

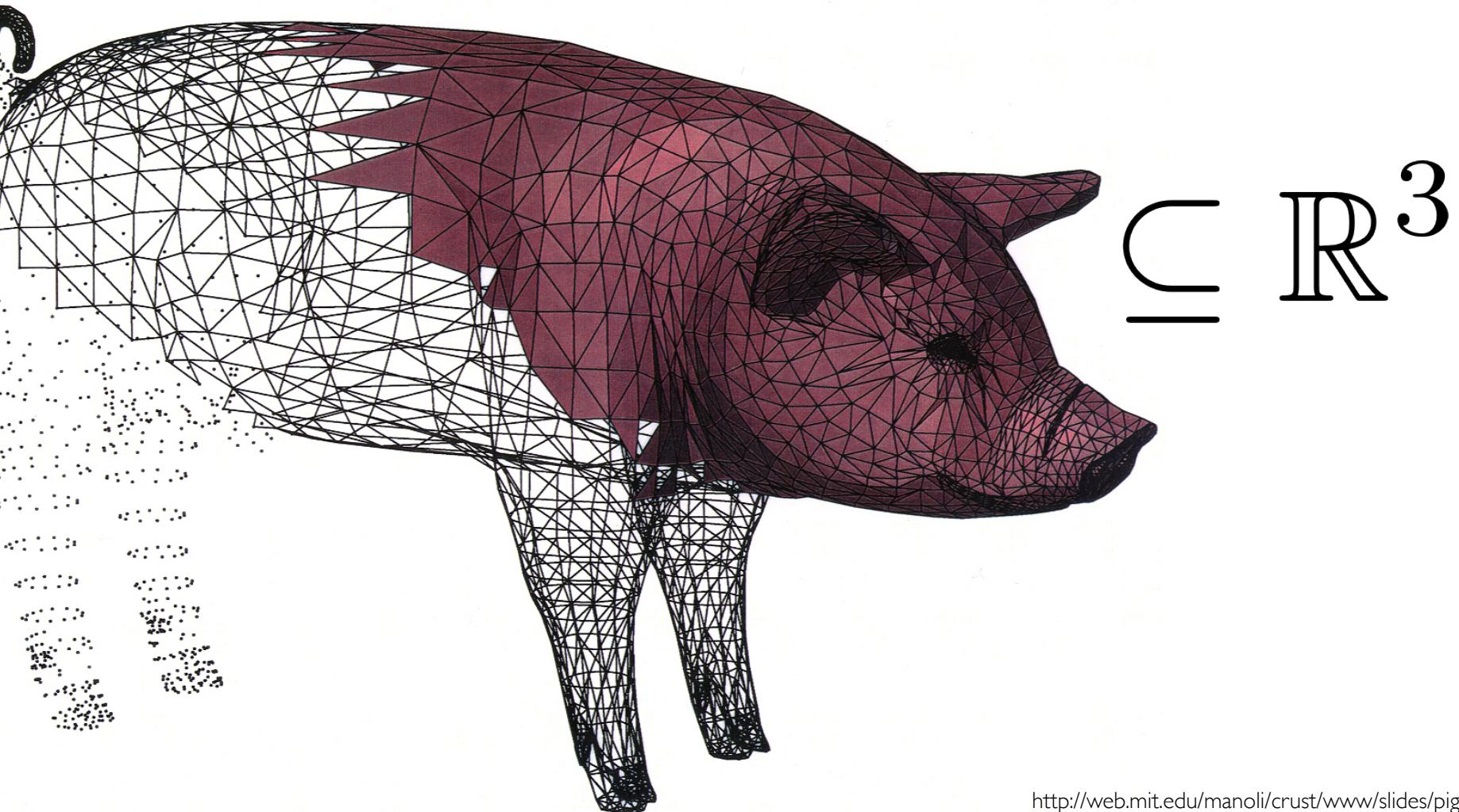
L2: Surfaces

Hao Su

Office Hour

- Check Piazza

Our Focus Today: Surface



Agenda

- Parameterized Surface
- Manifold
- Differential Map
- Curvature

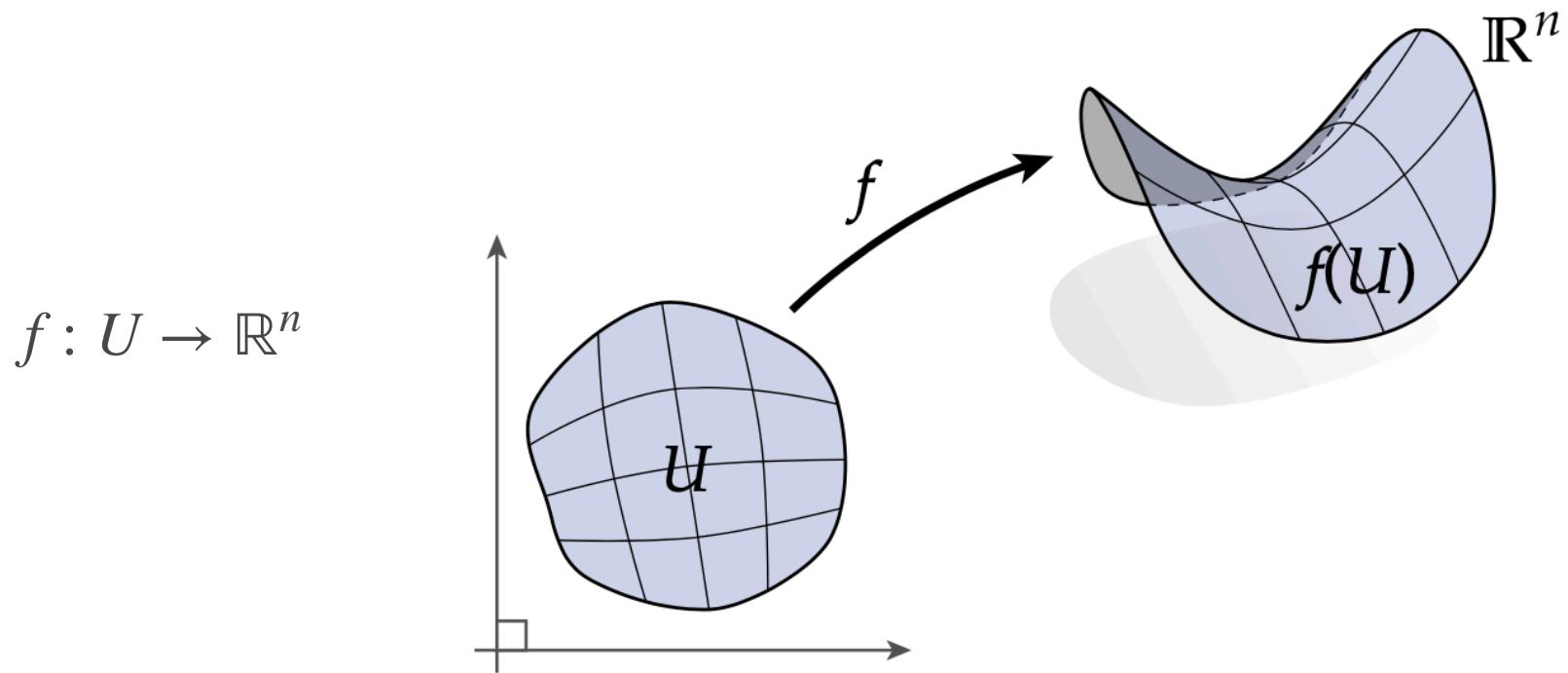


Lots of (sloppy) math!

Parameterized Surface

Parametrized Surface

A **parameterized surface** is a map from a two-dimensional region $U \subset \mathbb{R}^2$ into \mathbb{R}^n



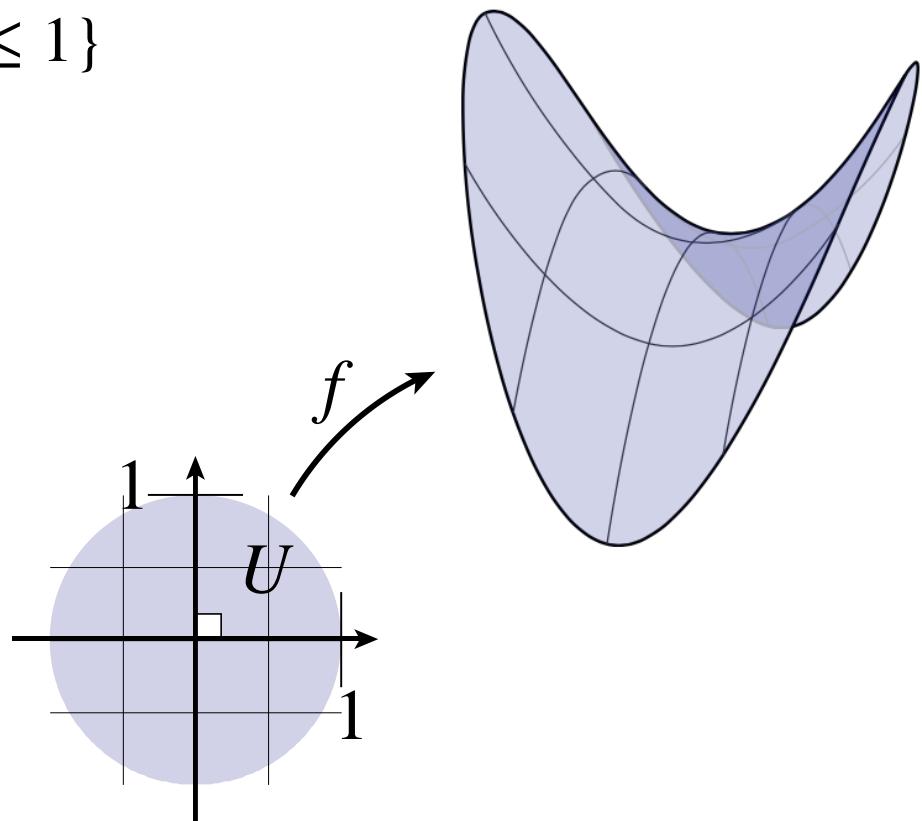
The set of points $f(U)$ is called the **image** of the parameterization.

Example

- Example: We can express a *saddle as a parameterized surface*:

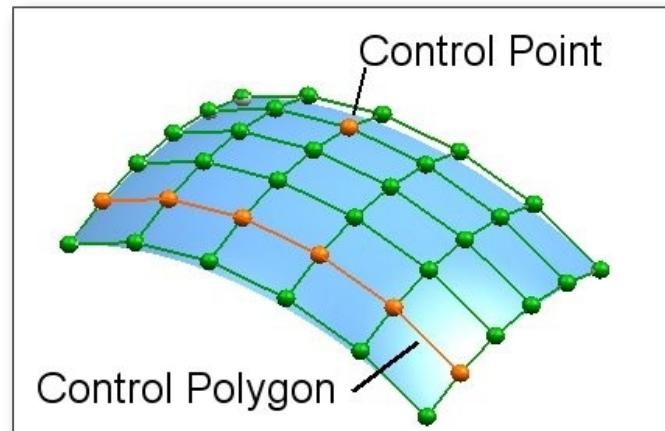
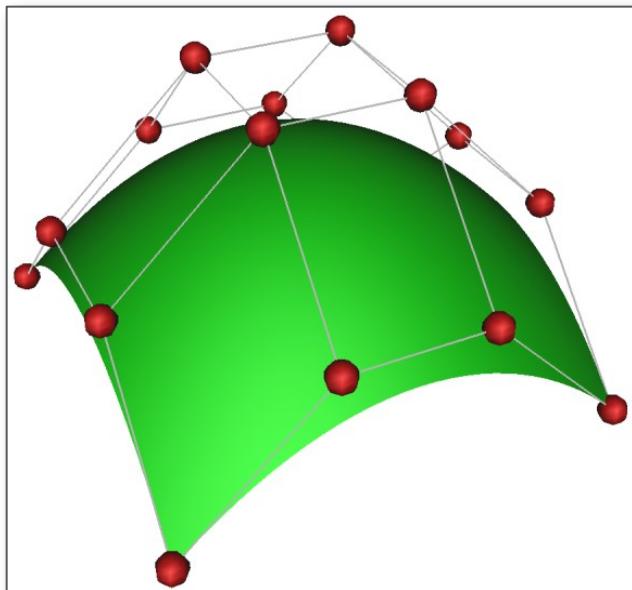
$$U := \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\}$$

$$f(u, v) = [u, v, u^2 - v^2]^T$$



Application: Bezier Surface, Spline Surface

- Smoothly “interpolate” between a set of points P_i

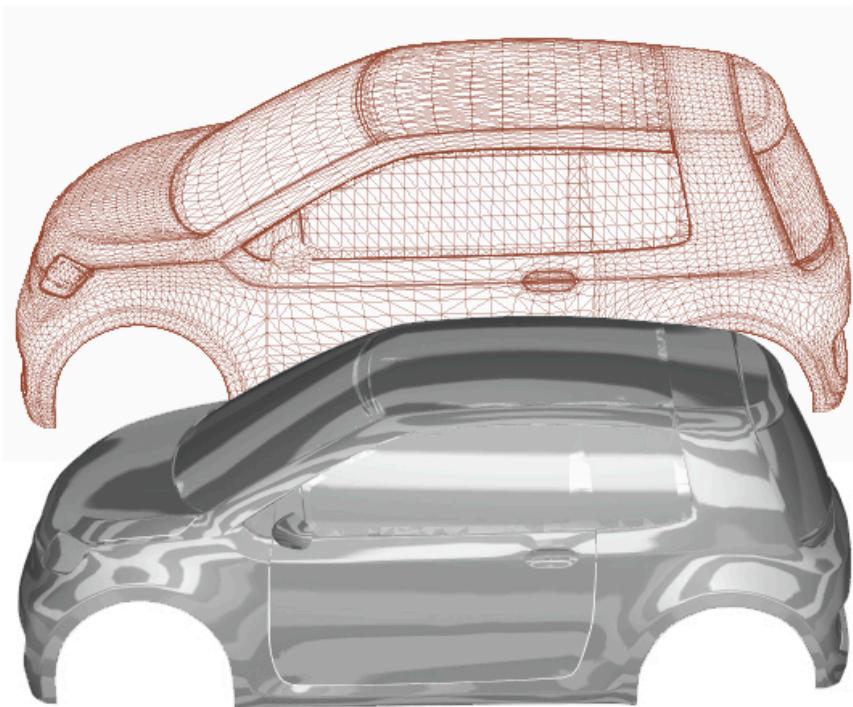


$$s(u, v) = \sum_{i=0}^m \sum_{j=0}^n \mathbf{p}_{i,j} B_i^m(u) B_j^n(v)$$

Application: Bezier Surface, Spline Surface

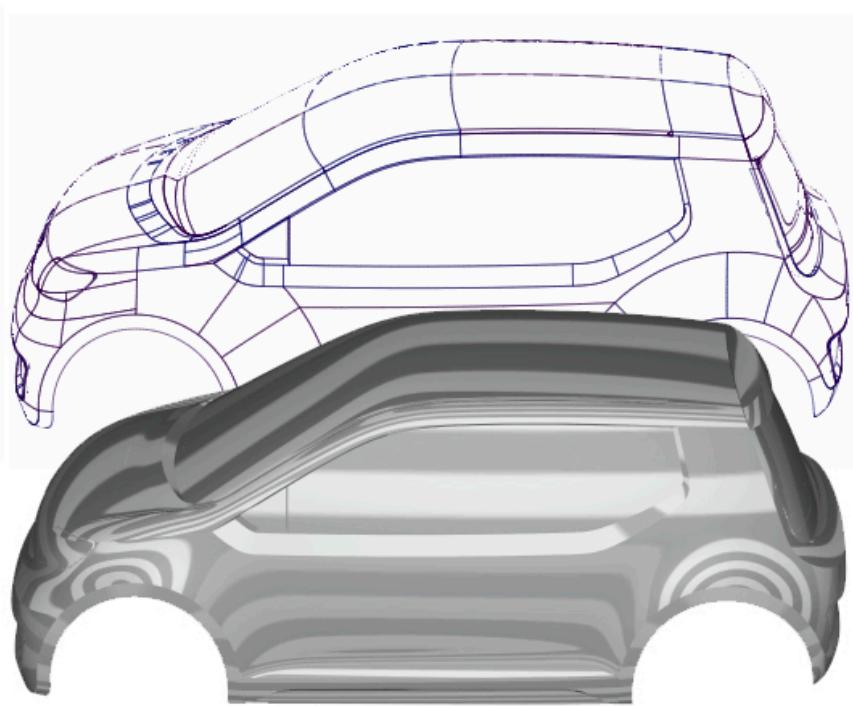
Widely used in design industry (e.g., car modeling)

Polygon model



Poor surface quality

NURBS model

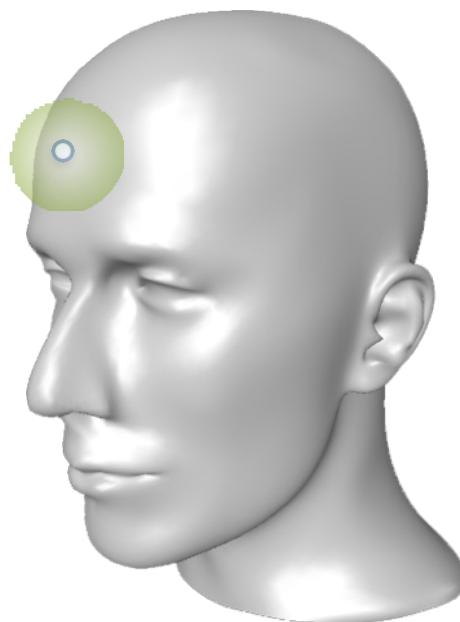


Pure, smooth highlights

(Differentiable) Manifold

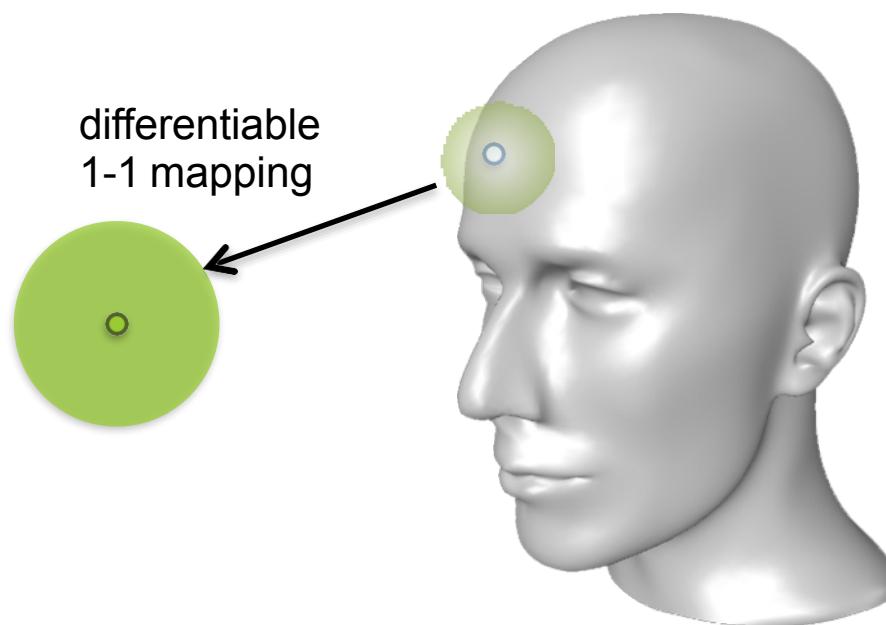
Smoothness as a Local Property

- Things that can be discovered by local observation: point + neighborhood



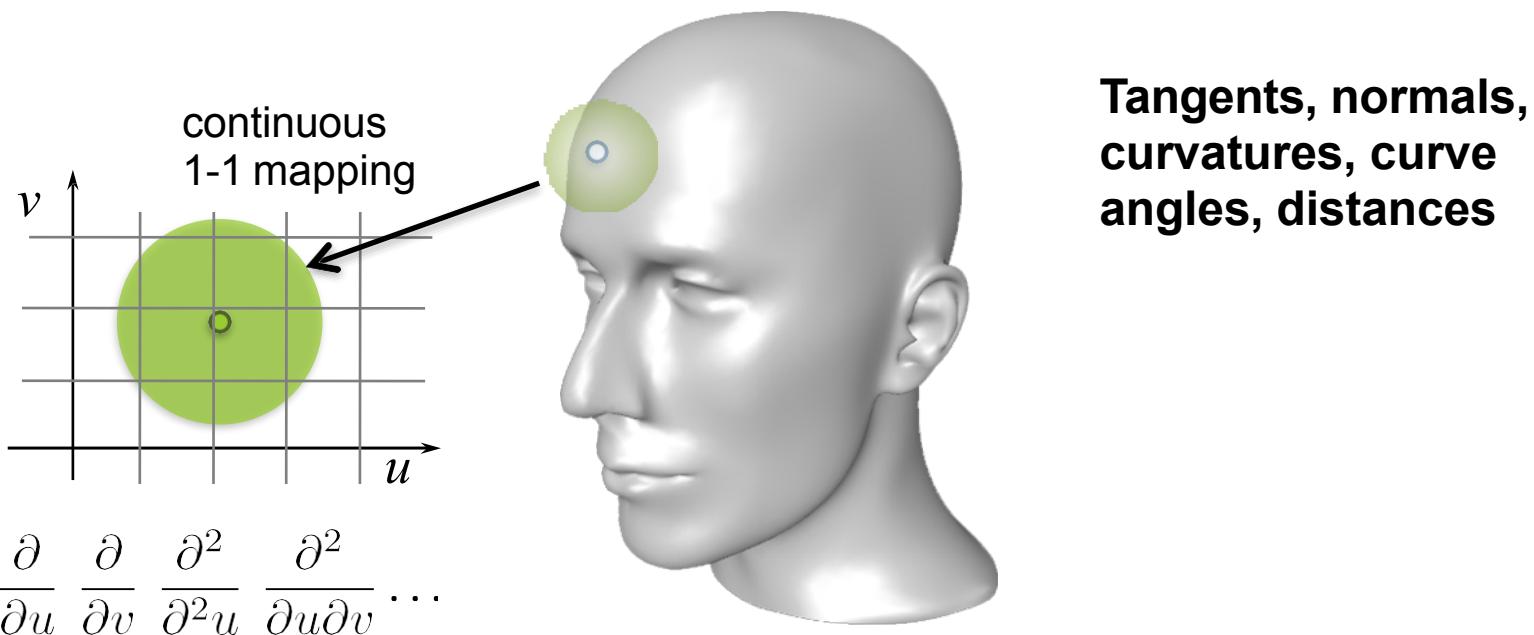
Local Smoothness

- Things that can be discovered by local observation: point + neighborhood



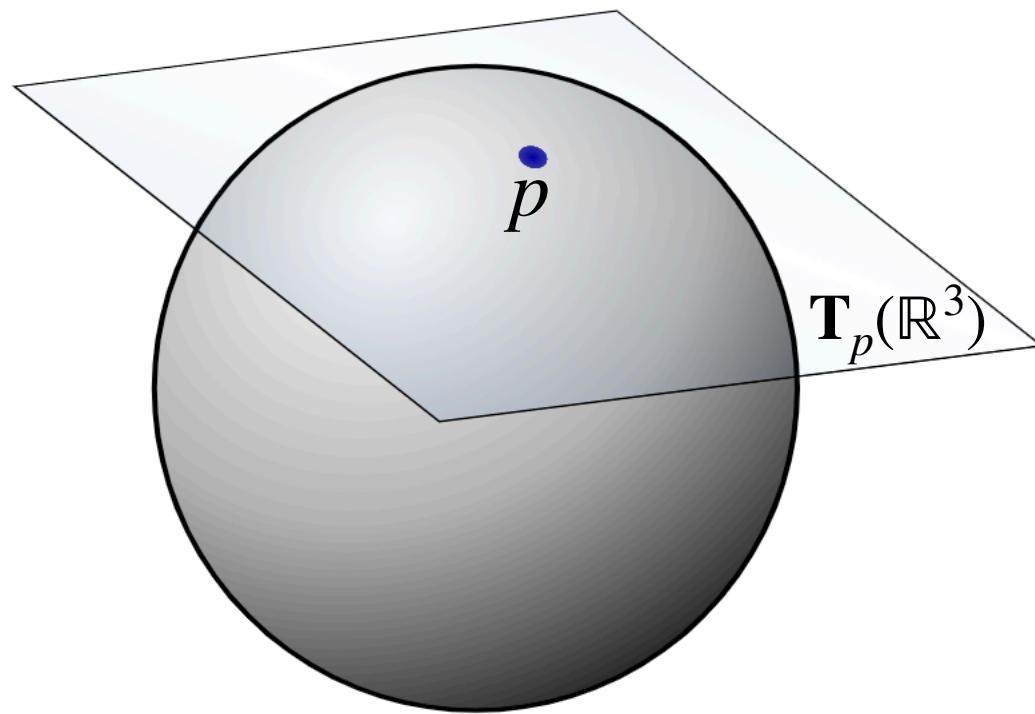
Local to Global

- Things that can be discovered by local observation: point + neighborhood



Tangent Plane

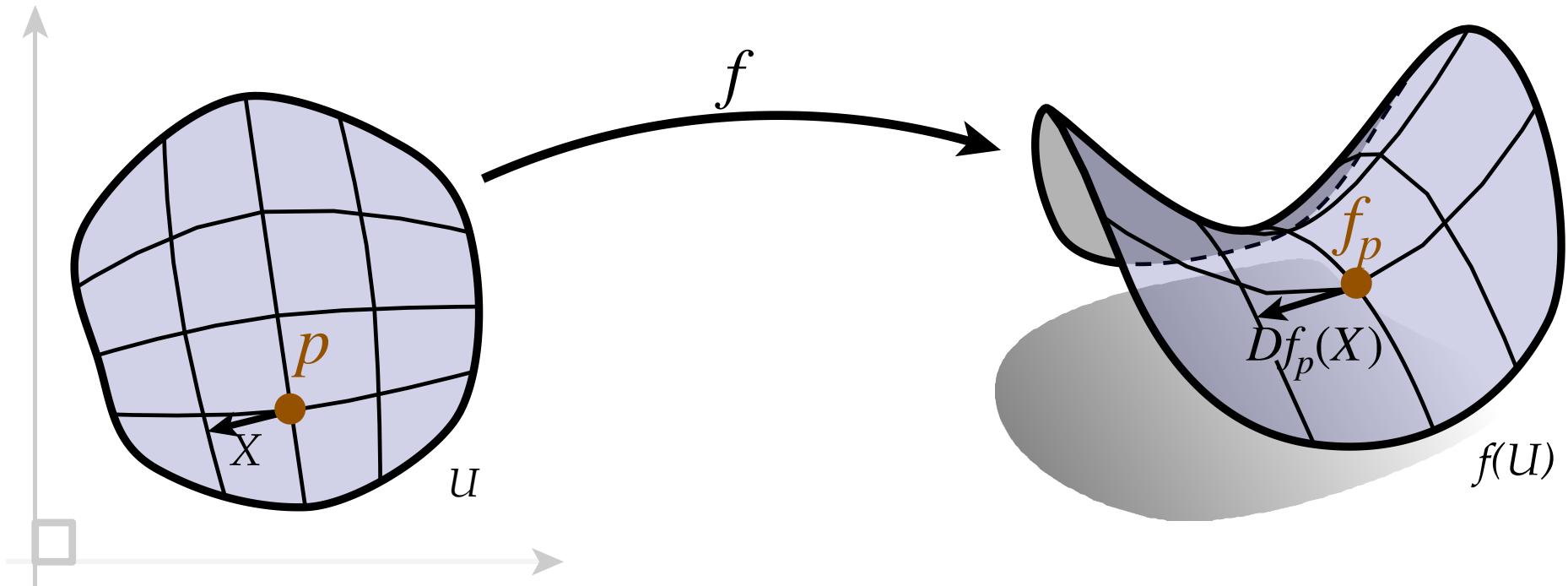
- One can attach to every point p a tangent plane \mathbf{T}_p
- Intuitively, it contains the possible directions in which one can tangentially pass through p .



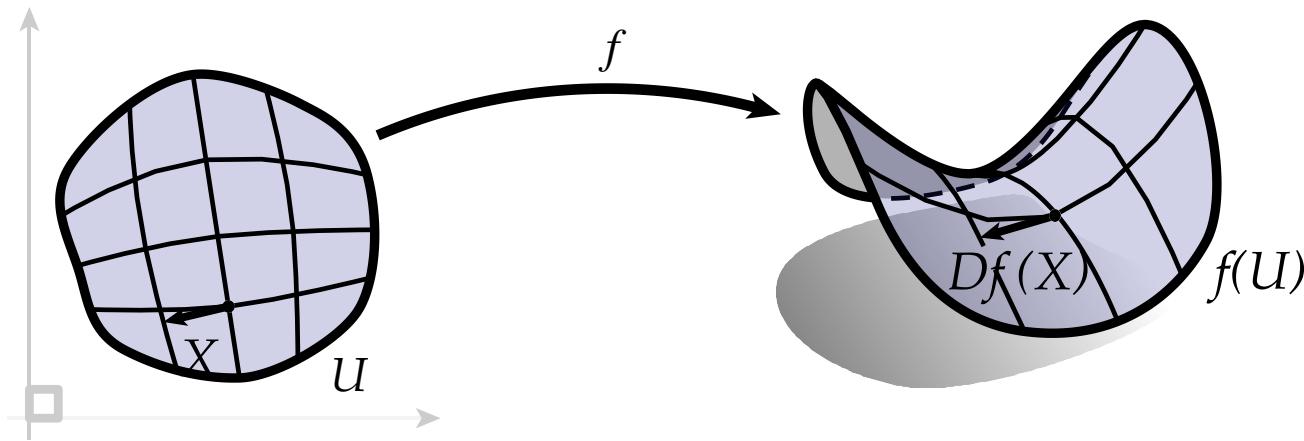
Differential Map

Differential of a Surface

- Relate the movement of point in the domain and on the surface



Differential of a Surface

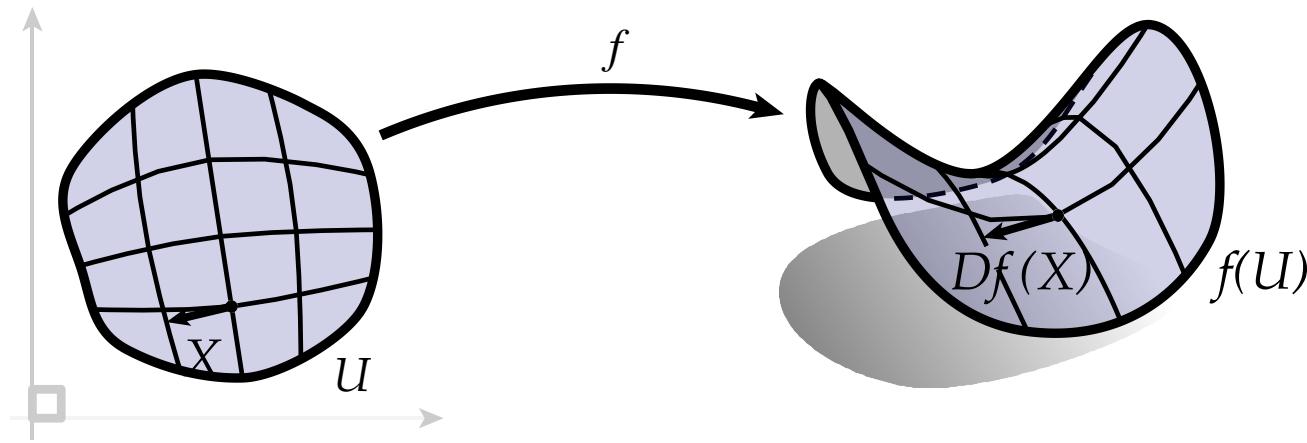


Total differential: $df = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv \implies \Delta f \approx \frac{\partial f}{\partial u} \Delta u + \frac{\partial f}{\partial v} \Delta v$

If point $p \in \mathbb{R}^2$ moves along vector $X = [u, v]^T$ by ϵ , the movement of f_p is:

$$\Delta f_p \approx \frac{\partial f}{\partial u}(\epsilon u) + \frac{\partial f}{\partial v}(\epsilon v) = \epsilon \left[\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v} \right] \begin{bmatrix} u \\ v \end{bmatrix}$$

Differential of a Surface



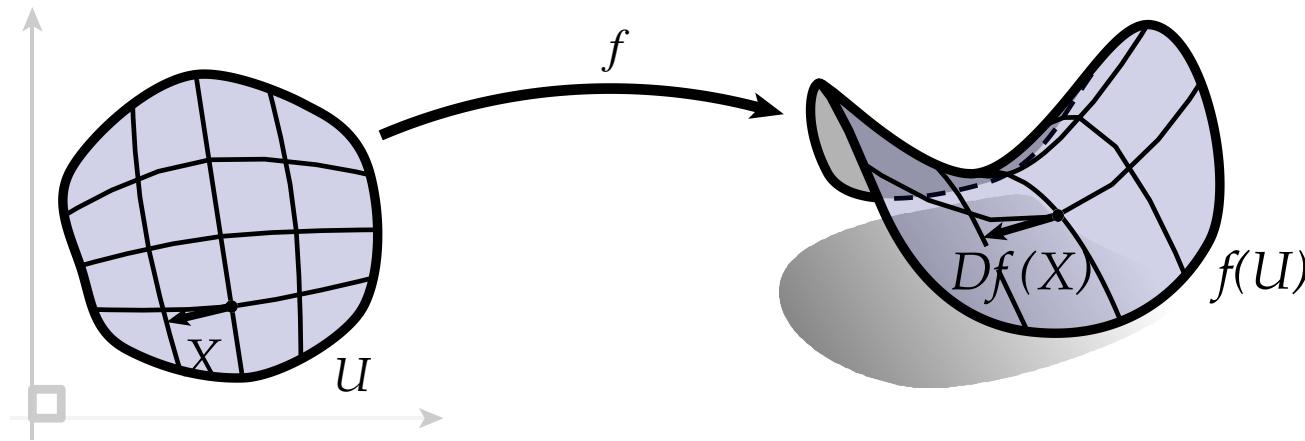
Total differential: $df = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv$

If point $p \in \mathbb{R}^2$ moves along vector $X = [u, v]^T$ by ϵ , the movement of f_p is:

$$\Delta f_p \approx \frac{\partial f}{\partial u}(\epsilon u) + \frac{\partial f}{\partial v}(\epsilon v) = \epsilon \left[\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v} \right] \begin{bmatrix} u \\ v \end{bmatrix}$$

$$Df_p := \left[\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v} \right] \in \mathbb{R}^{3 \times 2}$$

Differential of a Surface



Total differential: $df = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv$

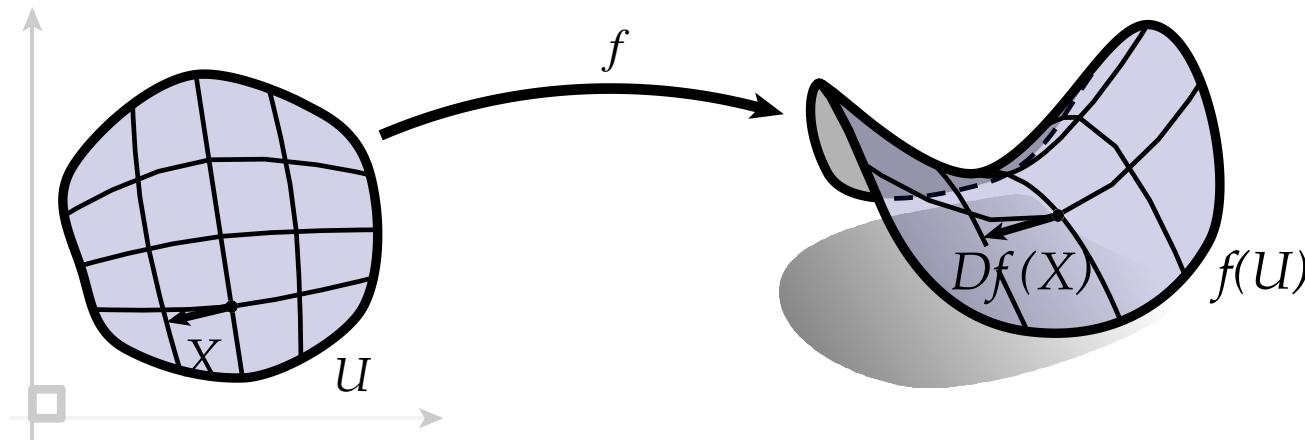
If point $p \in \mathbb{R}^2$ moves along vector $X = [u, v]^T$ by ϵ , the movement of f_p is:

$$\Delta f_p \approx \frac{\partial f}{\partial u}(\epsilon u) + \frac{\partial f}{\partial v}(\epsilon v) = \epsilon \left[\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v} \right] \begin{bmatrix} u \\ v \end{bmatrix} = \epsilon [Df_p]X$$

$$Df_p := \left[\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v} \right] \in \mathbb{R}^{3 \times 2}$$

Df_p : differential (Jacobian)
a linear map.

Differential of a Surface



Total differential: $df = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv$

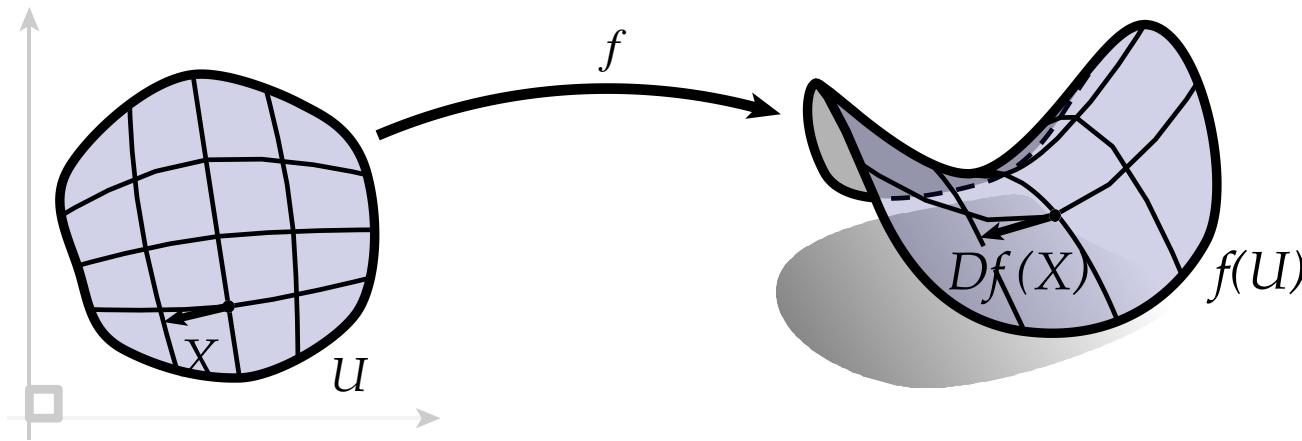
If point $p \in \mathbb{R}^2$ moves along vector $X = [u, v]^T$ by ϵ , the movement of f_p is:

$$\Delta f_p \approx \frac{\partial f}{\partial u}(\epsilon u) + \frac{\partial f}{\partial v}(\epsilon v) = \epsilon \left[\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v} \right] \begin{bmatrix} u \\ v \end{bmatrix} = \epsilon [Df_p] \boxed{X}$$

$$Df_p := \left[\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v} \right] \in \mathbb{R}^{3 \times 2}$$

movement **direction** in 2D domain

Differential of a Surface



Total differential: $df = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv$

If point $p \in \mathbb{R}^2$ moves along vector $X = [u, v]^T$ by ϵ , the movement of f_p is:

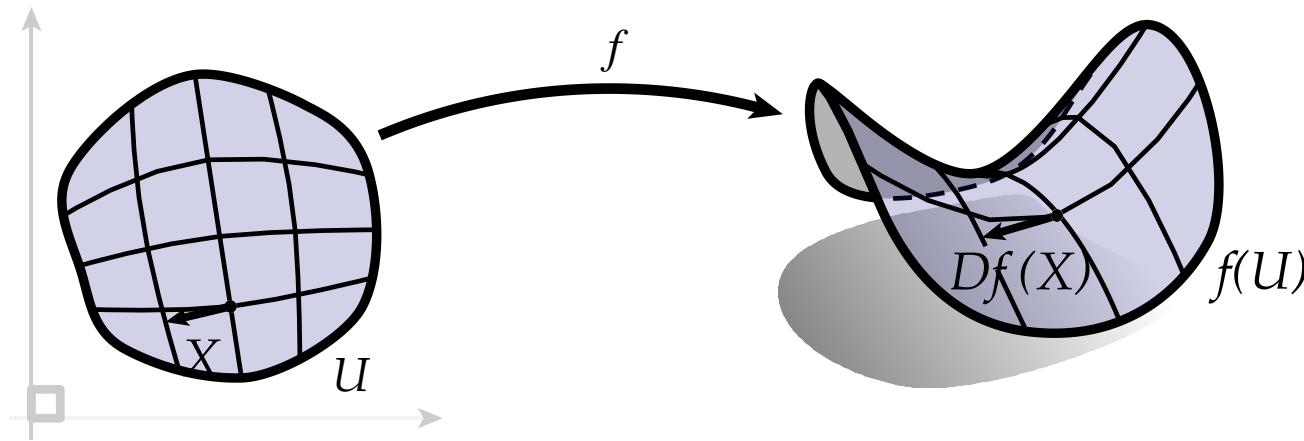
movement direction in 3D space

$$\Delta f_p \approx \frac{\partial f}{\partial u}(\epsilon u) + \frac{\partial f}{\partial v}(\epsilon v) = \epsilon \left[\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v} \right] \begin{bmatrix} u \\ v \end{bmatrix} = \epsilon [Df_p] X$$

$$Df_p := \left[\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v} \right] \in \mathbb{R}^{3 \times 2}$$

movement direction in 2D domain

Differential of a Surface



Intuitively, the *differential* of a parameterized surface tells us how tangent vectors on the domain get mapped to tangent vectors in space:

$$\Delta f_p \approx \frac{\partial f}{\partial u}(\epsilon u) + \frac{\partial f}{\partial v}(\epsilon v) = \epsilon \left[\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v} \right] \begin{bmatrix} u \\ v \end{bmatrix} = \epsilon [Df_p] \boxed{X}$$

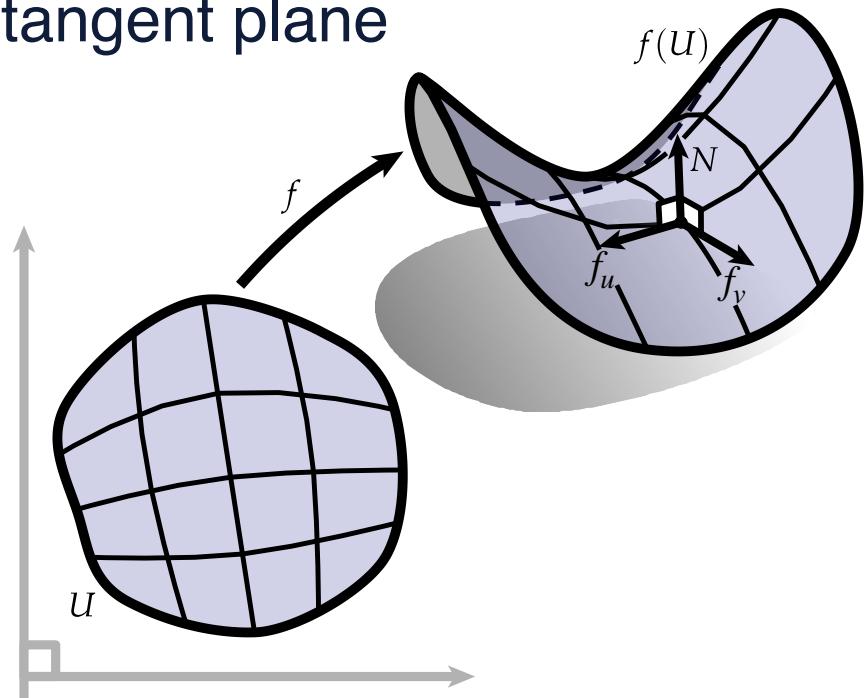
Tangent Plane

$$\Delta f_p \approx \frac{\partial f}{\partial u}(\epsilon u) + \frac{\partial f}{\partial v}(\epsilon v) = \epsilon \left[\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v} \right] \begin{bmatrix} u \\ v \end{bmatrix}$$

$\left[\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v} \right] \begin{bmatrix} u \\ v \end{bmatrix}$ is a vector in 3D tangent plane

Tangent plane at point $f(u, v)$ is spanned by

$$f_u = \frac{\partial f}{\partial u}, f_v = \frac{\partial f}{\partial v}$$

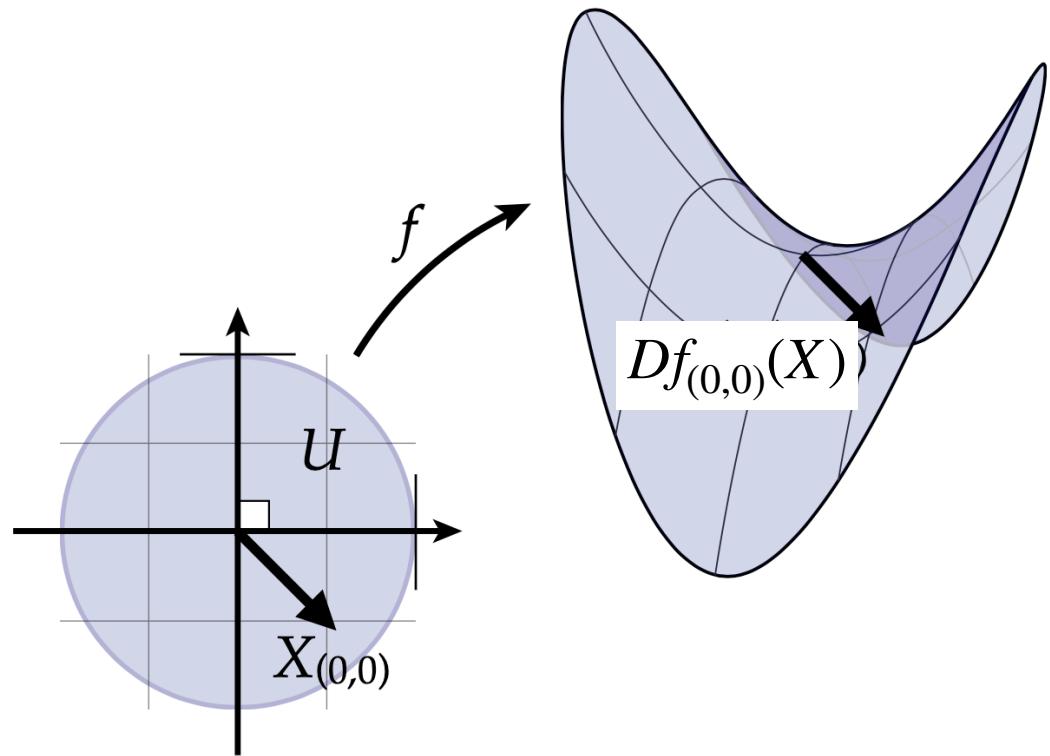


These vectors don't have to be orthogonal

An Example

$$f(u, v) = [u, v, u^2 - v^2]^T$$

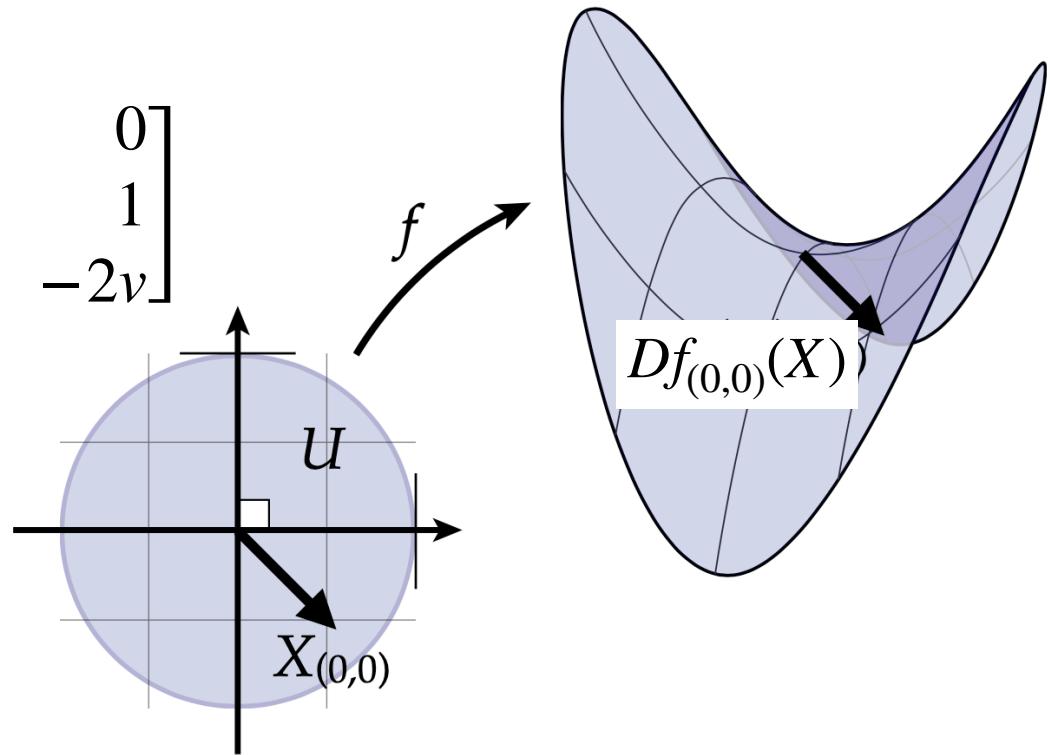
$$Df_p = \begin{bmatrix} \partial f_1 / \partial u & \partial f_1 / \partial v \\ \partial f_2 / \partial u & \partial f_2 / \partial v \\ \partial f_3 / \partial u & \partial f_3 / \partial v \end{bmatrix} =$$



An Example

$$f(u, v) = [u, v, u^2 - v^2]^T$$

$$Df_p = \begin{bmatrix} \partial f_1 / \partial u & \partial f_1 / \partial v \\ \partial f_2 / \partial u & \partial f_2 / \partial v \\ \partial f_3 / \partial u & \partial f_3 / \partial v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2u & -2v \end{bmatrix}$$



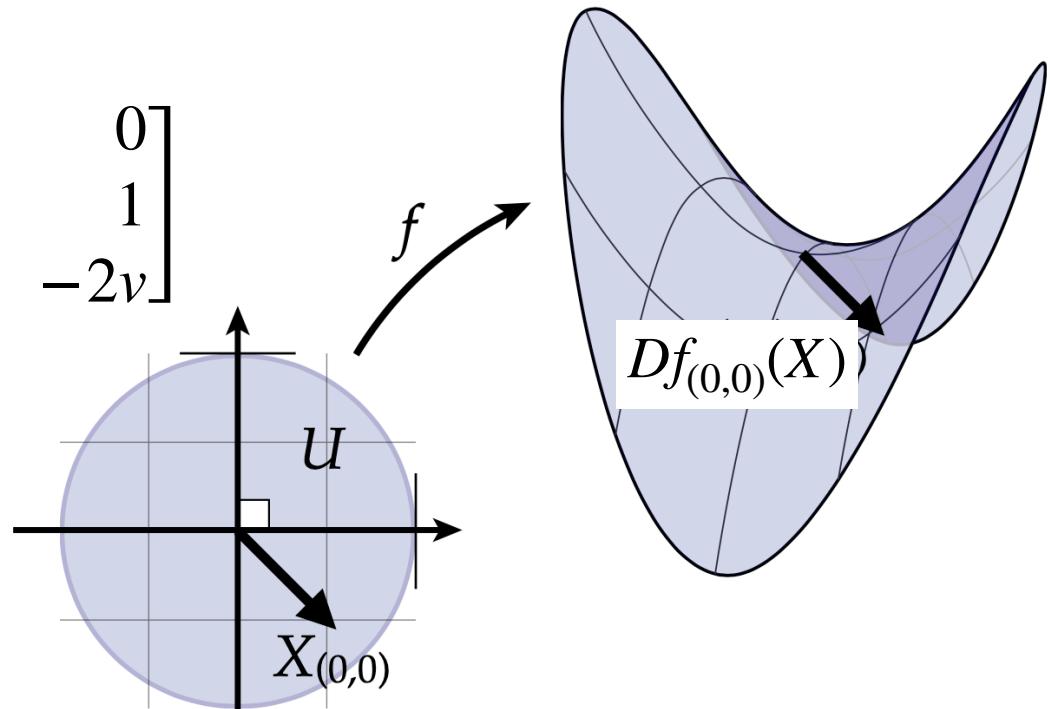
An Example

$$f(u, v) = [u, v, u^2 - v^2]^T$$

$$Df_p = \begin{bmatrix} \partial f_1 / \partial u & \partial f_1 / \partial v \\ \partial f_2 / \partial u & \partial f_2 / \partial v \\ \partial f_3 / \partial u & \partial f_3 / \partial v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2u & -2v \end{bmatrix}$$

$$X := \frac{3}{4}[1, -1]^T$$

$$Df(X) =$$



An Example

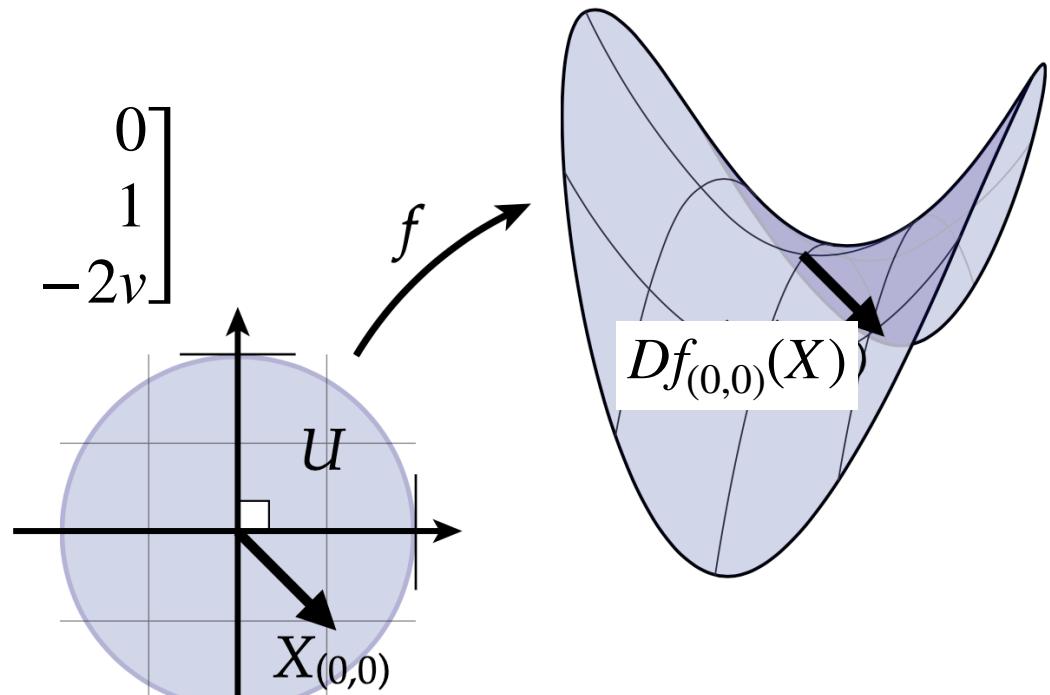
$$f(u, v) = [u, v, u^2 - v^2]^T$$

$$Df_p = \begin{bmatrix} \partial f_1 / \partial u & \partial f_1 / \partial v \\ \partial f_2 / \partial u & \partial f_2 / \partial v \\ \partial f_3 / \partial u & \partial f_3 / \partial v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2u & -2v \end{bmatrix}$$

$$X := \frac{3}{4}[1, -1]^T$$

$$Df(X) = \frac{3}{4}[1, -1, 2(u + v)]^T$$

$$\text{e.g., at } u = v = 0 : Df(X) = \left[\frac{3}{4}, -\frac{3}{4}, 0 \right]^T$$



An Example

$$f(u, v) = [u, v, u^2 - v^2]^T$$

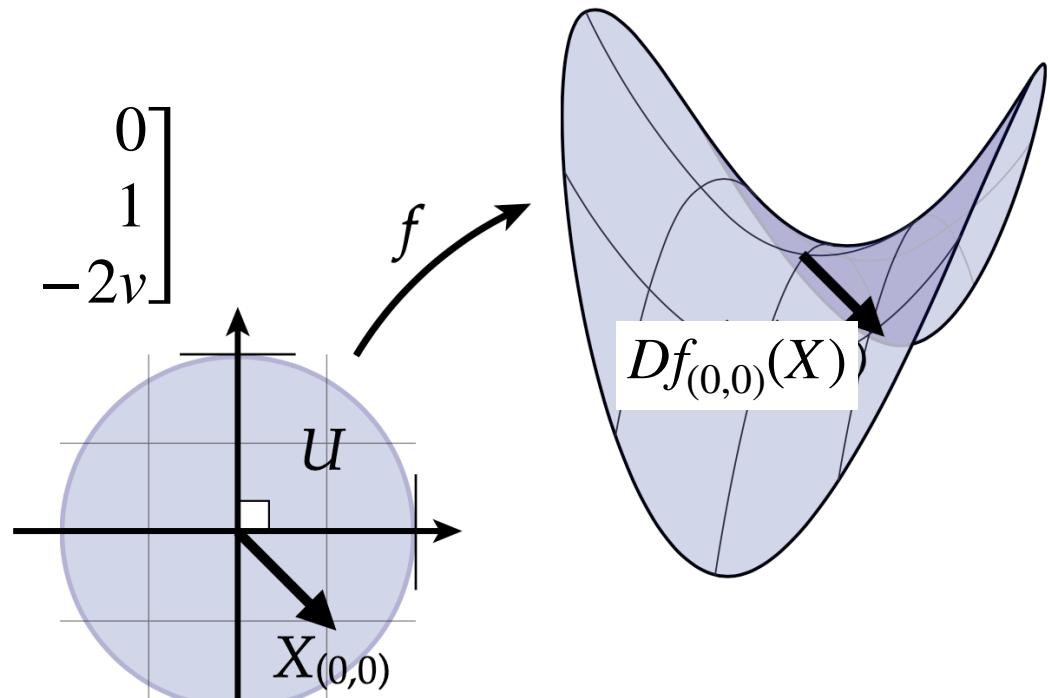
$$Df_p = \begin{bmatrix} \partial f_1 / \partial u & \partial f_1 / \partial v \\ \partial f_2 / \partial u & \partial f_2 / \partial v \\ \partial f_3 / \partial u & \partial f_3 / \partial v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2u & -2v \end{bmatrix}$$

$$X := \frac{3}{4}[1, -1]^T$$

$$Df(X) = \frac{3}{4}[1, -1, 2(u + v)]^T$$

$$\text{e.g., at } u = v = 0 : Df(X) = \left[\frac{3}{4}, -\frac{3}{4}, 0 \right]^T$$

at $u = v = 1$, tangent space is spanned by



An Example

$$f(u, v) = [u, v, u^2 - v^2]^T$$

$$Df_p = \begin{bmatrix} \partial f_1 / \partial u & \partial f_1 / \partial v \\ \partial f_2 / \partial u & \partial f_2 / \partial v \\ \partial f_3 / \partial u & \partial f_3 / \partial v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2u & -2v \end{bmatrix}$$

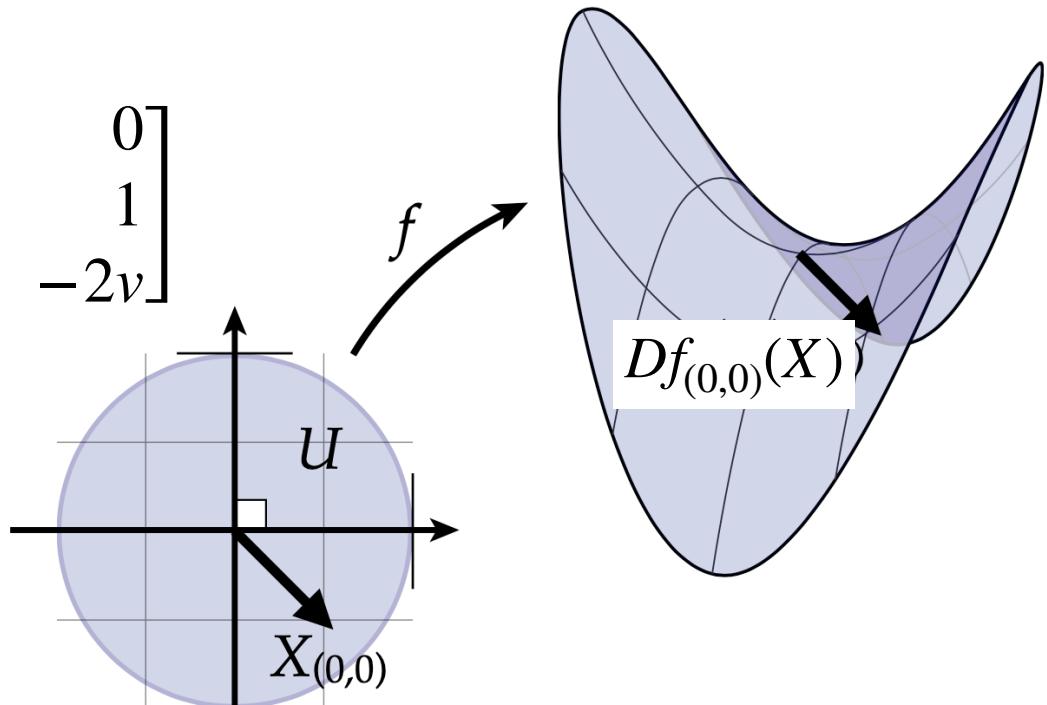
$$X := \frac{3}{4}[1, -1]^T$$

$$Df(X) = \frac{3}{4}[1, -1, 2(u + v)]^T$$

$$\text{e.g., at } u = v = 0 : Df(X) = \left[\frac{3}{4}, -\frac{3}{4}, 0 \right]^T$$

at $u = v = 1$, tangent space is spanned by

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}.$$



Summary of Differential Map

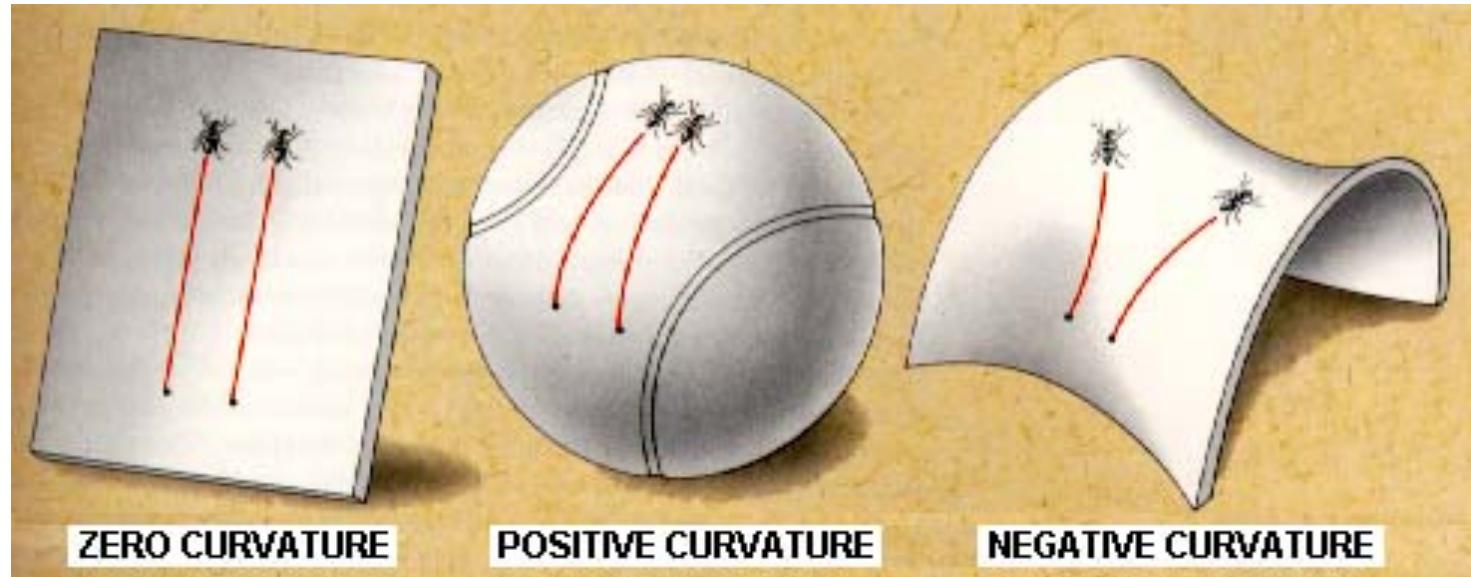
- Tells us the movement direction of point in 3D when the parameter changes in 2D
- Maps a vector in the tangent space of the domain to the tangent space of the surface
- Allows us to construct the bases of tangent plane
- Is a linear map

$$Df_p : \mathbf{T}_p(\mathbb{R}^2) \rightarrow \mathbf{T}_{f(p)}(\mathbb{R}^3)$$

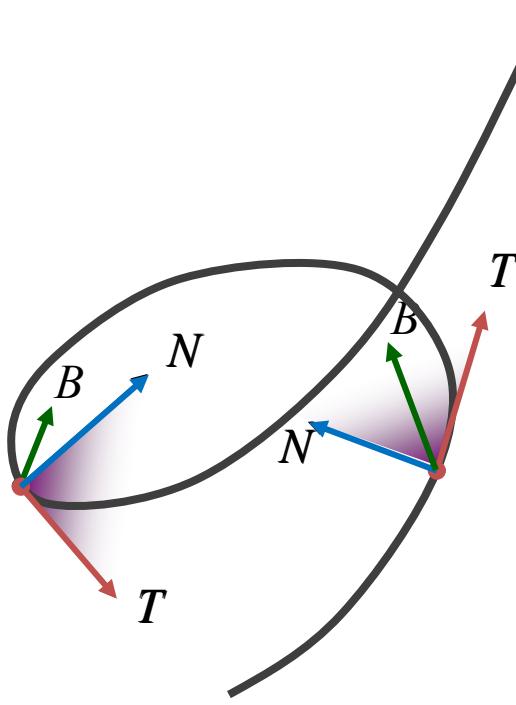
Curvature

Goal

Quantify how a surface **bends**.



Recall: Curvature of Curves



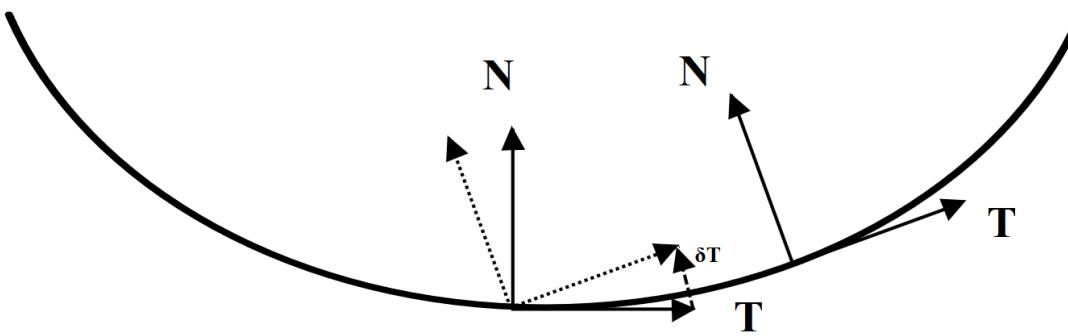
Theorem:

Curvature and torsion determine geometry
of a curve up to rigid motion.

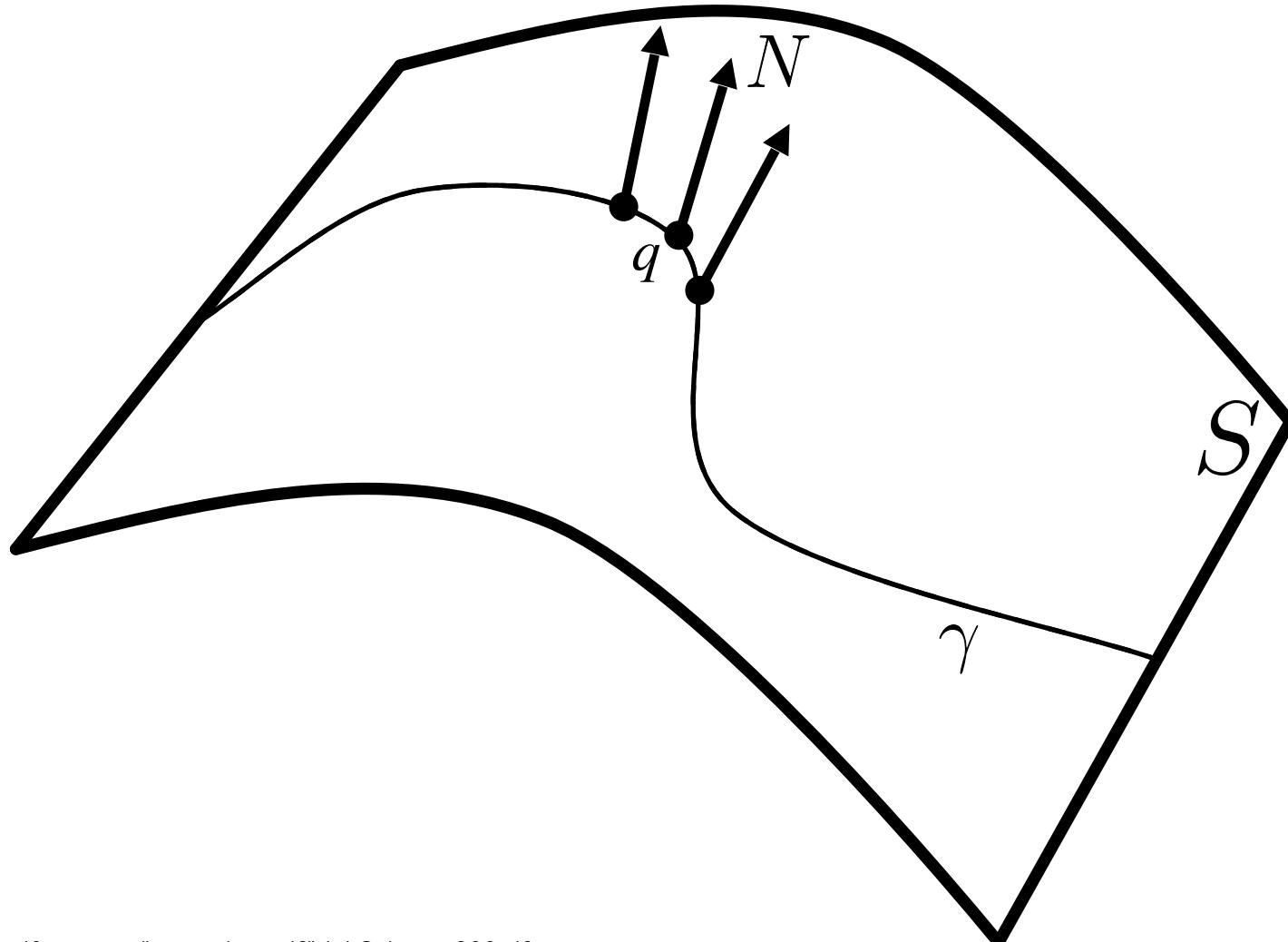


Can curvature/torsion of
a curve help us
understand surfaces?

Curves: Change of Normal Describes Curve Bending



Surfaces: Change of Normal Describes Surface Bending

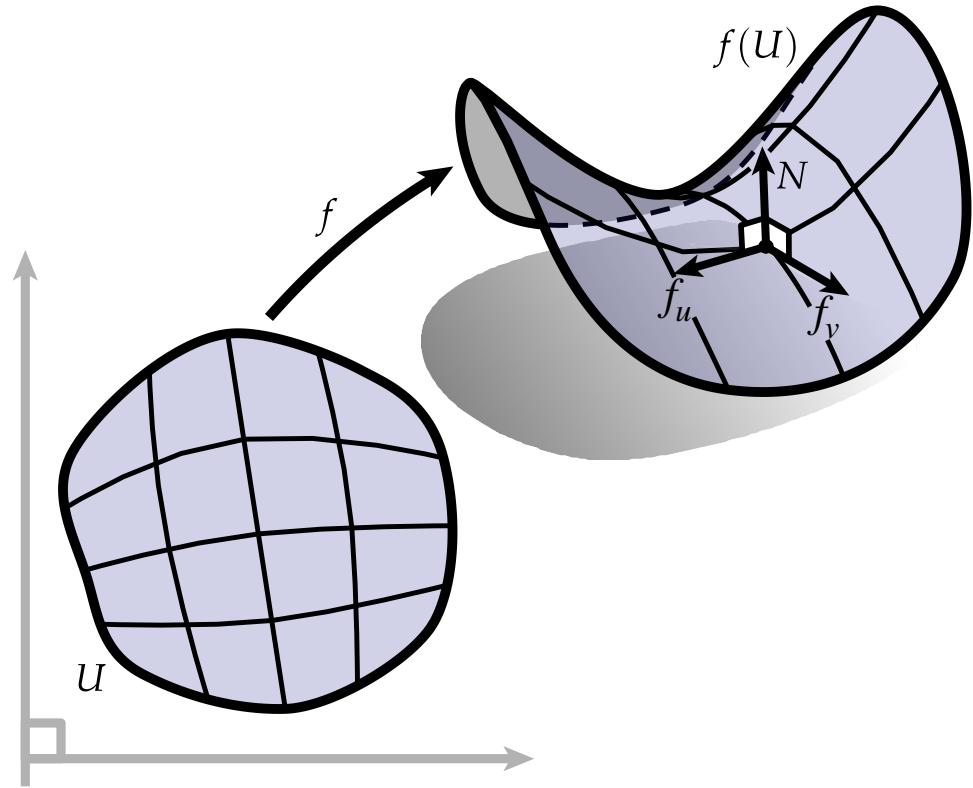


Surface Normals

$$f_u := \frac{\partial f}{\partial u}, \quad f_v := \frac{\partial f}{\partial v}$$

Surface normal:

$$N(u, v) = \frac{f_u \times f_v}{\|f_u \times f_v\|}$$



N also as a function of u, v

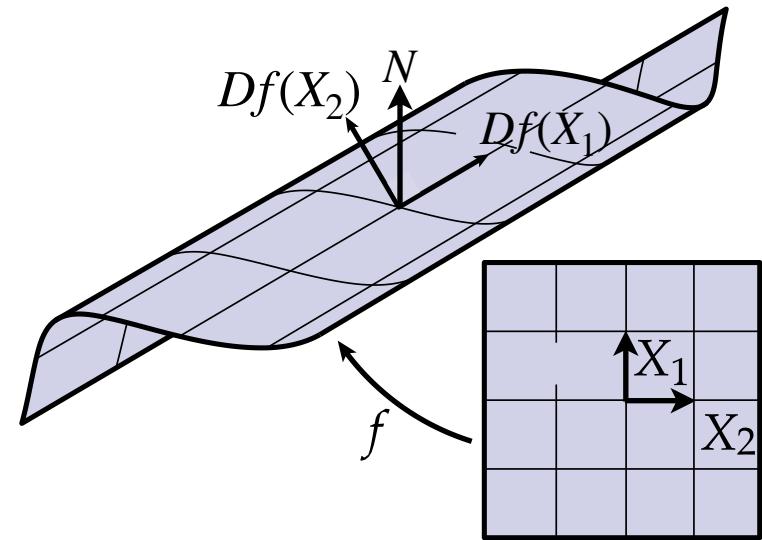
Example

Consider a nonstandard parameterization of the cylinder (sheared along z):

$$f(u, v) := [\cos(u), \sin(u), u + v]^T$$

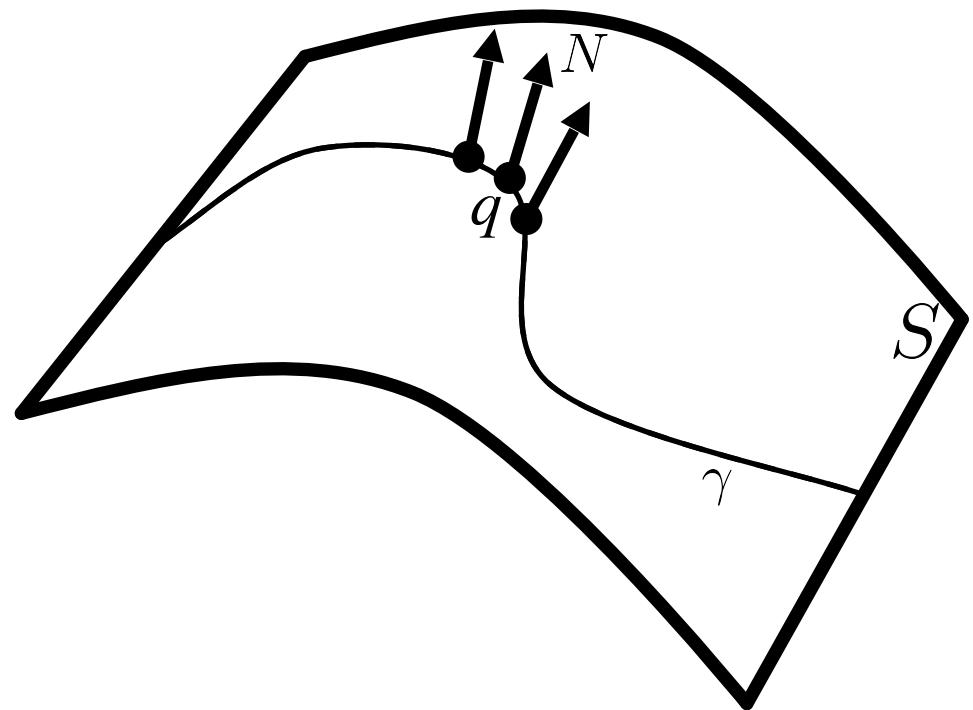
$$Df = \begin{bmatrix} -\sin(u) & 0 \\ \cos(u) & 0 \\ 1 & 1 \end{bmatrix}$$

$$N = \begin{bmatrix} -\sin(u) \\ \cos(u) \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(u) \\ \sin(u) \\ 0 \end{bmatrix}$$



Measure the Change of Normal

Assume q moves along a curve γ parameterized by arc-length: $q = \gamma(s)$, and the normal is $N(s)$ with unit norm

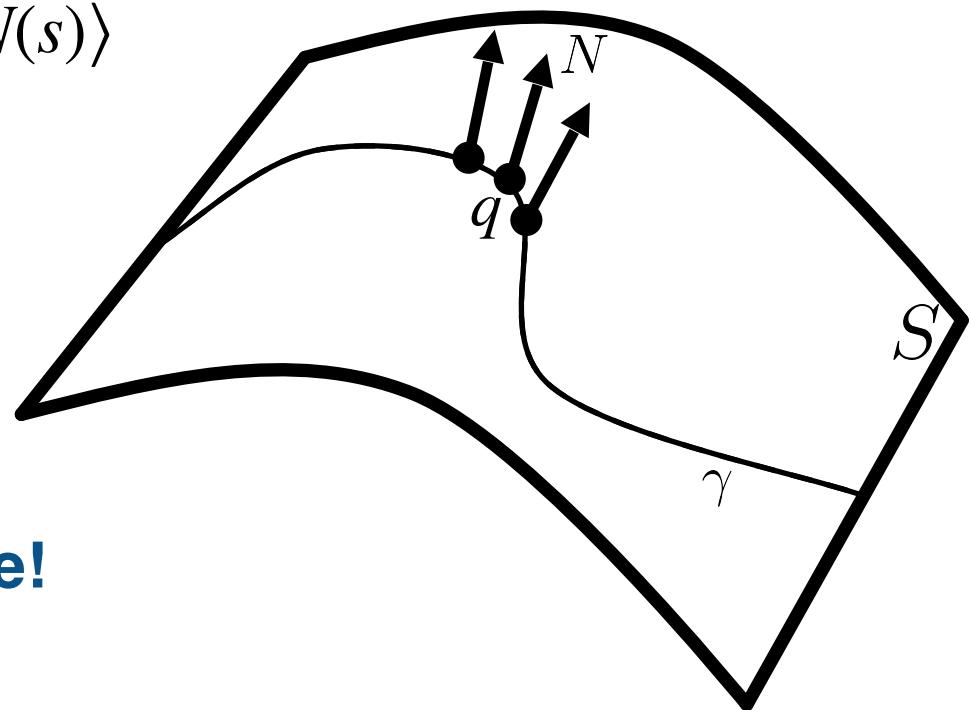


Measure the Change of Normal

Assume q moves along a curve γ parameterized by arc-length: $q = \gamma(s)$, and the normal is $N(s)$ with unit norm

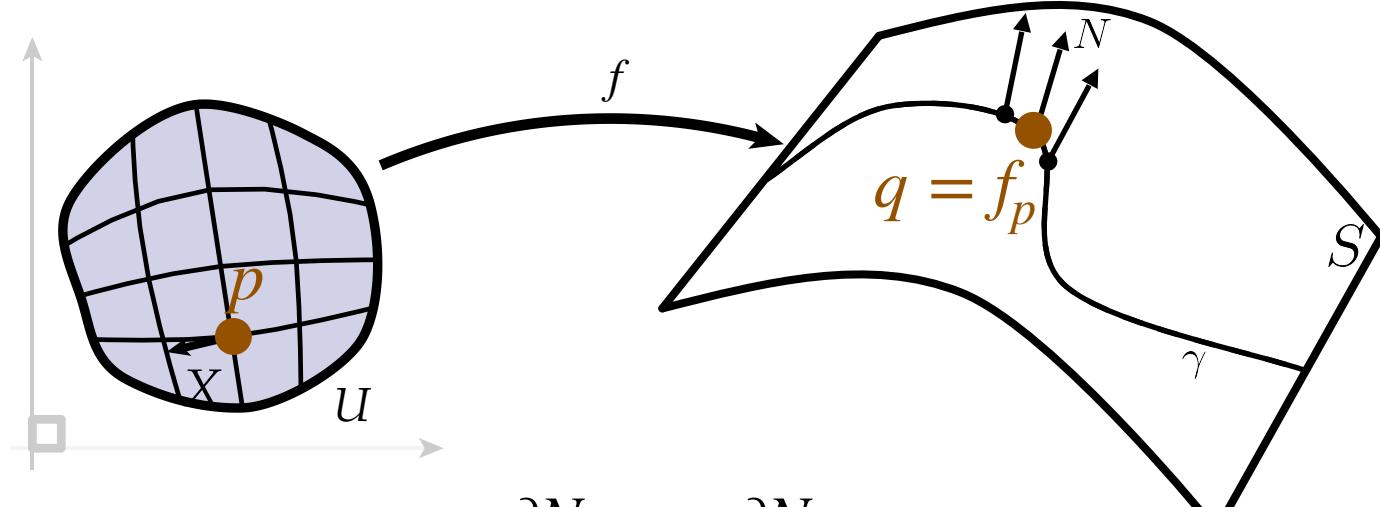
$$0 \equiv \frac{d}{ds} \langle N(s), N(s) \rangle = 2 \langle \dot{N}(s), N(s) \rangle$$

$$\dot{N}(s) \perp N(s)$$



**Local change of normal is
always in the tangent plane!**

Differential of Normal



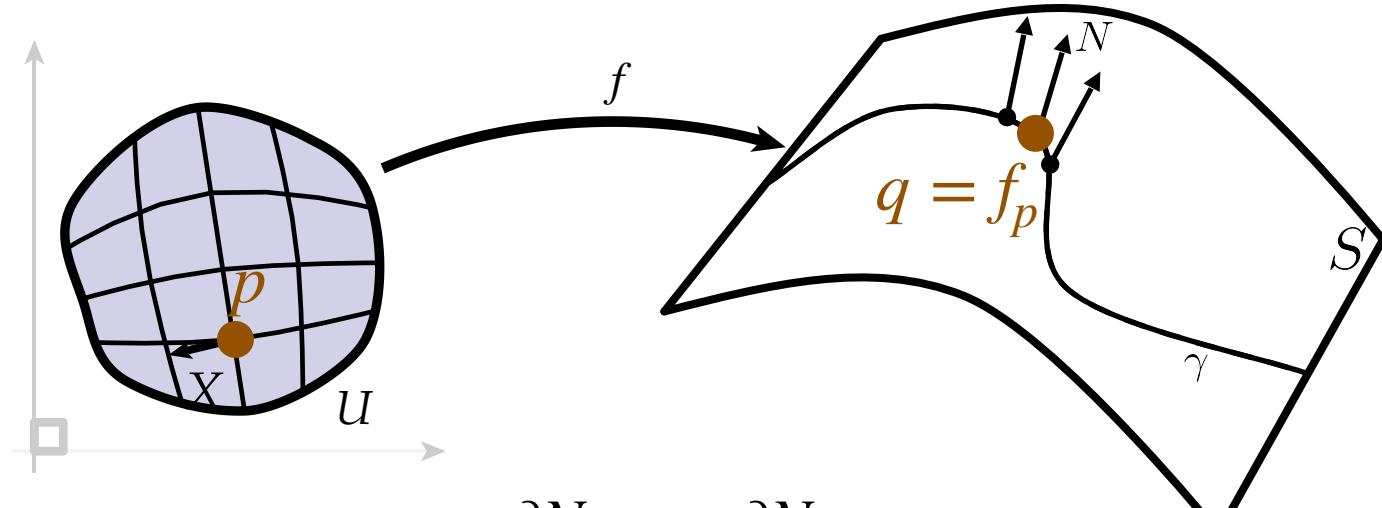
Total differential: $dN = \frac{\partial N}{\partial u} du + \frac{\partial N}{\partial v} dv$

If point $p \in \mathbb{R}^2$ moves along vector $X = [u, v]^T$ by ϵ , the movement of N_p is:

$$\Delta N_p = \frac{\partial N}{\partial u}(\epsilon u) + \frac{\partial N}{\partial v}(\epsilon v) = \epsilon \left[\frac{\partial N}{\partial u}, \frac{\partial N}{\partial v} \right] \begin{bmatrix} u \\ v \end{bmatrix} = \epsilon [DN_p]X$$

$$DN_p := \left[\frac{\partial N}{\partial u}, \frac{\partial N}{\partial v} \right] \in \mathbb{R}^{3 \times 2}$$

Differential of Normal



Total differential: $dN = \frac{\partial N}{\partial u} du + \frac{\partial N}{\partial v} dv$

If point $p \in \mathbb{R}^2$ moves along vector $X = [u, v]^T$ by ϵ , the movement of N_p is:

$$\Delta N_p = \frac{\partial N}{\partial u}(\epsilon u) + \frac{\partial N}{\partial v}(\epsilon v) = \epsilon \left[\frac{\partial N}{\partial u}, \frac{\partial N}{\partial v} \right] \begin{bmatrix} u \\ v \end{bmatrix} = \epsilon [DN_p]X$$

$$DN_p := \left[\frac{\partial N}{\partial u}, \frac{\partial N}{\partial v} \right] \in \mathbb{R}^{3 \times 2}$$

Note: $[DN_p]X \in \mathbf{T}_p(\mathbb{R}^3)$

Curvature $\vec{\kappa}$ of γ at p

- Recall we need the arc-length parameterization and measure the change of normal
- Recall that tangent vector $\|\mathbf{T}\| = 1$ under arc-length parameterization. So we need to scale X by μ so that:

$$\|Df_p[\mu X]\| = 1 \implies \mu = \frac{1}{\|Df_p X\|}$$

- As p moves with speed μX , the tangent is

$$Df_p[\mu X] = \frac{Df_p X}{\|Df_p X\|}$$

- the speed of normal change is:

$$DN_p[\mu X] = \frac{DN_p X}{\|Df_p X\|}$$

Curvature $\vec{\kappa}$ of γ at p

- As p moves with speed μX , the tangent is

$$Df_p[\mu X] = \frac{Df_p X}{\|Df_p X\|}$$

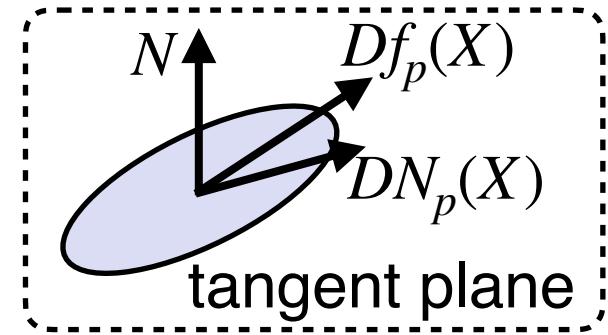
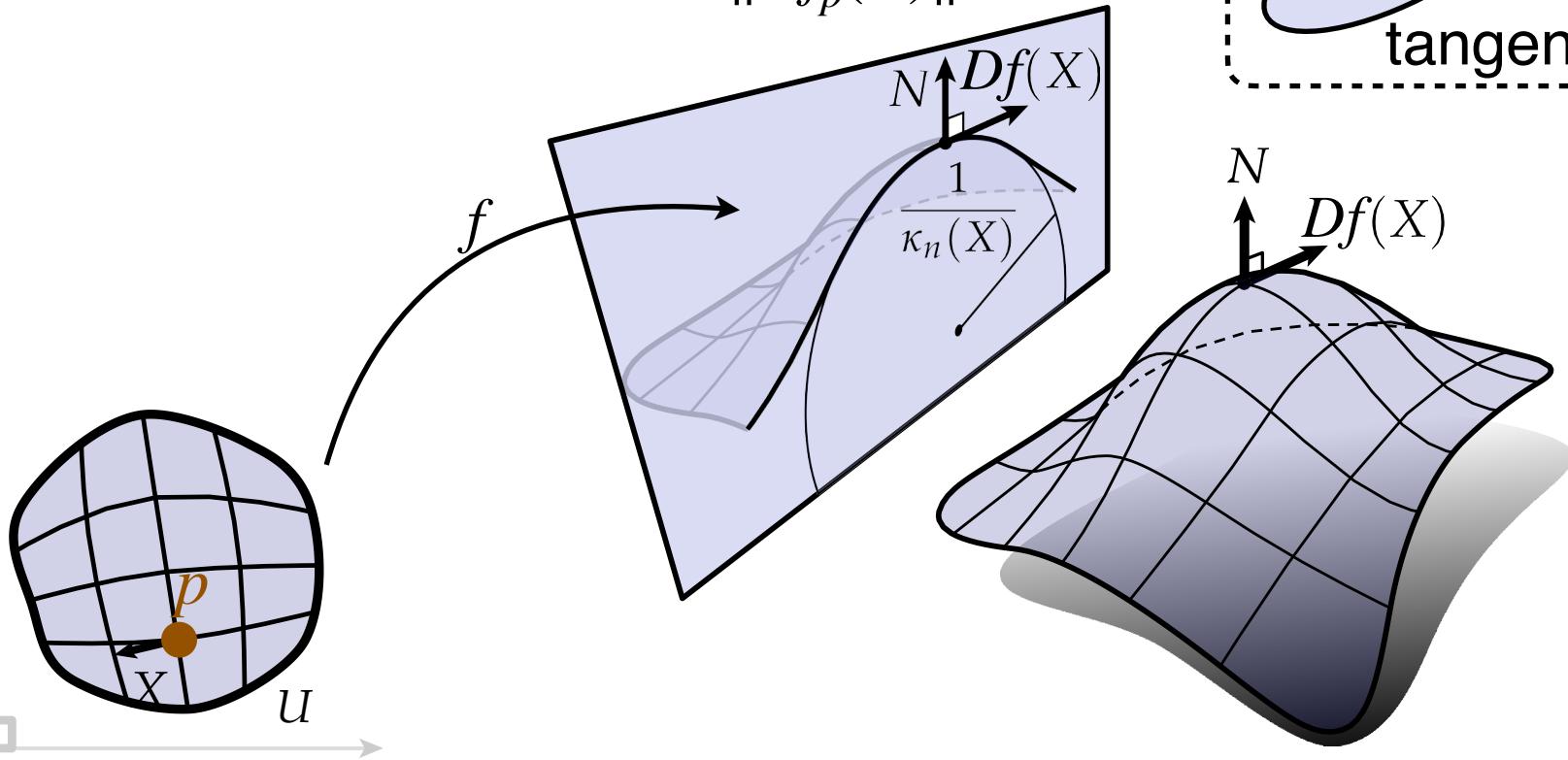
- The speed of normal change is:

$$DN_p[\mu X] = \frac{DN_p X}{\|Df_p X\|}$$

- We denote the speed of normal change as $\vec{\kappa}$ in this lecture (note that κ in the last lecture is a scalar, the norm of this vector)

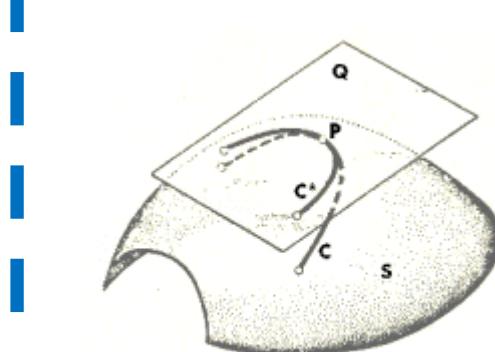
Directional Normal Curvature

$$\kappa_n(X) := \langle T, \vec{\kappa} \rangle = \frac{\langle Df_p(X), DN_p(X) \rangle}{\|Df_p(X)\|^2}$$



Note: κ_n is not the curvature κ of γ

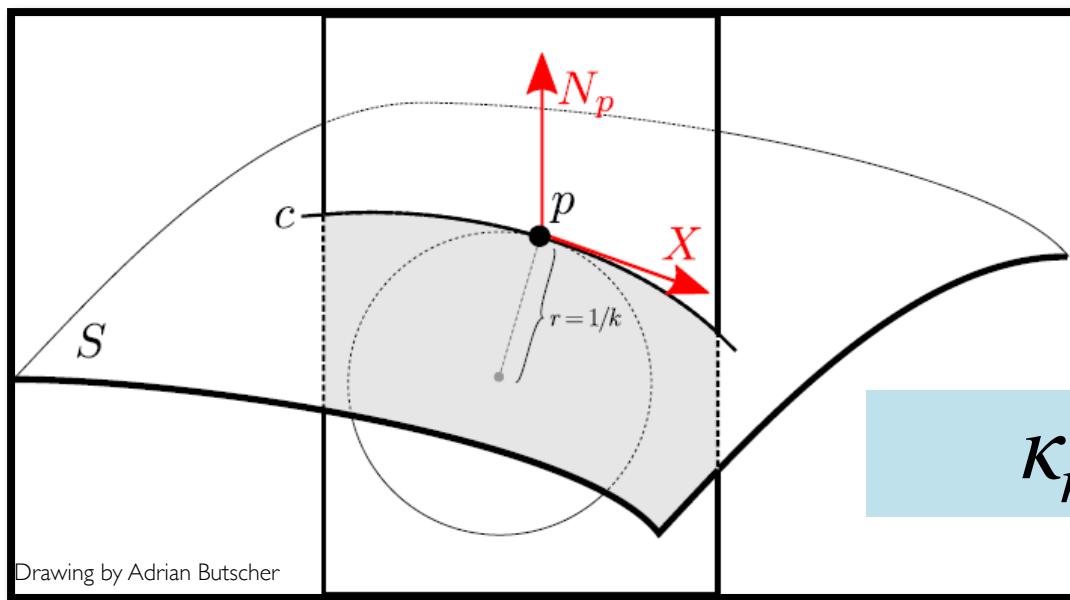
Relationship to Curvature of Curves



<http://www.solitaryroad.com/c335.html>

$$\kappa_g := \langle \vec{\kappa}, \mathbf{N} \times \mathbf{T} \rangle$$

(Geodesic curvature)



$$\kappa_n := \langle \vec{\kappa}, \mathbf{T} \rangle$$

Drawing by Adrian Butscher

Example

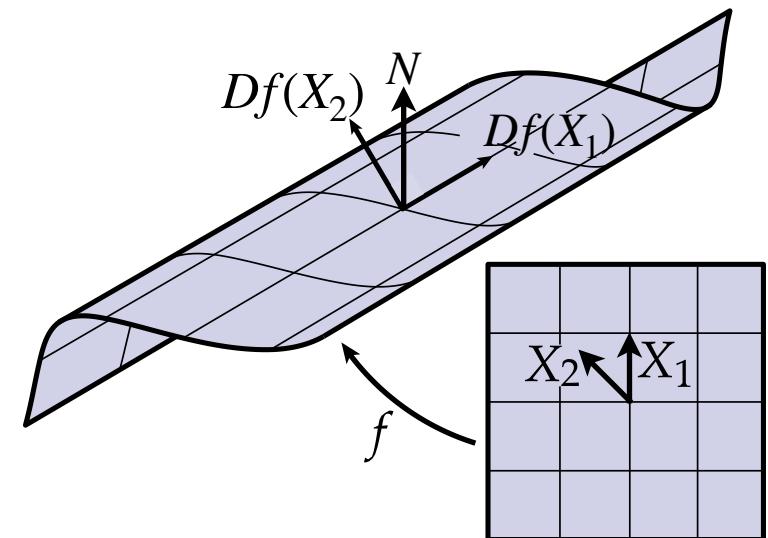
Consider a nonstandard parameterization of the cylinder (sheared along z):

$$f(u, v) := [\cos(u), \sin(u), u + v]^T$$

$$Df = \begin{bmatrix} -\sin(u) & 0 \\ \cos(u) & 0 \\ 1 & 1 \end{bmatrix}$$

$$N = \begin{bmatrix} \cos(u) \\ \sin(u) \\ 0 \end{bmatrix}$$

$$DN =$$



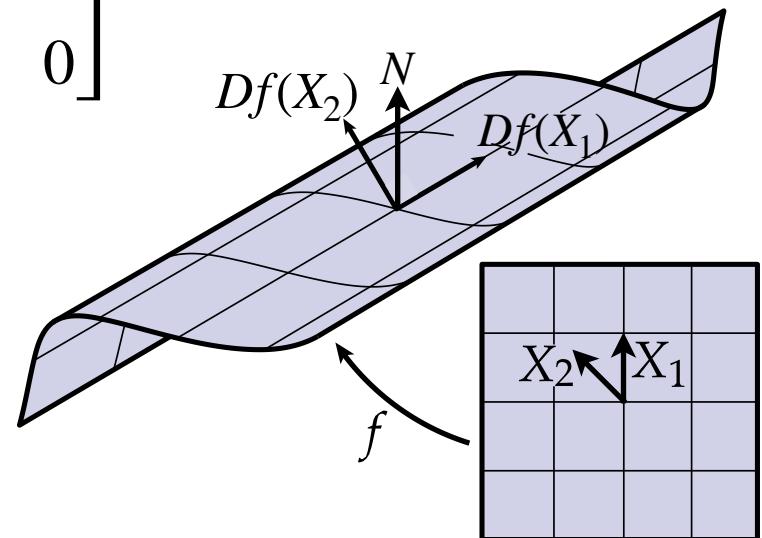
Example

Consider a nonstandard parameterization of the cylinder (sheared along z):

$$f(u, v) := [\cos(u), \sin(u), u + v]^T$$

$$N = \begin{bmatrix} \cos(u) \\ \sin(u) \\ 0 \end{bmatrix}$$

$$Df = \begin{bmatrix} -\sin(u) & 0 \\ \cos(u) & 0 \\ 1 & 1 \end{bmatrix}$$
$$DN = \begin{bmatrix} -\sin(u) & 0 \\ \cos(u) & 0 \\ 0 & 0 \end{bmatrix}$$



Example

Consider a nonstandard parameterization of the cylinder (sheared along z):

$$f(u, v) := [\cos(u), \sin(u), u + v]^T$$

$$N = \begin{bmatrix} \cos(u) \\ \sin(u) \\ 0 \end{bmatrix}$$

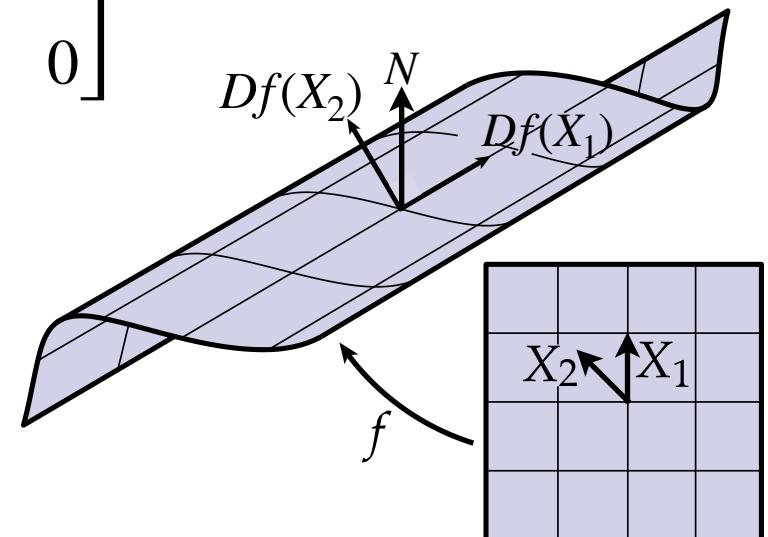
$$X_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\kappa_n(X_1) =$$

$$\kappa_n(X_2) =$$

$$Df = \begin{bmatrix} -\sin(u) & 0 \\ \cos(u) & 0 \\ 1 & 1 \end{bmatrix}$$

$$DN = \begin{bmatrix} -\sin(u) & 0 \\ \cos(u) & 0 \\ 0 & 0 \end{bmatrix}$$



Example

Consider a nonstandard parameterization of the cylinder (sheared along z):

$$f(u, v) := [\cos(u), \sin(u), u + v]^T$$

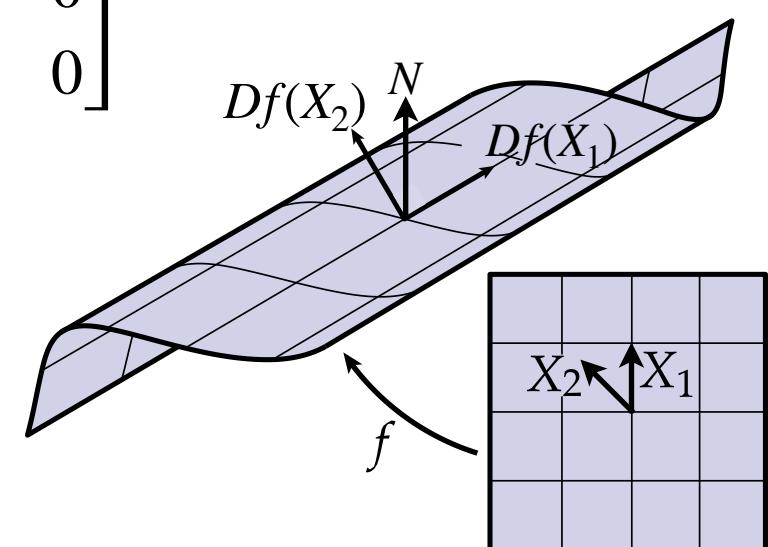
$$N = \begin{bmatrix} \cos(u) \\ \sin(u) \\ 0 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\kappa_n(X_1) = \frac{\langle Df(X_1), DN(X_1) \rangle}{\|Df(X_1)\|^2} = 0$$

$$\kappa_n(X_2) = \frac{\langle Df(X_2), DN(X_2) \rangle}{\|Df(X_2)\|^2} = 1$$

$$Df = \begin{bmatrix} -\sin(u) & 0 \\ \cos(u) & 0 \\ 1 & 1 \end{bmatrix}$$



Summary of Curvature

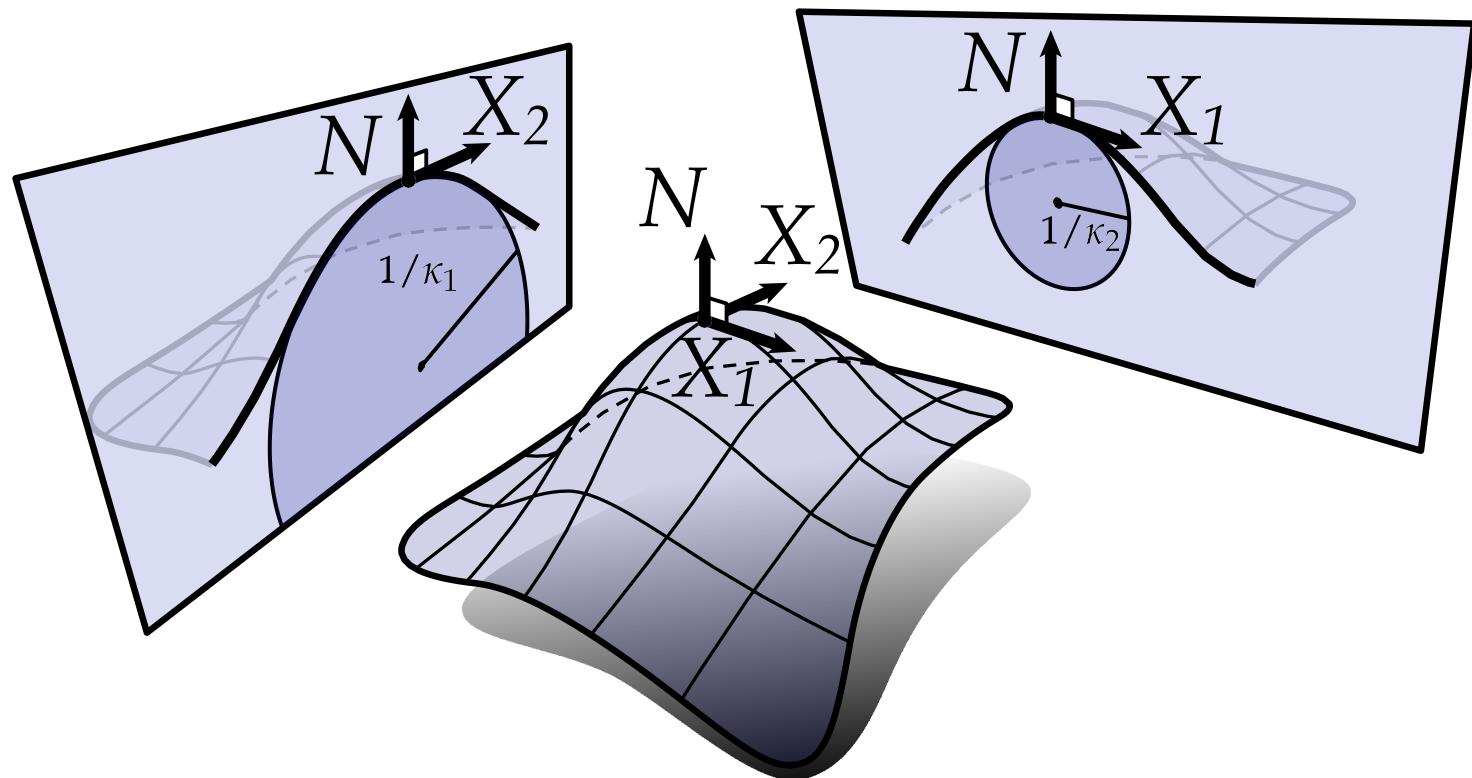
- Curvature quantifies the bending of surfaces
- Local change of normal (differential of normal) is always in the tangent plane
- Directional normal curvature quantifies how fast a surface bends along a direction

Principal Curvatures

Principal Curvatures

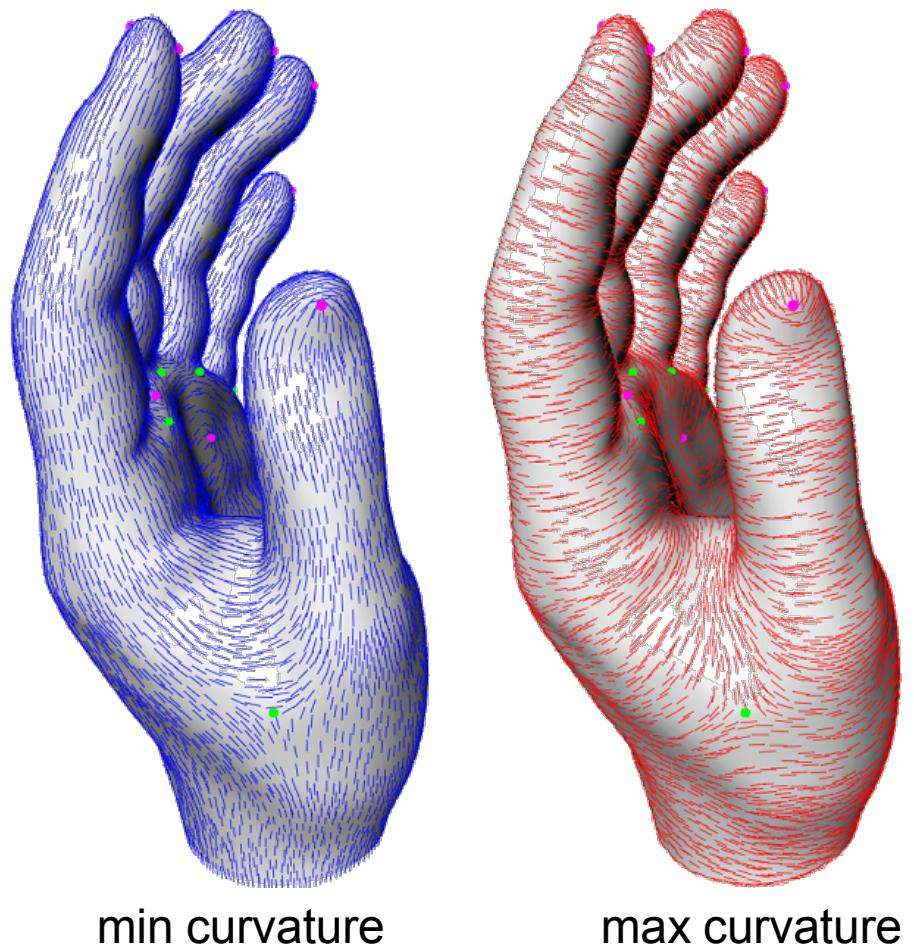
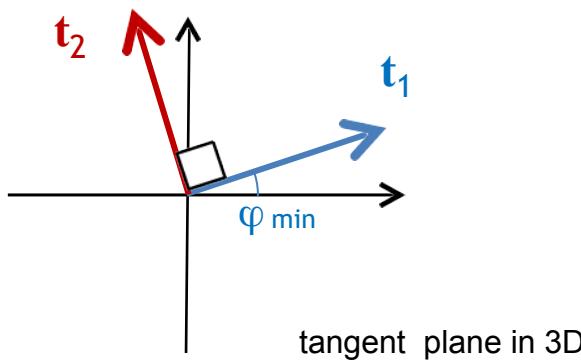
Minimal curvature: $\kappa_1 = \kappa_{\min} = \min_{\varphi} \kappa_n(\varphi)$

Maximal curvature: $\kappa_2 = \kappa_{\max} = \max_{\varphi} \kappa_n(\varphi)$

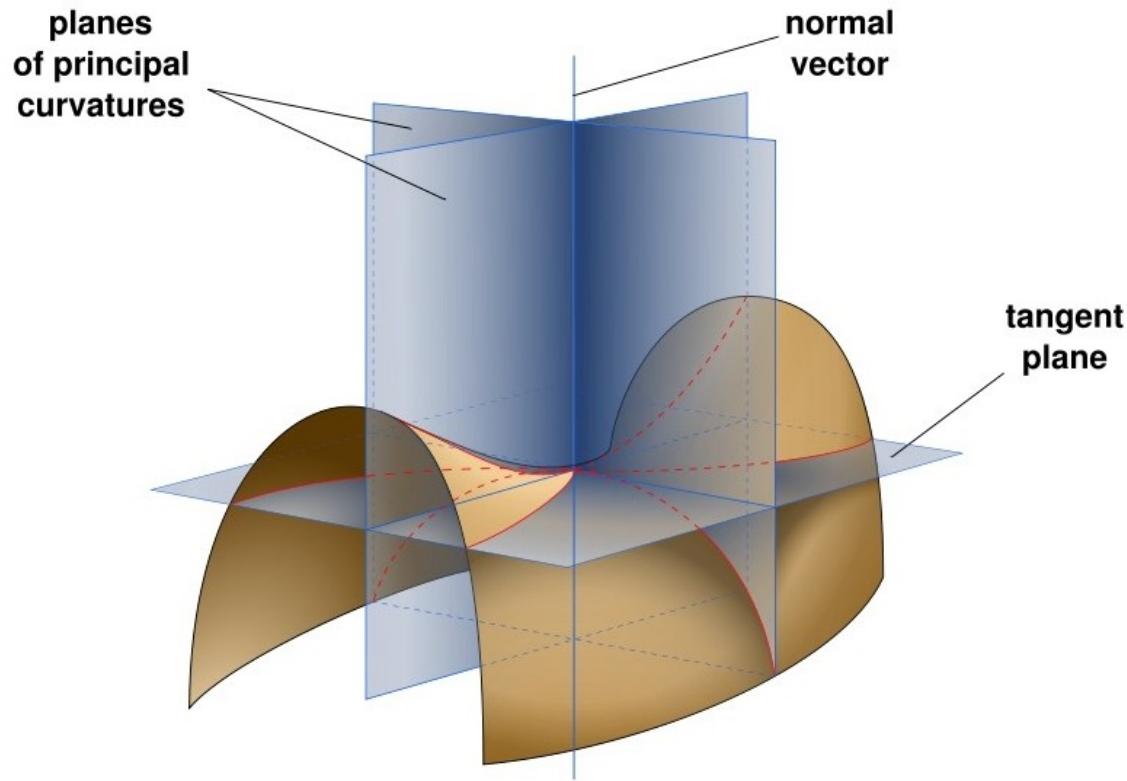


Principal Directions

Principal directions:
tangent vectors
corresponding to
 φ_{\max} and φ_{\min}



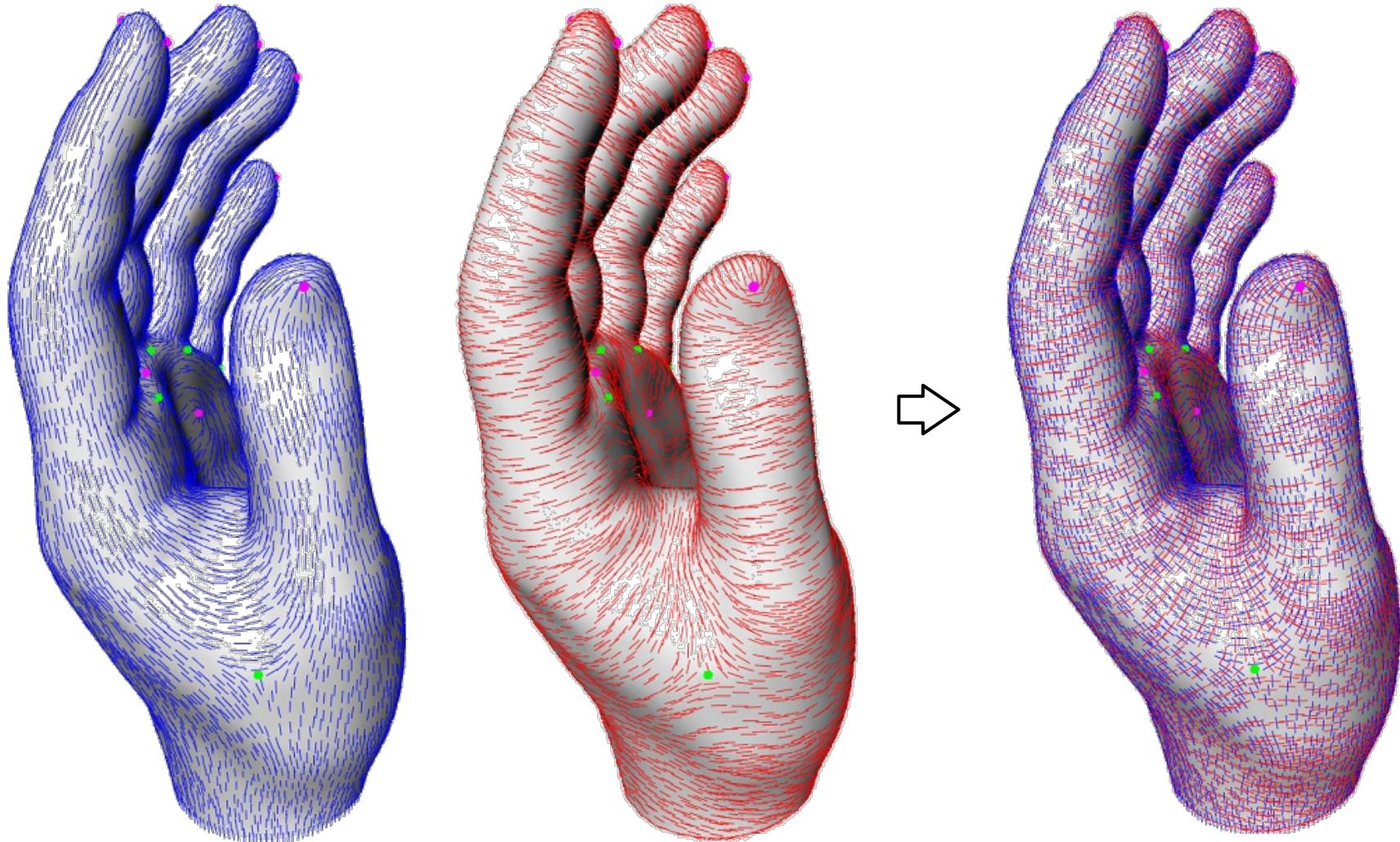
Principal Directions



Euler's Theorem: Planes of principal curvature are **orthogonal** and independent of parameterization.

$$\kappa_n(\varphi) = \kappa_1 \cos^2 \varphi + \kappa_2 \sin^2 \varphi, \quad \varphi = \text{angle with } t_1$$

Principal Directions



Summary of Principal Curvatures

- The direction that bends fastest / slowest are principal directions, which are orthogonal to each other
- The corresponding curvatures are principal curvatures

Shape Operator

Computation of Principal Directions

- Note that
 - $\forall X, DN_p X$ is in the tangent plane
 - $\forall X, Df_p X$ is also in the tangent plane
- So the column space of $DN_p \in \mathbb{R}^{3 \times 2}$ and $Df_p \in \mathbb{R}^{3 \times 2}$ are the same
- In other words,

Computation of Principal Directions

- Note that
 - $\forall X, DN_p X$ is in the tangent plane
 - $\forall X, Df_p X$ is also in the tangent plane
- So the column space of $DN_p \in \mathbb{R}^{3 \times 2}$ and $Df_p \in \mathbb{R}^{3 \times 2}$ are the same
- In other words, $\exists S \in \mathbb{R}^{2 \times 2}$ such that $DN_p = Df_p S$

Computation of Principal Directions

- Note that
 - $\forall X, DN_p X$ is in the tangent plane
 - $\forall X, Df_p X$ is also in the tangent plane
- So the column space of $DN_p \in \mathbb{R}^{3 \times 2}$ and $Df_p \in \mathbb{R}^{3 \times 2}$ are the same
- In other words, $\exists S \in \mathbb{R}^{2 \times 2}$ such that $DN_p = Df_p S$
- S is called the **shape operator**
- Principal directions are the *eigenvectors* of S
- Principal curvatures are the *eigenvalues* of S

Computation of Principal Directions

- Note that
 - $\forall X, DN_p X$ is in the tangent plane
 - $\forall X, Df_p X$ is also in the tangent plane
- So the column space of $DN_p \in \mathbb{R}^{3 \times 2}$ and $Df_p \in \mathbb{R}^{3 \times 2}$ are the same
- In other words, $\exists S \in \mathbb{R}^{2 \times 2}$ such that $DN_p = Df_p S$
- S is called the **shape operator**
- Principal directions are the *eigenvectors* of S
- Principal curvatures are the *eigenvalues* of S
- Note: S is not a symmetric matrix! Hence, eigenvectors are not orthogonal in \mathbb{R}^2 ; only orthogonal when mapped to \mathbb{R}^3

Example

Consider a nonstandard parameterization of the cylinder (sheared along z):

$$f(u, v) := [\cos(u), \sin(u), u + v]^T$$

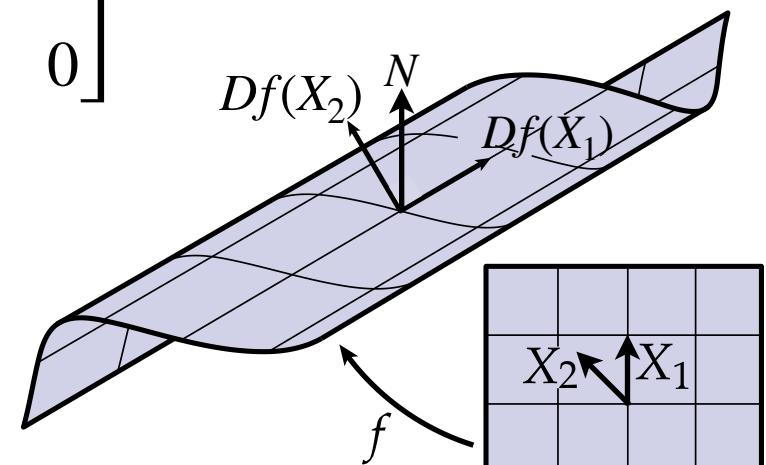
$$N = \begin{bmatrix} \cos(u) \\ \sin(u) \\ 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad X = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\kappa_n(X_1) = 0 \quad \kappa_n(X_2) = 1$$

$$DN_p = Df_p S \Rightarrow S = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

$$Df = \begin{bmatrix} -\sin(u) & 0 \\ \cos(u) & 0 \\ 1 & 1 \end{bmatrix}$$



Verify the eigens of S

Summary of Shape Operator

- A linear map between DN_p and Df_p
- The eigen-decomposition gives the principal curvature direction and values