

L2: Robot Geometry

Hao Su

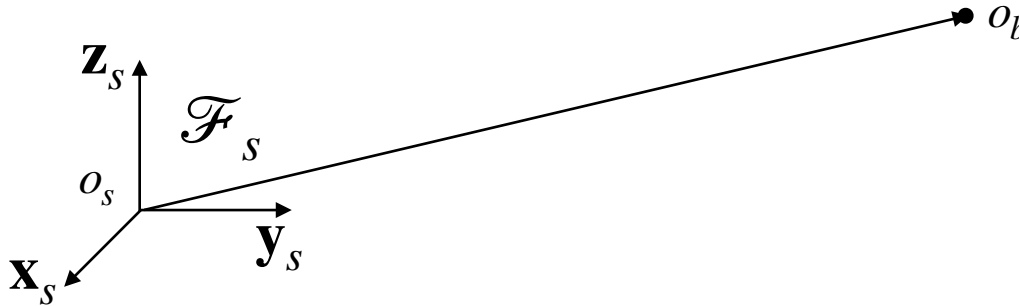
Ack: Slides prepared with the help of
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Agenda

- Rigid Transformation
- $(R_{s \rightarrow b}, \mathbf{t}_{s \rightarrow b})$ for Coordinate Transformation
- $(R_{s \rightarrow b}, \mathbf{t}_{s \rightarrow b})$ as a Linear Transformation

Rigid Transformation

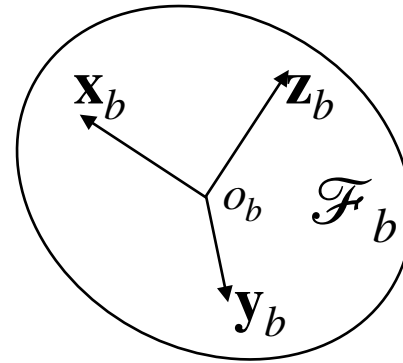
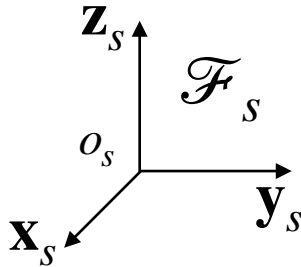
Notation Convention



- An observer **records** the position of any point in the space **using a frame** \mathcal{F}_s
- We use ordinary letters to denote points (e.g., p), and bold letters to denote **vectors** (e.g., \mathbf{v})
- When **writing equations**, we add a superscript to symbols to denote the recording frame, e.g.,

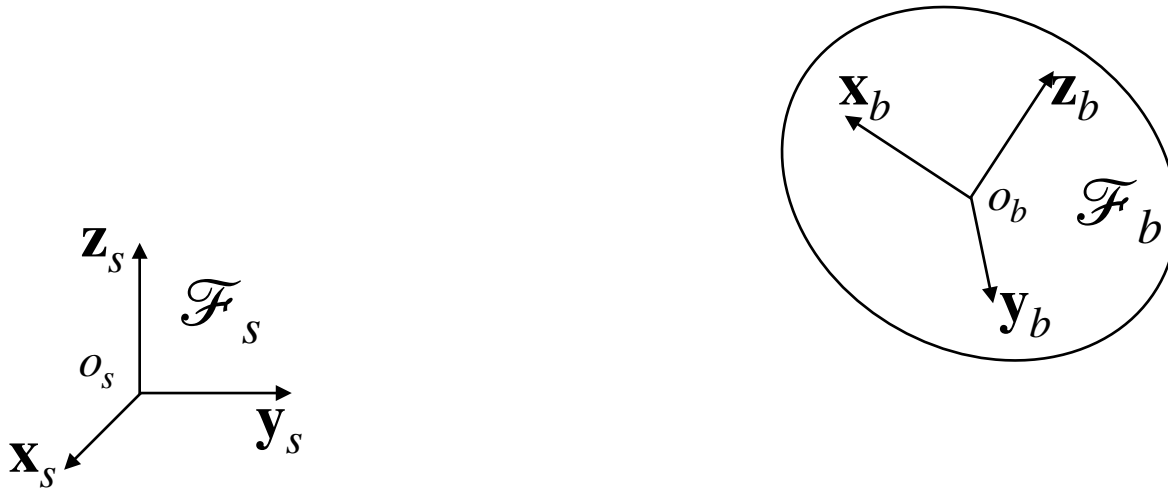
$$o_b^s = o_s^s + \mathbf{t}_{s \rightarrow b}^s$$

Rigid Transformation



- There is a rigid object, to which we bind a frame \mathcal{F}_b (body frame) tightly, so that \mathcal{F}_b moves along with the object

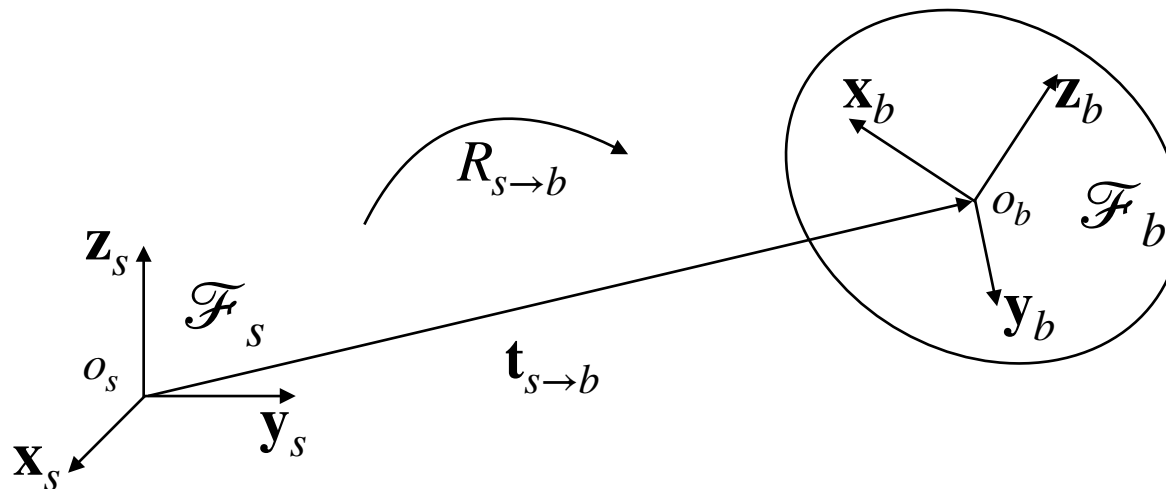
Rigid Transformation



- When talking about the pose of the *rigid* object, we ask:

How to **transform** \mathcal{F}_s so that it overlaps with \mathcal{F}_b ?

Rigid Transformation



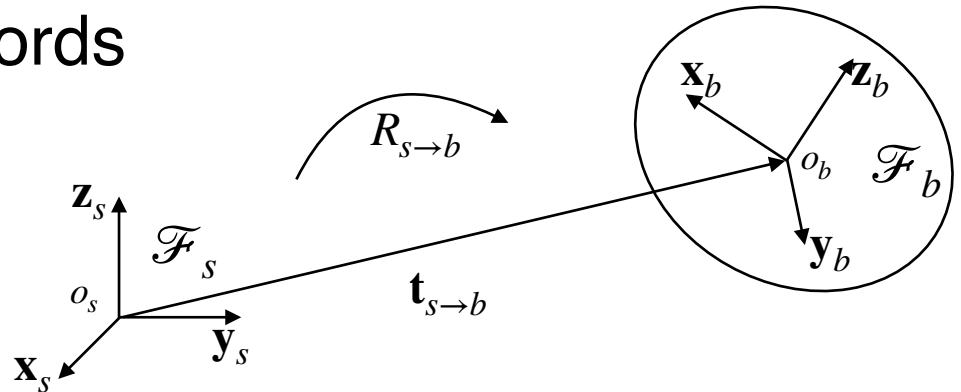
- We first translate \mathcal{F}_s by $\mathbf{t}_{s \rightarrow b}$ to align o_s and o_b
- And then rotate by $R_{s \rightarrow b}$ to align $\{\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i\}$ ($i = s$ or b)

Rigid Transformation

- Formally,
 - $o_b^s = o_s^s + \mathbf{t}_{s \rightarrow b}^s$
 - $[\mathbf{x}_b^s, \mathbf{y}_b^s, \mathbf{z}_b^s] = R_{s \rightarrow b}^s [\mathbf{x}_s^s, \mathbf{y}_s^s, \mathbf{z}_s^s]$

- Since the observer records everything using \mathcal{F}_s ,

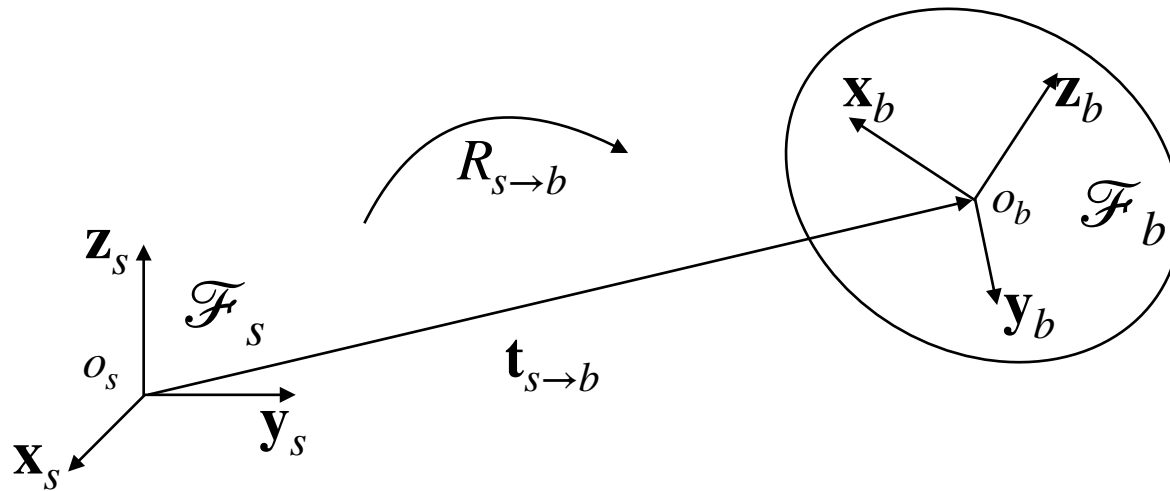
- $o_s^s = 0$
- $[\mathbf{x}_s^s, \mathbf{y}_s^s, \mathbf{z}_s^s] = I_{3 \times 3}$



- Therefore,
 - $\mathbf{t}_{s \rightarrow b}^s = o_b^s$
 - $R_{s \rightarrow b}^s = [\mathbf{x}_b^s, \mathbf{y}_b^s, \mathbf{z}_b^s] \in \mathbb{R}^{3 \times 3}$

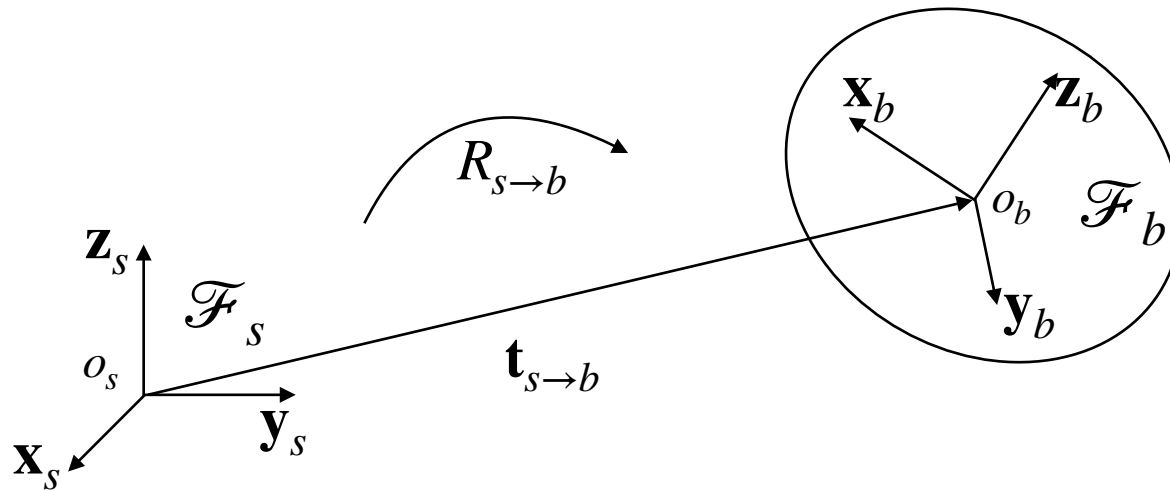
$(R_{s \rightarrow b}, \mathbf{t}_{s \rightarrow b})$ for **Coordinate Transformation**

Use Coordinate Transformation to Relate Coordinates in Frames



- Assume a second observer that records coordinates by \mathcal{F}_b
- Assume a point p on the body. Since \mathcal{F}_b moves along the body, its coordinate recorded in \mathcal{F}_b , denoted as p^b , should **never change**.

$(R_{s \rightarrow b}, \mathbf{t}_{s \rightarrow b})$ for Coordinate Transformation



- Imagine a process: \mathcal{F}_b moves from \mathcal{F}_s to the current location. This is how we define $(R_{s \rightarrow b}^s, \mathbf{t}_{s \rightarrow b}^s)$.
- Since p moves along \mathcal{F}_b , it is moved from the **initial position**, $p^s = p^b$, to the current location:

$$p^s = R_{s \rightarrow b}^s p^b + \mathbf{t}_{s \rightarrow b}^s$$

Homogenous Coordinates

- Homogeneous coordinate for 3D Space:

$$\tilde{x} := \begin{bmatrix} x \\ 1 \end{bmatrix} \in \mathbb{R}^4$$

- Homogeneous transformation matrix:

$$T_{s \rightarrow b}^s = \begin{bmatrix} R_{s \rightarrow b}^s & \mathbf{t}_{s \rightarrow b}^s \\ 0 & 1 \end{bmatrix}$$

- Coordinate transformation under linear form:

$$\tilde{x}^s = T_{s \rightarrow b}^s \tilde{x}^b$$

- Ignore \sim for simplicity in the future.

Homogenous Coordinates

- The coordinate transformation works for any choice of \mathcal{F}_s and \mathcal{F}_b
- As a general rule, we have:

$$x^1 = T_{1 \rightarrow 2}^1 x^2$$

Some Rules of Homogenous Coordinate Transformation

By $x^1 = T_{1 \rightarrow 2}^1 x^2$, we have $x^2 = T_{2 \rightarrow 1}^2 x^1$ and $x^3 = T_{3 \rightarrow 2}^3 x^2$.

Therefore, $x^3 = T_{3 \rightarrow 2}^3 T_{2 \rightarrow 1}^2 x^1$. But $x^3 = T_{3 \rightarrow 1}^3 x^1$

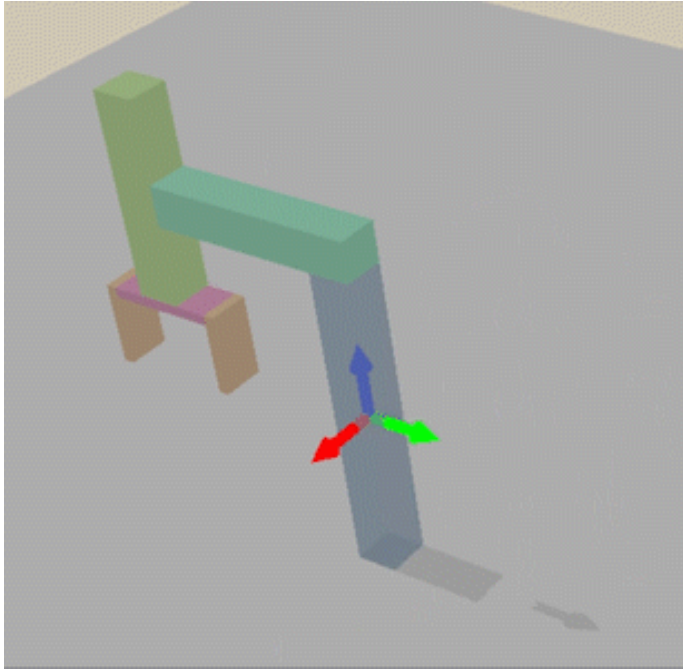
- Composition rule: $T_{3 \rightarrow 1}^3 = T_{3 \rightarrow 2}^3 T_{2 \rightarrow 1}^2$

By $x^1 = T_{1 \rightarrow 2}^1 x^2$, we have $x^2 = (T_{1 \rightarrow 2}^1)^{-1} x^1$

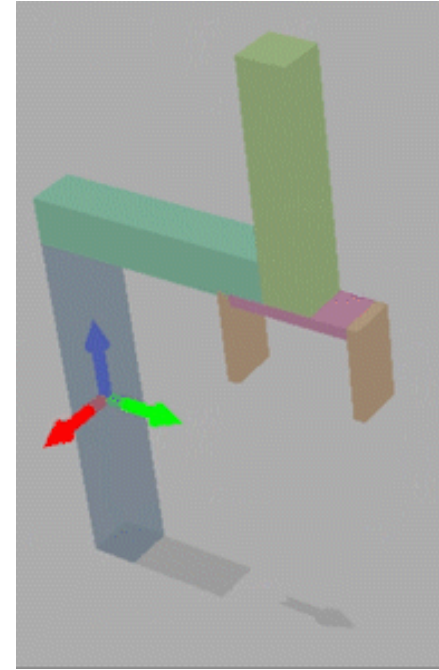
- Change of observer's frame: $T_{2 \rightarrow 1}^2 = (T_{1 \rightarrow 2}^1)^{-1}$

Example

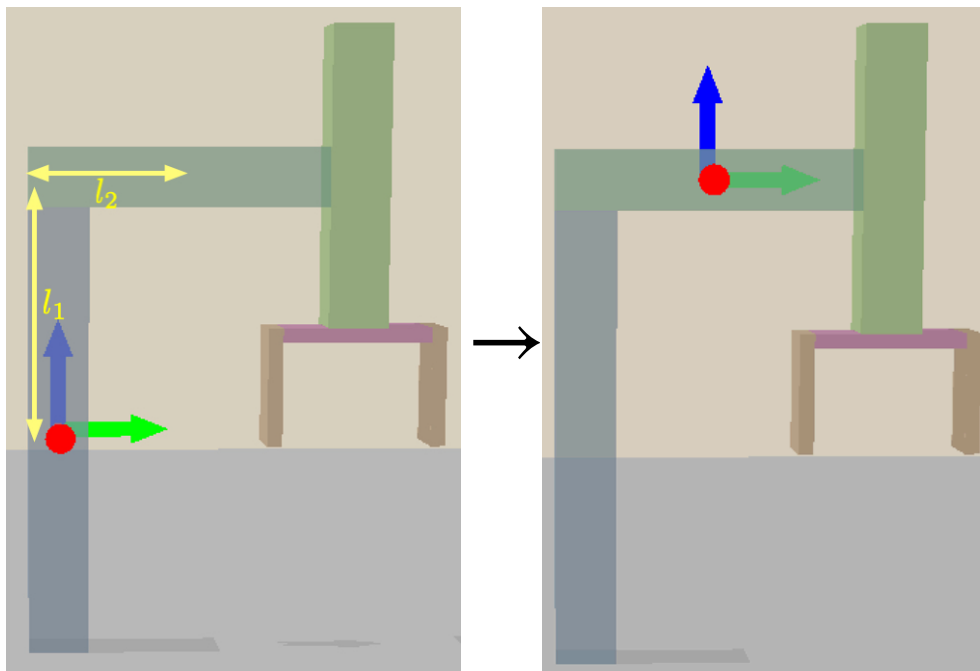
A simple 2 DoF robot arm



revolute (θ_1)



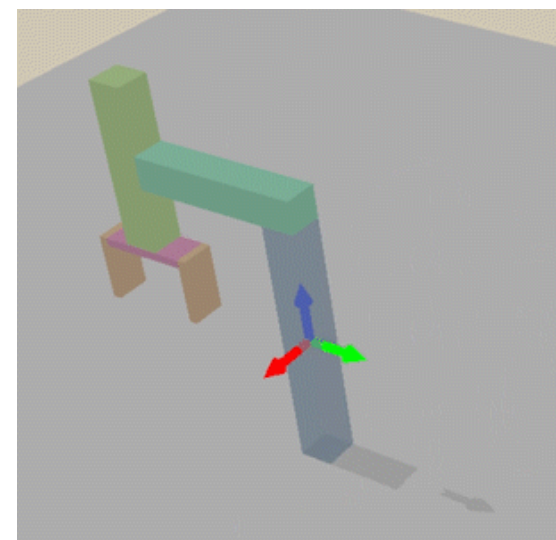
prismatic (θ_2)



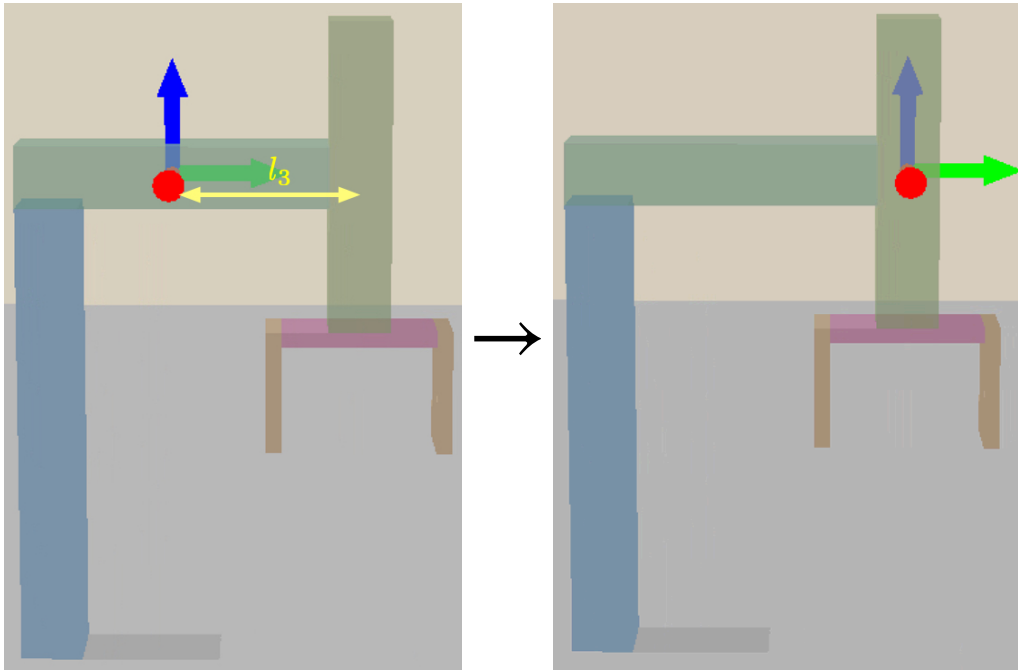
base

link1

$$T_{0 \rightarrow 1}^0 = \begin{bmatrix} \cos \theta_1 & -\sin \theta_1 & 0 & -l_2 \sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 & 0 & l_2 \cos \theta_1 \\ 0 & 0 & 1 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



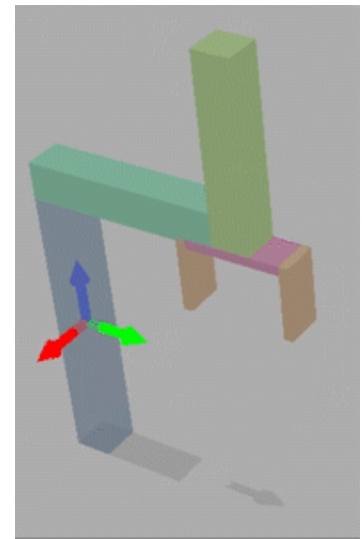
revolute (θ_1)



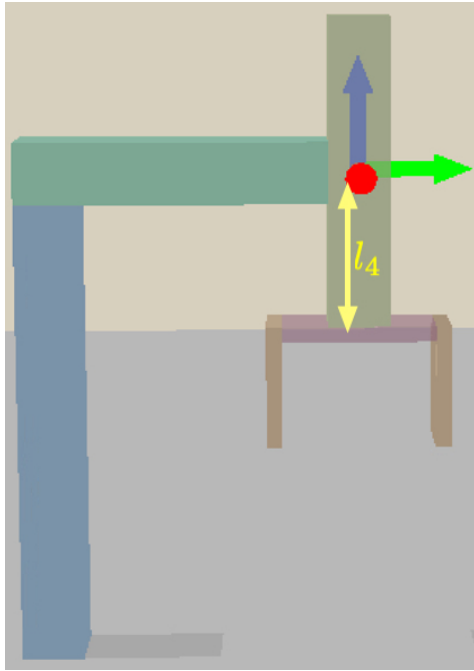
link1

link2

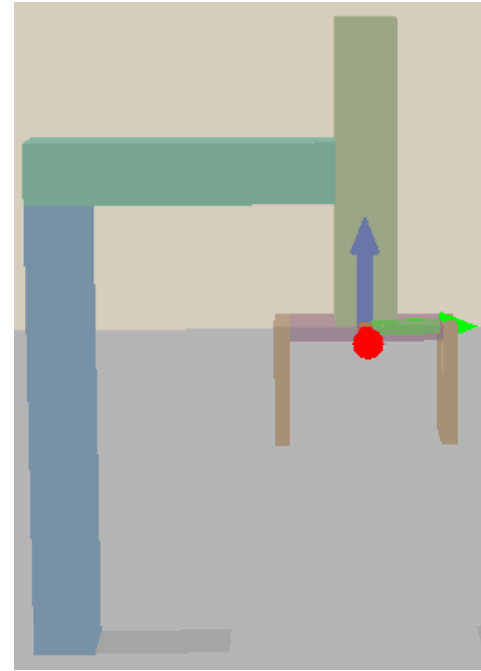
$$T_{1 \rightarrow 2}^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_3 \\ 0 & 0 & 1 & \theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



prismatic (θ_2)

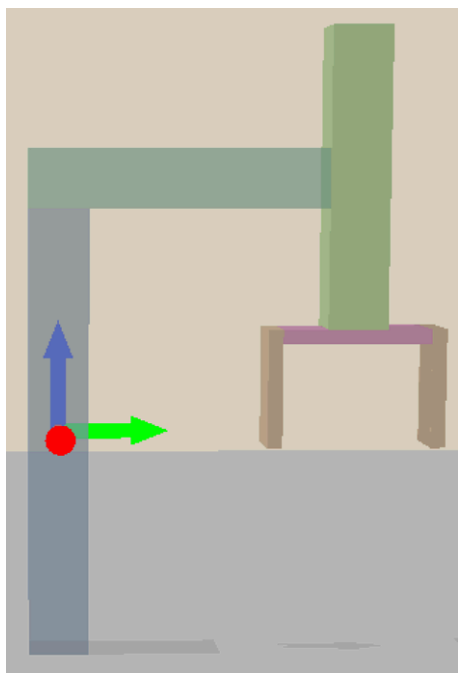


link2

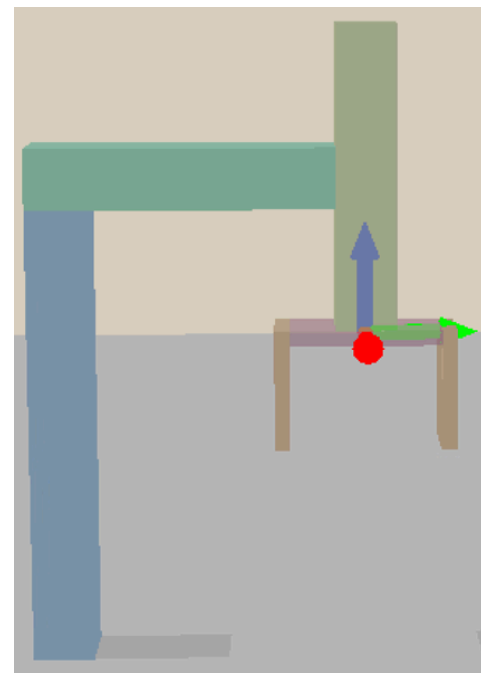


end_effector

$$T_{2 \rightarrow 3}^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -l_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



base

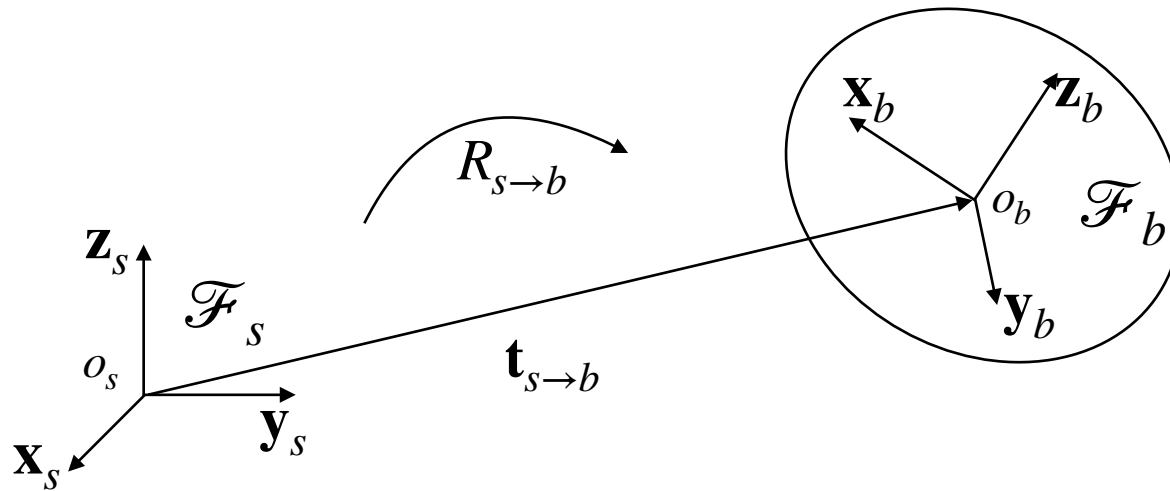


end_effector

$$T_{0 \rightarrow 3}^0 = T_{0 \rightarrow 1}^0 T_{1 \rightarrow 2}^1 T_{2 \rightarrow 3}^2 = \begin{bmatrix} \cos \theta_1 & -\sin \theta_1 & 0 & -\sin \theta_1(l_2 + l_3) \\ \sin \theta_1 & \cos \theta_1 & 0 & \cos \theta_1(l_2 + l_3) \\ 0 & 0 & 1 & l_1 - l_4 + \theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$(R_{s \rightarrow b}, \mathbf{t}_{s \rightarrow b})$ as a Linear
Transformation

$(R_{s \rightarrow b}, \mathbf{t}_{s \rightarrow b})$ as a Linear Transformation



- $(R_{s \rightarrow b}, \mathbf{t}_{s \rightarrow b})$ transforms any **point** in the *whole space* by the following equation:

$$\mathbf{x}'^s = R_{s \rightarrow b}^s \mathbf{x}^s + \mathbf{t}_{s \rightarrow b}^s$$

$(R_{s \rightarrow b}, \mathbf{t}_{s \rightarrow b})$ as a Linear Transformation

- Suppose $\mathcal{F}_p^s = \{p^s, (\mathbf{x}_p^s, \mathbf{y}_p^s, \mathbf{z}_p^s)\}$ is a frame at an arbitrary point p^s
- **Then, the new origin is: $p'^s = ?$**

$(R_{s \rightarrow b}, \mathbf{t}_{s \rightarrow b})$ as a Linear Transformation

- Suppose $\mathcal{F}_p^s = \{p^s, (\mathbf{x}_p^s, \mathbf{y}_p^s, \mathbf{z}_p^s)\}$ is a frame at an arbitrary point p^s
- **Then, the new origin is:** $p'^s = R_{s \rightarrow b}^s p^s + \mathbf{t}_{s \rightarrow b}^s$

$(R_{s \rightarrow b}, \mathbf{t}_{s \rightarrow b})$ as a Linear Transformation

- Suppose $\mathcal{F}_p^s = \{p^s, (\mathbf{x}_p^s, \mathbf{y}_p^s, \mathbf{z}_p^s)\}$ is a frame at an arbitrary point p^s
- **Then, the new origin is:** $p'^s = R_{s \rightarrow b}^s p^s + \mathbf{t}_{s \rightarrow b}^s$
- How about the bases vectors of the frame?
 - Assume three curves, $\gamma_x, \gamma_y, \gamma_z$, passing p^s at $t = 0$ with tangents $\mathbf{x}_p, \mathbf{y}_p, \mathbf{z}_p$

$(R_{s \rightarrow b}, \mathbf{t}_{s \rightarrow b})$ as a Linear Transformation

- Suppose $\mathcal{F}_p^s = \{p^s, (\mathbf{x}_p^s, \mathbf{y}_p^s, \mathbf{z}_p^s)\}$ is a frame at an arbitrary point p^s
- **Then, the new origin is:** $p'^s = R_{s \rightarrow b}^s p^s + \mathbf{t}_{s \rightarrow b}^s$
- How about the bases vectors of the frame?
 - Assume three curves, $\gamma_x, \gamma_y, \gamma_z$, passing p^s at $t = 0$ with tangents $\mathbf{x}_p, \mathbf{y}_p, \mathbf{z}_p$
 - Then, the new tangents after transformation are:
$$\frac{d}{dt} R_{s \rightarrow b}^s \gamma_x^s(0), \frac{d}{dt} R_{s \rightarrow b}^s \gamma_y^s(0), \frac{d}{dt} R_{s \rightarrow b}^s \gamma_z^s(0)$$

$(R_{s \rightarrow b}, \mathbf{t}_{s \rightarrow b})$ as a Linear Transformation

- Suppose $\mathcal{F}_p^s = \{p^s, (\mathbf{x}_p^s, \mathbf{y}_p^s, \mathbf{z}_p^s)\}$ is a frame at an arbitrary point p^s
- **Then, the new origin is:** $p'^s = R_{s \rightarrow b}^s p^s + \mathbf{t}_{s \rightarrow b}^s$
- How about the bases vectors of the frame?
 - Assume three curves, $\gamma_x, \gamma_y, \gamma_z$, passing p^s at $t = 0$ with tangents $\mathbf{x}_p, \mathbf{y}_p, \mathbf{z}_p$
 - Then, the new tangents after transformation are:
$$\frac{d}{dt} R_{s \rightarrow b}^s \gamma_x^s(0), \frac{d}{dt} R_{s \rightarrow b}^s \gamma_y^s(0), \frac{d}{dt} R_{s \rightarrow b}^s \gamma_z^s(0)$$
- **So the new frame is:** $\mathcal{F}_{p'}^s = \{p'^s, R_{s \rightarrow b}^s [\mathbf{x}_p^s, \mathbf{y}_p^s, \mathbf{z}_p^s]\}$

$$T_{1 \rightarrow 2}^s$$

- We have introduced the notations when the observer is recording via \mathcal{F}_s or \mathcal{F}_b
 - $T_{s \rightarrow b}^s$ (record the frame alignment from \mathcal{F}_s to \mathcal{F}_b)
 - By the change of observer's frame, we introduced $T_{b \rightarrow s}^b = (T_{s \rightarrow b}^s)^{-1}$
- Next, we define the notion of $T_{1 \rightarrow 2}^s$, which is how we **record** an arbitrary transformation from \mathcal{F}_1 to \mathcal{F}_2 in \mathcal{F}_s
 - $T_{1 \rightarrow 2}^s := T_{s \rightarrow 2}^s T_{1 \rightarrow s}^1$

Composition as a Homogeneous Linear Transformation

- Under the definition $T_{1 \rightarrow 2}^s := T_{s \rightarrow 2}^s T_{1 \rightarrow s}^1$, the composition rule is:

$$T_{1 \rightarrow 2}^s = T_{3 \rightarrow 2}^s T_{1 \rightarrow 3}^s$$

Change Observer's Frame with Similarity Transformation

- Given $T_{1 \rightarrow 2}^s$, what is $T_{1 \rightarrow 2}^b$?

$$T_{1 \rightarrow 2}^s T_{s \rightarrow 1}^s = T_{s \rightarrow 2}^s \quad \text{Composition as Linear Transformation}$$

$$T_{1 \rightarrow 2}^s T_{s \rightarrow b}^s T_{b \rightarrow 1}^b = T_{s \rightarrow b}^s T_{b \rightarrow 2}^b \quad \text{Composition as Coordinate Transformation}$$

$$T_{1 \rightarrow 2}^s T_{s \rightarrow b}^s T_{b \rightarrow 1}^b = T_{s \rightarrow b}^s T_{1 \rightarrow 2}^b T_{b \rightarrow 1}^b \quad \text{Composition as Linear Transformation}$$

$$T_{1 \rightarrow 2}^s T_{s \rightarrow b}^s = T_{s \rightarrow b}^s T_{1 \rightarrow 2}^b$$

$$T_{1 \rightarrow 2}^s = T_{s \rightarrow b}^s T_{1 \rightarrow 2}^b (T_{s \rightarrow b}^s)^{-1}$$

- Similarity Transformation changes the **superscript**

$$B = X^{-1}AX: \text{Similarity Transformation}$$

A Special Case

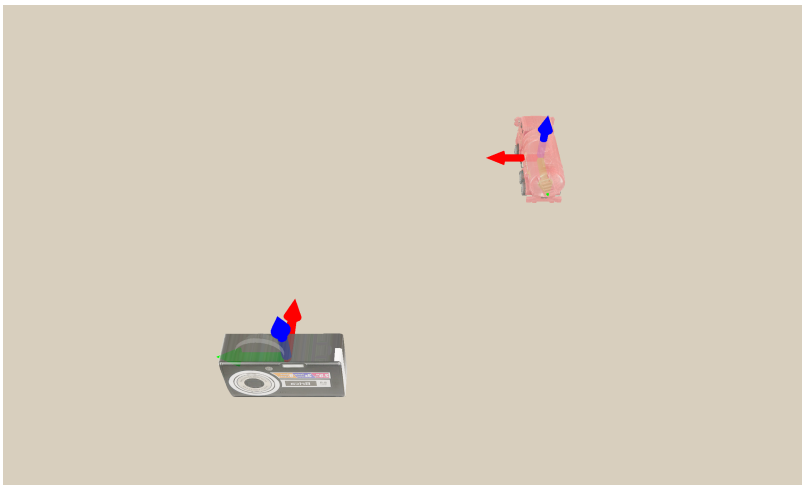
- By $T_{1 \rightarrow 2}^s = T_{s \rightarrow b}^s T_{1 \rightarrow 2}^b (T_{s \rightarrow b}^s)^{-1}$,
 - If $\mathcal{F}_1 = \mathcal{F}_s$ and $\mathcal{F}_2 = \mathcal{F}_b$, $T_{s \rightarrow b}^s = T_{s \rightarrow b}^b$!
- Therefore, we often see the abbreviated notations:
 - $T_b^s \equiv T_{s \rightarrow b}^s$
 - $T_{sb} \equiv T_{s \rightarrow b}^s$
 - $T_b \equiv T_{s \rightarrow b}^s$
- The above equation can therefore be written as:

$$T_{1 \rightarrow 2}^s = T_{s \rightarrow b} T_{1 \rightarrow 2}^b (T_{s \rightarrow b})^{-1}$$

Example

- Consider a camera with frame \mathcal{F}_c observing a red car
- Denote the current frame of the red car as \mathcal{F}_1

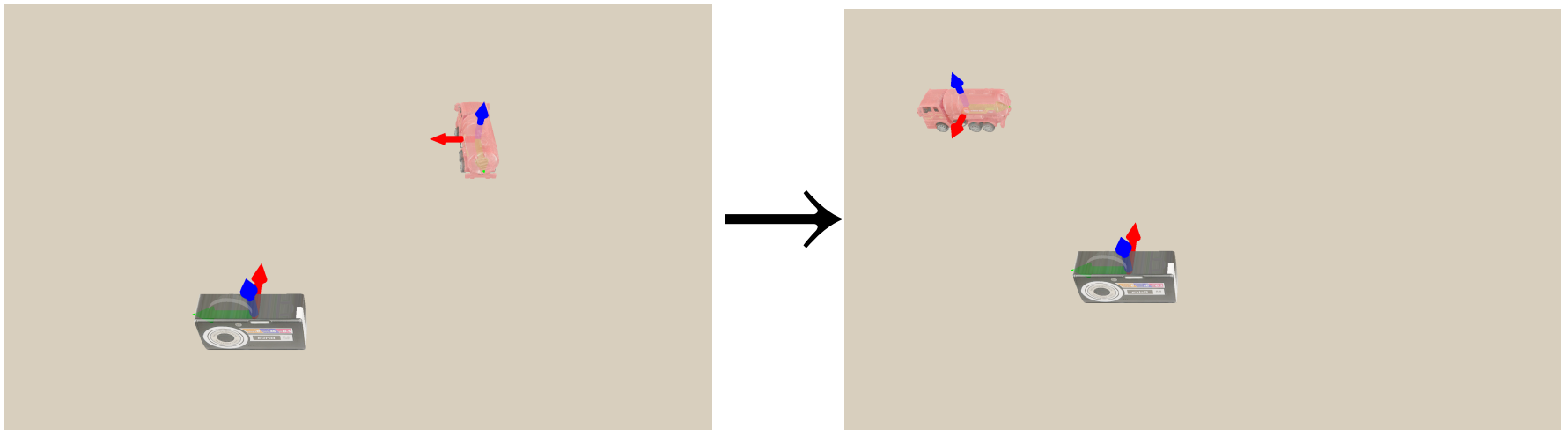
$$T_{c \rightarrow 1}^c = \begin{bmatrix} \cos \frac{\pi}{2} & -\sin \frac{\pi}{2} & 0 & l \\ \sin \frac{\pi}{2} & \cos \frac{\pi}{2} & 0 & -l \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Example

- Then the red car move to a new frame \mathcal{F}_2

$$T_{c \rightarrow 2}^c = \begin{bmatrix} \cos \pi & -\sin \pi & 0 & l \\ \sin \pi & \cos \pi & 0 & l \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

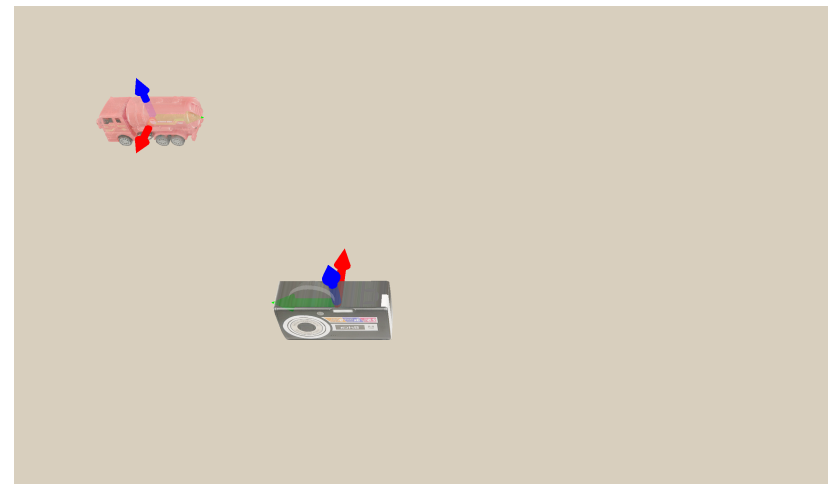
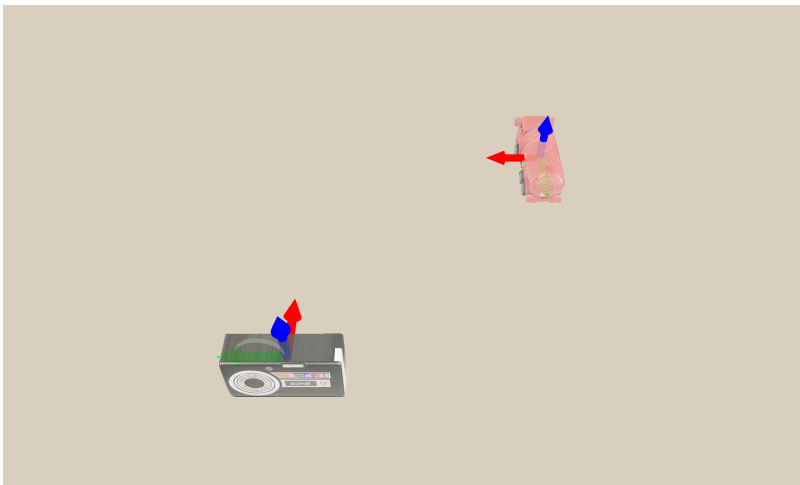


Example

- By the composition rule of coordinate transformation:

$$T_{c \rightarrow 2}^c = T_{c \rightarrow 1}^c T_{1 \rightarrow 2}^1$$

$$T_{1 \rightarrow 2}^1 = (T_{c \rightarrow 1}^c)^{-1} T_{c \rightarrow 2}^c = \begin{bmatrix} \cos \frac{\pi}{2} & -\sin \frac{\pi}{2} & 0 & 2l \\ \sin \frac{\pi}{2} & \cos \frac{\pi}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Example

- By the composition rule of coordinate transformation:

$$T_{c \rightarrow 2}^c = T_{c \rightarrow 1}^c T_{1 \rightarrow 2}^1$$

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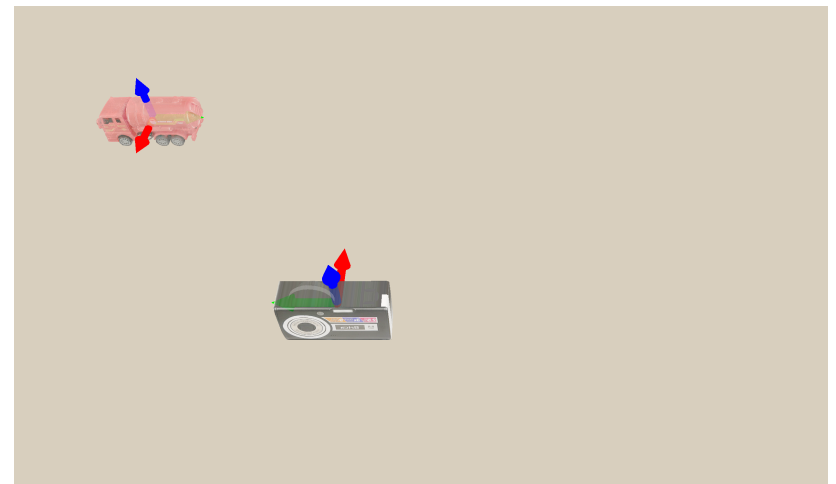
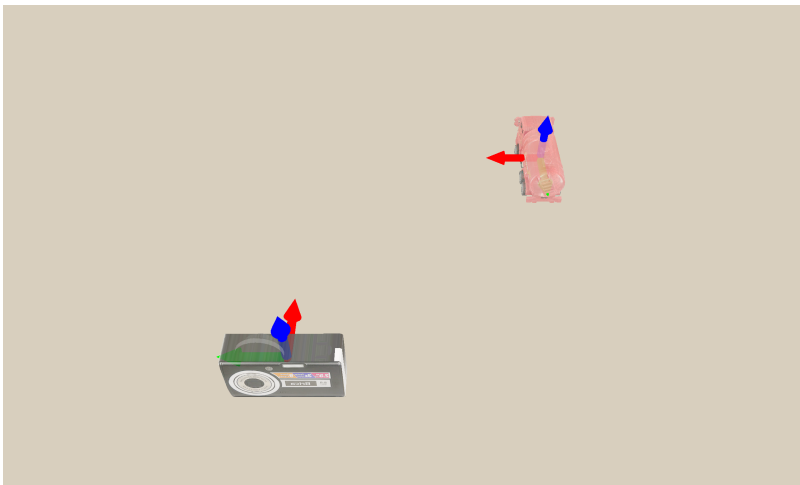
- The movement from \mathcal{F}_1 to \mathcal{F}_2 can also be represented as a linear transformation from \mathcal{F}_1 to \mathcal{F}_2 , recorded by frame c , denoted as $T_{1 \rightarrow 2}^c$

Example

- With similarity transformation:

$$T_{1 \rightarrow 2}^c = T_{c \rightarrow 1}^c T_{1 \rightarrow 2}^1 (T_{c \rightarrow 1}^c)^{-1} = \begin{bmatrix} \cos \frac{\pi}{2} & -\sin \frac{\pi}{2} & 0 & 0 \\ \sin \frac{\pi}{2} & \cos \frac{\pi}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

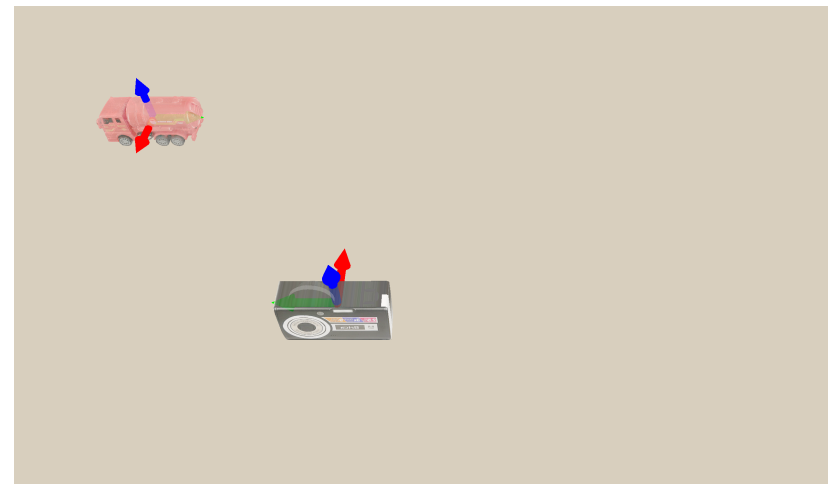
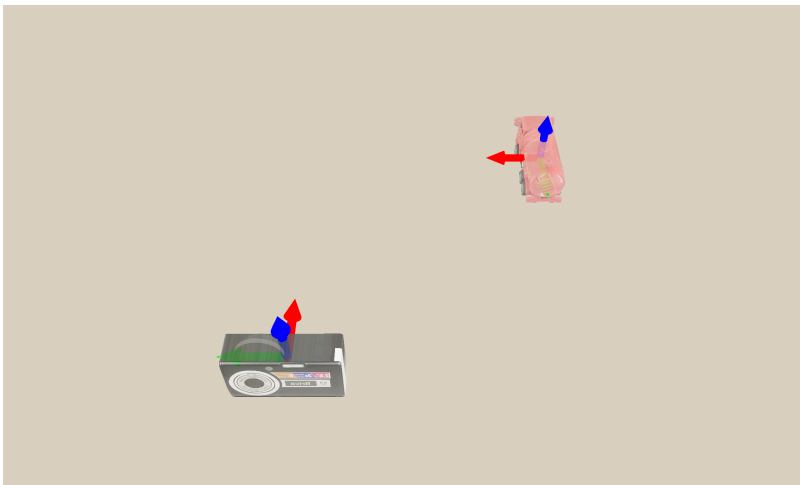
- Note: translation in $T_{1 \rightarrow 2}^c$ is all zero! Why?



Example

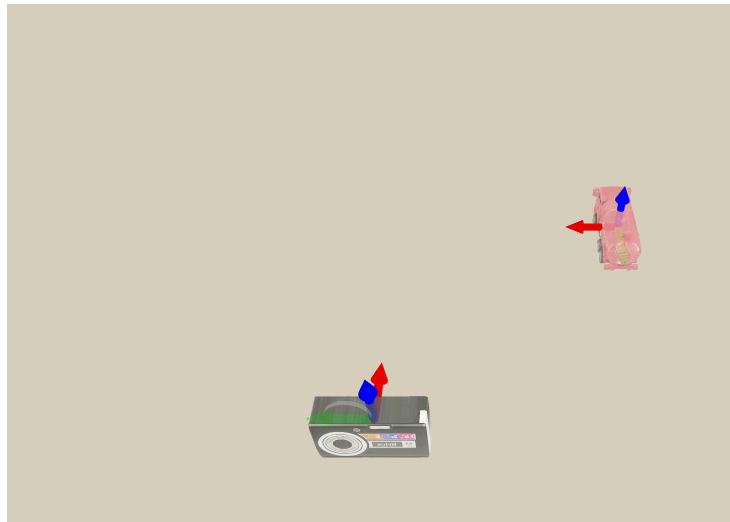
- Transformation $T_{1 \rightarrow 2}^c$ can be regarded as rotating about z-axis by 90 degree

$$T_{1 \rightarrow 2}^c = \begin{bmatrix} \cos \frac{\pi}{2} & -\sin \frac{\pi}{2} & 0 & 0 \\ \sin \frac{\pi}{2} & \cos \frac{\pi}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Example

- When observer is recording in the camera frame \mathcal{F}_c , the red car is rotated about the **z-axis** of camera frame c through +90 degree



Additional Notes by the Example

- $T_{1 \rightarrow 2}^s$ is **NOT** to record the transformation by first translating \mathcal{F}_1 to \mathcal{F}_2 and then rotating (this recording convention **only** works when $\mathcal{F}_1 = \mathcal{F}_s$). It is based on the rule $T_{1 \rightarrow 2}^s := T_{s \rightarrow 2}^s T_{1 \rightarrow s}^1$
- An observer chooses its way to decompose $T_{1 \rightarrow 2}$ into $R_{1 \rightarrow 2}$ and $\mathbf{t}_{1 \rightarrow 2}$ based upon its own frame choice
- We will discuss the “canonical” decomposition next week

Additional Notes by the Example

- The linear transformation view allows us to discuss the movement of bodies conveniently (without worrying about the change of observer):

$$T_{1 \rightarrow 2}^S = T_{3 \rightarrow 2}^S T_{1 \rightarrow 3}^S$$

- Suppose a body is moving. Then,

$$T_{t_0 \rightarrow t + \Delta t}^S = T_{t \rightarrow t + \Delta t}^S T_{t_0 \rightarrow t}^S$$

where t parameterizes time.

Summary

- Basic notation:
 - $T_{s \rightarrow b}^s$: Record the motion of frame alignment from \mathcal{F}_s to \mathcal{F}_b in \mathcal{F}_s
- Coordinate transformation
 - $T_{c \rightarrow a}^c = T_{c \rightarrow b}^c T_{b \rightarrow a}^b$: Composition for coordinate transformation
 - $T_{b \rightarrow s}^b = (T_{s \rightarrow b}^s)^{-1}$: Change of frame for \mathcal{F}_s to \mathcal{F}_b motion
- Linear transformation
 - $T_{1 \rightarrow 2}^s := T_{s \rightarrow 2}^s T_{1 \rightarrow s}^1$: Record the motion of frame alignment from \mathcal{F}_1 to \mathcal{F}_2 in \mathcal{F}_s
 - $T_{c \rightarrow a}^s = T_{b \rightarrow a}^b T_{c \rightarrow b}^b$: Composition as a linear transformation
- $T_{1 \rightarrow 2}^s = T_{s \rightarrow b} T_{1 \rightarrow 2}^b (T_{s \rightarrow b})^{-1}$: Change of frame for \mathcal{F}_1 -to- \mathcal{F}_2 motion