L5: Twist and Geometric Jacobian

Hao Su

Agenda

- Interpretation and Computation of Twist
- Example of Twist Computation
- Change of Coordinates for Twists
- Jacobian of Kinematics Chain
- Inverse Kinematics

Review: Twist

$$
\begin{aligned}
T^o_{s'\to b(t+\Delta t)} - T^o_{s'\to b(t)} &= T^o_{b(t)\to b(t+\Delta t)} T^o_{s'\to b(t)} - T^o_{s'\to b(t)} \\
&= e^{[\chi^o_{b(t)\to b(t+\Delta t)}]} T^o_{s'\to b(t)} - T^o_{s'\to b(t)} \\
&\approx [\chi^o_{b(t)\to b(t+\Delta t)}] T^o_{s'\to b(t)}
\end{aligned}
$$

- Divided by Δt and take the limit, we have

$$
\begin{aligned}
\dot{T}^o_{s'\to b(t)} &= \lim_{\Delta t\to 0} \left[\frac{\chi^o_{b(t)\to b(t+\Delta t)}}{\Delta t} \right] T^o_{s'\to b(t)} \\
&= [\xi^o_{b(t)}] T^o_{s'\to b(t)}
\end{aligned}
$$

- $\xi^o_{b(t)} := \lim_{\Delta t\to 0} \dfrac{\chi^o_{b(t)\to b(t+\Delta t)}}{\Delta t}$ is called "**twist**", the 6D instant velocity

Review: Twist

- Twist: $\xi^o_{b(t)} := \lim_{\Delta t \to 0} \frac{\chi^o_{b(t)\to b(t+\Delta t)}}{\Delta t}$
- $[\xi^o_{b(t)}] = \dot{T}^o_{s'\to b(t)}(T^o_{s'\to b(t)})^{-1}$
- Note: $\xi^o_{b(t)} \neq \dot{\chi}^o_{s'\to b(t)}$ for general $\chi^o_{s\to b(t)}(t)$ (verify by yourself)

Interpretation and Computation of Twist

- Let $\xi^{o}_{b(t)} = \begin{bmatrix} \nu^{o}_{b(t)} \\ \omega^{o}_{b(t)} \end{bmatrix} \in \mathbb{R}^6$, then $[\xi^{o}_{b(t)}] = \begin{bmatrix} [\omega^{o}_{b(t)}] & \nu^{o}_{b(t)} \\ 0 & 0 \end{bmatrix}$

Body Twist

- When $\mathcal{F}_o = \mathcal{F}_{b(t)}$, it is particularly easy to compute $\xi_{b(t)}^{b(t)}$
- Remarks
  - For body twist, when recording at time t, you should think it as ***first cloning the body frame and then record the movement using this cloned frame and keeping it static***
  - Body twist is “ego-centric” and sometimes simpler to specify for robotics. For example, if we take the gripper frame as the body frame. Using the body twist, we can express “move gripper forward” by a pure translation

Review: Linear Velocity from Twist

- The linear velocity of p^o caused by $T^o_{s'\to b(t)}$ at time t is

$$\mathbf{v}^o_p(t) = \lim_{\Delta t\to 0} \frac{T^o_{b(t)\to b(t+\Delta t)}p^o - p^o}{\Delta t} = \lim_{\Delta t\to 0} \frac{\exp([\chi^o_{b(t)\to b(t+\Delta t)}]) - I}{\Delta t} p^o$$

$$= \lim_{\Delta t\to 0} \frac{[\chi^o_{b(t)\to b(t+\Delta t)}]}{\Delta t} p^o = [\xi^o_{b(t)}]p^o$$

- Therefore, $\boxed{\mathbf{v}^o_p(t) = [\xi^o_{b(t)}]p^o}$

(Recall that, if a motion is a pure rotation, then $\mathbf{v}^o_p(t) = \omega^o_{b(t)} \times p^o$)

Body Twist Computation

- By $\mathbf{v}_p^o(t) = [\xi_{b(t)}^o]p^o$ and $[\xi_{b(t)}^o] = \begin{bmatrix} [\omega_{b(t)}^o] & \nu_{b(t)}^o \\ 0 & 0 \end{bmatrix}$, let $p_{origin}^{b(t)} = [0,0,0,1]^T$, which is the origin of body frame

$$\mathbf{v}_{origin}^{b(t)}(t) = [\xi_{b(t)}^{b(t)}]p_{origin}^{b(t)} = \nu^{b(t)}$$

- Note that $\mathbf{v}_{origin}^{b(t)}(t)$ is the linear velocity of the origin of the body frame
- Therefore, $\xi_{b(t)}^{b(t)}$ is composed by the linear velocity of the origin and an angular velocity around the axis (may not pass the origin)
- In practice, we often write down the body twist first and then obtain the twist in other frames by change of coordinate

Compute $\xi^o_{b(t)}$ from Angle-Axis

- Let $\xi^o_{b(t)} = \begin{bmatrix} \nu^o_{b(t)} \\ \omega^o_{b(t)} \end{bmatrix} \in \mathbb{R}^6$, then $[\xi^o_{b(t)}] = \begin{bmatrix} [\omega^o_{b(t)}] & \nu^o_{b(t)} \\ 0 & 0 \end{bmatrix}$

- The instant linear velocity can be decomposed into the rotation about the axis and translation along the axis

- Take a point $q^o \in \mathbb{R}^3$ on the axis,
  - By $\mathbf{v}^o_p(t) = [\xi^o_{b(t)}]p^o$, $\mathbf{v}^o_q = [\omega^o]q^o + \nu^o$
  - Since the only velocity of q^o is along $\hat{\omega}$, $\mathbf{v}^o_q = \mathbf{v}^o_\omega$
  - $\therefore \nu^o = -[\omega^o]q^o + \mathbf{v}^o_\omega$

- $\xi^o = \begin{bmatrix} -[\omega^o]q^o + \mathbf{v}^o_\omega \\ \omega^o \end{bmatrix}$ (all the symbols have $b(t)$ as the subscript)

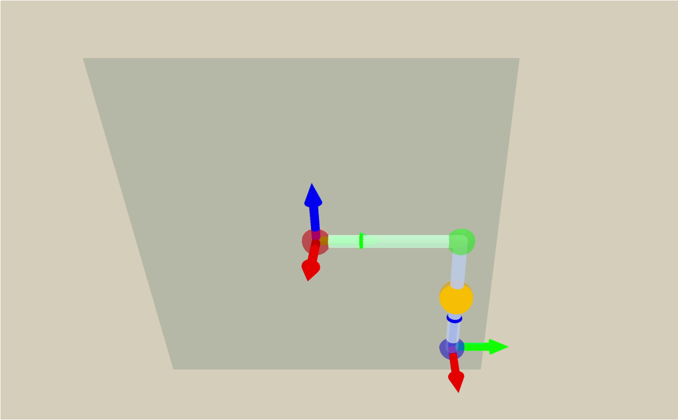
Example of Twist Computation

Example of Twist Computation

- Consider the example (last lecture), but now an orange point is fixed to the end-effector frame (blue sphere)

- What is the **velocity of orange point at** $t = 0$? Given the pose of end effector frame as below:

$$T^s_{s \to b(t)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha t) & -\sin(\alpha t) & 1 + \sin(\alpha t) \\ 0 & \sin(\alpha t) & \cos(\alpha t) & -\cos(\alpha t) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Example of Twist Computation

- The velocity of yellow point caused by the end-effector motion can be computed via twist
- Recall: $\mathbf{v}_p^s = [\xi_{b(t)}^s]p^s$

Example of Twist Computation

- The velocity of yellow point caused by the end-effector motion can be computed via twist

- Recall: $\mathbf{v}_p^s = [\xi_{b(t)}^s] p^s$

- By $T_{s\to b(t)}^s = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha t) & -\sin(\alpha t) & 1+\sin(\alpha t) \\ 0 & \sin(\alpha t) & \cos(\alpha t) & -\cos(\alpha t) \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\dot{T}_{s\to b(t)}^s = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\sin(\alpha t) & -\cos(\alpha t) & \cos(\alpha t) \\ 0 & \cos(\alpha t) & -\sin(\alpha t) & \sin(\alpha t) \\ 0 & 0 & 0 & 0 \end{bmatrix}$,

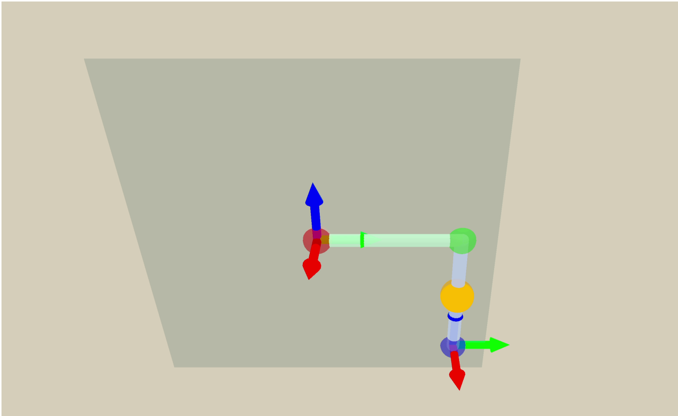
- we have $[\xi_{s\to b(t)}^s] = \dot{T}_{s\to b(t)}^s (T_{s\to b(t)}^s)^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & \alpha & 0 & -\alpha \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Example of Twist Computation

- The velocity of yellow point caused by the end-effector motion can be computed via twist
- Recall: $\mathbf{v}_p^s = [\xi_{b(t)}^s]p^s$

$$[\xi_{b(t)}^s] = \dot{T}_{s\to b(t)}^s (T_{s\to b(t)}^s)^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & \alpha & 0 & -\alpha \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

At $t = 0, p^s = \begin{bmatrix} 0 \\ 1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$

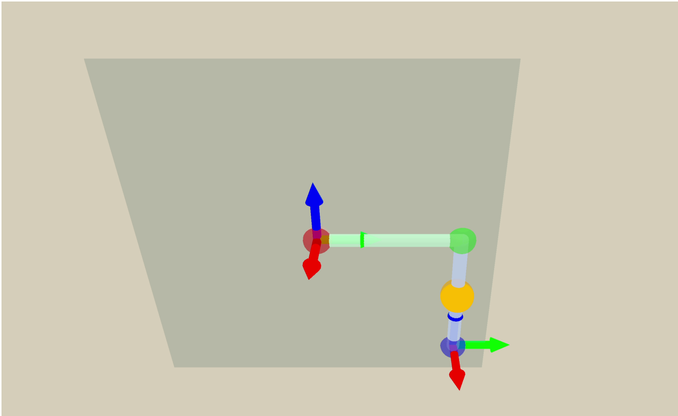


Example of Twist Computation

- The velocity of yellow point caused by the end-effector motion can be computed via twist
- Recall: $\mathbf{v}_p^s = [\xi_{b(t)}^s] p^s$

$$[\xi_{b(t)}^s] = \dot{T}_{s\to b(t)}^s (T_{s\to b(t)}^s)^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & \alpha & 0 & -\alpha \\ 0 & 0 & 0 & 0 \end{bmatrix}, p^s = \begin{bmatrix} 0 \\ 1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

$$\mathbf{v}_p^s = [\xi_{b(t)}^s] p^s = \begin{bmatrix} 0 \\ \frac{\alpha}{2} \\ 0 \\ 0 \end{bmatrix}$$

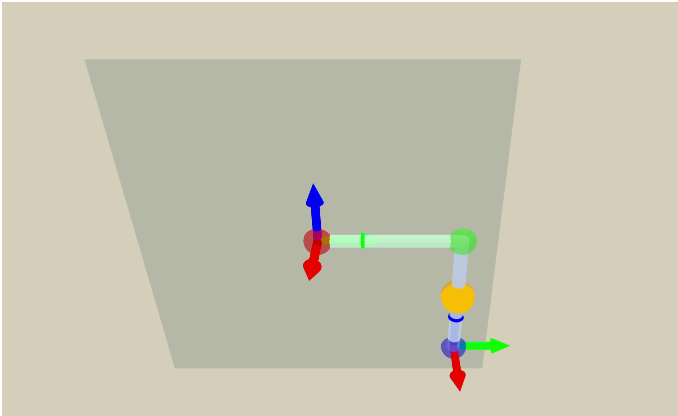


Example of Twist Computation

- We can verify this result by taking the derivative of $\frac{d}{dt}p^s(t)$

$$p^s(t) = \begin{bmatrix} 0 \\ 1 + \frac{1}{2}\sin(\alpha t) \\ -\frac{1}{2}\cos(\alpha t) \\ 1 \end{bmatrix}, \frac{d}{dt}p^s(t) = \begin{bmatrix} 0 \\ \frac{\alpha}{2}\cos(\alpha t) \\ \frac{\alpha}{2}\sin(\alpha t) \\ 0 \end{bmatrix}$$

$$\mathbf{v}^s_p = [\xi^s_{b(t)}]p^s = \begin{bmatrix} 0 \\ \frac{\alpha}{2} \\ 0 \\ 0 \end{bmatrix} = \frac{d}{dt}p^s(t)\bigg|_{t=0}$$

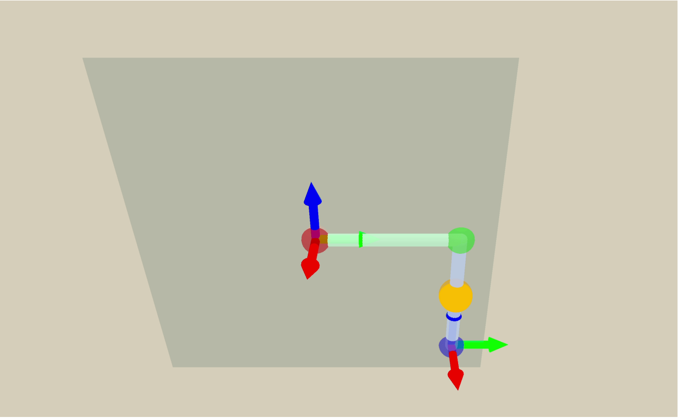


Example of Twist Computation

- What is the body twist of the end effector?

- In the body frame of the end effector (blue sphere), the origin of the frame, which is the blue sphere, has a constant linear velocity, which is always $[0,\alpha,0]$. The angular velocity is always $[\alpha,0,0]$.

So, $\xi_{b(t)}^{b(t)} = [0,\alpha,0,\alpha,0,0]^T$



Change of Coordinates for Twists

Review

- Recall that, the recordings by different observers are related by the similarity transformation:

$$T^{s_1}_{1\to 2} = T_{s_1\to s_2} T^{s_2}_{1\to 2} (T_{s_1\to s_2})^{-1}$$

Tricks in Recording Velocities

- If transformations could be recorded differently by observers, velocity should also be recorded differently

Relating 6D Velocities from Different Observers

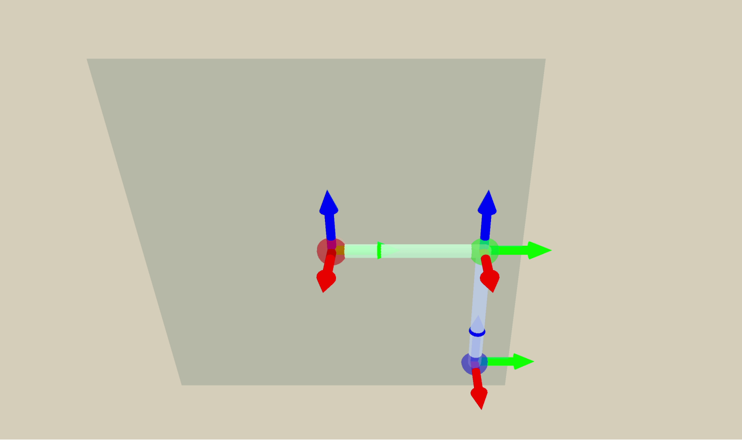
- Two observers record the same motion as $\xi^{s_1}_{b(t)}$ and $\xi^{s_2}_{b(t)}$
- **What is the relationship between $\xi^{s_1}_{b(t)}$ and $\xi^{s_2}_{b(t)}$?**

Example 1 of Change of Frame

- From our previous example, we know that the motion of the end-effector (blue sphere) can be recorded in the spatial frame (red sphere) as $\xi^s_{s\to b(t)}$:

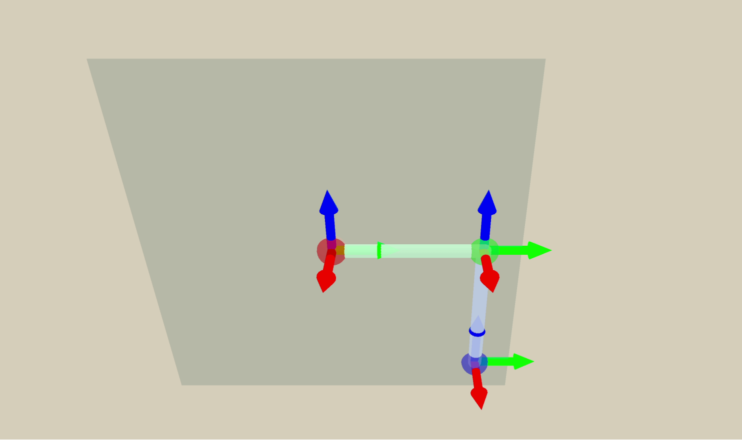
$$[\xi^s_{s\to b(t)}] = \dot{T}^s_{s\to b(t)}(T^s_{s\to b(t)})^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & \alpha & 0 & -\alpha \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xi^s_{s\to b(t)} = [0,0,-\alpha,\alpha,0,0]^T$$



Example 1 of Change of Frame

- From our previous example, we know that the motion of the end-effector (blue sphere) can be recorded in the spatial frame (red sphere) as $\xi^{s}_{s\to b(t)}$:
- Now we introduce a new frame $\mathcal{F}_o$, the frame of the green sphere. How can we record the same motion by $\mathcal{F}_o$ as $\xi^{o}_{s\to b(t)}$?

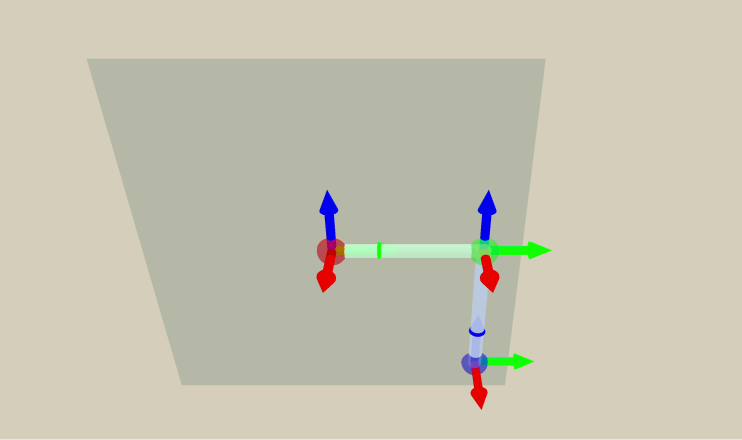


Example 1 of Change of Frame

- From our previous example, we know that the motion of the end-effector (blue sphere) can be recorded in the spatial frame (red sphere) as $\xi^s_{s\to b(t)}$:

- By simple inspection, we can find end-effector is rotating about the x-axis of $\mathscr{F}_o$ and the instant velocity along the axis is zero

$$\omega^o = [\alpha, 0, 0]^T$$
$$\hat{\omega}^o = [1, 0, 0]^T$$
$$q^o = [0, 0, 0]^T$$
$$\mathbf{v}^o_\omega = [0, 0, 0]^T$$



Example 1 of Change of Frame

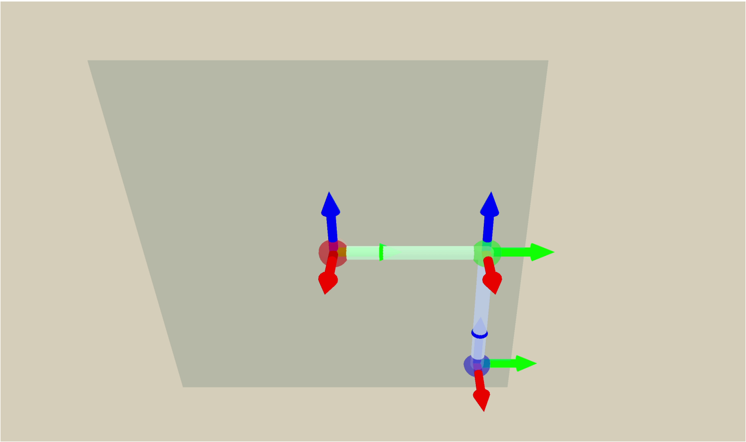
- From our previous example, we know that the motion of the end-effector (blue sphere) can be recorded in the spatial frame (red sphere) as $\xi^{s}_{s\rightarrow b(t)}$:

- By simple inspection, we can find end-effector is rotating about the x-axis of $\mathscr{F}_o$ and the instant velocity along the axis is zero

$$\omega^o = [\alpha,0,0]^T$$
$$\hat{\omega}^o = [1,0,0]^T$$
$$q^o = [0,0,0]^T$$
$$\mathbf{v}^o_\omega = [0,0,0]^T$$

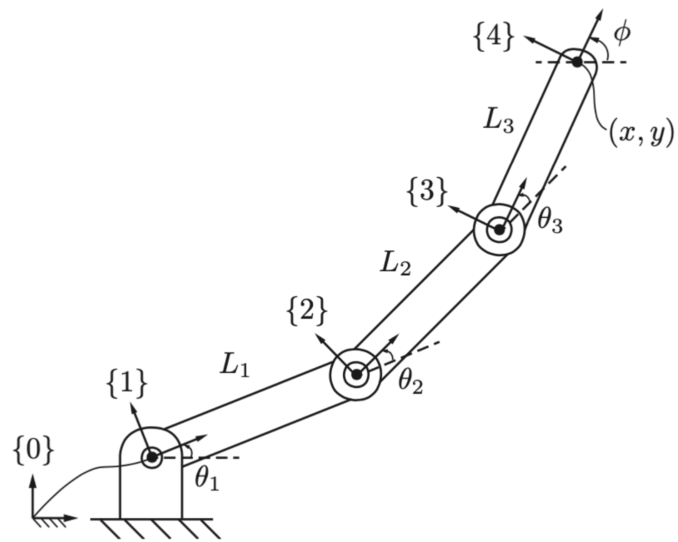
- Recall: $\xi^o = \begin{bmatrix} -[\omega^o]q^o + \mathbf{v}^o_\omega \\ \omega^o \end{bmatrix}$

- Thus we have $\xi^o_{s\rightarrow b(t)} = [0,0,0,\alpha,0,0]^T$



Example 2 of Change of Frame

- For the 3-link robot arm



- Given $\xi^3_{L_3(t)}$, what is $\xi^0_{L_3(t)}$? Assume the transformation is $T^0_{L_0 \to L_3(t)}$ at time t.

Change of Frame by Similarity Transformation

- For two observers, one records by $\mathscr{F}_{s_1}$ and the other by $\mathscr{F}_{s_2}$, then
  - $\dot{T}^{s_1}_{s' \to b(t)} = [\xi^{s_1}_{b(t)}] T^{s_1}_{s' \to b(t)}$
  - $\dot{T}^{s_2}_{s' \to b(t)} = [\xi^{s_2}_{b(t)}] T^{s_2}_{s' \to b(t)}$

Change of Frame by Similarity Transformation

$$[\xi_{b(t)}^{s_1}] = T_{s_1 \to s_2}[\xi_{b(t)}^{s_2}]T_{s_1 \to s_2}^{-1}$$

- When the observer's frame changes,
  - twist also conforms to the similarity transformation

Change of Frame by Similarity Transformation

- By $T^{s_1}_{s'\to b(t)} = T_{s_1\to s_2} T^{s_2}_{s'\to b(t)} (T_{s_1\to s_2})^{-1}$,

$$
\begin{aligned}
\dot{T}^{s_1}_{s'\to b(t)} = T_{s_1\to s_2}\dot{T}^{s_2}_{s'\to b(t)}(T_{s_1\to s_2})^{-1} &\Leftrightarrow [\xi^{s_1}_{b(t)}]T^{s_1}_{s'\to b(t)} = T_{s_1\to s_2}[\xi^{s_2}_{b(t)}]T^{s_2}_{s'\to b(t)}(T_{s_1\to s_2})^{-1} \\
&\Leftrightarrow [\xi^{s_1}_{b(t)}] = T_{s_1\to s_2}[\xi^{s_2}_{b(t)}]T^{s_2}_{s'\to b(t)}(T_{s_1\to s_2})^{-1}(T^{s_1}_{s'\to b(t)})^{-1} \\
&\Leftrightarrow [\xi^{s_1}_{b(t)}] = T_{s_1\to s_2}[\xi^{s_2}_{b(t)}]T^{s_2}_{s'\to b(t)}\{(T_{s_1\to s_2})^{-1}(T^{s_1}_{s'\to b(t)})^{-1}T_{s_1\to s_2}\}(T_{s_1\to s_2})^{-1} \\
&\Leftrightarrow [\xi^{s_1}_{b(t)}]T_{s_1\to s_2} = T_{s_1\to s_2}[\xi^{s_2}_{b(t)}]T^{b}_{s'\to b(t)}(T^{b}_{s'\to b(t)})^{-1}(T_{s_1\to s_2})^{-1} \\
&\Leftrightarrow [\xi^{s_1}_{b(t)}]T_{s_1\to s_2} = T_{s_1\to s_2}[\xi^{s_2}_{b(t)}]
\end{aligned}
$$

$$
\boxed{[\xi^{s_1}_{b(t)}] = T_{s_1\to s_2}[\xi^{s_2}_{b(t)}]T^{-1}_{s_1\to s_2}}
$$

Read by Yourself

Adjoint Matrix

$$[\xi_{b(t)}^{s_1}] = T_{s_1 \to s_2} [\xi_{b(t)}^{s_2}] T_{s_1 \to s_2}^{-1}$$

- $\xi_{b(t)}^{s_1}$ is linear w.r.t. $\xi_{b(t)}^{s_2}$
- We introduce a matrix $[\mathrm{Ad}_{T_{s_1 \to s_2}}] \in \mathbb{R}^{6 \times 6}$ to relate them:

$$\xi_{b(t)}^{s_1} = [Ad_{T_{s_1 \to s_2}}] \xi_{b(t)}^{s_2}$$

- Do computation based on the similarity transformation, and you can get

$$[\mathrm{Ad}_{T_{s_1 \to s_2}}] = \begin{bmatrix} R_{s_1 \to s_2} & [\mathbf{t}_{s_1 \to s_2}] R_{s_1 \to s_2} \\ 0 & R_{s_1 \to s_2} \end{bmatrix}$$

Spatial Twist and Body Twist

- If we observe the motion of the body
  - from $\mathscr{F}_s$, the velocity is $\xi^s_{b(t)}$ **(spatial twist)**
  - from the moving object $\mathscr{F}_b$, the velocity is $\xi^{b(t)}_{b(t)}$ **(body twist)**

Spatial Twist and Body Twist

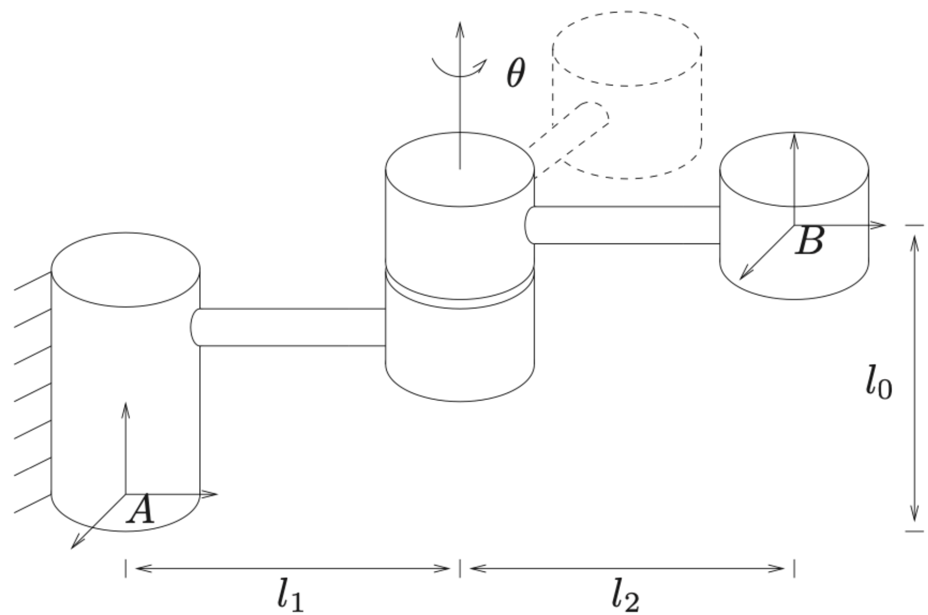
- By $\dot{T}^s_{s'\to b(t)} = [\xi^s_{b(t)}]T_{s'\to b(t)}$, $\boxed{[\xi^s_{b(t)}] = \dot{T}^s_{s'\to b(t)}(T^s_{s'\to b(t)})^{-1}}$
- Using the similarity transformation to change the frame, we have
  - $T^s_{s'\to b(t)}[\xi^{b(t)}_{b(t)}](T_{s'\to b(t)})^{-1} = \dot{T}^s_{s'\to b(t)}(T^s_{s'\to b(t)})^{-1}$
  - $\therefore \boxed{[\xi^{b(t)}_{b(t)}] = (T_{s'\to b(t)})^{-1}\dot{T}^s_{s'\to b(t)}}$

Example 3 of Change of Frame

- Given the motion of rigid-body

$$T_{A\to B(t)} = \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) & 0 & -l_2\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) & 0 & l_1 + l_2\cos\theta(t) \\ 0 & 0 & 1 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- What is the spatial twist?
- What is the body twist?



Example 3 of Change of Frame

- Given the motion of rigid-body

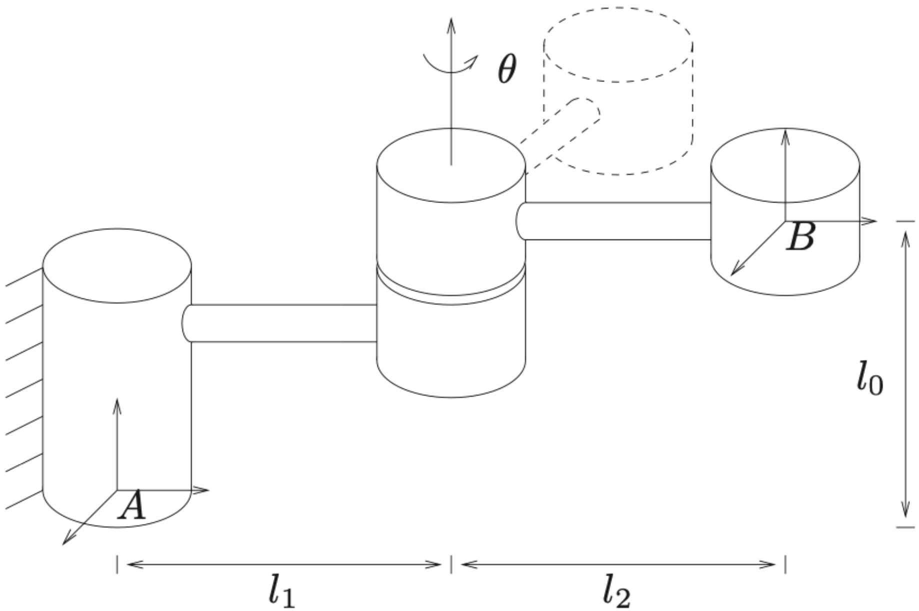
$$T_{A\to B(t)} = \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) & 0 & -l_2\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) & 0 & l_1 + l_2\cos\theta(t) \\ 0 & 0 & 1 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- $[\xi^A_{B(t)}] = \dot{T}_{A\to B(t)} T^{-1}_{A\to B(t)}$

$\xi^A_{B(t)} = [l_1, 0, 0, 0, 0, 1]^T$

- $[\xi^B_{B(t)}] = T^{-1}_{A\to B(t)} \dot{T}_{A\to B(t)}$

$\xi^{B(t)}_{B(t)} = [-l_2, 0, 0, 0, 0, 1]^T$



```python
import sympy as sp
from sympy import *

t = symbols("t")
l0 = symbols("l0")
l1 = symbols("l1")
l2 = symbols("l2")

T = Matrix(symarray('T', (4, 4)))
T[0, 0] = cos(t)
T[0, 1] = -sin(t)
T[0, 2] = 0
T[0, 3] = -l2 * sin(t)
T[1, 0] = sin(t)
T[1, 1] = cos(t)
T[1, 2] = 0
T[1, 3] = l1 + l2 * cos(t)
T[2, 0] = 0
T[2, 1] = 0
T[2, 2] = 1
T[2, 3] = l0
T[3, 0] = 0
T[3, 1] = 0
T[3, 2] = 0
T[3, 3] = 1

xi_s = sp.diff(T, t) @ sp.Inverse(T)
xi_s.simplify()

xi_b = sp.Inverse(T) @ sp.diff(T, t)
xi_b.simplify()
```

Example 3 of Change of Frame

- By $T_{A\to B(t)} = \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) & 0 & -l_2\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) & 0 & l_1 + l_2\cos\theta(t) \\ 0 & 0 & 1 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, we have

$$R_{A\to B(t)} = \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) & 0 \\ \sin\theta(t) & \cos\theta(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{t}_{A\to B(t)} = \begin{bmatrix} -l_2\sin\theta(t) \\ l_1 + l_2\cos\theta(t) \\ l_0 \end{bmatrix}.$$

- By $[\mathrm{Ad}_{T_{A\to B(t)}}] = \begin{bmatrix} R_{A\to B(t)} & [\mathbf{t}_{A\to B(t)}]R_{A\to B(t)} \\ 0 & R_{A\to B(t)} \end{bmatrix}$,

$$[\mathrm{Ad}_{T_{A\to B(t)}}] = \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) & 0 & -l_0\sin\theta(t) & -l_0\cos\theta(t) & l_1 + l_2\cos\theta(t) \\ \sin\theta(t) & \cos\theta(t) & 0 & l_0\cos\theta(t) & -l_0\sin\theta(t) & l_2\sin\theta(t) \\ 0 & 0 & 1 & -l_1\cos\theta(t) - l_2 & l_1\sin\theta(t) & 0 \\ 0 & 0 & 0 & \cos\theta(t) & -\sin\theta(t) & 0 \\ 0 & 0 & 0 & \sin\theta(t) & \cos\theta(t) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 3 of Change of Frame

By $\xi^{A}_{A\to B(t)} = [l_1, 0, 0, 0, 0, 1]^T$

$\xi^{B}_{A\to B(t)} = [-l_2, 0, 0, 0, 0, 1]^T$

$$[\mathrm{Ad}_{T_{s\to b}}] = \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) & 0 & -l_0\sin\theta(t) & -l_0\cos\theta(t) & l_1 + l_2\cos\theta(t) \\ \sin\theta(t) & \cos\theta(t) & 0 & l_0\cos\theta(t) & -l_0\sin\theta(t) & l_2\sin\theta(t) \\ 0 & 0 & 1 & -l_1\cos\theta(t) - l_2 & l_1\sin\theta(t) & 0 \\ 0 & 0 & 0 & \cos\theta(t) & -\sin\theta(t) & 0 \\ 0 & 0 & 0 & \sin\theta(t) & \cos\theta(t) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

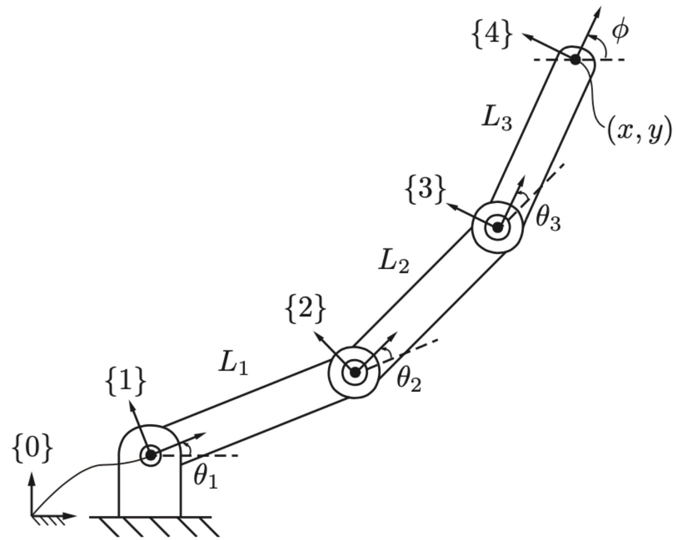
We can verify that $\xi^{A}_{A\to B(t)} = [\mathrm{Ad}_{T_{s\to b}}]\xi^{B}_{A\to B(t)}$

Summary

- Twist ξ denotes the 6D motion velocity
- Relationship with $\dot{T}$: $\dot{T}^o_{s' \to b(t)} = [\xi^o_{b(t)}] T^o_{s' \to b(t)}$
- Change of frame:
  - $[\xi^{s_1}_{b(t)}] = T_{s_1 \to s_2} [\xi^{s_2}_{b(t)}] T^{-1}_{s_1 \to s_2}$
  - $\xi^{s_1}_{b(t)} = [Ad_{T_{s_1 \to s_2}}] \xi^{s_2}_{b(t)}$
- Spatial twist: $[\xi^s_{b(t)}] = \dot{T}^s_{s' \to b(t)} (T^s_{s' \to b(t)})^{-1}$
- Body twist: $[\xi^{b(t)}_{b(t)}] = (T_{s' \to b(t)})^{-1} \dot{T}^s_{s' \to b(t)}$

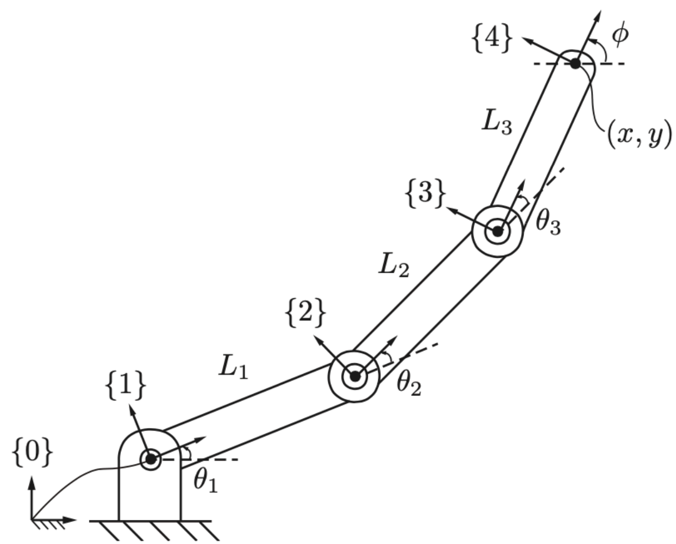
Jacobian of Kinematics Chain

Forward Kinematic Problem



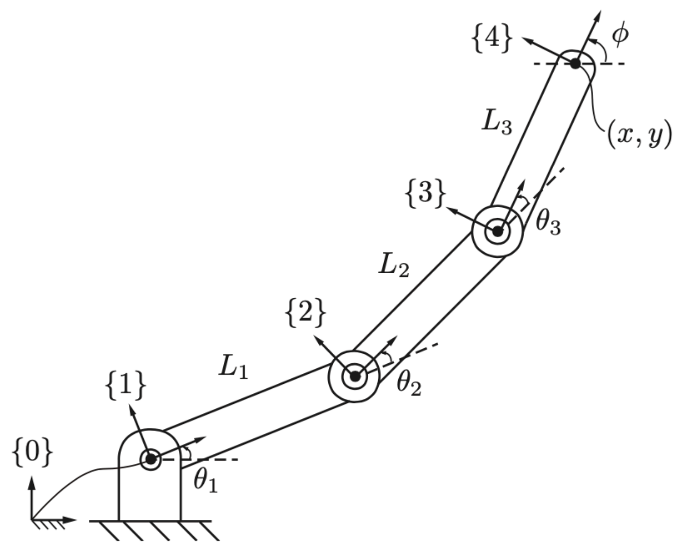
- Suppose that the arm moves
- How do I compute the velocity of the end-effector from the angular velocity of joints?

Spatial Frame Inverse Kinematics Problem



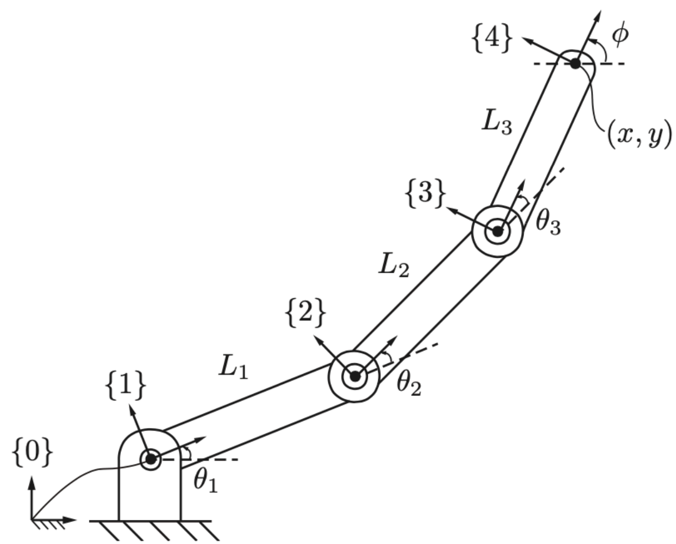
- If I specify the direction of the end-of-effector movement using the spatial frame, how can I change the joint angles?
- e.g. move to a pre-specified $T^s_{s\to e}$

Body Frame Inverse Kinematics Problem



- If I specify the direction of the end-of-effector movement using the body frame, how can I change the joint angles?
- e.g. move the end-effector forward along its link

Kinematic Equation



- We can solve the problems if we have $\xi_{e(t)} = f(\dot{\theta})$
- The language to describe the velocity of end-effector are
  - $\xi^{s}_{e(t)}$ for spatial frame query
  - $\xi^{e(t)}_{e(t)}$ for body frame query
- We will derive the f^s and $f^{e(t)}$

Spatial Geometric Jacobian

- Spatial Geometric Jacobian $J^s(\theta)$:

$$\xi^s_{e(t)} = J^s(\theta)\dot{\theta}$$

where $\theta \in \mathbb{R}^n$ (n joints), $J^s(\theta) \in \mathbb{R}^{6\times n}$, and the i-th column of $J(\theta)$ is ${}^{i}\hat{\xi}^s_{e(t)}$, the twist when the movement is caused only by the i-th joint **while all other joints stay static**

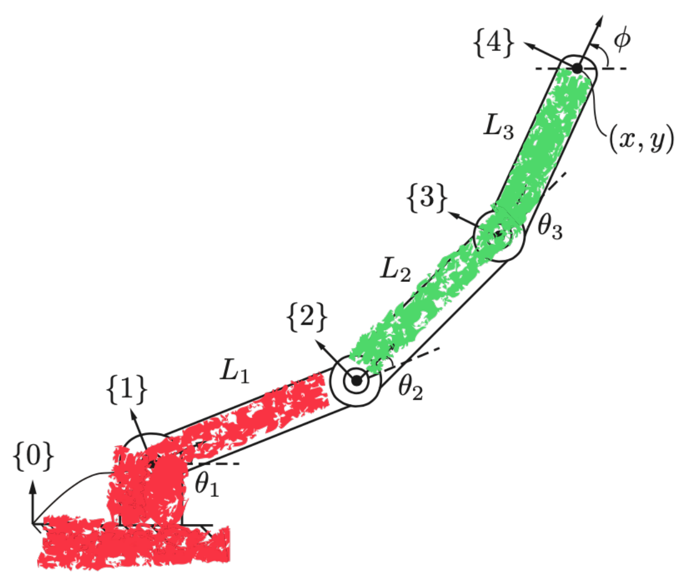
Spatial Geometric Jacobian

- Spatial Geometric Jacobian $J^s(\theta)$:

$$\xi^s_{e(t)} = J^s(\theta)\dot{\theta}$$

where $\theta \in \mathbb{R}^n$ (n joints), $J^s(\theta) \in \mathbb{R}^{6\times n}$, and the i-th column of $J(\theta)$ is ${}^i\hat{\xi}^s_{e(t)}$, the twist when the movement is caused only by the i-th joint **while all other joints stay static**

- For example, ${}^2\hat{\xi}^s_{e(t)}$ describes the motion of the green part, which is to revolute about Joint {2} (in this revolute joint, $\hat{\omega}^s$, q^s, and d^s are obvious).



Spatial Geometric Jacobian (Proof)

- First of all, $\dot{T}^s_{s'\to e(t)} = [\xi^s_{e(t)}]T^s_{s'\to e(t)}$
- Suppose that only θ_i can change at some $(\theta_1, \cdots, \theta_n)$. Let ${}^iM^s_{s'\to e(t)}(\theta_i) := T^s_{s'\to e(t)}(\theta_1, \cdots, \theta_n)$, then

$$ {}^i\dot{M}^s_{s'\to e(t)} = [{}^i\xi^s_{e(t)}]T^s_{s'\to e(t)} $$

- By total derivative,

$$ \dot{T}^s_{s'\to e(t)} = \sum_i \frac{\partial T^s_{s'\to e(t)}}{\partial \theta_i}\dot{\theta}_i = \sum_i {}^i\dot{M}^s_{s'\to e(t)} = \sum_i [{}^i\xi^s_{e(t)}]T^s_{s'\to e(t)} $$

- Therefore, $[\xi^s_{e(t)}] = \sum_i [{}^i\xi^s_{e(t)}] = \sum_i [{}^i\hat{\xi}^s_{e(t)}]\dot{\theta}_i$

Body Geometric Jacobian

- The previous proof works for any recoding frame. Simple substitution of $e(t)$ for s as the recording frame gives:
- Body Geometric Jacobian $J^{e(t)}(\theta)$:

$$\xi_{e(t)}^{e(t)} = J^{e(t)}(\theta)\dot{\theta}$$

where $J^{e(t)}(\theta) \in \mathbb{R}^{6 \times n}$, and the i-th column of $J(\theta)$ is ${}^{i}\hat{\xi}_{e(t)}^{e(t)}$, the twist when the movement is caused only by the i-th joint **while all other joints stay static**

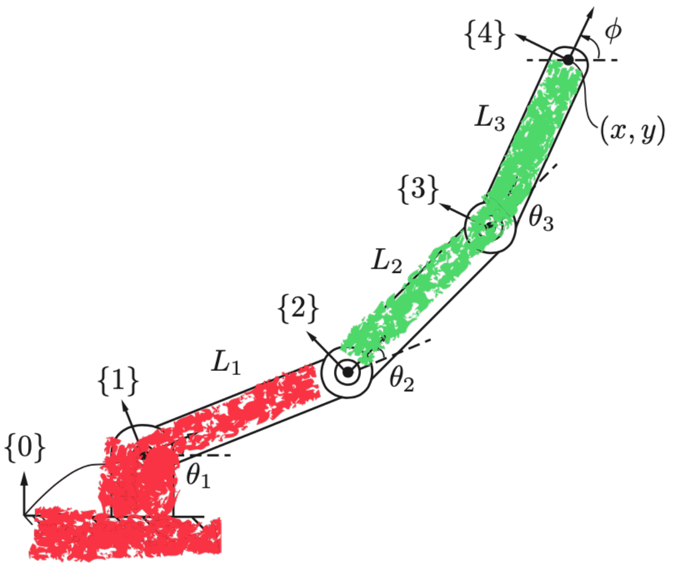
Body Geometric Jacobian

- Body Geometric Jacobian $J^{e(t)}(\theta)$:

$$\xi_{e(t)}^{e(t)} = J^{e(t)}(\theta)\dot{\theta}$$

where $J^{e(t)}(\theta) \in \mathbb{R}^{6\times n}$, and the i-th column of $J(\theta)$ is ${}^{i}\hat{\xi}_{e(t)}^{e(t)}$, the twist when the movement is caused only by the i-th joint **while all other joints stay static**

- For example, ${}^{2}\hat{\xi}_{e(t)}^{e(t)}$ describes the motion of the green part observed by $\mathscr{F}_s = \mathscr{F}_{\{0\}}$, which is to revolute about Joint {2}
- For this revolute joint, $\hat{\omega}^{e(t)}$, $q^{e(t)}$, and $d^{e(t)}$ can be computed using $T_{\{2\}\to\{4\}}$.



Computation of Geometric Jacobian

- Just need to know ${}^{i}\hat{\xi}^{o}_{e(t)}$ for the recording frame $\mathcal{F}_o$
- When computing ${}^{i}\hat{\xi}^{o}_{e(t)}$, only the i-th joint can move
- Therefore, we can view as it as a single-joint problem, as our Example 1

Computation of Geometric Jacobian

- Method 1:
  - Figure out ${}^{i}\hat{\xi}^{o}_{e(t)}$ for each joint by first computing $\hat{\omega}^{o}$, q^{o}, and d^{o} (as in the robot arm example with red/green colors)
- Method 2:
  - ${}^{i}\hat{\xi}^{o}_{e(t)} = [\mathrm{Ad}_{T_{o \to L_i}}]\hat{\xi}^{L_i}_{e(t)}$
  - Assume that the joint axis is aligned with the x-axis of $\hat{L}_i$
  - $\xi^{\hat{L}_i}_{e(t)} = [0,0,0,1,0,0]^T$ for prismatic joints
  - $\xi^{\hat{L}_i}_{e(t)} = [1,0,0,0,0,0]^T$ for prismatic joints

Inverse Kinematics

Inverse Kinematics

- Position query
  - Given the forward kinematics $T^s_{s\to e}(\theta)$ and the target pose $T_{target} = \mathbb{SE}(3)$, find θ that satisfies $T_{s\to e}(\theta) = T_{target}$
- Velocity query
  - Given the twist of the end-effector, find the angular velocity that satisfies $\xi_{target} = J(\theta)\dot{\theta}$
- May have multiple solutions, a unique solution or no solution

Null Space of Jacobian

- Consider the velocity query IK task
- Recall that $\xi = J(\theta)\dot{\theta}$ for an n-joint kinematic chain, where J is a $6 \times n$ matrix
- When $n > 6$, the joint space is projected to a lower-dimensional space and J must exist a null space
- As a result, IK may have infinite solutions (a special solution + any vector in the null space of J)
- The null space adds flexibility to make motion plans

Analytical Solution

- Try to solve the equation $T_{target} = T(\theta)$ and get an analytical solution for θ
    - e.g., solve θ_1 and θ_2 for $\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & -\sin\theta_1(l_2+l_3) \\ \sin\theta_1 & \cos\theta_1 & 0 & \cos\theta_1(l_2+l_3) \\ 0 & 0 & 1 & l_1 - l_4 + \theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = T_{target}$
- For robots with more than 3-DoF, analytical solution can be very complex
    - e.g., for a 6-DoF robot, you will need several pages to write down the formula
- Some useful libraries: IKFast, IKBT

Numerical Solution

- Solving a nonlinear optimization problem
- Standard numerical optimization algorithms can be utilized, e.g. Newton-Raphson and Levenberg-Marquardt
- Numerical IK leverages the geometric Jacobian
  $\xi = J(\theta)\dot{\theta}$

Levenberg–Marquardt Algorithm

- Error between the desired pose and the current one:

$$T_{err} = T_{target} T(\theta)^{-1} \in \mathbb{SE}(3)$$

- Calculate the corresponding screw:

$$\chi_{err} = \log(T_{err}) \in \mathfrak{se}(3)$$

- Recall that $\xi = J(\theta)\dot{\theta}$:

$$\xi \Delta t = J(\theta)\dot{\theta}\Delta t \Rightarrow \Delta\chi \approx J(\theta)\Delta\theta$$

Levenberg–Marquardt Algorithm

- In LM algorithm, we iteratively update θ
- In each iteration, we try to find a $\Delta\theta$ that minimizes:

$$S(\theta, \Delta\theta) = ||\chi_{err} - J(\theta)\Delta\theta||^2 + \lambda||\Delta\theta||^2$$

- λ term stabilizes the optimization
- Closed-form solution:

$$(J^{\mathrm{T}}J + \lambda I)\Delta\theta = J^{\mathrm{T}}\chi_{err}$$

- Solve $\Delta\theta$ and then update θ by: $\theta \leftarrow \theta + \Delta\theta$

Levenberg–Marquardt Algorithm

$$(J^{\mathrm{T}}J + \lambda I)\Delta\theta = J^{\mathrm{T}}\chi_{err}$$

- Damping factor $\lambda \geq 0$ is adjusted at each iteration:
- If $S(\theta, \Delta\theta)$ is decreasing, a smaller λ (e.g., $\lambda \leftarrow 0.1\lambda$) can be used.
  - closer to the Gauss–Newton algorithm
- Otherwise, a larger λ (e.g., $\lambda \leftarrow 10\lambda$) can be used.
  - closer to the gradient-descent algorithm

Read by Yourself

Levenberg–Marquardt Algorithm

- LM algorithm may converge to a local minima, initial θ_0 is very important:
  - Sampling multiple θ_0 may boost the performance
- In most cases, θ comes with limit constraints:
  - $l[i] \leq \theta[i] \leq r[i]$
  - A joint can only translate (or rotate) within the limit
  - Invalid state rejection
  - Clipping during the optimization iterations

Read by Yourself

Kinematic Singularity

Question: is it always possible to move the end-effector to any direction $\hat{\xi}$ for a robot with $\mathrm{DoF} \geq 6$

- **Kinematic singularity:**
  - A **robot configuration** where the robot's end-effector loses the ability to move in one direction instantaneously
- If $\mathrm{rank}(J(\theta)) < 6$ at some θ, by $\Delta\xi = J(\theta)\Delta\theta$, $\Delta\xi$ can only be in a linear space with dimension $\mathrm{rank}(J(\theta)) < 6$, losing its ability to move in some directions
- Note: Kinematic singularity does not mean that there exists a configuration that is not accessible (may get to the pose by some other motion trajectory)