Numerical Analysis (CS 450)
Homework Set 1
Due: February 12, 2014
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Problem 1

$$cond(f) = \frac{|relative\ change\ in\ solution|}{|relative\ change\ in\ input\ data|} = \frac{[f(x+\Delta x, y+\Delta y)-f(x,y)]/f(x,y)}{[(x+\Delta x, y+\Delta y)-(x,y)]/(x,y)}$$

$$= \frac{|(x+\Delta x)-(y+\Delta y)-(x-y)|/(x-y)}{(\Delta x, \Delta y)/(x,y)} = \frac{(\Delta x-\Delta y)(|x|+|y|)}{(x-y)(|\Delta x|+|\Delta y|)}$$

$$\geq \frac{\Delta x-\Delta y}{(|\Delta x|+|\Delta y|)\varepsilon} \geq \frac{1}{\varepsilon}$$

we can conclude that subtraction is sensitive when ε is small

Problem 2

Suppose a and b have the same sign. (ii) would be preferable since the result is guaranteed to lie in the interval [a; b], and no overflow would happen. The rounding process will do no harm to the problem, too.

For example,

a) $\beta = 10,\ t = 2,\ [L, U] = [-2,\ 2]$

b) $a = 5.7 \times 10^{-1}$, $b = 5.9 \times 10^{-1}$

c) The intermediate results for i and ii are as follows

i) $(a+b) \approx 1.2 \times 10^{-1}$ (i1)

$1.2 \times 10^{-1}/2.0 = 0.6 \times 10^{-1}$ (i2)

ii) $(b-a) = 0.2 \times 10^{-1}$ (ii1)

$(b-a)/2 = 0.1 \times 10^{-1}$ (ii2)

$a + 0.1 \times 10^{-1} = 5.8 \times 10^{-1}$ (ii3)

d) At step (i1) , according to the marks above, the problem will occur due to rounding.

Problem 3: Bessel recurrence vs. floating point

a) Please check problem3_a.py
For z = 20, we have the following result

n=2,	value=-0.160341	error=0
n=3,	value=-0.0989014	error=1.82415e-15
n=4,	value=0.130671	error=2.12408e-16
n=5,	value=0.15117	error=3.67211e-16
n=6,	value=-0.055086	error=7.55788e-16
n=7,	value=-0.184221	error=3.01328e-16
n=8,	value=-0.0738689	error=5.63611e-16
n=9,	value=0.125126	error=2.21821e-16
n=10,	value=0.186483	error=1.48837e-16
n=11,	value=0.0613563	error=4.52367e-16
n=12,	value=-0.118991	error=2.33259e-16
n=13,	value=-0.204145	error=0
n=14,	value=-0.146398	error=7.5836e-16
n=15,	value=-0.000812069	error=1.10948e-13
n=16,	value=0.14518	error=1.91181e-16
n=17,	value=0.2331	error=1.19072e-16
n=18,	value=0.25109	error=6.63242e-16
n=19,	value=0.218862	error=1.26818e-15
n=20,	value=0.164748	error=5.05419e-16
n=21,	value=0.110634	error=2.2579e-15
n=22,	value=0.0675829	error=3.69621e-15
n=23,	value=0.0380487	error=1.82369e-16
n=24,	value=0.0199291	error=3.13361e-15
n=25,	value=0.00978117	error=1.95089e-15
n=26,	value=0.00452381	error=1.91733e-15
n=27,	value=0.00198074	error=6.56848e-16
n=28,	value=0.000824178	error=1.31549e-16
n=29,	value=0.000326963	error=1.65799e-15
n=30,	value=0.000124015	error=2.40418e-15
n=31,	value=4.50828e-05	error=1.05215e-15
n=32,	value=1.57413e-05	error=5.81145e-15
n=33,	value=5.28924e-06	error=6.50179e-14
n=34,	value=1.71324e-06	error=3.26306e-14
n=35,	value=5.35784e-07	error=4.86134e-14
n=36,	value=1.62001e-07	error=4.06847e-14
n=37,	value=4.74202e-08	error=2.76307e-14
n=38,	value=1.34536e-08	error=4.67277e-14
n=39,	value=3.70356e-09	error=4.55629e-14
n=40,	value=9.90239e-10	error=5.53409e-14
n=41,	value=2.57401e-10	error=3.9969e-14
n=42,	value=6.51039e-11	error=1.07203e-14
n=43,	value=1.60356e-11	error=5.21885e-14
n=44,	value=3.84926e-12	error=2.64419e-14

n=45, value=9.01145e-13 error=2.00571e-14
n=46, value=2.05887e-13 error=3.53113e-14
n=47, value=4.59366e-14 error=3.95662e-14
n=48, value=1.00149e-14 error=4.25353e-14
n=49, value=2.13469e-15 error=4.85951e-14
n=50, value=4.45104e-16 error=2.37046e-14

b) Please check problem3_b.py
For z = 20, we have the following result

n=2,	value=-0.160341	error=0
n=3,	value=-0.0989014	error=1.82415e-15
n=4,	value=0.130671	error=2.12408e-16
n=5,	value=0.15117	error=7.34421e-16
n=6,	value=-0.055086	error=2.26736e-15
n=7,	value=-0.184221	error=4.51993e-16
n=8,	value=-0.0738689	error=1.8787e-15
n=9,	value=0.125126	error=4.43641e-16
n=10,	value=0.186483	error=5.9535e-16
n=11,	value=0.0613563	error=2.26184e-15
n=12,	value=-0.118991	error=5.83146e-16
n=13,	value=-0.204145	error=2.7192e-16
n=14,	value=-0.146398	error=1.51672e-15
n=15,	value=-0.000812069	error=1.96662e-13
n=16,	value=0.14518	error=3.82361e-16
n=17,	value=0.2331	error=2.38143e-16
n=18,	value=0.25109	error=2.21081e-16
n=19,	value=0.218862	error=5.07271e-16
n=20,	value=0.164748	error=1.17931e-15
n=21,	value=0.110634	error=2.50878e-16
n=22,	value=0.0675829	error=1.64276e-15
n=23,	value=0.0380487	error=5.65343e-15
n=24,	value=0.0199291	error=1.63644e-14
n=25,	value=0.00978117	error=6.01228e-14
n=26,	value=0.00452381	error=2.5117e-13
n=27,	value=0.00198074	error=1.19524e-12
n=28,	value=0.000824178	error=6.37739e-12
n=29,	value=0.000326963	error=3.77686e-11
n=30,	value=0.000124015	error=2.4639e-10
n=31,	value=4.50828e-05	error=1.75942e-09
n=32,	value=1.57413e-05	error=1.36796e-08
n=33,	value=5.28924e-06	error=1.15281e-07
n=34,	value=1.71324e-06	error=1.0488e-06
n=35,	value=5.3579e-07	error=1.02644e-05

n=36,	value=1.62019e-07	error=0.000107724
n=37,	value=4.74775e-08	error=0.00120889
n=38,	value=1.36483e-08	error=0.0144685
n=39,	value=4.38591e-09	error=0.184244
n=40,	value=3.45678e-09	error=2.49086
n=41,	value=9.44122e-09	error=35.6791
n=42,	value=3.52522e-08	error=540.476
n=43,	value=1.38618e-07	error=8643.39
n=44,	value=5.60806e-07	error=145691
n=45,	value=2.32893e-06	error=2.58441e+06
n=46,	value=9.91936e-06	error=4.81786e+07
n=47,	value=4.33001e-05	error=9.42606e+08
n=48,	value=0.000193591	error=1.93304e+10
n=49,	value=0.000885938	error=4.1502e+11
n=50,	value=0.00414751	error=9.31806e+12

c) the truncation error in (1) due to loss of precision caused by cancellation cannot be bounded anymore when n reaches round 30

d) Yes. When we calculate the result from 50, the number have less precision bits than it needs in the computer. Then we used the number with bits lost to do the calculation. The error will be accumulated during computing and then the precision would definitely lost. (Please check problem3_d.py)

Problem 4: Gaussian elimination and partial pivoting

a) Please check problem4_a.py
b) Please check problem4_b.py
c) Please check problem4_c{1,2,3}.py
    1) a ‘random' matrix
        - condition number for matrix A : 2032.7
        - residual from un-pivoted solve: 9.57203e-11
        - error from un-pivoted solve: 4.5902e-11
        - residual from partially-pivoted solve:6.66282e-11
        - error from partially-pivoted solve: 2.56879e-10
        - residual from np.linalg.solve: 1.48289e-13
        - error from np.linalg.solve: 8.26606e-13

        This is not a well conditioned matrix, Gaussian elimination with partial pivoting is more accurate than Gaussian elimination without pivoting.

    2) the matrix given by
        - condition number for matrix A : 1.02
        - un-pivoted solves failed
        - residual from partially-pivoted solve3.89423e-14
        - error from partially-pivoted solve: 7.90039e-15
        - residual from np.linalg.solve: 3.92022e-14
        - error from np.linalg.solve: 7.95229e-15

        This is a well conditioned matrix. From the result, we can find that unpivoted case would be possible to fail solving the problem. However, partially-pivoted Gaussian elimination could solve the problem somewhat well.

    3) the matrix given by
        - condition number for matrix A : 1.30228
        - residual from un-pivoted solve: 6.38225e-08
        - error from un-pivoted solve: 6.42032e-08
        - residual from partially-pivoted solve4.67187e-15
        - error from partially-pivoted solve: 4.53856e-15
        - residual from np.linalg.solve: 4.51273e-15
        - error from np.linalg.solve: 4.35569e-15

        This is a well conditioned matrix. Both Gaussian unpivoted and partially-pivoted Gaussian elimination could solve the problem. For this matrix, Gaussian elimination with partial pivoting is more accurate than Gaussian elimination without pivoting.

Problem 5: Scaling a linear system

a) Please check problem5_a.py

   relative residuals: 2.01239e-15

   relative error: 9.28687e-14

   cond(A): 2032.7

b) Please check problem5_b.py

   relative residuals: 1

   relative error: 9.28687e-14

   cond(DA): 2032.7

c) Please check problem5_c.py

   relative residuals: 49.2039

   relative error: 1.54848e-13

   cond(DA): 11523.4

d) Please check problem5_d.py

   relative residuals: 4967.92

   relative error: 8.31082e-14

   cond(DA): 633521

e) Please check problem5_e.py

   relative residuals: 3.79728e+14

   relative error: 9.83454e-14

   cond(DA): 2.56716e+31

The scaling of case in (c) gives the worst accuracy.

In this case, relative residuals is still small comparing to case (d) and (e) .

Condition number is positively correlated with relative residual but not necessarily correlated with relative error.