Graph Theory Fall 2021

Assignment 3

Due at 5:00 pm on Friday, September 17

Questions with a ($\star$) are each worth 1 bonus point for 453 students.

1. Let $n \geq 3$ be given and let C_n be the n-cycle whose vertices are $\{0,1,2,\dots,n-1\}$. Recall that ab is an edge if and only if $a = b + 1 \bmod n$ or $b = a + 1 \bmod n$.
   a. How many vertices does $C_n \times C_n$ have?
   b. Show that $C_n \times C_n$ is 4-regular. One way is to let (p, q) be any vertex of $C_n \times C_n$ and explicitly show that (p, q) is adjacent to exactly four vertices.
   c. How many edges does $C_n \times C_n$ have?

2. We discuss Cartesian products of regular graphs more generally.
   a. Let G be an a-regular graph and H be a b-regular graph. Show that $G \times H$ is an $(a + b)$-regular graph.
   b. ($\star$) Let G be an a-regular graph. Show that G^k (this is the k-fold cartesian product $G \times G \times \cdots \times G$) is a ka-regular graph. If you use mathematical induction, you may assume that G^k is isomorphic to $G \times G^{k-1}$.

3. Let Q_3 be the 3-cube with vertex set
$$V = \{000,001,010,011,100,101,110,111\}.$$
   a. Give an example of two vertices such that if we delete them both, the graph G that remains is a 6-cycle.
   b. Suppose we start with Q_3 and produce the graph H by identifying vertices 000 and 011; we call the resulting vertex w. What is the degree of w in H?
   c. ($\star$) What is the smallest number of edges we could delete from Q_3 so that the resulting graph is disconnected? Justify your answer.

4. Let G be a graph with at least one edge.
    a. Show that for any $k \geq 1$, if W is a walk in G of length k, then there exists a walk W' in G of length $k+1$. This shows that G cannot have a longest walk.
    b. Show that there is some upper bound b on the length of a path in G. This means that if W is a walk of length $k > b$, then a vertex must be repeated in W. This shows that G must have a longest path.