UC San Diego

L2: Robot Geometry

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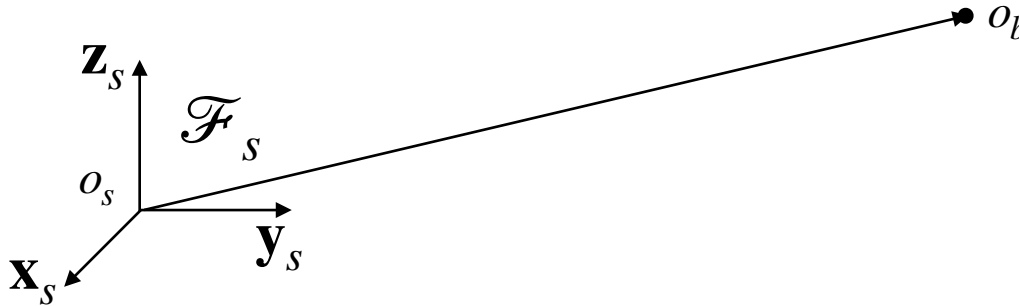
Agenda

- Rigid Transformation

- Definition of $\mathbb{SO}(3)$ and $\mathbb{SE}(3)$

- Multi-Link Rigid-Body Geometry

Rigid Transformation

Notation Convention

- An observer **records** the position of any point in the space **using a frame** $\mathscr{F}_s$
- We use ordinary letters to denote points (e.g., p), and bold letters to dente **vectors (e.g., v)**
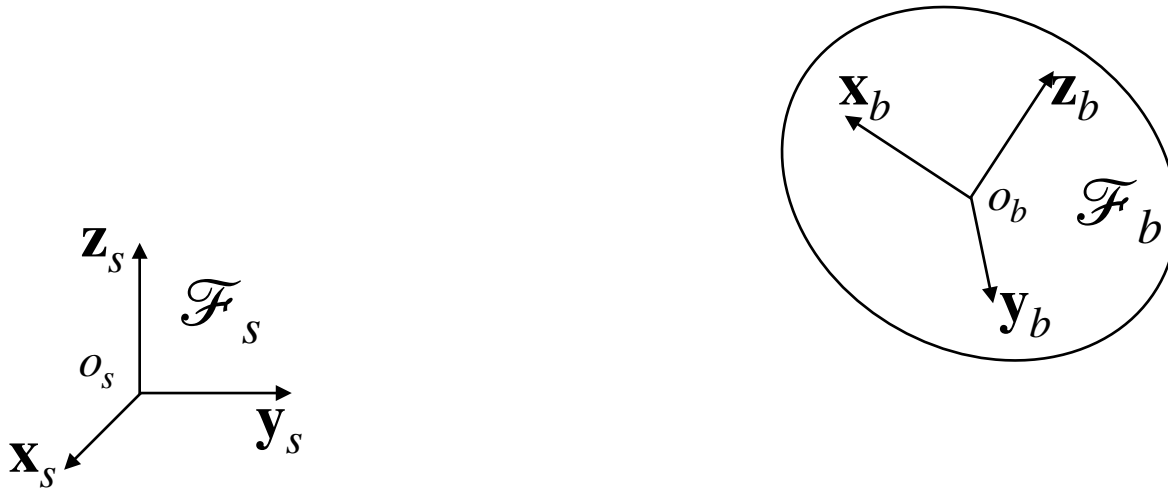- When **writing equations**, we add a superscript to symbols to denote the recording frame, e.g.,

$$o_b^s = o_s^s + \mathbf{t}_{s \to b}^s$$

Rigid Transformation

$\mathbf{z}_s$

$\mathscr{F}_s$

o_s

$\mathbf{y}_s$

$\mathbf{x}_s$

$\mathbf{x}_b$ $\mathbf{z}_b$

o_b $\mathscr{F}_b$

$\mathbf{y}_b$
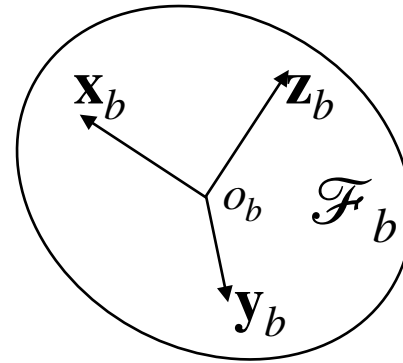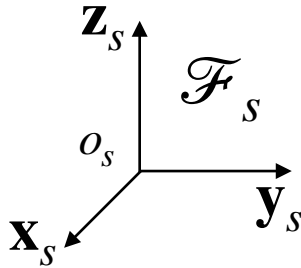
- There is a rigid object, to which we bind a frame $\mathscr{F}_b$ (body frame) tightly, so that $\mathscr{F}_b$ moves along with the object
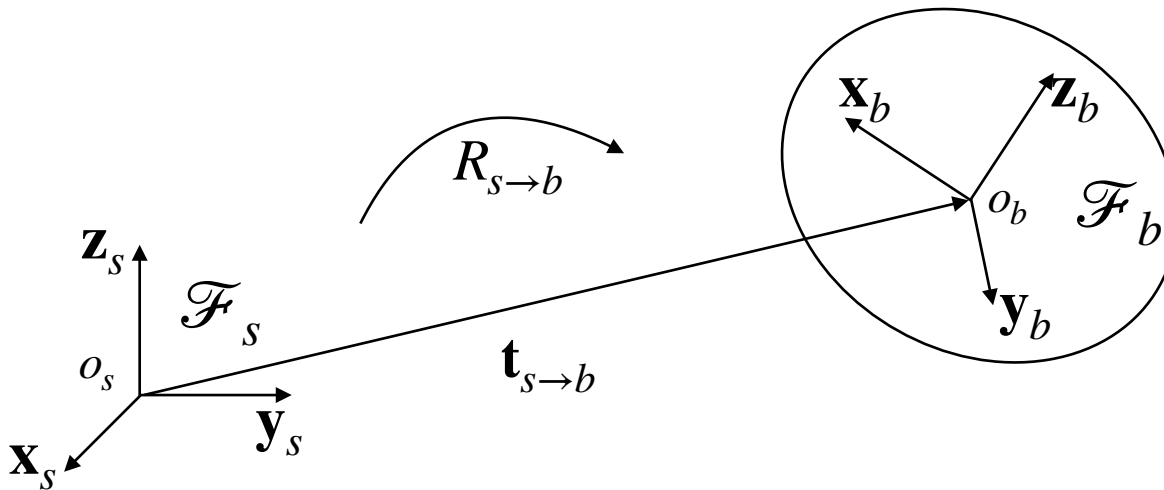
Rigid Transformation

$$\mathbf{x}_b \quad \mathbf{z}_b$$

$$o_b \quad \mathscr{F}_b$$

$$\mathbf{y}_b$$

$$\mathbf{z}_s$$

$$\mathscr{F}_s$$

$$o_s$$

$$\mathbf{y}_s$$

$$\mathbf{x}_s$$

- When talking about the pose of the *rigid* object, we ask:

  How to **transform** $\mathscr{F}_s$ so that it overlaps with $\mathscr{F}_b$?
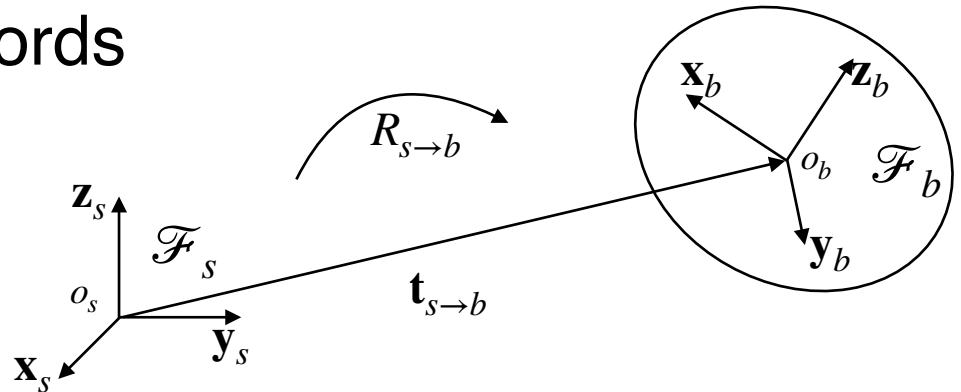
Rigid Transformation

- We first translate $\mathscr{F}_s$ by $\mathbf{t}_{s \to b}$ to align o_s and o_b

- And then rotate by $R_{s \to b}$ to align $\{\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i\}$ ($i = s$ or b)

Rigid Transformation

- Formally,
  - $o_b^s = o_s^s + \mathbf{t}_{s \to b}^s$
  - $[\mathbf{x}_b^s, \mathbf{y}_b^s, \mathbf{z}_b^s] = R_{s \to b}^s [\mathbf{x}_s^s, \mathbf{y}_s^s, \mathbf{z}_s^s]$

- Since the observer records everything using $\mathscr{F}_s$,
  - $o_s^s = 0$
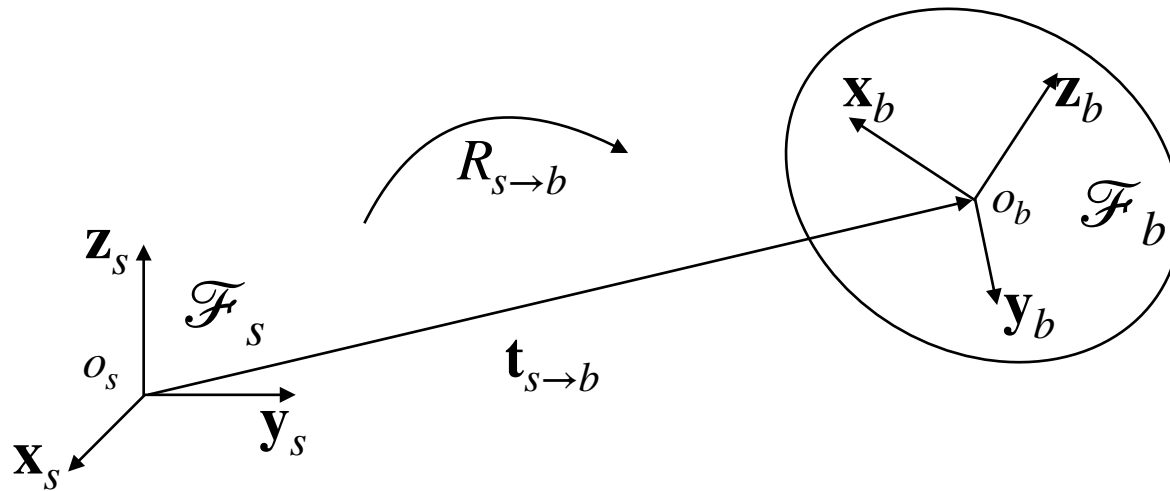 - $[\mathbf{x}_s^s, \mathbf{y}_s^s, \mathbf{z}_s^s] = I_{3 \times 3}$

- Therefore,
  - $\mathbf{t}_{s \to b}^s = o_b^s$
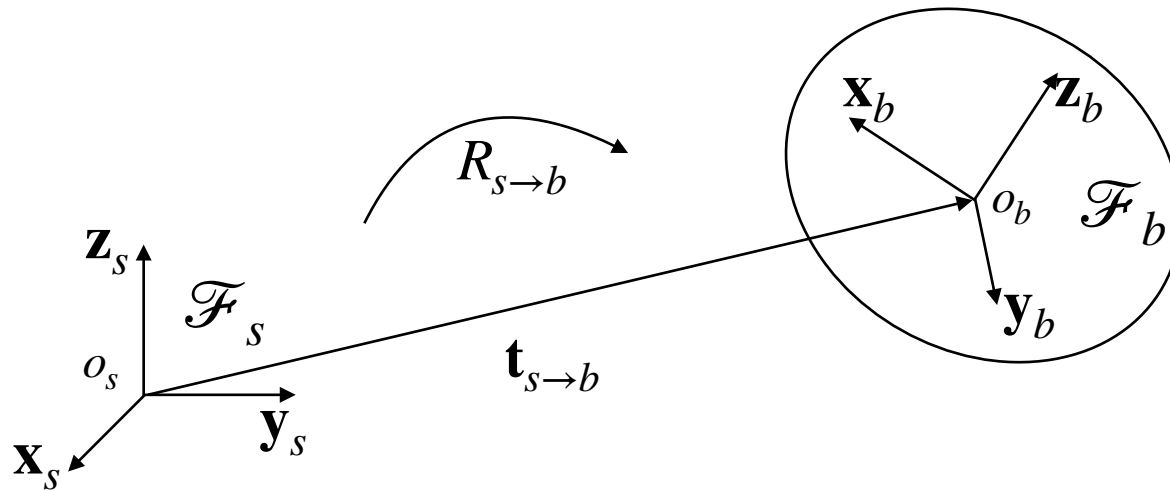 - $R_{s \to b}^s = [\mathbf{x}_b^s, \mathbf{y}_b^s, \mathbf{z}_b^s] \in \mathbb{R}^{3 \times 3}$

$(R_{s \rightarrow b}, \mathbf{t}_{s \rightarrow b})$ for Coordinate Transformation

Use Coordinate Transformation to Relate Coordinates in Frames

- Assume a second observer that records coordinates by $\mathscr{F}_b$
- Assume a point p on the body. Since $\mathscr{F}_b$ moves along the body, its coordinate recorded in $\mathscr{F}_b$, denoted as p^b, should **never change**.

$(R_{s \to b}, \mathbf{t}_{s \to b})$ for Coordinate Transformation

- Imagine a process: $\mathscr{F}_b$ moves from $\mathscr{F}_s$ to the current location. This is how we define $(R^s_{s \to b}, \mathbf{t}^s_{s \to b})$.
- Since p moves along $\mathscr{F}_b$, it is moved from the **initial position**, $p^s = p^b$, to the current location:

$$p^s = R^s_{s \to b} p^b + \mathbf{t}^s_{s \to b}$$

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Homogenous Coordinates

- Homogeneous coordinate for 3D Space:

$$\tilde{x} := \begin{bmatrix} x \\ 1 \end{bmatrix} \in \mathbb{R}^4$$

- Homogeneous transformation matrix:

$$T_{s \to b}^s = \begin{bmatrix} R_{s \to b}^s & \mathbf{t}_{s \to b}^s \\ 0 & 1 \end{bmatrix}$$

- Coordinate transformation under linear form:

$$\tilde{x}^s = T_{s \to b}^s \tilde{x}^b$$

- Ignore ˜ for simplicity in the future.

Homogenous Coordinates

- The coordinate transformation works for any choice of $\mathscr{F}_s$ and $\mathscr{F}_b$

- As a general rule, we have:

$$x^1 = T^1_{1 \to 2} x^2$$

Some Rules of Homogenous Coordinate Transformation

By $x^1 = T^1_{1\to2}x^2$, we have $x^2 = T^2_{2\to1}x^1$ and $x^3 = T^3_{3\to2}x^2$.

Therefore, $x^3 = T^3_{3\to2}T^2_{2\to1}x^1$. But $x^3 = T^3_{3\to1}x^1$

- Composition rule: $T^3_{3\to1} = T^3_{3\to2}T^2_{2\to1}$
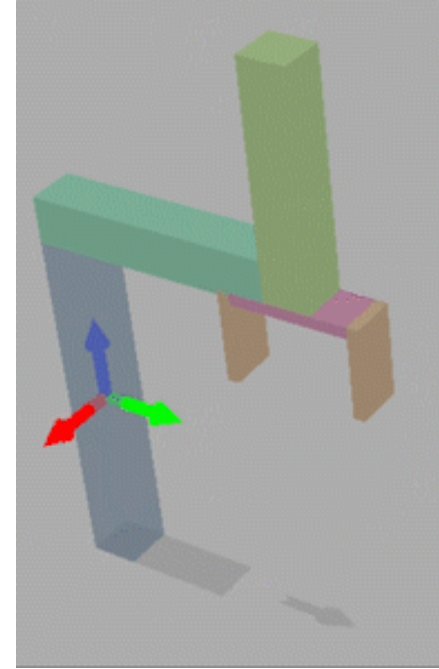
By $x^1 = T^1_{1\to2}x^2$, we have $x^2 = (T^1_{1\to2})^{-1}x^1$

- Change of observer's frame: $T^2_{2\to1} = (T^1_{1\to2})^{-1}$

Example
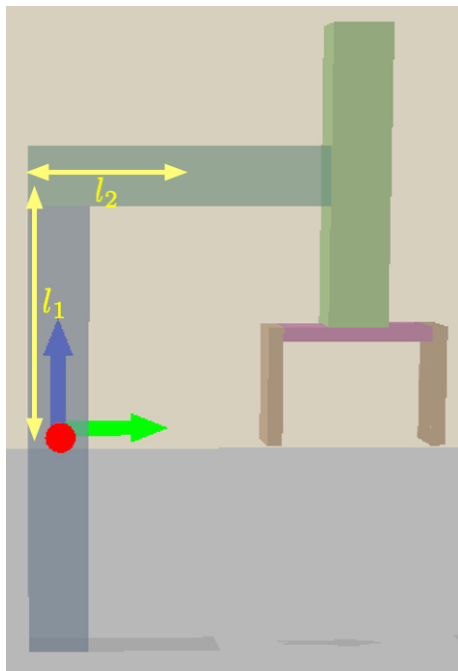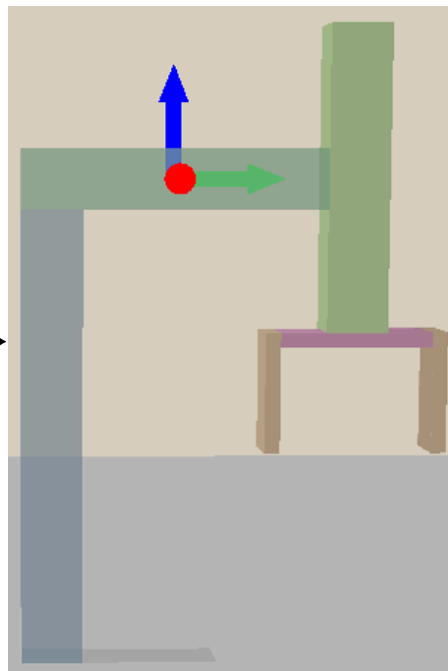
A simple 2 DoF robot arm
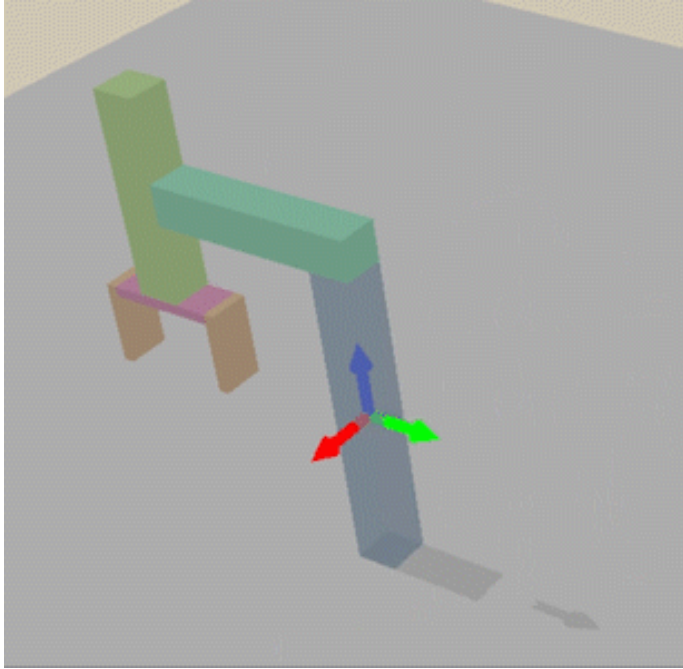
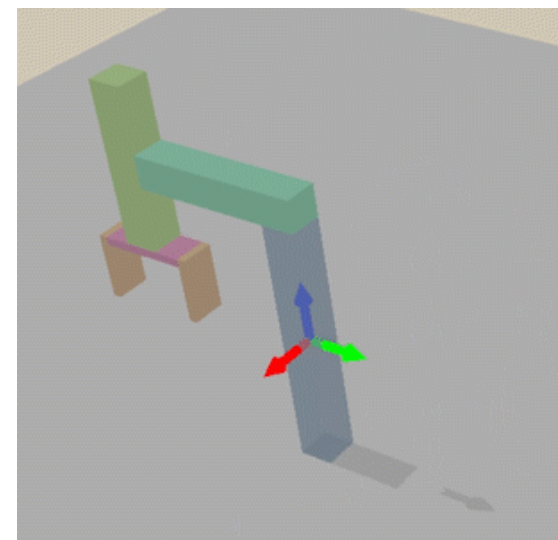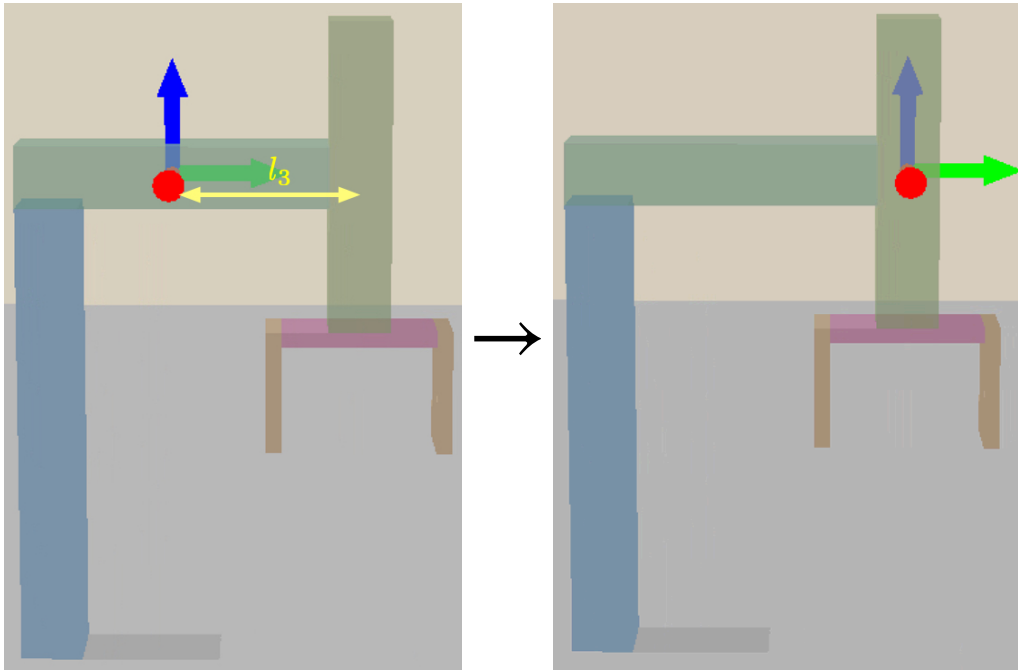revolute (θ_1)

prismatic (θ_2)

base

link1

revolute (θ_1)

$$T_{0 \to 1}^0 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & -l_2\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 & 0 & l_2\cos\theta_1 \\ 0 & 0 & 1 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
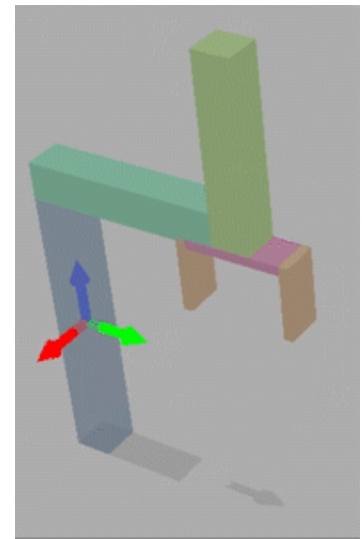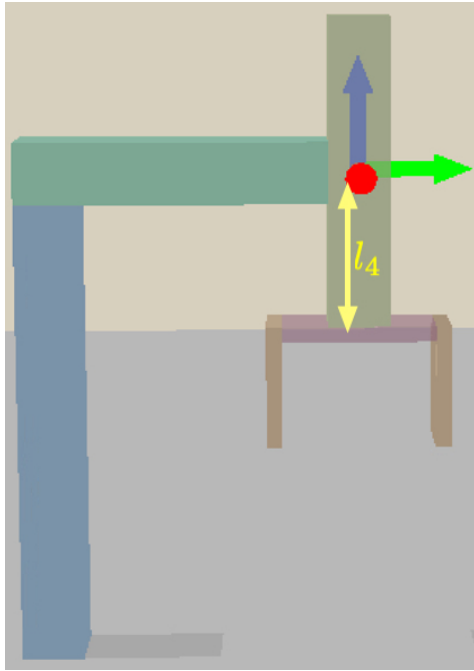
link1 → link2

$$T^1_{1 \to 2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_3 \\ 0 & 0 & 1 & \theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

prismatic (θ_2)
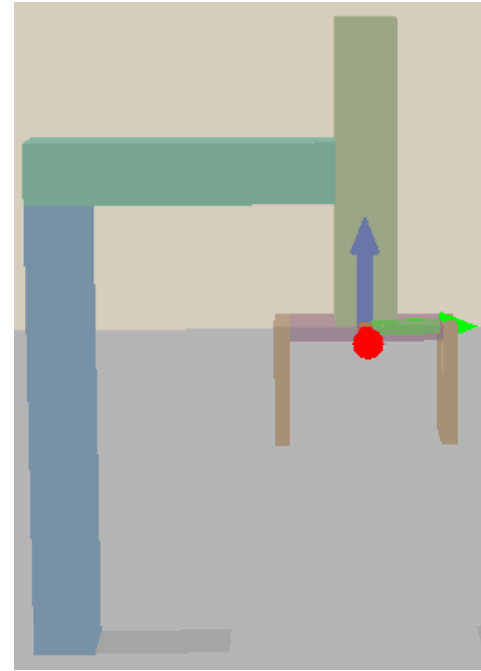
link2 $\rightarrow$ end_effector

$$T_{2\to3}^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -l_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
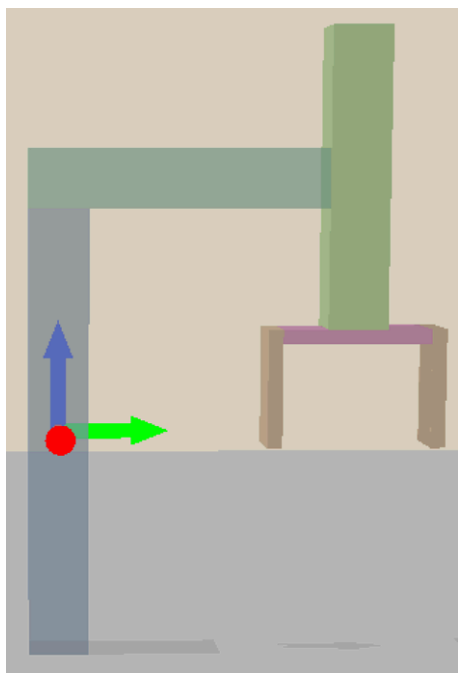
base $\rightarrow$ end_effector
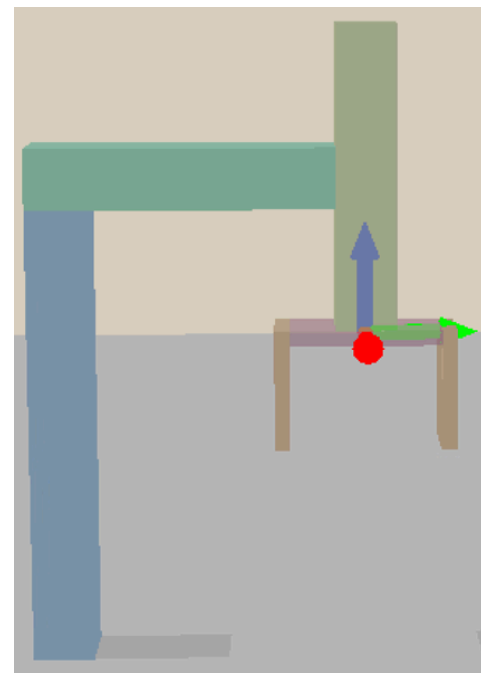
$$T^0_{0\rightarrow3} = T^0_{0\rightarrow1}T^1_{1\rightarrow2}T^2_{2\rightarrow3} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & -\sin\theta_1(l_2 + l_3) \\ \sin\theta_1 & \cos\theta_1 & 0 & \cos\theta_1(l_2 + l_3) \\ 0 & 0 & 1 & l_1 - l_4 + \theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$(R_{s \to b}, \mathbf{t}_{s \to b})$ as a Linear Transformation

$(R_{s \to b}, \mathbf{t}_{s \to b})$ as a Linear Transformation
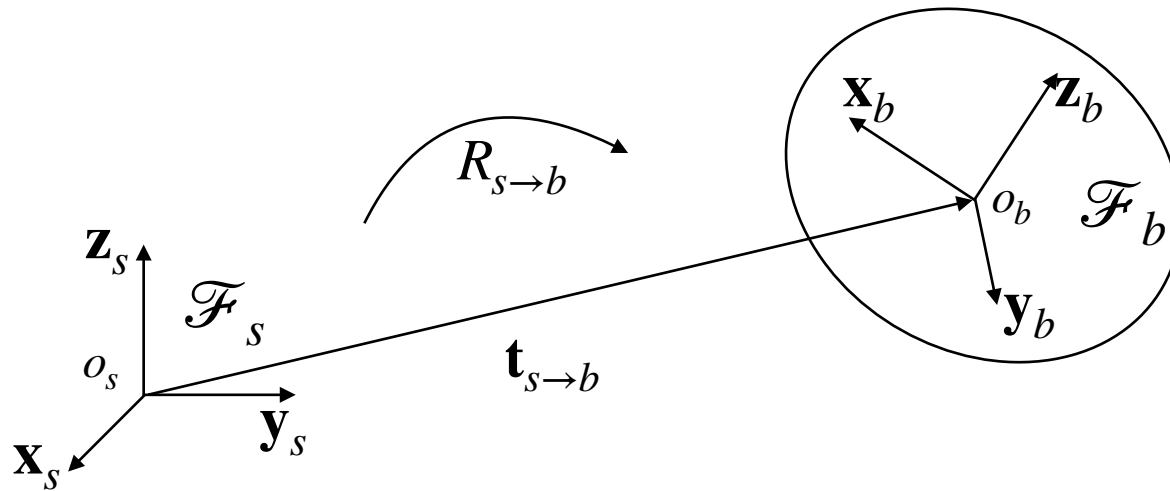
- $(R_{s \to b}, \mathbf{t}_{s \to b})$ transforms any **point** in the *whole space* by the following equation:

$$x'^s = R^s_{s \to b} x^s + \mathbf{t}^s_{s \to b}$$

$(R_{s \rightarrow b}, \mathbf{t}_{s \rightarrow b})$ as a Linear Transformation

- Suppose $\mathscr{F}_p^s = \{p^s, (\mathbf{x}_p^s, \mathbf{y}_p^s, \mathbf{z}_p^s)\}$ is a frame at an arbitrary point p^s
- **Then, the new origin is:** $p'^s = ?$

$(R_{s\to b}, \mathbf{t}_{s\to b})$ as a Linear Transformation

- Suppose $\mathscr{F}_p^s = \{p^s, (\mathbf{x}_p^s, \mathbf{y}_p^s, \mathbf{z}_p^s)\}$ is a frame at an arbitrary point p^s
- **Then, the new origin is:** $p'^s = R_{s\to b}^s p^s + \mathbf{t}_{s\to b}^s$

$(R_{s \to b}, \mathbf{t}_{s \to b})$ as a Linear Transformation

- Suppose $\mathscr{F}_p^s = \{p^s, (\mathbf{x}_p^s, \mathbf{y}_p^s, \mathbf{z}_p^s)\}$ is a frame at an arbitrary point p^s
- **Then, the new origin is:** $p'^s = R_{s \to b}^s p^s + \mathbf{t}_{s \to b}^s$
- How about the bases vectors of the frame?
  - Assume three curves, $\gamma_x, \gamma_y, \gamma_z$, passing p^s at $t = 0$ with tangents $\mathbf{x}_p, \mathbf{y}_p, \mathbf{z}_p$

$(R_{s \to b}, \mathbf{t}_{s \to b})$ as a Linear Transformation

- Suppose $\mathscr{F}_p^s = \{p^s, (\mathbf{x}_p^s, \mathbf{y}_p^s, \mathbf{z}_p^s)\}$ is a frame at an arbitrary point p^s
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- How about the bases vectors of the frame?
  - Assume three curves, $\gamma_x, \gamma_y, \gamma_z$, passing p^s at $t = 0$ with tangents $\mathbf{x}_p, \mathbf{y}_p, \mathbf{z}_p$
  - Then, the new tangents after transformation are:
    $$\frac{d}{dt} R_{s \to b}^s \gamma_x^s(0), \ \frac{d}{dt} R_{s \to b}^s \gamma_y^s(0), \ \frac{d}{dt} R_{s \to b}^s \gamma_z^s(0)$$

$(R_{s \to b}, \mathbf{t}_{s \to b})$ as a Linear Transformation

- Suppose $\mathscr{F}_p^s = \{p^s, (\mathbf{x}_p^s, \mathbf{y}_p^s, \mathbf{z}_p^s)\}$ is a frame at an arbitrary point p^s
- **Then, the new origin is:** $p'^s = R_{s \to b}^s p^s + \mathbf{t}_{s \to b}^s$
- How about the bases vectors of the frame?
  - Assume three curves, $\gamma_x, \gamma_y, \gamma_z$, passing p^s at $t = 0$ with tangents $\mathbf{x}_p, \mathbf{y}_p, \mathbf{z}_p$
  - Then, the new tangents after transformation are:
  $$\frac{d}{dt} R_{s \to b}^s \gamma_x^s(0), \ \frac{d}{dt} R_{s \to b}^s \gamma_y^s(0), \ \frac{d}{dt} R_{s \to b}^s \gamma_z^s(0)$$

- **So the new frame is:** $\mathscr{F}_{p'}^s = \{p'^s, R_{s \to b}^s [\mathbf{x}_p^s, \mathbf{y}_p^s, \mathbf{z}_p^s]\}$

$$T^s_{1 \rightarrow 2}$$

- We have introduced the notations when the observer is recoding by $\mathscr{F}_s$ or $\mathscr{F}_b$

  - $T^s_{s \rightarrow b}$ (record the frame alignment from $\mathscr{F}_1$ to $\mathscr{F}_2$)

  - By the change of observer's frame, we introduced $T^b_{b \rightarrow s} = (T^s_{s \rightarrow b})^{-1}$

- Next, we define the notion of $T^s_{1 \rightarrow 2}$, which is how we **record** an arbitrary transformation from $\mathscr{F}_1$ to $\mathscr{F}_2$ in $\mathscr{F}_s$

  - $T^s_{1 \rightarrow 2} := T^s_{s \rightarrow 2} T^1_{1 \rightarrow s}$

Composition as a Homogeneous Linear Transformation

- Under the definition $T^s_{1\to 2} := T^s_{s\to 2} T^1_{1\to s}$, the composition rule is:

$$T^s_{1\to 2} = T^s_{3\to 2} T^s_{1\to 3}$$

Change Observer's Frame with Similarity Transformation

- Given $T^s_{1 \to 2}$, what is $T^b_{1 \to 2}$?

$$T^s_{1 \to 2} T^s_{s \to 1} = T^s_{s \to 2}$$ Composition as Linear Transformation

$$T^s_{1 \to 2} T^s_{s \to b} T^b_{b \to 1} = T^s_{s \to b} T^b_{b \to 2}$$ Composition as Coordinate Transformation

$$T^s_{1 \to 2} T^s_{s \to b} T^b_{b \to 1} = T^s_{s \to b} T^b_{1 \to 2} T^b_{b \to 1}$$ Composition as Linear Transformation

$$T^s_{1 \to 2} T^s_{s \to b} = T^s_{s \to b} T^b_{1 \to 2}$$

$$T^s_{1 \to 2} = T^s_{s \to b} T^b_{1 \to 2} (T^s_{s \to b})^{-1}$$

- Similarity Transformation changes the **superscript**

$$B = X^{-1} A X \text{: Similarity Transformation}$$

A Special Case

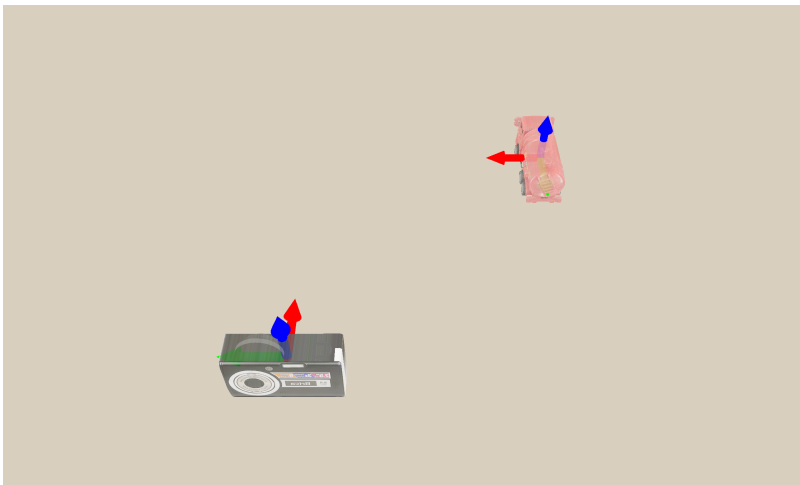- By $T^s_{1\to2} = T^s_{s\to b}T^b_{1\to2}(T^s_{s\to b})^{-1}$,

  - If $\mathscr{F}_1 = \mathscr{F}_s$ and $\mathscr{F}_2 = \mathscr{F}_b$, $T^s_{s\to b} = T^b_{s\to b}$!

- Therefore, we often see the abbreviated notations:

  - $T^s_b \equiv T^s_{s\to b}$

  - $T_{sb} \equiv T^s_{s\to b}$

  - $T_b \equiv T^s_{s\to b}$

- The above equation can therefore be written as:

$$T^s_{1\to2} = T_{s\to b}T^b_{1\to2}(T_{s\to b})^{-1}$$

Example

- Consider a camera with frame $\mathscr{F}_c$ observing a red car
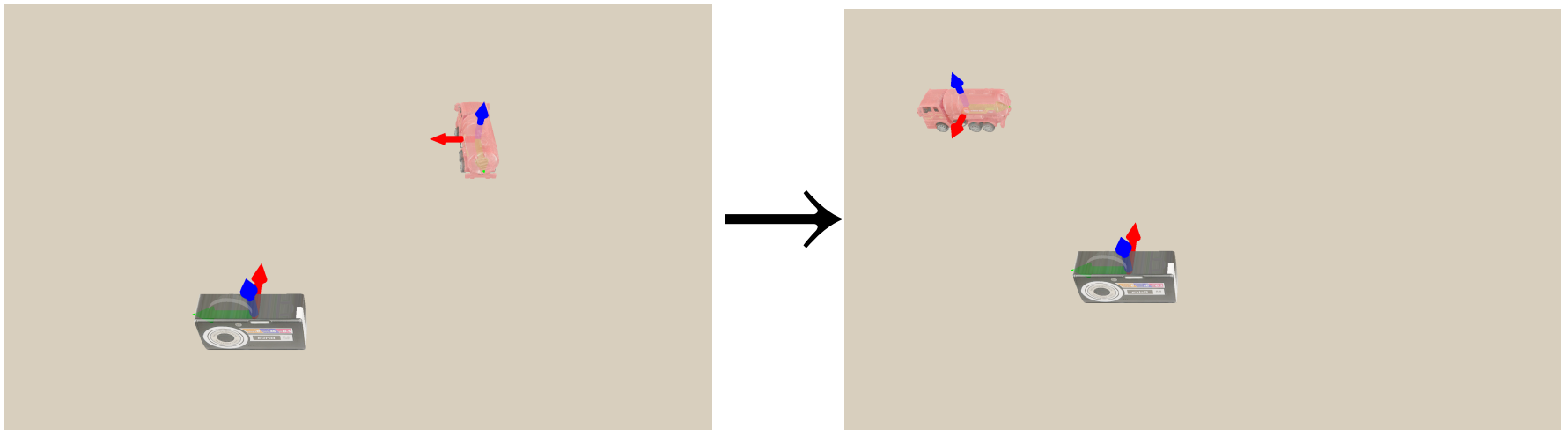- Denote the current frame of the red car as $\mathscr{F}_1$

$$T^c_{c \to 1} = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 & l \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 & -l \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

- Then the red car move to a new frame $\mathscr{F}_2$

$$T^c_{c \to 2} = \begin{bmatrix} \cos \pi & -\sin \pi & 0 & l \\ \sin \pi & \cos \pi & 0 & l \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
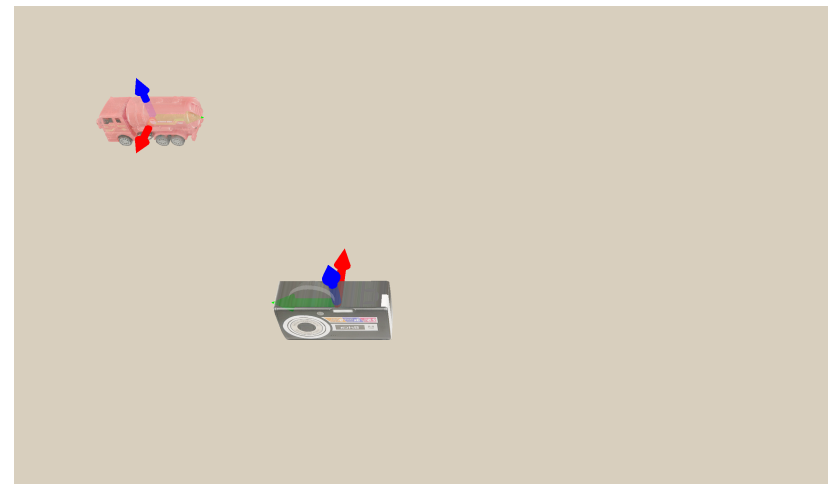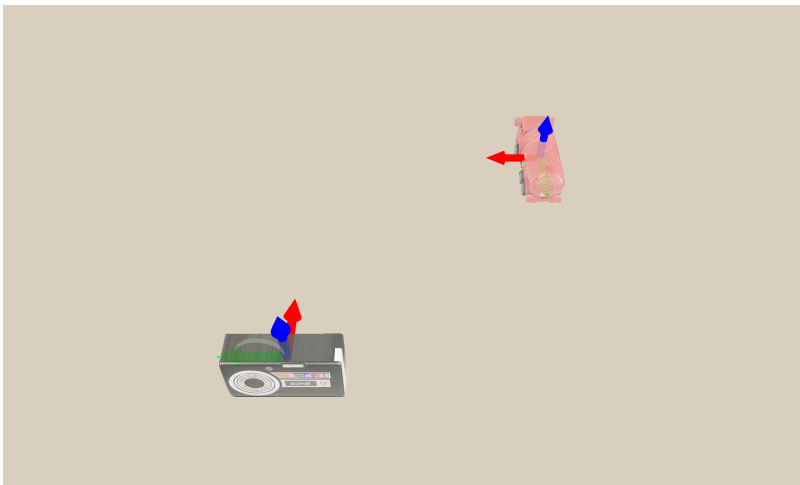
Example

- By the composition rule of coordinate transformation:

$$T^c_{c \to 2} = T^c_{c \to 1} T^1_{1 \to 2}$$

$$T^1_{1 \to 2} = (T^c_{c \to 1})^{-1} T^c_{c \to 2} = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 & 2l \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

- By the composition rule of coordinate transformation:

$$T^c_{c \to 2} = T^c_{c \to 1} T^1_{1 \to 2}$$

$$T^1_{1 \to 2} = (T^c_{c \to 1})^{-1} T^c_{c \to 2} = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 & 2l \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- The movement from $\mathscr{F}_1$ to $\mathscr{F}_2$ can also be represented as a linear transformation from $\mathscr{F}_1$ to $\mathscr{F}_2$, recorded by frame c, denoted as $T^c_{1 \to 2}$

Example

- With similarity transformation:

$$T_{1\to2}^c = T_{c\to1}^c T_{1\to2}^1 (T_{c\to1}^c)^{-1} = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 & 0 \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
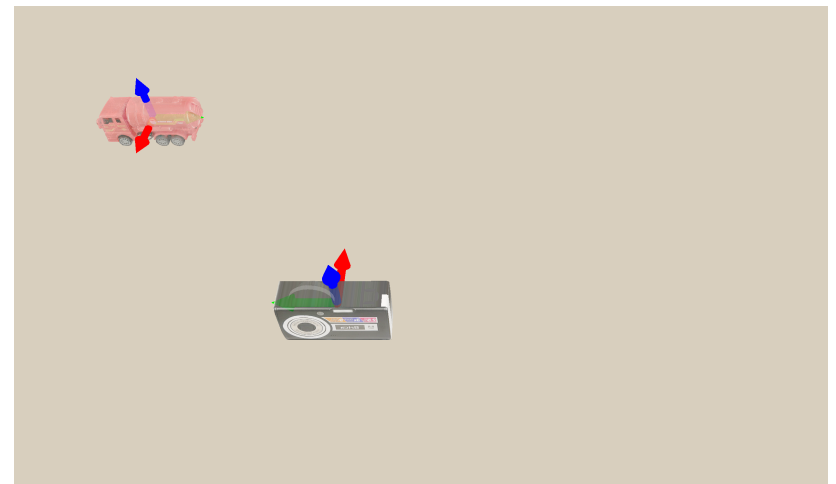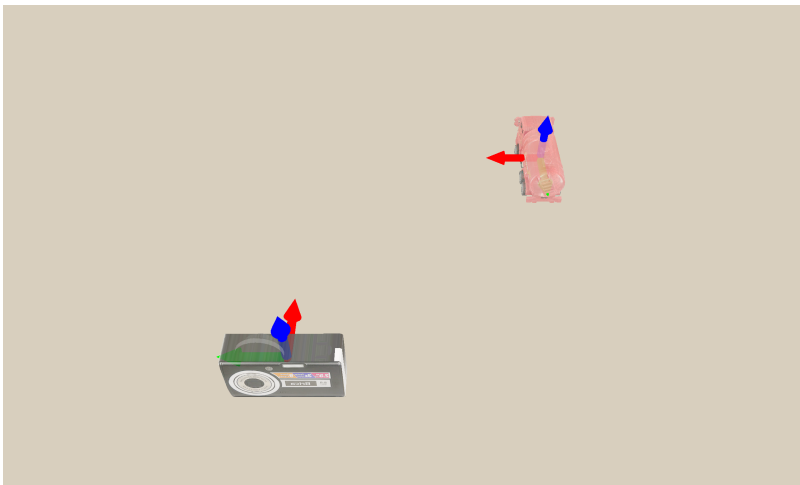
- Note: translation in $T_{1\to2}^c$ is all zero! Why?

Example

- Transformation $T^c_{1 \to 2}$ can be regarded as rotating about z-axis by 90 degree

$$T^c_{1 \to 2} = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 & 0 \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
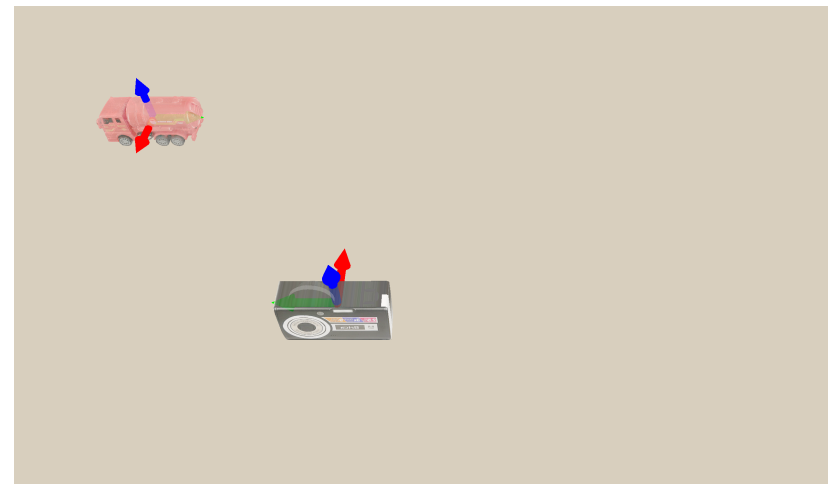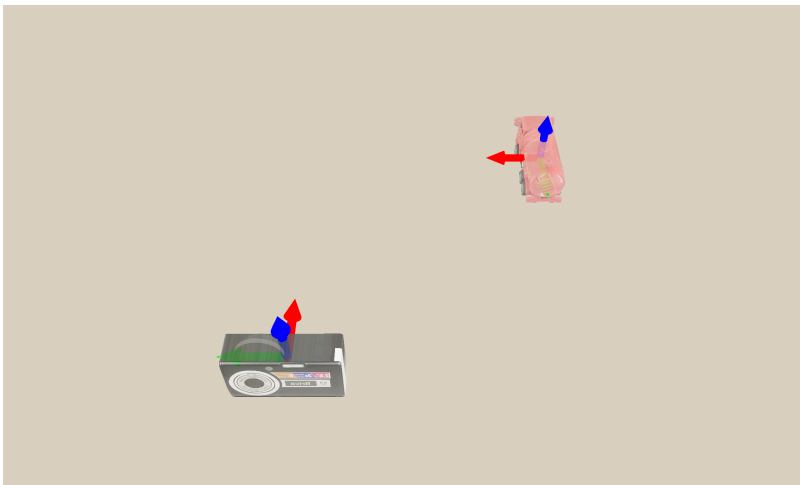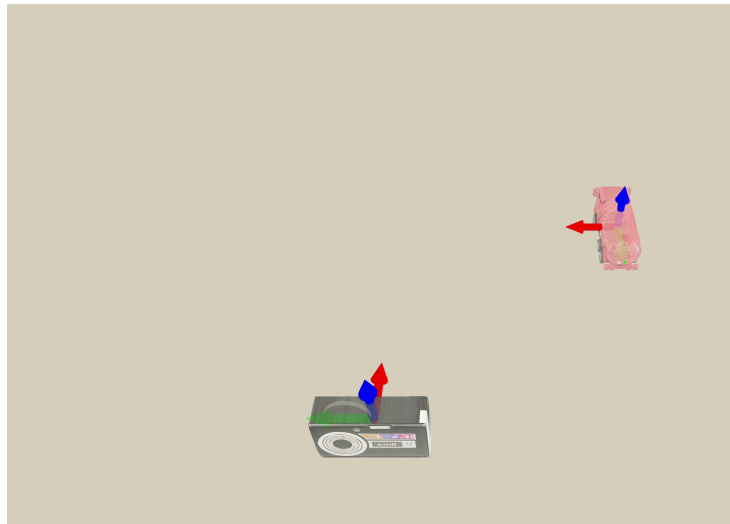
Example

- When observer is recording in the camera frame $\mathscr{F}_c$, the red car is rotated about the z-axis of camera frame c through +90 degree

Additional Notes by the Example

- $T_{1 \to 2}^s$ is **NOT** to record the transformation by first translating $\mathcal{F}_1$ to $\mathcal{F}_2$ and then rotating (this recording convention **only** works when $\mathcal{F}_1 = \mathcal{F}_s$). It is based on the rule $T_{1 \to 2}^s := T_{s \to 2}^s T_{1 \to s}^1$

- An observer chooses its way to decompose $T_{1 \to 2}$ into $R_{1 \to 2}$ and $\mathbf{t}_{1 \to 2}$ based upon its own frame choice

- We will discuss the "canonical" decomposition next week

Additional Notes by the Example

- The linear transformation view allows us to discuss the movement of bodies conveniently (without worrying about the change of observer):

$$T^s_{1\to2} = T^s_{3\to2}T^s_{1\to3}$$

- Suppose a body is moving. Then,

$$T^s_{t_0\to t+\Delta t} = T^s_{t\to t+\Delta t}T^s_{t_0\to t}$$

where t parameterizes time.

Summary

- Basic notation:

    - $T^s_{s \to b}$: Record the motion of frame alignment from $\mathscr{F}_s$ to $\mathscr{F}_b$ in $\mathscr{F}_s$

- Coordinate transformation

    - $T^c_{c \to a} = T^{\color{red}c}_{c \to b} T^{\color{red}b}_{b \to a}$: Composition for coordinate transformation

    - $T^b_{b \to s} = (T^s_{s \to b})^{-1}$: Change of frame for $\mathscr{F}_s$ to $\mathscr{F}_b$ motion

- Linear transformation

    - $T^s_{1 \to 2} := T^s_{s \to 2} T^1_{1 \to s}$: Record the motion of frame alignment from $\mathscr{F}_1$ to $\mathscr{F}_2$ in $\mathscr{F}_s$

    - $T^s_{c \to a} = T^{\color{blue}s}_{b \to a} T^{\color{blue}s}_{c \to b}$: Composition as a linear transformation

- $T^s_{1 \to 2} = T_{s \to b} T^b_{1 \to 2} (T_{s \to b})^{-1}$: Change of frame for $\mathscr{F}_1$ to $\mathscr{F}_2$ motion