Water Contact Simplification

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Based on literature, force on generated foot pad can be calculated as follow:

$$F_w = C_D^* \rho \int_{-R}^{-R+2sR} \sqrt{(R^2 - z^2)}(2gh(z) + v(z)|v(z)|)dz \tag{1}$$

$$h(z) = (y_{water} - y_{BF})(1 - \frac{z+R}{2sR}) \tag{2}$$

$$v(z) = \vec{V}(z)' \cdot \vec{n} \tag{3}$$

$$s = \frac{y_{water} - y_{BF}}{y_{TF} - y_{BF}} \tag{4}$$

Equation could be more simple if position of center of foot (y_c),foot angle (θ), normal linear and angular velocity at center of foot(v_n,$\dot{\theta}$). Therefore, change of variables is:

$$(y_{TF}, y_{BF}) \Leftrightarrow (y_c, \theta) \tag{5}$$

$$y_{TF} = y_c + R\ \sin(\theta) \tag{6}$$

$$y_{BF} = y_c - R\ \sin(\theta) \tag{7}$$

now we can rewrite

$$h(z) = (y_{water} - y_c) - z\mathrm{sin}\theta \tag{8}$$

$$v(z) = v_n + \dot{\theta} z \tag{9}$$

$$s = 0.5 + \frac{y_{water} - y_c}{2R\mathrm{sin}\theta} \tag{10}$$

Upper boundary for integral can be simplified this way:

$$r_u = -R + 2sR \tag{11}$$

$$r_u = -R + 2\left(0.5 + \frac{y_{water} - y_c}{2R\sin\theta}\right)R \tag{12}$$

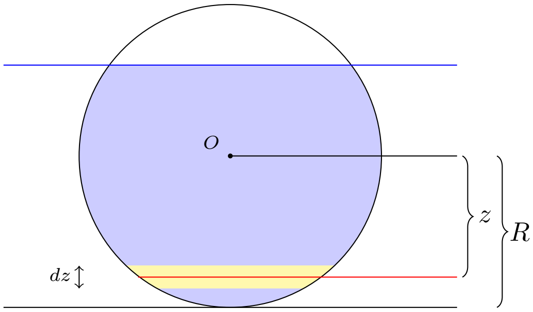
$$r_u = \frac{(y_{water} - y_c)}{\mathrm{sin}\theta} \tag{13}$$

Integral can be seen as two less complicated integral:

$$F_w = C_D^* \rho \int_{-R}^{-R+2sR} \sqrt{(R^2 - z^2)}(2gh(z) + v(z)|v(z)|)dz \tag{14}$$

$$= C_D^* \rho \int_{-R}^{-R+2sR} \sqrt{(R^2 - z^2)}(2gh(z))dz \tag{15}$$

$$+ C_D^* \rho \int_{-R}^{-R+2sR} \sqrt{(R^2 - z^2)}(v(z)|v(z)|)dz \tag{16}$$



We are going to use analytic solution to following integrals.

$$I_k(r) = \int_{-R}^{r} (az+b)\ \sqrt{R^2 - z^2} dz \tag{17}$$

$$I_b(r) = \int_{-R}^{r} (az+b)^2 \sqrt{R^2 - z^2} dz \tag{18}$$

for sake of simplicity we define following function for $-R < r < R$.

$$l(r) = \sqrt{R^2 - r^2} \tag{19}$$

$$\phi(r) = \arcsin(\frac{r}{R}) \tag{20}$$

The analytic solution to integral in (32) is as follow:

$$\int_{-R}^{r} (az+b) l(z) dz = \frac{\pi b R^2}{4} - \frac{a l(r)^3}{3} + \frac{b R^2 \phi(r)}{2} + \frac{b r l(r)}{2} \tag{21}$$

$$= \frac{bR^2}{2}\left(\phi(r) + \frac{\pi}{2}\right) + \frac{br}{2} l(r) - \frac{a}{3} l(r)^3 \tag{22}$$

$$\int_{-R}^{r} (az+b)^2 l(z) dz = \frac{\pi R^2}{16}(a^2R^2 + 4b^2) + \frac{R^2 \phi(r)}{8}(a^2R^2 + 4b^2) + \frac{r l(r)}{8}(a^2R^2 + 4b^2) - a l(r)^3 (\frac{ar}{4} + \frac{2b}{3}) \tag{23}$$

$$= \frac{1}{8}(a^2R^2 + 4b^2)\left(\left(\phi(r) + \frac{\pi}{2}\right) R^2 + r l(r)\right) - a l(r)^3 (\frac{ar}{4} + \frac{2b}{3}) \tag{24}$$

Now we are going to deal with absolute in following integral

$$\tilde{I}_b = \int_{-R}^{r} v(z)|v(z)|\ l(z) dz \tag{25}$$

luckily $v(z)$ only change once over z; let assume that it happens at r_0. we first define:

$$v(r_0) = 0 \tag{26}$$

$$v_n + \dot{\theta} r_0 = 0 \tag{27}$$

$$\Rightarrow r_0 = \frac{v_n}{\dot{\theta}}\ ,\ (\ \dot{\theta} \neq 0) \tag{28}$$

$$S(z) = \operatorname{sign}(v(z)) \tag{29}$$

$$r_0^+ = r_o + \epsilon \tag{30}$$

$$r_0^- = r_o - \epsilon \tag{31}$$

we can rewrite the integral as:

$$\begin{aligned}
\tilde{I}_b &= S(r_0^-) \int_{-R}^{r_0} v(z)^2 \; l(z)dz + S(r_0^+) \int_{r_0}^{r} v(z)^2 \; l(z)dz && (32)\\
&= S(r_0^-) \int_{-R}^{r_0} v(z)^2 \; l(z)dz + S(r_0^+) \left(\int_{-R}^{r} v(z)^2 \; l(z)dz) - \int_{-R}^{r_0} v(z)^2 \; l(z)dz \right) && (33)\\
&= \big((S(r_0^-) - S(r_0^+)\big) \int_{-R}^{r_0} v(z)^2 \; l(z)dz + S(r_0^+) \int_{-R}^{r} v(z)^2 \; l(z)dz) && (34)\\
&= \big((S(r_0^-) - S(r_0^+)\big) \, I_b(r_0) + S(r_0^+) I_b(r) && (35)
\end{aligned}$$