Graph Theory Fall 2021

Assignment 8

Due at 5:00 pm on Friday, December 10

There's only one question, with four parts.

Let G be a graph having Laplacian matrix

$$L = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 4 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 3 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 4 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 2 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 3 \end{bmatrix}$$

Recall that the eigenvalues of L will take the form

$$0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \lambda_5 \leq \lambda_6 \leq \lambda_7 \leq \lambda_8 \leq \lambda_9 \leq \lambda_{10}$$

We will associate with each eigenvalue λ_i its corresponding eigenvector $\mathbf{v}_i$. For instance,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}; \;\; \lambda_1 = 0.$$

A. Use a matrix calculator (Wolfram Alpha is good for this) to find λ_2 and λ_3 together with eigenvectors $\mathbf{v}_2$ and $\mathbf{v}_3$. You're on the right track if your values of λ_2 and λ_3 are about 0.83 and 1.18, respectively. Also, check that the entries of $\mathbf{v}_2$ and $\mathbf{v}_3$ sum to 0 (this is a consequence of the eigenvectors corresponding to different eigenvalues of L being orthogonal.)

B. Let

$$\mathbf{x} = \frac{1}{\sqrt{\mathbf{v}_2 \cdot \mathbf{v}_2}}\, \mathbf{v}_2 \text{ and } \mathbf{y} = \frac{1}{\sqrt{\mathbf{v}_3 \cdot \mathbf{v}_3}}\, \mathbf{v}_3;$$

these vectors $\mathbf{x}$ and $\mathbf{y}$ are unit vectors.

Letting x_i and y_i be the i^{th} coordinates of $\mathbf{x}$ and $\mathbf{y}$, respectively, plot the points

$$(x_1, y_1), (x_2, y_2), \dots, (x_{10}, y_{10})$$

in $\mathbb{R}^2$ and for each edge ij of G, draw the segment joining (x_i, y_i) to (x_j, y_j).

The result should be a "nice" drawing of G in the sense that adjacent vertices are drawn close together.

C. Do the same process in part A to find λ_9 and λ_{10} along with corresponding eigenvectors $\mathbf{v}_9$ and $\mathbf{v}_{10}$. You should expect λ_9 and λ_{10} to be about 4.75 and 5.73, respectively.

D. Repeat the process in part B where

$$\mathbf{x} = \frac{1}{\sqrt{\mathbf{v}_9 \cdot \mathbf{v}_9}}\, \mathbf{v}_9 \text{ and } \mathbf{y} = \frac{1}{\sqrt{\mathbf{v}_{10} \cdot \mathbf{v}_{10}}}\, \mathbf{v}_{10}.$$

This time, the resulting drawing should have adjacent vertices drawn far apart and so vertices that are drawn close together could be assigned the same color in a graph coloring. This should give you an indication of the chromatic number of G. What is this number?