Machine Learning

Assignment #1 (Linear Algebra)

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Problem 1

Let two vectors a=(1 2 3)^{T}and b=(-8 1 2)^{T}answer the following equations:

(1) Compute the ℓ_2 norm of a and b

Answer：

$$||a|| = \sqrt{\sum_i^n (a_i)^2} = \sqrt{(a_1)^2 + (a_2)^2 + \cdots + (a_n)^2} = \sqrt{1+4+9} = \sqrt{14}$$

Foe the same reason

$$||b|| = \sqrt{64+1+4} = \sqrt{69}$$

(2) Calculate the Euclidean distance between a and b

Answer：

$$||a-b|| = ||b-a|| = \sqrt{\sum_i^n (b_i - a_i)^2} == \sqrt{81+1+1} = \sqrt{83}$$

(3) Are a and b orthogonal?

Answer：

Yes.

$$\because a \times b = 0 \rightarrow a \perp b$$
$$\because a \times b = (-8+2+6) = 0$$
$$\therefore a \perp b$$

Problem 2

Suppose $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$, answer the following questions:

(1) Calculate A^{-1} and $\det(A)$.

Answer:

$$
\begin{aligned}
det(A) &= \begin{vmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{vmatrix} \\
&= 1 \cdot \begin{vmatrix} -5 & 3 \\ -6 & 4 \end{vmatrix} + 3 \cdot \begin{vmatrix} 3 & 3 \\ 6 & 4 \end{vmatrix} + 3 \cdot \begin{vmatrix} 3 & -5 \\ 6 & -6 \end{vmatrix} \\
&= -20 + 18 + 3 \times (12 - 18) + 3 \times (-18 + 30) \\
&= -2 - 18 + 36 = 16
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
&\because |A| \neq 0 \\
&\therefore A^{-1} = \frac{1}{|A|} \begin{bmatrix} A_{11} & A_{21} & \dots & A_{m1} \\ A_{12} & A_{22} & \dots & A_{12} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{mn} \end{bmatrix} = \begin{bmatrix} -1/8 & -3/8 & 3/8 \\ 3/8 & -7/8 & 3/8 \\ 3/4 & -3/4 & 1/4 \end{bmatrix}
\end{aligned}
$$

(2) The Rank of A is?

Answer:

$$rank(A) = 3$$

(3) The trace of A is?

Answer:

$$tr(A) = (1 - 5 + 4) = 0$$

(4) Calculate $A + A^T$

Answer:

$$A + A^T = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 6 \\ -3 & -5 & -6 \\ 3 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 9 \\ 0 & -10 & -3 \\ 9 & -3 & 8 \end{bmatrix}$$

(5) Is A an orthogonal matrix? State your reason.

Answer:

$$
\begin{aligned}
&\because A^T A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 & 6 \\ -3 & -5 & -6 \\ 3 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 46 & -54 & 36 \\ -54 & 70 & -48 \\ 36 & -48 & 34 \end{bmatrix} \neq I \\
&\therefore A^T \text{is not an orthogonal matrix.}
\end{aligned}
$$

(6) Calculate all the eigenvalue λ and corresponding eigenvectors of A.

Answer:

$$
\begin{aligned}
&[\lambda E - A] = \begin{vmatrix} \lambda - 1 & 3 & -3 \\ -3 & \lambda + 5 & -3 \\ -6 & 6 & \lambda - 4 \end{vmatrix} = 0 \\
&(\lambda + 2)^2 (\lambda - 4) = 0 \\
&\rightarrow \lambda_{1,2} = -2, \lambda_3 = 4 \\
&a_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, a_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, a_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}
\end{aligned}
$$

(7) Diagonalize the matrix A.

Answer:

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

(8) Calculate the $\ell_{2,1}$ norm $\|A\|_{2,1}$ and the Frobenius norm (i.e. ℓ_2 norm) $\|A\|_F$

Answer:

$$\begin{aligned}\|A\|_{2,1} &= \sqrt{46}+\sqrt{70}+\sqrt{34} \\ \|A\|_F &= \sqrt{46+70+34} = \sqrt{150}\end{aligned}$$

(9) Calculate the nuclear norm $\|A\|_*$ and the spectral norm $\|A\|_2$

Answer:

$$\begin{vmatrix} \lambda-46 & 54 & -36 \\ 54 & \lambda-70 & 48 \\ -36 & 48 & \lambda-34 \end{vmatrix} = 0$$

$$\lambda^3 - 150\lambda^2 + 648\lambda - 256 = 0$$

$$\begin{aligned}\lambda_1 &= 4 \\ \lambda_2 &= 73+9\sqrt{65} \\ \lambda_3 &= 73-9\sqrt{65}\end{aligned}$$

$$\begin{aligned}\|A\|_* &\approx 14.727922061357859 \\ \|A\|_2 = \sqrt{\max\left(A^T A\right)} &\approx 12.064838156174618\end{aligned}$$

Problem 3

Please give some proper steps to show how you get the answer. Let $x = (x_1, x_2, x_3)^T$ and

$$\begin{cases} 2x_1 + 2x_2 + 3x_3 = 1 \\ x_1 - x_2 = -1 \\ -x_1 + 2x_2 + x_3 = 2 \end{cases}$$

Answer the following questions:

(1) Solve the linear equations

Answer:

$$\begin{cases} x_1 = -1 \\ x_2 = 0 \\ x_3 = 1 \end{cases}$$

(2) Write it into matrix form(i.e. $Ax = b$) and we will use the same A and b in the following questions.

Answer:

$$\begin{bmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

(3) The Rank of A is?

Answer:

$$\begin{aligned}&\because\ |A| \neq 0 \\ &\therefore\ rank(A) = 3\end{aligned}$$

(4) Calculate A^{-1} and $\det(A)$

Answer:

$$det(A) = -1$$

$$A^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 4 & -5 & -6 \\ -3 & 3 & 4 \end{bmatrix}$$

(5) Use (4) to solve the linear equations

Answer:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{det(A)} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \frac{1}{-1} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

(6) Calculate the inner product and outer product of x and b.(i.e. $\langle x, b \rangle$ and $x \otimes b$)

Answer:

$$\langle x, b \rangle = 1$$

$$x \otimes b^T = \begin{bmatrix} -1 & 1 & -2 \\ 0 & 0 & 0 \\ 1 & -1 & 2 \end{bmatrix}$$

(7) Calculate the ℓ_1, ℓ_2 and ℓ_∞ norm of b

Answer:

$$\ell_1 = 4$$
$$\ell_2 = \sqrt{6}$$
$$\ell_\infty = 2$$

(8) Suppose $y = (y_1, y_2, y_3)^T$, calculate $y^T Ay, \nabla_y y^T Ay$

Answer:

$$y^T Ay = y^T \begin{bmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{bmatrix} y = \begin{bmatrix} 2y_1 + y_2 - y_3 \\ 2y_1 - y_2 + y_3 \\ 3y_1 + y_3 \end{bmatrix} y = 2y_1^2 - y_2^2 + y_3^2 + 3y_1y_2 + 2y_1y_3 + y_2y_3$$

$$\nabla_y y^T Ay = \begin{bmatrix} 4y_1 + 3y_2 + 2y_3 \\ -2y_2 + 3y_1 + y_3 \\ 2y_3 + 2y_1 + y_2 \end{bmatrix}$$

(9) We add one linear equation $-x_1 + 2x_2 + x_3 = 2$ into linear equations above. Write it into matrix form(i.e. $A_1 x = b$)

Answer:

$$\begin{cases} 2x_1 + 2x_2 + 3x_3 = 1 \\ x_1 - x_2 = -1 \\ -x_1 + 2x_2 + x_3 = 2 \\ -x_1 + 2x_2 + x_3 = 2 \end{cases}$$

Therefore:

$$\begin{bmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

(10) The rank of A_1 is?

Answer:

$$rank(A) = 3$$

(11) Could these linear equations $A_1x = b$ be solved? State reasons.

Answer:

Yes.

$$\because\ rank(A) = 3 < 4$$
$$\therefore\ A_1x = b \text{ can be solved.}$$