( \frac{16mc}{\hbar^2} \right)^{1/3} + c \left( \frac{\hbar^2}{16mc} \right)^{2/3} = 3c \left( \frac{\hbar^2}{16mc} \right)^{2/3}$$

**Problem 3 (4.2).** Using the fact that the energy uncertainty of a free wave packet is  $\Delta E = p_0 \Delta p / m$  and the time uncertainty is  $\Delta t = m \Delta x / p_0$ . Then

$$\Delta E = \frac{\hbar}{2dm} p_0$$

$$\Delta E \Delta t = \sqrt{1 + \left( \frac{t\hbar}{2md^2} \right)^2} \frac{\hbar}{2}$$

**Problem 4 (5.3).**

*Claim.*  $L_i = L_i^\dagger$

*Proof.* Using the fact  $[x_i, p_j] = i\hbar \delta_{ij}$  ( $x_i, p_j$  commute when  $i \neq j$ )

$$L_i^\dagger = \epsilon_{ijk} p_k^\dagger x_j^\dagger = \epsilon_{ijk} p_k x_j = \epsilon_{ijk} x_j p_k = L_i \quad \square$$

*Claim.*  $\langle \psi | L_i^2 | \psi \rangle \geq 0$ .

*Proof.* Since  $L_i = L_i^\dagger$ , we have

$$\begin{aligned} \langle \psi | L_i^2 | \psi \rangle &= \langle L_i^\dagger \psi | L_i \psi \rangle \\ &= \langle L_i \psi | L_i \psi \rangle \geq 0 \end{aligned} \quad \square$$

By the claims above, we can derive that

$$\langle \psi | \mathbf{L}^2 | \psi \rangle = 0 \Leftrightarrow \sum_i \langle \psi | L_i^2 | \psi \rangle = 0 \Leftrightarrow \langle \psi | L_i^2 | \psi \rangle = \langle L_i \psi | L_i \psi \rangle = 0 \Leftrightarrow |L_i \psi\rangle = |0\rangle \Rightarrow \langle \psi | L_i | \psi \rangle = 0$$

note that we can derive  $|L_i \psi\rangle = |0\rangle$  if  $|L_i \psi\rangle$  is indeed a continuous function under any representations.