L3: Surfaces (II)

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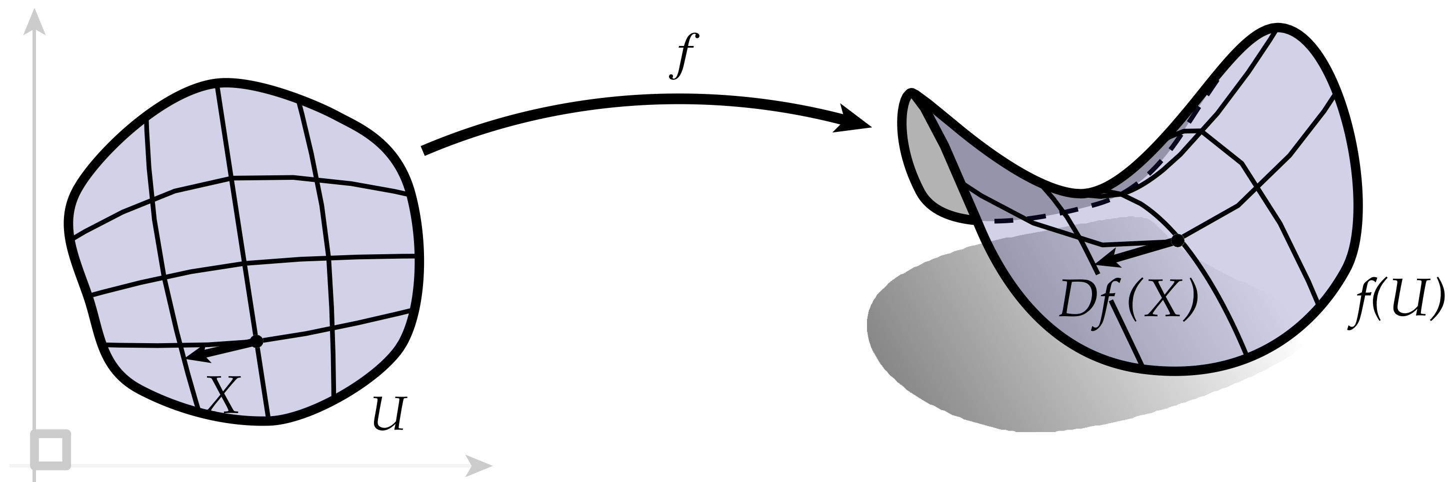
Agenda

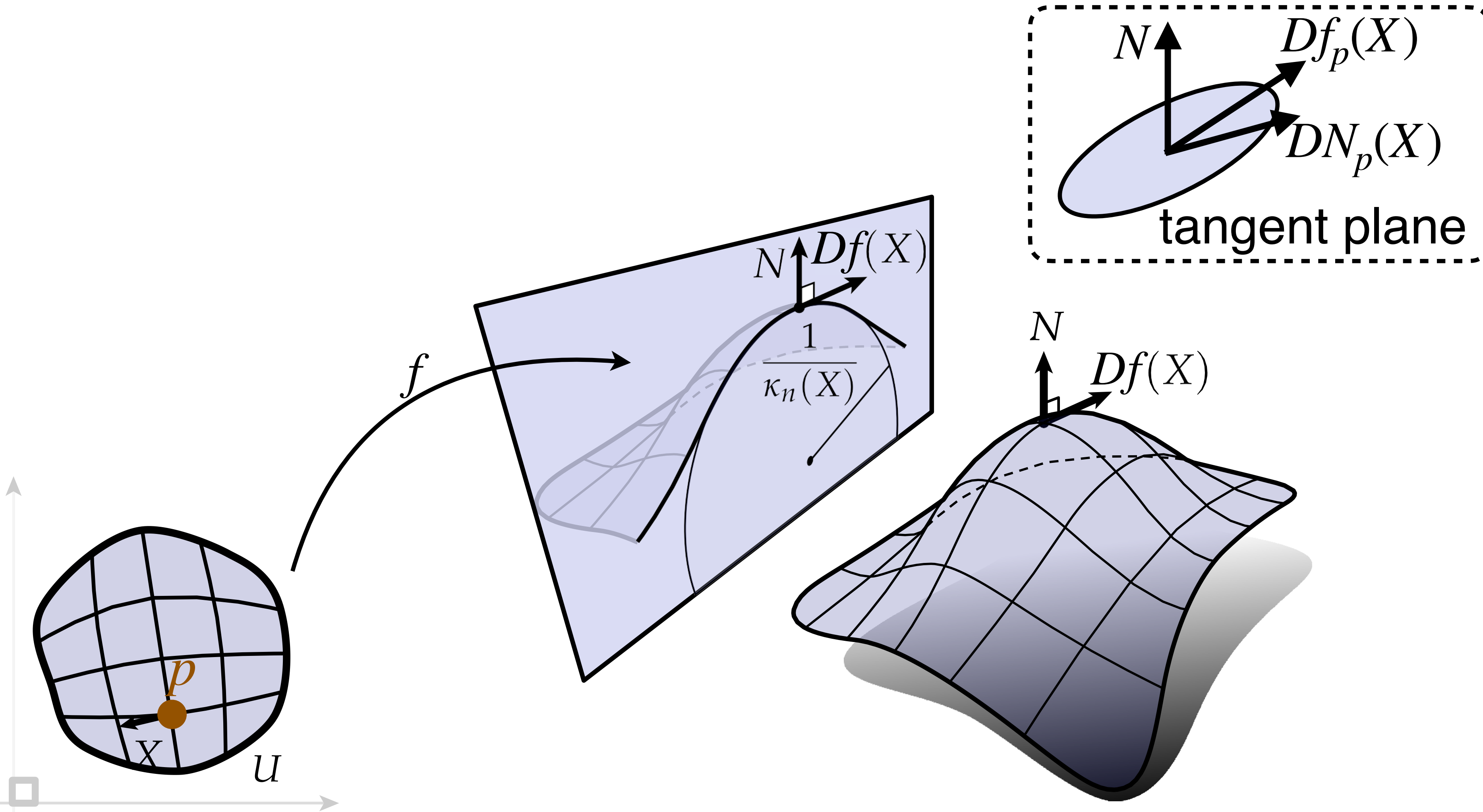
- Shape Operator
- First Fundamental Form
- Fundamental Theorem of Surfaces
- Gaussian and Mean Curvature

Warm Up (Review)

Differential Map



Directional Normal Curvature

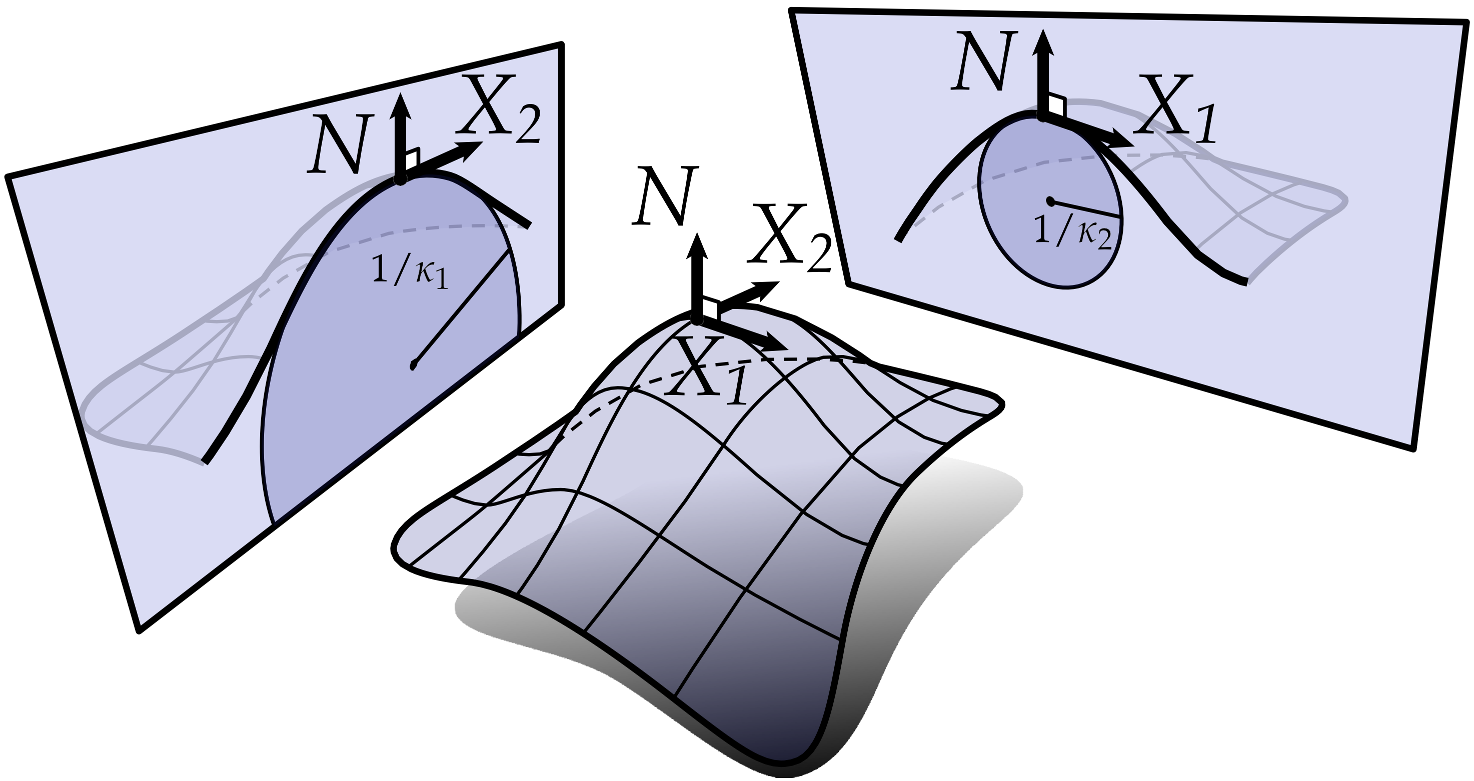


Note: κ_n is not the curvature κ of γ

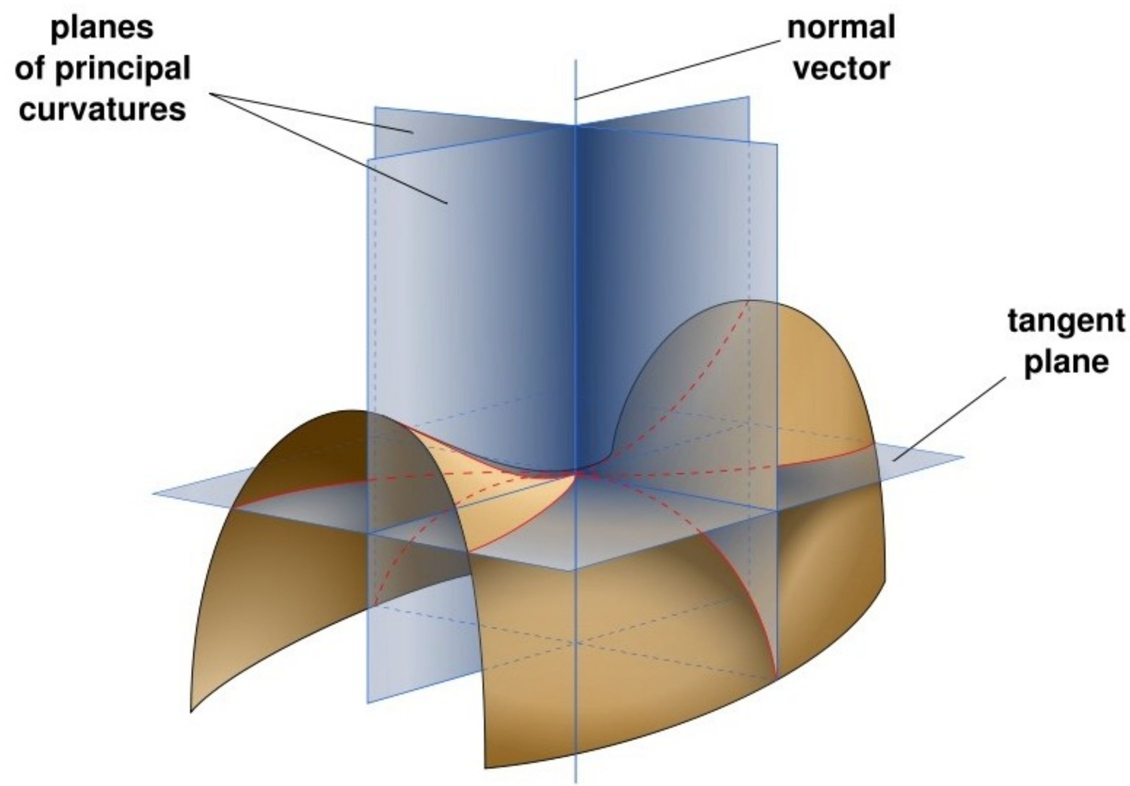
Principal Curvatures

Maximal curvature: $\kappa_1 = \kappa_{\max} = \max\limits_{\varphi} \ \kappa_n(\varphi)$

Minimal curvature: $\kappa_2 = \kappa_{\min} = \min\limits_{\varphi} \ \kappa_n(\varphi)$



Principal Directions



Euler's Theorem: Planes of principal curvature are **orthogonal** and independent of parameterization.

$$\kappa_n(\varphi) = \kappa_1 \cos^2 \varphi + \kappa_2 \sin^2 \varphi, \qquad \varphi = \text{angle with } t_1$$

Shape Operator

Shape Operator

- Note that
  - $\forall X,\ DN_p X$ is in the tangent plane
  - $\forall X,\ Df_p X$ is also in the tangent plane
- So the column space of $DN_p \in \mathbb{R}^{3\times 2}$ and $Df_p \in \mathbb{R}^{3\times 2}$ are the same
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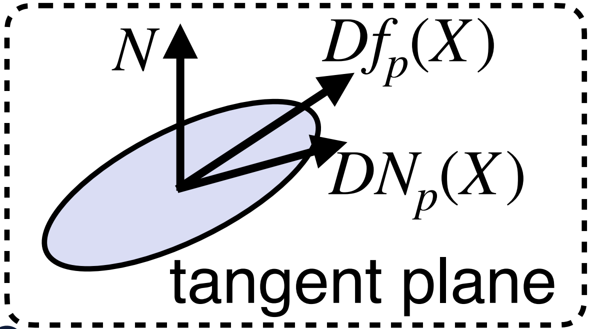

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- S is called the **shape operator**

A Linear Map That Tells Us Normal Change

$$\because \quad DN_p = Df_p S,$$

$$\therefore \quad \forall X \in \mathbf{T}_p(\mathbb{R}^2),\ \boxed{[DN_p]X} = [Df_p]SX$$

- Interpretation:
  - When p moves along X, we want to know the direction of normal change $\overrightarrow{d} \in \mathbb{R}^3$

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- This ***linear map*** S predicts the normal change when p moves along any direction!

Computation of Principal Directions

- Principal directions are the *eigenvectors* of S
- Principal curvatures are the *eigenvalues* of S
- Note: S is not a symmetric matrix! Hence, eigenvectors are not orthogonal in $\mathbb{R}^2$; only orthogonal when mapped to $\mathbb{R}^3$

Example

Consider a nonstandard parameterization of the cylinder (sheared along z):

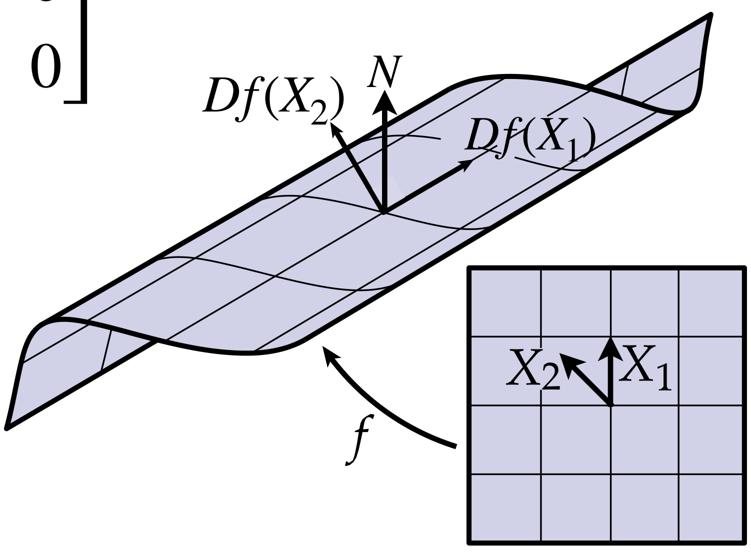
$$f(u,v) := [\cos(u), \sin(u), u+v]^T \qquad Df = \begin{bmatrix} -\sin(u) & 0 \\ \cos(u) & 0 \\ 1 & 1 \end{bmatrix}$$

$$N = \begin{bmatrix} \cos(u) \\ \sin(u) \\ 0 \end{bmatrix} \qquad DN = \begin{bmatrix} -\sin(u) & 0 \\ \cos(u) & 0 \\ 0 & 0 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\kappa_n(X_1) = 0 \qquad \kappa_n(X_2) = 1$$

$$DN_p = Df_p S \quad \Rightarrow \quad S = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$



Verify the eigens of S

Summary of Shape Operator

- A linear map between movement of point and movement of normal change

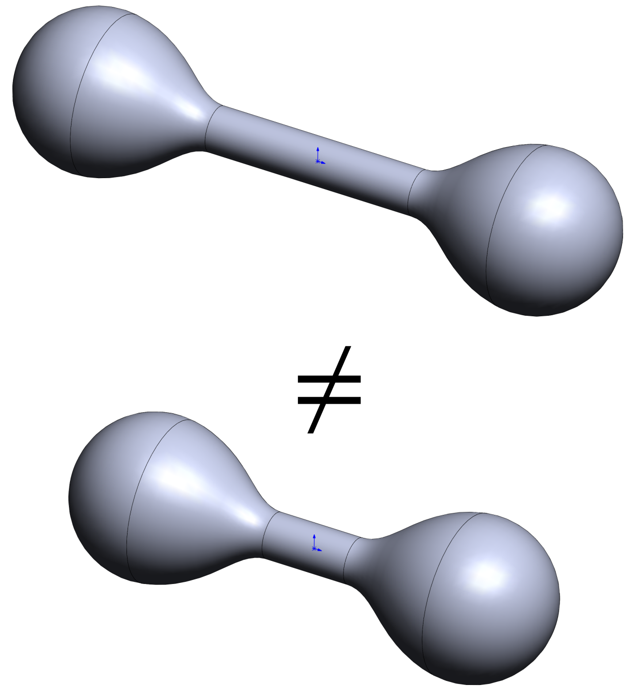
- The eigen-decomposition gives the principal curvature direction and values

First Fundamental Form

First Claim

Curvature **completely** determines *local* surface geometry.

Does Curvature Uniquely Determine Global Geometry?



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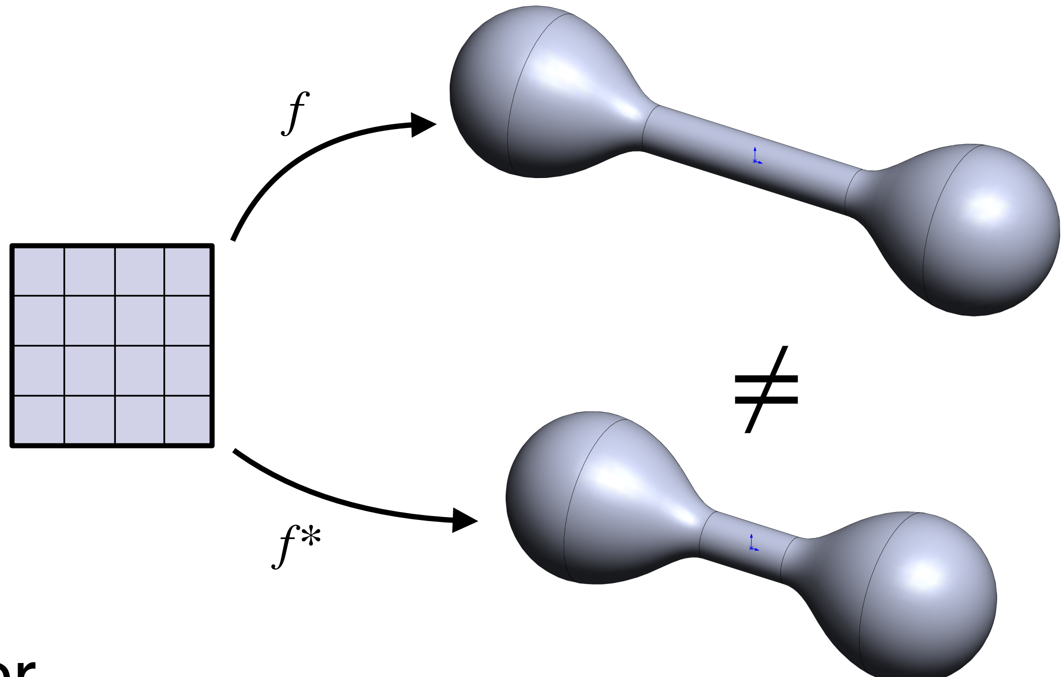
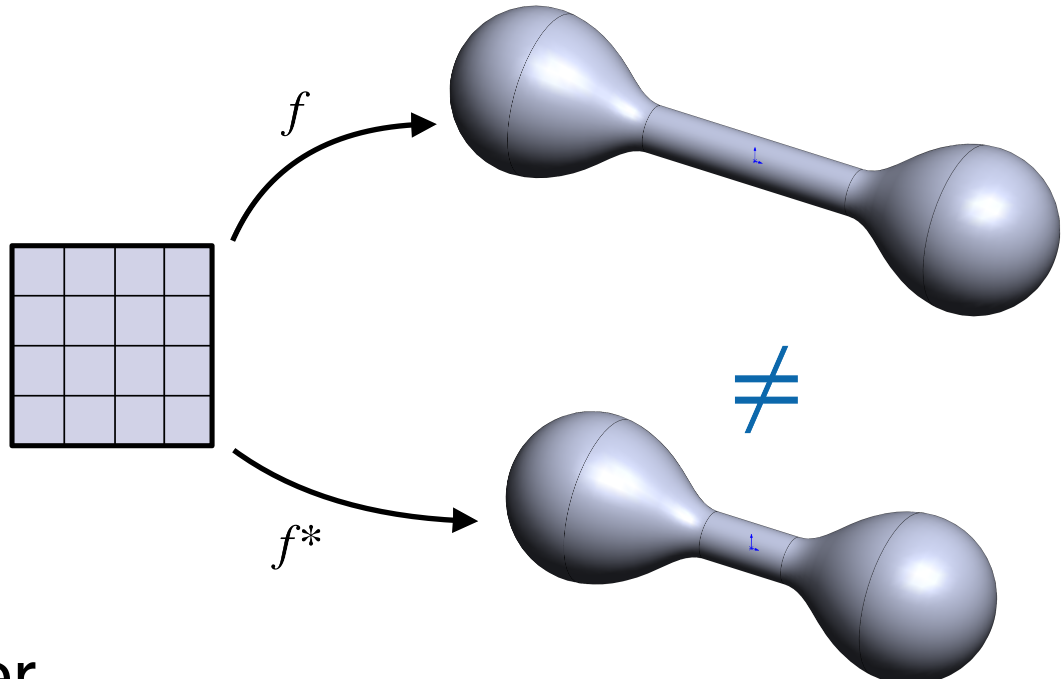


However,
$\exists f$ and f^* such that (principal) curvature value and directions are the same for any pair (f_p, f^*_p), $\forall p \in U$

Does Curvature Uniquely Determine Global Geometry?



However,

Insufficient to Uniquely Determine the Surface 😢

Other than measuring how the surface bends, we should also measure **length** and **angle!**

First Fundamental Form

- Defined as the inner product in $\mathbf{T}_p(\mathbb{R}^3)$:

$$\mathbf{I}_p(X, Y) = \langle Df_p X, Df_p Y \rangle$$

$$\Rightarrow \mathbf{I}_p(X, Y) = X^T (Df_p^T Df_p) Y$$

- $\mathbf{I}$: First fundament form, given p, we obtain a bilinear **function**

- $\mathbf{I}_p$ is dependent on both p and f

Arc-length by $\mathbf{I}(X, Y)$

- Suppose a point $p \in U$ is moving with velocity $X(t)$

$$\gamma(t) = f(p(t)) = f(p_0 + \int_0^t X(t)dt)$$

$$\Rightarrow \gamma'(t) = Df_{p(t)}[X(t)]$$

- So:

$$s(t) = \int_0^t \|\gamma'(t)\| dt = \int_0^t \sqrt{\langle Df_{p(t)}X(t), Df_{p(t)}X(t)\rangle} dt$$

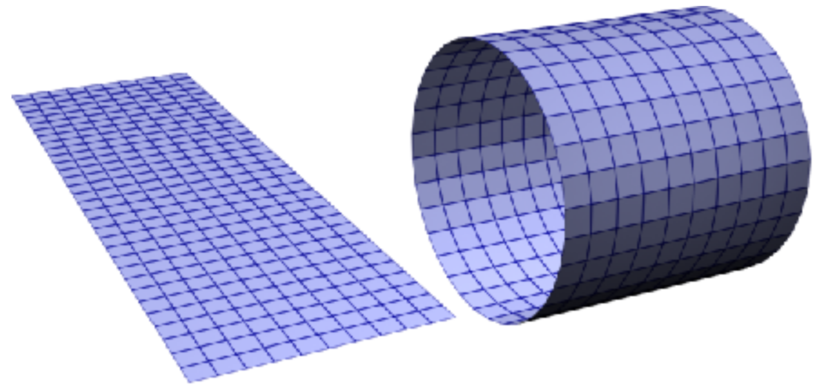
$$= \int_0^t \sqrt{\mathbf{I}_{p(t)}(X(t), X(t))} dt$$

Arc-length by $\mathbf{I}(X, Y)$

$$s(t) = \int_0^t \sqrt{\mathbf{I}_{p(t)}(X(t), X(t))}\ dt$$

With $\mathbf{I}$, we have completely determined curve length within the surface without referring to f

Local Isometric Surfaces



For two surfaces M and M^*,

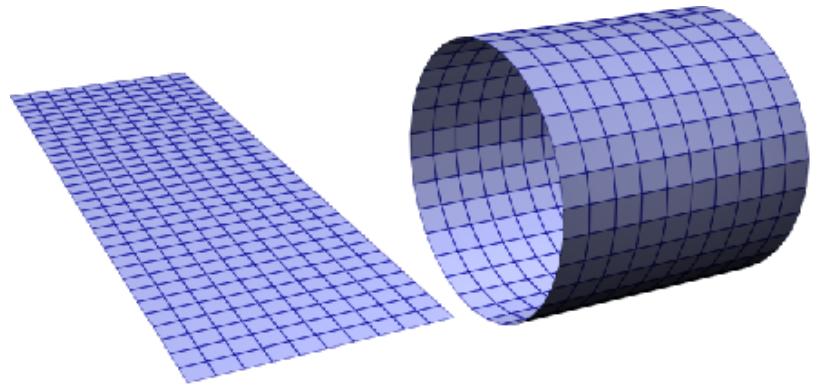
- If there exists parameterizations $f(U) = M$ and $f^*(U) = M^*$
- such that $\mathbf{I}_p = \mathbf{I}_p^*, \forall p \in U$
- Then the two surfaces are locally isometric

Preserve length between corresponding curves!

Reading: P39 of TS

Figure credit: Kenneth Lloyd Patrick Rose

Local Isometric Surfaces



Verify by yourself:

$$f(u, v) = [u, v, 0]^T, \;\; f^*(u, v) = [\cos u, \sin u, v]^T$$

on $U = \{(u, v) : u \in (0, 2\pi), v \in (0, 1)\}$

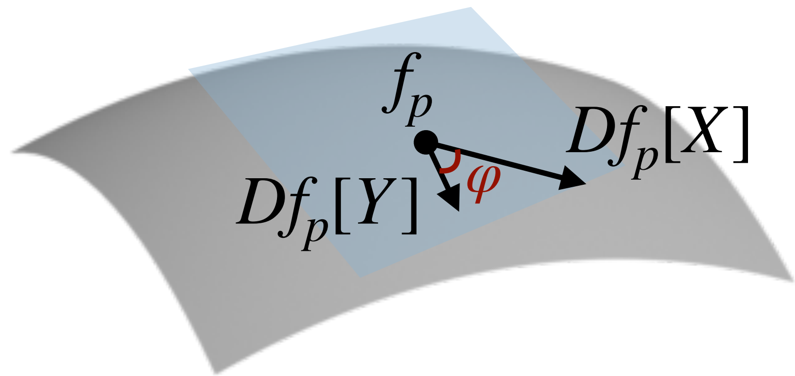
Reading: P39 of TS

Figure credit: Kenneth Lloyd Patrick Rose

Angle of Curves by $\mathbf{I}(X, Y)$

- Suppose a point $p \in U$ is moving with velocity X at time t_0

- Its tangent is $Df_p[X]$



- Given a vector (e.g., maximal principal direction) $Df_p[Y] \in \mathbf{T}_{f_p}(\mathbb{R}^3)$

- The angle φ between the tangent and the vector is:

$$\cos\varphi = \langle \frac{Df_pX}{\|Df_pX\|}, \frac{Df_pY}{\|Df_pY\|} \rangle = \frac{\mathbf{I}(X, Y)}{\sqrt{\mathbf{I}(X, X)}\sqrt{\mathbf{I}(Y, Y)}}$$

Angle of Curves by $\mathbf{I}(X, Y)$

$$\cos\varphi = \frac{\mathbf{I}(X, Y)}{\sqrt{\mathbf{I}(X, X)}\sqrt{\mathbf{I}(Y, Y)}}$$

With $\mathbf{I}$, we have completely determined angles within the surface without referring to f

Summary of First Fundamental Form

- Is a bilinear function over movement directions (velocities) in the tangent space of $\mathbf{T}_p(\mathbb{R}^2)$
- Induced by the inner product in the tangent space at surface point $f(p)$
- Completely determines curve lengths and angles within the surface

Fundamental Theorem of Surfaces

First and Second Fundamental Forms

- First fundamental form (angle and length):

$$\mathbf{I}(X,Y) = \langle Df_p X, Df_p Y \rangle$$

- Second fundamental form (bending):

$$\mathbf{II}(X,Y) = \langle DN_p X, Df_p Y \rangle$$

- Recall the definition of normal curvature:

$$\kappa_n(X) := \frac{\langle DN_p X, Df_p X \rangle}{\langle Df_p X, Df_p X \rangle} = \frac{\mathbf{II}(\mathbf{X},\mathbf{Y})}{\mathbf{I}(\mathbf{X},\mathbf{Y})}$$

Uniqueness Result

Theorem:

A smooth surface is determined up to rigid motion by its first and second fundamental forms.

Note: compatible first and second fundamental forms have to satisfy the Gauss-Codazzi condition (just FYI)

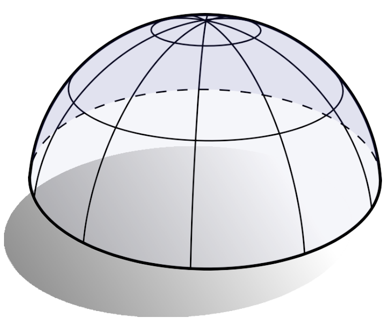
Gaussian and Mean Curvature

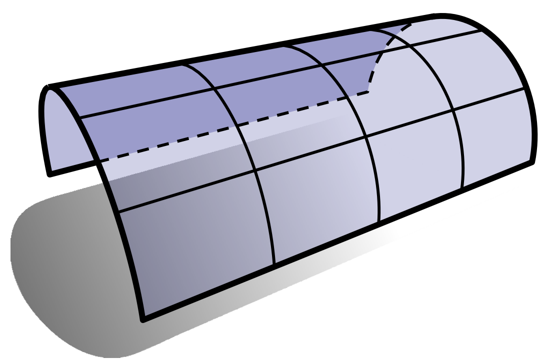
Gaussian and Mean Curvature

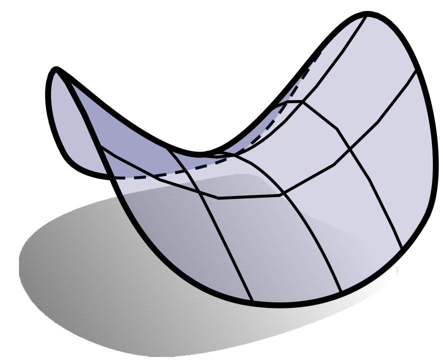
- Gaussian and mean curvature also fully describe local bending:

$$\text{Gaussian:} \quad K := \kappa_1 \kappa_2$$

$$\text{mean:} \quad H := \frac{1}{2}(\kappa_1 + \kappa_2)$$







$K > 0$

$H \neq 0$

"developable" $\boxed{K = 0}$

$H \neq 0$

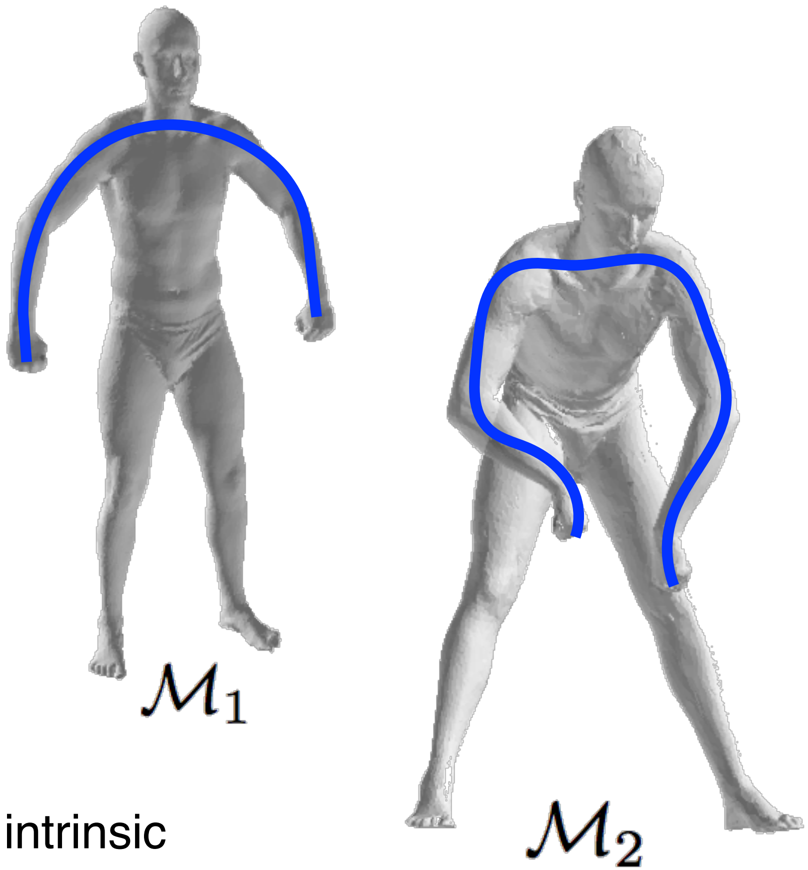
$K < 0$

"minimal" $\boxed{H = 0}$

Gauss's Theorema Egregium

The Gaussian curvature of an embedded smooth surface in $\mathbb{R}^3$ is invariant under the local isometries.

Isometric Invariance



geodesic = intrinsic

isometry = length-preserving transform