L7-2: Basic Concepts of Rigid-Body Dynamics

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Spring, 2021

Agenda

- Angular Momentum and Rotational Inertia
- Torque
- Kinetic Energy
- Change of Frame for Various Quantities

click to jump to the section.

Angular Momentum and Rotational Inertia

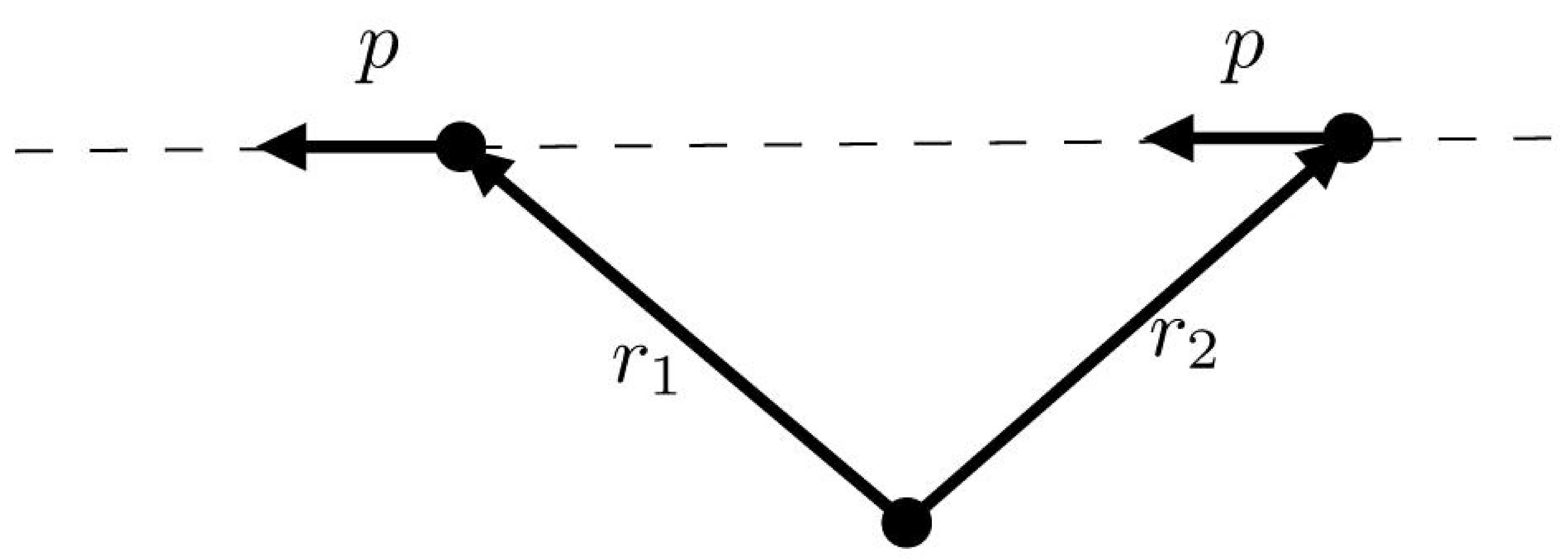
Notations

$\boldsymbol{a}^o = \dot{\boldsymbol{v}}^o_{s(t)\rightarrow b(t)}$	acceleration
m	mass
$\boldsymbol{p}^o_{s(t)\rightarrow b(t)} = m\boldsymbol{v}^o_{s(t)\rightarrow b(t)}$	momentum (state of movement)

Angular Momentum of Point Mass

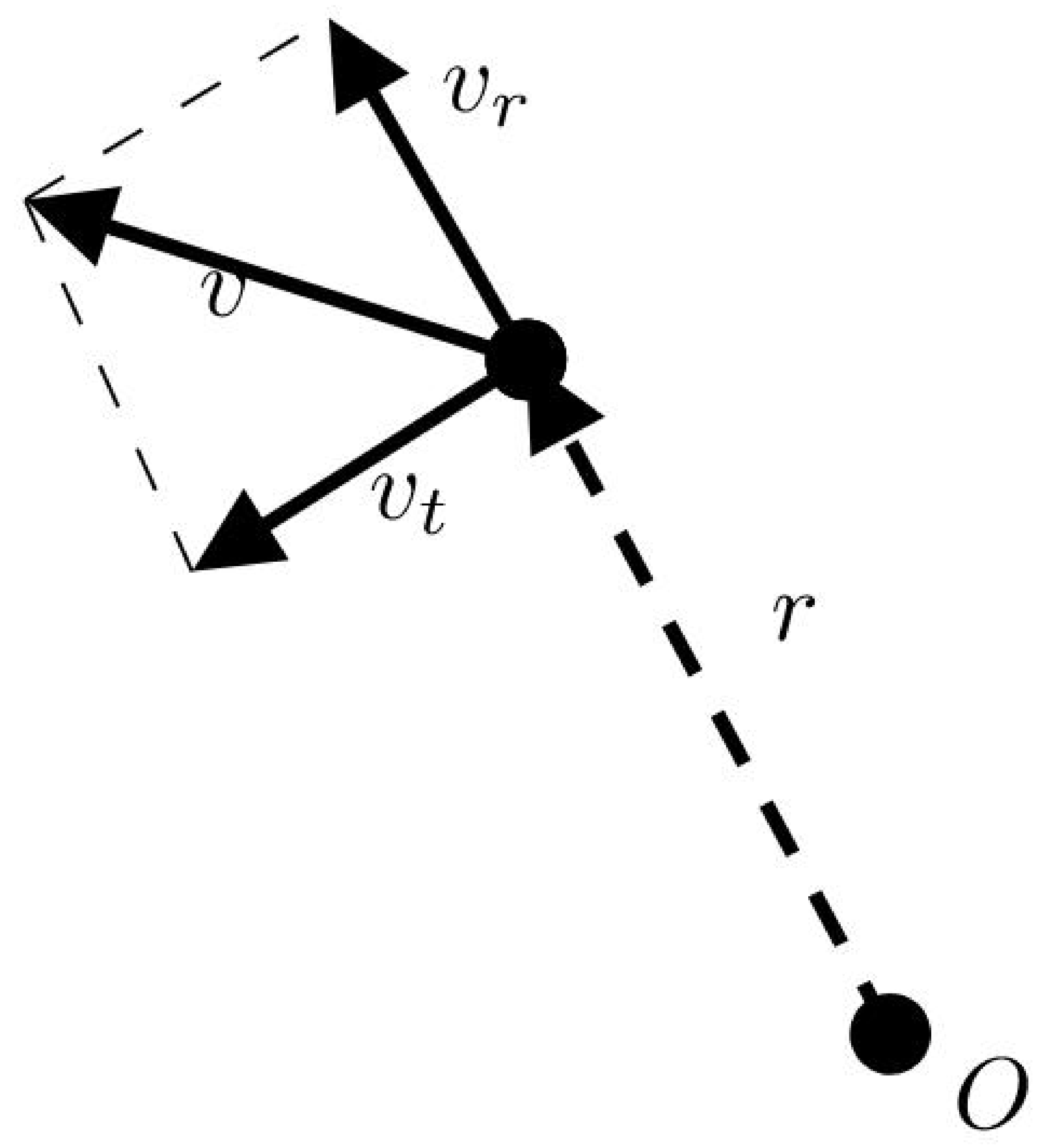
- Assume a point mass m that has a momentum $\boldsymbol{p}^o$
- Assume a vector from *the origin of the observer's frame* O to the point mass $\boldsymbol{r}^o$
- Angular momentum:

$$\boldsymbol{L}^o = \boldsymbol{r}^o \times \boldsymbol{p}^o$$



Rotational Inertia Preparation

$\boldsymbol{v}$ can be decomposed into tangential velocity $\boldsymbol{v}_t$ and radial velocity $\boldsymbol{v}_r$



$$\boldsymbol{r} \times \boldsymbol{v} = \boldsymbol{r} \times (\boldsymbol{v}_t + \boldsymbol{v}_r) = \boldsymbol{r} \times \boldsymbol{v}_t = \boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r})$$

Rotational Inertia of Point Mass

$$\begin{aligned}\boldsymbol{L}^o &= \boldsymbol{r}^o \times \boldsymbol{p}^o = \boldsymbol{r}^o \times (m\boldsymbol{v}^o) = m\boldsymbol{r}^o \times (\boldsymbol{\omega}^o \times \boldsymbol{r}^o)\\ &= -m\boldsymbol{r}^o \times (\boldsymbol{r}^o \times \boldsymbol{\omega}^o) = -m[\boldsymbol{r}^o][\boldsymbol{r}^o]\boldsymbol{\omega}^o\end{aligned}$$

Angular momentum depends on the choice of the observer's frame!

Rotational Inertia of Point Mass

$$
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- Recall that a momentum, such as $\boldsymbol{p}$, is a product of inertia and velocity

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- Recall that a momentum, such as $\boldsymbol{p}$, is a product of inertia and velocity
- We define the rotational inertia similarly. The rotation inertia for a point mass is

$$\boldsymbol{I}^o = -m[\boldsymbol{r}^o][\boldsymbol{r}^o] = \begin{bmatrix} m(r_y^2 + r_z^2) & -mr_xr_y & -mr_xr_z \\ -mr_xr_y & m(r_x^2 + r_z^2) & -mr_yr_z \\ -mr_xr_z & -mr_yr_z & m(r_x^2 + r_y^2) \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

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- Then,

$$\boldsymbol{L}^o = \boldsymbol{I}^o\boldsymbol{\omega}^o$$

Angular Momentum and Inertia of Rigid Body

- Let us view rigid body as *a system of particles* whose relative positions are fixed (no deformation).
- Define the angular momentum of a body by aggregating from volume elements:

$$\boldsymbol{L}^o = \int_{x^o \in B} \mathrm{d}\{\boldsymbol{r}^o(x) \times \boldsymbol{p}^o(x^o)\} = \int_{x^o \in B} \mathrm{d}\{\boldsymbol{r}^o(x) \times m(x^o)\boldsymbol{v}^o(x^o)\}$$

- One more step:

$$\boldsymbol{L}^o = \int_{x^o \in B} -\mathrm{d}\{m^o(x^o)[\boldsymbol{r}^o(x^o)][\boldsymbol{r}^o(x^o)]\boldsymbol{\omega}^o\} = \left(\int_{x^o \in B} -\mathrm{d}\{m(x^o)[\boldsymbol{r}^o(x^o)][\boldsymbol{r}^o(x^o)]\}\right)\boldsymbol{\omega}^o$$

Angular Momentum and Inertia of Rigid Body

- Particularly, if we choose the origin of the observer's frame O at the *center of mass*:

$$\boldsymbol{L}^b = \boldsymbol{I}^b \boldsymbol{\omega}^b \qquad \text{(body angular momentum)}$$

where

$$\boldsymbol{I}^b = \int_{x^b \in B} -\mathrm{d}V \{\rho(x^b)[\boldsymbol{r}^b(x^b)][\boldsymbol{r}^b(x^b)]\} \qquad \text{(body inertia)}$$

and center of mass

$$x^o_{cm} = \frac{\int \boldsymbol{r}^o \rho d\mathbf{V}}{\int \rho d\mathbf{V}} \qquad \text{(center of mass)}$$

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- Since $\mathcal{F}_{b(t)}$ is tightly binded to the body, $\boldsymbol{I}^b$ does not change w.r.t. time and is a basic property of the object.

Computation of Rigid Body Inertia

$$\boldsymbol{I}^b = \int_{x^b \in B} -\mathrm{d}V \rho(\boldsymbol{x}^b)[\boldsymbol{r}^b(\boldsymbol{x}^b)][\boldsymbol{r}^b(\boldsymbol{x}^b)]$$
$$= \begin{bmatrix} \int \rho(r_y^2 + r_z^2) d\boldsymbol{V} & -\int \rho r_x r_y d\boldsymbol{V} & -\int \rho r_x r_z d\boldsymbol{V} \\ -\int \rho r_x r_y d\boldsymbol{V} & \int \rho(r_x^2 + r_z^2) d\boldsymbol{V} & -\int \rho r_y r_z d\boldsymbol{V} \\ -\int \rho r_x r_z d\boldsymbol{V} & -\int \rho r_y r_z d\boldsymbol{V} & \int \rho(r_y^2 + r_x^2) d\boldsymbol{V} \end{bmatrix}$$

- Given uniform density, the integral can be computed analytically for watertight meshes

Fast Inertia Computation

- Divergence theorem! Let $\boldsymbol{F}: \mathbb{R}^3 \to \mathbb{R}^3$, $\int_V \nabla \cdot \boldsymbol{F} dV = \oint_S \boldsymbol{F} \cdot \boldsymbol{n} dS$
- An example: a term of $\boldsymbol{I}$, which is $-\rho \int_{\boldsymbol{V}} \boldsymbol{r}_y \boldsymbol{r}_z d\boldsymbol{V}$
  Let $\boldsymbol{F}(\boldsymbol{r}_x, \boldsymbol{r}_y, \boldsymbol{r}_z) = \begin{bmatrix} \boldsymbol{r}_x \boldsymbol{r}_y \boldsymbol{r}_z & 0 & 0 \end{bmatrix}^T$

$$\nabla \cdot \boldsymbol{F} = \boldsymbol{r}_y \boldsymbol{r}_z$$

  The integral becomes

$$\oint_S \boldsymbol{F} \cdot \boldsymbol{n} dS$$

  Now we only need to do 2D integral over triangles.

Read by yourself

Mass Properties

- Observe $\boldsymbol{I}^b = \int_{\boldsymbol{r}^b \in B} -\mathrm{d}\boldsymbol{V} \rho(\boldsymbol{r}^b)[\boldsymbol{r}^b][\boldsymbol{r}^b]$
- Although the origin is always at the center of mass, if we change the orientation of body frame axes, $\boldsymbol{I}^b$ may change!
- How will it change, then?

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- If we rotate the frame by R^T and obtain a new frame b', then

$$\boldsymbol{I}^{b'} = \int_{\boldsymbol{r}^b \in B} -\mathrm{d}\boldsymbol{V}\rho(\boldsymbol{r}^b)[R\boldsymbol{r}^b][R\boldsymbol{r}^b] = \int_{\boldsymbol{r}^b \in B} -\mathrm{d}\boldsymbol{V}\rho(\boldsymbol{r}^b)R[\boldsymbol{r}^b][\boldsymbol{r}^b]R^T = R\boldsymbol{I}^b R^T$$

where the second equality follows $[Rr] = R[r]R^T$ for $R \in \mathbb{SO}^3$. Again, **similarity transformation**!

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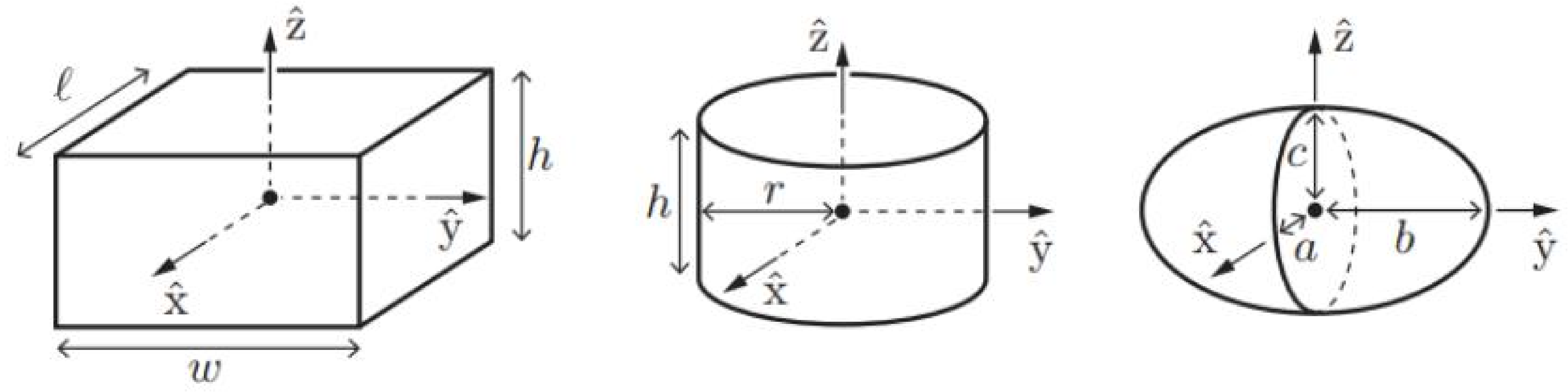
Conclusion: Rigid-transformation does not change the eigen properties of $\boldsymbol{I}^b$

Mass Properties

- $\boldsymbol{I}^b$ admits eigen-decomposition
  - The eigenvectors are called **principal axes**.
  - The eigenvalues (I_1, I_2, I_3) are called the **principal moments of inertia**.



rectangular parallelepiped: volume $= abc$, $\mathcal{I}_{xx} = \mathfrak{m}(w^2+h^2)/12$, $\mathcal{I}_{yy} = \mathfrak{m}(\ell^2+h^2)/12$, $\mathcal{I}_{zz} = \mathfrak{m}(\ell^2+w^2)/12$

circular cylinder: volume $= \pi r^2 h$, $\mathcal{I}_{xx} = \mathfrak{m}(3r^2+h^2)/12$, $\mathcal{I}_{yy} = \mathfrak{m}(3r^2+h^2)/12$, $\mathcal{I}_{zz} = \mathfrak{m}r^2/2$

ellipsoid: volume $= 4\pi abc/3$, $\mathcal{I}_{xx} = \mathfrak{m}(b^2+c^2)/5$, $\mathcal{I}_{yy} = \mathfrak{m}(a^2+c^2)/5$, $\mathcal{I}_{zz} = \mathfrak{m}(a^2+b^2)/5$

(from: https://www.cnblogs.com/21207-iHome/p/7765508.html)

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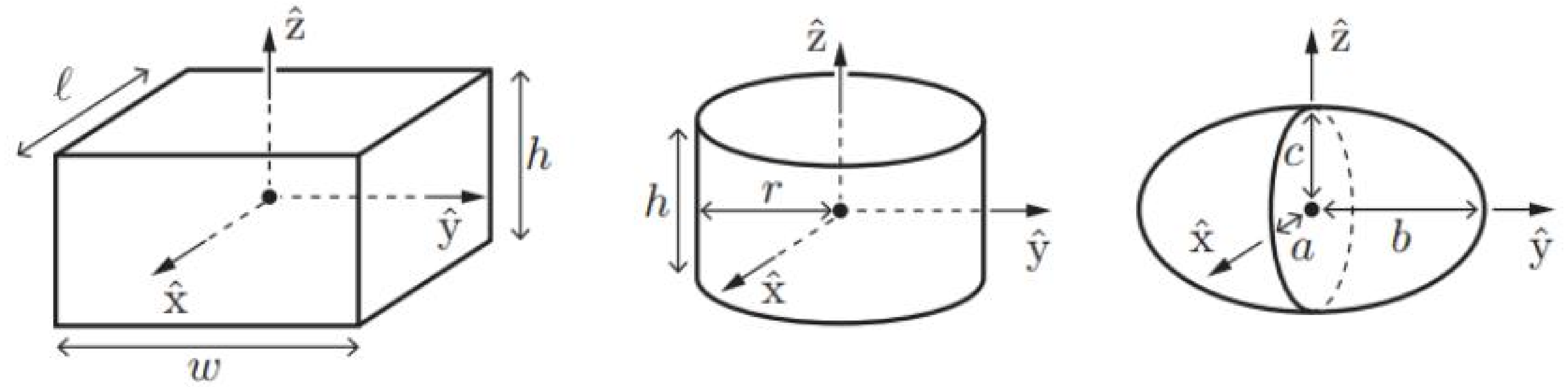

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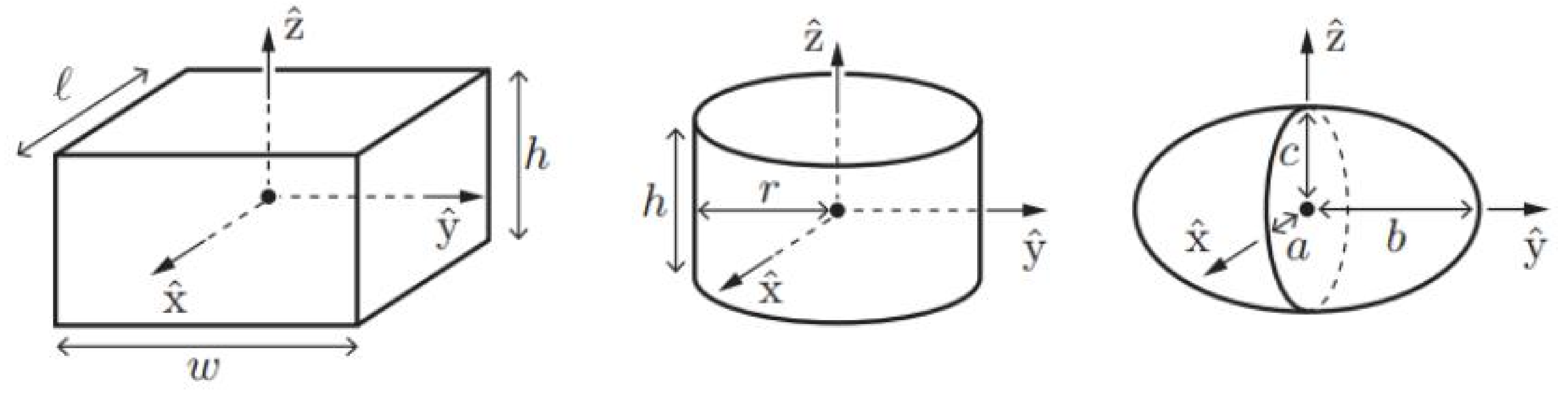
- x_{cm} and principal axes form a **body frame** that is intrinsic to the object
- **x_{cm}, principal axes, m, I_1, I_2, I_3 fully determine the behavior of a rigid body under external forces**



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Quiz

Suppose an object is moving in space (rotating and translating), which of the following quantities may change during the motion. (Assume all quantities are measured w.r.t. a static spatial frame)

- A. principal axes (observed from the spatial frame)
- B. x_{cm} (observed from the spatial frame)
- C. m
- D. I_1, I_2, I_3

Torque

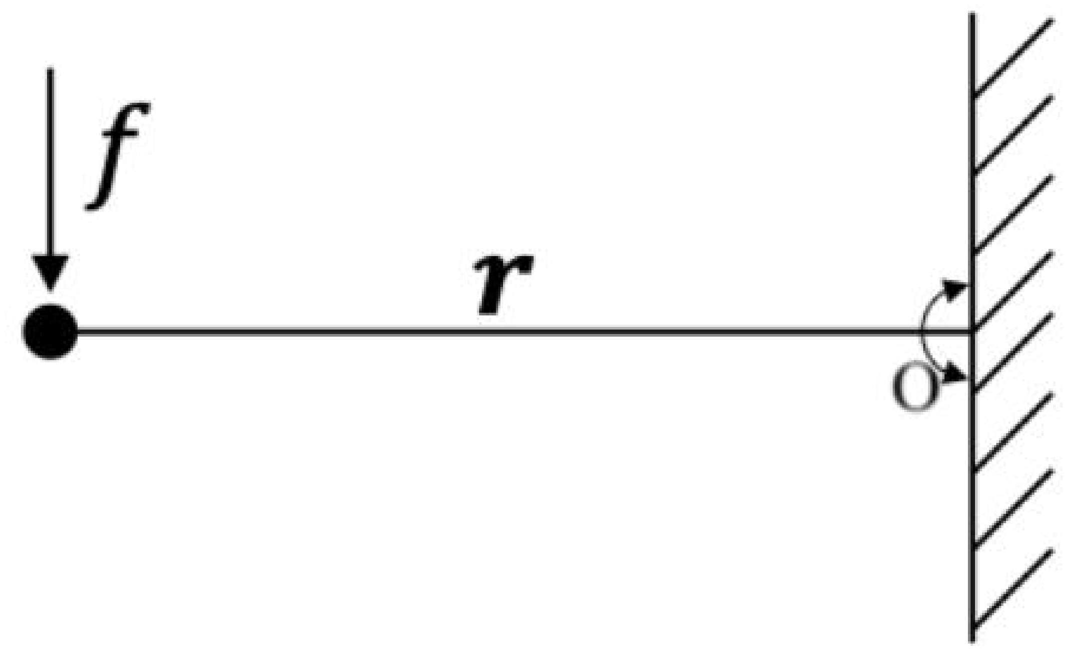
Torque

- Consider a simple example on the right.
- Recall how we define the angular momentum $\boldsymbol{L}^o$ for *point mass*:

$$\boldsymbol{L}^o = \boldsymbol{r}^o \times \boldsymbol{p}^o = \boldsymbol{r}^o \times (m\boldsymbol{v}^o) \tag{1}$$

- We have also derived that

$$\boldsymbol{L}^o = \boldsymbol{I}^o \boldsymbol{\omega}^o \tag{2}$$



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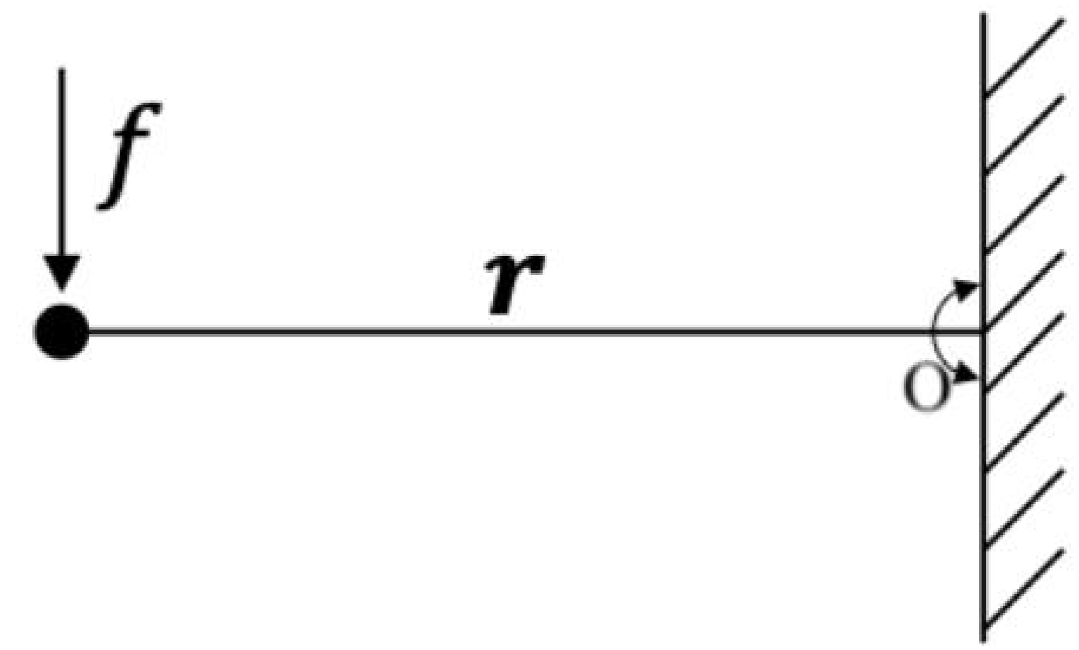
- We use the time derivative of $\boldsymbol{L}^o$ to define **torque**, denoted by $\boldsymbol{\tau}^o$
  1. By (1), $\boldsymbol{\tau}^o = \dot{\boldsymbol{L}}^o = \dot{\boldsymbol{r}}^o \times (m\boldsymbol{v}^o) + \boldsymbol{r}^o \times \boldsymbol{f}^o = \boldsymbol{r}^o \times \boldsymbol{f}^o$, because $\dot{\boldsymbol{r}}^o \parallel \boldsymbol{v}^o$
  2. By (2), $\boldsymbol{\tau} = \frac{\mathrm{d}(\boldsymbol{I}^o\boldsymbol{\omega}^o)}{\mathrm{d}t}$



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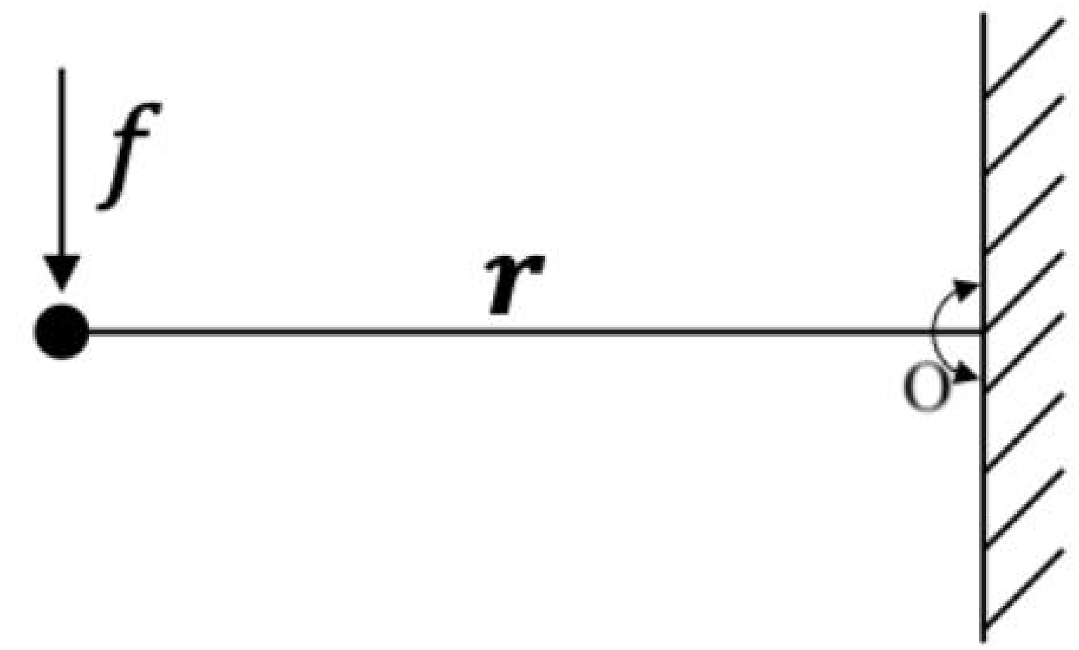
  2. By (2), $\boldsymbol{\tau} = \frac{\mathrm{d}(\boldsymbol{I}^o\boldsymbol{\omega}^o)}{\mathrm{d}t}$
- Torque describes how fast the angular momentum changes (from 2). Torque also relates the change with the cause: an external power input (from 1).



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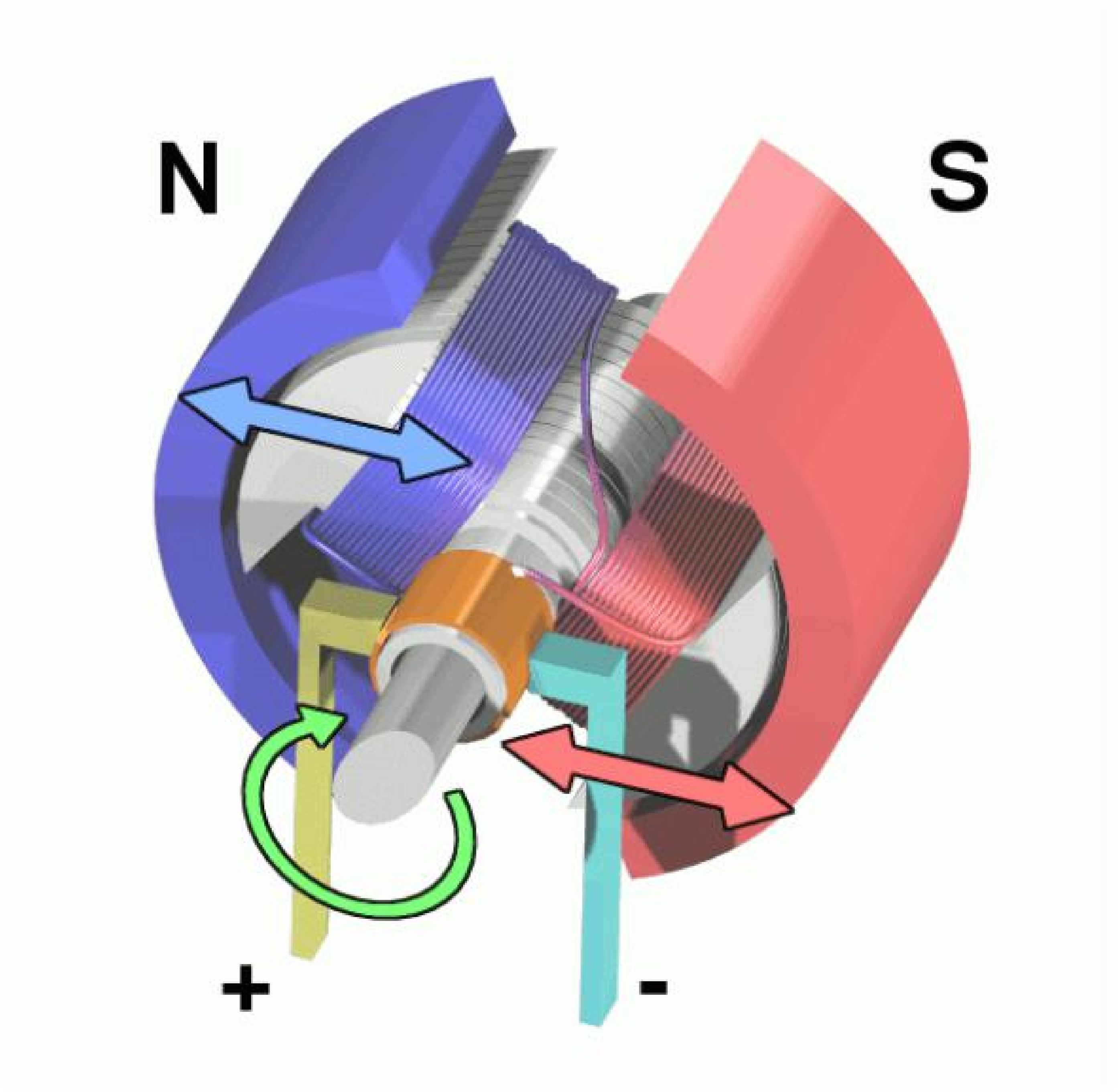
Torque

- In the example of point mass, we showed the equality of two torque computations
  - the change rate of $\boldsymbol{L}$
  - the input to the system
- For general rigid-body systems, the equality is also true
- For robotic manipulation, torque is the most common description of system input



https://en.wikipedia.org/wiki/Electric_motor

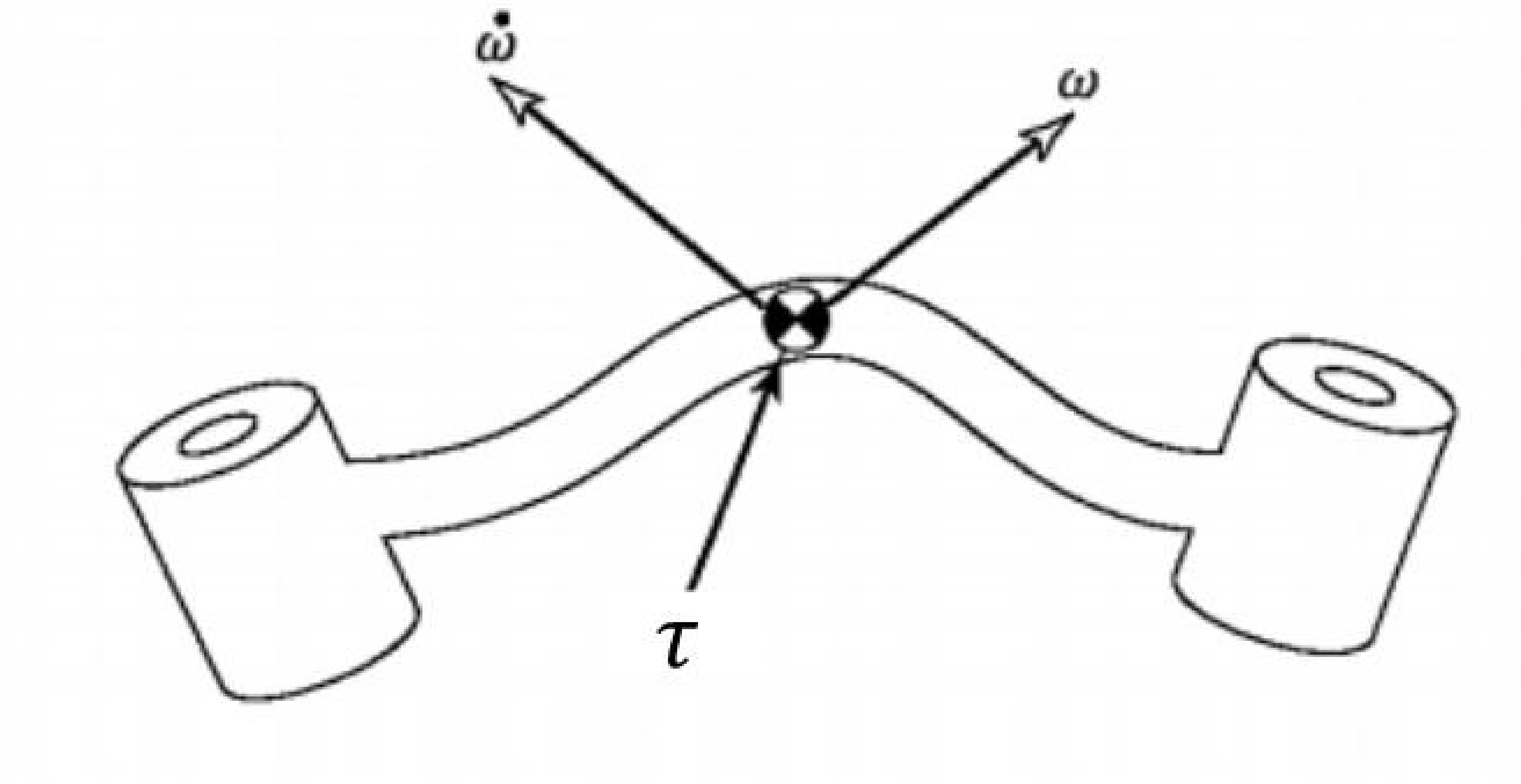
Euler Equation

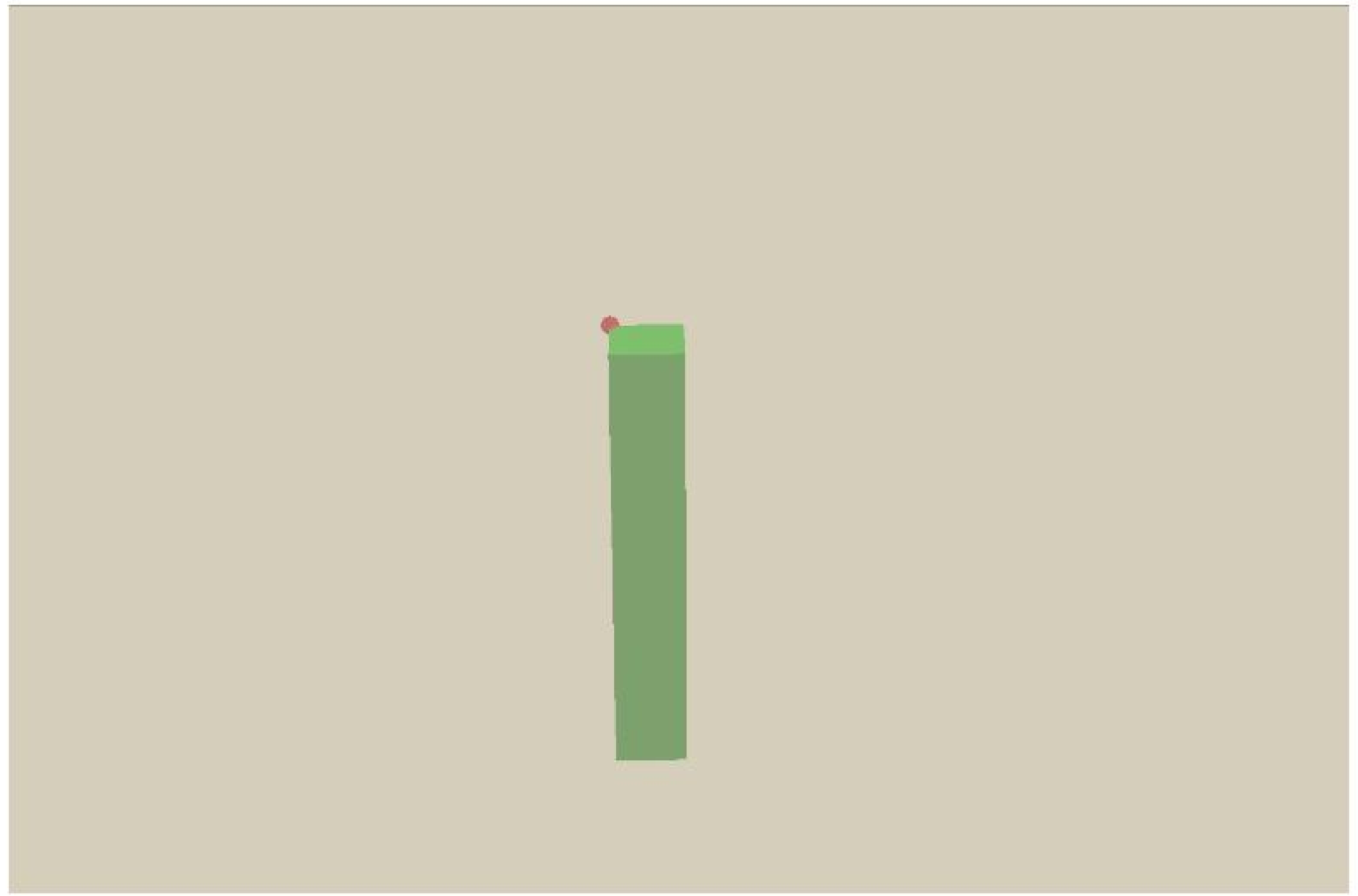
$$\boldsymbol{\tau}^b = \frac{\mathrm{d}\boldsymbol{L}^b}{\mathrm{d}t} = \frac{\mathrm{d}(\boldsymbol{I}^b\boldsymbol{\omega}^b)}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{I}^b}{\mathrm{d}t}\boldsymbol{\omega}^b + \boldsymbol{I}^b\frac{\mathrm{d}\boldsymbol{\omega}^b}{\mathrm{d}t} = \boldsymbol{\omega}^b \times \boldsymbol{I}^b\boldsymbol{\omega}^b + \boldsymbol{I}^b\dot{\boldsymbol{\omega}}^b$$

- We used $\dot{\boldsymbol{I}}^b = [\boldsymbol{\omega}]\boldsymbol{I}^b$ without proof
- An observation (example): Even if there is no torque input, if the object has a non-zero angular velocity $\boldsymbol{\omega}^b$, then it may still have an angular acceleration $\dot{\boldsymbol{\omega}}^b$
  - When $\boldsymbol{\omega}^b \nparallel \boldsymbol{I}^b\boldsymbol{\omega}^b$, i.e., $\boldsymbol{\omega}^b$ is not along the eigenvector of $\boldsymbol{I}^b$, $\boldsymbol{\omega}^b$ will **NOT** keep unchanged
  - $\boldsymbol{\omega}^b$ will not converge ($\dot{\omega}^b$ will never be zero). Its trajectory will form a periodic curve
  - $\boldsymbol{L}^b = \boldsymbol{I}^b\boldsymbol{\omega}^b$ is conserved

Euler Equation

$$\boldsymbol{\tau}^b = \boldsymbol{\omega}^b \times \boldsymbol{I}^b \boldsymbol{\omega}^b + \boldsymbol{I}^b \dot{\boldsymbol{\omega}}^b \qquad \text{(angular motion)}$$





A numerical experiment in SAPIEN for $\boldsymbol{\tau}^b = 0$

(this is illustrative and there are numerical errors)

Kinetic Energy

Kinetic Energy for Point Mass

- If a point mass m is moving with velocity $\boldsymbol{v}^o_{s(t)\to b(t)}$ ($s(t)$ is an inertia frame and the origin of $b(t)$ is the point), then the kinetic energy of the point mass is

$$T_{s(t)\to b(t)} = \frac{1}{2} m \|\boldsymbol{v}^o_{s(t)\to b(t)}\|^2 \qquad \text{(kinetic energy)}$$

Observer-Independence of Kinetic Energy

- Note that we omit observer's frame when describing kinetic energy, because it is independent of the observer's frame.
- We prove by showing that $\|\boldsymbol{v}^{o_1}_{s(t)\to b(t)}\| = \|\boldsymbol{v}^{o_2}_{s(t)\to b(t)}\|$

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- We prove by showing that $\|\boldsymbol{v}^{o_1}_{s(t)\to b(t)}\| = \|\boldsymbol{v}^{o_2}_{s(t)\to b(t)}\|$
- Proof:

$$\begin{aligned}\boldsymbol{v}^{o_1}_{s(t)\to b(t)} &= \boldsymbol{v}^{o_1}_{b(t)} - \boldsymbol{v}^{o_1}_{s(t)} = [\boldsymbol{\xi}^{o_1}_{b(t)} - \boldsymbol{\xi}^{o_1}_{s(t)}]p^{o_1} = T^{-1}_{o_2\to o_1}[\boldsymbol{\xi}^{o_2}_{b(t)} - \boldsymbol{\xi}^{o_2}_{s(t)}]T_{o_2\to o_1}p^{o_1} \\ &= T^{-1}_{o_2\to o_1}[\boldsymbol{\xi}^{o_2}_{b(t)} - \boldsymbol{\xi}^{o_2}_{s(t)}]p^{o_2} = T^{-1}_{o_2\to o_1}(\boldsymbol{v}^{o_2}_{b(t)} - \boldsymbol{v}^{o_2}_{s(t)}) = T^{-1}_{o_2\to o_1}\boldsymbol{v}^{o_2}_{s(t)\to b(t)} \\ &= \begin{bmatrix} R^T_{o_2\to o_1} & -R^T_{o_2\to o_1}t_{o_2\to o_1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}^{o_2}_{s(t)\to b(t)}|_{3\times 1} \\ 0 \end{bmatrix} = \begin{bmatrix} R_{o_1\to o_2}\boldsymbol{v}^{o_2}_{s(t)\to b(t)}|_{3\times 1} \\ 0 \end{bmatrix}\end{aligned}$$

$$\therefore \|\boldsymbol{v}^{o_1}_{s(t)\to b(t)}\| = \|\boldsymbol{v}^{o_2}_{s(t)\to b(t)}\|$$

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\boldsymbol{v}^{o_1}_{s(t)\to b(t)} &= \boldsymbol{v}^{o_1}_{b(t)} - \boldsymbol{v}^{o_1}_{s(t)} = [\boldsymbol{\xi}^{o_1}_{b(t)} - \boldsymbol{\xi}^{o_1}_{s(t)}]p^{o_1} = T^{-1}_{o_2\to o_1}[\boldsymbol{\xi}^{o_2}_{b(t)} - \boldsymbol{\xi}^{o_2}_{s(t)}]T_{o_2\to o_1}p^{o_1} \\
&= T^{-1}_{o_2\to o_1}[\boldsymbol{\xi}^{o_2}_{b(t)} - \boldsymbol{\xi}^{o_2}_{s(t)}]p^{o_2} = T^{-1}_{o_2\to o_1}(\boldsymbol{v}^{o_2}_{b(t)} - \boldsymbol{v}^{o_2}_{s(t)}) = T^{-1}_{o_2\to o_1}\boldsymbol{v}^{o_2}_{s(t)\to b(t)} \\
&= \begin{bmatrix} R^T_{o_2\to o_1} & -R^T_{o_2\to o_1}t_{o_2\to o_1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}^{o_2}_{s(t)\to b(t)}|_{3\times 1} \\ 0 \end{bmatrix} = \begin{bmatrix} R_{o_1\to o_2}\boldsymbol{v}^{o_2}_{s(t)\to b(t)}|_{3\times 1} \\ 0 \end{bmatrix}
\end{aligned}
$$

$$\therefore \|\boldsymbol{v}^{o_1}_{s(t)\to b(t)}\| = \|\boldsymbol{v}^{o_2}_{s(t)\to b(t)}\|$$

- We have also derived

$$\boldsymbol{v}^{o_1}_{s(t)\to b(t)} = R_{o_1\to o_2}\boldsymbol{v}^{o_2}_{s(t)\to b(t)} \text{ for } \boldsymbol{v} \in \mathbb{R}^3 \quad \text{(change of frame for velocities)}$$

Kinetic Energy for Rigid Body

- Integrate kinetic energy of every point mass over the body
- We choose the body frame $\mathcal{F}_{b(t)}$ to start the derivation. Using the independence of observer's frame, we derive the formula to compute the energy in other frames.
- *The origin of our body frame is always at the center of mass of the body*

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- *The origin of our body frame is always at the center of mass of the body*
- The velocity of a body point $\boldsymbol{r}^{b(t)}$ is

$$\begin{aligned}\boldsymbol{v}_{s(t)\to b(t)}^{b(t)} &= [\boldsymbol{\xi}_{s(t)\to b(t)}^{b(t)}]\boldsymbol{r}^{b(t)} \\ &= \begin{bmatrix} [\boldsymbol{\omega}_{s(t)\to b(t)}^{b(t)}] & \dot{\boldsymbol{t}}_{s(t)\to b(t)}^{b(t)} \\ 0 & 0 \end{bmatrix} \boldsymbol{r}^{b(t)}\end{aligned}$$

Kinetic Energy for Rigid Body

Therefore,

$$\begin{aligned}
T_{s(t)\to b(t)} &= \int_{\boldsymbol{r}^b \in B} \frac{1}{2}\rho(x)\mathrm{d}V \|\boldsymbol{v}^{b(t)}_{s(t)\to b(t)}\|^2 = \int_{\boldsymbol{r}^b \in B} \frac{1}{2}\rho(x)\mathrm{d}V \|[\boldsymbol{\omega}^{b(t)}_{s(t)\to b(t)}]\boldsymbol{r}^{b(t)} + \dot{\boldsymbol{t}}^{b(t)}_{s(t)\to b(t)}\|^2 \\
&= (\text{some derivations using } [\boldsymbol{\omega}]\boldsymbol{r} = -[\boldsymbol{r}]\boldsymbol{\omega}) \\
&= \frac{1}{2}m\|\dot{\boldsymbol{t}}^{b(t)}_{s(t)\to b(t)}\|^2 + \frac{1}{2}(\boldsymbol{\omega}^{b(t)}_{s(t)\to b(t)})^T \boldsymbol{I}^b \boldsymbol{\omega}^{b(t)}_{s(t)\to b(t)} \\
&= \frac{1}{2}(\boldsymbol{\xi}^{b(t)}_{s(t)\to b(t)})^T \mathfrak{M}^b \boldsymbol{\xi}^{b(t)}_{s(t)\to b(t)}
\end{aligned}$$

where

$$\mathfrak{M}^b = \begin{bmatrix} m\mathrm{Id}_{3\times 3} & 0 \\ 0 & \boldsymbol{I}^b \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

Kinetic Energy for Rigid Body

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$$\frac{1}{2}(\boldsymbol{\xi}^1)^T \mathfrak{M}^1 \boldsymbol{\xi}^1 = \frac{1}{2}(\boldsymbol{\xi}^2)^T \mathfrak{M}^2 \boldsymbol{\xi}^2$$

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  - Recall that $\boldsymbol{\xi}^1 = [\mathrm{Ad}_{1\to 2}]\boldsymbol{\xi}^2$, and we conclude that

$$\mathfrak{M}^2 = [\mathrm{Ad}_{1\to 2}]^{\mathrm{T}}\mathfrak{M}^1[\mathrm{Ad}_{1\to 2}] \qquad \text{(change of frame)}$$

Change of Observer's Frame for Rotational Inertia Matrix

- A side-product of introducing $\mathfrak{M}^o$ is that we can compute the inertia matrix in other frames conveniently
- We derived the change of frame formula for different body frames. **What about frame change between general observer's frames?**

Change of Observer's Frame for Rotational Inertia Matrix

- A side-product of introducing $\mathfrak{M}^o$ is that we can compute the inertia matrix in other frames conveniently
- We derived the change of frame formula for different body frames. **What about frame change between general observer's frames?**
- One can verify that,

$$\textbf{the bottom-right } 3 \times 3 \textbf{ block of } \mathfrak{M}^2 \textbf{ is the rotational inertial matrix in } \mathcal{F}_2$$

Change of Frame for Various Quantities

Motivating Example: Grasp Problem

- Consider the right grasp problem

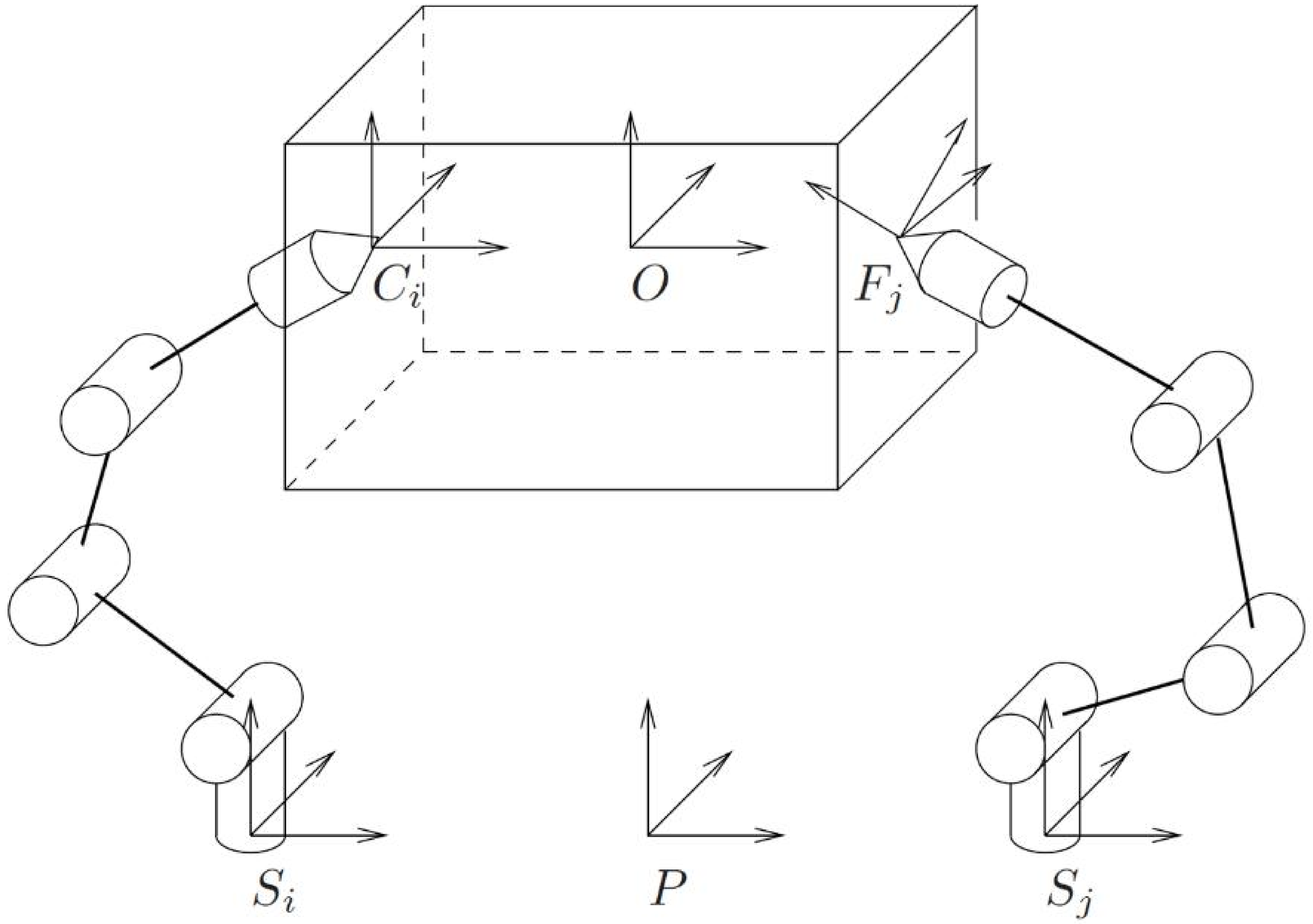
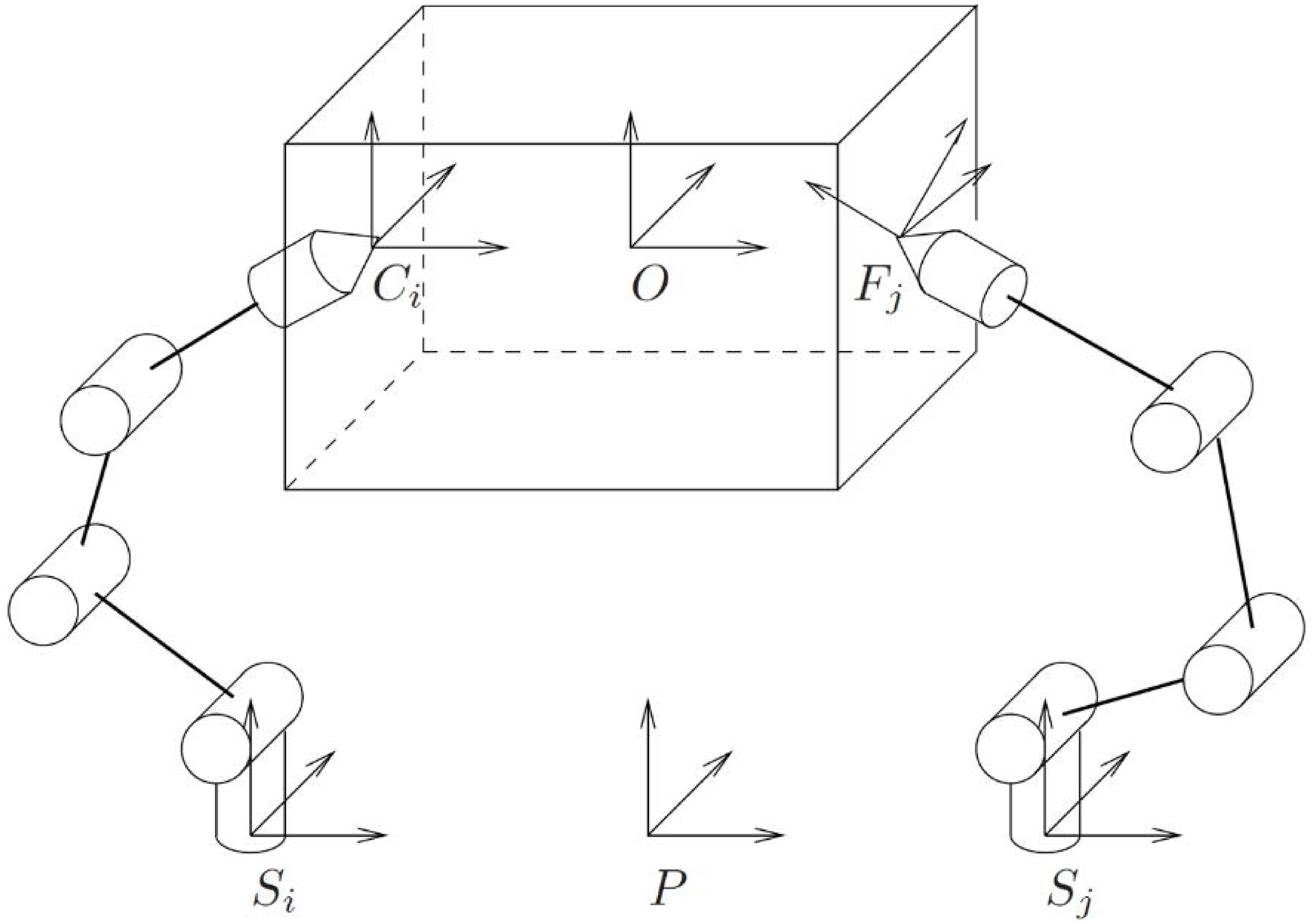


Figure 5.14: Grasp coordinate frames.

Motivating Example: Grasp Problem

- Consider the right grasp problem
  - Assume that we are grasping this box using two arms



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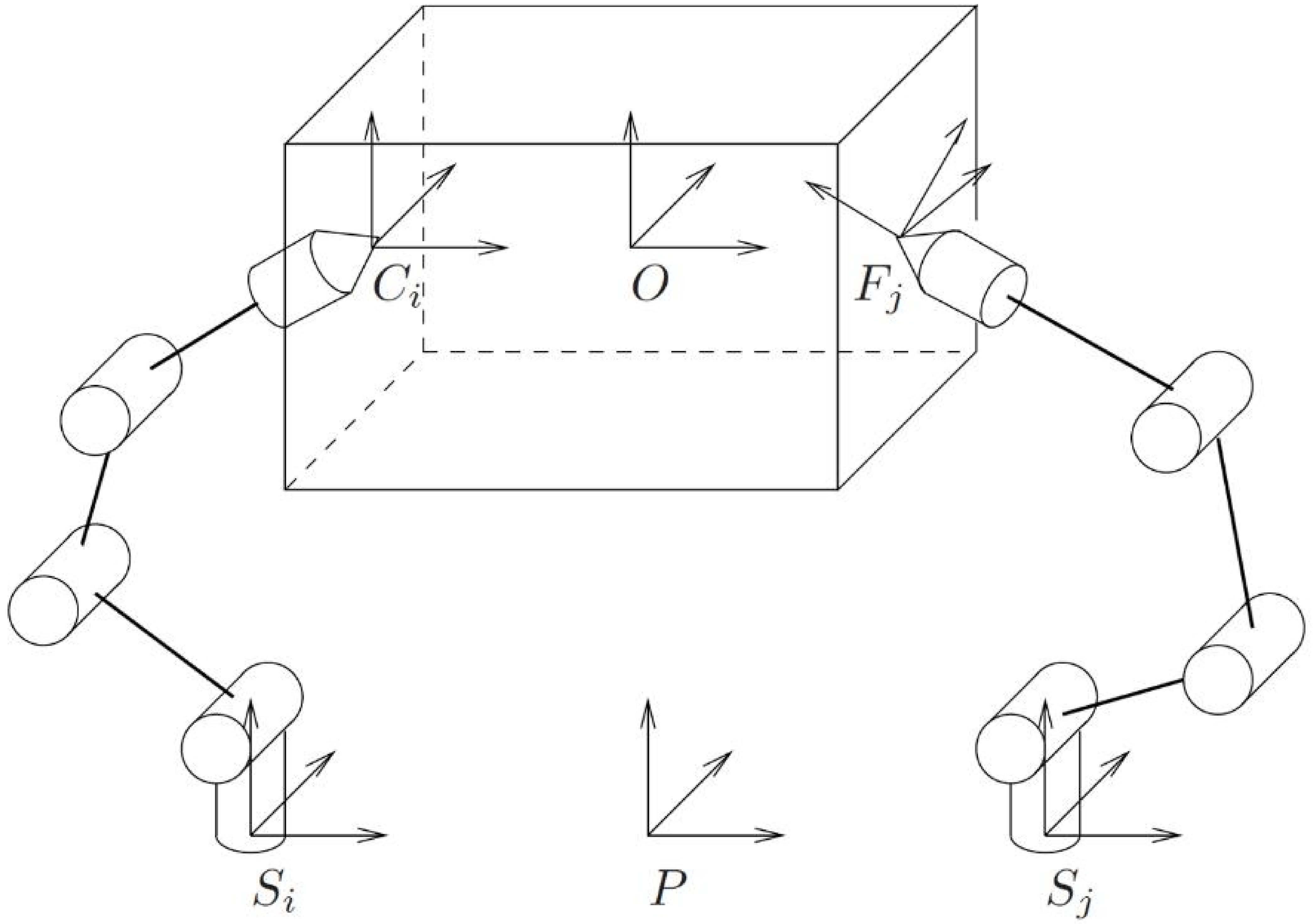

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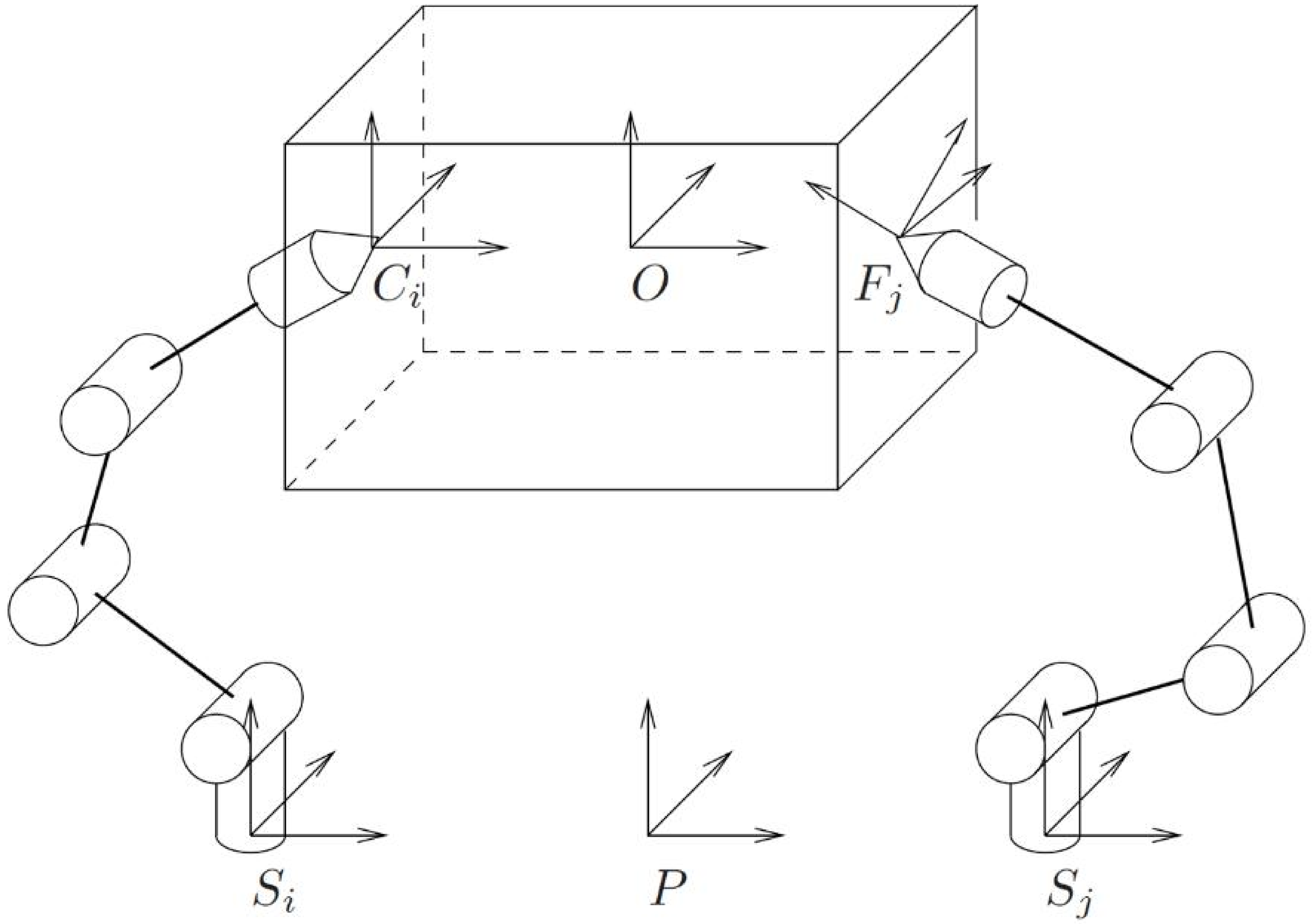

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Motivating Example: Grasp Problem

- Consider the right grasp problem
  - Assume that we are grasping this box using two arms
  - We apply torques at each joint through the installed motors
  - These torques will be passed to the tips of the fingers.
  - The contact area will create certain force and torque at the contact point
    - force: pressure and friction
    - torque: e.g., anti-twisting friction force caused by the area contact

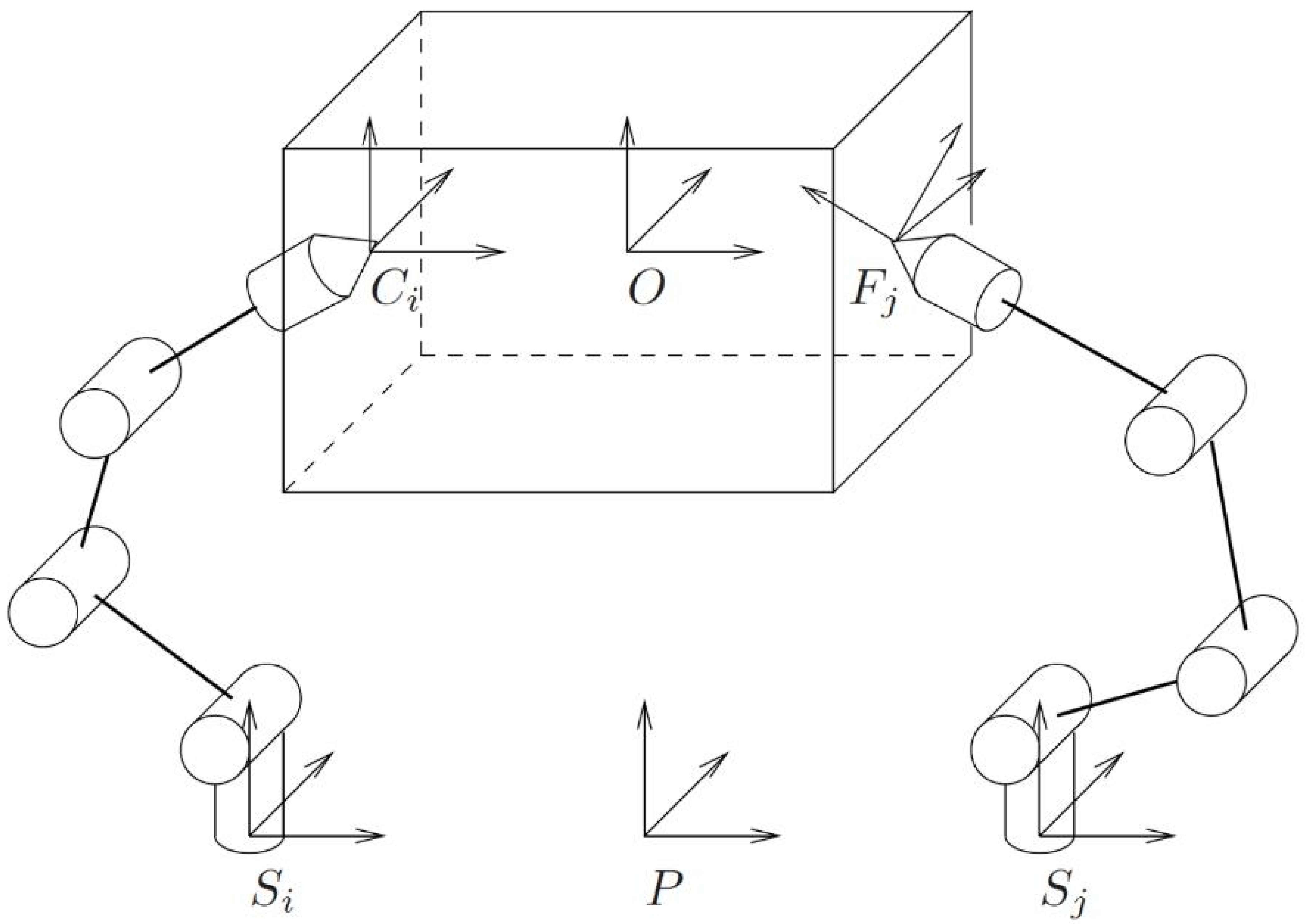


Figure 5.14: Grasp coordinate frames.

Contact Coordinate Frame

- We build a **contact frame** C_i at each contact point
- The z-axis of the frame points inward along *surface normal*
- When recording force and torque at the contact point, it is natural to set C_i as the *observer's frame*, i.e.,

$$\boldsymbol{F}^{C_i} = \begin{bmatrix} \boldsymbol{f}^{C_i} \\ \boldsymbol{\tau}^{C_i} \end{bmatrix}$$

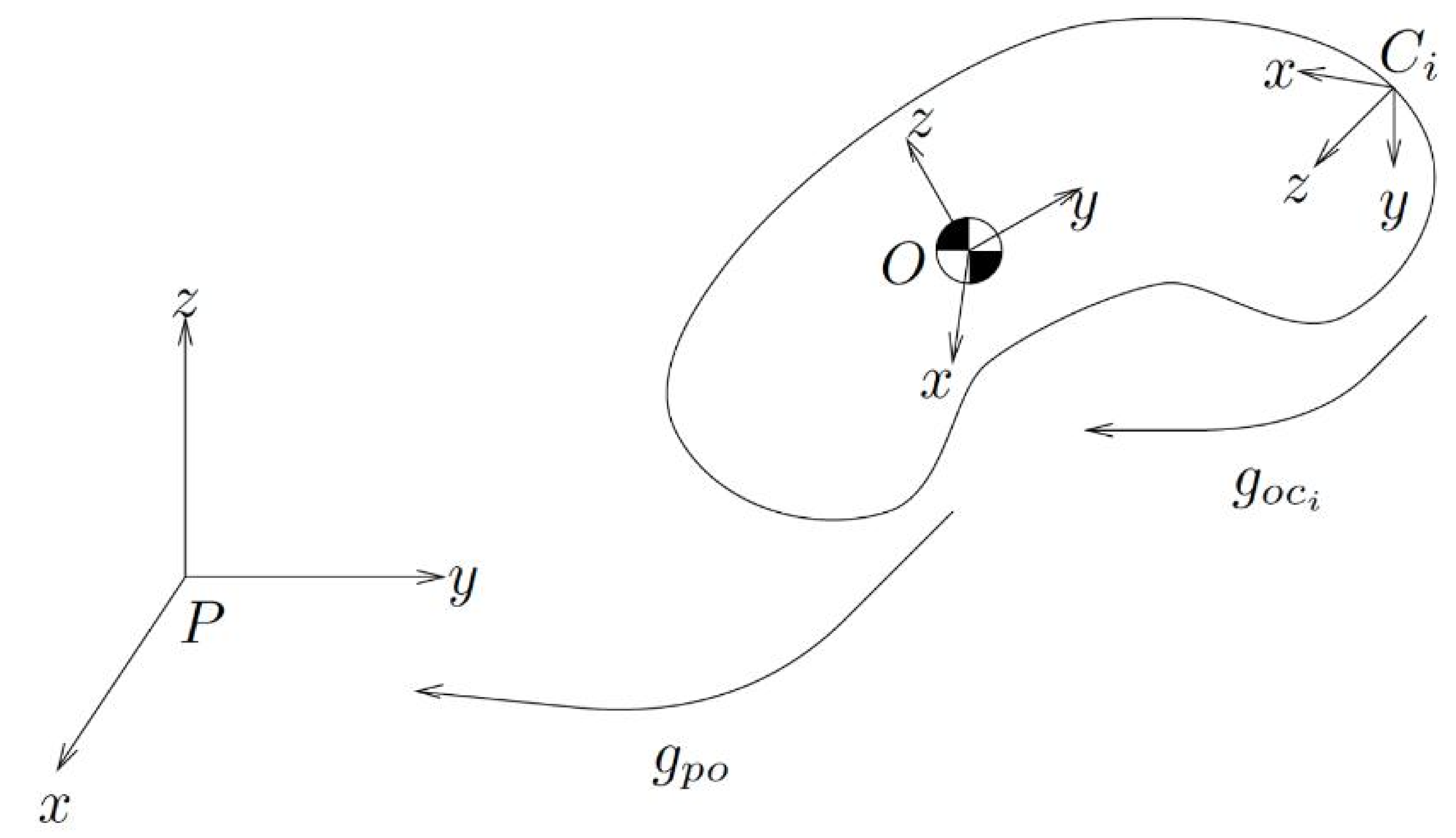


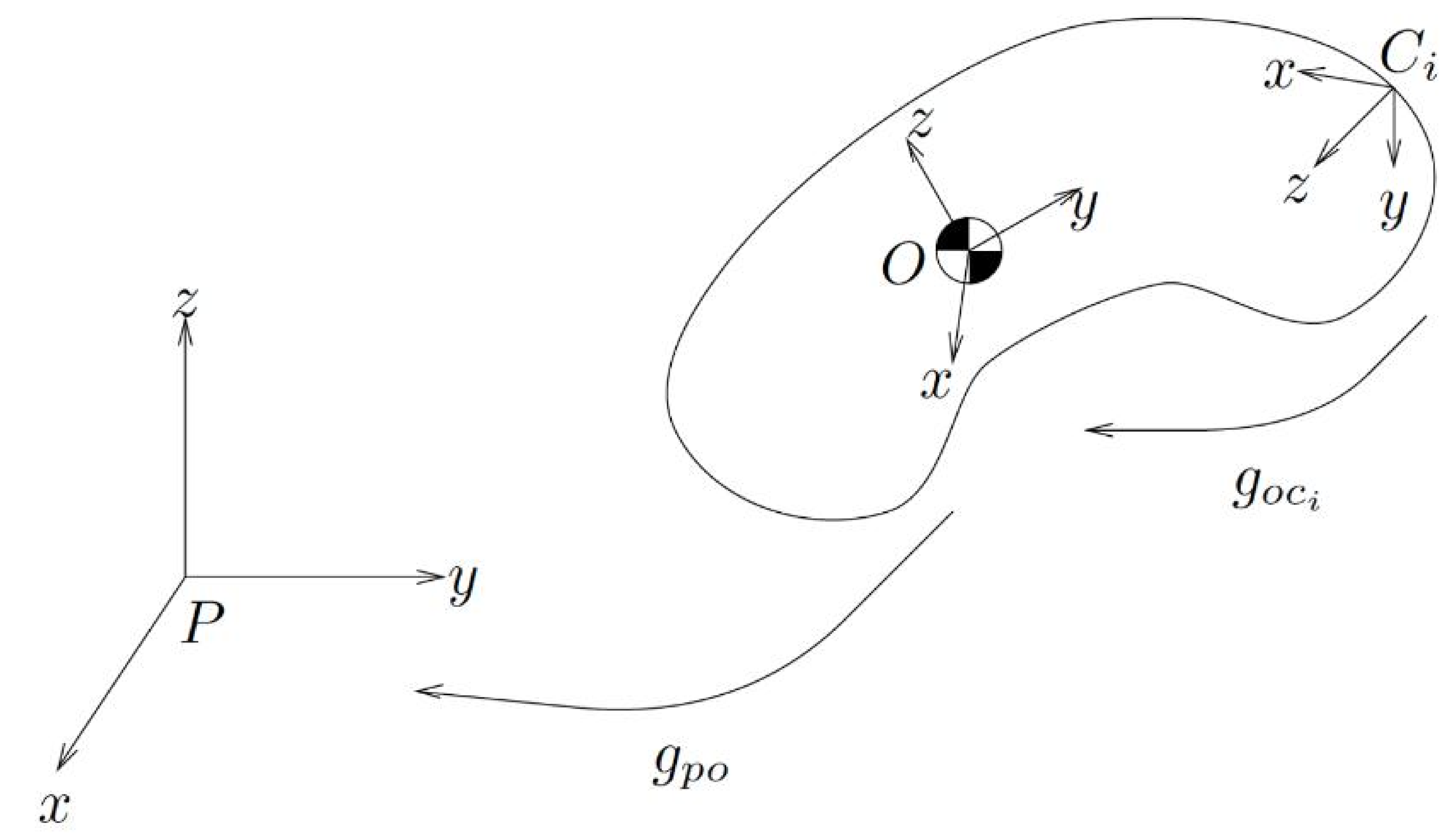
Figure 5.2: Coordinate frames for contact and object forces.

Some Kinds of Contact Forces

Contact Type	Frictionless point contact	Point contact with friction	Soft-finger
$\boldsymbol{F}^C$	$\begin{bmatrix} 0 \\ 0 \\ f_z \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} f_x \\ f_y \\ f_z \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} f_x \\ f_y \\ f_z \\ 0 \\ 0 \\ \tau_z \end{bmatrix}$

Adding Forces and Torques

- Suppose we have calculated $\boldsymbol{F}^{C_i}$ at each contact (will learn later)
- What is the combined force and torque?



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- What is the combined force and torque?
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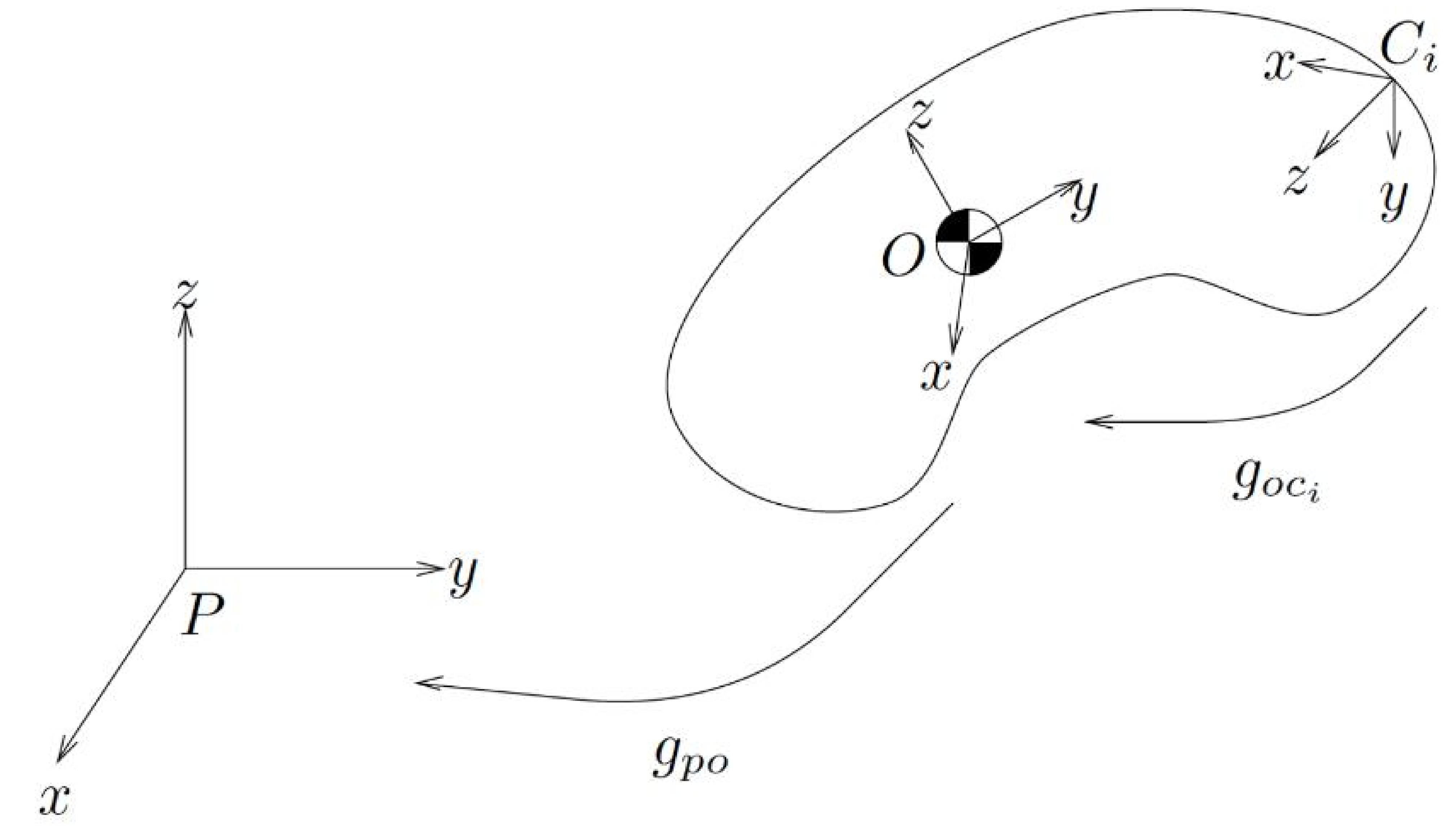

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Adding Forces and Torques

- Suppose we have calculated $\boldsymbol{F}^{C_i}$ at each contact (will learn later)
- What is the combined force and torque?
- We **cannot** directly add forces and torques recorded using different observer frames
- However, we can change all to the same frame (e.g., body frame) and add together!

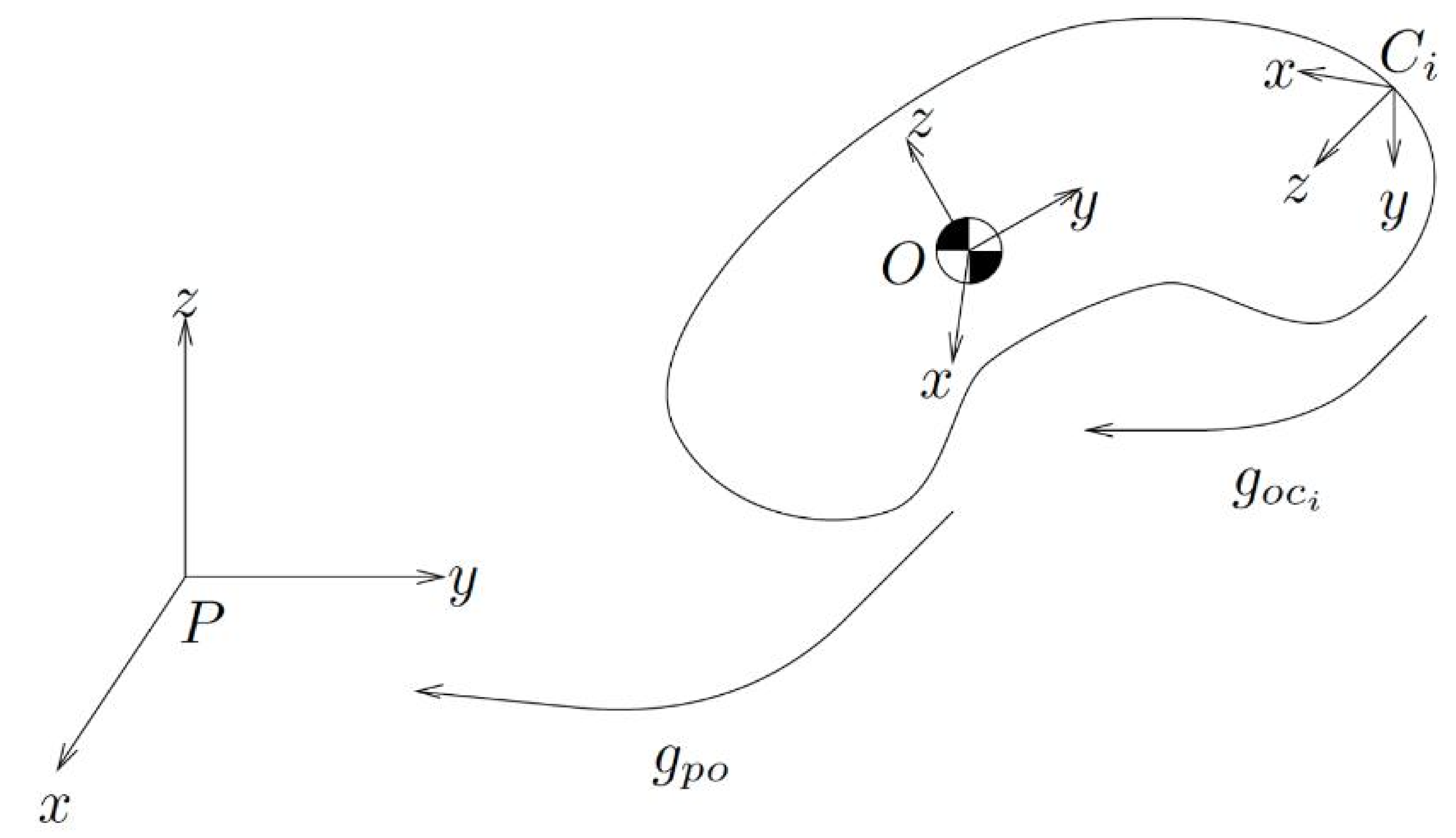

Figure 5.2: Coordinate frames for contact and object forces.

Change of Observer's Frame for Force and Torque

Consider the question of changing the observer's frame for force and torque

- We would relate
  - $\boldsymbol{f}^1$ and $\boldsymbol{f}^2$
  - $\boldsymbol{\tau}^1 = \boldsymbol{r}^1 \times \boldsymbol{f}^1$ and $\boldsymbol{\tau}^2 = \boldsymbol{r}^2 \times \boldsymbol{f}^2$
- Note that

$$\begin{aligned} \boldsymbol{r}^2 &= R_{2\to1}\boldsymbol{r}^1 + \boldsymbol{t}_{2\to1} \\ \boldsymbol{f}^2 &= R_{2\to1}\boldsymbol{f}^1 \end{aligned}$$

- Plug in the definition, and we derive that

$$\begin{bmatrix} \boldsymbol{f}^2 \\ \boldsymbol{\tau}^2 \end{bmatrix} = \begin{bmatrix} R_{2\to1} & 0 \\ [\boldsymbol{t}_{2\to1}]R_{2\to1} & R_{2\to1} \end{bmatrix} \begin{bmatrix} \boldsymbol{f}^1 \\ \boldsymbol{\tau}^1 \end{bmatrix} = (\mathrm{Ad}_{1\to2})^{\mathrm{T}} \begin{bmatrix} \boldsymbol{f}^1 \\ \boldsymbol{\tau}^1 \end{bmatrix}$$

Change of Observer's Frame for Force and Torque

- Define $\boldsymbol{F}^o = \begin{bmatrix} \boldsymbol{f}^o \\ \boldsymbol{\tau}^o \end{bmatrix}$, then formula for change of frame is:

$$\boldsymbol{F}^2 = (\mathrm{Ad}_{1\to 2})^{\mathrm{T}} \boldsymbol{F}^1 \qquad \text{(change of frame)}$$

- Using definitions and frame change equations, it is easy to verify that the following equation to compute the **power** of the system input (change rate of kinetic energy):

$$(\boldsymbol{F}^b)^T \boldsymbol{\xi}^b = (\boldsymbol{F}^o)^T \boldsymbol{\xi}^o = \frac{\mathrm{d}T}{\mathrm{d}t} \qquad \text{(system input power)}$$

Solution to Adding Forces and Torques

$$\boldsymbol{F}^{b} = \sum_{i=1}^{k} [\mathrm{Ad}_{\mathrm{C_i \to b}}]^{\mathrm{T}} \boldsymbol{F}^{\mathrm{C_i}}$$

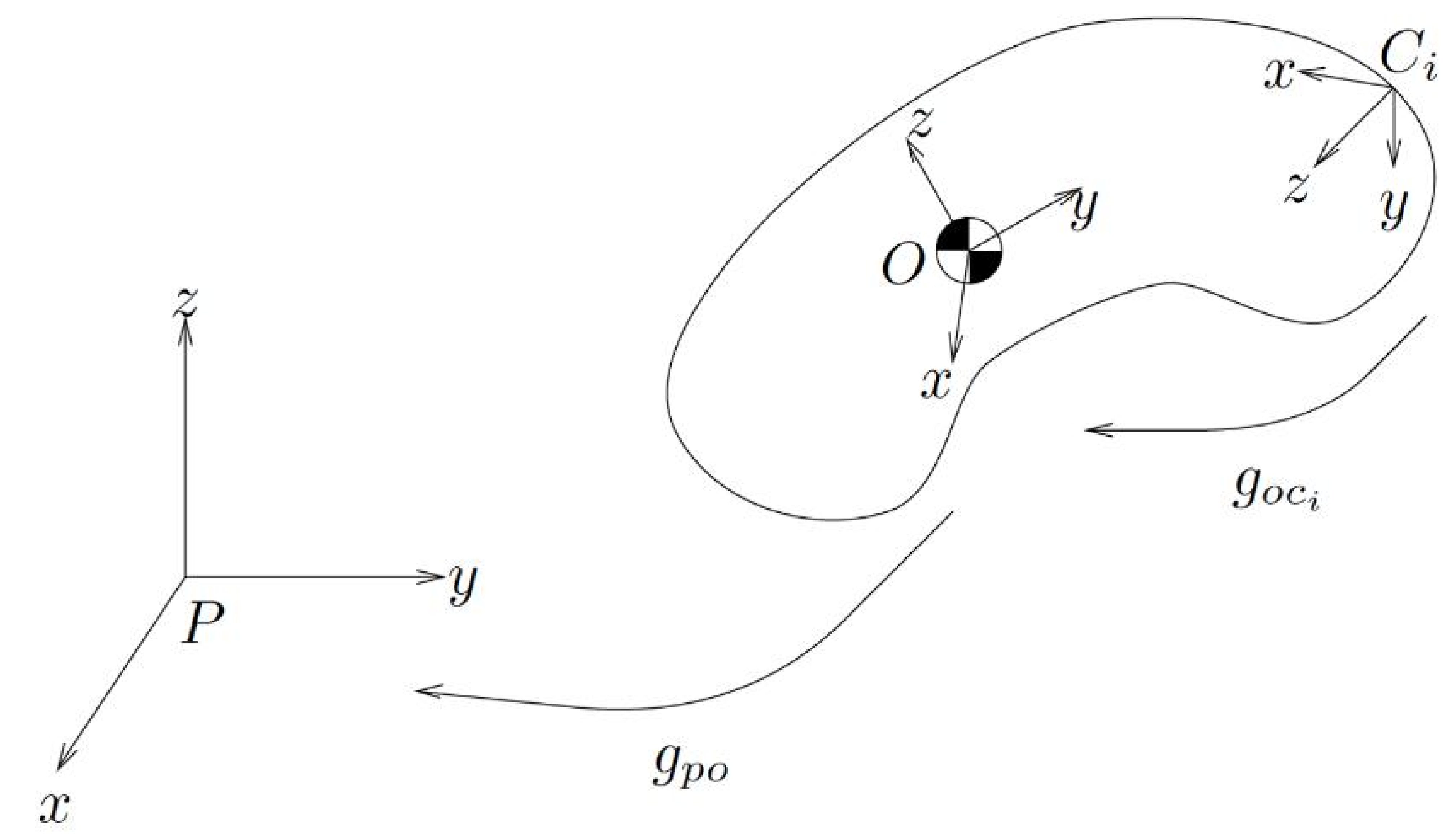

Figure 5.2: Coordinate frames for contact and object forces.

Change of Observer's Frame for Momentum and Angular Momentum

Consider the question of changing the observer's frame for momentum and angular momentum

- We would relate
  - $\boldsymbol{p}^1 = m\boldsymbol{v}^1$ and $\boldsymbol{p}^2 = m\boldsymbol{v}^2$
  - $\boldsymbol{L}^1 = \boldsymbol{r}^1 \times m\boldsymbol{v}^1$ and $\boldsymbol{L}^2 = \boldsymbol{r}^2 \times m\boldsymbol{v}^2$
- Note that

$$\boldsymbol{r}^2 = R_{2\to1}\boldsymbol{r}^1 + \boldsymbol{t}_{2\to1}$$
$$\boldsymbol{v}^2 = R_{2\to1}\boldsymbol{v}^1$$

- The same derivation as force and torque pair, and we get $\begin{bmatrix}\boldsymbol{p}^2\\ \boldsymbol{L}^2\end{bmatrix} = (\mathrm{Ad}_{1\to2})^{\mathrm{T}}\begin{bmatrix}\boldsymbol{p}^1\\ \boldsymbol{L}^1\end{bmatrix}$

Read by Yourself

Change of Observer's Frame for Momentum and Angular Momentum

- Define $\boldsymbol{P}^o = \begin{bmatrix} \boldsymbol{p}^o \\ \boldsymbol{L}^o \end{bmatrix}$, and the formula for change of frame is:

$$\boldsymbol{P}^2 = (\mathrm{Ad}_{1\to 2})^{\mathrm{T}} \boldsymbol{P}^1 \qquad \text{(change of frame)}$$

- Note: similar to linear momentum that $\boldsymbol{p}^o = \frac{\mathrm{d}T}{\mathrm{d}\boldsymbol{v}^o}$ for translation-only motion, it is straight-forward to verify that

$$\boldsymbol{P}^o = \frac{dT}{d\boldsymbol{\xi}^o} = \mathfrak{M}^o \xi^o \qquad \text{(system input power)}$$

Summary

- We have learned basic concepts for body motion dynamics
  - Properties of objects: mass, rotational inertia
  - Motion state: momentum, angular momentum
  - Action: force, torque
  - Energy perspective: kinetic energy
- We have also introduced various equations for changing the observer's frame