

L4: Screw and Twist

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Agenda

- Screw (6D representation of rigid motion)
- Twist (6D representation of rigid motion velocity)

Rigid Transformation and $\mathbb{SE}(3)$

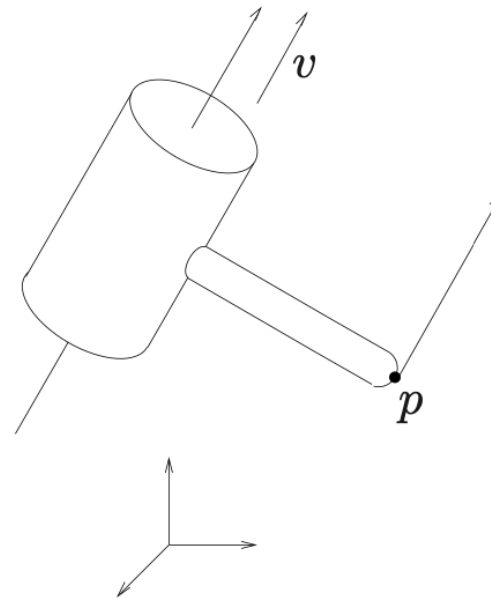
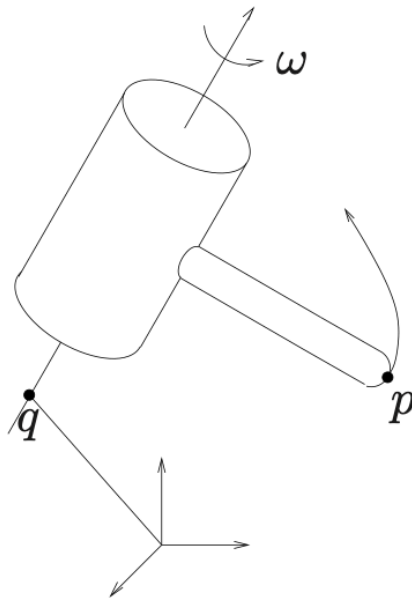
The Set of Rigid Transformations

- $\text{SE}(3) := \left\{ T = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}, R \in \text{SO}(3), t \in \mathbb{R}^3 \right\}$
- $\text{SE}(3)$: “Special Euclidean Group”
- “Group”: closed under matrix multiplication and other conditions of group
- “Euclidean”: R and t
- “Special”: $\det(R) = 1$
- 6 DoF

- Recall Euler's Theorem about $\text{SO}(3)$:
 - Any rotation in $\text{SO}(3)$ is equivalent to rotation about a fixed axis $\hat{\omega} \in \mathbb{R}^3$ through a positive angle θ
- Similar results for $\text{SE}(3)$: Screw Parameterization
- (In your mind, think T as a linear transformation)

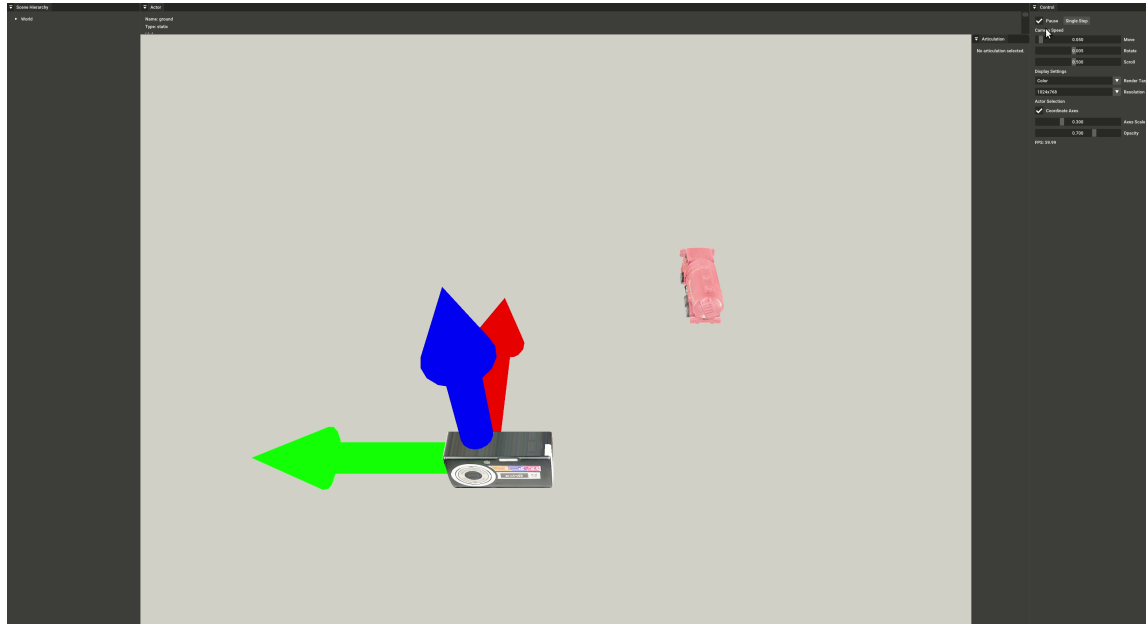
Screw Motion Theorem

- Any rigid body motion is equivalent to rotating about one axis while also translating along the axis
- **The axis may not pass the origin**



Screw Motion Theorem

- Any rigid body motion is equivalent to rotating about one axis while also translating along the axis
- Recall our question of “canonical” rigid transformation decomposition—by sharing rotation axis and translation direction, we identify the decomposition



Review: Lie algebra of $\mathbb{SO}(3)$

- Motion interpretation

$\hat{\omega}$: motion direction

- Exponential coordinate

$\vec{\theta} = \hat{\omega}\theta \in \mathbb{R}^3$ (rot vector)

- Exponential map

$R = \exp([\hat{\omega}]\theta) \in \mathbb{SO}(3)$

- Tangent space at $R = I$

$[\hat{\omega}]\theta \in \mathfrak{so}(3)$

Goal: The Lie Algebra of $\mathbb{SE}(3)$

- Motion interpretation

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- Motion interpretation

$\hat{\xi}$: 6D motion direction

- Exponential coordinate

$\chi = \hat{\xi}\theta \in \mathbb{R}^6$ (screw)

- Exponential map

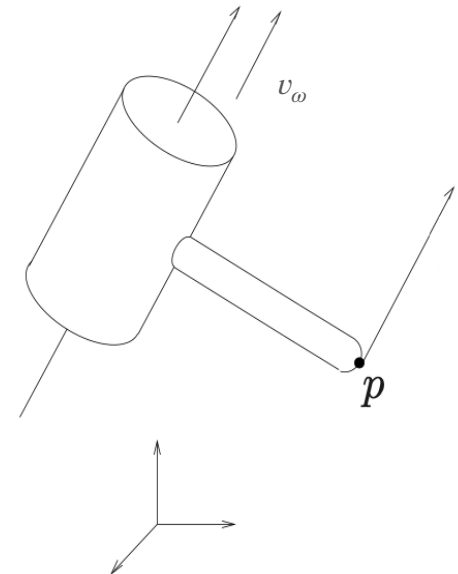
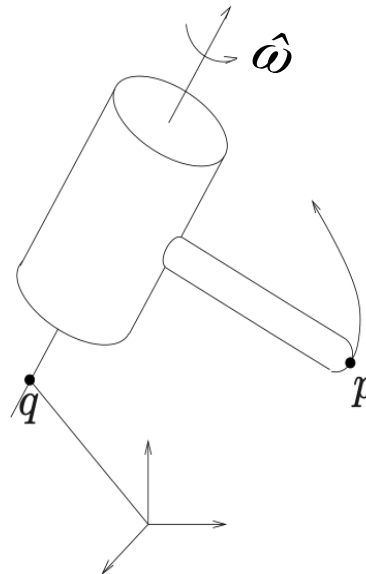
$T = \exp([\hat{\xi}]\theta) \in \mathbb{SE}(3)$

- Tangent space at $T = I$

$[\hat{\xi}]\theta \in \mathfrak{se}(3)$

An Imaged Motion for $T \in \mathbb{SE}(3)$

- Transforming by $T \iff$ **rotating** about one axis while also **translating** along the axis
- Assume an arbitrary point q on the axis, a **unit** vector $\hat{\omega}$ denoting axis, and the angle θ
- Assume the translation along $\hat{\omega}$ is d_ω



Screw Parameterization

- In $\mathbb{SO}(3)$, we have

$$\text{Rot}(\hat{\omega}, \theta)x = (I + \theta[\hat{\omega}] + \frac{\theta^2}{2!}[\hat{\omega}]^2 + \frac{\theta^3}{3!}[\hat{\omega}]^3 + \dots)x$$

- In $\mathbb{SE}(3)$, we have a similar result ($x \in \mathbb{R}^4$ by homogeneous coordinate):

$$\text{Trans}(\hat{\omega}, \theta, q, d_\omega)x = (I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots)x$$

$$\text{where } A = \begin{bmatrix} [\hat{\omega}\theta] & -[\hat{\omega}\theta]q + d_\omega \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4 \times 4}$$

Screw Parameterization

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- Similar to $\text{Rot}(\hat{\omega}, \theta) \equiv e^{[\hat{\omega}]\theta}$, we have $\text{Trans}(\hat{\omega}, \theta, q, d_{\omega}) = e^A$
- We try to align the form of $\text{Rot}(\hat{\omega}, \theta)$ and $T = \text{Trans}(\hat{\omega}, \theta, q, d_{\omega})$:
 - Notice that the power of e is the product of a matrix that corresponds to motion direction and a scalar, we factor A similarly

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 - Notice that the power of e is the product of a matrix that corresponds to motion direction and a scalar, we factor A similarly
 - Let $A = \begin{bmatrix} [\hat{\omega}] & d \\ 0 & 0 \end{bmatrix} \theta$, where $d = \frac{-[\hat{\omega}\theta]q + d_{\omega}}{\theta}$, then $T = \exp \left(\begin{bmatrix} [\hat{\omega}] & d \\ 0 & 0 \end{bmatrix} \theta \right)$

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 - A special case:
 - **When the motion is translation-only, define $\hat{\omega} = 0$, $\theta = \|d_{\omega}\|$, and**

$$d = \frac{d_{\omega}}{\|d_{\omega}\|}$$

Screw Parameterization

$$\text{Trans}(\hat{\omega}, \theta, q, d_{\omega})x = (I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots)x, \text{ where } A = \begin{bmatrix} [\hat{\omega}\theta] & -[\hat{\omega}\theta]q + d_{\omega} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4 \times 4}$$

- Let $A = \begin{bmatrix} [\hat{\omega}] & d \\ 0 & 0 \end{bmatrix} \theta$, where $d = \frac{-[\hat{\omega}\theta]q + d_{\omega}}{\theta}$, then $T = \exp \left(\begin{bmatrix} [\hat{\omega}] & d \\ 0 & 0 \end{bmatrix} \theta \right)$
- The following rule introduces $\hat{\xi}$ so that $T = \exp \left(\begin{bmatrix} [\hat{\omega}] & d \\ 0 & 0 \end{bmatrix} \theta \right) \equiv e^{[\hat{\xi}]\theta}$:
 - $\hat{\xi} = \begin{bmatrix} d \\ \hat{\omega} \end{bmatrix} \in \mathbb{R}^6$ and $[\hat{\xi}]\theta = \begin{bmatrix} [\hat{\omega}] & d \\ 0 & 0 \end{bmatrix} \theta$

Screw Parameterization

- To sum up, $\text{Trans}(\hat{\omega}, \theta, q, d_\omega) = e^{[\hat{\xi}]\theta}$, where
 - $\hat{\xi} = \begin{bmatrix} d \\ \hat{\omega} \end{bmatrix} \in \mathbb{R}^6$ ($\hat{\omega} = 0$ and $d = \frac{d_\omega}{\|\hat{\omega}\|}$ if translation-only)
 - $d = \frac{-[\hat{\omega}\theta]q + d_\omega}{\theta}$,
 - $[\hat{\xi}]\theta = \begin{bmatrix} [\hat{\omega}] & d \\ 0 & 0 \end{bmatrix} \theta$
- $\chi = \hat{\xi}\theta = \begin{bmatrix} d \\ \hat{\omega} \end{bmatrix} \theta = \begin{bmatrix} -[\hat{\omega}]q\theta + d_\omega \\ \hat{\omega}\theta \end{bmatrix}$ is called **screw**, or **exponential coordinate**
- Introducing the inverse function of $T = e^{[\chi]}$, $\chi = \log(T)$
- $\hat{\xi}$ is called **unit twist**, which describes **motion direction**

Generate $T \in \mathbb{SE}(3)$ from $\hat{s}\theta$

- Recall **Rodrigues Formula** for rotations:

$$e^{[\hat{\omega}]\theta} = I + [\hat{\omega}]\sin \theta + [\hat{\omega}]^2(1 - \cos \theta)$$

- Similarly, using Taylor's expansion definition of exp,

$$e^{[\hat{\xi}]\theta} = I + [\hat{\xi}] + \frac{1 - \cos \theta}{\theta^2}[\hat{\xi}]^2 + \frac{\theta - \sin \theta}{\theta^3}[\hat{\xi}]^3$$

- Further computation gives

$$e^{[\hat{\xi}\theta]} = \begin{bmatrix} e^{[\hat{\omega}\theta]} & (I - e^{[\hat{\omega}\theta]})(\hat{\omega} \times d) + \hat{\omega}\hat{\omega}^T d\theta \\ 0 & 1 \end{bmatrix}$$

Local Structure of $\mathbb{SE}(3)$

- Definition of Matrix Exponential:

$$e^{[\hat{\xi}]^{\theta}} = I + \theta[\hat{\xi}] + \frac{\theta^2}{2!}[\hat{\xi}]^2 + \frac{\theta^3}{3!}[\hat{\xi}]^3 + \dots$$

- When $\theta \approx 0$, $e^{[\hat{\xi}]^{\theta}} = I + \theta[\hat{\xi}] + o(\theta[\hat{\xi}])$
- $\forall T \in \mathbb{SE}(3), e^{\theta[\hat{\xi}]}T \approx T + \theta[\hat{\xi}]T$ when $\theta \approx 0$
 - Implies that $\mathbb{SE}(3)$ has a linear local structure (differentiable manifold)

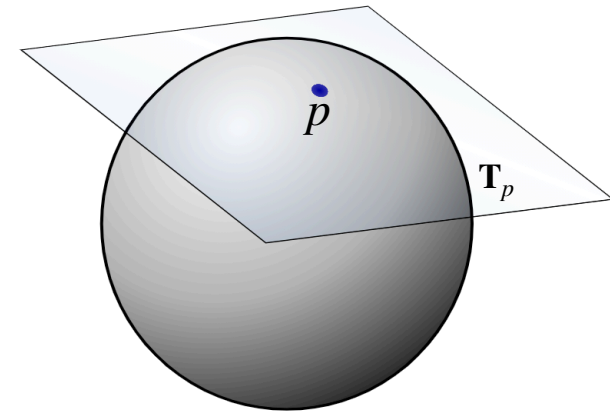
Local Structure of $\mathbb{SE}(3)$

- By $e^{[\hat{\xi}]\theta} = I + \theta[\hat{\xi}] + o(\theta[\hat{\xi}])$ when $\theta \approx 0$,

$$e^{[\chi]} - I = [\chi] + o([\chi])$$

- Interpretation:

- $[\chi]$ is a linear subspace of $\mathbb{R}^{4 \times 4}$
- $e^{[\chi]} \rightarrow I$ as $[\chi] \rightarrow 0$
- Any local movement in $\mathbb{SE}(3)$ around I , which is $e^{[\chi]} - I$, can be approximated by some small $[\chi]$
- The set of $[\chi]$ forms the tangent space of $\mathbb{SE}(3)$ at I



Lie algebra $\mathfrak{se}(3)$ of $\mathbb{SE}(3)$

- The set of $[\chi]$ forms the tangent space of $\mathbb{SE}(3)$ at I
 - Ex: What is the tangent space at any $T \in \mathbb{SE}(3)$?
- We give this set a name, the “Lie algebra of $\mathbb{SE}(3)$ ”
 - $\mathfrak{se}(3) := \left\{ \begin{bmatrix} S & t \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4 \times 4} : S^T = -S \right\}$

The Lie algebra of $\mathbb{SE}(3)$

- Motion interpretation

$\hat{\omega}$: motion direction

- Exponential coordinate

$$\vec{\theta} = \hat{\omega}\theta \in \mathbb{R}^3$$

- Exponential map

$$R = \exp([\hat{\omega}]\theta) \in \mathbb{SO}(3)$$

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$$\chi = \hat{\xi}\theta \in \mathbb{R}^6$$

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Compute $T \in \mathbb{SE}(3)$ from $\hat{\xi}\theta$

- Recall **Rodrigues Formula** for rotations:

$$e^{[\hat{\omega}]\theta} = I + [\hat{\omega}]\sin \theta + [\hat{\omega}]^2(1 - \cos \theta)$$

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- Further computation gives

$$e^{[\hat{\xi}\theta]} = \begin{bmatrix} e^{[\hat{\omega}\theta]} & (I - e^{[\hat{\omega}\theta]})(\hat{\omega} \times d) + \hat{\omega}\hat{\omega}^T d\theta \\ 0 & 1 \end{bmatrix}$$

Compute $\hat{\xi}\theta$ from $T \in \mathbb{SE}(3)$

- First, determine $\hat{\omega}\theta \in so(3)$ from the $SO(3)$ rotation
- The translation component of T is t , then d in

$$\hat{\xi}\theta = \begin{bmatrix} d \\ \hat{\omega} \end{bmatrix} \theta \text{ can be calculated as follow } (\theta \neq 0):$$

$$d = \left(\frac{1}{\theta} I - \frac{1}{2} [\hat{\omega}] + \left(\frac{1}{\theta} - \frac{1}{2} \cot \frac{\theta}{2} \right) [\hat{\omega}]^2 \right) t$$

- $t \perp \hat{\omega} \iff \frac{1}{\theta} (I + [\hat{\omega}]^2) t = 0$, and there is no $\frac{1}{\theta}$ term in d

Read Motion Parameters from $\hat{\xi}\theta$

- Let us extract $\hat{\omega}$, q , θ , d_ω from $\hat{\xi}\theta$
 - $\hat{\omega}$: we can directly read from $\hat{\xi}$
 - $q = [\hat{\omega}]^\dagger (\hat{\omega}\hat{\omega}^T - I)d$, where d can be read from $\hat{\xi}$
 - θ : we can directly read
 - $d_\omega = \hat{\omega}\hat{\omega}^T d\theta$

Summary

- **Exponential map:** $T = \text{Trans}(\hat{\omega}, \theta, q, d_{\omega}) = e^{[s]}$
- **Screw:** $\chi = \begin{bmatrix} -[\hat{\omega}]q\theta + d_{\omega} \\ \hat{\omega}\theta \end{bmatrix}$ is the displacement of the 6D motion
- **Unit twist:** $\hat{\xi} = \begin{bmatrix} d \\ \hat{\omega} \end{bmatrix} \in \mathbb{R}^6$ so that $\chi = \hat{\xi}\theta$, the direction of the 6D motion

Example of Screw Computation

Q: What is the screw $\chi = \hat{\xi}\theta$ given $T(\theta) = e^{[\hat{\xi}]\theta}$?

$$T(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha t) & -\sin(\alpha t) & \sin(\alpha t) \\ 0 & \sin(\alpha t) & \cos(\alpha t) & 1 - \cos(\alpha t) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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- Recall that given $R \in \mathbb{SO}(3)$, we can compute θ and $[\hat{\omega}]$ (L3 P29)

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$$\bullet \quad \theta = \alpha t, [\hat{\omega}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \text{ thus } \hat{\omega} = [1, 0, 0]^T$$

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- $\theta = \alpha t, [\hat{\omega}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \hat{\omega} = [1, 0, 0]^T$

- Recall that $d = (\frac{1}{\theta}I - \frac{1}{2}[\hat{\omega}] + (\frac{1}{\theta} - \frac{1}{2}\cot\frac{\theta}{2})(\hat{\omega}^2))t$

- With some calculation, we get $d = [0, 1, 0]^T$

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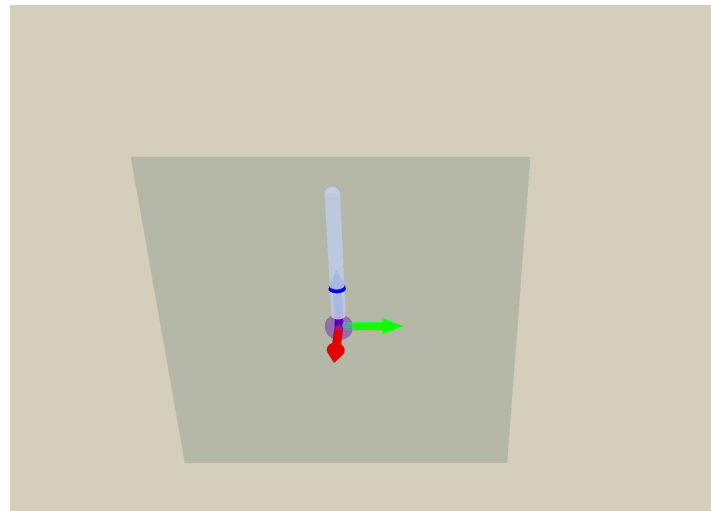
- $\hat{\xi}\theta = \begin{bmatrix} d \\ \hat{\omega} \end{bmatrix} \theta = [0, 1, 0, 1, 0, 0]^T \alpha t$, so $\chi = \hat{\xi}\theta = [0, \alpha t, 0, \alpha t, 0, 0]^T$

Assume $T(\theta)$ describes the relative transformation of a body frame relative to spatial frame: $T_{s \rightarrow b}^s(\theta) \equiv T(\theta)$

$$T(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha t) & -\sin(\alpha t) & \sin(\alpha t) \\ 0 & \sin(\alpha t) & \cos(\alpha t) & 1 - \cos(\alpha t) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- $\chi_{s \rightarrow b}^s = \hat{\xi}_{s \rightarrow b}^s \theta_{s \rightarrow b}^s = [0, \alpha t, 0, \alpha t, 0, 0]^T$

$\chi_{s \rightarrow b}^s$ represents the linear transformation of rotating about a fixed axis



R: x-axis
G: y-axis
B: z-axis

- $\chi_{s \rightarrow b}^s = \hat{\xi}_{s \rightarrow b}^s \theta_{s \rightarrow b}^s = [0, \alpha t, 0, \alpha t, 0, 0]^T$

$\chi_{s \rightarrow b}^s$ should represent the linear transformation of rotating about a fixed axis.
Can we decode this information from $[0, \alpha t, 0, \alpha t, 0, 0]^T$?

Q: What are $\hat{\omega}$, q , θ , d_ω for $T(\theta) = e^{[\hat{\xi}] \theta}$, where $\hat{\xi} \theta = [0, \alpha t, 0, \alpha t, 0, 0]^T$?

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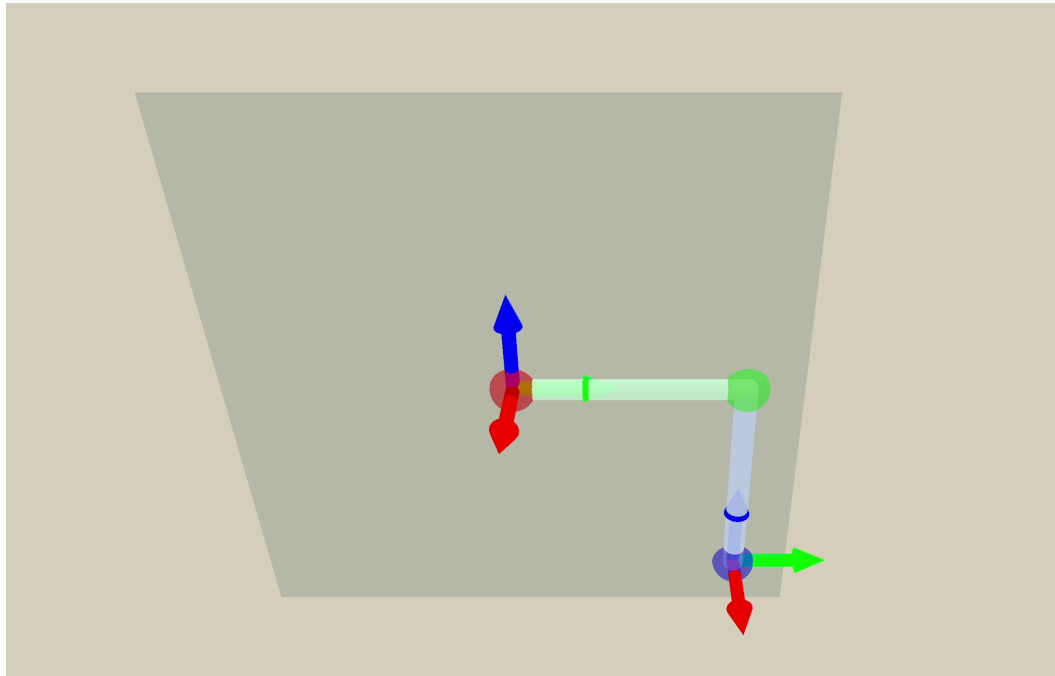
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Q: What are $\hat{\omega}$, q , θ , d_ω for $T(\theta) = e^{[\hat{\xi}] \theta}$, where $\hat{\xi} \theta = [0, \alpha t, 0, \alpha t, 0, 0]^T$?

- Recall: $q = [\hat{\omega}]^\dagger (\hat{\omega} \hat{\omega}^T - I) d$
- With $\hat{\omega} = [1, 0, 0]^T$, $d = [0, 1, 0]^T$, we have $q = [0, 0, 1]^T$
- Recall: $d_\omega = \hat{\omega} \hat{\omega}^T d \theta$
- With $\theta = \alpha t$, we have $d_\omega = [0, 0, 0]^T$

Now, let's consider another case:

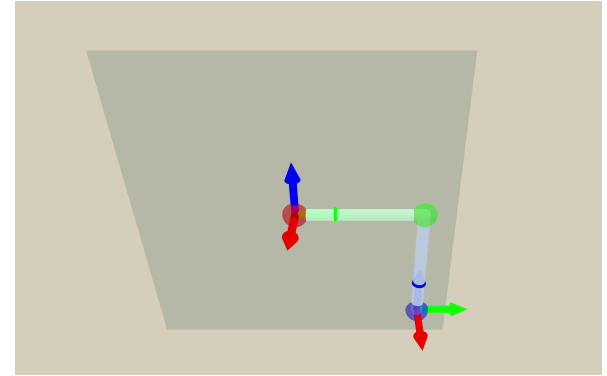
- A robot has two links (green stick and blue stick) connected by a revolute joint (green sphere). The end-effector (blue sphere) is connected to the end of the second link. The spatial frame is at the red sphere (static).
- What is the screw $\chi_{s \rightarrow e}^s(t)$ of the end-effector in the spatial frame?



What is the screw $\chi_{s \rightarrow e}^s(t)$ of the end-effector in the spatial frame?

- Write down $T_{s \rightarrow e}^s$ by observation

$$T(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha t) & -\sin(\alpha t) & 1 + \sin(\alpha t) \\ 0 & \sin(\alpha t) & \cos(\alpha t) & -\cos(\alpha t) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



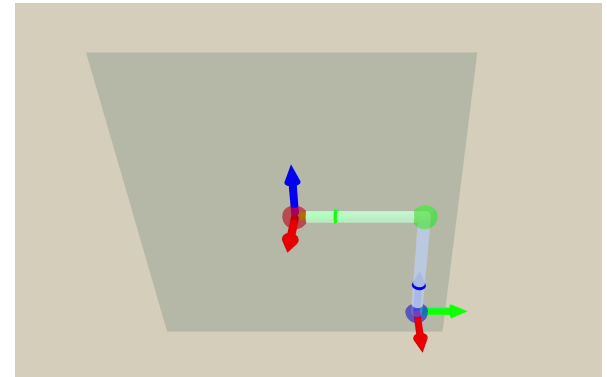
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- Similar as before, $\theta = \alpha t$, $\hat{\omega} = [1, 0, 0]^T$

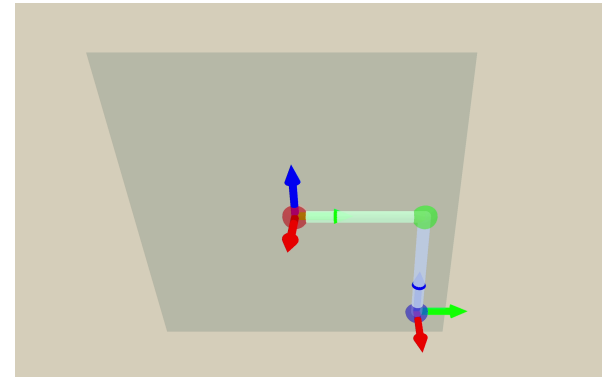
- $d = (\frac{1}{\theta}I - \frac{1}{2}[\hat{\omega}] + (\frac{1}{\theta} - \frac{1}{2} \cot \frac{\theta}{2})[\hat{\omega}]^2)t$



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- Similar as before, $\theta = \alpha t$, $\hat{\omega} = [1, 0, 0]^T$

$$d = \left(\frac{1}{\theta} I - \frac{1}{2} [\hat{\omega}] + \left(\frac{1}{\theta} - \frac{1}{2} \cot \frac{\theta}{2} \right) [\hat{\omega}]^2 \right) t$$

$$d = \begin{bmatrix} 0 \\ \frac{(\sin(\theta) + 1) \cot(\frac{\theta}{2}) - \cos(\theta)}{2} \\ -\frac{\sin(\theta) + \cos(\theta) \cot(\frac{\theta}{2}) + 1}{2} \end{bmatrix}, \text{ so complex!}$$

What is the screw $\chi_{s \rightarrow e}^s(t)$ of the end-effector in the spatial frame?

- Screw $\chi_{s \rightarrow e}^s(t)$ is a function of time, since $\theta = \alpha t$

$$d = \begin{bmatrix} 0 \\ \frac{(\sin(\theta) + 1)\cot(\frac{\theta}{2}) - \cos(\theta)}{2} \\ -\frac{\sin(\theta) + \cos(\theta)\cot(\frac{\theta}{2}) + 1}{2} \end{bmatrix}$$

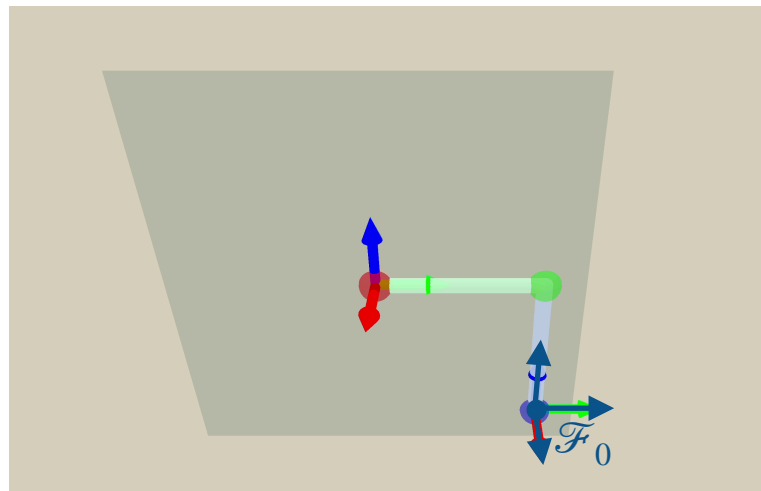
- Even for a simple motion, the screw representation can be very complex
- Is there a better way of representing $T_{s \rightarrow e}^s(t)$ by screw?

Is there a better way of representing $T_{s \rightarrow e}^s(t)$ by screw?

- Here we define a fixed auxiliary frame \mathcal{F}_0 and decompose $T_{s \rightarrow e}^s(t)$ into a composition of transformations

$$T_{s \rightarrow e}^s(t) = e^{[\chi_{s \rightarrow 0}^s]} e^{[\chi_{0 \rightarrow e}^0]} \text{ (coord. trans. composition rule)}$$

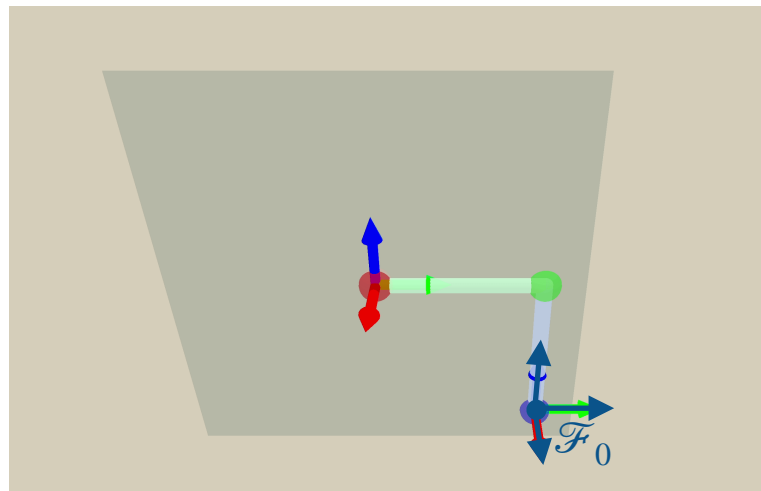
- $\mathcal{F}_0 = \{p_0^s, (x_0^s, y_0^s, z_0^s)\}: p_0^s = [0, 1, -1]^T$ and (x_p^s, y_p^s, z_p^s) has the same direction as \mathcal{F}_s
- Note that at $t = 0$, \mathcal{F}_0 aligns with \mathcal{F}_e



Is there a better way of representing $T_{s \rightarrow e}^s(t)$ by screw?

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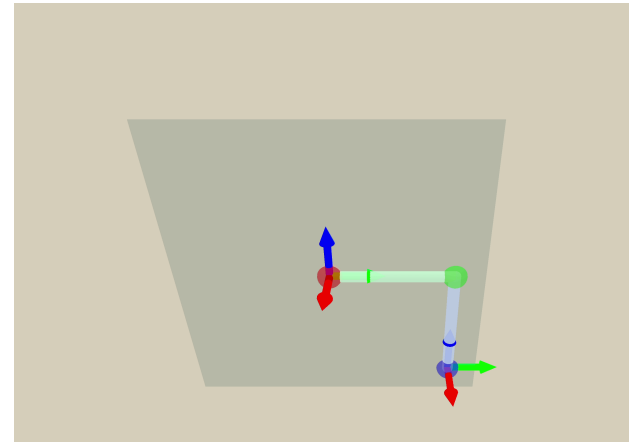
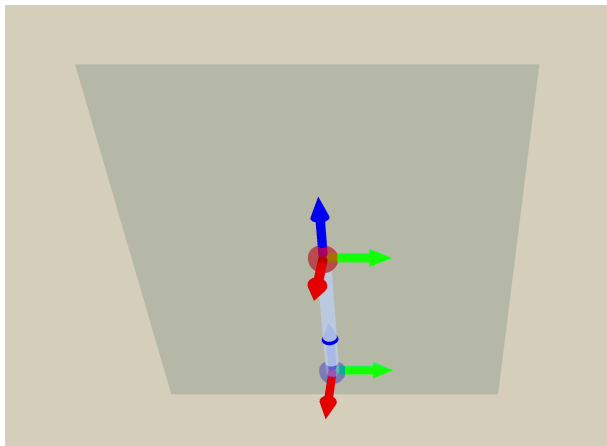
$$T_{s \rightarrow e}^s(t) = e^{[\chi_{s \rightarrow 0}^s]} e^{[\chi_{0 \rightarrow e}^0]} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha t) & -\sin(\alpha t) & \sin(\alpha t) \\ 0 & \sin(\alpha t) & \cos(\alpha t) & 1 - \cos(\alpha t) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Is there a better way of representing $T_{s \rightarrow e}^s(t)$ by screw?

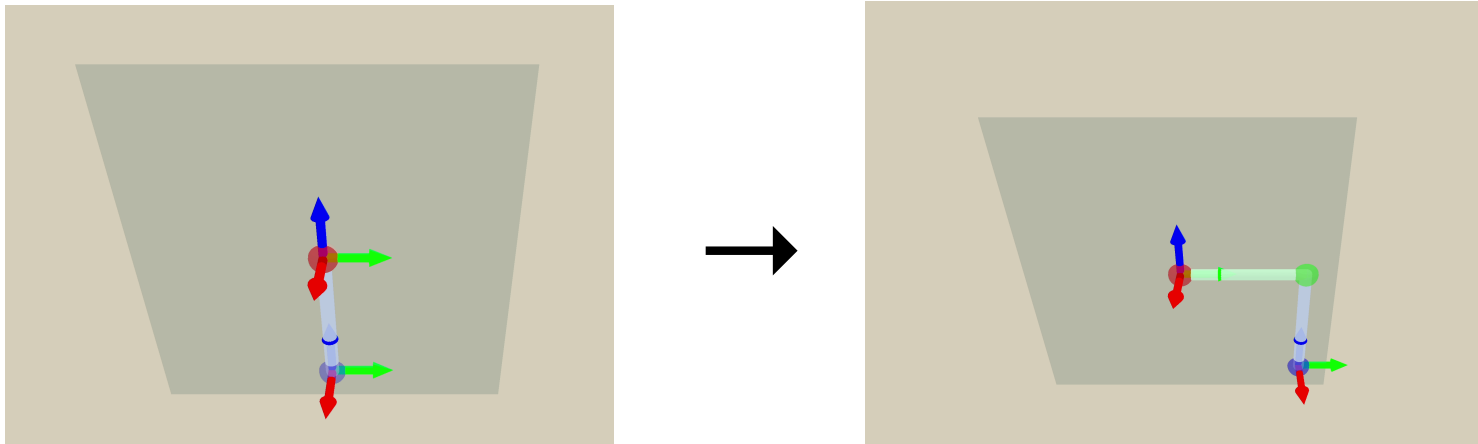
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- By simple inspection, $[\chi_{s \rightarrow 0}^s] = [0, 1, -1, 0, 0, 0]^T$
- As calculated in the previous example, $[\chi_{0 \rightarrow e}^0] = [0, \alpha t, 0, \alpha t, 0, 0]^T$



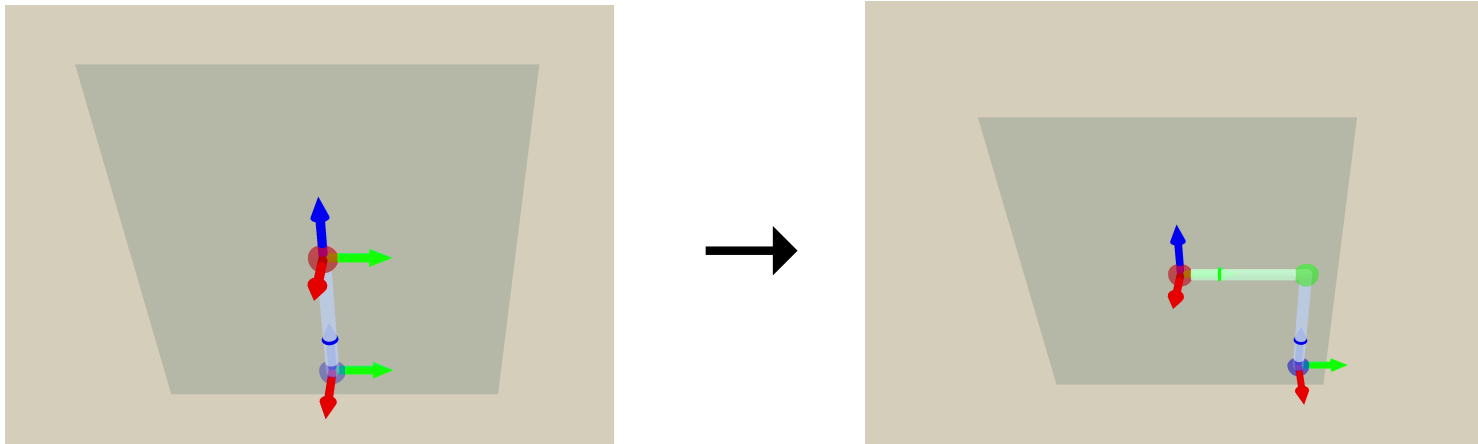
Is there a better way of representing $T_{s \rightarrow e}^s(t)$ by screw?

- Decomposing $T_{s \rightarrow e}^s(t)$ into two screw $e^{[\chi_{s \rightarrow 0}^s]}e^{[\chi_{0 \rightarrow e}^0]}$ makes things easier!
- Why we select $\mathcal{F}_0 = \{p_0^s, (x_0^s, y_0^s, z_0^s)\}: p_0^s = [0, 1, -1]^T$ and (x_p^s, y_p^s, z_p^s) r ?



Is there a better way of representing $T_{s \rightarrow e}^s(t)$ by screw?

- Decomposing $T_{s \rightarrow e}^s(t)$ into two screw $e^{[\chi_{s \rightarrow 0}^s]}e^{[\chi_{0 \rightarrow e}^0]}$ makes things easier
- Why we select $\mathcal{F}_0 = \{p_0^s, (x_0^s, y_0^s, z_0^s)\}$, where $p_0^s = [0, 1, -1]^T$ and (x_p^s, y_p^s, z_p^s) represent the same direction as \mathcal{F}_s ?
- Observation: \mathcal{F}_0 aligns with \mathcal{F}_e at $t = 0$



Is there a better way of representing $T_{s \rightarrow e}^s(t)$ by screw?

- Observation: \mathcal{F}_0 aligns with \mathcal{F}_e at $t = 0$

$$T_{s \rightarrow e}^s(t) = e^{[\chi_{s \rightarrow 0}^s]} e^{[\chi_{0 \rightarrow e}^0]} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha t) & -\sin(\alpha t) & \sin(\alpha t) \\ 0 & \sin(\alpha t) & \cos(\alpha t) & 1 - \cos(\alpha t) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$e^{[\chi_{0 \rightarrow e}^0]}$ is an identity matrix at $t = 0$

For motion of rotating about a fixed axis (common for revolute joint in real robot), screw will be very simple when it starts with an identity matrix

Libraries based on Screw Theory

- https://github.com/NxRLab/ModernRobotics/blob/master/packages/Python/modern_robotics/core.py
- <https://petercorke.github.io/robotics-toolbox-python/intro.html#>

Twist (6D Velocity Parameterization)

Setup

- Let us first parameterize the motion of a body frame by time:
 - An observer associated to \mathcal{F}_o records the motion as $T_{s' \rightarrow b(t)}^o$, where the body frame is at $\mathcal{F}_{b(t)}$.

Twist

$$\begin{aligned}
 T_{s' \rightarrow b(t+\Delta t)}^o - T_{s' \rightarrow b(t)}^o &= T_{b(t) \rightarrow b(t+\Delta t)}^o T_{s' \rightarrow b(t)}^o - T_{s' \rightarrow b(t)}^o \\
 &= e^{[\chi_{b(t) \rightarrow b(t+\Delta t)}^o]} T_{s' \rightarrow b(t)}^o - T_{s' \rightarrow b(t)}^o \\
 &\approx [\chi_{b(t) \rightarrow b(t+\Delta t)}^o] T_{s' \rightarrow b(t)}^o
 \end{aligned}$$

- Divided by Δt and take the limit, we have

$$\begin{aligned}
 \dot{T}_{s' \rightarrow b(t)}^o &= \lim_{\Delta t \rightarrow 0} \left[\frac{\chi_{b(t) \rightarrow b(t+\Delta t)}^o}{\Delta t} \right] T_{s' \rightarrow b(t)}^o \\
 &= [\xi_{b(t)}^o] T_{s' \rightarrow b(t)}^o
 \end{aligned}$$

- $\xi_{b(t)}^o := \lim_{\Delta t \rightarrow 0} \frac{\chi_{b(t) \rightarrow b(t+\Delta t)}^o}{\Delta t}$ is called “**twist**”, the 6D instant velocity

Twist

- Twist: $\xi_{b(t)}^o := \lim_{\Delta t \rightarrow 0} \frac{\chi_{b(t) \rightarrow b(t+\Delta t)}^o}{\Delta t}$
- $[\xi_{b(t)}^o] = \dot{T}_{s' \rightarrow b(t)}^o (T_{s' \rightarrow b(t)}^o)^{-1}$
- Note: $\xi_{b(t)}^o \neq \dot{\chi}_{s' \rightarrow b(t)}^o$ for general $\chi_{s \rightarrow b(t)}^o(t)$ (verify by yourself)

Linear Velocity from Twist

- The linear velocity of p^o caused by $T_{s' \rightarrow b(t)}^o$ at time t is

$$\begin{aligned}\mathbf{v}_p^o(t) &= \lim_{\Delta t \rightarrow 0} \frac{T_{b(t) \rightarrow b(t+\Delta t)}^o p^o - p^o}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\exp([\chi_{b(t) \rightarrow b(t+\Delta t)}^o]) - I}{\Delta t} p^o \\ &= \lim_{\Delta t \rightarrow 0} \frac{[\chi_{b(t) \rightarrow b(t+\Delta t)}^o]}{\Delta t} p^o = [\xi_{b(t)}^o] p^o\end{aligned}$$

- Therefore, $\boxed{\mathbf{v}_p^o(t) = [\xi_{b(t)}^o] p^o}$

(Recall that, if a motion is a pure rotation, then $\mathbf{v}_p^o(t) = \omega_{b(t)}^o \times p^o$)

Interpretation and Computation of Twist

- Let $\xi_{b(t)}^o = \begin{bmatrix} \nu_{b(t)}^o \\ \omega_{b(t)}^o \end{bmatrix} \in \mathbb{R}^6$, then $[\xi_{b(t)}^o] = \begin{bmatrix} [\omega_{b(t)}^o] & \nu_{b(t)}^o \\ 0 & 0 \end{bmatrix}$

Body Twist

- When $\mathcal{F}_o = \mathcal{F}_{b(t)}$, it is particularly easy to compute $\xi_{b(t)}^{b(t)}$
- Remarks
 - For body twist, when recording at time t , you should think it as ***first cloning the body frame and then record the movement with this cloned frame keeping static***
 - Body twist is “ego-centric” and sometimes simpler to specify for robotics. For example, if we take the gripper frame as the body frame. Using the body twist, we can express “move gripper forward” by a pure translation

Body Twist Computation

- By $\mathbf{v}_p^o(t) = [\xi_{b(t)}^o] p^o$ and $[\xi_{b(t)}^o] = \begin{bmatrix} [\omega_{b(t)}^o] & \nu_{b(t)}^o \\ 0 & 0 \end{bmatrix}$, let $p_{origin}^{b(t)} = [0,0,0,1]^T$, which is the origin of body frame

$$\mathbf{v}_{origin}^{b(t)}(t) = [\xi_{b(t)}^{b(t)}] p_{origin}^{b(t)} = \nu^{b(t)}$$

- Note that $\mathbf{v}_{origin}^{b(t)}(t)$ is the linear velocity of the origin of the body frame
- Therefore, $\xi_{b(t)}^{b(t)}$ is composed by the linear velocity of the origin and the angular velocity
- In practice, we often write down the body twist first and then obtain the twist in other frames by change of coordinate (next lecture)

Compute $\xi_{b(t)}^o$ from Angle-Axis

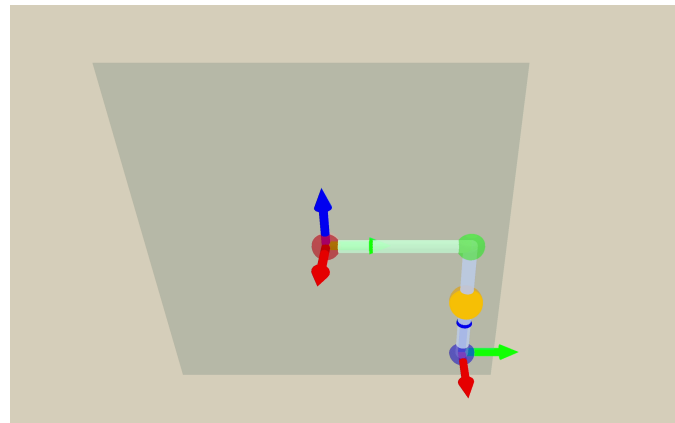
- Let $\xi_{b(t)}^o = \begin{bmatrix} \nu_{b(t)}^o \\ \omega_{b(t)}^o \end{bmatrix} \in \mathbb{R}^6$, then $[\xi_{b(t)}^o] = \begin{bmatrix} [\omega_{b(t)}^o] & \nu_{b(t)}^o \\ 0 & 0 \end{bmatrix}$
- The instant linear velocity can be decomposed into the rotation about the axis and translation along the axis
- Take a point $q^o \in \mathbb{R}^3$ on the axis,
 - By $\mathbf{v}_p^o(t) = [\xi_{b(t)}^o]p^o$, $\mathbf{v}_q^o = [\omega^o]q^o + \nu^o$
 - Since the only velocity of q^o is along $\hat{\omega}$, $\mathbf{v}_q^o = \mathbf{v}_\omega^o$
 - $\therefore \nu^o = -[\omega^o]q^o + \mathbf{v}_\omega^o$
- $\xi^o = \begin{bmatrix} -[\omega^o]q^o + \mathbf{v}_\omega^o \\ \omega^o \end{bmatrix}$ (all the symbols have $b(t)$ as the subscript)

Example of Twist Computation

Example of Twist Computation

- Consider the previous example, but now a orange point is fixed to the end-effector frame (blue sphere)
- What is the **velocity of orange point at $t = 0$** ? Given the pose of end effector frame as below:

$$T_{s \rightarrow b(t)}^s = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha t) & -\sin(\alpha t) & 1 + \sin(\alpha t) \\ 0 & \sin(\alpha t) & \cos(\alpha t) & -\cos(\alpha t) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Example of Twist Computation

- The velocity of yellow point caused by the end-effector motion can be computed via twist
- Recall: $\mathbf{v}_p^s = [\xi_{b(t)}^s] p^s$

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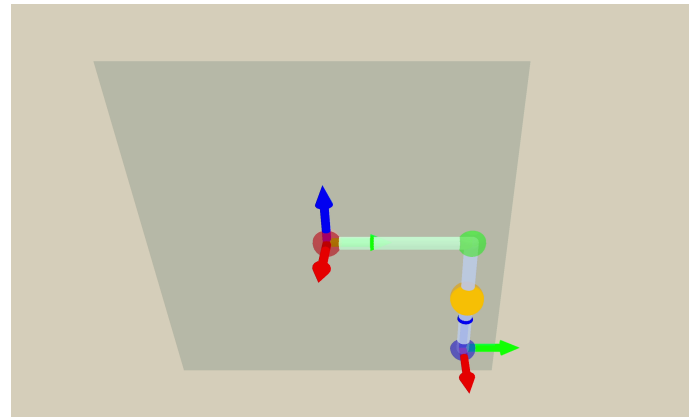
- we have $[\xi_{s \rightarrow b(t)}^s] = \dot{T}_{s \rightarrow b(t)}^s (T_{s \rightarrow b(t)}^s)^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & \alpha & 0 & -\alpha \\ 0 & 0 & 0 & 0 \end{bmatrix}$

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- The velocity of yellow point caused by the end-effector motion can be computed via twist
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$$\text{At } t = 0, p^s = \begin{bmatrix} 0 \\ 1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

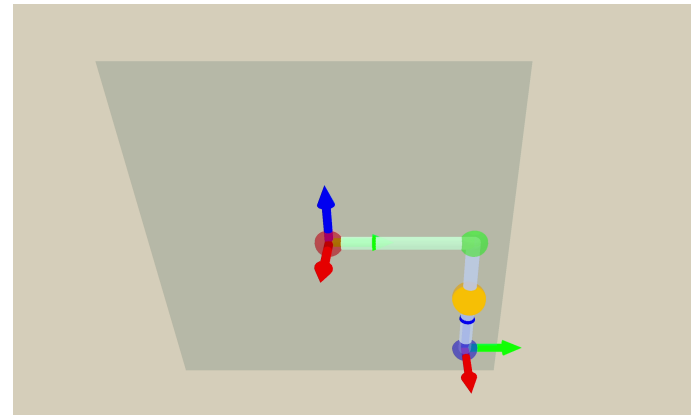


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- Recall: $\mathbf{v}_p^s = [\xi_{b(t)}^s] p^s$

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$$\mathbf{v}_p^s = [\xi_{b(t)}^s] p^s = \begin{bmatrix} 0 \\ \frac{\alpha}{2} \\ 0 \\ 0 \end{bmatrix}$$

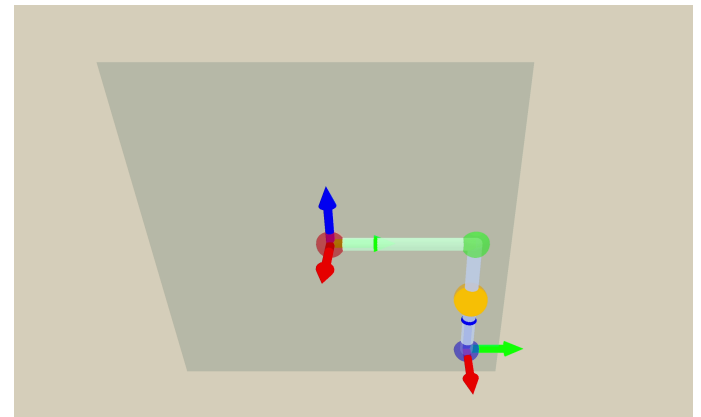


Example of Twist Computation

- We can verify this result by taking the derivative of $\frac{d}{dt}p^s(t)$

$$p^s(t) \begin{bmatrix} 0 \\ 1 + \frac{\alpha}{2} \sin(t) \\ -\frac{\alpha}{2} \cos(t) \\ 1 \end{bmatrix}, \frac{d}{dt}p^s(t) = \begin{bmatrix} 0 \\ \frac{\alpha}{2} \cos(t) \\ \frac{\alpha}{2} \sin(t) \\ 0 \end{bmatrix}$$

$$\mathbf{v}_p^s = [\xi_{b(t)}^s]p^s = \begin{bmatrix} 0 \\ \frac{\alpha}{2} \\ 0 \\ 0 \end{bmatrix} = \frac{d}{dt}p^s(t) \big|_{t=0}$$



Example of Twist Computation

- What is the body twist of the end effector?
- In the body frame of the end effector (blue sphere), the blue sphere has a constant linear velocity, which is always $[0, \alpha, 0]$. The angular velocity is always $[\alpha, 0, 0]$.

$$\text{So, } \xi_{b(t)}^{b(t)} = [0, \alpha, 0, \alpha, 0, 0]^T$$

