UC San Diego

L2: Robot Geometry

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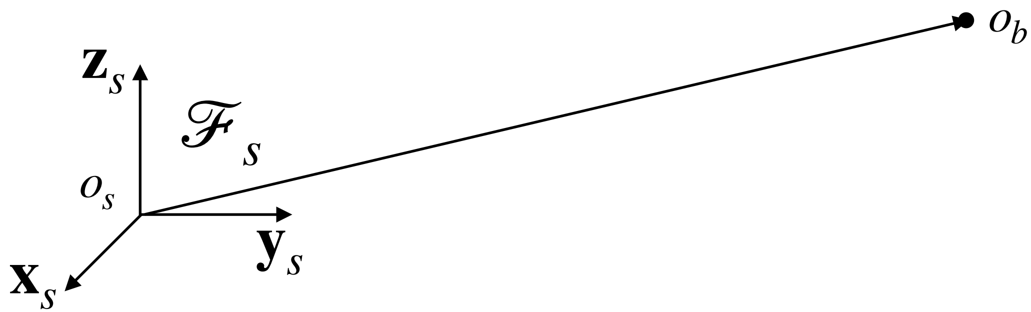
Agenda

- Rigid Transformation
- $\mathbb{SO}(3)$ and $\mathbb{SE}(3)$
- Multi-Link Rigid-Body Geometry

Rigid Transformation

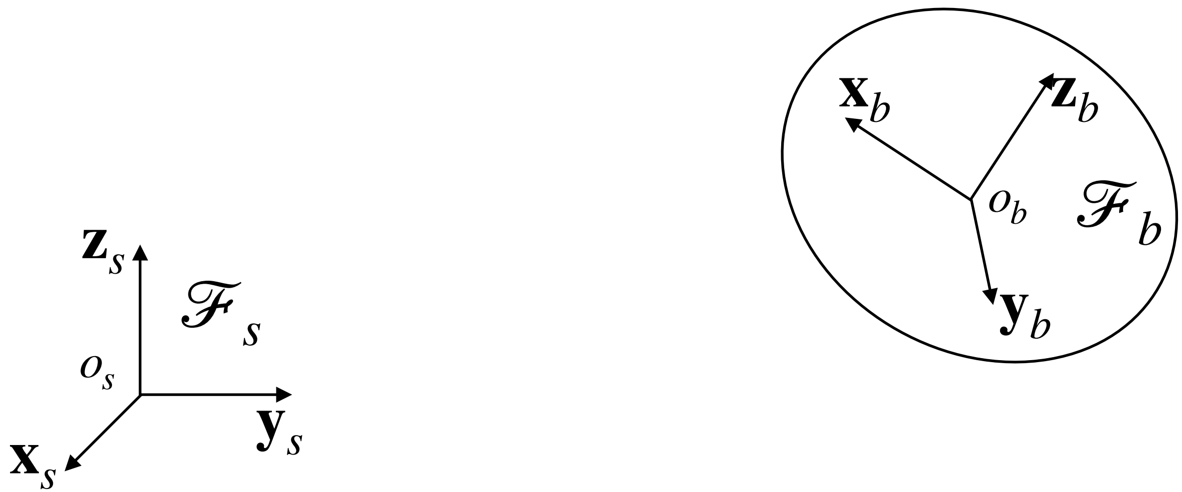
Notation Convention



- An observer **records** the position of any point in the space **using a frame** $\mathscr{F}_s$
- We use ordinary letters to denote points (e.g., p), and bold letters to dente **vectors (e.g., $\mathbf{v}$)**
- When **writing equations**, we add a superscript to symbols to denote the recording frame, e.g.,

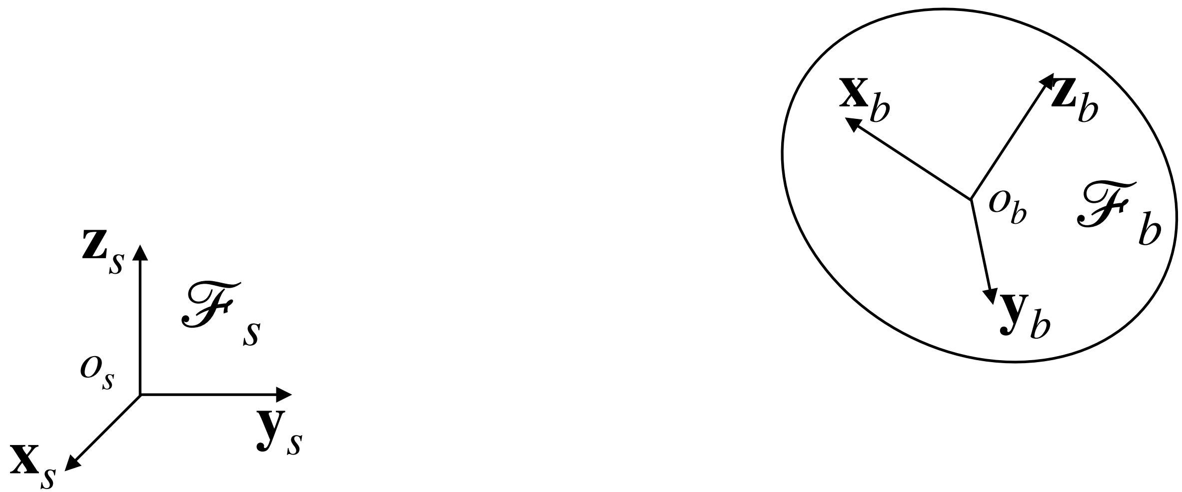
$$o_b^s = o_s^s + \mathbf{t}_{s\to b}^s$$

Rigid Transformation



- There is a rigid object, to which we bind a frame $\mathscr{F}_b$ (body frame) tightly, so that $\mathscr{F}_b$ moves along with the object

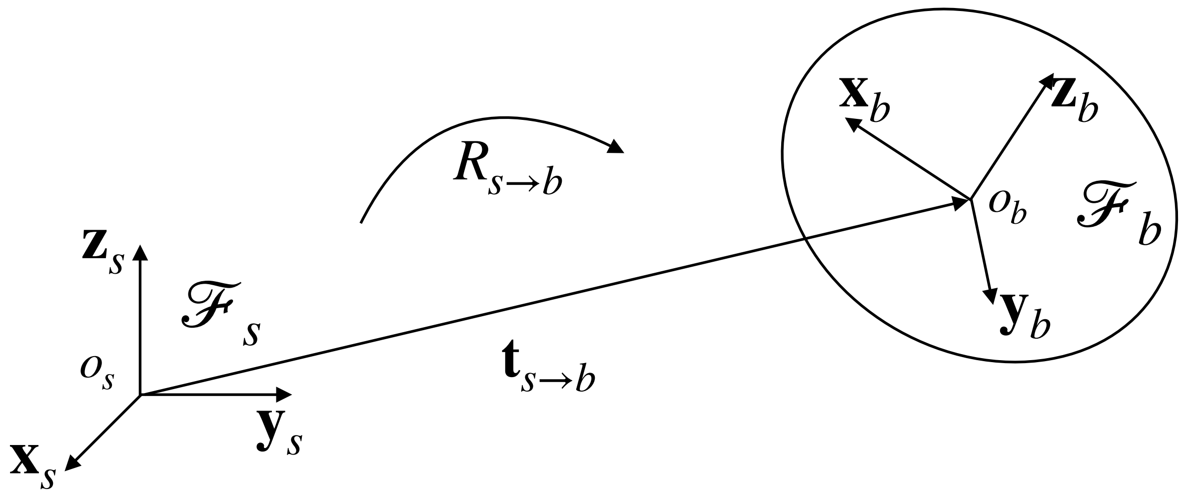
Rigid Transformation



- When talking about the pose of the *rigid* object, we ask:

How to **transform** $\mathscr{F}_s$ so that it overlaps with $\mathscr{F}_b$?

Rigid Transformation



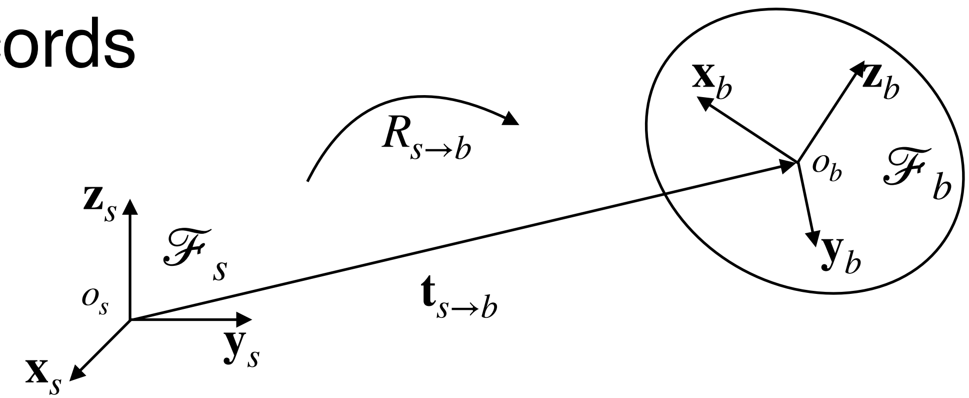
- We first translate $\mathscr{F}_s$ by $\mathbf{t}_{s \to b}$ to align o_s and o_b

- And then rotate by $R_{s \to b}$ to align $\{\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i\}$ ($i = s \text{ or } b$)

Rigid Transformation

- Formally,
  - $o_b^s = o_s^s + \mathbf{t}_{s\to b}^s$
  - $[\mathbf{x}_b^s, \mathbf{y}_b^s, \mathbf{z}_b^s] = R_{s\to b}^s[\mathbf{x}_s^s, \mathbf{y}_s^s, \mathbf{z}_s^s]$

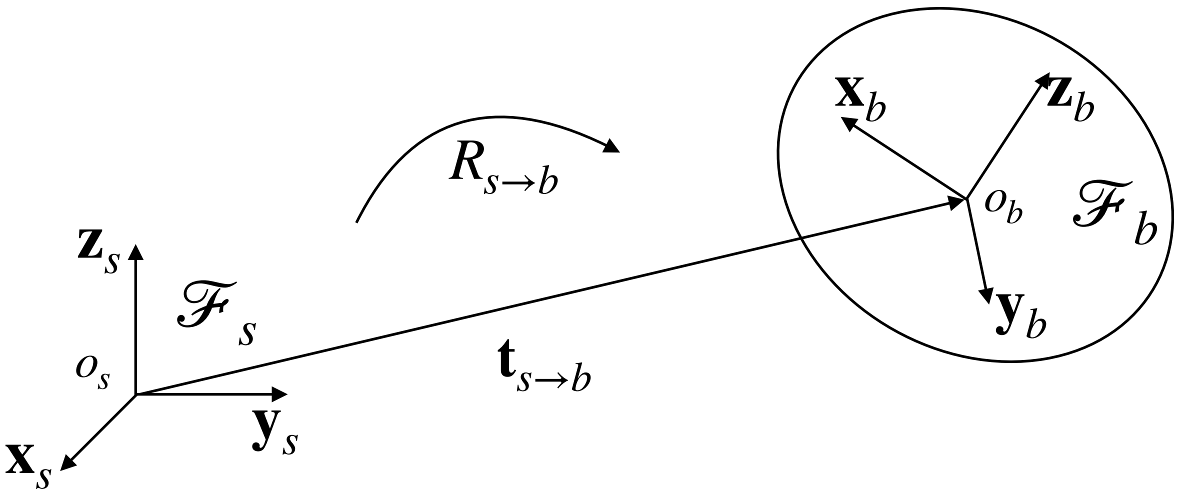
- Since the observer records everything using $\mathscr{F}_s$,
  - $o_s^s = 0$
  - $[\mathbf{x}_s^s, \mathbf{y}_s^s, \mathbf{z}_s^s] = I_{3\times 3}$



- Therefore,
  - $\mathbf{t}_{s\to b}^s = o_b^s$
  - $R_{s\to b}^s = [\mathbf{x}_b^s, \mathbf{y}_b^s, \mathbf{z}_b^s] \in \mathbb{R}^{3\times 3}$

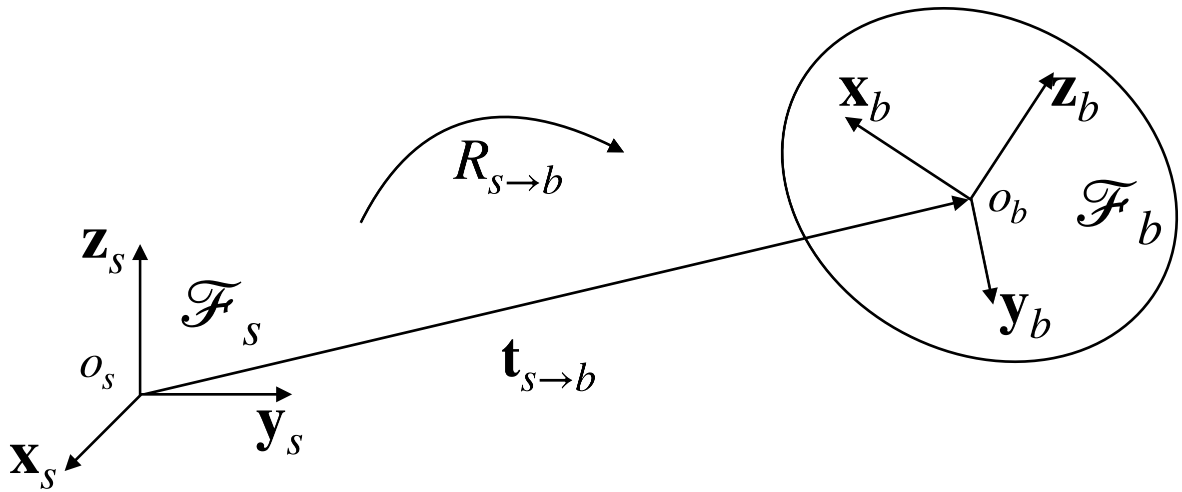
$(R_{s \to b}, \mathbf{t}_{s \to b})$ for Coordinate Transformation

Use Coordinate Transformation to Relate Coordinates in Frames



- Assume a second observer that records coordinates by $\mathscr{F}_b$
- Assume a point p on the body. Since $\mathscr{F}_b$ moves along the body, its coordinate recorded in $\mathscr{F}_b$, denoted as p^b, should **never change**.

$(R_{s\to b}, \mathbf{t}_{s\to b})$ for Coordinate Transformation



- Imagine a process: $\mathscr{F}_b$ moves from $\mathscr{F}_s$ to the current location. This is how we define $(R^s_{s\to b}, \mathbf{t}^s_{s\to b})$.
- Since p moves along $\mathscr{F}_b$, it is moved from the **initial position**, $p^s = p^b$, to the current location:

$$p^s = R^s_{s\to b} p^b + \mathbf{t}^s_{s\to b}$$

Homogenous Coordinates

- Homogeneous coordinate for 3D Space:

$$\tilde{x} := \begin{bmatrix} x \\ 1 \end{bmatrix} \in \mathbb{R}^4$$

- Homogeneous transformation matrix:

$$T^s_{s\to b} = \begin{bmatrix} R^s_{s\to b} & \mathbf{t}^s_{s\to b} \\ 0 & 1 \end{bmatrix}$$

- Coordinate transformation under linear form:

$$\tilde{x}^s = T^s_{s\to b}\tilde{x}^b$$

- Ignore $\tilde{}$ for simplicity in the future.

Homogenous Coordinates

- The coordinate transformation works for any choice of $\mathscr{F}_s$ and $\mathscr{F}_b$
- As a general rule, we have:

$$x^1 = T^1_{1\to 2} x^2$$

Some Rules of Homogenous Coordinate Transformation

By $x^1 = T^1_{1\to2}x^2$, we have $x^2 = T^2_{2\to1}x^1$ and $x^3 = T^3_{3\to2}x^2$.

Therefore, $x^3 = T^3_{3\to2}T^2_{2\to1}x^1$. But $x^3 = T^3_{3\to1}x^1$

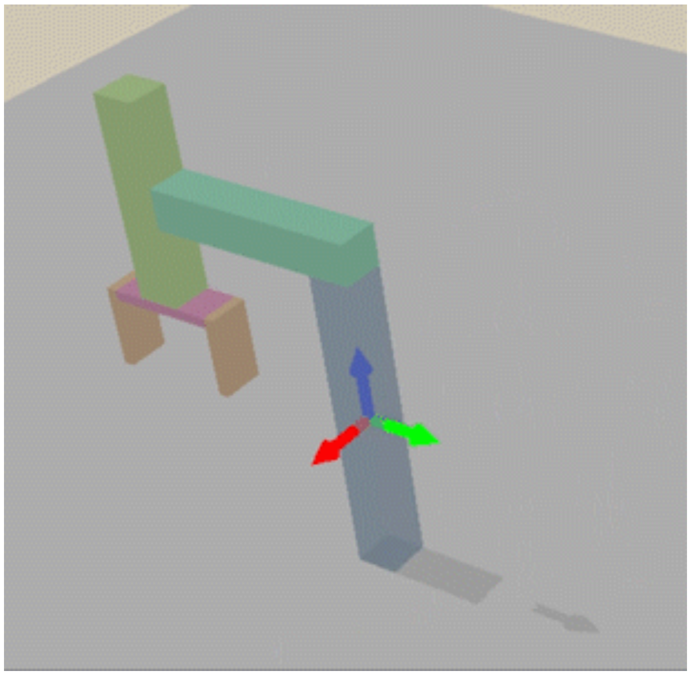
- Composition rule: $T^3_{3\to1} = T^3_{3\to2}T^2_{2\to1}$

By $x^1 = T^1_{1\to2}x^2$, we have $x^2 = (T^1_{1\to2})^{-1}x^1$

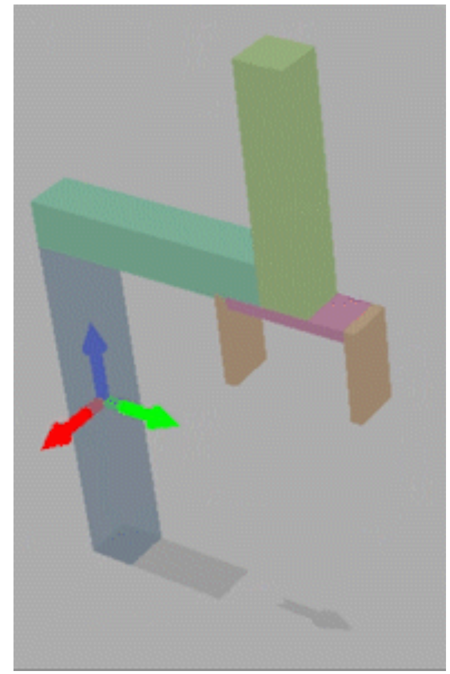
- Change of observer's frame: $T^2_{2\to1} = (T^1_{1\to2})^{-1}$

Example

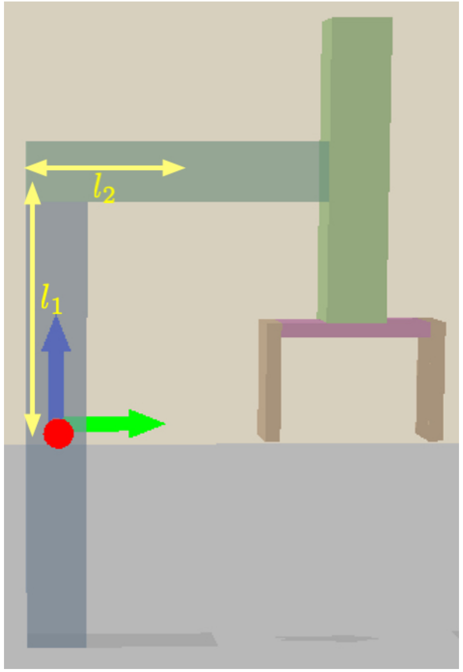
A simple 2 DoF robot arm



revolute (θ_1)



prismatic (θ_2)



base

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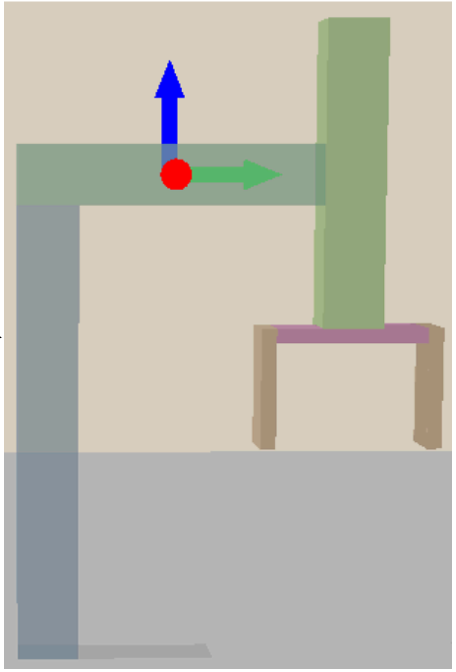
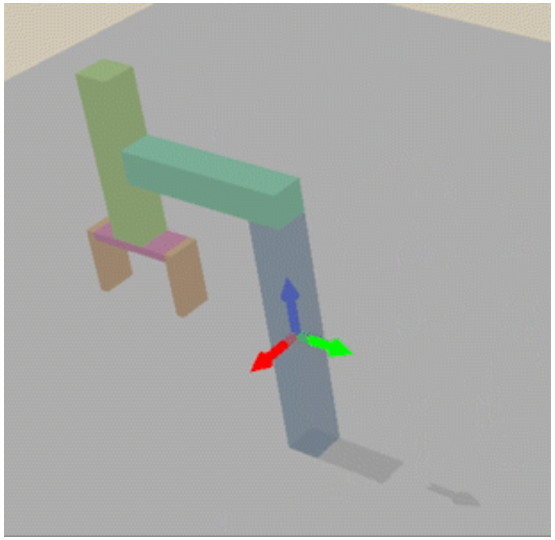
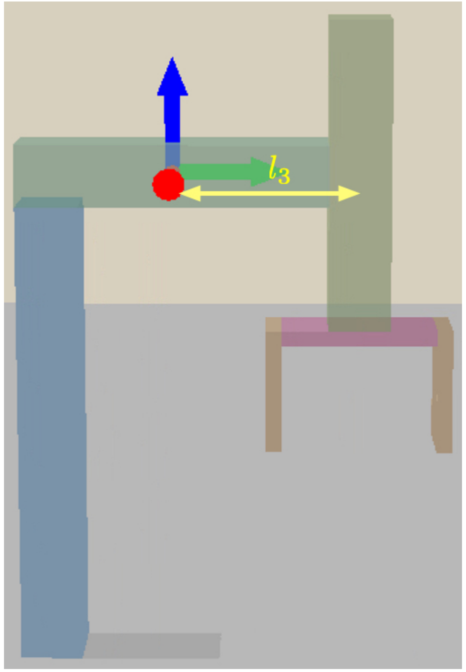


link1



revolute (θ_1)

$$T^0_{0\to 1} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & -l_2\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 & 0 & l_2\cos\theta_1 \\ 0 & 0 & 1 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



link1

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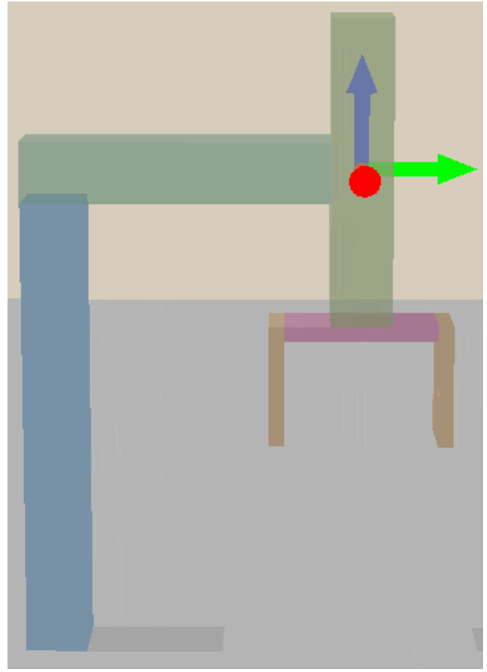
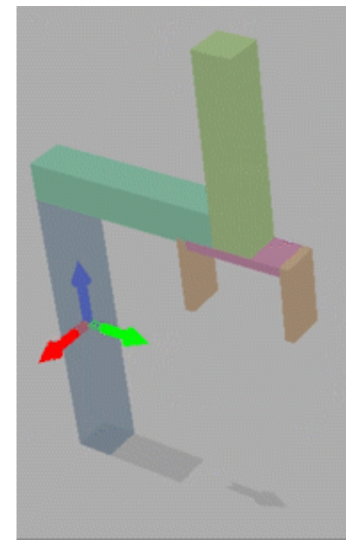
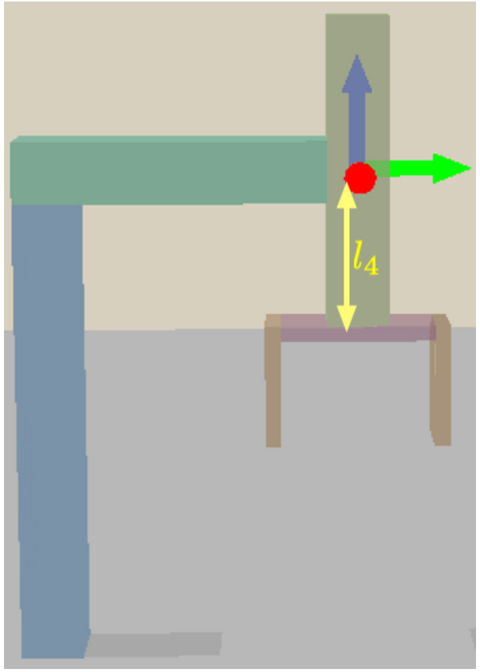


link2



prismatic (θ_2)

$$T^1_{1\to 2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_3 \\ 0 & 0 & 1 & \theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



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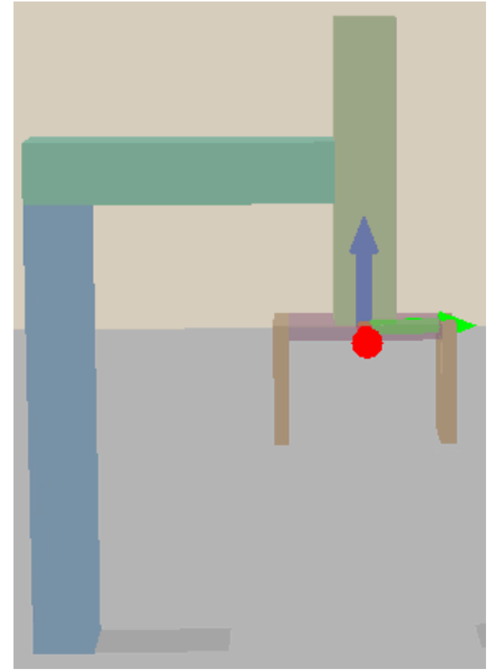
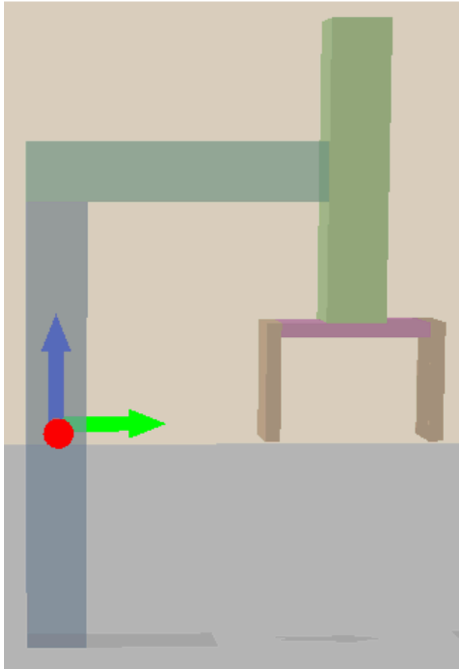


end_effector

$$T^2_{2\to 3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -l_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$


base

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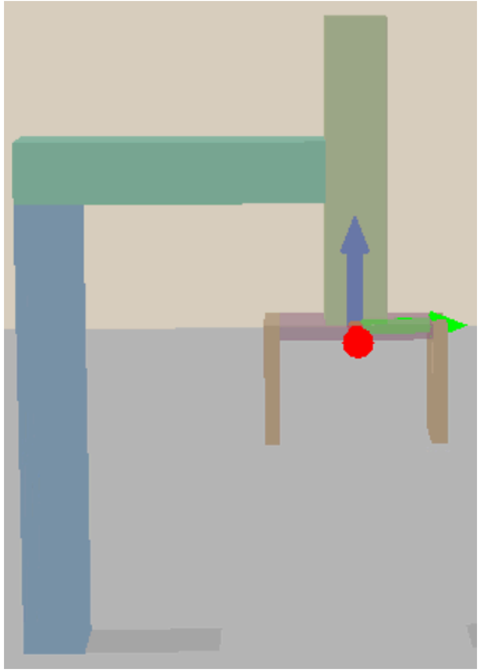
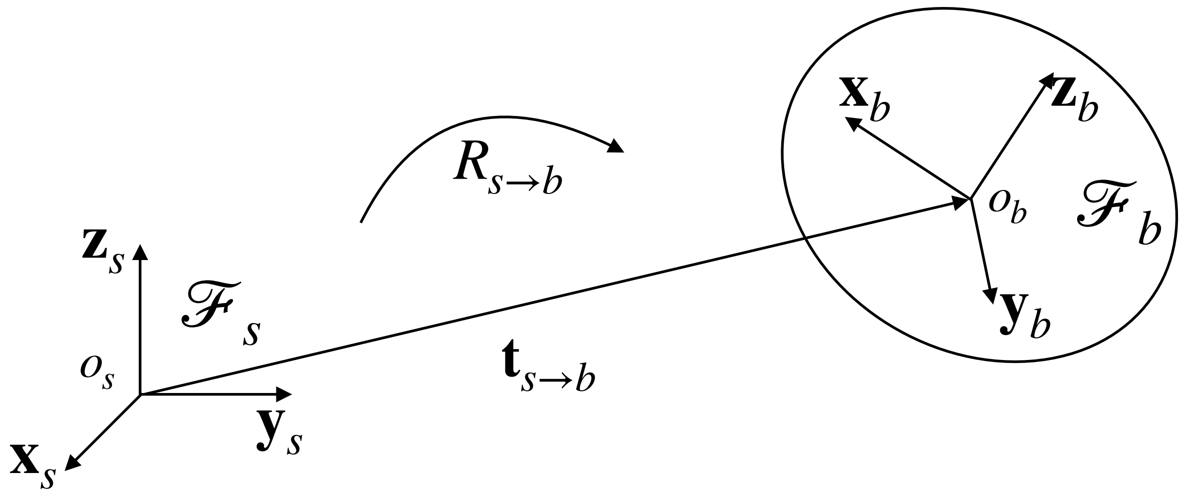

end_effector

$$T^0_{0\to 3} = T^0_{0\to 1} T^1_{1\to 2} T^2_{2\to 3} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & -\sin\theta_1(l_2+l_3) \\ \sin\theta_1 & \cos\theta_1 & 0 & \cos\theta_1(l_2+l_3) \\ 0 & 0 & 1 & l_1 - l_4 + \theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$(R_{s\to b}, \mathbf{t}_{s\to b})$ as a Linear Transformation

$(R_{s\to b}, \mathbf{t}_{s\to b})$ as a Linear Transformation



- $(R_{s\to b}, \mathbf{t}_{s\to b})$ transforms any **point** in the *whole space* by the following equation:

$$x'^s = R^s_{s\to b} x^s + \mathbf{t}^s_{s\to b}$$

$(R_{s\to b}, \mathbf{t}_{s\to b})$ as a Linear Transformation

- Suppose $\mathscr{F}_p^s = \{p^s, (\mathbf{x}_p^s, \mathbf{y}_p^s, \mathbf{z}_p^s)\}$ is a frame at an arbitrary point p^s
- **Then, the new origin is:** $p'^s = ?$

$(R_{s\to b}, \mathbf{t}_{s\to b})$ as a Linear Transformation

- Suppose $\mathscr{F}_p^s = \{p^s, (\mathbf{x}_p^s, \mathbf{y}_p^s, \mathbf{z}_p^s)\}$ is a frame at an arbitrary point p^s
- **Then, the new origin is:** $p'^s = R_{s\to b}^s p^s + \mathbf{t}_{s\to b}^s$

$(R_{s\to b}, \mathbf{t}_{s\to b})$ as a Linear Transformation

- Suppose $\mathscr{F}_p^s = \{p^s, (\mathbf{x}_p^s, \mathbf{y}_p^s, \mathbf{z}_p^s)\}$ is a frame at an arbitrary point p^s
- **Then, the new origin is:** $p'^s = R_{s\to b}^s p^s + \mathbf{t}_{s\to b}^s$
- How about the bases vectors of the frame?
  - Assume three curves, $\gamma_x, \gamma_y, \gamma_z$, passing p^s at $t = 0$ with tangents $\mathbf{x}_p, \mathbf{y}_p, \mathbf{z}_p$

$(R_{s\to b}, \mathbf{t}_{s\to b})$ as a Linear Transformation

- Suppose $\mathscr{F}_p^s = \{p^s, (\mathbf{x}_p^s, \mathbf{y}_p^s, \mathbf{z}_p^s)\}$ is a frame at an arbitrary point p^s
- **Then, the new origin is:** $p'^s = R^s_{s\to b} p^s + \mathbf{t}^s_{s\to b}$
- How about the bases vectors of the frame?
  - Assume three curves, $\gamma_x, \gamma_y, \gamma_z$, passing p^s at $t = 0$ with tangents $\mathbf{x}_p, \mathbf{y}_p, \mathbf{z}_p$
  - Then, the new tangents after transformation are:

$$\frac{d}{dt} R^s_{s\to b}\gamma^s_x(0),\ \frac{d}{dt} R^s_{s\to b}\gamma^s_y(0),\ \frac{d}{dt} R^s_{s\to b}\gamma^s_z(0)$$

$(R_{s\to b}, \mathbf{t}_{s\to b})$ as a Linear Transformation

- Suppose $\mathscr{F}_p^s = \{p^s, (\mathbf{x}_p^s, \mathbf{y}_p^s, \mathbf{z}_p^s)\}$ is a frame at an arbitrary point p^s
- **Then, the new origin is:** $p'^s = R_{s\to b}^s p^s + \mathbf{t}_{s\to b}^s$
- How about the bases vectors of the frame?
  - Assume three curves, $\gamma_x, \gamma_y, \gamma_z$, passing p^s at $t = 0$ with tangents $\mathbf{x}_p, \mathbf{y}_p, \mathbf{z}_p$
  - Then, the new tangents after transformation are:

$$\frac{d}{dt}R_{s\to b}^s\gamma_x^s(0),\ \frac{d}{dt}R_{s\to b}^s\gamma_y^s(0),\ \frac{d}{dt}R_{s\to b}^s\gamma_z^s(0)$$

- **So the new frame is:** $\mathscr{F}_{p'}^s = \{p'^s, R_{s\to b}^s[\mathbf{x}_p^s, \mathbf{y}_p^s, \mathbf{z}_p^s]\}$

$T^s_{1\to 2}$

- We have introduced the notations when the observer is recoding by $\mathscr{F}_s$ or $\mathscr{F}_b$
  - $T^s_{s\to b}$ (record the frame alignment from $\mathscr{F}_1$ to $\mathscr{F}_2$)
  - By the change of observer's frame, we introduced $T^b_{b\to s} = (T^s_{s\to b})^{-1}$
- Next, we define the notion of $T^s_{1\to 2}$, which is how we **record** an arbitrary transformation from $\mathscr{F}_1$ to $\mathscr{F}_2$ in $\mathscr{F}_s$
  - $T^s_{1\to 2} := T^s_{s\to 2} T^1_{1\to s}$

Composition as a Homogeneous Linear Transformation

- Under the definition $T^s_{1\to 2} := T^s_{s\to 2} T^1_{1\to s}$, the composition rule is:

$$T^s_{1\to 2} = T^s_{3\to 2} T^s_{1\to 3}$$

Change Observer's Frame with Similarity Transformation

- Given $T^s_{1\to2}$, what is $T^b_{1\to2}$?

$$T^s_{1\to2}T^s_{s\to1} = T^s_{s\to2}$$ Composition as Linear Transformation

$$T^s_{1\to2}T^s_{s\to b}T^b_{b\to1} = T^s_{s\to b}T^b_{b\to2}$$ Composition as Coordinate Transformation

$$T^s_{1\to2}T^s_{s\to b}T^b_{b\to1} = T^s_{s\to b}T^b_{1\to2}T^b_{b\to1}$$ Composition as Linear Transformation

$$T^s_{1\to2}T^s_{s\to b} = T^s_{s\to b}T^b_{1\to2}$$

$$T^s_{1\to2} = T^s_{s\to b}T^b_{1\to2}(T^s_{s\to b})^{-1}$$

- Similarity Transformation changes the **superscript**

$B = X^{-1}AX$: Similarity Transformation

A Special Case

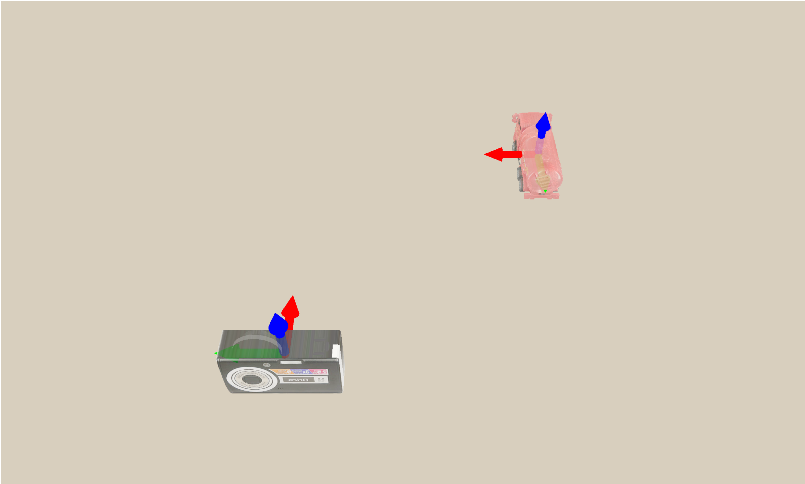
- By $T^s_{1\to 2} = T^s_{s\to b} T^b_{1\to 2} (T^s_{s\to b})^{-1}$,
  - If $\mathscr{F}_1 = \mathscr{F}_s$ and $\mathscr{F}_2 = \mathscr{F}_b$, $T^s_{s\to b} = T^b_{s\to b}$!
- Therefore, we often see the abbreviated notations:
  - $T^s_b \equiv T^s_{s\to b}$
  - $T_{sb} \equiv T^s_{s\to b}$
  - $T_b \equiv T^s_{s\to b}$
- The above equation can therefore be written as:

$$T^s_{1\to 2} = T_{s\to b} T^b_{1\to 2} (T_{s\to b})^{-1}$$

Example

- Consider a camera with frame $\mathscr{F}_c$ observing a red car
- Denote the current frame of the red car as $\mathscr{F}_1$

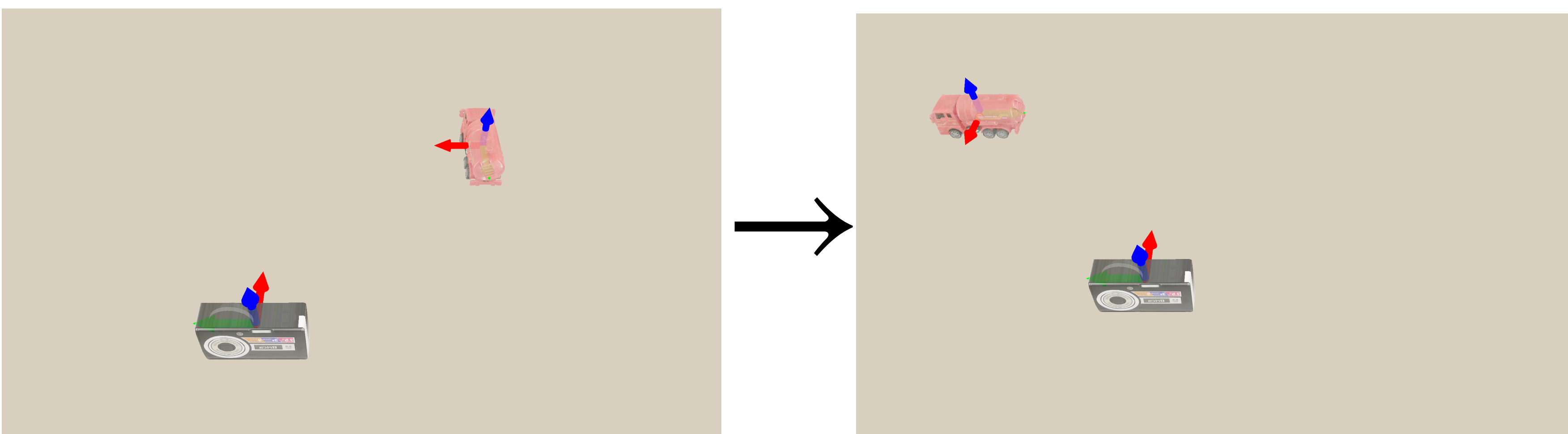
$$T^c_{c\to 1} = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 & l \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 & -l \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Example

- Then the red car move to a new frame $\mathscr{F}_2$

$$T^{c}_{c\to 1} = \begin{bmatrix} \cos\pi & -\sin\pi & 0 & l \\ \sin\pi & \cos\pi & 0 & -l \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

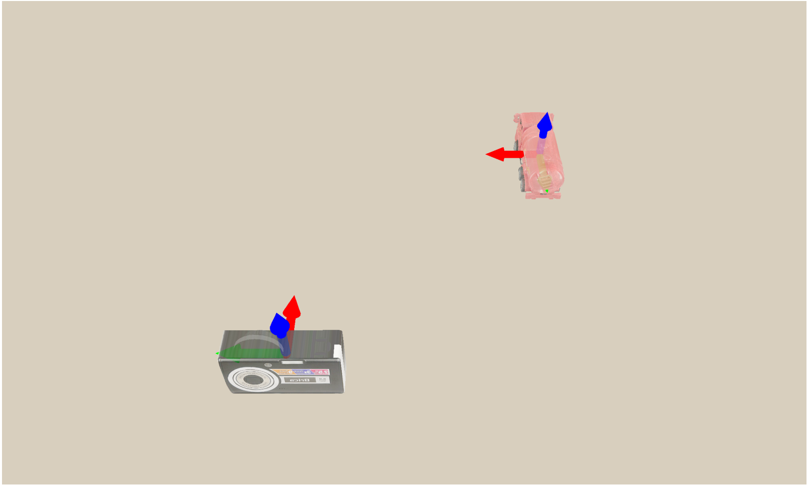


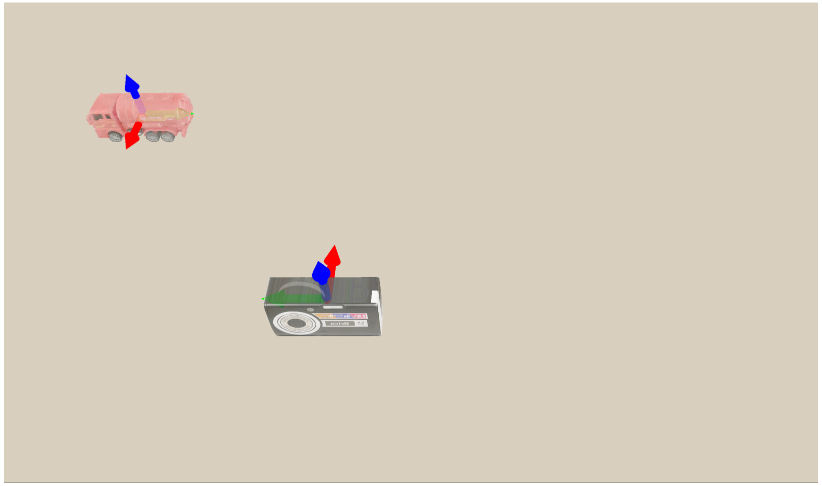
Example

- By the composition rule of coordinate transformation:

$$T^c_{c\to 2} = T^c_{c\to 1} T^1_{1\to 2}$$

$$T^1_{1\to 2} = (T^c_{c\to 1})^{-1} T^c_{c\to 2} = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 & 2l \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$





Example

- By the composition rule of coordinate transformation:
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$$T^1_{1\to 2} = (T^c_{c\to 1})^{-1} T^c_{c\to 2} = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 & 2l \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

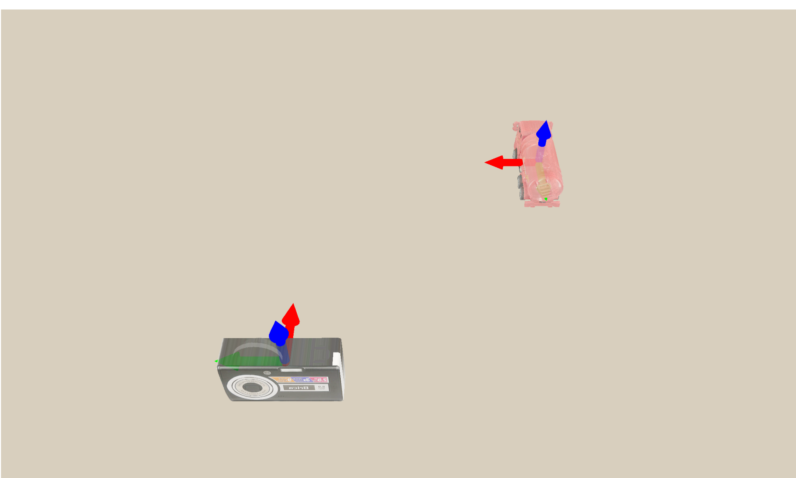
- The movement from $\mathscr{F}_1$ to $\mathscr{F}_2$ can also be represented as a linear transformation from $\mathscr{F}_1$ to $\mathscr{F}_2$, recorded by frame c, denoted as $T^c_{1\to 2}$

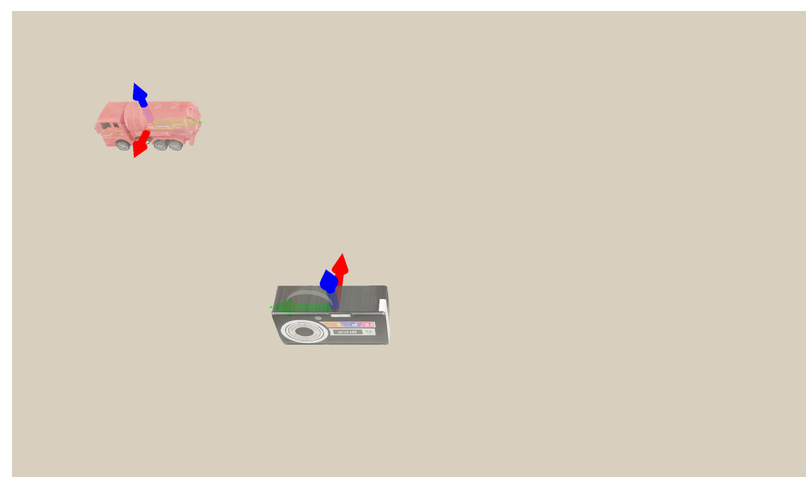
Example

- With similarity transformation:

$$T^{c}_{1\to 2} = T^{c}_{c\to 1} T^{1}_{1\to 2} (T^{c}_{c\to 1})^{-1} = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 & 0 \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Note: translation in $T^{c}_{1\to 2}$ is all zero! Why?

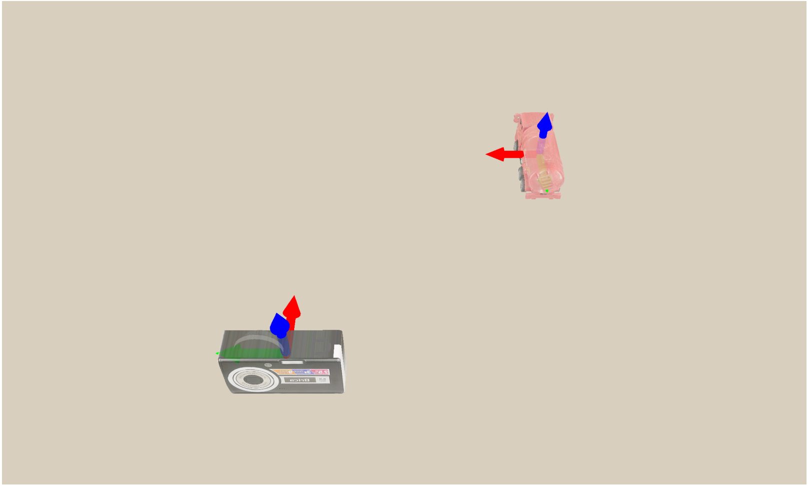


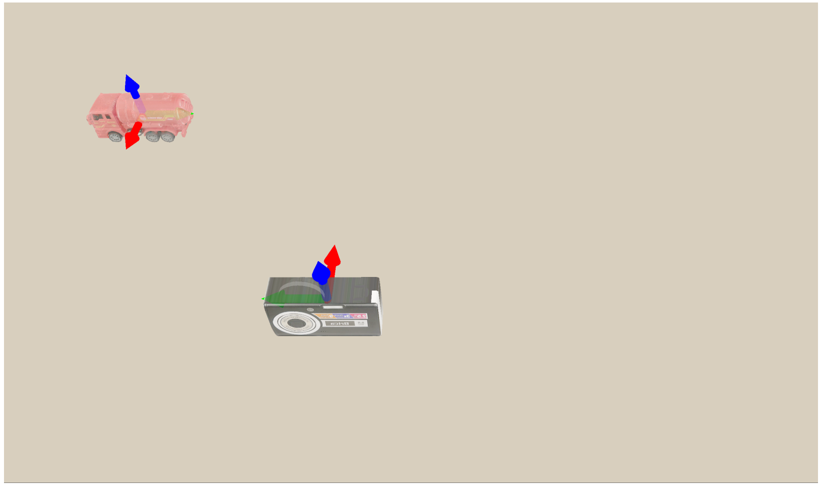


Example

- Transformation $T^c_{1\to 2}$ can be regarded as rotating about z-axis by 90 degree

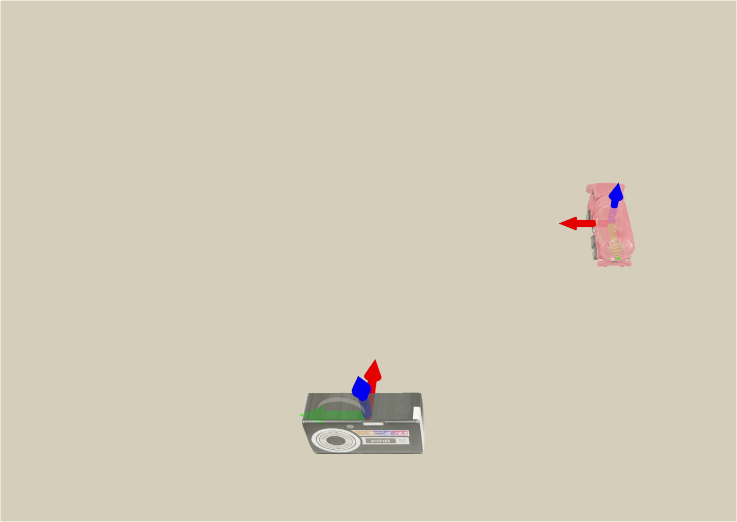
$$T^c_{1\to 2} = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 & 0 \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$





Example

- When observer is recording in the camera frame $\mathscr{F}_c$, the red car is rotated about the z-axis of camera frame c through +90 degree



Additional Notes by the Example

- $T^s_{1\to2}$ is **NOT** to record the transformation by first translating $\mathscr{F}_1$ to $\mathscr{F}_2$ and then rotating (this recording convention **only** works when $\mathscr{F}_1 = \mathscr{F}_s$). It is based on the rule $T^s_{1\to2} := T^s_{s\to2}T^1_{1\to s}$
- An observer chooses its way to decompose $T_{1\to2}$ into $R_{1\to2}$ and $\mathbf{t}_{1\to2}$ based upon its own frame choice
- We will discuss the "canonical" decomposition next week

Additional Notes by the Example

- The linear transformation view allows us to discuss the movement of bodies conveniently (without worrying about the change of observer):

$$T^s_{1\to 2} = T^s_{3\to 2} T^s_{1\to 3}$$

- Suppose a body is moving. Then,

$$T^s_{t_0\to t+\Delta t} = T^s_{t\to t+\Delta t} T^s_{t_0\to t}$$

where t parameterizes time.

Summary

- Basic notation:
  - $T^s_{s\to b}$: Record the motion of frame alignment from $\mathcal{F}_s$ to $\mathcal{F}_b$ in $\mathcal{F}_s$
- Coordinate transformation
  - $T^c_{c\to a} = T^c_{c\to b} T^b_{b\to a}$: Composition for coordinate transformation
  - $T^b_{b\to s} = (T^s_{s\to b})^{-1}$: Change of frame for $\mathcal{F}_s$ to $\mathcal{F}_b$ motion
- Linear transformation
  - $T^s_{1\to 2} := T^s_{s\to 2} T^1_{1\to s}$: Record the motion of frame alignment from $\mathcal{F}_1$ to $\mathcal{F}_2$ in $\mathcal{F}_s$
  - $T^s_{c\to a} = T^s_{b\to a} T^s_{c\to b}$: Composition as a linear transformation
- $T^s_{1\to 2} = T_{s\to b} T^b_{1\to 2} (T_{s\to b})^{-1}$: Change of frame for $\mathcal{F}_1$ to $\mathcal{F}_2$ motion

$\mathbb{SO}(3)$ and $\mathbb{SE}(3)$

$\mathbb{SO}(3)$: The Space of Rotations

- $\mathbb{SO}(n) = \{R \in \mathbb{R}^{n \times n} : \det(R) = 1, RR^T = I\}$
- $\mathbb{SO}(n)$: "Special Orthogonal Group"

- "Group": roughly, closed under matrix multiplication
- "Orthogonal": $RR^T = I$
- "Special": $\det(R) = 1$

- $\mathbb{SO}(2)$: 2D rotations, 1 DoF
- $\mathbb{SO}(3)$: 3D rotations, 3 DoF

$\mathbb{SE}(3)$: The Space of Rigid Transformations

- $\mathbb{SE}(3) := \left\{ T = \begin{bmatrix} R & \mathbf{t} \\ 0 & 1 \end{bmatrix}, R \in \mathbb{SO}(3), \mathbf{t} \in \mathbb{R}^3 \right\}$
- $\mathbb{SE}(3)$: "Special Euclidean Group"

- "Group": roughly, closed under matrix multiplication
- "Euclidean": R and $\mathbf{t}$
- "Special": $\det(R) = 1$

- 6 DoF

- We need some theoretical understanding of $\mathbb{SO}(3)$ and $\mathbb{SE}(3)$
    - The topological structure
    - The parameterization
    - The differentiable properties

Multi-Link Rigid-Body Geometry

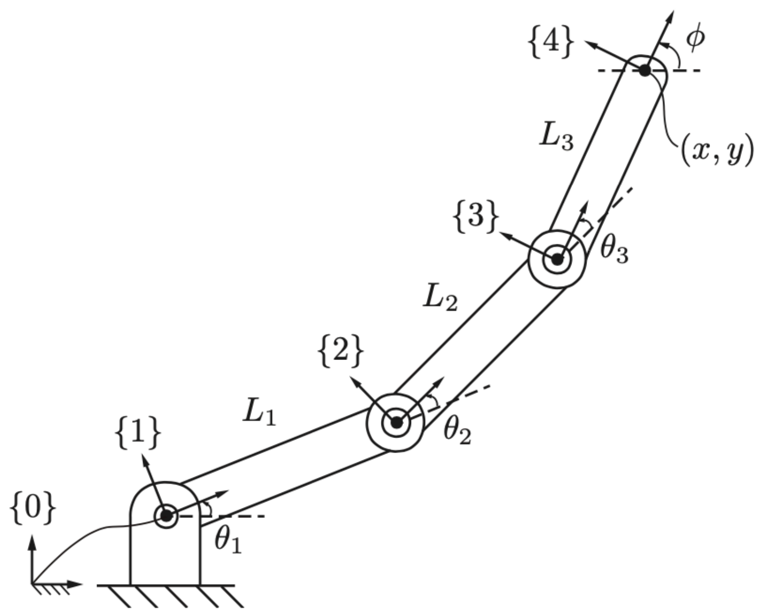
Link and Joint

Link:

- **Links** are the rigid-body connected in sequence

Joint:

- **Joints** are the connectors between links. They determine the DoF of motion between adjacent links

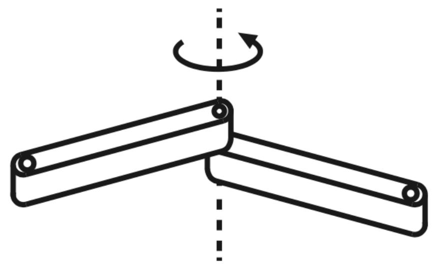


Base Link and End-Effector Link

- Base link / root link:
  - The 0-th link of the robot
  - Regarded as the “fixed” reference
  - The spatial frame $\mathcal{F}_s$ is attached to it
- End-effector link
  - The last link
  - e.g., the gripper
  - A frame $\mathcal{F}_e$ is attached to it

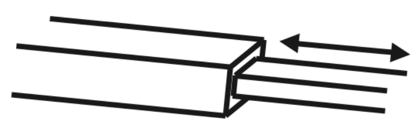
Two Common Joint Types

- Revolute/Hinge/Rotational joint



Revolute
(R)

- Prismatic/Translational joint



Prismatic
(P)

Kinematics: The Basic Geometry Calculation Task

- Kinematics: Describing the motion of bodies (position and velocity)
- Kinematics **does not consider** how to achieve motion via force

Kinematics Configuration

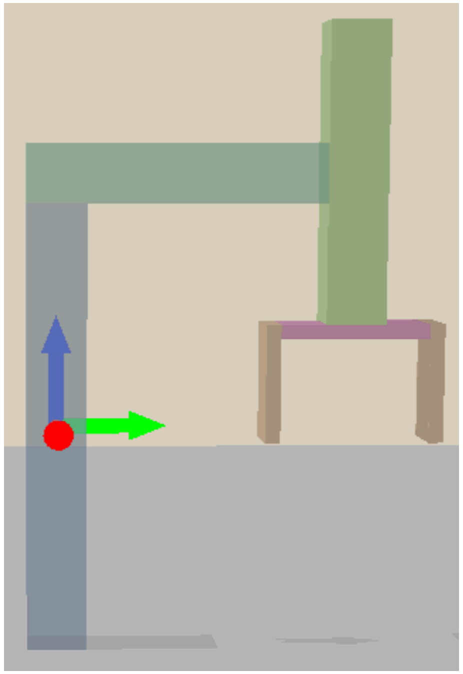
- Assuming frames are assigned to each link, we can parameterize **the pose of each joint**
  - Using the relative **angle** and **translation** between adjacent frames
- Two representations of the pose of the end-effector
  - **Joint space:** The space in which each coordinate is a vector of joint poses (**angles** around **joint axis**)
  - **Cartesian space:** The space of the rigid transformations of the end-effector by $(R_{s\to e}, \mathbf{t}_{s\to e})$, where $\mathcal{F}_e$ is the end-effector frame

Kinematics Equations

- Map the joint space coordinate $\theta \in \mathbb{R}^n$ to Cartesian space transformation $T \in \mathbb{SE}(3)$:

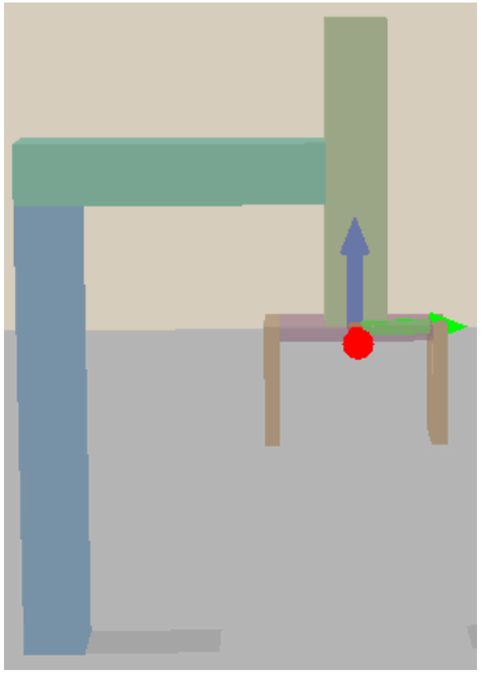
$$T_{s \to e} = f(\theta)$$

- Calculated by composing transformations along the kinematic chain



base

$\rightarrow$



end_effector

$$T^0_{0\to 3} = T^0_{0\to 1} T^1_{1\to 2} T^2_{2\to 3} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & -\sin\theta_1(l_2+l_3) \\ \sin\theta_1 & \cos\theta_1 & 0 & \cos\theta_1(l_2+l_3) \\ 0 & 0 & 1 & l_1 - l_4 + \theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

How to Relate the Motion in Joint Space and Cartesian Space?

- Q1: If the robot moves by $\Delta\theta$ in the joint space, how much will it move in the Cartesian space? (Forward Kinematics)
- Q2: If the robot would move the end-effector by Δx in the Cartesian space, how shall it change its joint poses? (Inverse Kinematic)
- Suppose we parameterize (R, t) by the **angles** around **axis**, we need to derive the differential map

- We will study the differentiability of rotation and rigid transformations