UC San Diego

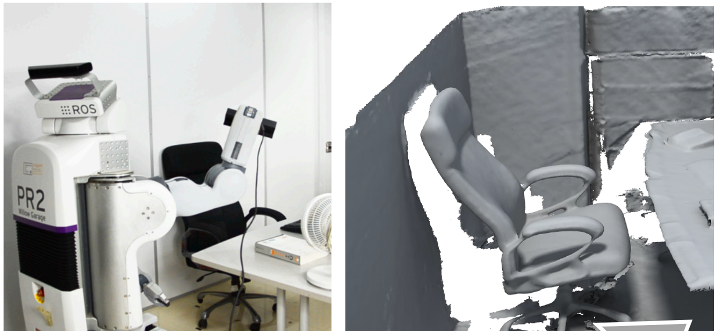
Machine Learning Meets Geometry

L1: Introduction

Agenda

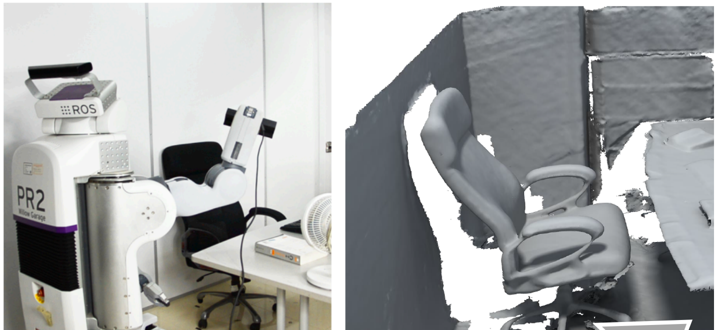
- Overview of Topics
- Logistics
- Curve Theory

Geometry Understanding is Important



Robotics

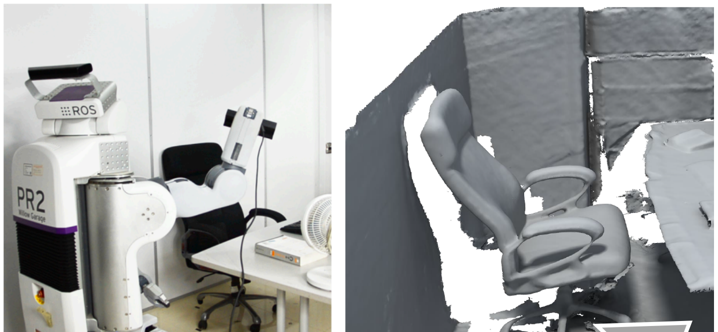
Geometry Understanding is Important



Robotics

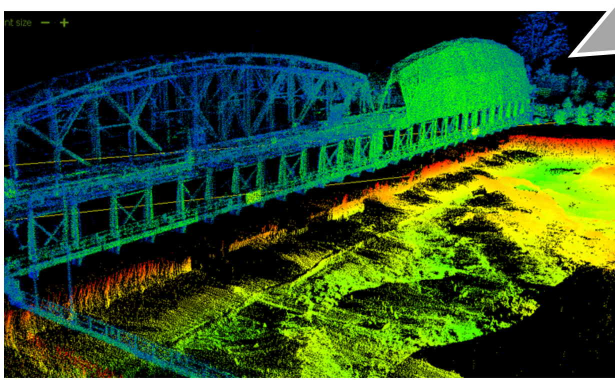
Augmented Reality

Geometry Understanding is Important



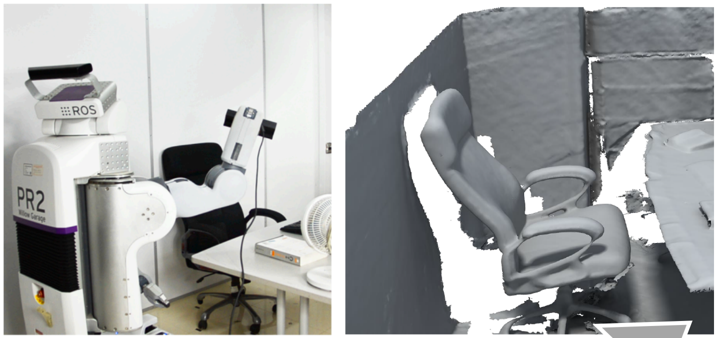
Robotics

Augmented Reality

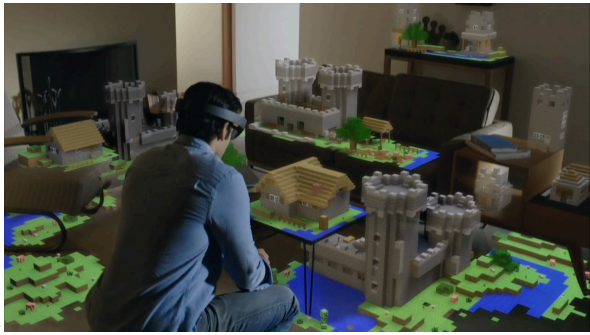


Autonomous driving

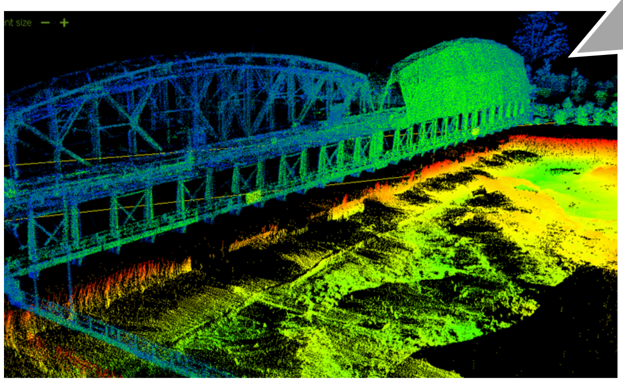
Geometry Understanding is Important



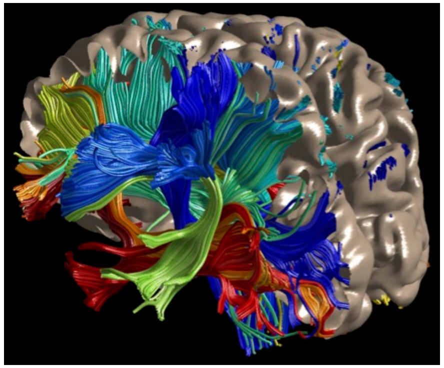
Robotics



Augmented Reality



Autonomous driving



Medical Image Processing

Learning Knowledge of Geometry from Data

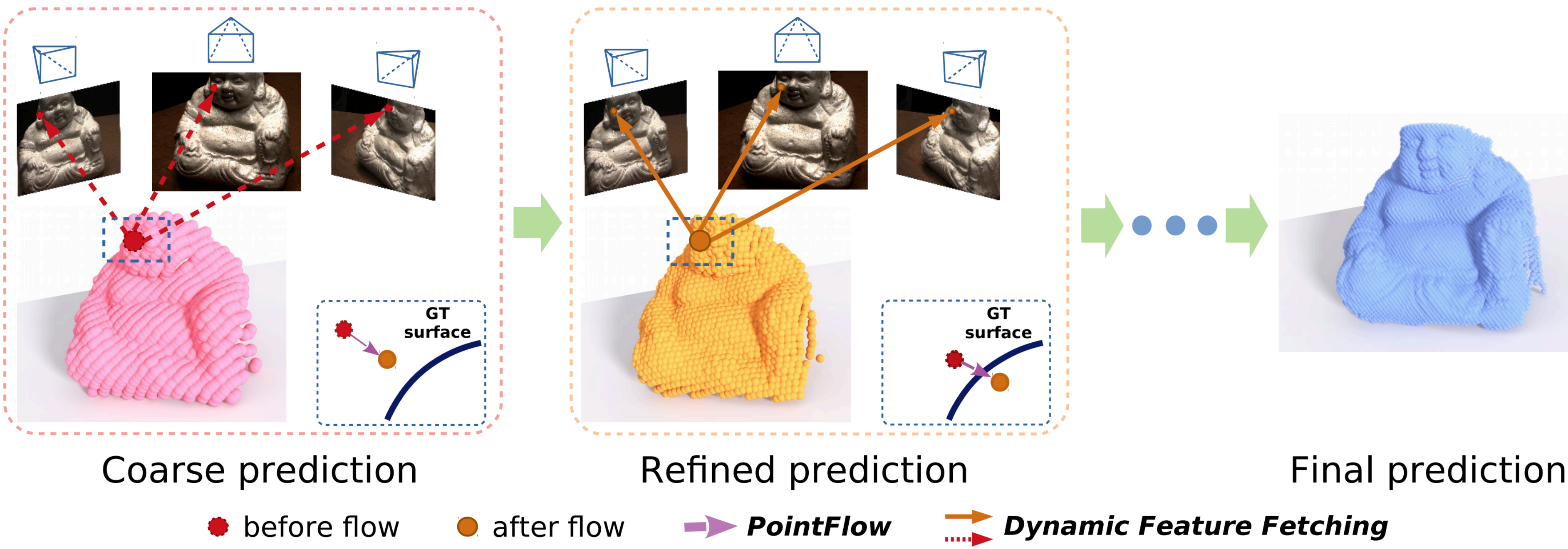
A priori knowledge of the 3D world

Tasks Covered in This Course

- 3D reconstruction from a single image

Tasks Covered in This Course

- 3D reconstruction from multiple views

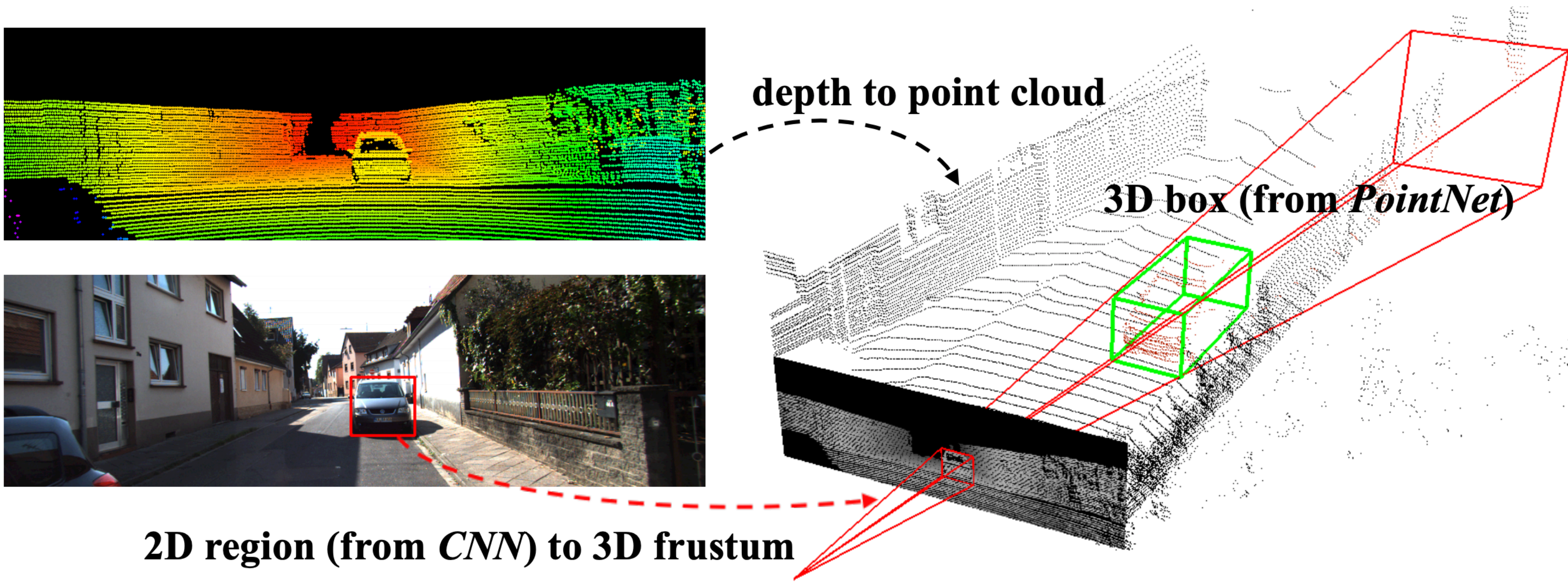


Tasks Covered in This Course

- Object classification

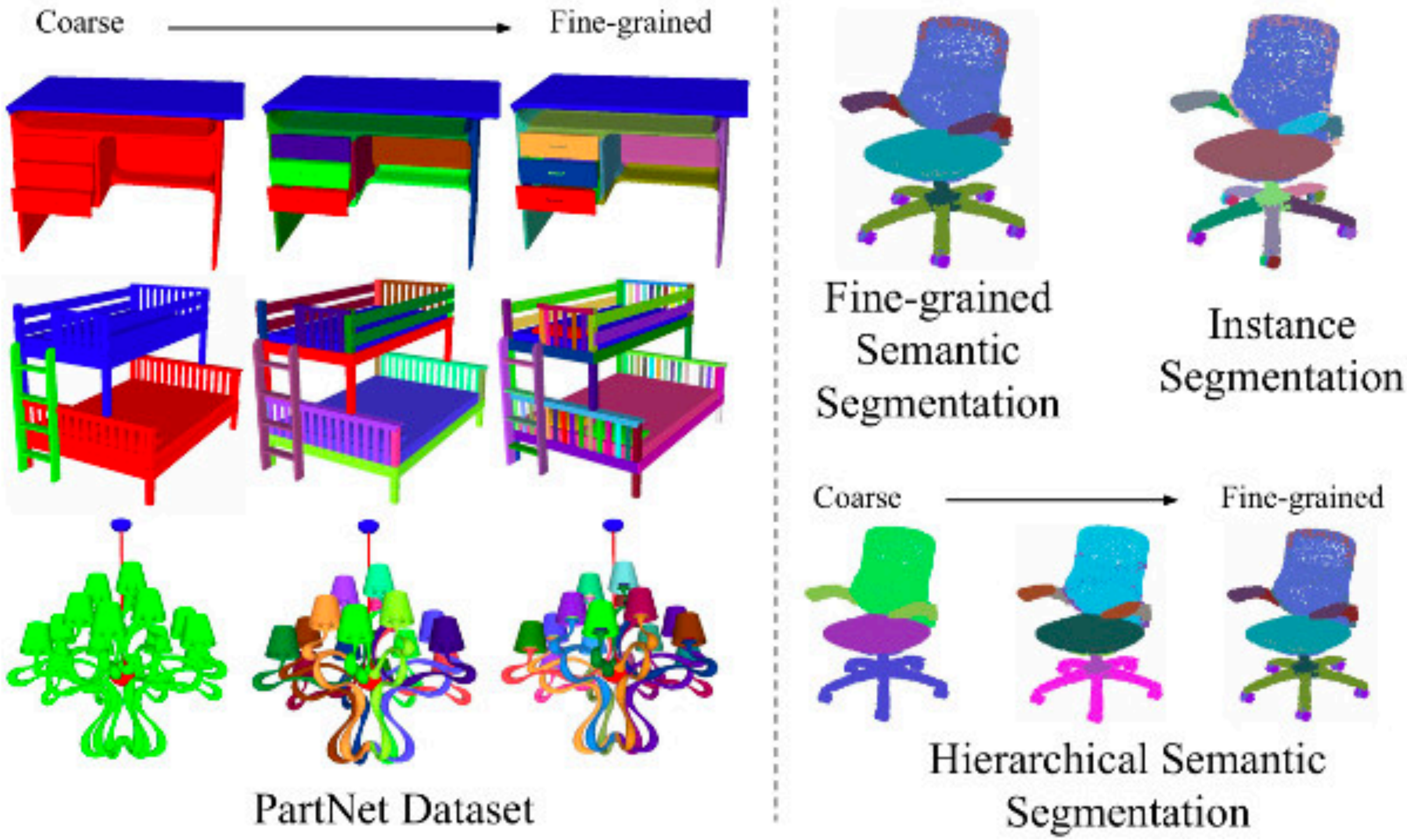
Tasks Covered in This Course

- Object detection



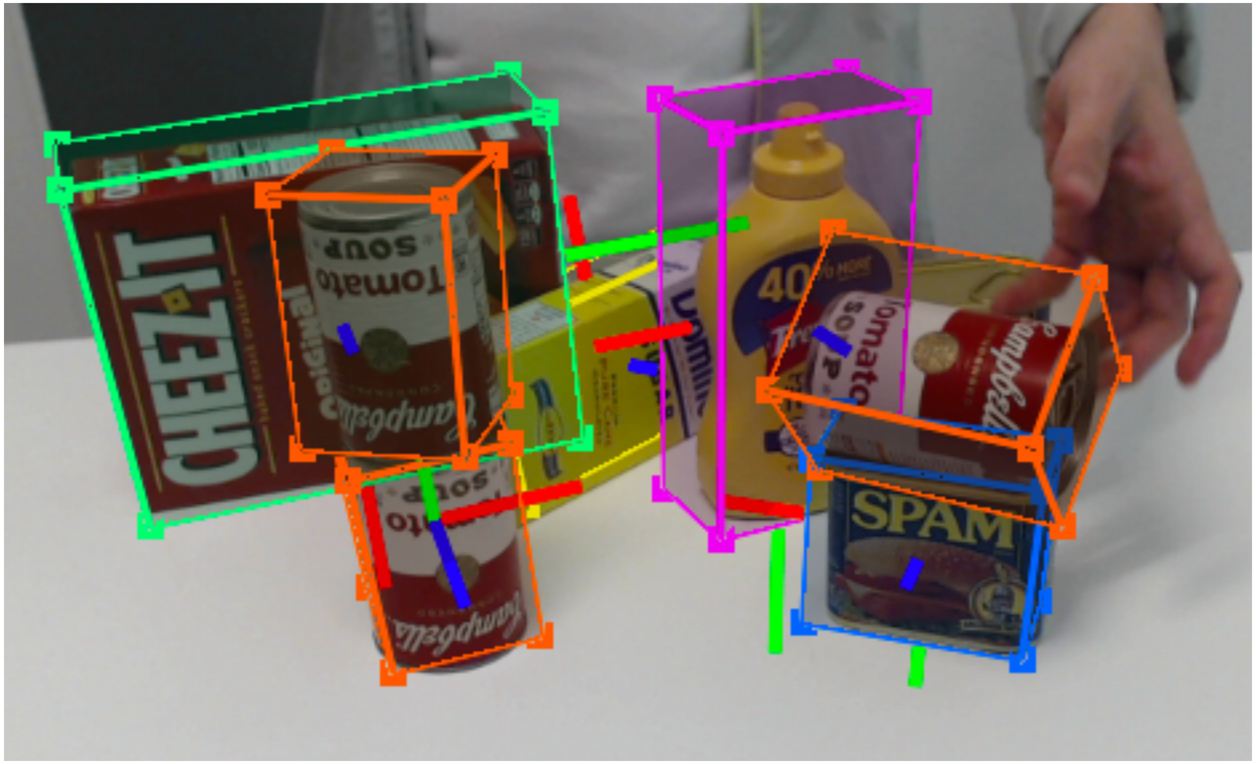
Tasks Covered in This Course

- Part segmentation



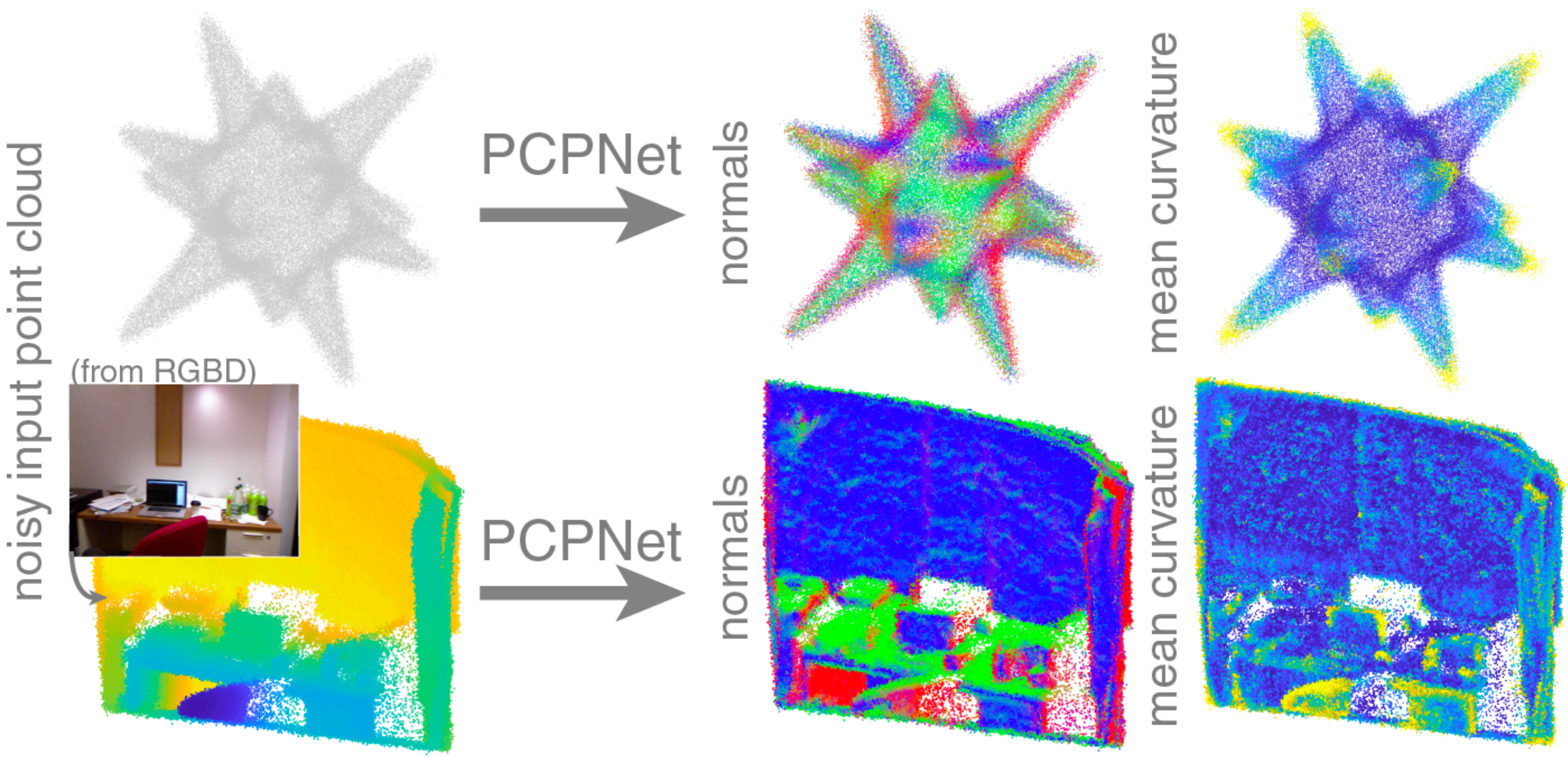
Tasks Covered in This Course

- 6D pose estimation



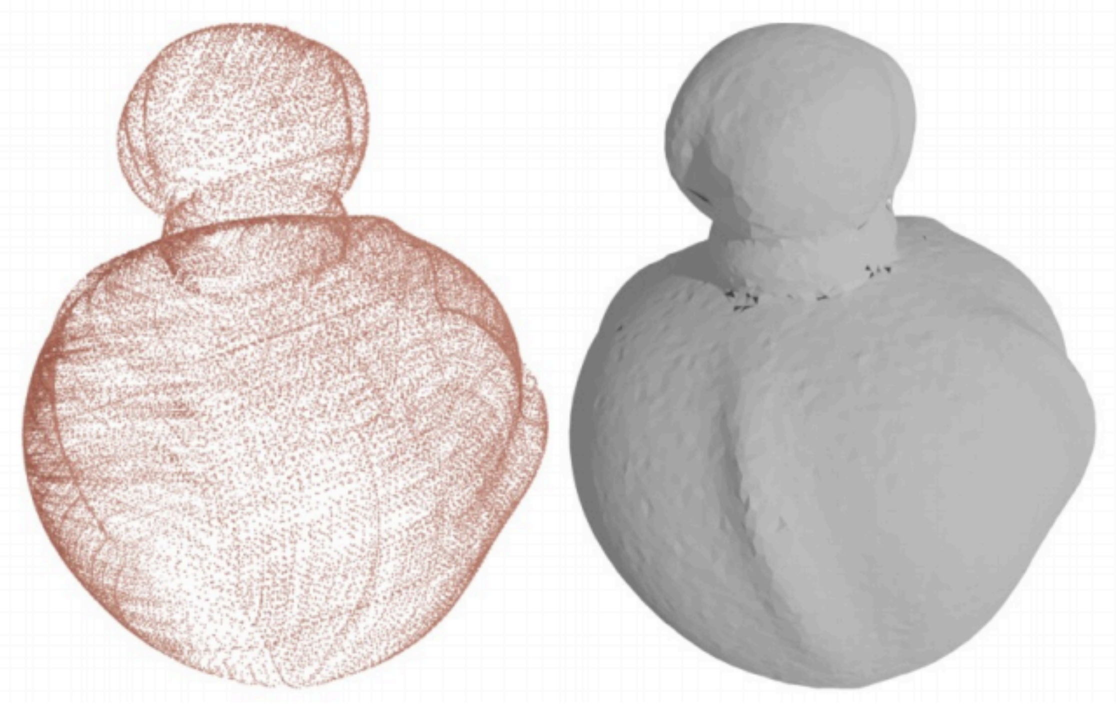
Tasks Covered in This Course

- Local geometric property estimation



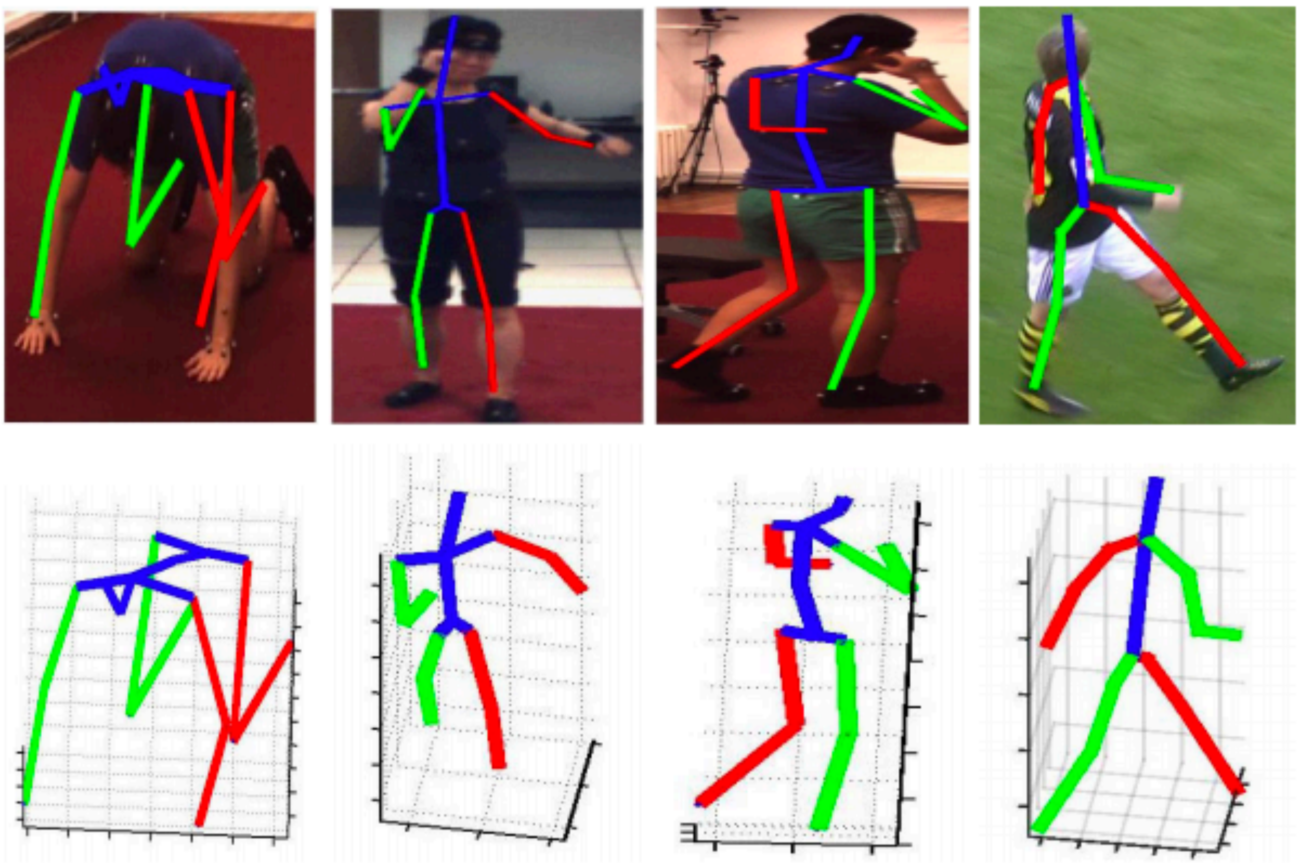
Tasks Covered in This Course

- Mesh reconstruction



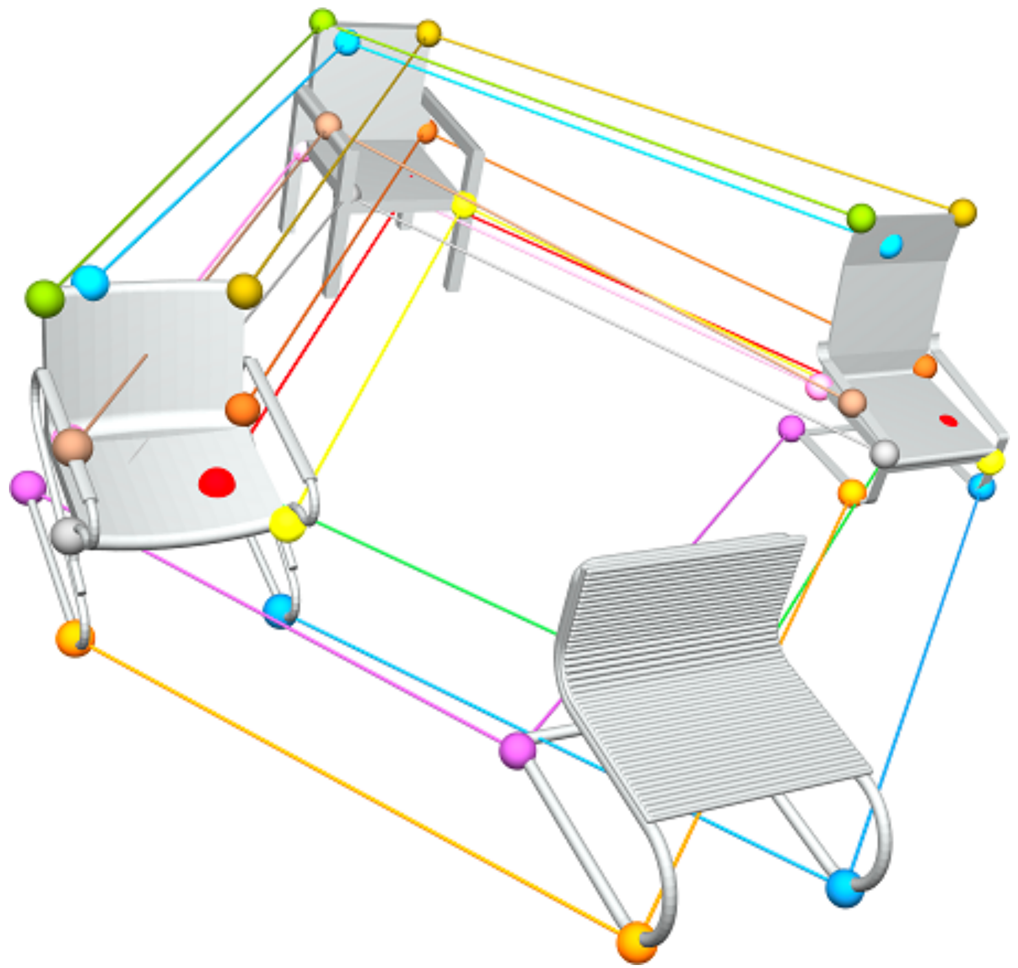
Tasks Covered in This Course

- Human pose estimation

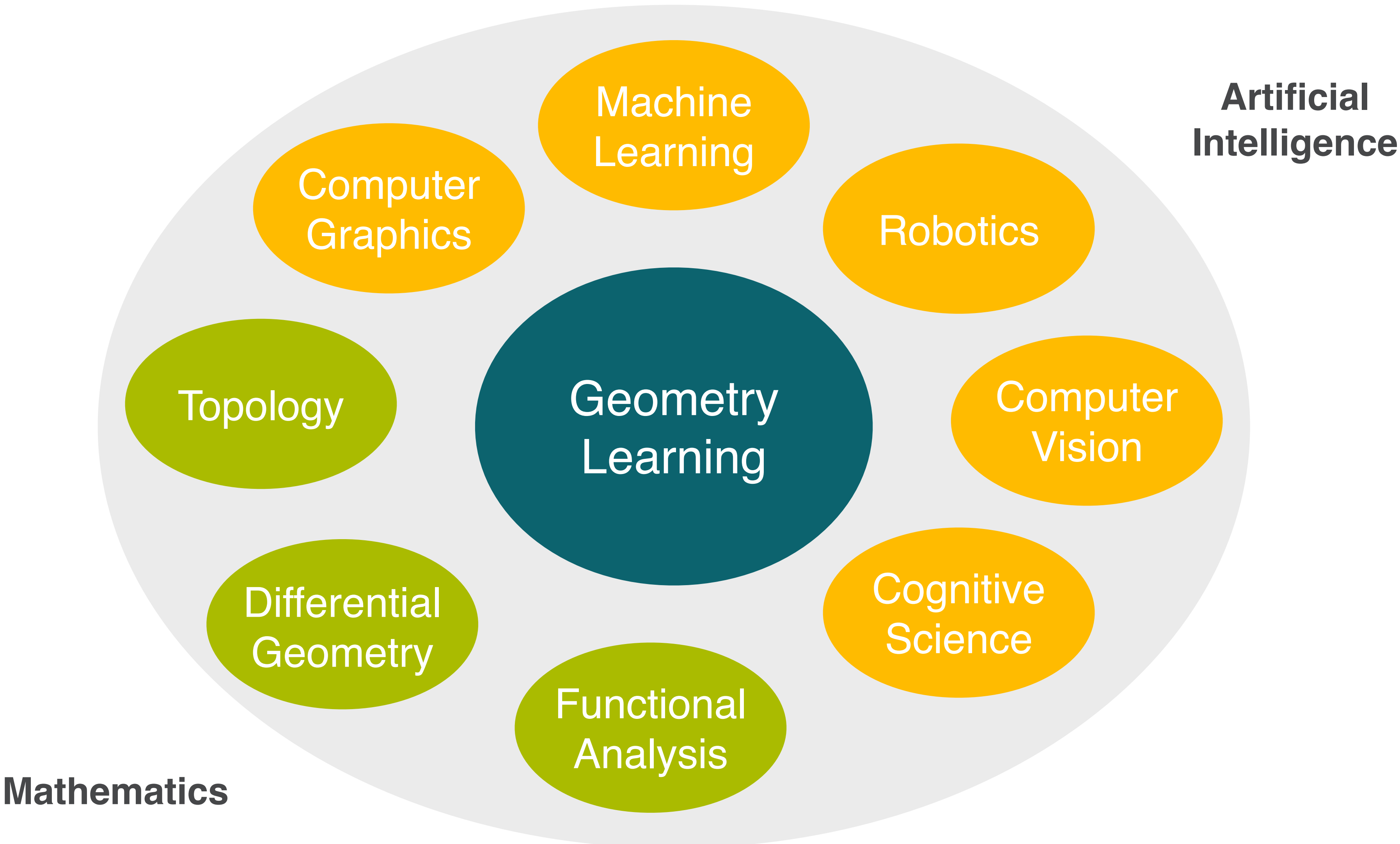


Tasks Covered in This Course

- Shape correspondences



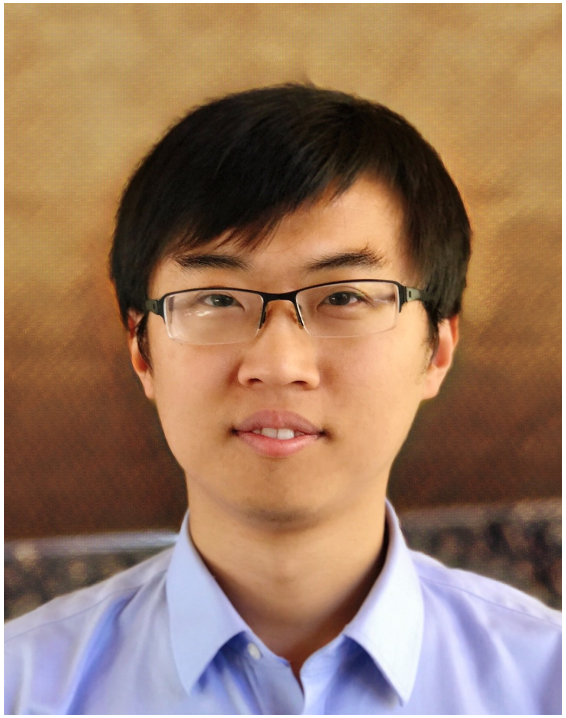
Highly Interdisciplinary Field



Course Logistic

Instructors

Instructor: Hao Su



TA: Fanbo Xiang

Teaching Goal

- State-of-the-art
  - **Enable** you to read and replicate recent 3D papers in top CV/CG conferences (not industry job oriented)
- Hands-on
  - **Heavy** programming assignments to exercise what are taught in class
- Foundational
  - Theory problems are **proof based**
  - Programming problems ask you to **implement low-level modules from scratch**

Pre-requisite

- **Skilled** in Linear Algebra
- **Familiar** with Multi-variable Calculus
- **Familiar** with Probability and Numerical Methods
- **Strong** programming skills
  - Familiar with Linux Toolchain
  - Familiar with python, numpy, and pytorch
- Course/project experiences in computer vision or deep learning

Background Check

- On Piazza now (HW0)
  - Visible to enrolled and waitlist students

- 5 points in your final grade

- **Mandatory!** We will not grade your subsequent homeworks without seeing your HW0.

- **Due: 1/12/2021**

- If you are in the waitlist and intend to enroll, you need to submit HW0 by this deadline

Assignments

- 4 assignments and 1 final project
  - HW0: due week 2 (5 points)
  - HW1: due week 4 (20 points)
  - HW2: due week 6 (20 points)
  - HW3: due week 8 (20 points)
  - Final project: final week (35 points)
  - No mid-term/final exams

- HW0-HW3: theory problems + programming

- Late policy: 15% grade reduction for each 12 hours late. No acceptance 72 hours after the due date.

3D Recognition Competition

- HW0-HW3: build individual modules
- Final project: integrate modules and test new ideas. Score by performance ranking.
- Online evaluation system will be set up
- We allow you to see homework (through Piazza) and attend the competition *even if you* ***audit*** *the course*
- We estimate **>=15 hrs per week** (out of class) solid time commitment

Course Resources

- Course website: https://geoml.github.io/
  - Collaboration policy
  - Lecture Slides
  - Office hour and location

- Piazza
  - Homework release
  - Discussions

- Homework submission: Gradescope

Curve

- Definition of curve
- Describing the shape of curves by calculus

Parameterized Curves

Intuition

- A particle is moving in space
- At time t its position is given by

$$\gamma(t) = (x(t), y(t))$$

Example

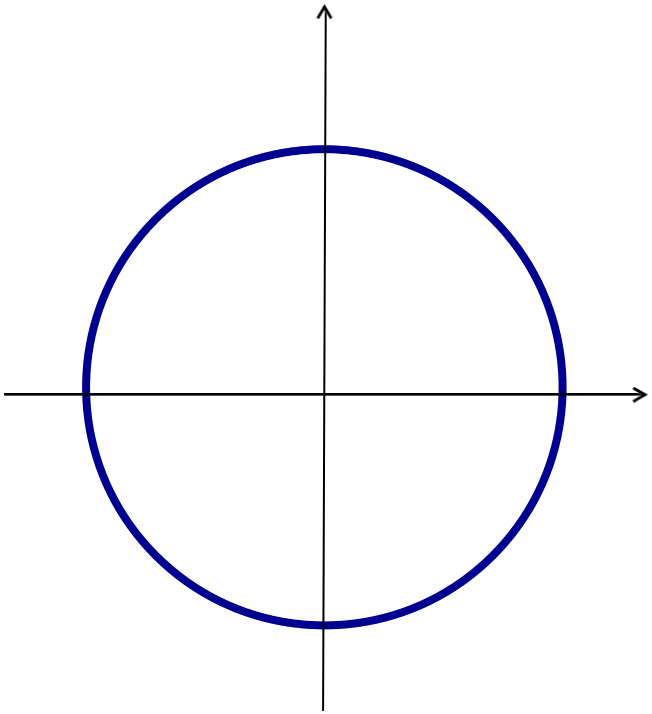
Explicit curve/circle in 2D

$$\mathbf{p} : \mathbb{R} \to \mathbb{R}^2$$

$$t \mapsto \mathbf{p}(t) = (x(t), y(t))$$

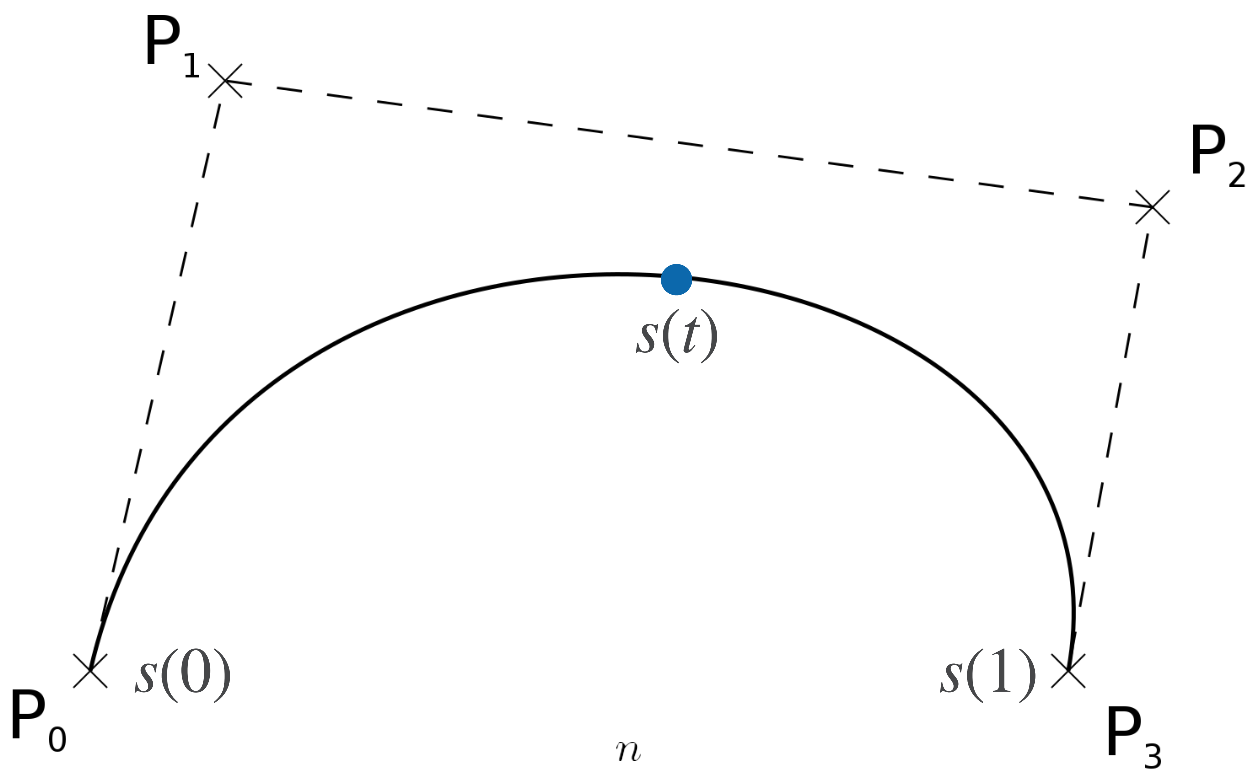
$$\mathbf{p}(t) = r\,(\cos(t), \sin(t))$$

$$t \in [0, 2\pi)$$



Application: Bezier Curves, Splines

- Smoothly "interpolate" between *a set of* points P_i
- Widely used in design (e.g., in your Powerpoint)

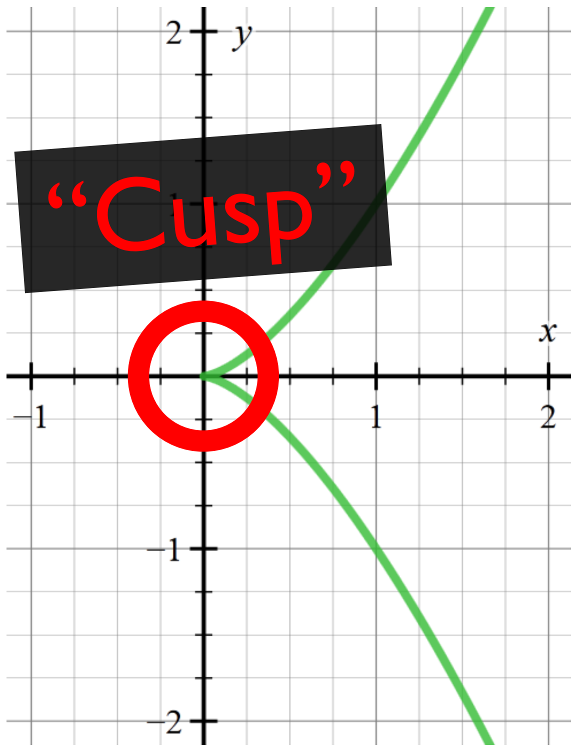


$$s(t) = \sum_{i=0}^{n} \mathbf{p}_i B_i^n(t)$$

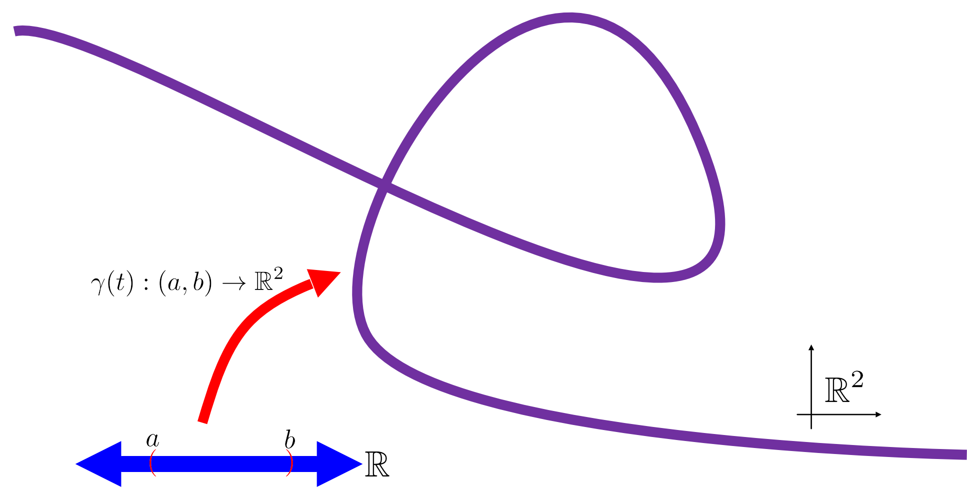
One-dimensional "Manifold"

Set of points that locally looks like a line.

Negative Examples of Manifolds



$$f(t) = (t^2, t^3)$$

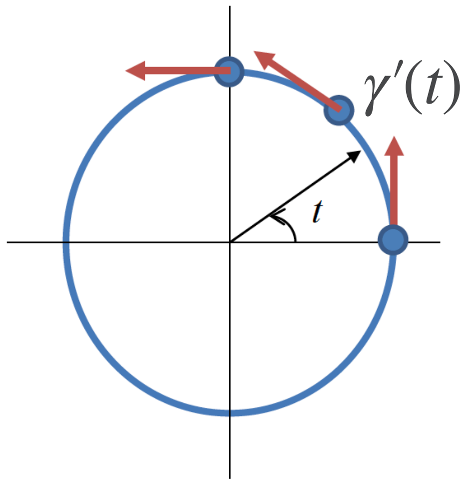


Tangent

- $\gamma'(t) = (x'(t), y'(t)) \in \mathbb{R}^2$ is the tangent vector of the curve at t

Quiz: Tangent of a Circle

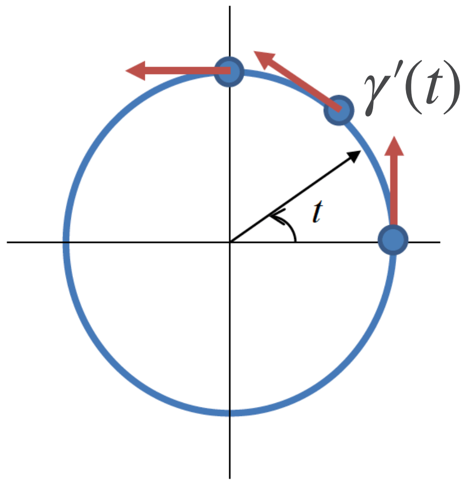
- $\gamma'(t) = (x'(t), y'(t)) \in \mathbb{R}^2$ is the tangent vector of the curve at t



$$\gamma(t) = (\cos(t), \sin(t))$$

Quiz: Tangent of a Circle

- $\gamma'(t) = (x'(t), y'(t)) \in \mathbb{R}^2$ is the tangent vector of the curve at t



$$\gamma(t) = (\cos(t), \sin(t))$$

$$\gamma'(t) = (-\sin(t), \cos(t))$$

$\gamma'(t)$ - direction of movement

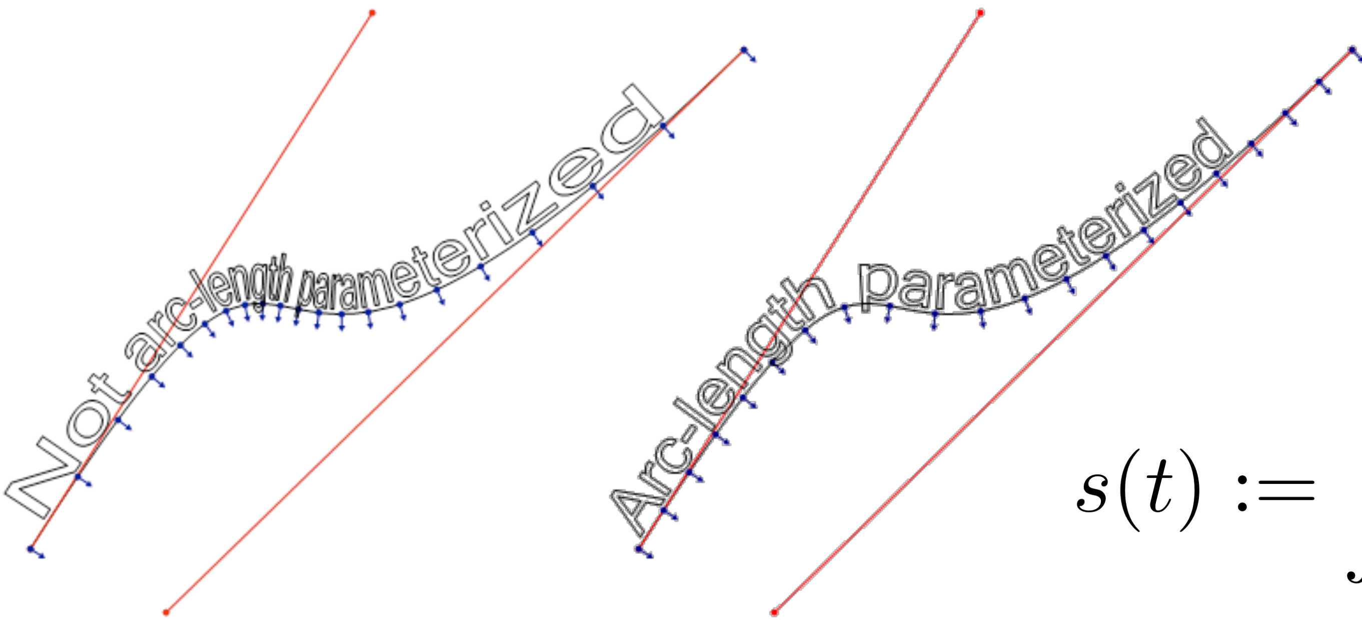
$||\gamma'(t)||$ - speed of movement

Arc Length

$$\int_a^b \|\gamma'(t)\|\,dt$$

Parameterization by Arc Length

http://www.planetclegg.com/projects/WarpingTextToSplines.html



$$s(t) := \int_{t_0}^{t} \|\gamma'(t)\|\, dt$$

$$t(s) := \text{inverse of } s(t)$$

$$\bar{\gamma}(s) = \gamma(t(s))$$

Constant-speed parameterization

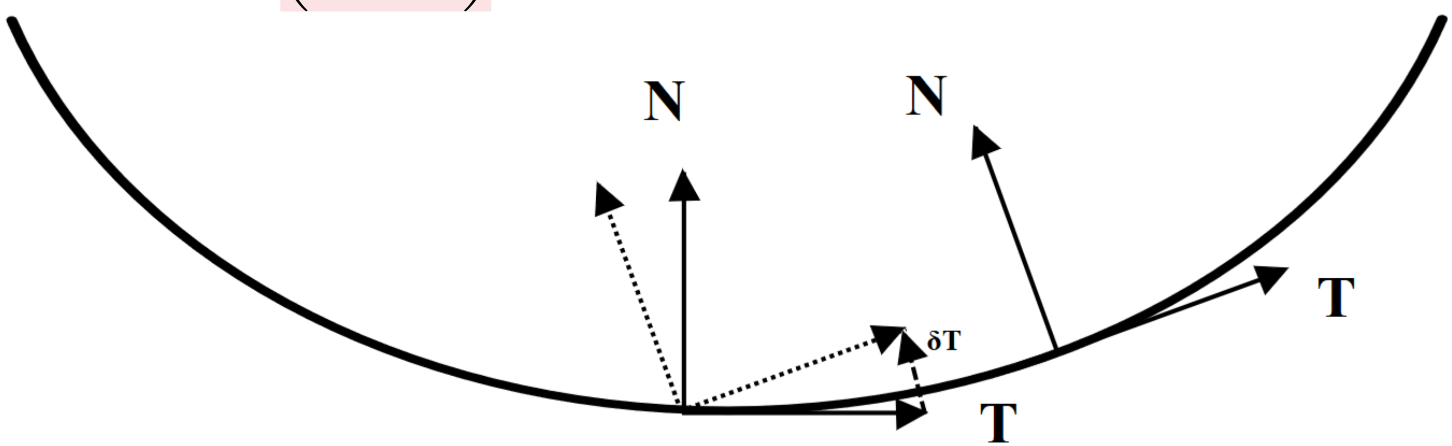
Moving Frame in 2D

$$T(s) := \gamma'(s)$$

$$\implies \quad \text{(on board) } \|T(s)\| \equiv 1$$

$$N(s) := JT(s)$$

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



https://en.wikipedia.org/wiki/Frenet%E2%80%93Serret_formulas

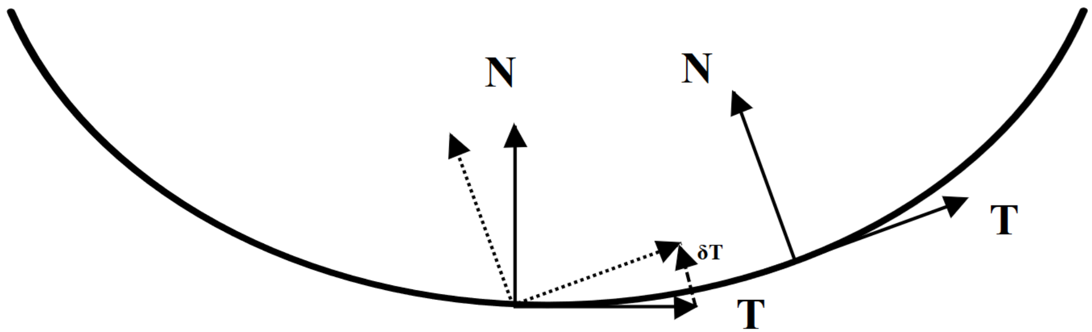
Derivation of $\|T(s)\| \equiv 1$

(See notes)

Turtles All The Way Down

On the board:

$$\frac{d}{ds}\begin{pmatrix} T(s) \\ N(s) \end{pmatrix} := \begin{pmatrix} 0 & k(s) \\ -k(s) & 0 \end{pmatrix}\begin{pmatrix} T(s) \\ N(s) \end{pmatrix}$$



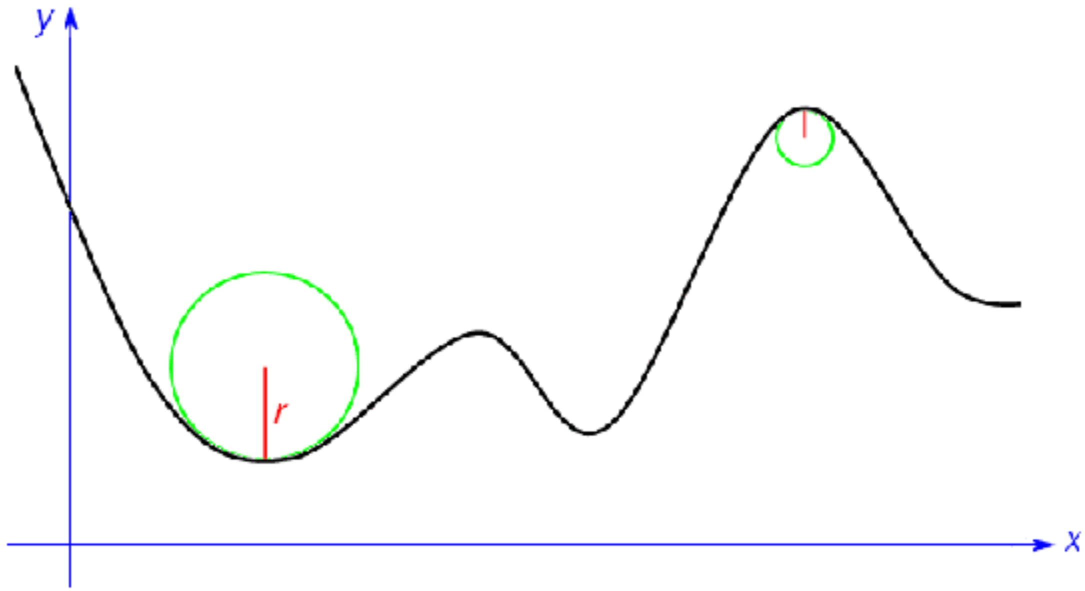
https://en.wikipedia.org/wiki/Frenet%E2%80%93Serret_formulas

Use coordinates *from* the curve to express its shape!

$$\frac{d}{ds}\begin{pmatrix} T(s) \\ N(s) \end{pmatrix} := \begin{pmatrix} 0 & k(s) \\ -k(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \end{pmatrix}$$

(See notes)

Radius of Curvature



$$r(s) := \frac{1}{k(s)}$$

https://www.quora.com/What-is-the-base-difference-between-radius-of-curvature-and-radius-of-gyration

Invariance is Important

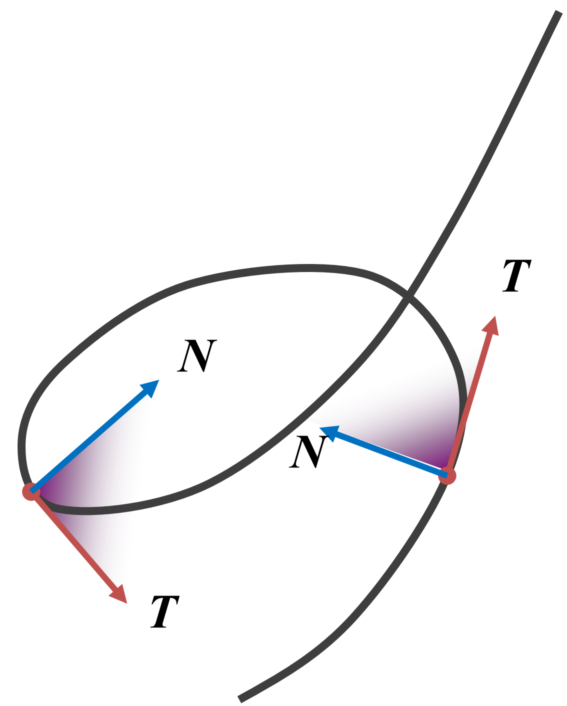
Fundamental theorem of the local theory of plane curves:

$\kappa(s)$ characterizes a **planar curve** up to rigid motion.

3D Curves

- Osculating Plane

The plane determined by the unit tangent and normal vectors $T(s)$ and $N(s)$ is called the *osculating plane* at s



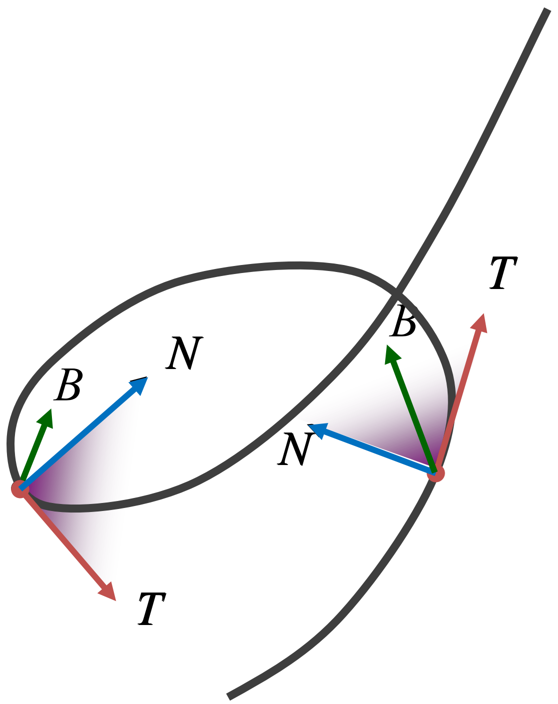
Self-reading

The Binormal Vector

For points s, s.t. $\kappa(s) \neq 0$, the *binormal vector* $\boldsymbol{B}(s)$ is defined as:

$$\boldsymbol{B}(s) = \boldsymbol{T}(s) \times \boldsymbol{N}(s)$$

The binormal vector defines the osculating plane



Self-reading

$\mathbf{T}'(s)$

- Already used it to define the curvature:

$$\mathbf{T}'(s) = \kappa(s)\boxed{\mathbf{N}(s)}$$

Unit vector

- Orthogonal to $\mathbf{T}(s)$ (the same derivation as 2D curve)

- Since along the direction of $\mathbf{N}(s)$, also orthogonal to $\mathbf{B}(s)$

Self-reading

$\mathbf{N}'(s)$

We know: $\langle \mathbf{N}(s), \mathbf{N}(s) \rangle$=1

From the lemma $\longrightarrow \langle \mathbf{N}'(s), \mathbf{N}(s) \rangle = 0$

(Derivative orthogonal to itself)

We know: $\langle \mathbf{N}(s), \mathbf{T}(s) \rangle = 0$

From the lemma $\longrightarrow \langle \mathbf{N}'(s), \mathbf{T}(s) \rangle = \langle -\mathbf{N}(s), \mathbf{T}'(s) \rangle$

From the definition $\longrightarrow \kappa(s) = \langle \mathbf{N}(s), \mathbf{T}'(s) \rangle$

$\longrightarrow \langle \mathbf{N}'(s), \mathbf{T}(s) \rangle = -\kappa(s)$

Self-reading

The Torsion

- From previous slide:

$$\langle \mathbf{N}'(s), \mathbf{N}(s) \rangle = 0$$

$$\langle \mathbf{N}'(s), \mathbf{T}(s) \rangle = -\kappa(s)$$

The remaining component of $\mathbf{N}'(s)$ is along $\mathbf{B}(s)$ direction:

$$\langle \mathbf{N}'(s), \mathbf{B}(s) \rangle = \tau(s)$$

Now we can express $N'(s)$ as

$$\mathbf{N}'(s) = -\kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{B}(s)$$

Self-reading

Perspective from Measuring Normal Change

$$\mathbf{N}'(s) = -\kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{B}(s)$$

- Curvature indicates how much the normal changes in the direction tangent to the curve
- Torsion indicates how much normal changes in the direction orthogonal to the osculating plane of the curve
- Curvature is always positive but torsion can be negative

Self-reading

$\mathbf{B}'(s)$ (Examine by Yourself)

We know: $\langle \mathbf{B}(s), \mathbf{B}(s) \rangle = 1$

From the lemma $\longrightarrow$ $\langle \mathbf{B}'(s), \mathbf{B}(s) \rangle = 0$

We know: $\langle \mathbf{B}(s), \mathbf{T}(s) \rangle = 0$, $\langle \mathbf{B}(s), \mathbf{N}(s) \rangle = 0$

From the lemma $\longrightarrow$

$\langle \mathbf{B}'(s), \mathbf{T}(s) \rangle = \langle -\mathbf{B}(s), \mathbf{T}'(s) \rangle = \langle -\mathbf{B}(s), \kappa(s)\mathbf{N}(s) \rangle = 0$

From the lemma $\longrightarrow$

$\langle \mathbf{B}'(s), \mathbf{N}(s) \rangle \ = \langle -\mathbf{B}(s), \mathbf{N}'(s) \rangle = -\tau(s)$

Now we express $\mathbf{B}'(s)$ as:

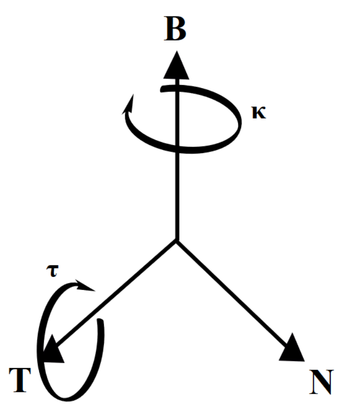
$$\mathbf{B}'(s) = -\tau(s)\mathbf{N}(s)$$

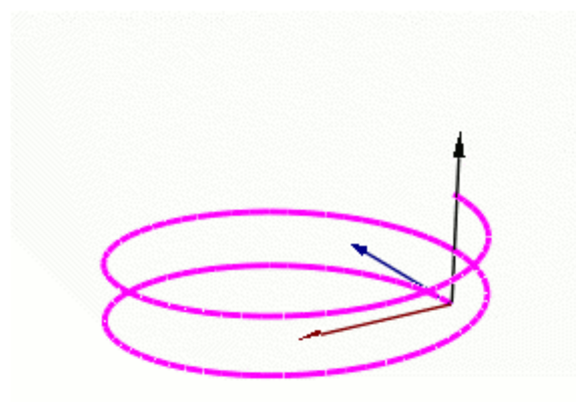
Self-reading

Frenet Frame: Curves in $\mathbb{R}^3$

- Binormal:
  - Curvature: In-plane motion
  - Torsion: Out-of-plane motion

$$\frac{d}{ds}\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix}$$





Self-reading

Fundamental theorem of the local theory of space curves:

Curvature and torsion characterize a 3D curve up to rigid motion.

Summary

- Curve is a map from an interval to $\mathbb{R}^n$
- Tangent describes the moving direction
- The derivative of tangent under arc-length parameterization is normal
- Curvature and torsion both characterize the change of normal direction, uniquely describing the shape of a curve (up to rigid transformation)
- Tangent, normal, and binormal form a moving frame (Frenet frame)