Dynamics

Dynamics

- The dynamics of a robot describes how the robot moves in response to these actuator forces.

- Mechanics in physics describes how forces applied to objects result in displacement.

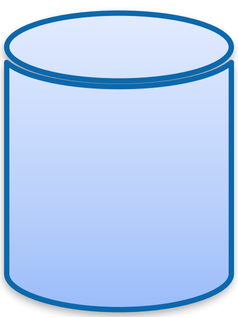
High-level Idea of Robot Control

- Goal: Calculate a sequence of control signals to realize a trajectory.

- Pipeline:
  - Dynamics: Model the dynamics of the task.
  - Motion Planning: Obtain a trajectory in state space.
  - Control: Calculate the control signals

Cylinder Example: From Translation to Rigid Motion

- Consider the motion of a cylinder.
- Rotation need to be considered.



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Cylinder Example: From Translation to Rigid Motion

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- Rotation need to be considered.

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Review of SE(3)

- $a \times b = (a)^{\wedge} b = \hat{a} b, \hat{a} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.$
- $\begin{bmatrix} v \\ \omega \end{bmatrix}$ is the twist representation of $\begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \in R^{4 \times 4}$.
- $\begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} = \left(\begin{bmatrix} v \\ \omega \end{bmatrix} \right)^{\wedge}$

Review of SE(3)

- $g_{ab}(t)$ give the the configuration of frame B relative to frame A. So $q_a = g_{ab}(t)q_b$, where q_a is the coordinates of q in frame A, q_b is similar.

- $g_{ab}(t) = \big(p, R\big), q_a = Rq_b + p$.

- Body velocity V^b_{ab} satisfies $\hat{V}^b_{ab} = g^{-1}_{ab}\dot{g}_{ab}$ which maps q_b to the velocity in frame B.

- Spatial velocity V^s_{ab} satisfies $\hat{V}^s_{ab} = \dot{g}_{ab}g^{-1}_{ab}$ which maps q_a to the velocity in frame A.

Agenda

- **Wrench (Generalized Force)**
- Kinetic Energy
- Generalized Inertia Matrix
- Lagrangian Dynamics
- Lagrangian of Robot Manipulator

Generalized Force (Wrench)

- Twist representation of velocity $V = [v, \omega]$ where v is the linear velocity and ω is rotational velocity.
- Merge the moment and force into a single 6D vector called the **generalized force** or **wrench**.
- $F = [f, \tau]$ where f is a linear force and τ is a torque.

Question: Wrench in different frames?

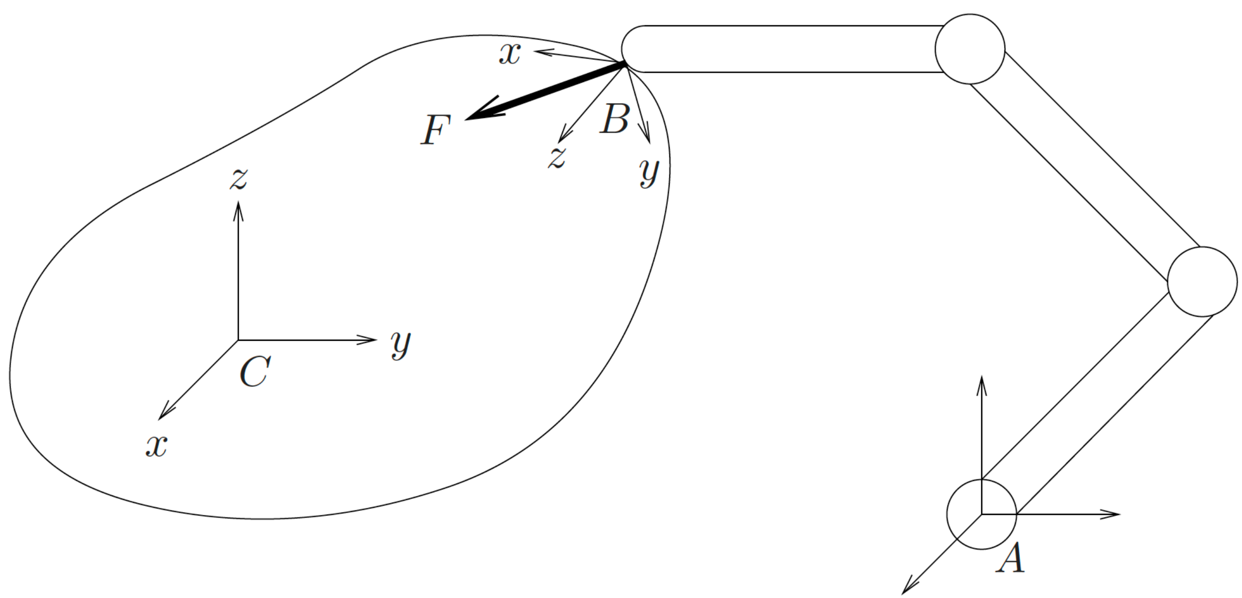
- Consider two frames B and C(End-effector and Base).
- Configuration of frame C relative to B is g_{bc}.
- V^b_{ab} is the velocity relative to inertia reference frame A represented in frame B, F^b is the force in frame B.
- Similarly, we have V^c_{ab}, F^c.

Tool: Energy Conservation

- The **power** is given by $V \cdot F = v \cdot f + \tau \cdot \omega$.
- The **work** is given by $W = \int_{t_1}^{t_2} V \cdot F dt$.
- The power generated by F and V must be the same regardless of the frame.

Solution: Forces in different frames

- $V_{ab}^{b} = Ad_{g_{bc}} V_{ab}^{c}$.
- $\left(V_{ab}^{c}\right)^{T} \boxed{F^{c}} = \left(V_{ab}^{b}\right)^{T} F^{b} = \left(Ad_{g_{bc}} V_{ab}^{c}\right)^{T} F^{b} = \left(V_{ab}^{c}\right)^{T} \boxed{\left(Ad_{g_{bc}}^{T} F^{b}\right)}$.
- So $F^{c} = Ad_{g_{bc}}^{T} F^{b}$.



Agenda

- Wrench (Generalized Force)
- **Kinetic Energy**
- Generalized Inertia Matrix
- Lagrangian Dynamics
- Lagrangian of Robot Manipulator

Kinetic energy - point mass

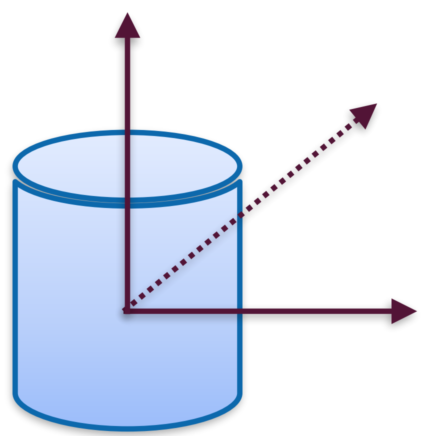
- If the object with mass m is moving in velocity v in an inertia frame, the the kinetic energy is $K = \frac{1}{2}m\|v\|^2$.

Kinetic energy - rigid body

- Integrate kinetic energy of every point mass over the body.

Kinetic energy - rigid body

- $g = \left(p, R\right)$ is the configuration of the body frame which is fixed at the mass center of the object relative to an **inertial frame** (spatial frame).
- $r(x) \in R^3$ be the coordinates of a body point x **relative to the body frame.**
- The coordinates of a body point x relative to the inertial frame is $p + Rr(x)$ and the velocity is $\dot{p} + \dot{R}r(x)$.



Kinetic energy - rigid body

- Kinetic energy is $K = \dfrac{1}{2}\displaystyle\int_{x \in O} \rho(x)\|\dot{p} + \dot{R}r(x)\|^2 dx$.

Kinetic energy - rigid body

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- We will show that $K = \frac{1}{2}m\|\dot{p}\|^2 + \frac{1}{2}\left(\omega^b\right)^T I^b \omega^b$

Translational energy Rotational energy

Kinetic energy - rigid body

- Kinetic energy is $K = \frac{1}{2}\int_{x \in O} \rho(x) \|\dot{p} + \dot{R}r(x)\|^2 dx$.

- We will show that $K = \frac{1}{2} m \|\dot{p}\|^2 + \frac{1}{2}\left(\omega^b\right)^T I^b \omega^b$

m

I^b

$\dot{p}$

ω^b

Kinetic energy - rigid body

0 due to body frame originate at mass center

$$
\begin{aligned}
K &= \frac{1}{2}\int_{x\in O} \rho(x)\|\dot{p} + \dot{R}r(x)\|^2 dx \\
&= \frac{1}{2}\int_{x\in O} \rho(x)\left(\|\dot{p}\|^2 + 2\dot{p}^T\dot{R}r(x) + \|\dot{R}r(x)\|^2\right) dx \\
&= \frac{1}{2}\int_{x\in O} \rho(x)\|\dot{p}\|^2 dx + \dot{p}^T\dot{R}\boxed{\int_{x\in O} \rho(x)r(x)dx} + \frac{1}{2}\int_{x\in O} \rho(r)\|\dot{R}r\|^2 dx \\
&= \frac{1}{2}\int_{x\in O} \rho(x)\|\dot{p}\|^2 dx + \frac{1}{2}\int_{x\in O} \rho(x)\|\dot{R}\cdot r(x)\|^2 dx
\end{aligned}
$$

K contains two terms: translational energy and rotational energy.

Rotational energy

- We know $\hat{\omega}^b = R^T \dot{R}$ is the rotational velocity in the body frame.

$$\frac{1}{2}\int_{x\in O} \rho(x)\|\dot{R}\cdot r(x)\|^2 dx = \frac{1}{2}\int_{x\in O} \rho(x)\left(\dot{R}r(x)\right)^T\left(\dot{R}r(x)\right)dx$$

$$= \frac{1}{2}\int_{x\in O} \rho(x)\left(R\hat{\omega}^b r(x)\right)^T\left(R\hat{\omega}^b r(x)\right)dx$$

$$\hat{\omega}^b = R^T\dot{R} \Rightarrow R\hat{\omega}^b = RR^T\dot{R} = \dot{R}$$

Rotational energy

- We know $\hat{\omega}^b = R^T \dot{R}$ is the rotational velocity in the body frame.

$$\frac{1}{2}\int_{x\in O} \rho(x)\|\dot{R}\cdot r(x)\|^2 dx = \frac{1}{2}\int_{x\in O} \rho(x)\left(\dot{R}r(x)\right)^T \left(\dot{R}r(x)\right) dx$$

$$= \frac{1}{2}\int_{x\in O} \rho(x)\left(R\hat{\omega}^b r(x)\right)^T \left(R\hat{\omega}^b r(x)\right) dx$$

$$\hat{a}b = a \times b = -\,b \times a = -\,\hat{b}a$$

Rotational energy

- We know $\hat{\omega}^b = R^T \dot{R}$ is the rotational velocity in the body frame.

$$
\begin{aligned}
\frac{1}{2}\int_{x\in O} \rho(x)\|\dot{R}\cdot r(x)\|^2 dx &= \frac{1}{2}\int_{x\in O} \rho(x)\left(\dot{R}r(x)\right)^T \left(\dot{R}r(x)\right) dx \\
&= \frac{1}{2}\int_{x\in O} \rho(x)\left(R\hat{\omega}^b r(x)\right)^T \left(R\hat{\omega}^b r(x)\right) dx \\
&= \frac{1}{2}\int_{x\in O} \rho(x)\left(R\hat{r}(x)\omega^b\right)^T \left(R\hat{r}(x)\omega^b\right) dx \\
&= \frac{1}{2}\left(\omega^b\right)^T \left(\int_{x\in O} \rho(x)\hat{r}^T(x)\hat{r}(x)dx\right)\omega^b
\end{aligned}
$$

Rotational inertia matrix

- We can define **rotational inertia matrix** I of a rigid body that determines the torque needed for a desired angular acceleration about a rotational axis.

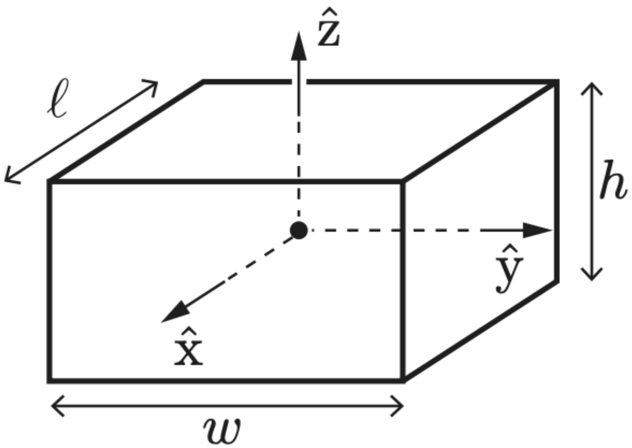
$$I = \int_{x \in O} \rho(x) \hat{r}^T(x) \hat{r}(x) dx$$

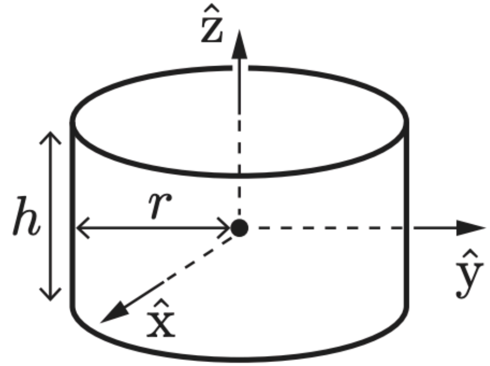
Rotational inertia matrix in body frame

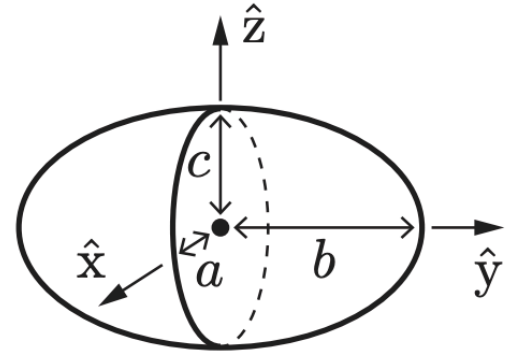
$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

- If $r(x)$ is the coordinates in body frame, then we get rotational inertia matrix I^b in body frame.

Example







rectangular parallelepiped:

$$\text{volume} = abc,$$
$$\mathcal{I}_{xx} = \mathfrak{m}(w^2 + h^2)/12,$$
$$\mathcal{I}_{yy} = \mathfrak{m}(\ell^2 + h^2)/12,$$
$$\mathcal{I}_{zz} = \mathfrak{m}(\ell^2 + w^2)/12$$

circular cylinder:

$$\text{volume} = \pi r^2 h,$$
$$\mathcal{I}_{xx} = \mathfrak{m}(3r^2 + h^2)/12,$$
$$\mathcal{I}_{yy} = \mathfrak{m}(3r^2 + h^2)/12,$$
$$\mathcal{I}_{zz} = \mathfrak{m}r^2/2$$

ellipsoid:

$$\text{volume} = 4\pi abc/3,$$
$$\mathcal{I}_{xx} = \mathfrak{m}(b^2 + c^2)/5,$$
$$\mathcal{I}_{yy} = \mathfrak{m}(a^2 + c^2)/5,$$
$$\mathcal{I}_{zz} = \mathfrak{m}(a^2 + b^2)/5$$

The principal axes and the inertia about the principal axes for uniform-density bodies

Rotation energy

- We can define rotational energy with rotational inertia matrix.

$$
\begin{aligned}
&\frac{1}{2}\int_{x\in O}\rho(x)\|\dot{R}\cdot r(x)\|^2 dx \\
&= \frac{1}{2}\left(\omega^b\right)^T\left(\int_{x\in O}\rho(x)\hat{r}^T(x)\hat{r}(x)dx\right)\omega^b \\
&= \frac{1}{2}\left(\omega^b\right)^T I^b\omega^b
\end{aligned}
$$

Kinetic energy - rigid body

$$
\begin{aligned}
K &= \frac{1}{2}\int_{x\in O} \rho(x)\|\dot{p} + \dot{R}r(x)\|^2 dx \\
&= \frac{1}{2}\int_{x\in O} \rho(x)\|\dot{p}\|^2 dx + \frac{1}{2}\int_{x\in O} \rho(x)\|\dot{R}\cdot r(x)\|^2 dx \\
&= \frac{1}{2} m\|\dot{p}\|^2 + \frac{1}{2}\left(\omega^b\right)^T I^b \omega^b
\end{aligned}
$$

Agenda

- Wrench (Generalized Force)
- Kinetic Energy
- **Generalized Inertia Matrix**
- Lagrangian Dynamics
- Lagrangian of Robot Manipulator

Generalized inertia matrix

- $M = \begin{bmatrix} mI_{3\times3} & 0 \\ 0 & I^b \end{bmatrix} \in R^{6\times6}$ is the generalized inertia matrix.
- $\hat{V}^b = g^{-1}\dot{g}$ is the velocity of the object in body frame.
- Kinetic energy is

$$K = \frac{1}{2}m\|\dot{p}\|^2 + \frac{1}{2}(\omega^b)^T I^b \omega^b$$
$$= \frac{1}{2}\left(V^b\right)^T M \left(V^b\right)$$

Inertia matrix in different frames

- The energy should be independent of frames.
- Consider two frames B and C. Let $g_{bc} = (p_{bc}, R_{bc})$ be the configuration of frame C relative to B.
- Let V_b, M_b be the velocity and inertia in frame B.

$$V_b = Ad_{g_{bc}} V_c$$

$$\frac{1}{2}(V_c)^T M_c V_c = \frac{1}{2}(V_b)^T M_b V_b$$

- $M_c = Ad_{g_{bc}}^T M_b Ad_{g_{bc}}$

Agenda

- Wrench (Generalized Force)
- Kinetic Energy
- Generalized Inertia Matrix
- **Lagrangian Dynamics**
- Lagrangian of Robot Manipulator

Newton’s Laws -> Lagrangian

- An equivalent systems to Newton’s laws.
- Allows us to calculate **rotation** through rotational energy more conveniently than Newton’s approach.
- Derived from the perspective of energy.

Potential energy

- **Potential energy** is the energy held by an object because of its position relative to other objects.
  - Gravitational potential energy: $V(h) = mgh$

Lagrangian

- **Lagrangian** L of a system is the difference between the kinetic and potential energy of the system:

$$L(q, \dot{q}) = K(q, \dot{q}) - V(q)$$

, where K is the kinetic energy and V is the potential energy of the system.

Lagrangian Equations

- The equations of motion for a mechanical system with generalized coordinates $q \in R^m$ and Lagrangian L are given by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \Upsilon_i, i = 1, \cdots, m$$

- , where Υ_i is the external force acting on the i-th generalized coordinate.

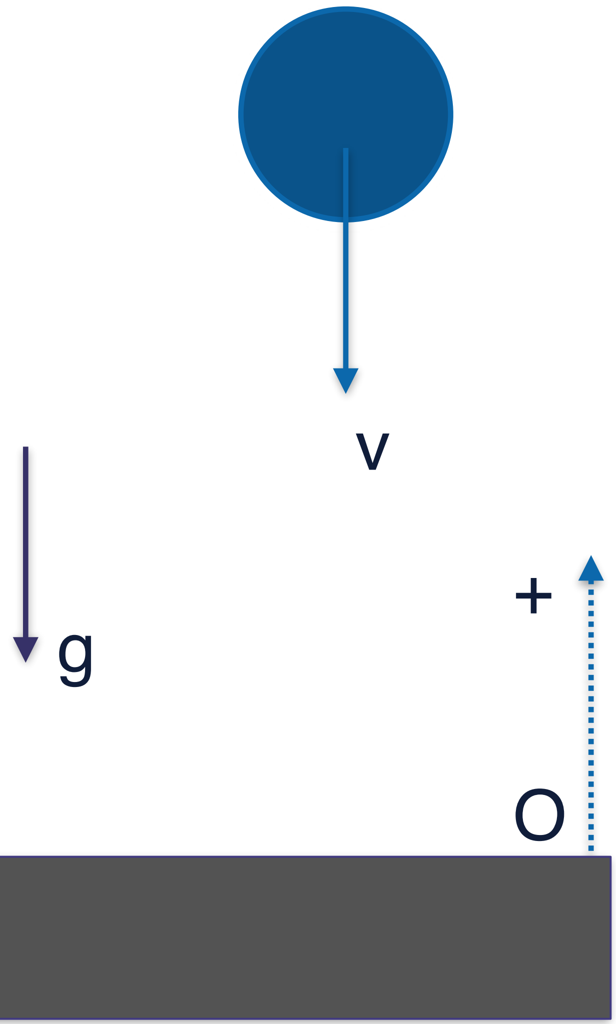
Lagrangian Equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \Upsilon_i, i = 1, \cdots, m$$

- Comments:
  - If q is the position and $V \equiv 0$, $L = \frac{1}{2}m\|v\|^2$, then this equation just a restatement of Newton's laws in generalized coordinates.

Example: Falling Ball

- $K = \frac{1}{2}mv^2, V = mgh, L = K - V$
- $F = \frac{d}{dt}\frac{\partial L}{\partial v} - \frac{\partial L}{\partial h}$
- $F = ma + mg$ or $a = \frac{F}{m} - g$



Lagrangian Equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \Upsilon_i, i = 1, \cdots, m$$

- We can use Lagrangian to infer force.
- We can use Lagrangian to connect force to acceleration.

Example: From Lagrangian to Newton-Euler

- Assume $V \equiv 0$ (no potential energy)
- We have $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \Upsilon_i, i = 1, \cdots, m$
- Consider $L = \frac{1}{2} m \|\dot{p}\|^2 + \frac{1}{2} \left(\omega^b\right)^T I^b \omega^b$

Newton-Euler equations - I

- We can relate the force and acceleration represented in the **spatial frame** from Lagrangian with energy.

- **Linear**
  - We have $\dot{p} = v^s$.
  - $$f^s = \frac{d}{dt}\frac{\partial L}{\partial v^s} = \frac{d}{dt}\frac{\partial\left(\frac{1}{2}m\|v^s\|^2\right)}{\partial v^s},$$
  - $$f^s = \frac{d}{dt}\left(mv^s\right).$$

Newton-Euler equations - I

- **Rotation**
  - We have $\omega^s = R\omega^b, I^s = RI^bR^T$
  - $\frac{1}{2}\left(\omega^b\right)^T I^b\omega^b = \frac{1}{2}\left(\omega^s\right)^T I^s\omega^s.$

$$\tau^s = \frac{d}{dt}\frac{\partial L}{\partial \omega^s} = \frac{d}{dt}\frac{\partial\left(\frac{1}{2}\left(\omega^s\right)^T I^s\omega^s\right)}{\partial \omega^s} = \frac{d\left(I^s\omega^s\right)}{dt}$$

$$\begin{aligned}\tau^s &= RI^bR^T\dot{\omega}^s + \dot{R}I^bR^T\omega^s + RI^b\dot{R}^T\omega^s \\ &= I^s\dot{\omega}^s + \dot{R}R^TI^s\omega^s + I^sR\dot{R}^T\omega^s \\ &= I^s\dot{\omega}^s + \omega^s \times I^s\omega^s - \boxed{I^s\omega^s \times \omega^s} \\ &= \boxed{I^s\dot{\omega}^s + \omega^s \times I^s\omega^s}\end{aligned}$$

0 due to $\omega^s \times \omega^s = 0$

Newton-Euler equations - I

- Combine two parts together, we can get **Newton-Euler equations** in spatial frame:

$$\begin{bmatrix} mI_{3\times3} & 0 \\ 0 & I^s \end{bmatrix} \begin{bmatrix} \dot{v}^s \\ \dot{\omega}^s \end{bmatrix} + \begin{bmatrix} 0 \\ \omega^s \times I^s \omega^s \end{bmatrix} = \begin{bmatrix} f^s \\ \tau^s \end{bmatrix}$$

Newton-Euler equations - II

- We can relate the force and acceleration represented in the **body frame** from Lagrangian with energy.
- **Linear**
    - We have $f^b = R^T f^s, v^s = Rv^b, \hat{\omega}^b = R^T \dot{R}$.
    - $f^s = \dfrac{d\left(mv^s\right)}{dt} = \dfrac{d}{dt}\left(mRv^b\right) = Rm\dot{v}^b + \dot{R}mv^b$
    - $f^b = m\dot{v}^b + \boxed{\omega^b \times mv^b}$

centripetal force

Newton-Euler equations - II

- **Rotation**
  - We have $\tau^b = R^T \tau^s, \omega^s = R\omega^b, I^s = RI^bR^T$
  - $\tau^s = I^s\dot{\omega}^s + \omega^s \times I^s\omega^s$
  - So $\tau^b = I^b\dot{\omega}^b + \omega^b \times I^b\omega^b$

Newton-Euler equations - II

- Combine two parts together, we can get **Newton-Euler equations** in body frame:

$$\begin{bmatrix} mI_{3\times3} & 0 \\ 0 & I^b \end{bmatrix} \begin{bmatrix} \dot{v}^b \\ \dot{\omega}^b \end{bmatrix} + \begin{bmatrix} \omega^b \times mv^b \\ \omega^b \times I^b \omega^b \end{bmatrix} = F^b$$

Agenda

- Wrench (Generalized Force)
- Kinetic Energy
- Generalized Inertia Matrix
- Lagrangian Dynamics
- **Lagrangian of Robot Manipulator**

Potential energy of robot

- Consider open-chain robot manipulator with n joints.
- Let m_i is the mass of the i-th link and g is the gravitational constant.
- The total potential energy is given by the sum of the contributions from each link:

$$V(\theta) = \sum_{i=1}^{n} V_i(\theta) = \sum_{i=1}^{n} m_i g h_i(\theta).$$

Kinetic energy of robot

- Let L_i be the frame attached to the center of mass of the i-th link, S be the base frame of the robot.
- The velocity of the i-th link in body frame is $V_{sl_i}^b = J_{sl_i}^b(\theta)\dot{\theta}$, where $J_{sl_i}^b(\theta)$ is the body Jacobian corresponding to g_{sl_i}.
- Let M_i be generalized inertia matrix of the i-th link.

Kinetic energy of robot

- $K_i(\theta, \dot{\theta}) = \frac{1}{2}\left(V^b_{sl_i}\right)^T M_i V^b_{sl_i} = \frac{1}{2}\dot{\theta}^T J_i(\theta)^T M_i J_i(\theta)\dot{\theta}$.
- $K(\theta, \dot{\theta}) = \sum_{i=1}^{n} K_i(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta} M(\theta)\dot{\theta}$,
- where $M(\theta) = \sum_{i=1}^{n} J_i(\theta)^T M_i J_i(\theta)$ is the **manipulator inertia matrix**.

Lagrangian equations of robot

- The Lagrangian equations of robots are given by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = \Upsilon_i, i = 1, \cdots, m,$$

- where Υ_i represents the actuator torque and other nonconservative, generalized forces acting on the i-th joint $\theta \in R^m$.

Calculation

- $K(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}M(\theta)\dot{\theta}$, $V(\theta) = \sum_{i=1}^{n} m_i g h_i(\theta)$
- $L = K - V$
- $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = \Upsilon_i, i = 1, \cdots, m,$

- After substitution and reorganization:

$$M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + N(\theta, \dot{\theta}) = \Upsilon,$$

- where Υ is joint actuation.

Explanation

$$M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + N(\theta, \dot{\theta}) = \Upsilon$$

- $M(\theta)\ddot{\theta}$ is inertial forces which depend on the acceleration of the joints.

- $C(\theta, \dot{\theta})$ is the **Coriolis matrix** for the manipulator; the vector $C(\theta, \dot{\theta})\dot{\theta}$ gives the **Coriolis and centrifugal force** in the equations of motion.

- $-N(\theta, \dot{\theta})$ to be any other forces which act on the i-th generalized coordinate, including conservative forces arising from a potential as well as frictional forces.

Summary

- Inertia matrix
- Kinetic energy
- Lagrangian theory