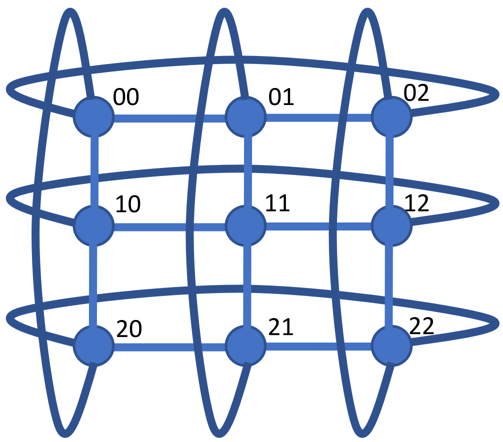
Graph Theory Fall 2021

Assignment 5

Due at 5:00 pm on Monday, October 11

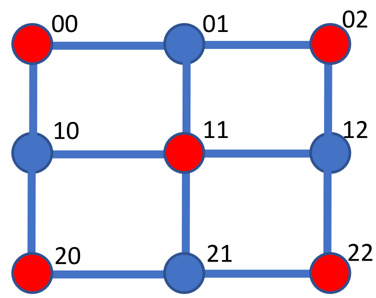
Questions with a ($\star$) are each worth 1 bonus point for 453 students.

1. The graph $G = C_3 \times C_3$ is drawn below:



   a. Show that G is Eulerian. The suggested way is to use the characterization we discussed in class. The not-so-easy way is to actually produce an Euler circuit; if you choose this latter route, it suffices to list the vertices in order as they'd be encountered in the circuit.
   b. Show that G is Hamiltonian. Here, you'll want to produce a Hamilton cycle. Again, it suffices to list the vertices in order as they'd be encountered in the cycle.

2. Now, for $n, q \geq 2$, consider the grid graph $P_{n,q} = P_n \times P_q$.
    a. Show that if either n or q is at least 3, then $P_{n,q}$ is not Eulerian. The easy way is to consider what happens along the boundary.
    b. Show that for any $q \geq 2$, $P_{2,q}$ is Hamiltonian
    c. Show that $P_{4,4}$ is Hamiltonian. The most straightforward way is to produce a Hamilton cycle.
    d. Show that $P_{3,3}$ is not Hamiltonian. The drawing below may suggest a clever approach:

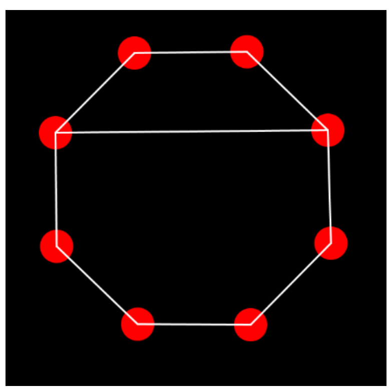


    e. If n and q are both odd, show that $P_{n,q}$ is not Hamiltonian. (Use the same clever argument from part d).
    f. (⋆) If either n or q is even, show that $P_{n,q}$ is Hamiltonian.

3. Recall that if a graph G has no links (this means every edge of G is a bridge), then there is a relationship among k, m, and n. Here, k is the number of components, m is the number of edges, and n is the number of vertices in the graph G.

   We now consider when G has links.

   a. What is the relationship among k, m, n if deleting any 1 link of G turns every other link into a bridge (Example: G is a cycle)? Consider the effect of deleting a link.
   b. Now, suppose G is a graph where you can arrive at an all-bridge graph by deleting ℓ links, one at a time, taking care not to delete any edges that become bridges in this process. What is the relationship among k, m, n now?
   c. ($\star$) Show that for any graph G, there is a unique value ℓ that works in part b. (What would happen if there were two such values ℓ_1 and ℓ_2?)
   d. What is the value of ℓ in the "theta" graph (C_8 with an extra edge) below?



   e. What is the value of ℓ for K_5, the complete graph on 5 vertices?