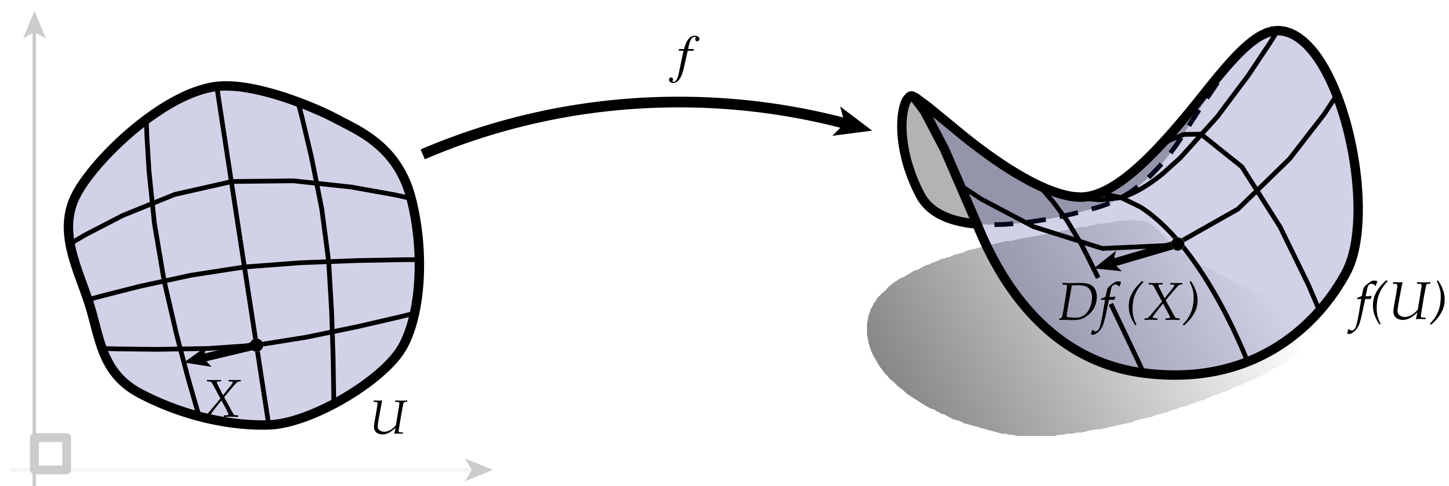
L3: Surfaces (II)

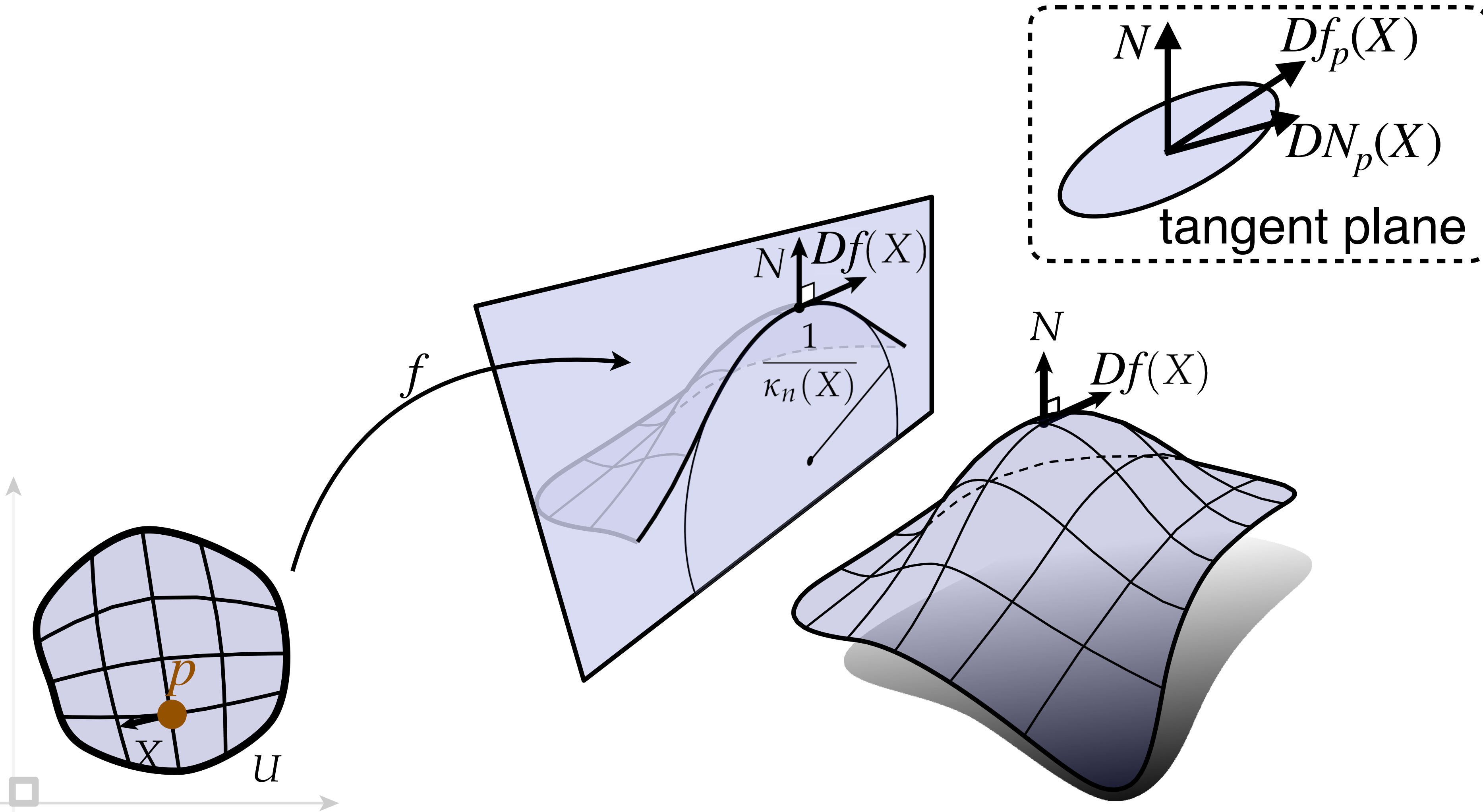
Hao Su

Warm Up (Review)

Differential Map



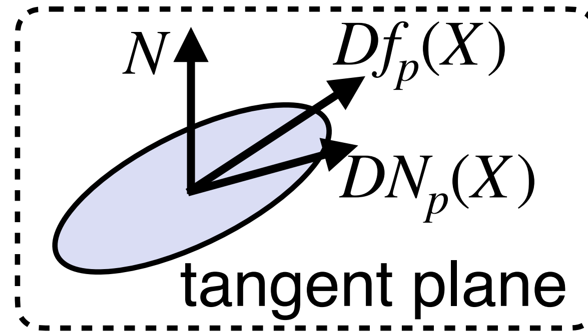
Directional Normal Curvature

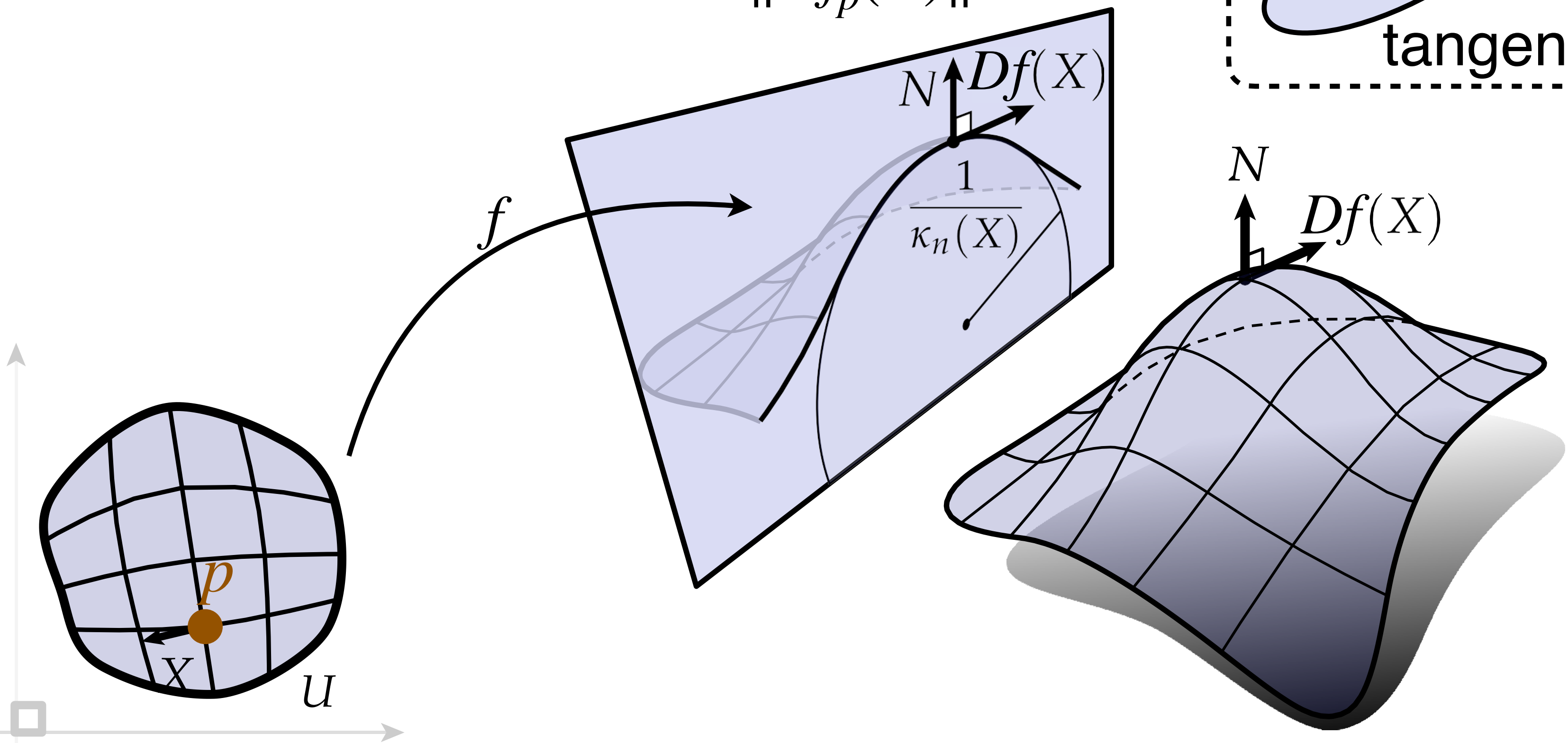


Note: κ_n is not the curvature κ of γ

Directional Normal Curvature

$$\kappa_n(X) := \langle \mathbf{T}, \vec{\kappa} \rangle = \frac{\langle Df_p(X), DN_p(X) \rangle}{\|Df_p(X)\|^2}$$



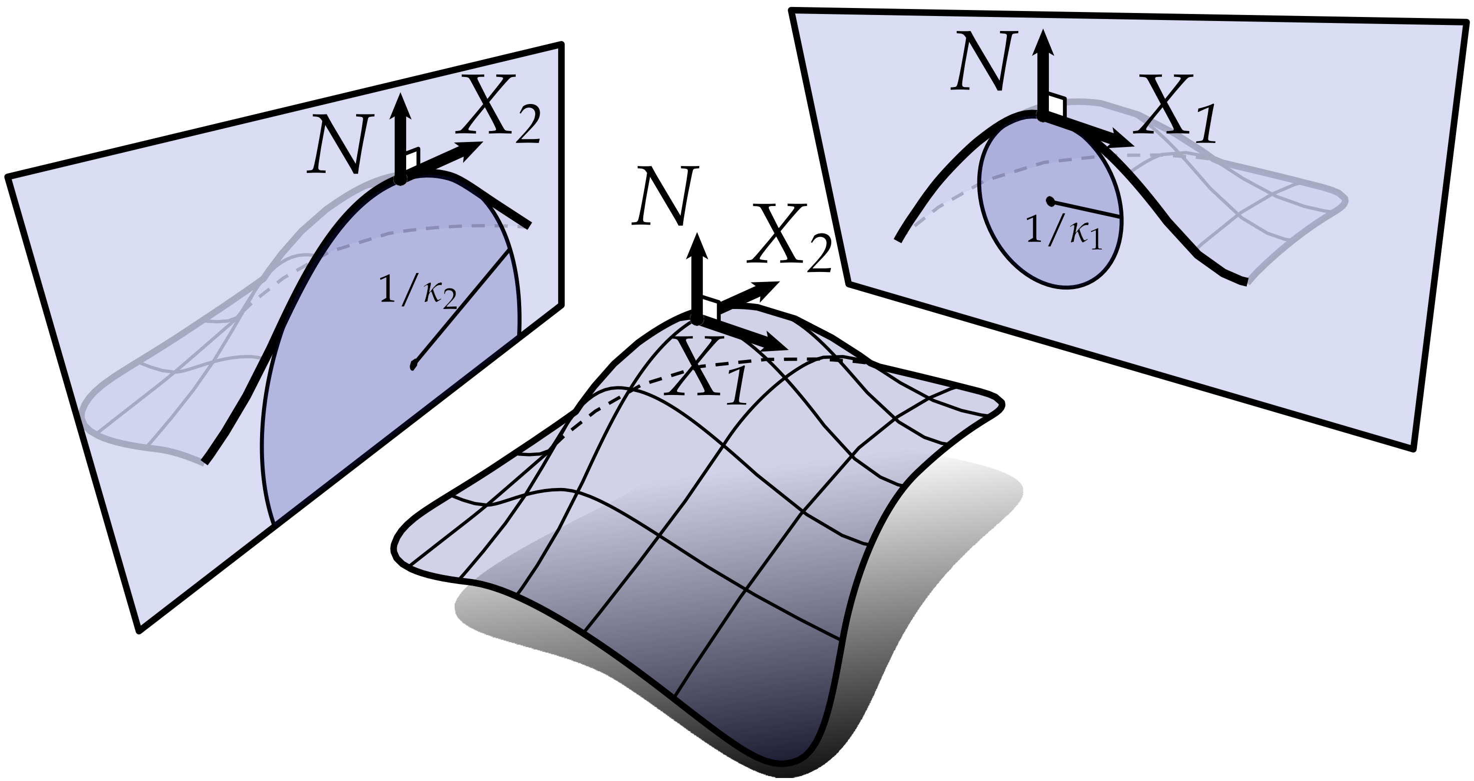


Note: κ_n is not the curvature κ of γ

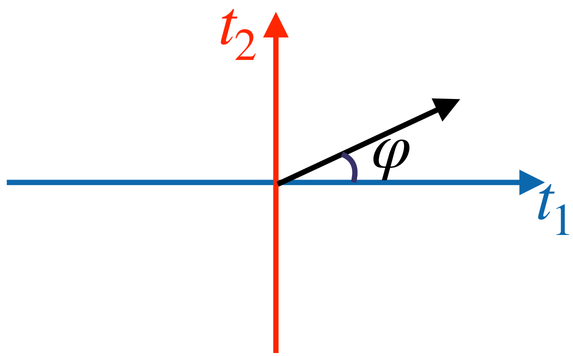
Principal Curvatures

Maximal curvature: $\kappa_1 = \kappa_{\max} = \max_{\varphi} \ \kappa_n(\varphi)$

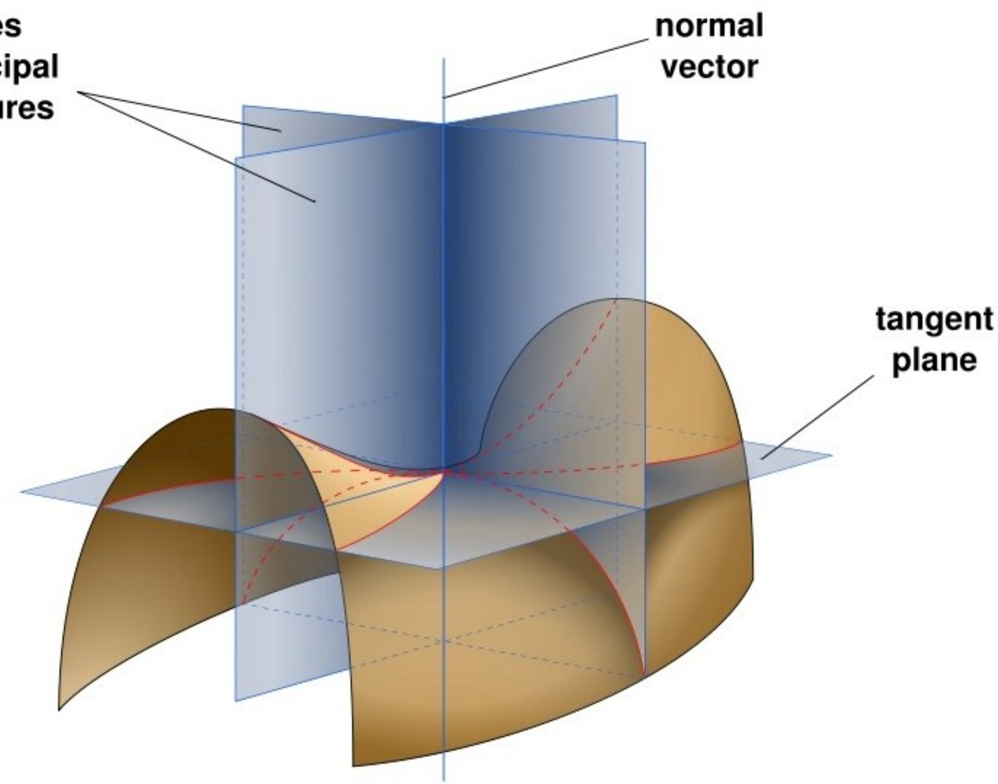
Minimal curvature: $\kappa_2 = \kappa_{\min} = \min_{\varphi} \ \kappa_n(\varphi)$



Principal Directions



Tangent plane with principal directions (t_1 and t_2) as axes



Euler's Theorem: Planes of principal curvature are orthogonal and independent of parameterization.

$$\kappa_n(\varphi) = \kappa_1 \cos^2 \varphi + \kappa_2 \sin^2 \varphi, \qquad \varphi = \text{angle with } t_1$$

Agenda

- Shape Operator
- First Fundamental Form
- Fundamental Theorem of Surfaces
- Gaussian and Mean Curvature

Shape Operator

Shape Operator

- Note that
  - $\forall X$, $DN_p X$ is in the tangent plane
  - $\forall X$, $Df_p X$ is also in the tangent plane
- So the column space of $DN_p \in \mathbb{R}^{3\times 2}$ and $Df_p \in \mathbb{R}^{3\times 2}$ are the same
- In other words,

Shape Operator

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- In other words, $\exists S \in \mathbb{R}^{2\times 2}$ such that $DN_p = Df_p S$

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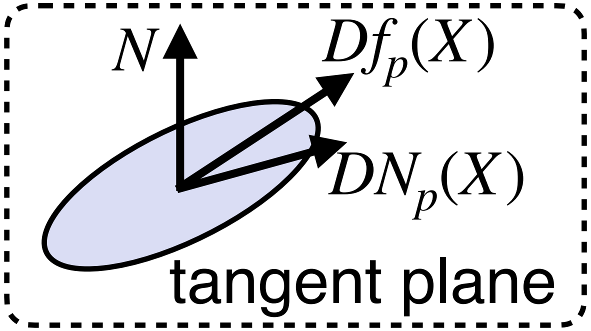

- So the column space of $DN_p \in \mathbb{R}^{3\times 2}$ is a subspace of the column space of $Df_p \in \mathbb{R}^{3\times 2}$

- In other words, $\exists S \in \mathbb{R}^{2\times 2}$ such that $DN_p = Df_p S$

- S is called the **shape operator**

A Linear Map That Tells Us Normal Change

$$\because \qquad DN_p = Df_p S,$$

$$\therefore \qquad \forall X \in \mathbf{T}_p(\mathbb{R}^2),\ \boxed{[DN_p]X} = [Df_p]SX$$

- Interpretation:
  - When p moves along X, we want to know the direction of normal change $\overrightarrow{d} \in \mathbb{R}^3$

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- This ***linear map*** S predicts the normal change when p moves along any direction!

Computation of Principal Directions

- Principal directions are the *eigenvectors* of S

- Principal curvatures are the *eigenvalues* of S

- Note: S is not a symmetric matrix! Hence, eigenvectors are not orthogonal in $\mathbb{R}^2$; only orthogonal when mapped to $\mathbb{R}^3$

Example

Consider a nonstandard parameterization of the cylinder (sheared along z):

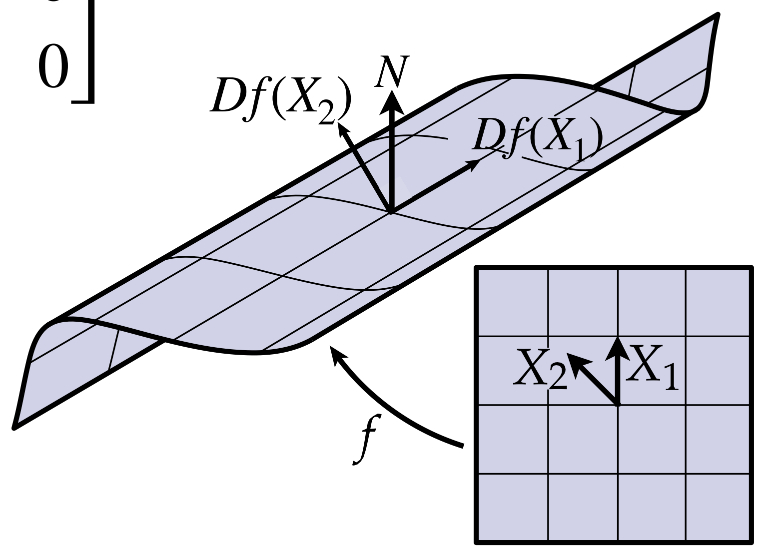
$$f(u,v) := [\cos(u), \sin(u), u+v]^T \qquad Df = \begin{bmatrix} -\sin(u) & 0 \\ \cos(u) & 0 \\ 1 & 1 \end{bmatrix}$$

$$N = \begin{bmatrix} \cos(u) \\ \sin(u) \\ 0 \end{bmatrix} \qquad DN = \begin{bmatrix} -\sin(u) & 0 \\ \cos(u) & 0 \\ 0 & 0 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\kappa_n(X_1) = 0 \qquad \kappa_n(X_2) = 1$$

$$DN_p = Df_p S \quad \Rightarrow \quad S = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$



Verify the eigens of S

Summary of Shape Operator

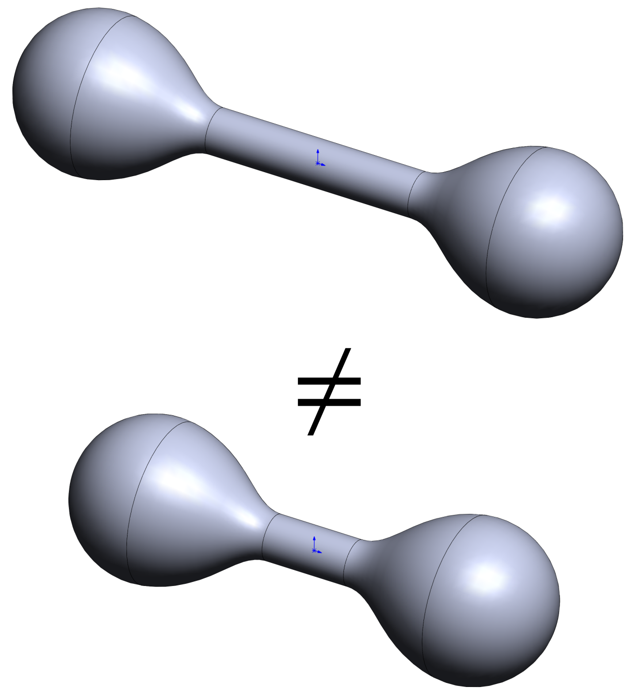
- A linear map between movement of point and movement of normal change
- The eigen-decomposition gives the principal curvature direction and values

First Fundamental Form

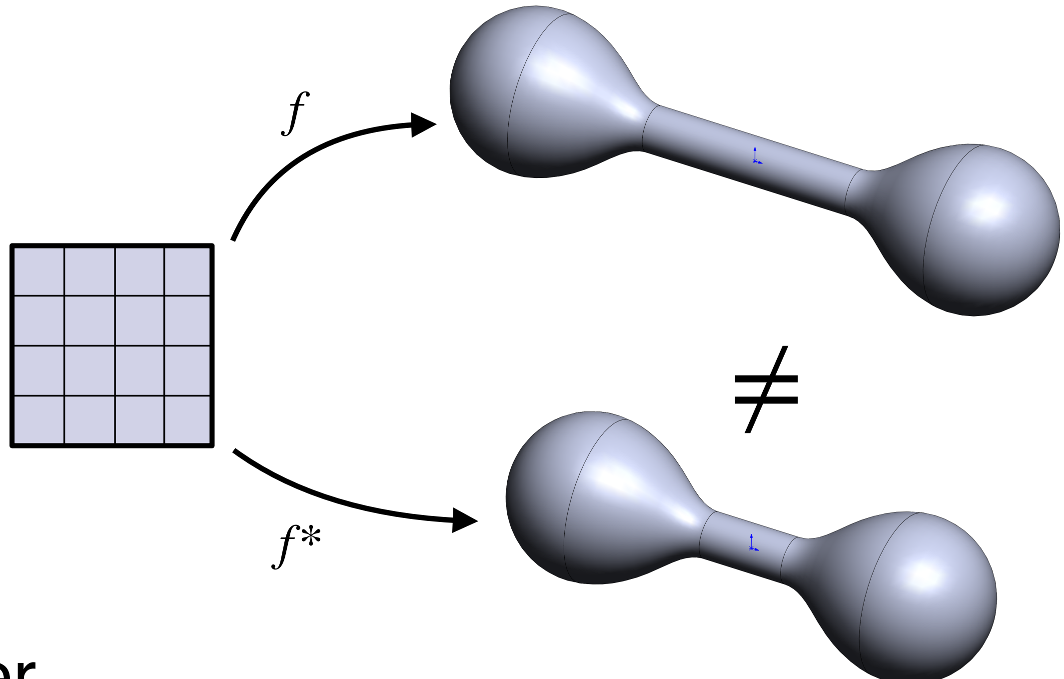
First Claim

Curvature
completely determines
local surface geometry.

Does Curvature Uniquely Determine Global Geometry?



Does Curvature Uniquely Determine Global Geometry?

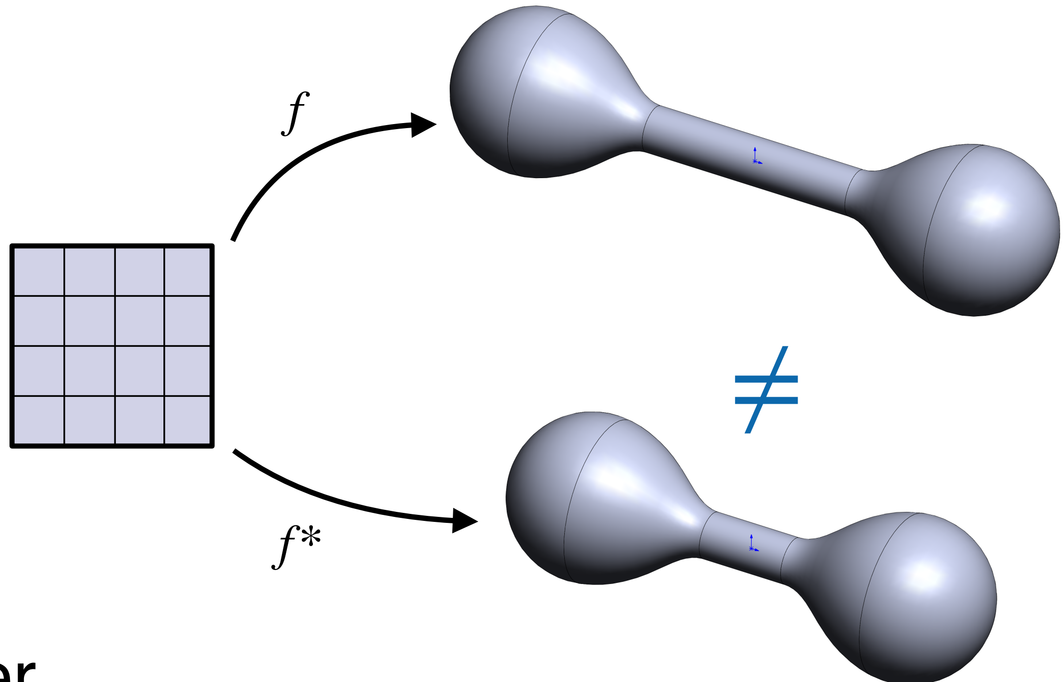


However,

$\exists f$ and f^* such that:

(principal) curvature value and directions are the same for any pair $(f(p), f^*(p))$, $\forall p \in U$

Does Curvature Uniquely Determine Global Geometry?



However,

Curvature is Insufficient to Determine Surface Globally

Other than measuring how the surface bends, we should also measure **length** and **angle!**

First Fundamental Form

- Defined as the inner product in $\mathbf{T}_p(\mathbb{R}^3)$:

$$\mathbf{I}_p(X, Y) = \langle Df_p X, Df_p Y \rangle$$

$$\Rightarrow \mathbf{I}_p(X, Y) = X^T (Df_p^T Df_p) Y$$

- $\mathbf{I}$: First fundament form, given p, we obtain a bilinear **function**

- $\mathbf{I}_p$ is dependent on both p and f

Arc-length by $\mathbf{I}(X, Y)$

- Suppose a point $p \in U$ is moving with velocity $X(t)$

$$\gamma(t) = f(p(t)) = f(p_0 + \int_0^t X(t)dt)$$

$$\Rightarrow \gamma'(t) = Df_{p(t)}[X(t)]$$

- So:

$$s(t) = \int_0^t \|\gamma'(t)\| dt = \int_0^t \sqrt{\langle Df_{p(t)}X(t), Df_{p(t)}X(t)\rangle} dt$$

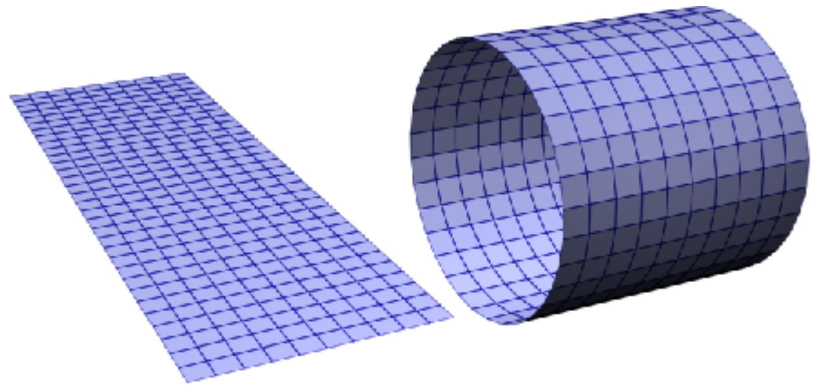
$$= \int_0^t \sqrt{\mathbf{I}_{p(t)}(X(t), X(t))} dt$$

Arc-length by $\mathbf{I}(X, Y)$

$$s(t) = \int_0^t \sqrt{\mathbf{I}_{p(t)}(X(t), X(t))}\ dt$$

With $\mathbf{I}$, we have completely determined curve length within the surface without referring to f

Local Isometric Surfaces



For two surfaces M and M^*,

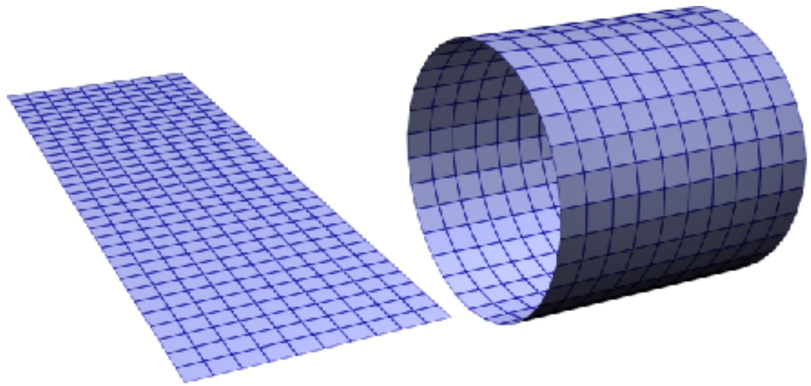
- If there exists parameterizations $f(U) = M$ and $f^*(U) = M^*$
- such that $\mathbf{I}_p = \mathbf{I}_p^*, \forall p \in U$
- Then the two surfaces are locally isometric

Preserve length between corresponding curves!

Reading: P39 of TS

Figure credit: Kenneth Lloyd Patrick Rose

Local Isometric Surfaces



Verify by yourself:

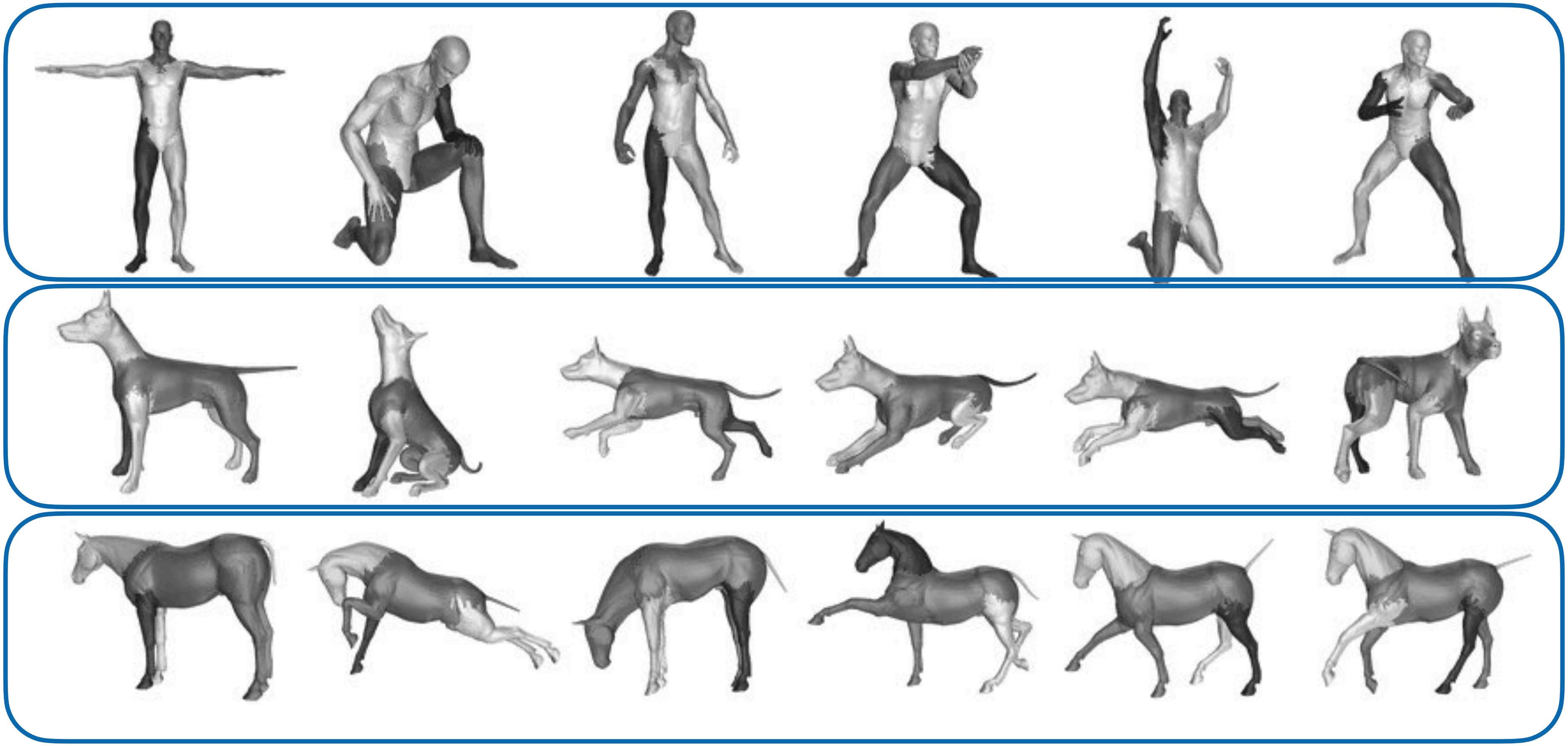
$$f(u,v) = [u, v, 0]^T, \;\; f^*(u,v) = [\cos u, \sin u, v]^T$$

on $U = \{(u,v) : u \in (0,2\pi), v \in (0,1)\}$

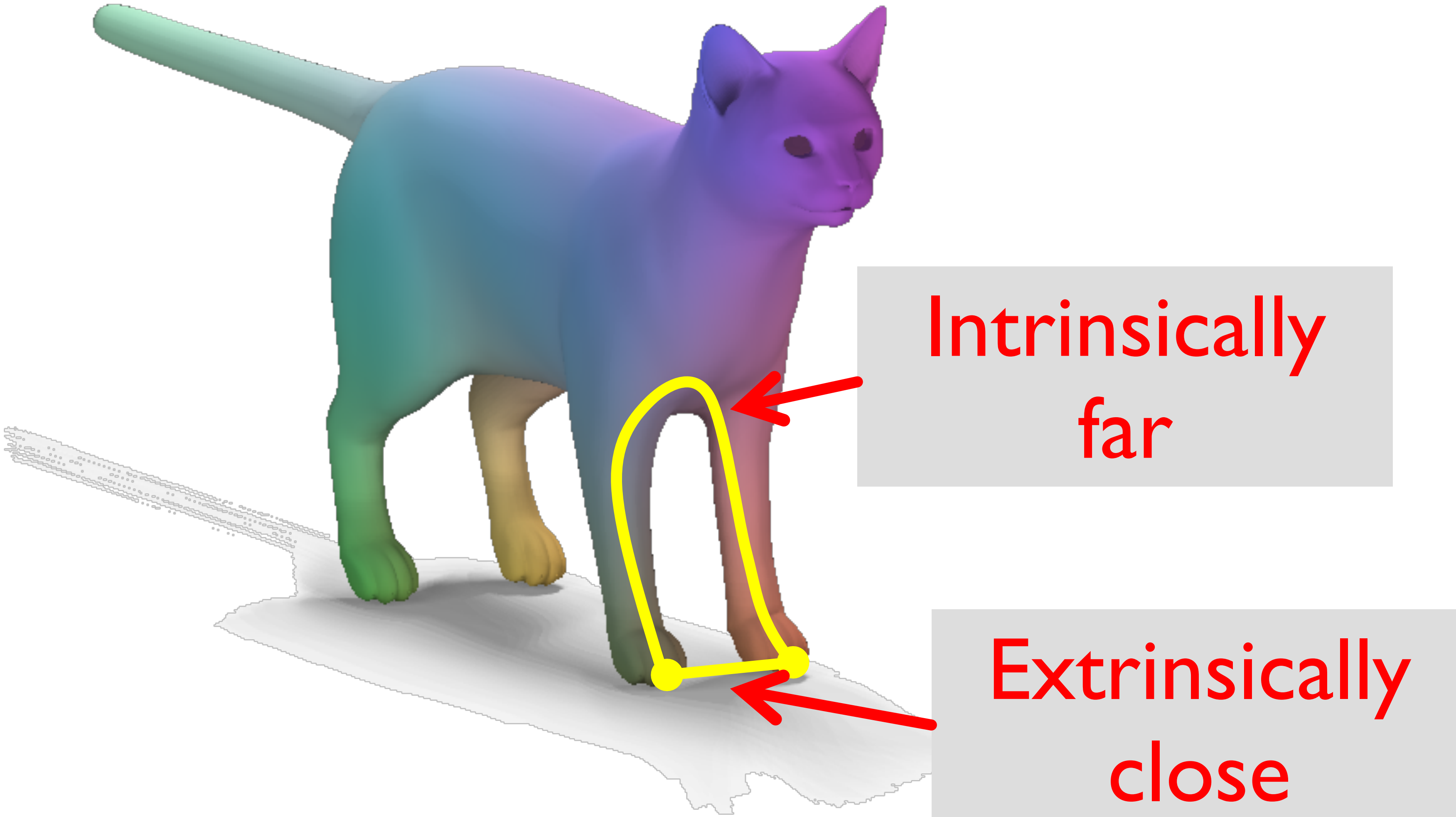
Reading: P39 of TS

Figure credit: Kenneth Lloyd Patrick Rose

Shape Classification by Isometry

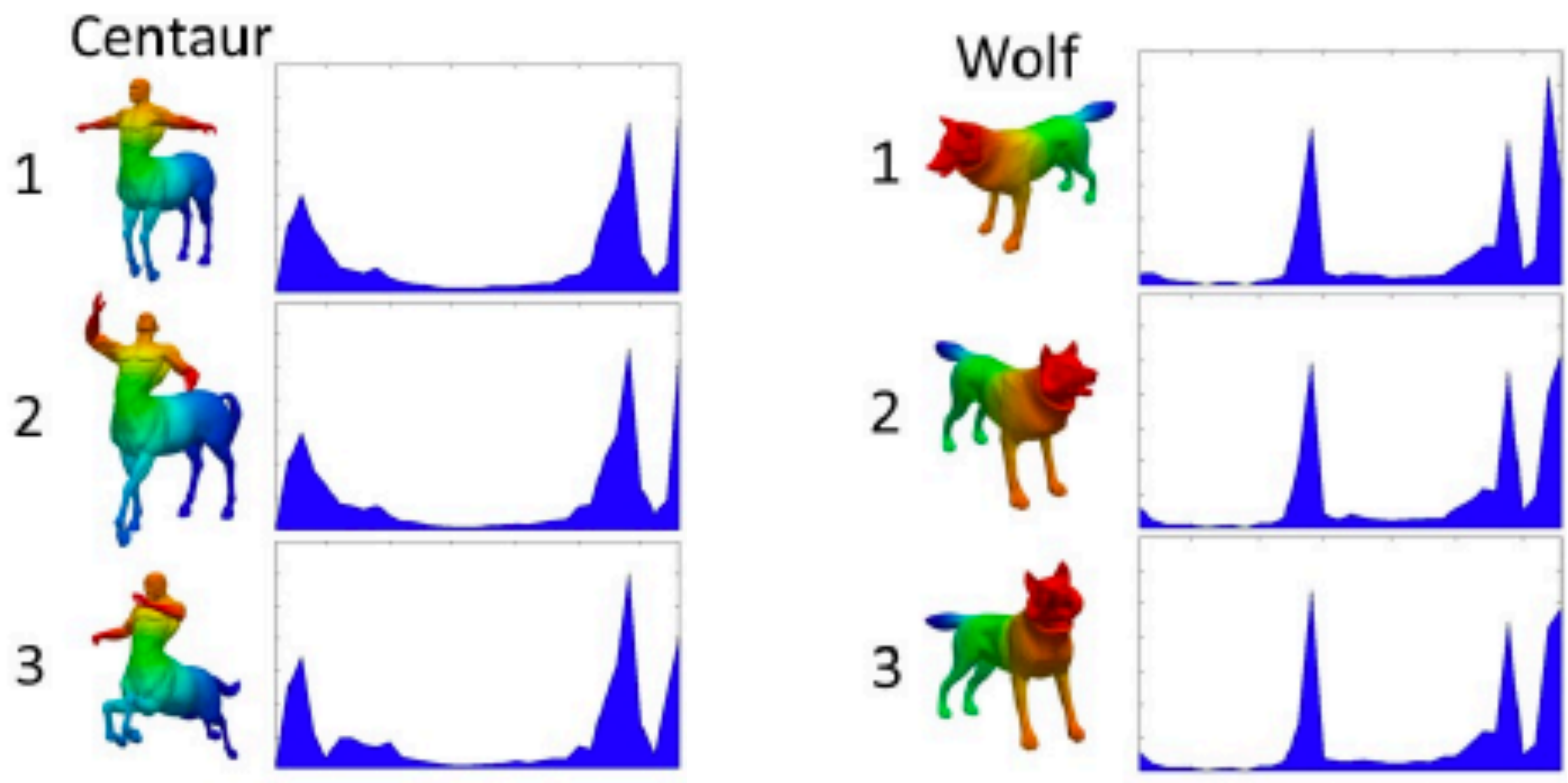


Geodesic Distances



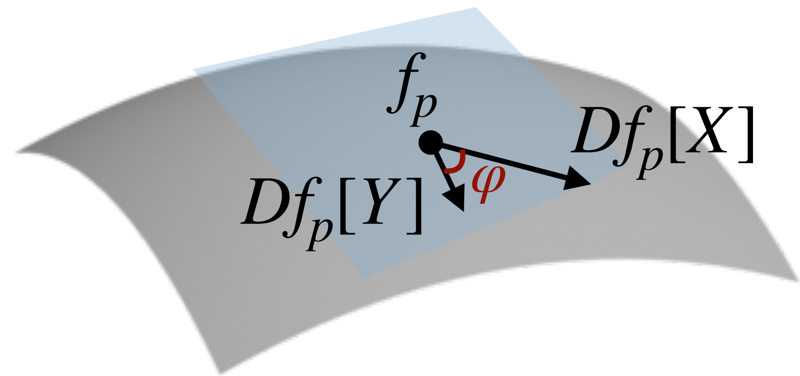
Distance Distribution Descriptor

- Compute distribution of distances for point pairs randomly picked on the surface



Angle of Curves by $\mathbf{I}(X, Y)$

- Given a vector (e.g., maximal principal direction) $Df_p[Y] \in \mathbf{T}_{f_p}(\mathbb{R}^3)$



- The angle φ between the tangent and the vector is:

$$\cos\varphi = \langle \frac{Df_pX}{\|Df_pX\|}, \frac{Df_pY}{\|Df_pY\|} \rangle = \frac{\mathbf{I}(X, Y)}{\sqrt{\mathbf{I}(X, X)}\sqrt{\mathbf{I}(Y, Y)}}$$

Angle of Curves by $\mathbf{I}(X, Y)$

$$\cos\varphi = \frac{\mathbf{I}(X, Y)}{\sqrt{\mathbf{I}(X, X)}\sqrt{\mathbf{I}(Y, Y)}}$$

With $\mathbf{I}$, we have completely determined angles within the surface without referring to f

Summary of First Fundamental Form

- Is a bilinear function over movement directions (velocities) in the tangent space of $\mathbf{T}_p(\mathbb{R}^2)$
- Induced by the inner product in the tangent space at surface point $f(p)$
- Completely determines curve lengths and angles within the surface

Fundamental Theorem of Surfaces

First and Second Fundamental Forms

- First fundamental form (angle and length):

$$\mathbf{I}(X, Y) = \langle Df_p X, Df_p Y \rangle$$

- Second fundamental form (bending):

$$\mathbf{II}(X, Y) = \langle DN_p X, Df_p Y \rangle$$

- Recall the definition of normal curvature:

$$\kappa_n(X) := \frac{\langle DN_p X, Df_p X \rangle}{\langle Df_p X, Df_p X \rangle} = \frac{\mathbf{II}(\mathbf{X}, \mathbf{Y})}{\mathbf{I}(\mathbf{X}, \mathbf{Y})}$$

Uniqueness Result

Theorem:

A smooth surface is determined up to rigid motion by its first and second fundamental forms.

Note: compatible first and second fundamental forms have to satisfy the Gauss-Codazzi condition (just FYI)

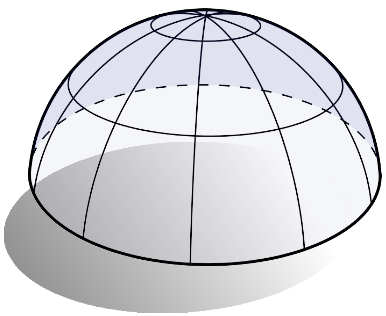
Gaussian and Mean Curvature

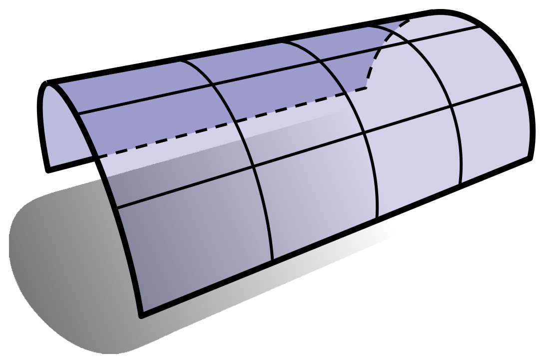
Gaussian and Mean Curvature

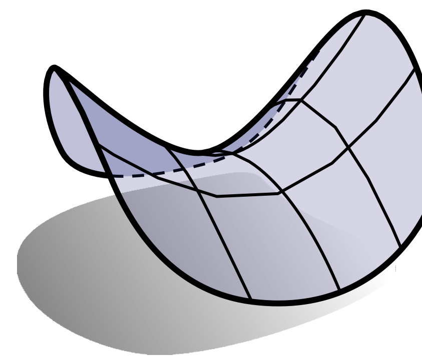
- Gaussian and mean curvature also fully describe local bending:

$$\text{Gaussian:} \quad K := \kappa_1 \kappa_2$$

$$\text{mean:} \quad H := \frac{1}{2}(\kappa_1 + \kappa_2)$$







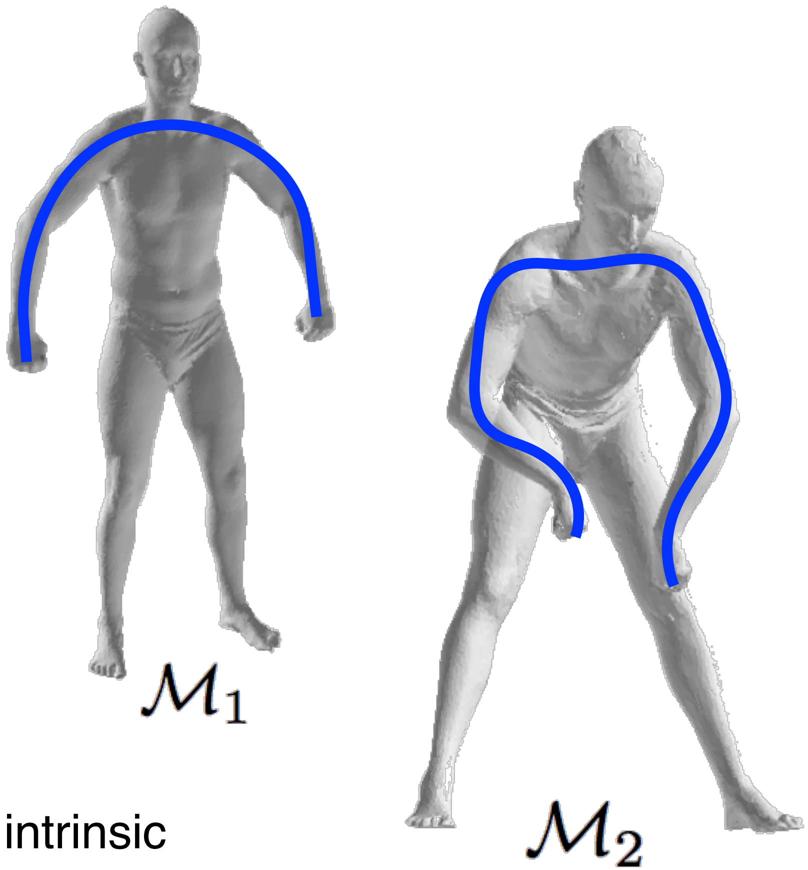
$K > 0$ | *"developable"* $\boxed{K = 0}$ | $K < 0$

$H \neq 0$ | $H \neq 0$ | *"minimal"* $\boxed{H = 0}$

Gauss's Theorema Egregium

The Gaussian curvature of an embedded smooth surface in $\mathbb{R}^3$ is invariant under the local isometries.

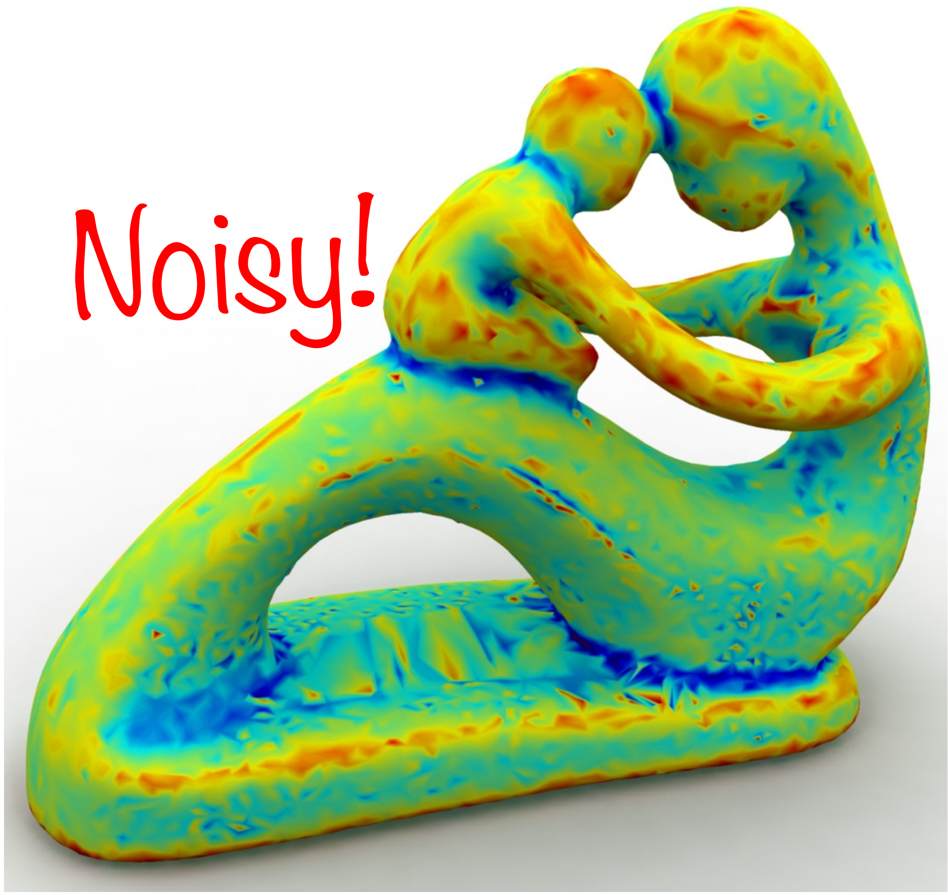
Isometric Invariance



geodesic = intrinsic

isometry = length-preserving transform

End of the Story?

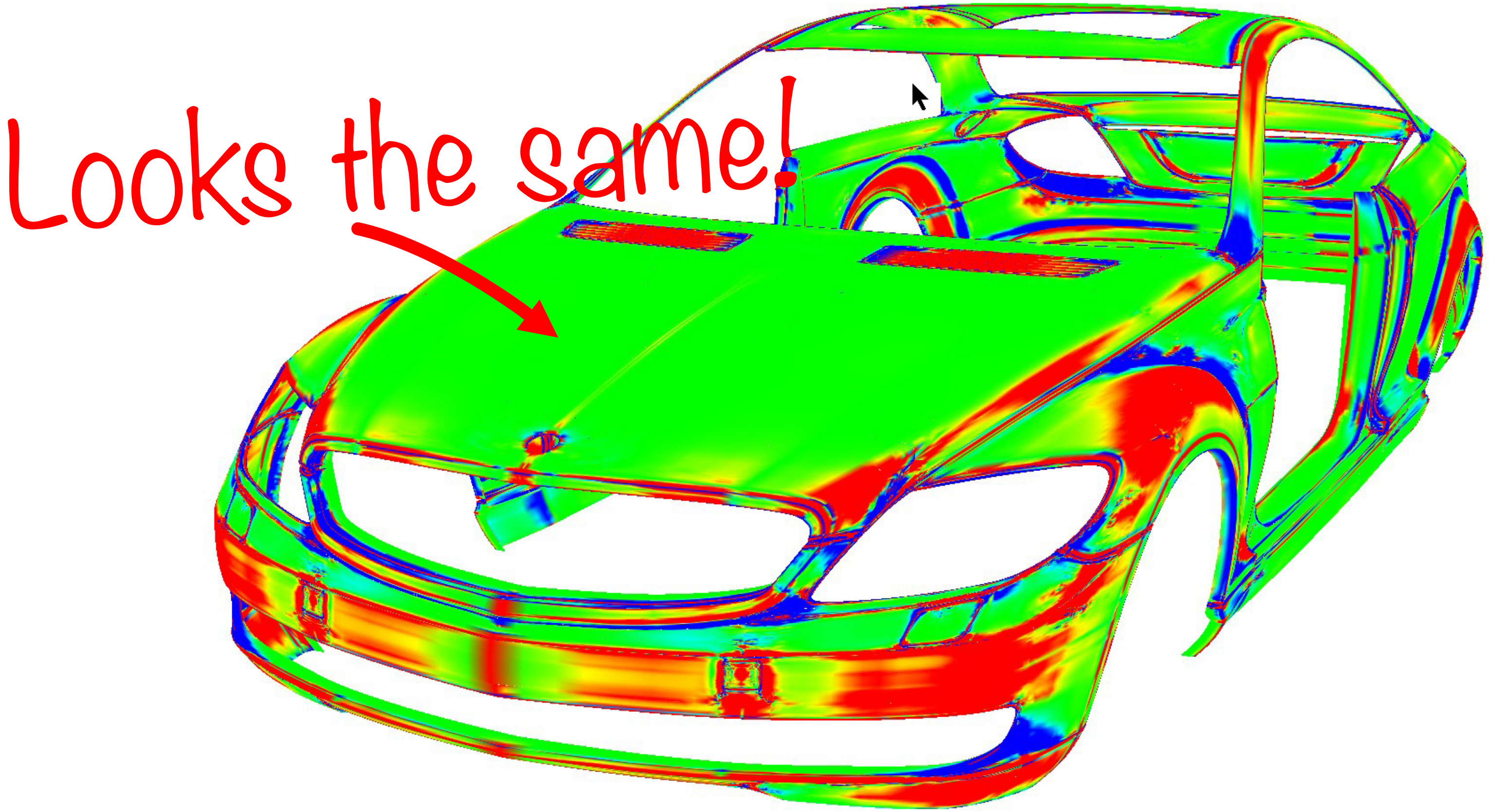


Noisy!

$$K = \kappa_1 \kappa_2$$

Second derivative quantity

End of the Story?



http://www.integrityware.com/images/MerceedesGaussianCurvature.jpg

Non-unique

Summary of Gaussian and Mean Curvatures

- $K = \kappa_1\kappa_2$ and $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ are Gaussian and mean curvatures
- Locally isometric surfaces are invariant measured by Gaussian curvature
- Gaussian curvatures are vulnerable to noises in practice and not informative
- Stronger shape descriptors are needed