

# **L5: Twist and Geometric Jacobian**

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# Agenda

- Interpretation and Computation of Twist
- Example of Twist Computation
- Change of Coordinates for Twists
- Jacobian of Kinematics Chain
- Inverse Kinematics

# **Interpretation and Computation of Twist**

- Let  $\xi_{b(t)}^o = \begin{bmatrix} \nu_{b(t)}^o \\ \omega_{b(t)}^o \end{bmatrix} \in \mathbb{R}^6$ , then  $[\xi_{b(t)}^o] = \begin{bmatrix} [\omega_{b(t)}^o] & \nu_{b(t)}^o \\ 0 & 0 \end{bmatrix}$

# Body Twist

- When  $\mathcal{F}_o = \mathcal{F}_{b(t)}$ , it is particularly easy to compute  $\xi_{b(t)}^{b(t)}$
- Remarks
  - For body twist, when recording at time  $t$ , you should think it as ***first cloning the body frame and then record the movement with this cloned frame keeping static***
  - Body twist is “ego-centric” and sometimes simpler to specify for robotics. For example, if we take the gripper frame as the body frame. Using the body twist, we can express “move gripper forward” by a pure translation

# Body Twist Computation

- By  $\mathbf{v}_p^o(t) = [\xi_{b(t)}^o] p^o$  and  $[\xi_{b(t)}^o] = \begin{bmatrix} [\omega_{b(t)}^o] & \nu_{b(t)}^o \\ 0 & 0 \end{bmatrix}$ , let  $p_{origin}^{b(t)} = [0,0,0,1]^T$ , which is the origin of body frame

$$\mathbf{v}_{origin}^{b(t)}(t) = [\xi_{b(t)}^{b(t)}] p_{origin}^{b(t)} = \nu^{b(t)}$$

- Note that  $\mathbf{v}_{origin}^{b(t)}(t)$  is the linear velocity of the origin of the body frame
- Therefore,  $\xi_{b(t)}^{b(t)}$  is composed by the linear velocity of the origin and the angular velocity
- In practice, we often write down the body twist first and then obtain the twist in other frames by change of coordinate (next lecture)

# Compute $\xi_{b(t)}^o$ from Angle-Axis

- Let  $\xi_{b(t)}^o = \begin{bmatrix} \nu_{b(t)}^o \\ \omega_{b(t)}^o \end{bmatrix} \in \mathbb{R}^6$ , then  $[\xi_{b(t)}^o] = \begin{bmatrix} [\omega_{b(t)}^o] & \nu_{b(t)}^o \\ 0 & 0 \end{bmatrix}$
- The instant linear velocity can be decomposed into the rotation about the axis and translation along the axis
- Take a point  $q^o \in \mathbb{R}^3$  on the axis,
  - By  $\mathbf{v}_p^o(t) = [\xi_{b(t)}^o]p^o$ ,  $\mathbf{v}_q^o = [\omega^o]q^o + \nu^o$
  - Since the only velocity of  $q^o$  is along  $\hat{\omega}$ ,  $\mathbf{v}_q^o = \mathbf{v}_\omega^o$
  - $\therefore \nu^o = -[\omega^o]q^o + \mathbf{v}_\omega^o$
- $\xi^o = \begin{bmatrix} -[\omega^o]q^o + \mathbf{v}_\omega^o \\ \omega^o \end{bmatrix}$  (all the symbols have  $b(t)$  as the subscript)

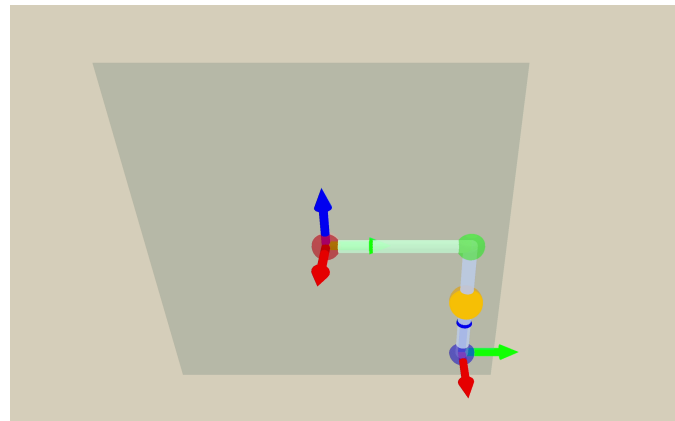
# **Example of Twist Computation**



# Example of Twist Computation

- Consider the previous example, but now a orange point is fixed to the end-effector frame (blue sphere)
- What is the **velocity of orange point at  $t = 0$** ? Given the pose of end effector frame as below:

$$T_{s \rightarrow b(t)}^s = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha t) & -\sin(\alpha t) & 1 + \sin(\alpha t) \\ 0 & \sin(\alpha t) & \cos(\alpha t) & -\cos(\alpha t) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



# Example of Twist Computation

- The velocity of yellow point caused by the end-effector motion can be computed via twist
- Recall:  $\mathbf{v}_p^s = [\xi_{b(t)}^s] p^s$

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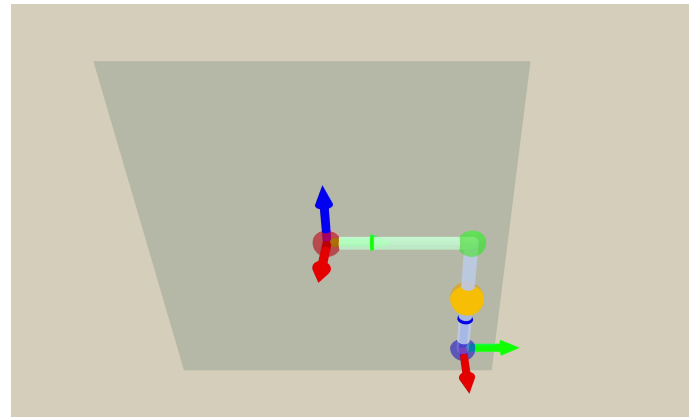
- we have  $[\xi_{s \rightarrow b(t)}^s] = \dot{T}_{s \rightarrow b(t)}^s (T_{s \rightarrow b(t)}^s)^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & \alpha & 0 & -\alpha \\ 0 & 0 & 0 & 0 \end{bmatrix}$

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$$[\xi_{b(t)}^s] = \dot{T}_{s \rightarrow b(t)}^s (T_{s \rightarrow b(t)}^s)^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & \alpha & 0 & -\alpha \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{At } t = 0, p^s = \begin{bmatrix} 0 \\ 1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

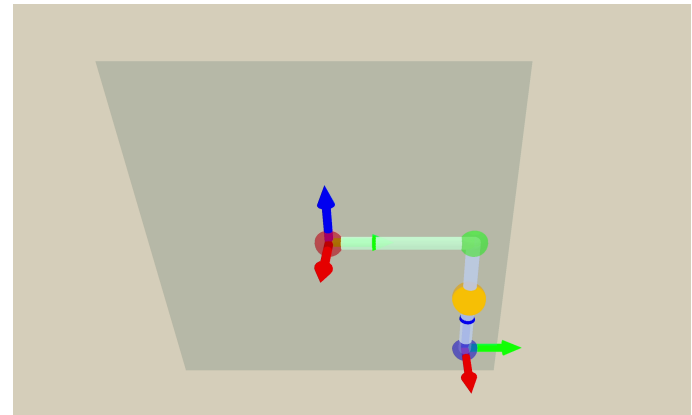


# Example of Twist Computation

- The velocity of yellow point caused by the end-effector motion can be computed via twist
- Recall:  $\mathbf{v}_p^s = [\xi_{b(t)}^s] p^s$

$$[\xi_{b(t)}^s] = \dot{T}_{s \rightarrow b(t)}^s (T_{s \rightarrow b(t)}^s)^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & \alpha & 0 & -\alpha \\ 0 & 0 & 0 & 0 \end{bmatrix}, p^s = \begin{bmatrix} 0 \\ 1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

$$\mathbf{v}_p^s = [\xi_{b(t)}^s] p^s = \begin{bmatrix} 0 \\ \frac{\alpha}{2} \\ 0 \\ 0 \end{bmatrix}$$

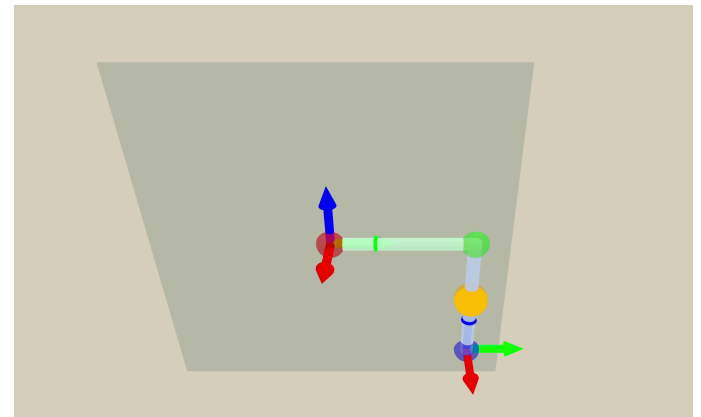


# Example of Twist Computation

- We can verify this result by taking the derivative of  $\frac{d}{dt}p^s(t)$

$$p^s(t) \begin{bmatrix} 0 \\ 1 + \frac{\alpha}{2} \sin(t) \\ -\frac{\alpha}{2} \cos(t) \\ 1 \end{bmatrix}, \frac{d}{dt}p^s(t) = \begin{bmatrix} 0 \\ \frac{\alpha}{2} \cos(t) \\ \frac{\alpha}{2} \sin(t) \\ 0 \end{bmatrix}$$

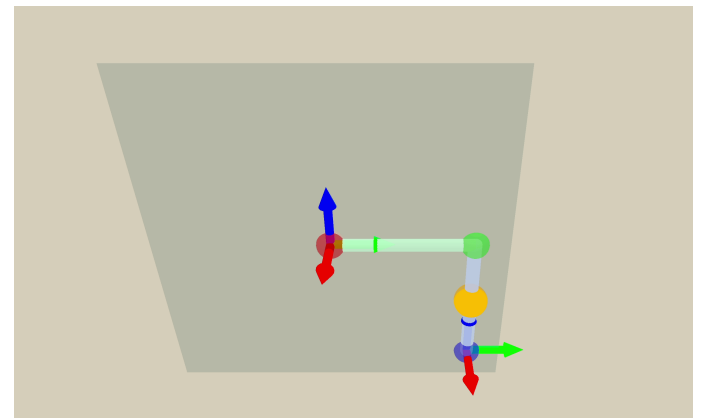
$$\mathbf{v}_p^s = [\xi_{b(t)}^s]p^s = \begin{bmatrix} 0 \\ \frac{\alpha}{2} \\ 0 \\ 0 \end{bmatrix} = \frac{d}{dt}p^s(t) \big|_{t=0}$$



# Example of Twist Computation

- What is the body twist of the end effector?
- In the body frame of the end effector (blue sphere), the blue sphere has a constant linear velocity, which is always  $[0, \alpha, 0]$ . The angular velocity is always  $[\alpha, 0, 0]$ .

So,  $\xi_{b(t)}^{b(t)} = [0, \alpha, 0, \alpha, 0, 0]^T$



# **Change of Coordinates for Twists**



# *Review*

- Recall that, the recordings by different observers are related by the similarity transformation:

$$T_{1 \rightarrow 2}^{s_1} = T_{s_1 \rightarrow s_2} T_{1 \rightarrow 2}^{s_2} (T_{s_1 \rightarrow s_2})^{-1}$$

# Tricks in Recording Velocities

- If transformations could be recorded differently by observers, velocity should also be recorded differently

# Relating 6D Velocities from Different Observers

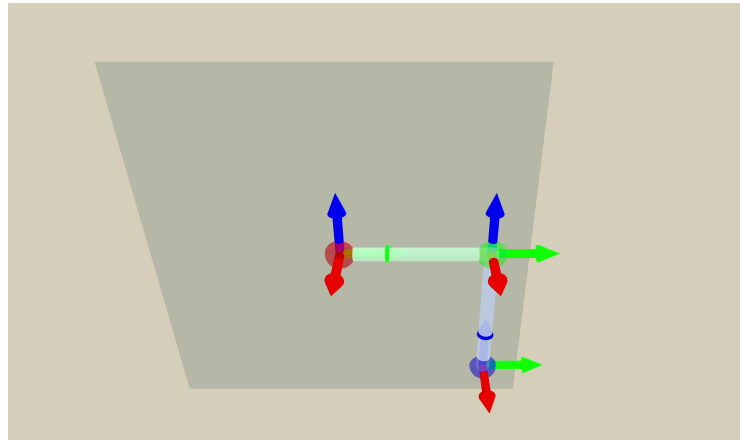
- Two observers record the same motion as  $\xi_{b(t)}^{s_1}$  and  $\xi_{b(t)}^{s_2}$
- **What is the relationship between  $\xi_{b(t)}^{s_1}$  and  $\xi_{b(t)}^{s_2}$  ?**

# Example 1 of Change of Frame

- From our previous example, we know that the motion of the end-effector (blue sphere) can be recorded in the spatial frame (red sphere) as  $\xi_{s \rightarrow b(t)}^s$ :

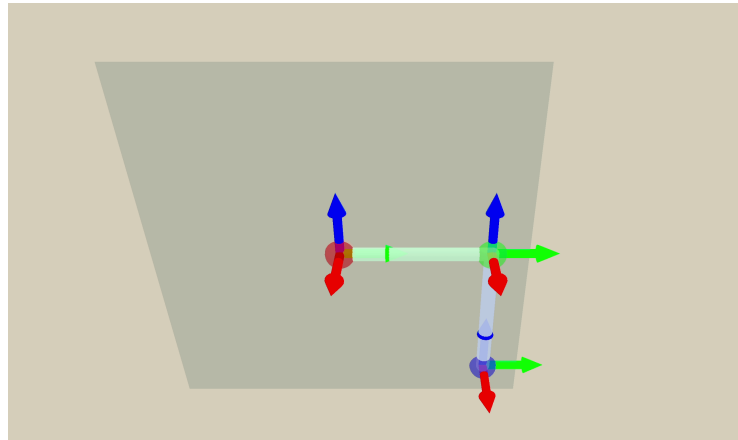
$$[\xi_{s \rightarrow b(t)}^s] = \dot{T}_{s \rightarrow b(t)}^s (T_{s \rightarrow b(t)}^s)^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & \alpha & 0 & -\alpha \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xi_{s \rightarrow b(t)}^s = [0, 0, -\alpha, \alpha, 0, 0]^T$$



# Example 1 of Change of Frame

- From our previous example, we know that the motion of the end-effector (blue sphere) can be recorded in the spatial frame (red sphere) as  $\xi_{s \rightarrow b(t)}^s$ :
- Now we introduce a new frame  $\mathcal{F}_o$ , the frame of the green sphere. How can we record the same motion by  $\mathcal{F}_o$  as  $\xi_{s \rightarrow b(t)}^o$ ?



# Example 1 of Change of Frame

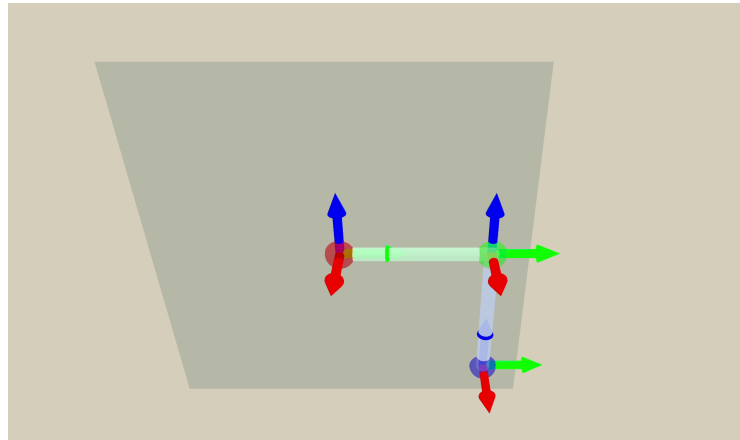
- From our previous example, we know that the motion of the end-effector (blue sphere) can be recorded in the spatial frame (red sphere) as  $\xi_{s \rightarrow b(t)}^s$ :
- By simple inspection, we can find end-effector is rotating about the x-axis of  $\mathcal{F}_o$  and the instant velocity along the axis is zero

$$\omega^o = [\alpha, 0, 0]^T$$

$$\hat{\omega}^o = [1, 0, 0]^T$$

$$q^o = [0, 0, 0]^T$$

$$\mathbf{v}_\omega^o = [0, 0, 0]^T$$



# Example 1 of Change of Frame

- From our previous example, we know that the motion of the end-effector (blue sphere) can be recorded in the spatial frame (red sphere) as  $\xi_{s \rightarrow b(t)}^s$ :
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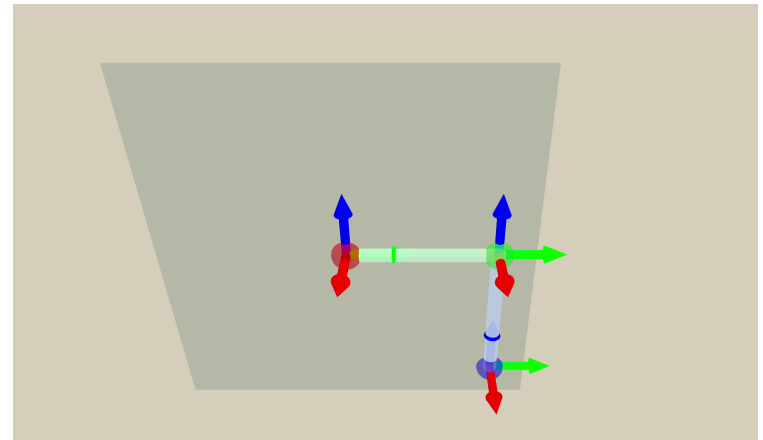
$$\omega^o = [\alpha, 0, 0]^T$$

$$\hat{\omega}^o = [1, 0, 0]^T$$

$$q^o = [0, 0, 0]^T$$

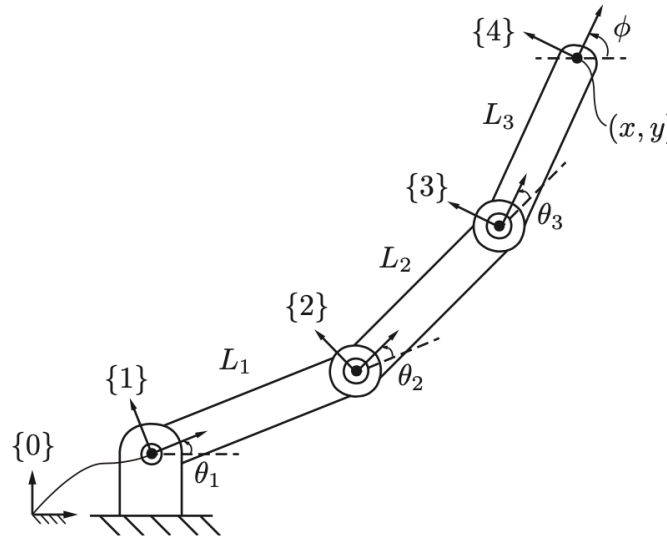
$$\mathbf{v}_\omega^o = [0, 0, 0]^T$$

- Recall:  $\xi^o = \begin{bmatrix} -[\omega^o]q^o + \mathbf{v}_\omega^o \\ \omega^o \end{bmatrix}$
- Thus we have  $\xi_{s \rightarrow b(t)}^o = [0, 0, 0, \alpha, 0, 0]^T$



# Example 2 of Change of Frame

- For the 3-link robot arm



- Given  $\xi_{L_3(t)}^3$ , what is  $\xi_{L_3(t)}^0$ ? Assume the transformation is  $T_{L_0 \rightarrow L_3(t)}^0$  at time  $t$ .



# Change of Frame by Similarity Transformation

- For two observers, one records by  $\mathcal{F}_{s_1}$  and the other by  $\mathcal{F}_{s_2}$ , then

- $\dot{T}_{s' \rightarrow b(t)}^{s_1} = [\xi_{b(t)}^{s_1}] T_{s' \rightarrow b(t)}^{s_1}$

- $\dot{T}_{s' \rightarrow b(t)}^{s_2} = [\xi_{b(t)}^{s_2}] T_{s' \rightarrow b(t)}^{s_2}$

# Change of Frame by Similarity Transformation

$$[\xi_{b(t)}^{s_1}] = T_{s_1 \rightarrow s_2} [\xi_{b(t)}^{s_2}] T_{s_1 \rightarrow s_2}^{-1}$$

- When the observer's frame changes,
  - twist also conforms to the similarity transformation

# Change of Frame by Similarity Transformation

- By  $T_{s' \rightarrow b(t)}^{s_1} = T_{s_1 \rightarrow s_2} T_{s' \rightarrow b(t)}^{s_2} (T_{s_1 \rightarrow s_2})^{-1}$ ,

$$\begin{aligned}
 \dot{T}_{s' \rightarrow b(t)}^{s_1} &= T_{s_1 \rightarrow s_2} \dot{T}_{s' \rightarrow b(t)}^{s_2} (T_{s_1 \rightarrow s_2})^{-1} \Leftrightarrow [\xi_{b(t)}^{s_1}] T_{s' \rightarrow b(t)}^{s_1} = T_{s_1 \rightarrow s_2} [\xi_{b(t)}^{s_2}] T_{s' \rightarrow b(t)}^{s_2} (T_{s_1 \rightarrow s_2})^{-1} \\
 &\Leftrightarrow [\xi_{b(t)}^{s_1}] = T_{s_1 \rightarrow s_2} [\xi_{b(t)}^{s_2}] T_{s' \rightarrow b(t)}^{s_2} (T_{s_1 \rightarrow s_2})^{-1} (T_{s' \rightarrow b(t)}^{s_1})^{-1} \\
 &\Leftrightarrow [\xi_{b(t)}^{s_1}] = T_{s_1 \rightarrow s_2} [\xi_{b(t)}^{s_2}] T_{s' \rightarrow b(t)}^{s_2} \{ (T_{s_1 \rightarrow s_2})^{-1} (T_{s' \rightarrow b(t)}^{s_1})^{-1} T_{s_1 \rightarrow s_2} \} (T_{s_1 \rightarrow s_2})^{-1} \\
 &\Leftrightarrow [\xi_{b(t)}^{s_1}] T_{s_1 \rightarrow s_2} = T_{s_1 \rightarrow s_2} [\xi_{b(t)}^{s_2}] T_{s' \rightarrow b(t)}^b (T_{s' \rightarrow b(t)}^b)^{-1} (T_{s_1 \rightarrow s_2})^{-1} \\
 &\Leftrightarrow [\xi_{b(t)}^{s_1}] T_{s_1 \rightarrow s_2} = T_{s_1 \rightarrow s_2} [\xi_{b(t)}^{s_2}]
 \end{aligned}$$

$$[\xi_{b(t)}^{s_1}] = T_{s_1 \rightarrow s_2} [\xi_{b(t)}^{s_2}] T_{s_1 \rightarrow s_2}^{-1}$$

# Adjoint Matrix

$$[\xi_{b(t)}^{s_1}] = T_{s_1 \rightarrow s_2} [\xi_{b(t)}^{s_2}] T_{s_1 \rightarrow s_2}^{-1}$$

- $\xi_{b(t)}^{s_1}$  is linear w.r.t.  $\xi_{b(t)}^{s_2}$
- We introduce a matrix  $[\text{Ad}_{T_{s_1 \rightarrow s_2}}] \in \mathbb{R}^{6 \times 6}$  to relate them:

$$\xi_{b(t)}^{s_1} = [\text{Ad}_{T_{s_1 \rightarrow s_2}}] \xi_{b(t)}^{s_2}$$

- Do computation based on the similarity transformation, and you can get

$$[\text{Ad}_{T_{s_1 \rightarrow s_2}}] = \begin{bmatrix} R_{s_1 \rightarrow s_2} & [\mathbf{t}_{s_1 \rightarrow s_2}] R_{s_1 \rightarrow s_2} \\ 0 & R_{s_1 \rightarrow s_2} \end{bmatrix}$$

# Spatial Twist and Body Twist

- If we observe the motion of the body
  - \_ from  $\mathcal{F}_s$ , the velocity is  $\xi_{b(t)}^s$  (**spatial twist**)
  - \_ from the moving object  $\mathcal{F}_b$ , the velocity is  $\xi_{b(t)}^{b(t)}$  (**body twist**)

# Spatial Twist and Body Twist

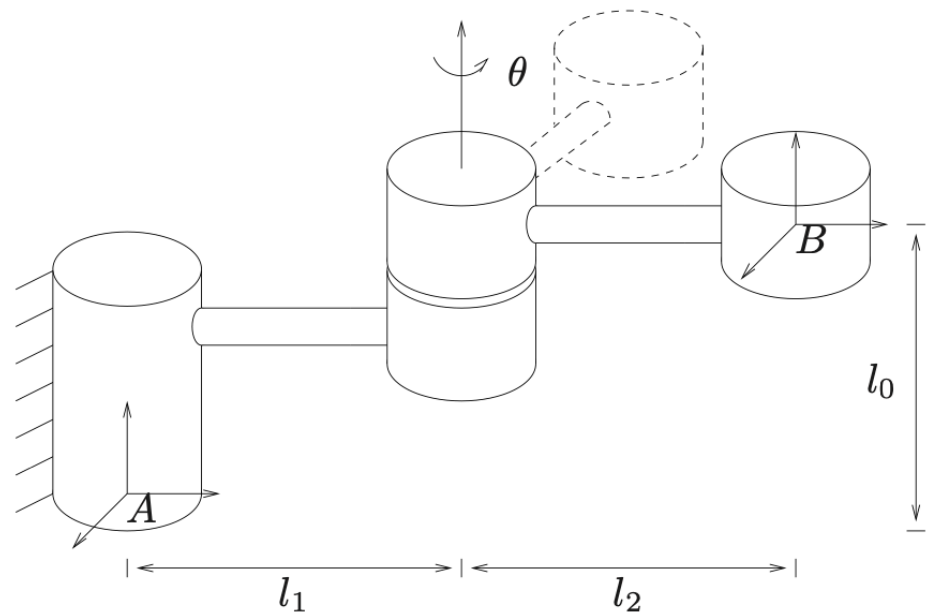
- By  $\dot{T}_{s' \rightarrow b(t)}^s = [\xi_{b(t)}^s] T_{s' \rightarrow b(t)}^s$ ,  $[\xi_{b(t)}^s] = \dot{T}_{s' \rightarrow b(t)}^s (T_{s' \rightarrow b(t)}^s)^{-1}$
- Using the similarity transformation to change the frame, we have
  - $T_{s' \rightarrow b(t)}^s [\xi_{b(t)}^{b(t)}] (T_{s' \rightarrow b(t)}^s)^{-1} = \dot{T}_{s' \rightarrow b(t)}^s (T_{s' \rightarrow b(t)}^s)^{-1}$
  - $\therefore [\xi_{b(t)}^{b(t)}] = (T_{s' \rightarrow b(t)}^s)^{-1} \dot{T}_{s' \rightarrow b(t)}^s$

# Example 3 of Change of Frame

- Given the motion of rigid-body

$$T_{A \rightarrow B(t)} = \begin{bmatrix} \cos \theta(t) & -\sin \theta(t) & 0 & -l_2 \sin \theta(t) \\ \sin \theta(t) & \cos \theta(t) & 0 & l_1 + l_2 \cos \theta(t) \\ 0 & 0 & 1 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- What is the spatial twist?
- What is the body twist?



# Example 3 of Change of Frame

- Given the motion of rigid-body

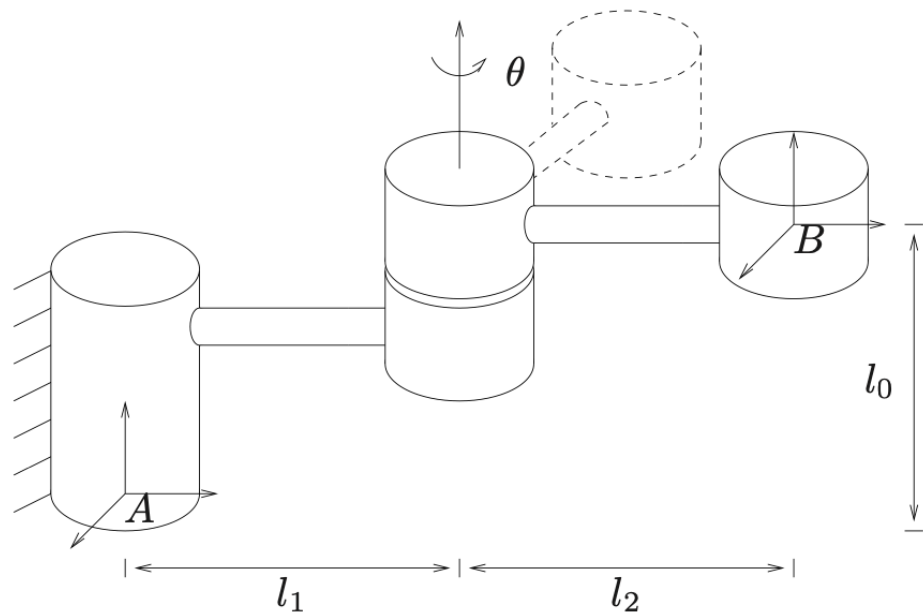
$$T_{A \rightarrow B(t)} = \begin{bmatrix} \cos \theta(t) & -\sin \theta(t) & 0 & -l_2 \sin \theta(t) \\ \sin \theta(t) & \cos \theta(t) & 0 & l_1 + l_2 \cos \theta(t) \\ 0 & 0 & 1 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\cdot [\xi_{B(t)}^A] = \dot{T}_{A \rightarrow B(t)} T_{A \rightarrow B(t)}^{-1}$$

$$\xi_{B(t)}^A = [l_1, 0, 0, 0, 0, 1]^T$$

$$\cdot [\xi_{B(t)}^B] = T_{A \rightarrow B(t)}^{-1} \dot{T}_{A \rightarrow B(t)}$$

$$\xi_{B(t)}^{B(t)} = [-l_2, 0, 0, 0, 0, 1]^T$$





```

import sympy as sp
from sympy import *

t = symbols("t")
l0 = symbols("l0")
l1 = symbols("l1")
l2 = symbols("l2")

T = Matrix(symarray('T', (4, 4)))
T[0, 0] = cos(t)
T[0, 1] = -sin(t)
T[0, 2] = 0
T[0, 3] = -l2 * sin(t)
T[1, 0] = sin(t)
T[1, 1] = cos(t)
T[1, 2] = 0
T[1, 3] = l1 + l2 * cos(t)
T[2, 0] = 0
T[2, 1] = 0
T[2, 2] = 1
T[2, 3] = l0
T[3, 0] = 0
T[3, 1] = 0
T[3, 2] = 0
T[3, 3] = 1

xi_s = sp.diff(T, t) @ sp.Inverse(T)
xi_s.simplify()

xi_b = sp.Inverse(T) @ sp.diff(T, t)
xi_b.simplify()

```

# Example 3 of Change of Frame

• By  $T_{A \rightarrow B(t)} = \begin{bmatrix} \cos \theta(t) & -\sin \theta(t) & 0 & -l_2 \sin \theta(t) \\ \sin \theta(t) & \cos \theta(t) & 0 & l_1 + l_2 \cos \theta(t) \\ 0 & 0 & 1 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ , we have

$$R_{A \rightarrow B(t)} = \begin{bmatrix} \cos \theta(t) & -\sin \theta(t) & 0 \\ \sin \theta(t) & \cos \theta(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{t}_{A \rightarrow B(t)} = \begin{bmatrix} -l_2 \sin \theta(t) \\ l_1 + l_2 \cos \theta(t) \\ l_0 \end{bmatrix}.$$

• By  $[\text{Ad}_{T_{A \rightarrow B(t)}}] = \begin{bmatrix} R_{A \rightarrow B(t)} & [\mathbf{t}_{A \rightarrow B(t)}]R_{A \rightarrow B(t)} \\ 0 & R_{A \rightarrow B(t)} \end{bmatrix}$ ,

$$[\text{Ad}_{T_{A \rightarrow B(t)}}] = \begin{bmatrix} \cos \theta(t) & -\sin \theta(t) & 0 & -l_0 \sin \theta(t) & -l_0 \cos \theta(t) & l_1 + l_2 \cos \theta(t) \\ \sin \theta(t) & \cos \theta(t) & 0 & l_0 \cos \theta(t) & -l_0 \sin \theta(t) & l_2 \sin \theta(t) \\ 0 & 0 & 1 & -l_1 \cos \theta(t) - l_2 & l_1 \sin \theta(t) & 0 \\ 0 & 0 & 0 & \cos \theta(t) & -\sin \theta(t) & 0 \\ 0 & 0 & 0 & \sin \theta(t) & \cos \theta(t) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

# Example 3 of Change of Frame

By  $\xi_{A \rightarrow B(t)}^A = [l_1, 0, 0, 0, 0, 1]^T$

$\xi_{A \rightarrow B(t)}^B = [-l_2, 0, 0, 0, 0, 1]^T$

$$[\text{Ad}_{T_{s \rightarrow b}}] = \begin{bmatrix} \cos \theta(t) & -\sin \theta(t) & 0 & -l_0 \sin \theta(t) & -l_0 \cos \theta(t) & l_1 + l_2 \cos \theta(t) \\ \sin \theta(t) & \cos \theta(t) & 0 & l_0 \cos \theta(t) & -l_0 \sin \theta(t) & l_2 \sin \theta(t) \\ 0 & 0 & 1 & -l_1 \cos \theta(t) - l_2 & l_1 \sin \theta(t) & 0 \\ 0 & 0 & 0 & \cos \theta(t) & -\sin \theta(t) & 0 \\ 0 & 0 & 0 & \sin \theta(t) & \cos \theta(t) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

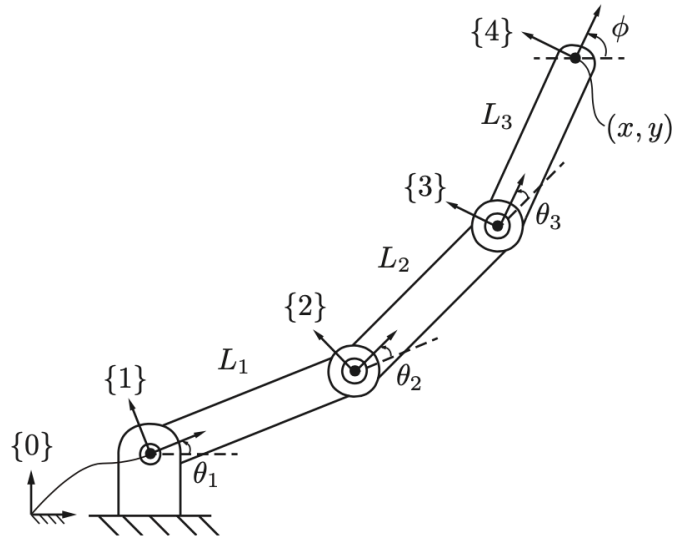
We can verify that  $\xi_{A \rightarrow B(t)}^A = [\text{Ad}_{T_{s \rightarrow b}}] \xi_{A \rightarrow B(t)}^B$

# Summary

- Twist  $\xi = \dot{\chi}$  denotes the 6D motion velocity
- Relationship with  $\dot{T}$ :  $\dot{T}_{s' \rightarrow b(t)}^o = [\dot{\chi}_{b(t)}^o] T_{s' \rightarrow b(t)}^o$
- Change of frame:
  - $[\xi_{b(t)}^{s_1}] = T_{s_1 \rightarrow s_2} [\xi_{b(t)}^{s_2}] T_{s_1 \rightarrow s_2}^{-1}$
  - $\xi_{b(t)}^{s_1} = [Ad_{T_{s_1 \rightarrow s_2}}] \xi_{b(t)}^{s_2}$
- Spatial twist:  $[\xi_{b(t)}^s] = \dot{T}_{s' \rightarrow b(t)}^s (T_{s' \rightarrow b(t)}^s)^{-1}$
- Body twist:  $[\xi_{b(t)}^{b(t)}] = (T_{s' \rightarrow b(t)})^{-1} \dot{T}_{s' \rightarrow b(t)}^s$

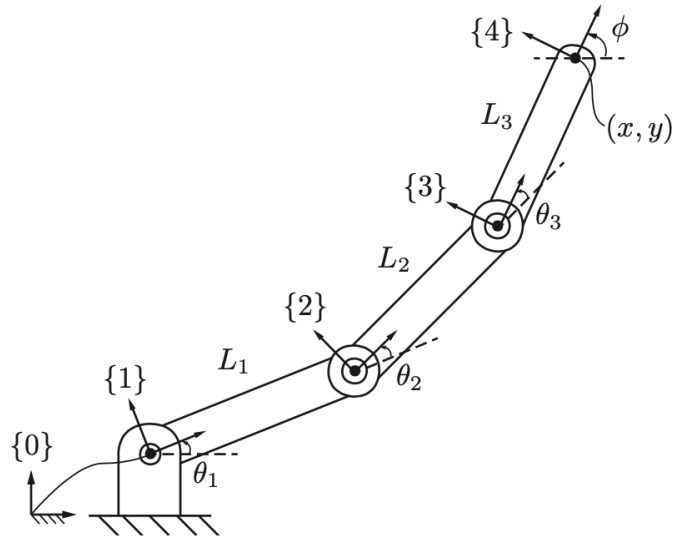
# **Jacobian of Kinematics Chain**

# Forward Kinematic Problem



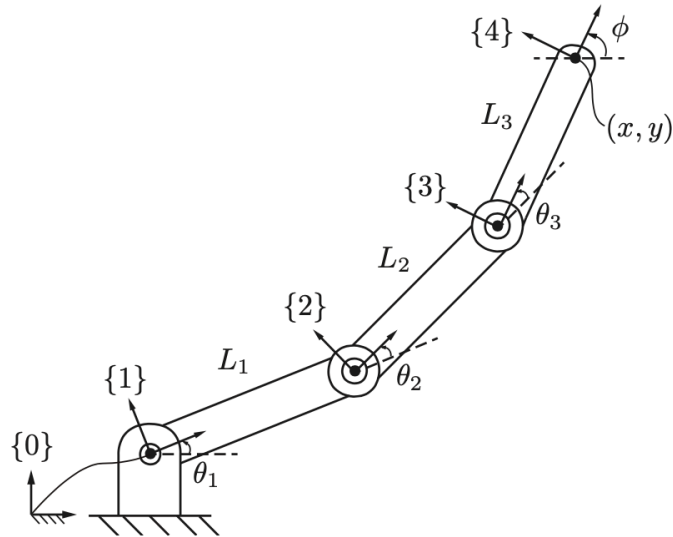
- Suppose that the arm moves
- How do I compute the velocity of the end-effector from the angular velocity of joints?

# Spatial Frame Inverse Kinematics Problem



- If I specify the direction of the end-of-effector movement using the spatial frame, how can I change the joint angles?
- e.g. move to a pre-specified  $T_{s \rightarrow e}^s$

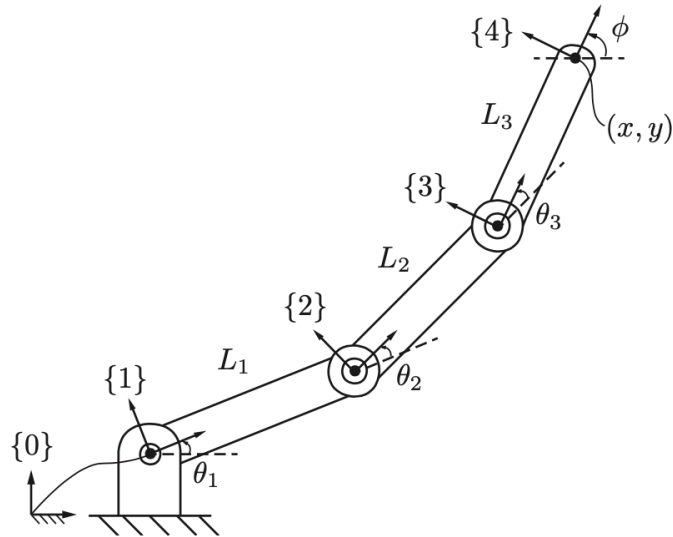
# Body Frame Inverse Kinematics Problem



- If I specify the direction of the end-of-effector movement using the body frame, how can I change the joint angles?
- e.g. move the end-effector forward along its link



# Kinematic Equation



- We can solve the problems if we have  $\xi_{e(t)} = f(\dot{\theta})$
- The language to describe the velocity of end-effector are
  - $\xi_{e(t)}^s$  for spatial frame query
  - $\xi_{e(t)}^{e(t)}$  for body frame query
- We will derive the  $f^s$  and  $f^{e(t)}$

# Spatial Geometric Jacobian

- Spatial Geometric Jacobian  $J^s(\theta)$ :

$${}^{\xi^s}_{\xi_{e(t)}} = J^s(\theta)\dot{\theta}$$

where  $\theta \in \mathbb{R}^n$  (n joints),  $J^s(\theta) \in \mathbb{R}^{6 \times n}$ , and the  $i$ -th column of  $J(\theta)$  is  ${}^i\hat{\xi}^s_{\xi_{e(t)}}$ , the twist when the movement is caused only by the  $i$ -th joint **while all other joints stay static**

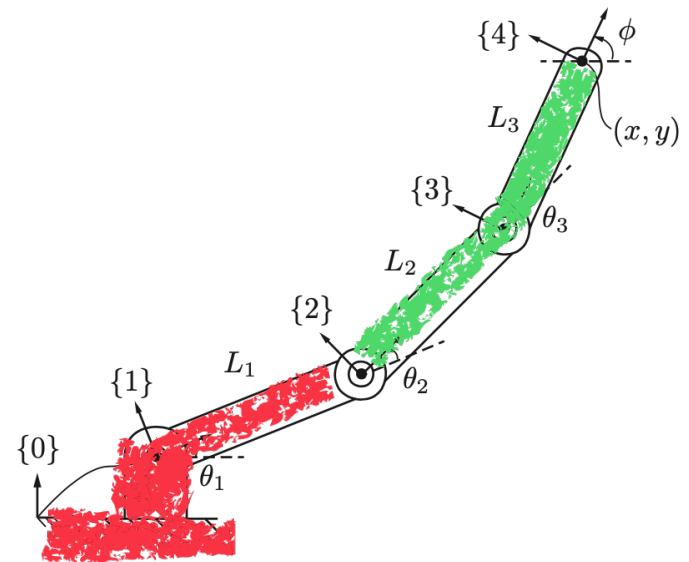
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- For example,  ${}^2\hat{\xi}_{e(t)}^s$  describes the motion of the green part, which is to revolute about Joint {2} (in this revolute joint,  $\hat{\omega}^s$ ,  $q^s$ , and  $d^s$  are obvious).



# Spatial Geometric Jacobian (Proof)

- First of all,  $\dot{T}_{s' \rightarrow e(t)}^s = [\xi_{e(t)}^s] T_{s' \rightarrow e(t)}^s$
- Suppose that only  $\theta_i$  can change at some  $(\theta_1, \dots, \theta_n)$ . Let  ${}^i M_{s' \rightarrow e(t)}^s(\theta_i) := T_{s' \rightarrow e(t)}^s(\theta_1, \dots, \theta_n)$ , then

$${}^i \dot{M}_{s' \rightarrow e(t)}^s = [{}^i \xi_{e(t)}^s] T_{s' \rightarrow e(t)}^s$$

- By total derivative,

$$\dot{T}_{s' \rightarrow e(t)}^s = \sum_i \frac{\partial T_{s' \rightarrow e(t)}^s}{\partial \theta_i} \dot{\theta}_i = \sum_i {}^i \dot{M}_{s' \rightarrow e(t)}^s = \sum_i [{}^i \xi_{e(t)}^s] T_{s' \rightarrow e(t)}^s$$

- Therefore,  $[\xi_{e(t)}^s] = \sum_i [{}^i \xi_{e(t)}^s] = \sum_i [{}^i \hat{\xi}_{e(t)}^s] \dot{\theta}_i$

# Body Geometric Jacobian

- The previous proof works for any recording frame. Simple substitution of  $e(t)$  for  $s$  as the recording frame gives:
- Body Geometric Jacobian  $J^{e(t)}(\theta)$ :

$$\xi_{e(t)}^{e(t)} = J^{e(t)}(\theta)\dot{\theta}$$

where  $J^{e(t)}(\theta) \in \mathbb{R}^{6 \times n}$ , and the  $i$ -th column of  $J(\theta)$  is  ${}^i\hat{\xi}_{e(t)}^{e(t)}$ , the twist when the movement is caused only by the  $i$ -th joint **while all other joints stay static**

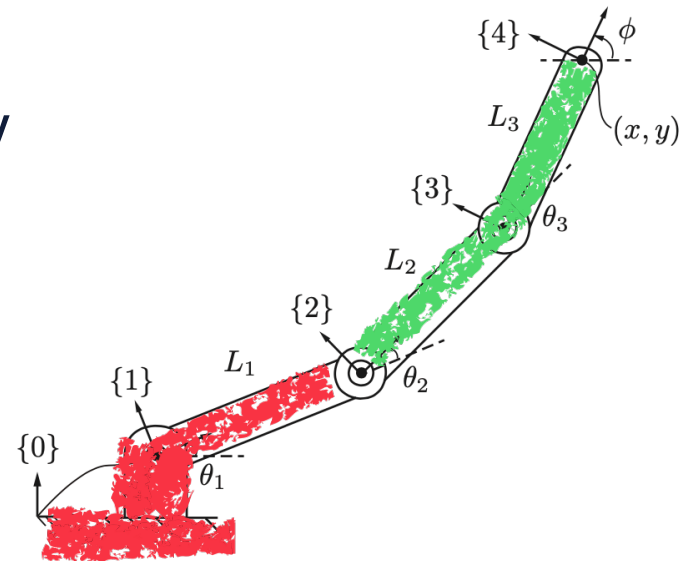
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where  $J^{e(t)}(\theta) \in \mathbb{R}^{6 \times n}$ , and the  $i$ -th column of  $J(\theta)$  is  $\hat{\xi}_{e(t)}^{e(t)}$ , the twist when the movement is caused only by the  $i$ -th joint **while all other joints stay static**

- For example,  $\hat{\xi}_{e(t)}^{e(t)}$  describes the motion of the green part observed by  $\mathcal{F}_s = \mathcal{F}_{\{0\}}$ , which is to revolute about Joint  $\{2\}$
- For this revolute joint,  $\hat{\omega}^{e(t)}$ ,  $q^{e(t)}$ , and  $d^{e(t)}$  can be computed using  $T_{\{2\} \rightarrow \{4\}}$ .



# Computation of Geometric Jacobian

- Just need to know  ${}^i\hat{\xi}_o^{e(t)}$  for the recording frame  $\mathcal{F}_o$
- When computing  ${}^i\hat{\xi}_o^{e(t)}$ , only the  $i$ -th joint can move
- Therefore, we can view as it as a single-joint problem, as our Example 1

# Computation of Geometric Jacobian

- Method 1:
  - Figure out  ${}^i\hat{\xi}_{e(t)}^o$  for each joint by first computing  $\hat{\omega}^o$ ,  $q^o$ , and  $d^o$  (as in the robot arm example with red/green colors)
- Method 2:
  - ${}^i\hat{\xi}_{e(t)}^o = [\text{Ad}_{T_{o \rightarrow L_i}}] \hat{\xi}_{e(t)}^{L_i}$
  - Assume that the joint axis is aligned with the x-axis of  $\hat{L}_i$
  - $\hat{\xi}_{e(t)}^{L_i} = [0,0,0,1,0,0]^T$  for prismatic joints
  - $\hat{\xi}_{e(t)}^{L_i} = [1,0,0,0,0,0]^T$  for revolute joints



# Inverse Kinematics

# Inverse Kinematics

- Position query
  - Given the forward kinematics  $T_{s \rightarrow e}^s(\theta)$  and the target pose  $T_{target} = \mathbb{SE}(3)$ , find  $\theta$  that satisfies  $T_{s \rightarrow e}(\theta) = T_{target}$
- Velocity query
  - Given the twist of the end-effector, find the angular velocity that satisfies  $\xi_{target} = J(\theta)\dot{\theta}$
- May have multiple solutions, a unique solution or no solution

# Null Space of Jacobian

- Consider the velocity query IK task
- Recall that  $\xi = J(\theta)\dot{\theta}$  for an  $n$ -joint kinematic chain, where  $J$  is a  $6 \times n$  matrix
- When  $n > 6$ , the joint space is projected to a lower-dimensional space and  $J$  must exist a null space
- As a result, IK may have infinite solutions (a special solution + any vector in the null space of  $J$ )
- The null space adds flexibility to make motion plans

# Analytical Solution

- Try to solve the equation  $T_{target} = T(\theta)$  and get an analytical solution for  $\theta$

- e.g., solve  $\theta_1$  and  $\theta_2$  for

$$\begin{bmatrix} \cos \theta_1 & -\sin \theta_1 & 0 & -\sin \theta_1(l_2 + l_3) \\ \sin \theta_1 & \cos \theta_1 & 0 & \cos \theta_1(l_2 + l_3) \\ 0 & 0 & 1 & l_1 - l_4 + \theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = T_{target}$$

- For robots with more than 3-DoF, analytical solution can be very complex
  - e.g., for a 6-DoF robot, you will need several pages to write down the formula
- Some useful libraries: IKFast, IKBT

# Numerical Solution

- Solving a nonlinear optimization problem
- Standard numerical optimization algorithms can be utilized, e.g. Newton-Raphson and Levenberg-Marquardt
- Numerical IK leverages the geometric Jacobian  
 $\xi = J(\theta)\dot{\theta}$

# Levenberg–Marquardt Algorithm

- Error between the desired pose and the current one:

$$T_{err} = T_{target}T(\theta)^{-1} \in \mathbb{SE}(3)$$

- Calculate the corresponding screw:

$$\chi_{err} = \log(T_{err}) \in \mathfrak{se}(3)$$

- Recall that  $\xi = J(\theta)\dot{\theta}$ :

$$\xi\Delta t = J(\theta)\dot{\theta}\Delta t \Rightarrow \Delta\chi \approx J(\theta)\Delta\theta$$

# Levenberg–Marquardt Algorithm

- In LM algorithm, we iteratively update  $\theta$
- In each iteration, we try to find a  $\Delta\theta$  that minimizes:

$$S(\theta, \Delta\theta) = \|\chi_{err} - J(\theta)\Delta\theta\|^2 + \lambda\|\Delta\theta\|^2$$

- $\lambda$  term stabilizes the optimization
- Closed-form solution:

$$(J^T J + \lambda I)\Delta\theta = J^T \chi_{err}$$

- Solve  $\Delta\theta$  and then update  $\theta$  by:  $\theta \leftarrow \theta + \Delta\theta$

# Levenberg–Marquardt Algorithm

$$(J^T J + \lambda I) \Delta \theta = J^T \chi_{err}$$

- Damping factor  $\lambda \geq 0$  is adjusted at each iteration:
- If  $S(\theta, \Delta \theta)$  is decreasing, a smaller  $\lambda$  (e.g.,  $\lambda \leftarrow 0.1\lambda$ ) can be used.
  - closer to the Gauss–Newton algorithm
- Otherwise, a larger  $\lambda$  (e.g.,  $\lambda \leftarrow 10\lambda$ ) can be used.
  - closer to the gradient-descent algorithm



# Levenberg–Marquardt Algorithm

- LM algorithm may converge to a local minima, initial  $\theta_0$  is very important:
  - Sampling multiple  $\theta_0$  may boost the performance
- In most cases,  $\theta$  comes with limit constraints:
  - $l[i] \leq \theta[i] \leq r[i]$
  - A joint can only translate (or rotate) within the limit
  - Invalid state rejection
  - Clipping during the optimization iterations

# Kinematic Singularity

**Question:** is it always possible to move the end-effector to any direction  $\hat{\xi}$  for a robot with  $\text{DoF} \geq 6$

- **Kinematic singularity:**
  - A **robot configuration** where the robot's end-effector loses the ability to move in one direction instantaneously
- If  $\text{rank}(J(\theta)) < 6$  at some  $\theta$ , by  $\Delta\xi = J(\theta)\Delta\theta$ ,  $\Delta\xi$  can only be in a linear space with dimension  $\text{rank}(J(\theta)) < 6$ , losing its ability to move in some directions
- Note: Kinematic singularity does not mean that there exists a configuration that is not accessible (may get to the pose by some other motion trajectory)