L4: Forward Kinematics & Inverse Kinematics

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Agenda

- Rigid Transformation and $\mathbb{SE}(3)$
- Jacobian of Kinematics Chain
- Inverse Kinematics

Rigid Transformation and $\mathbb{SE}(3)$

The Set of Rigid Transformations

- $\mathbb{SE}(3) := \left\{ T = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}, R \in \mathbb{SO}(3), t \in \mathbb{R}^3 \right\}$
- $\mathbb{SE}(3)$: “Special Euclidean Group”

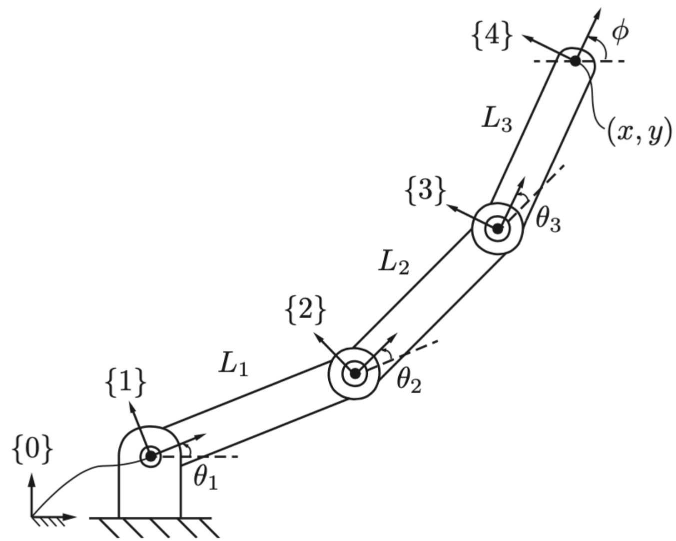
- “Group”: closed under matrix multiplication and other conditions of group
- “Euclidean”: R and t
- “Special”: $\det(R) = 1$

- 6 DoF

Conditions of Group: https://en.wikipedia.org/wiki/Group_(mathematics)#Definition

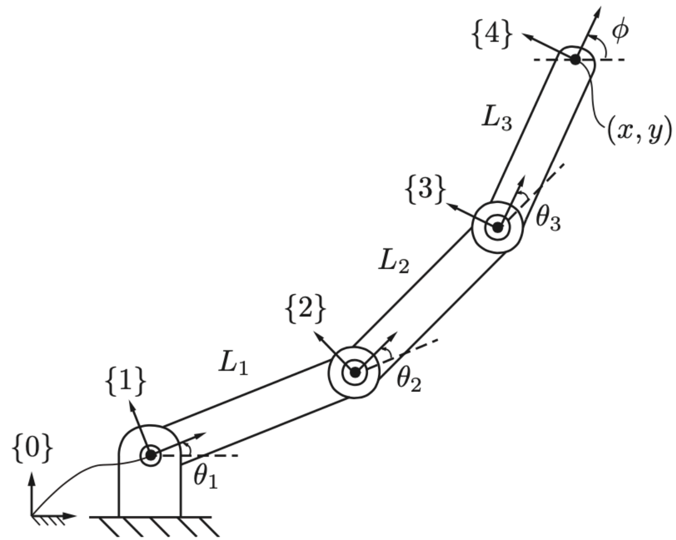
Jacobian of Kinematics Chain

Forward Kinematic Problem



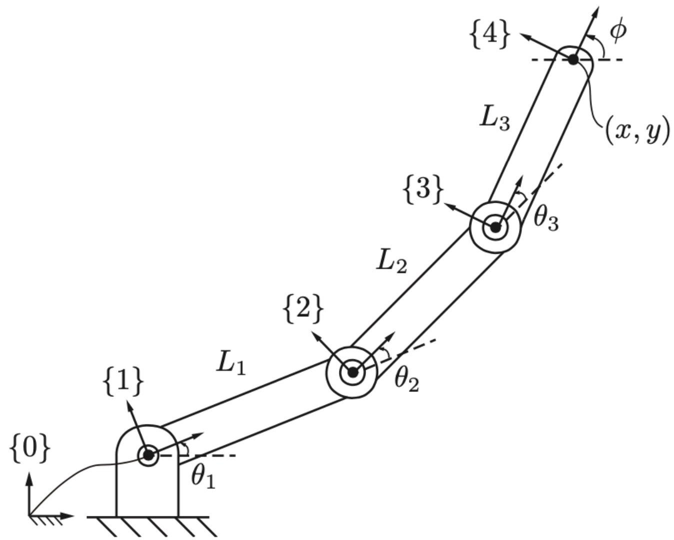
- Suppose that the arm moves
- How do I compute the velocity of the end-effector from the angular velocity of joints?

Spatial Frame Inverse Kinematics Problem



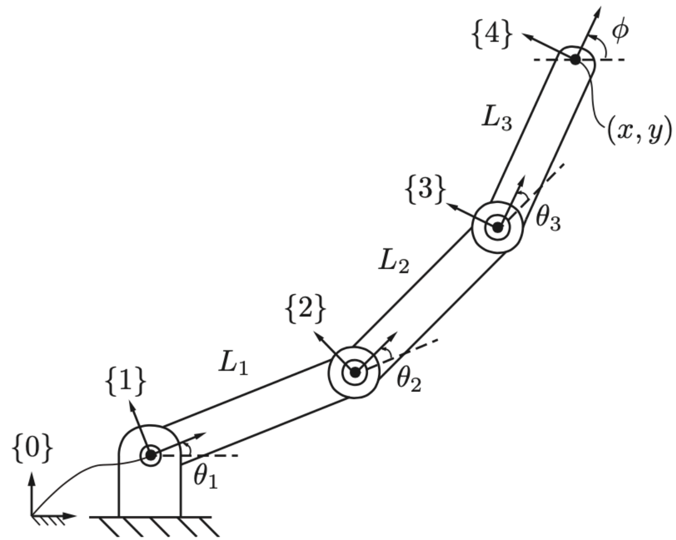
- If I specify the direction of the end-of-effector movement using the spatial frame, how can I change the joint angles?
- e.g. move to a pre-specified $T^s_{s\to e}$

Body Frame Inverse Kinematics Problem



- If I specify the direction of the end-of-effector movement using the body frame, how can I change the joint angles?
- e.g. move the end-effector forward along its link

Kinematic Equation



- We can solve the problems if we have $\xi_{e(t)} = f(\dot{\theta})$
- The language to describe the velocity of end-effector are
  - $\xi^{s}_{e(t)}$ for spatial frame query
  - $\xi^{e(t)}_{e(t)}$ for body frame query
- We will derive the f^{s} and $f^{e(t)}$

Spatial Geometric Jacobian

- Spatial Geometric Jacobian $J^s(\theta)$:

$$\xi^s_{e(t)} = J^s(\theta)\dot{\theta}$$

where $\theta \in \mathbb{R}^n$ (n joints), $J^s(\theta) \in \mathbb{R}^{6\times n}$, and the i-th column of $J(\theta)$ is ${}^i\hat{\xi}^s_{e(t)}$, the twist when the movement is caused only by the i-th joint **while all other joints stay static**

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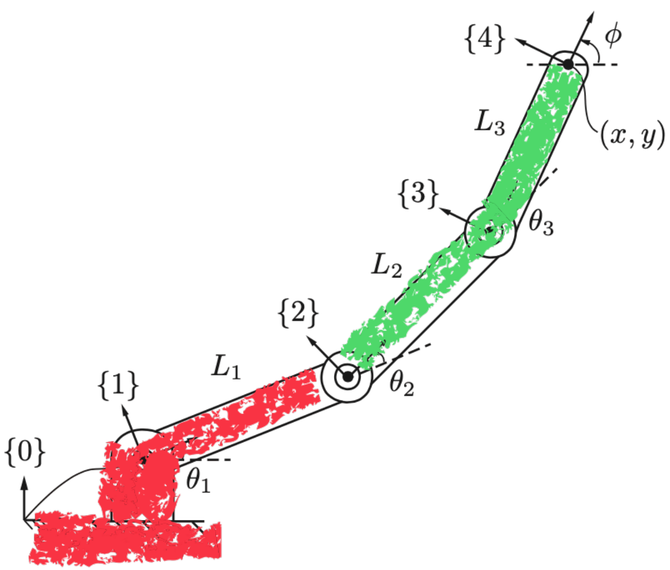
- Spatial Geometric Jacobian $J^s(\theta)$:

$$\xi^s_{e(t)} = J^s(\theta)\dot{\theta}$$

where $\theta \in \mathbb{R}^n$ (n joints), $J^s(\theta) \in \mathbb{R}^{6\times n}$

- For example, ${}^2\hat{\xi}^s_{e(t)}$ describes the motion of the green part, which is to revolute about Joint {2}



Body Geometric Jacobian

- Body Geometric Jacobian $J^{e(t)}(\theta)$:

$$\xi_{e(t)}^{e(t)} = J^{e(t)}(\theta)\dot{\theta}$$

where $J^{e(t)}(\theta) \in \mathbb{R}^{6\times n}$

Inverse Kinematics

Inverse Kinematics

- Position query
  - Given the forward kinematics $T^s_{s\to e}(\theta)$ and the target pose $T_{target} = \mathbb{SE}(3)$, find θ that satisfies $T_{s\to e}(\theta) = T_{target}$
- Velocity query
  - Given the twist of the end-effector, find the angular velocity that satisfies $\xi_{target} = J(\theta)\dot{\theta}$
- May have multiple solutions, a unique solution or no solution

Null Space of Jacobian

- Consider the velocity query IK task
- Recall that $\xi = J(\theta)\dot{\theta}$ for an n-joint kinematic chain, where J is a $6 \times n$ matrix
- When $n > 6$, the joint space is projected to a lower-dimensional space and J must exist a null space
- As a result, IK may have infinite solutions (a special solution + any vector in the null space of J)
- The null space adds flexibility to make motion plans

Analytical Solution

- Try to solve the equation $T_{target} = T(\theta)$ and get an analytical solution for θ
  - e.g., solve θ_1 and θ_2 for $\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & -\sin\theta_1(l_2+l_3) \\ \sin\theta_1 & \cos\theta_1 & 0 & \cos\theta_1(l_2+l_3) \\ 0 & 0 & 1 & l_1 - l_4 + \theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = T_{target}$
- For robots with more than 3-DoF, analytical solution can be very complex
  - e.g., for a 6-DoF robot, you will need several pages to write down the formula
- Some useful libraries: IKFast, IKBT

Numerical Solution

- Solving a nonlinear optimization problem
- Standard numerical optimization algorithms can be utilized, e.g. Newton-Raphson and Levenberg-Marquardt
- Numerical IK leverages the geometric Jacobian
  $\xi = J(\theta)\dot{\theta}$

Kinematic Singularity

Question: is it always possible to move the end-effector to any direction $\hat{\xi}$ for a robot with $\mathrm{DoF} \geq 6$

- **Kinematic singularity:**
  - A **robot configuration** where the robot's end-effector loses the ability to move in one direction instantaneously
- If $\mathrm{rank}(J(\theta)) < 6$ at some θ, by $\Delta\xi = J(\theta)\Delta\theta$, $\Delta\xi$ can only be in a linear space with dimension $\mathrm{rank}(J(\theta)) < 6$, losing its ability to move in some directions
- Note: Kinematic singularity does not mean that there exists a configuration that is not accessible (may get to the pose by some other motion trajectory)