Graph Theory Fall 2021

Assignment 6

Due at 5:00 pm on Wednesday, October 27

Questions with a ($\star$) are each worth 1 bonus point for 453 students.

1. Show that for any $k \geq 3$, if a tree T has fewer than k leaves, then the maximum degree $\Delta(T)$ among the vertices of T must satisfy $\Delta(T) < k$. It can help to consider the summations

$$n = \sum_{j=1}^{\infty} n_j \; ; \qquad 2(n-1) = \text{total degree} = \sum_{j=1}^{\infty} j n_j .$$

   The phrase "T has fewer than k leaves" means $n_1 < k$.

   The two sums can be combined into the single sum

$$\sum_{j=1}^{n} (2-j) n_j = 2$$

   It suffices to show that $n_j = 0$ for all $j \geq k$.

2. Let (T, r) be a rooted tree. Recall that the level of a vertex x is $L(x) = D(r, x)$. Also, the **height** of a rooted tree H is the maximum of the levels of its vertices.
   a. Show that if r is on the unique u, v-path, then $D(u, v) = L(u) + L(v)$.
   b. Show that if $L(u) + L(v) = D(u, v)$, then r must be on the unique u, v-path.
   c. Show that for any two vertices u and v, $D(u, v) \leq 2H$.
   d. Show that if $D(u, v) = 2H$, then u and v must be non-parents. Equivalently, you can show that if either u or v is a parent, then $D(u, v) < 2H$.

3. Suppose (T, r) is a rooted q-ary tree where every parent has exactly q children; such a tree is said to be **saturated**.
    a. Show that T has bq edges for some integer b.
    b. Find a formula for the number of vertices of T in terms of b, q.
    c. Find a formula for the number of non-parents in terms of b, q.

4. Suppose (T, r) is a rooted tree with exactly 10^{12} edges. Recall that a lower bound or an upper bound on H is **tight** if there exists an example T where that bound is attained.
    a. Find tight lower and upper bounds for H, the height of T.
    b. Find tight lower and upper bounds for H if T is a saturated rooted binary tree. Recall that **saturated** means every parent has the maximum allowed number of children; here, that number is 2.