MATH104 SU22 HW#7

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Remark. $V_\epsilon(x)$ is used to denote a ϵ-neighborhood

$$V_\epsilon(x) = B_\epsilon(x) \setminus \{x\}$$

1. (a) To let $|x(1-x)| < 1$, we can derive that $x(1-x) \le 0.5$ and thus $x \in ((1-\sqrt{5})/2, (1+\sqrt{5})/2)$. Thus, $f_n(x)$ converges pointwise in this open interval and converges uniformly in any closed interval in this interval.

   (b) $f_n(x)$ converges pointwise on $\mathbb{R}$, and it converges on $\{-1, 1\}$ and any closed interval in $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

   (c) $S_n(x)$ converges pointwise on $((1-\sqrt{5})/2, (1+\sqrt{5})/2)$, and converges uniformly on any closed interval in $((1-\sqrt{5})/2, (1+\sqrt{5})/2)$.

2. Since $f_n \to f$ converges uniformly on $(a,b)\cap\mathbb{Q}$, we know f is a function defined on $(a,b)\cap\mathbb{Q}$. Extend the function f as $g : [a,b] \to \mathbb{R}$ by

$$g(x) = \begin{cases} f(x) & x \in (a,b)\cap\mathbb{Q} \\ \lim_{r\to x} f(r), r \in \mathbb{Q} & \text{otherwise} \end{cases}$$

Claim. Limit

$$\lim_{r\to x} f(r), r \in \mathbb{Q}$$

exists.

Proof. Consider any sequence $a_n \to x$ where $a_n \in (a,b)\cap\mathbb{Q}$. $\forall\epsilon$, $\exists N \in \mathbb{N}$ s.t. $\forall n \ge N$ we have

$$|f_n(r) - f(r)| < \frac{\epsilon}{3}$$

$\forall r \in (a,b)\cap\mathbb{Q}$. Then, since f_N continuous on $[a,b]$, $\forall\epsilon > 0$, $\exists\delta > 0$ s.t.

$$|x-y| \in V_\delta(0) \Rightarrow |f_N(x) - f_N(y)| \le \frac{\epsilon}{3}$$

For this $\delta > 0$, since a_n Cauchy, we can find $N' \in \mathbb{N}$ s.t. $\forall m, n > N'$, we have

$$|a_n - a_m| < \delta$$

Therefore

$$|f_N(a_n) - f_N(a_m)| < \frac{\epsilon}{3}$$

Thus $\forall\epsilon$, $\exists N, N' \in \mathbb{N}$ s.t. $\forall m, n > N'$ we have

$$\begin{aligned} |f(a_n) - f(a_m)| &\le |f(a_n) - f_N(a_n)| + |f(a_m) - f_N(a_m)| + |f_N(a_n) - f_N(a_m)| \\ &\le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

Thus we know that $\forall a_n \to x$ and $a_n \in \mathbb{Q}$, $f(a_n)$ also Cauchy, then limit $f(a_n)$ exists.

To show $f(a_n)$ converges to the unique value for any $a_n \to x$, the scheme is similar as above. Therefore

$$\lim_{r\to x} f(r)$$

exists. □

Claim. $f_n \to g$ uniformly on $[a,b]$.

Proof. We already know that $f_n \to g$ uniformly on $(a,b)\cap\mathbb{Q}$. Now consider any $x \in [a,b] \setminus ((a,b)\cap\mathbb{Q})$. $\forall\epsilon$ $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$ we have

$$|f(r) - f_n(r)| < \frac{\epsilon}{3}$$

$\forall r \in (a,b)\cap\mathbb{Q}$. Also, since $g(x)$ is defined as

$$g(x) = \lim_{r\to x} f(r)$$

Then $\forall\epsilon$, $\exists\delta_1 > 0$ s.t.

$$r \in V_{\delta_1}(x), r \in (a,b)\cap\mathbb{Q} \Rightarrow |f(r) - g(x)| < \frac{\epsilon}{3}$$

Also, by the continuity of f_n, $\forall\epsilon$, $\exists\delta_2 > 0$ s.t.

$$r \in V_{\delta_2}(x), r \in (a,b)\cap\mathbb{Q} \Rightarrow |f_n(x) - f_n(r)| < \frac{\epsilon}{3}$$

Hence, $\forall\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $\exists\delta_1, \delta_2 > 0$, $r \in V_{\min(\delta_1,\delta_2)}(x), r \in (a,b)\cap\mathbb{Q}$ we have

$$\begin{aligned}|f_n(x) - g(x)| &\leq |f_n(x) - f_n(r)| + |f_n(r) - f(r)| + |f(r) - g(x)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon\end{aligned}$$

Thus $f_n \to g$ uniformly. □

Therefore f_n converges uniformly on $[a,b]$.

3. Let $|f|$ and $|g|$ bounded by $M > 0$, then $\forall\epsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$ we have

$$\begin{aligned}|f_n(x) - f(x)| &< \frac{\epsilon}{2M} \\ |g_n(x) - g(x)| &< \frac{\epsilon}{4M}\end{aligned}$$

Since $|f|$ bounded by M and $f_n \to f$ uniformly, $\exists N' \in \mathbb{N}$ s.t. $\forall n \geq N'$ $|f_n(x)|$ bounded by $2M$. Thus, $\forall\epsilon > 0$, $\exists N, N' \in \mathbb{N}$ s.t. $\forall n \geq \max(N, N')$ we have

$$\begin{aligned}|f_n(x)g_n(x) - f(x)g(x)| &\leq |f_n(x)||g_n(x) - g(x)| + |g(x)||f(x) - f_n(x)| \\ &< 2M\frac{\epsilon}{4M} + M\frac{\epsilon}{2M} = \epsilon\end{aligned}$$

$\forall x \in \mathbb{R}$. Thus $f_n g_n \to fg$ uniformly on $\mathbb{R}$.

4. Since

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$\forall x \in (-1,1)$, it is easy to prove that $f_n' \to f'$ and $f_n'' \to f''$ and hence

$$f''(x) = \frac{2}{(1-x)^3} = \sum_{n=1}^{\infty} n(n+1)x^{n-1}$$

Thus we have

$$\frac{2x}{(1-x)^3} = \sum_{n=1}^{\infty} n(n+1)x^n$$

Substitute x to $x-1$, we have

$$\sum_{n=1}^{\infty} n(n+1)(x-1)^n = \frac{2(x-1)}{(2-x)^3}$$

By root test, we have convergence radius $\rho = 1$, then $f(x)$ converges on $(0, 2)$.

5. The Taylor expansion for $\ln(1+x)$ appears to be

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n$$

Thus

$$\ln(1-1/3) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^{-n}}{n} = -\sum_{n=1}^{\infty} \frac{3^{-n}}{n}$$

Therefore

$$\sum_{k=1}^{\infty} \frac{3^{-k}}{k} = \ln \frac{3}{2}$$

6. (a) Let a_n decrease to 0 and

$$\sum_{k=1}^{n} b_k$$

is bounded forall n. Then

$$\sum a_k b_k$$

converges.

(b) Use $f_n(x)$ to denote

$$\sum_{k=1}^{n} \frac{(-1)^k}{x^4 + k}$$

Since it is a alternative series and $1/(x^4+n) \to 0$ as $n \to 0$, then it converges pointwisely $\forall x \in \mathbb{R}$.

Claim. $\forall \epsilon$, $\exists N \in \mathbb{N}$ s.t. $\forall n > m \geq \frac{1}{N}$, we have

$$|f_n(x) - f_m(x)| < \epsilon$$

$\forall x \in \mathbb{R}$.

Proof. $\forall \epsilon > 0$, we can pick $N \in \mathbb{N}$ s.t. $\forall n > m \geq N$, we have $1/N < \epsilon$ and

$$\begin{aligned} f_n(x) - f_m(x) &= \sum_{k=m+1}^{n} \frac{(-1)^k}{x^4 + k} \\ &= (-1)^(m+1) \left(\frac{1}{x^4 + m + 1} - \frac{1}{x^4 + m + 2} + \cdots \right) \\ \Rightarrow |f_n(x) - f_m(x)| &= \frac{1}{x^4 + m + 1} - \frac{1}{x^4 + m + 2} + \cdots \\ &\leq \frac{1}{x^4 + m + 1} \leq \frac{1}{m + 1} \\ &< \frac{1}{N} < \epsilon \end{aligned}$$

$\forall x$. □

Therefore $f_n(x)$ converges uniformly on $\mathbb{R}$.
However, $f_n(x)$ does not converge absolutely.

7. Note that f bounded on $[0, 1]$ and continuous on $[c, 1]$ $\forall c \in (0, 1)$, then f integrable on any $[c, 1]$. Now consider the integrability of f on $[0, c]$. Let P be any partition of $[0, c]$, we have

$$\begin{aligned} U(f, P) - L(f, P) &= (\sup\{f(x) | x \in I_k\} - \inf\{f(x) | x \in I_k\}) l_k \\ &\leq 2\sqrt{a_k} l_k \leq 2c\sqrt{c} \end{aligned}$$

Thus, $\forall \epsilon > 0$, $\exists c < (\epsilon/2)^{2/3}$ s.t.

$$U(f, P) - L(f, P) < \epsilon$$

which means f integrable on $[0, c]$ (c depends on ϵ). Thus, f integrable on $[0, 1]$ since it is both integrable on $[0, c]$ and $[c, 1]$.

8. Let $P = \{a_0, a_1, \ldots, a_n\}$, then

$$g(x) = g(a_k), \; x \in I_k$$

Since

$$L(g, P) \leq L(g) \leq U(g) \leq U(g, P)$$

we have

$$L(g, P) = L(g) = U(g) = U(g, P) = \int_a^b g(x) \, \mathrm{d}x$$

9. Let $P = \{0, 1/N, 2/N, \ldots, 1\}$, $\forall \epsilon > 0$, $\forall N > \{1, 3/\epsilon\}$ we have

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^{N} \left[\left(\frac{i}{N} \right)^2 - \left(\frac{(i-1)}{N} \right)^2 \right] \frac{1}{N} \\ &= \sum_{i=1}^{N} \frac{1}{N^3} (2i + 1) \\ &= \frac{N^2 + 2N}{N^3} < \frac{3N^2}{N^3} < \epsilon \end{aligned}$$

which shows that f integrable.