L8: Basic Concepts of Rigid-Body Dynamics

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Some Further Clarifications on Velocity

- Consider we observe the motion of a moving body $b(t)$ from a moving observer's frame $o(t)$, and the recording of motion is relative to $s(t)$.
- **velocity** observed from an arbitrary frame (e.g., point linear velocity, angular velocity, and twist) is:

$$\boldsymbol{v}_{s(t_0)\to b(t_0)}^{o(t_0)} = \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{p}_{\color{red}{s(t)\to b(t)}}^{o(t_0)} \bigg|_{t=t_0}, \qquad \text{where } \boldsymbol{p}_{s(t)\to b(t)}^{o(t_0)} = \boldsymbol{p}_{o(t_0)\to b(t)}^{o(t_0)} - \boldsymbol{p}_{o(t_0)\to s(t)}^{o(t_0)}$$

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$$\boldsymbol{v}^{o(t_0)}_{s(t_0)\to b(t_0)} = \frac{\mathrm{d}}{\mathrm{d}t} p^{o(t_0)}_{s(t)\to b(t)}\bigg|_{t=t_0}, \qquad \text{where } p^{o(t_0)}_{s(t)\to b(t)} = p^{o(t_0)}_{o(t_0)\to b(t)} - p^{o(t_0)}_{o(t_0)\to s(t)}$$

- Note: above is a general rule of **taking derivative of coordinate w.r.t. time** (so that body velocity/twist is non-zero).
  - e.g., when we derive the body-frame Euler equation last lecture, body inertia is defined by body frame coordinates, more precisely, $I^{b(t_0)}_{b(t)} = \int \mathrm{d}V \rho(r^{b(t_0)}_{b(t)\to x(t)})[r^{b(t_0)}_{b(t)\to x(t)}][r^{b(t_0)}_{b(t)\to x(t)}]$
  - Therefore, while I is an invariant w.r.t. t (since $r^{b(t_0)}_{b(t)\to x(t)}\Big|_{t=t_0}$ is invariant), its derivative (taken only w.r.t. the subscript) is non-zero.
  - This is exactly the case that body-frame coordinate is a constant for body points, but they have velocity.

Agenda

- Kinetic Energy
- Change of Frame for Various Quantities

click to jump to the section.

Kinetic Energy

Kinetic Energy for Point Mass

- If a point mass m is moving with velocity $\boldsymbol{v}^o_{s(t)\to b(t)}$ ($s(t)$ is an inertia frame and the origin of $b(t)$ is the point), then the kinetic energy of the point mass is

$$T_{s(t)\to b(t)} = \frac{1}{2}m\|\boldsymbol{v}^o_{s(t)\to b(t)}\|^2 \qquad \text{(kinetic energy)}$$

Observer-Independence of Kinetic Energy

- Note that we omit observer's frame when describing kinetic energy, because it is independent of the observer's frame.
- We prove by showing that $\|\boldsymbol{v}^{o_1}_{s(t)\to b(t)}\| = \|\boldsymbol{v}^{o_2}_{s(t)\to b(t)}\|$

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- Proof:

$$\begin{aligned}\boldsymbol{v}^{o_1}_{s(t)\to b(t)} &= \boldsymbol{v}^{o_1}_{b(t)} - \boldsymbol{v}^{o_1}_{s(t)} = [\boldsymbol{\xi}^{o_1}_{b(t)} - \boldsymbol{\xi}^{o_1}_{s(t)}]p^{o_1} = T^{-1}_{o_2\to o_1}[\boldsymbol{\xi}^{o_2}_{b(t)} - \boldsymbol{\xi}^{o_2}_{s(t)}]T_{o_2\to o_1}p^{o_1} \\ &= T^{-1}_{o_2\to o_1}[\boldsymbol{\xi}^{o_2}_{b(t)} - \boldsymbol{\xi}^{o_2}_{s(t)}]p^{o_2} = T^{-1}_{o_2\to o_1}(\boldsymbol{v}^{o_2}_{b(t)} - \boldsymbol{v}^{o_2}_{s(t)}) = T^{-1}_{o_2\to o_1}\boldsymbol{v}^{o_2}_{s(t)\to b(t)} \\ &= \begin{bmatrix} R^T_{o_2\to o_1} & -R^T_{o_2\to o_1}t_{o_2\to o_1} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \boldsymbol{v}^{o_2}_{s(t)\to b(t)}\big|_{3\times 1} \\ 0 \end{bmatrix} = \begin{bmatrix} R_{o_1\to o_2}\boldsymbol{v}^{o_2}_{s(t)\to b(t)}\big|_{3\times 1} \\ 0 \end{bmatrix}\end{aligned}$$

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$$\therefore \|\boldsymbol{v}^{o_1}_{s(t)\to b(t)}\| = \|\boldsymbol{v}^{o_2}_{s(t)\to b(t)}\|$$

- We have also derived

$$\boldsymbol{v}^{o_1}_{s(t)\to b(t)} = R_{o_1\to o_2}\boldsymbol{v}^{o_2}_{s(t)\to b(t)} \text{ for } \boldsymbol{v} \in \mathbb{R}^3 \quad \text{(change of frame for velocities)}$$

Kinetic Energy for Rigid Body

- Integrate kinetic energy of every point mass over the body
- We choose the body frame $\mathcal{F}_{b(t)}$ to start the derivation. Using the independence of observer's frame, we derive the formula to compute the energy in other frames.
- *The origin of our body frame is always at the center of mass of the body*

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- Integrate kinetic energy of every point mass over the body
- We choose the body frame $\mathcal{F}_{b(t)}$ to start the derivation. Using the independence of observer's frame, we derive the formula to compute the energy in other frames.
- *The origin of our body frame is always at the center of mass of the body*
- The velocity of a body point $\boldsymbol{r}^{b(t)}$ is

$$
\begin{aligned}
\boldsymbol{v}_{s(t)\to b(t)}^{b(t)} &= [\boldsymbol{\xi}_{s(t)\to b(t)}^{b(t)}]\boldsymbol{r}^{b(t)} \\
&= \begin{bmatrix} [\boldsymbol{\omega}_{s(t)\to b(t)}^{b(t)}] & \dot{\boldsymbol{t}}_{s(t)\to b(t)}^{b(t)} \\ 0 & 0 \end{bmatrix} \boldsymbol{r}^{b(t)}
\end{aligned}
$$

Kinetic Energy for Rigid Body

Therefore,

$$\begin{aligned}
T_{s(t)\to b(t)} &= \int_{\boldsymbol{r}^b \in B} \frac{1}{2}\rho(x)\mathrm{d}V \|\boldsymbol{v}^{b(t)}_{s(t)\to b(t)}\|^2 = \int_{\boldsymbol{r}^b \in B} \frac{1}{2}\rho(x)\mathrm{d}V \|[\boldsymbol{\omega}^{b(t)}_{s(t)\to b(t)}]\boldsymbol{r}^{b(t)} + \dot{\boldsymbol{t}}^{b(t)}_{s(t)\to b(t)}\|^2 \\
&= (\text{some derivations using } [\boldsymbol{\omega}]\boldsymbol{r} = -[\boldsymbol{r}]\boldsymbol{\omega}) \\
&= \frac{1}{2}m\|\dot{\boldsymbol{t}}^{b(t)}_{s(t)\to b(t)}\|^2 + \frac{1}{2}(\boldsymbol{\omega}^{b(t)}_{s(t)\to b(t)})^T \boldsymbol{I}^b \boldsymbol{\omega}^{b(t)}_{s(t)\to b(t)} \\
&= \frac{1}{2}(\boldsymbol{\xi}^{b(t)}_{s(t)\to b(t)})^T \mathfrak{M}^b \boldsymbol{\xi}^{b(t)}_{s(t)\to b(t)}
\end{aligned}$$

where

$$\mathfrak{M}^b = \begin{bmatrix} m\mathrm{Id}_{3\times 3} & 0 \\ 0 & \boldsymbol{I}^b \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

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$$\frac{1}{2}(\boldsymbol{\xi}^1)^T\mathfrak{M}^1\boldsymbol{\xi}^1 = \frac{1}{2}(\boldsymbol{\xi}^2)^T\mathfrak{M}^2\boldsymbol{\xi}^2$$

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$$\frac{1}{2}(\boldsymbol{\xi}^1)^T \mathfrak{M}^1 \boldsymbol{\xi}^1 = \frac{1}{2}(\boldsymbol{\xi}^2)^T \mathfrak{M}^2 \boldsymbol{\xi}^2$$

  - Recall that $\boldsymbol{\xi}^1 = [\mathrm{Ad}_{1\to 2}]\boldsymbol{\xi}^2$, and we conclude that

$$\mathfrak{M}^2 = [\mathrm{Ad}_{1\to 2}]^{\mathrm{T}} \mathfrak{M}^1 [\mathrm{Ad}_{1\to 2}] \qquad \text{(change of frame)}$$

Change of Observer's Frame for Rotational Inertia Matrix

- A side-product of introducing $\mathfrak{M}^o$ is that we can compute the inertia matrix in other frames conveniently
- We derived the change of frame formula for different body frames. **What about frame change between general observer's frames?**

Change of Observer's Frame for Rotational Inertia Matrix

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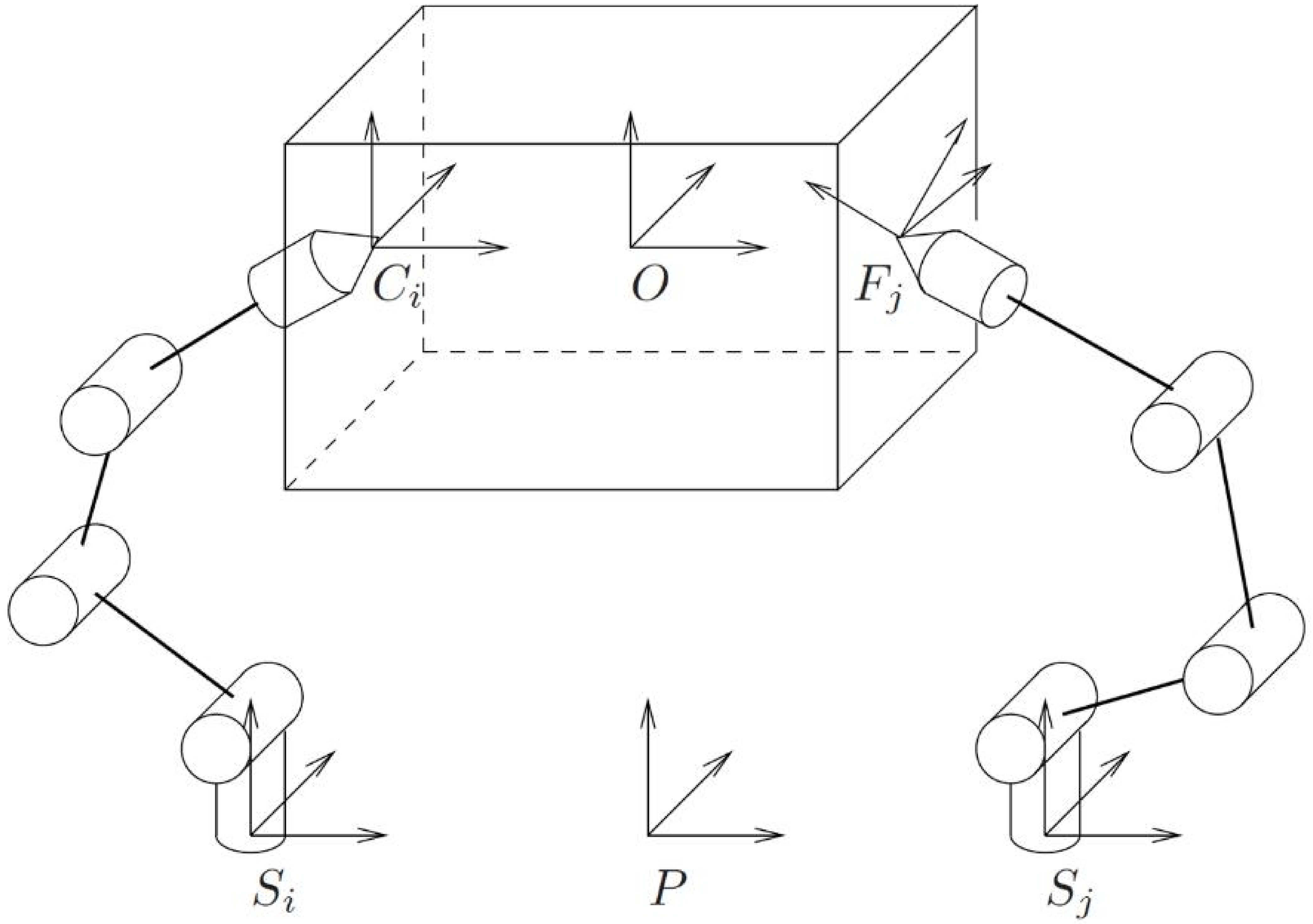
- We derived the change of frame formula for different body frames. **What about frame change between general observer's frames?**
- One can verify that,

$$\textbf{the bottom-right } 3 \times 3 \textbf{ block of } \mathfrak{M}^2 \textbf{ is the rotational inertial matrix in } \mathcal{F}_2$$

Change of Frame for Various Quantities

Motivating Example: Grasp Problem

- Consider the right grasp problem



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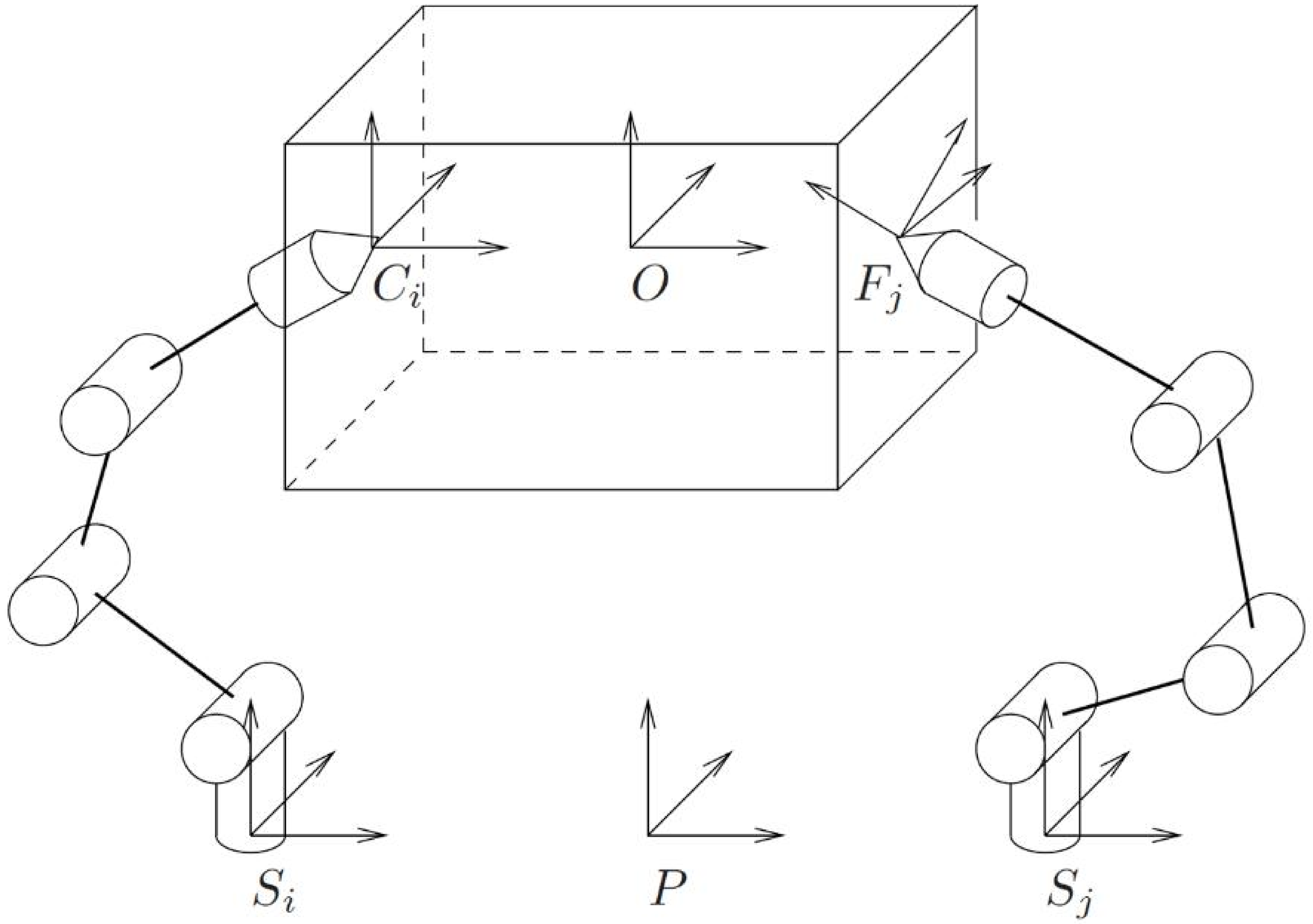

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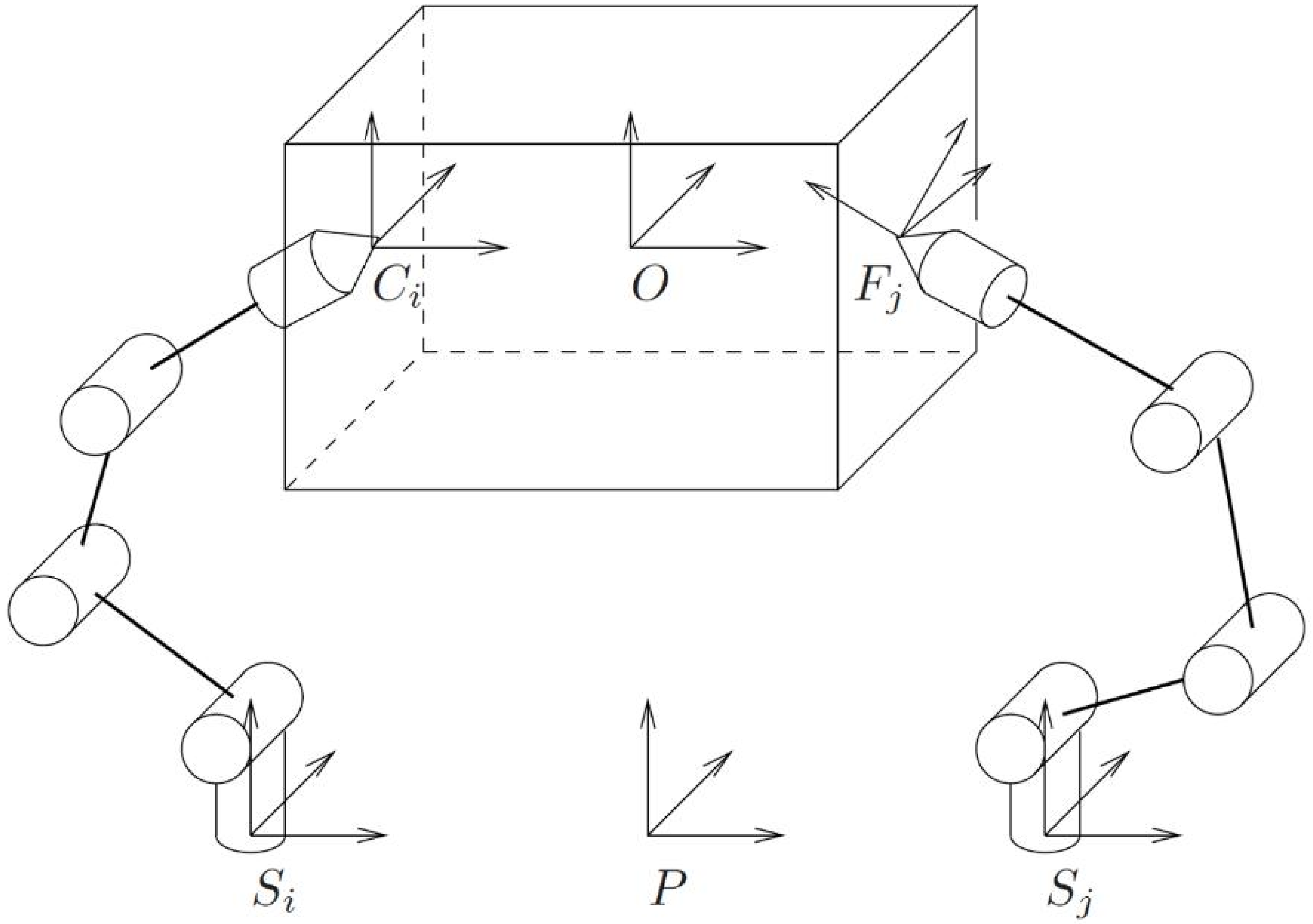

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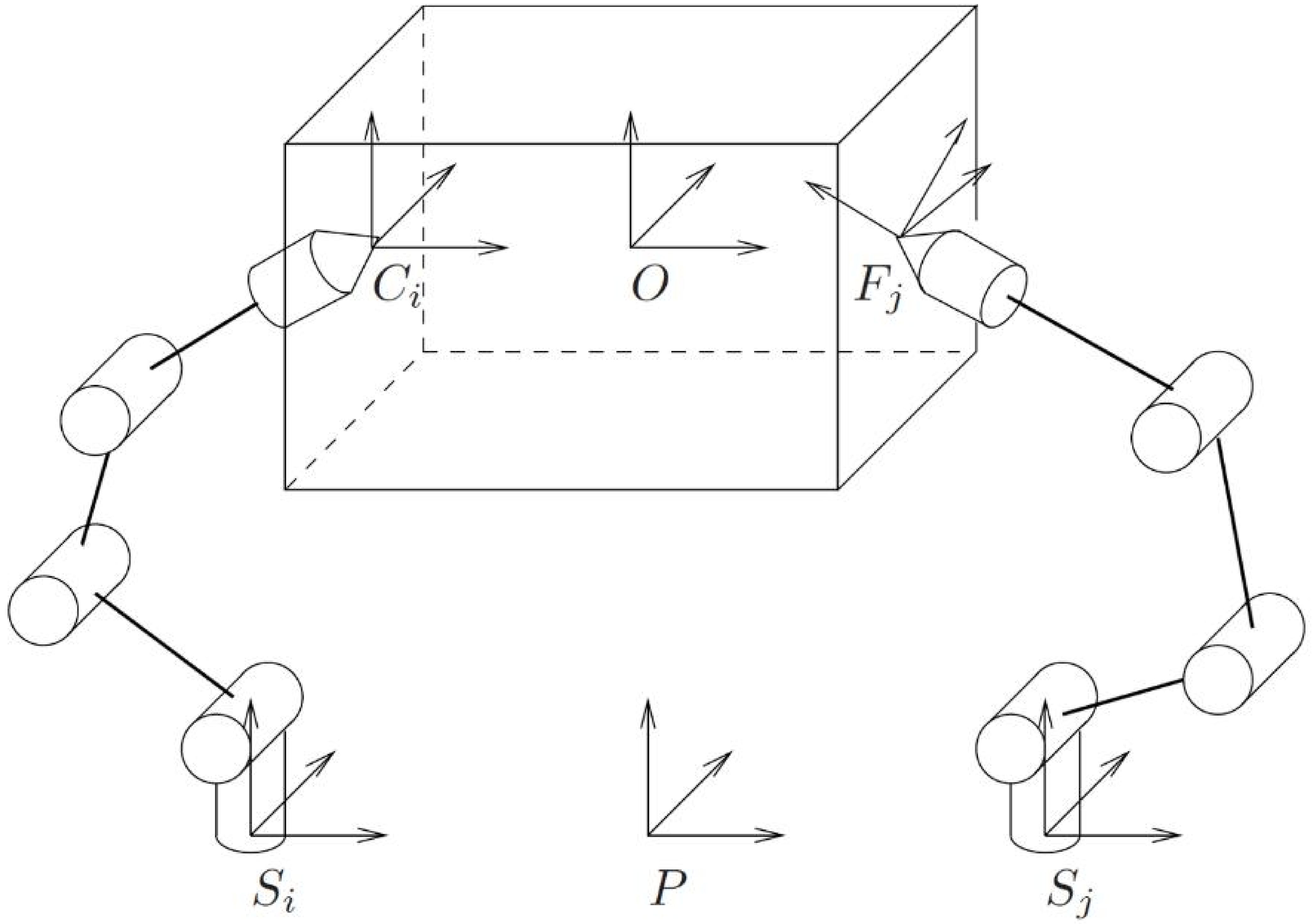

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  - Assume that we are grasping this box using two arms
  - We apply torques at each joint through the installed motors
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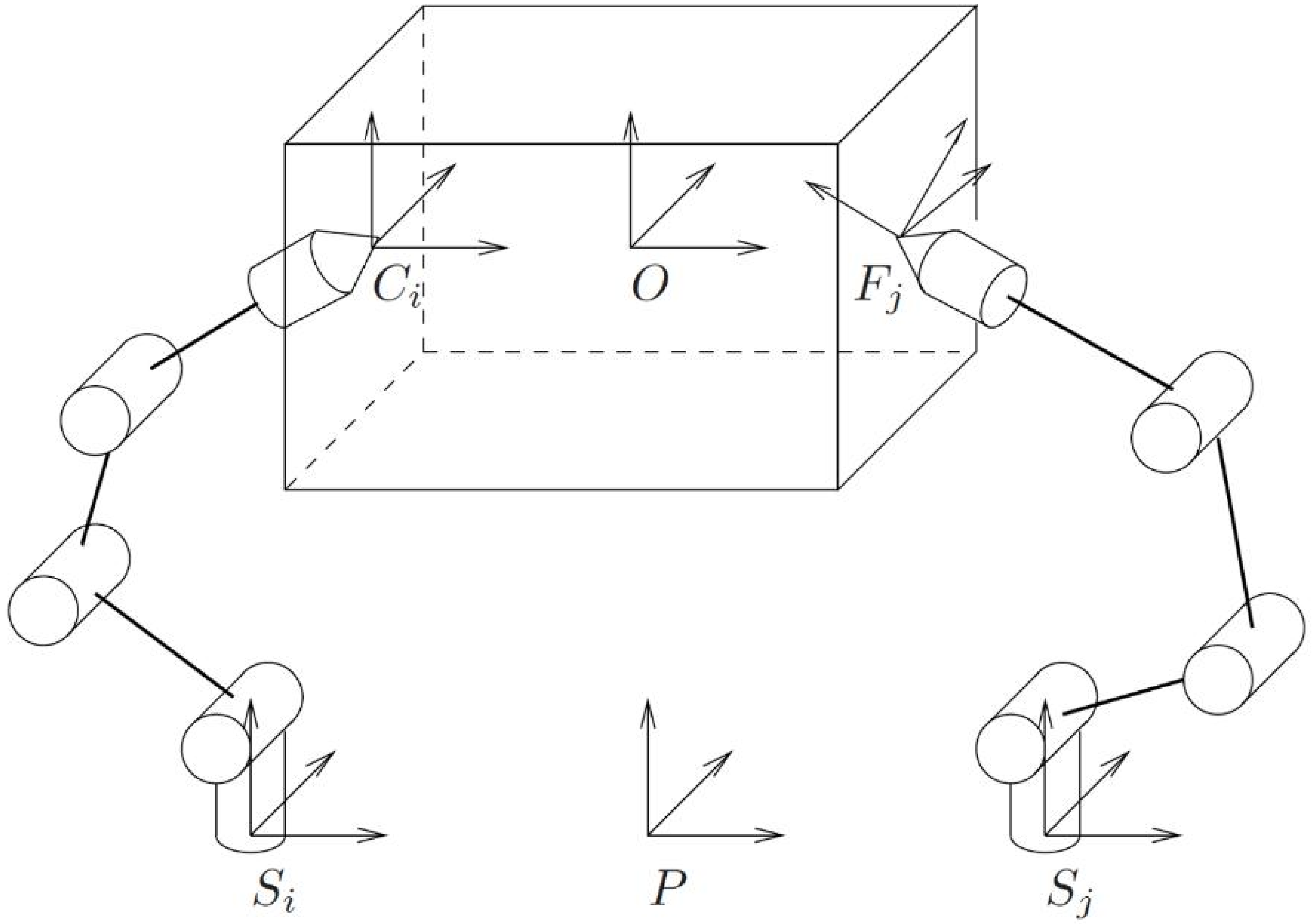
  - The contact area will create certain force and torque at the contact point
    - force: pressure and friction
    - torque: e.g., anti-twisting friction force caused by the area contact



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Contact Coordinate Frame

- We build a **contact frame** C_i at each contact point
- The z-axis of the frame points inward along *surface normal*
- When recording force and torque at the contact point, it is natural to set C_i as the *observer's frame*, i.e.,

$$\boldsymbol{F}^{C_i} = \begin{bmatrix} \boldsymbol{f}^{C_i} \\ \boldsymbol{\tau}^{C_i} \end{bmatrix}$$

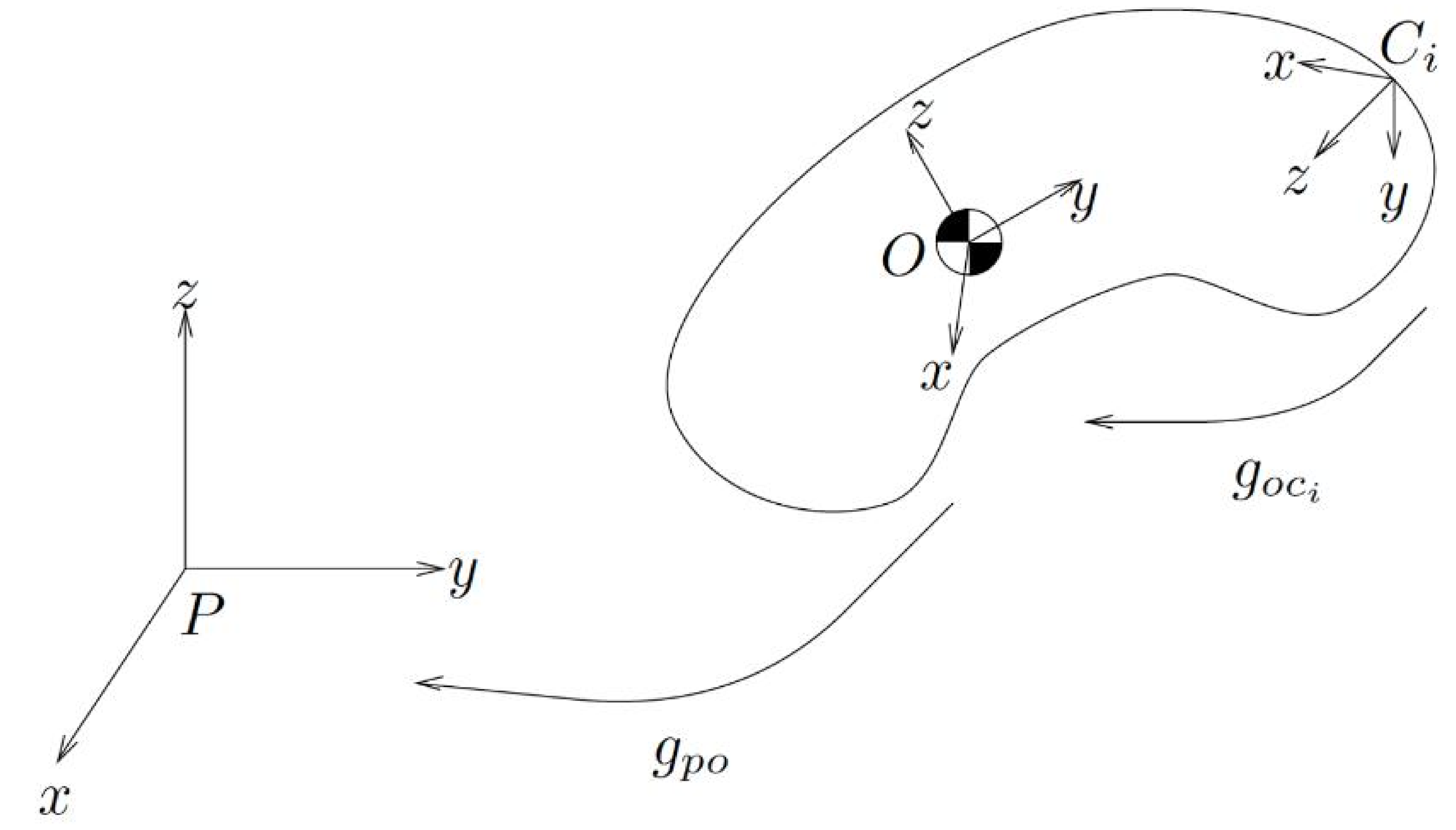


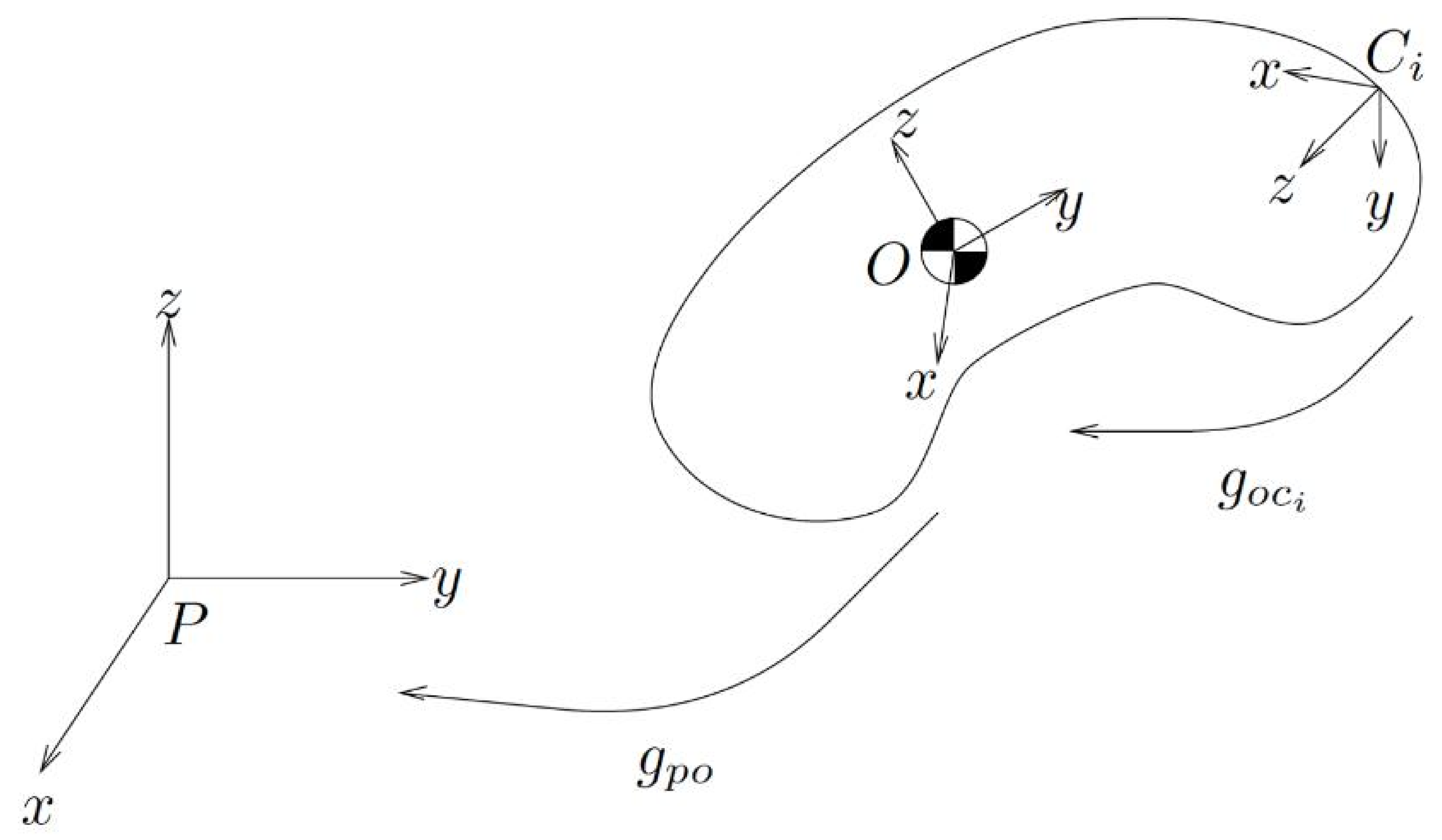
Figure 5.2: Coordinate frames for contact and object forces.

Some Kinds of Contact Forces

Contact Type	Frictionless point contact	Point contact with friction	Soft-finger
$\boldsymbol{F}^C$	$\begin{bmatrix} 0 \\ 0 \\ f_z \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} f_x \\ f_y \\ f_z \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} f_x \\ f_y \\ f_z \\ 0 \\ 0 \\ \tau_z \end{bmatrix}$

Adding Forces and Torques

- Suppose we have calculated $\boldsymbol{F}^{C_i}$ at each contact (will learn later)
- What is the combined force and torque?



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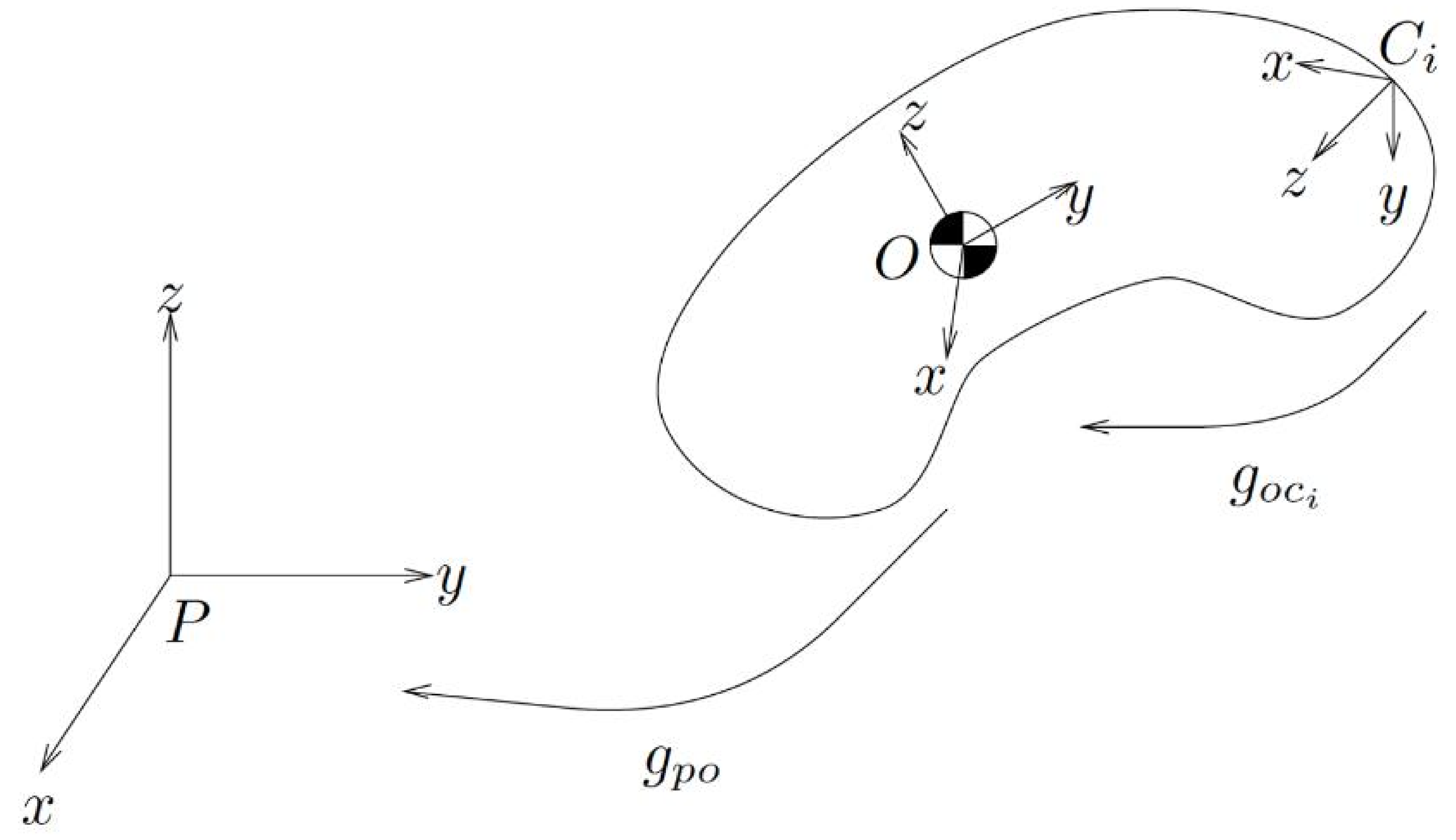

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Adding Forces and Torques

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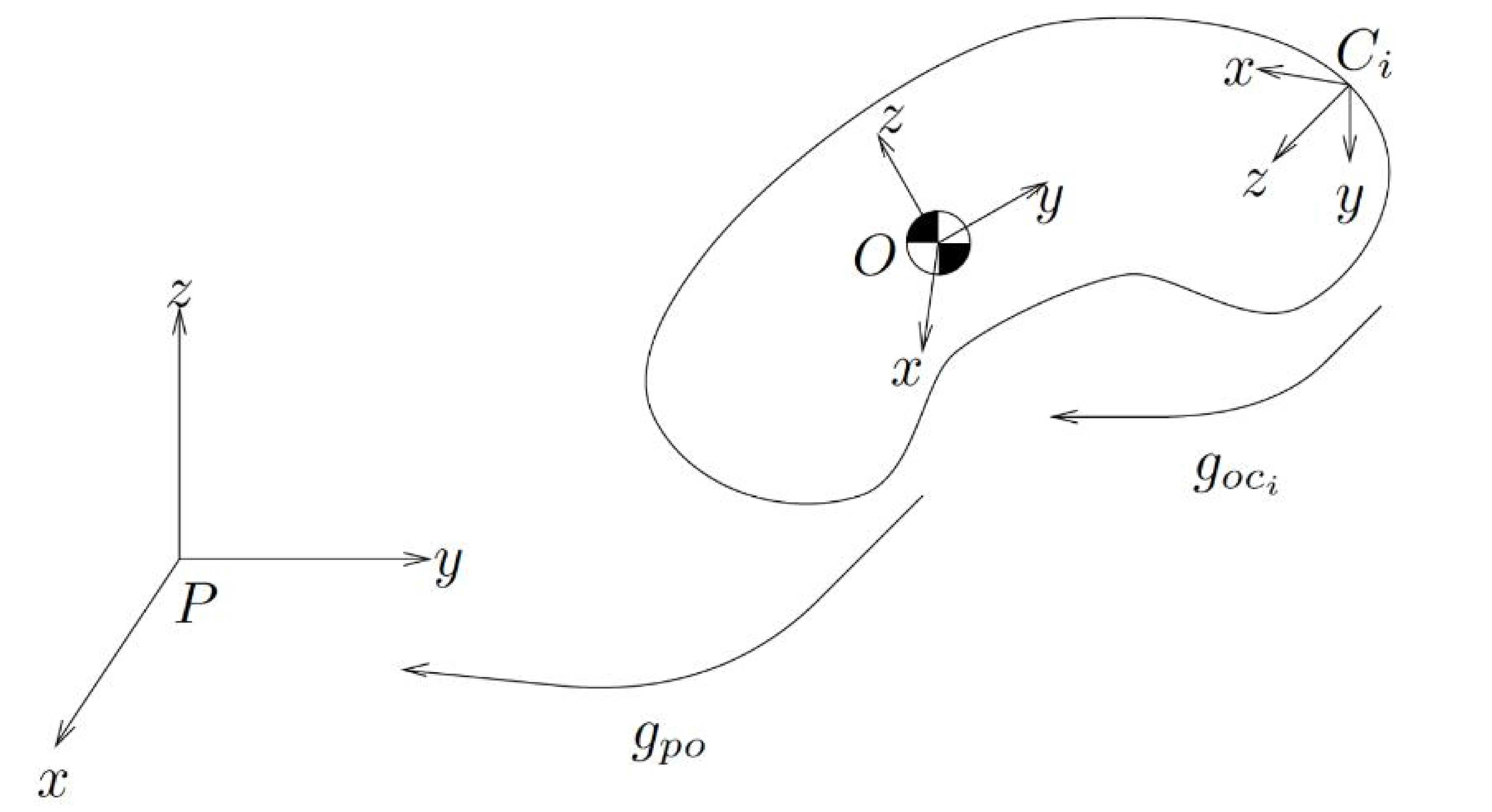
- What is the combined force and torque?
- We **cannot** directly add forces and torques recorded using different observer frames
- However, we can change all to the same frame (e.g., body frame) and add together!



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Change of Observer's Frame for Force and Torque

Consider the question of changing the observer's frame for force and torque

- We would relate
  - $\boldsymbol{f}^1$ and $\boldsymbol{f}^2$
  - $\boldsymbol{\tau}^1 = \boldsymbol{r}^1 \times \boldsymbol{f}^1$ and $\boldsymbol{\tau}^2 = \boldsymbol{r}^2 \times \boldsymbol{f}^2$
- Note that

$$
\begin{aligned}
\boldsymbol{r}^2 &= R_{2\to1}\boldsymbol{r}^1 + \boldsymbol{t}_{2\to1} \\
\boldsymbol{f}^2 &= R_{2\to1}\boldsymbol{f}^1
\end{aligned}
$$

- Plug in the definition, and we derive that

$$
\begin{bmatrix} \boldsymbol{f}^2 \\ \boldsymbol{\tau}^2 \end{bmatrix} = \begin{bmatrix} R_{2\to1} & 0 \\ [\boldsymbol{t}_{2\to1}]R_{2\to1} & R_{2\to1} \end{bmatrix} \begin{bmatrix} \boldsymbol{f}^1 \\ \boldsymbol{\tau}^1 \end{bmatrix} = (\mathrm{Ad}_{1\to2})^{\mathrm{T}} \begin{bmatrix} \boldsymbol{f}^1 \\ \boldsymbol{\tau}^1 \end{bmatrix}
$$

Change of Observer's Frame for Force and Torque

- Define $\boldsymbol{F}^o = \begin{bmatrix} \boldsymbol{f}^o \\ \boldsymbol{\tau}^o \end{bmatrix}$, then formula for change of frame is:

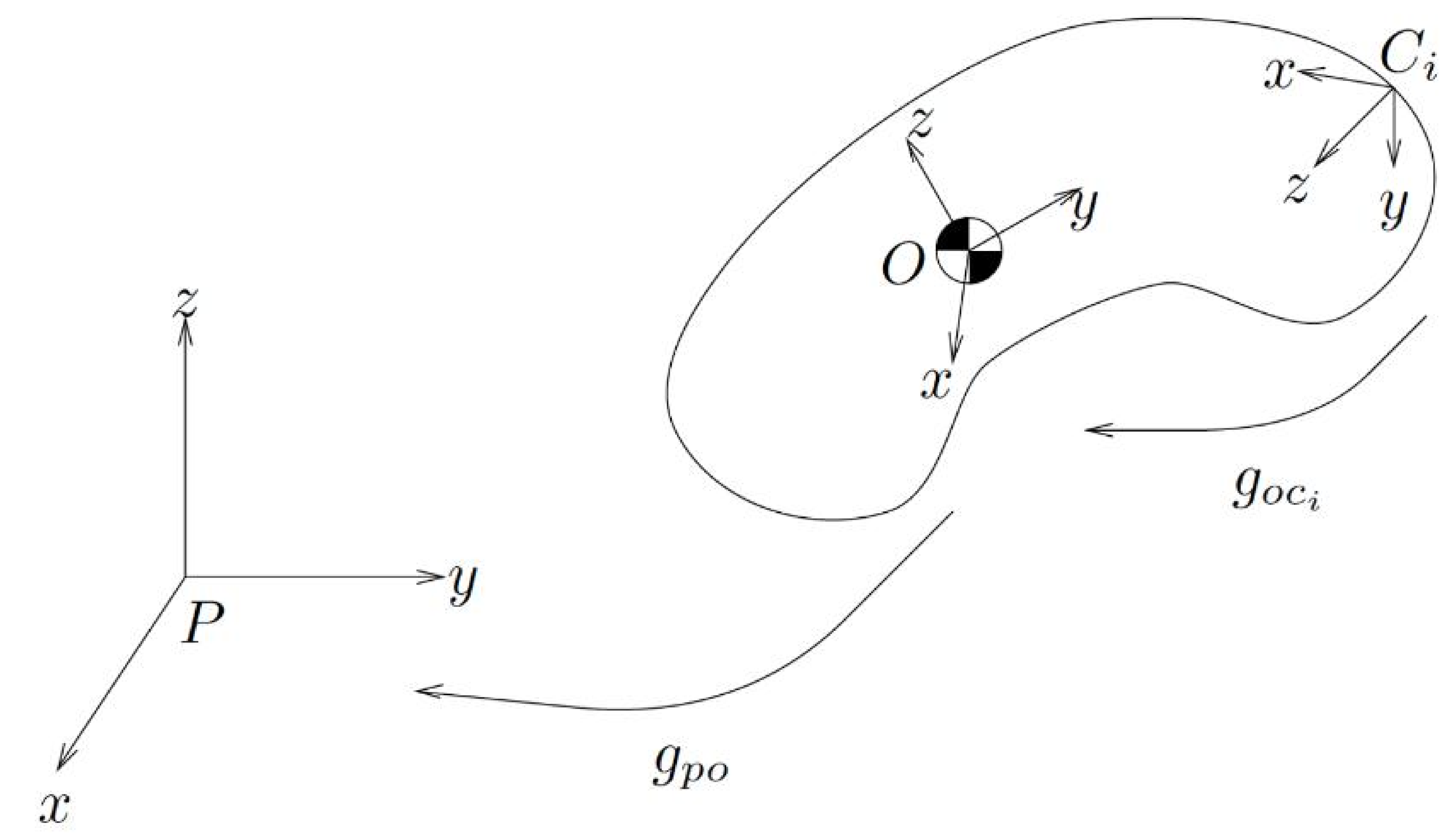
$$\boldsymbol{F}^2 = (\mathrm{Ad}_{1\to 2})^{\mathrm{T}} \boldsymbol{F}^1 \qquad \text{(change of frame)}$$

- Using definitions and frame change equations, it is easy to verify that the following equation to compute the **power** of the system input (change rate of kinetic energy):

$$(\boldsymbol{F}^b)^T \boldsymbol{\xi}^b = (\boldsymbol{F}^o)^T \boldsymbol{\xi}^o = \frac{\mathrm{d}T}{\mathrm{d}t} \qquad \text{(system input power)}$$

Solution to Adding Forces and Torques

$$\boldsymbol{F}^b = \sum_{i=1}^{k} [\mathrm{Ad}_{\mathrm{C_i \to b}}]^{\mathrm{T}} \boldsymbol{F}^{\mathrm{C_i}}$$



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Change of Observer's Frame for Momentum and Angular Momentum

Consider the question of changing the observer's frame for momentum and angular momentum

- We would relate
  - $\boldsymbol{p}^1 = m\boldsymbol{v}^1$ and $\boldsymbol{p}^2 = m\boldsymbol{v}^2$
  - $\boldsymbol{L}^1 = \boldsymbol{r}^1 \times m\boldsymbol{v}^1$ and $\boldsymbol{L}^2 = \boldsymbol{r}^2 \times m\boldsymbol{v}^2$
- Note that

$$\begin{aligned} \boldsymbol{r}^2 &= R_{2\to 1}\boldsymbol{r}^1 + \boldsymbol{t}_{2\to 1} \\ \boldsymbol{v}^2 &= R_{2\to 1}\boldsymbol{v}^1 \end{aligned}$$

- The same derivation as force and torque pair, and we get $\begin{bmatrix} \boldsymbol{p}^2 \\ \boldsymbol{L}^2 \end{bmatrix} = (\mathrm{Ad}_{1\to 2})^{\mathrm{T}} \begin{bmatrix} \boldsymbol{p}^1 \\ \boldsymbol{L}^1 \end{bmatrix}$

Read by Yourself

Change of Observer's Frame for Momentum and Angular Momentum

- Define $\boldsymbol{P}^o = \begin{bmatrix} \boldsymbol{p}^o \\ \boldsymbol{L}^o \end{bmatrix}$, and the formula for change of frame is:

$$\boldsymbol{P}^2 = (\mathrm{Ad}_{1\to 2})^{\mathrm{T}} \boldsymbol{P}^1 \qquad \text{(change of frame)}$$

- Note: similar to linear momentum that $\boldsymbol{p}^o = \frac{\mathrm{d}T}{\mathrm{d}\boldsymbol{v}^o}$ for translation-only motion, it is straight-forward to verify that

$$\boldsymbol{P}^o = \frac{dT}{d\boldsymbol{\xi}^o} = \mathfrak{M}^o \xi^o \qquad \text{(system input power)}$$

Summary

- We have learned basic concepts for body motion dynamics
  - Properties of objects: mass, rotational inertia
  - Motion state: momentum, angular momentum
  - Action: force, torque
  - Energy perspective: kinetic energy
- We have also introduced various equations for changing the observer's frame

Forward and Inverse Dynamics

Dynamics Example: Grasp

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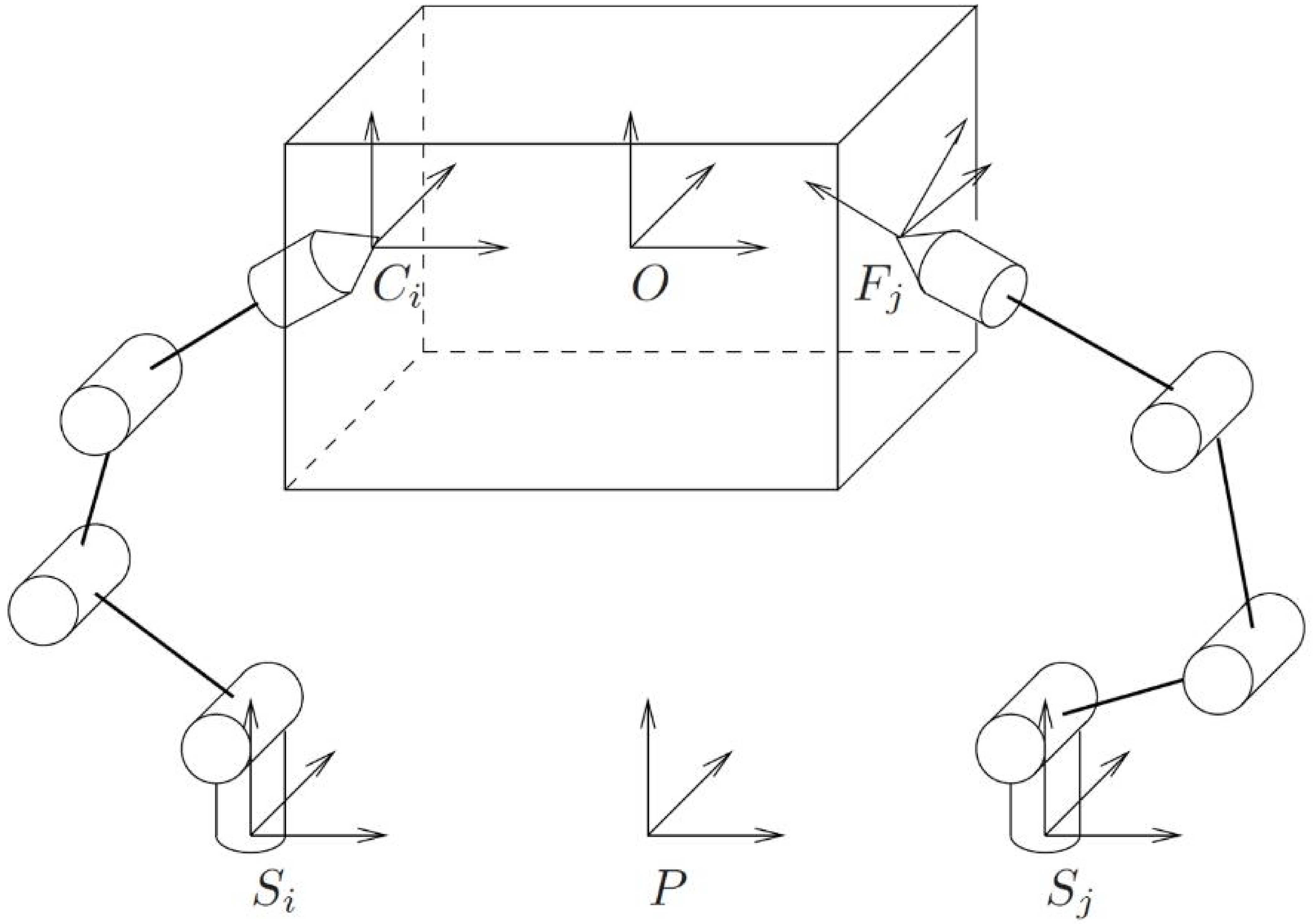

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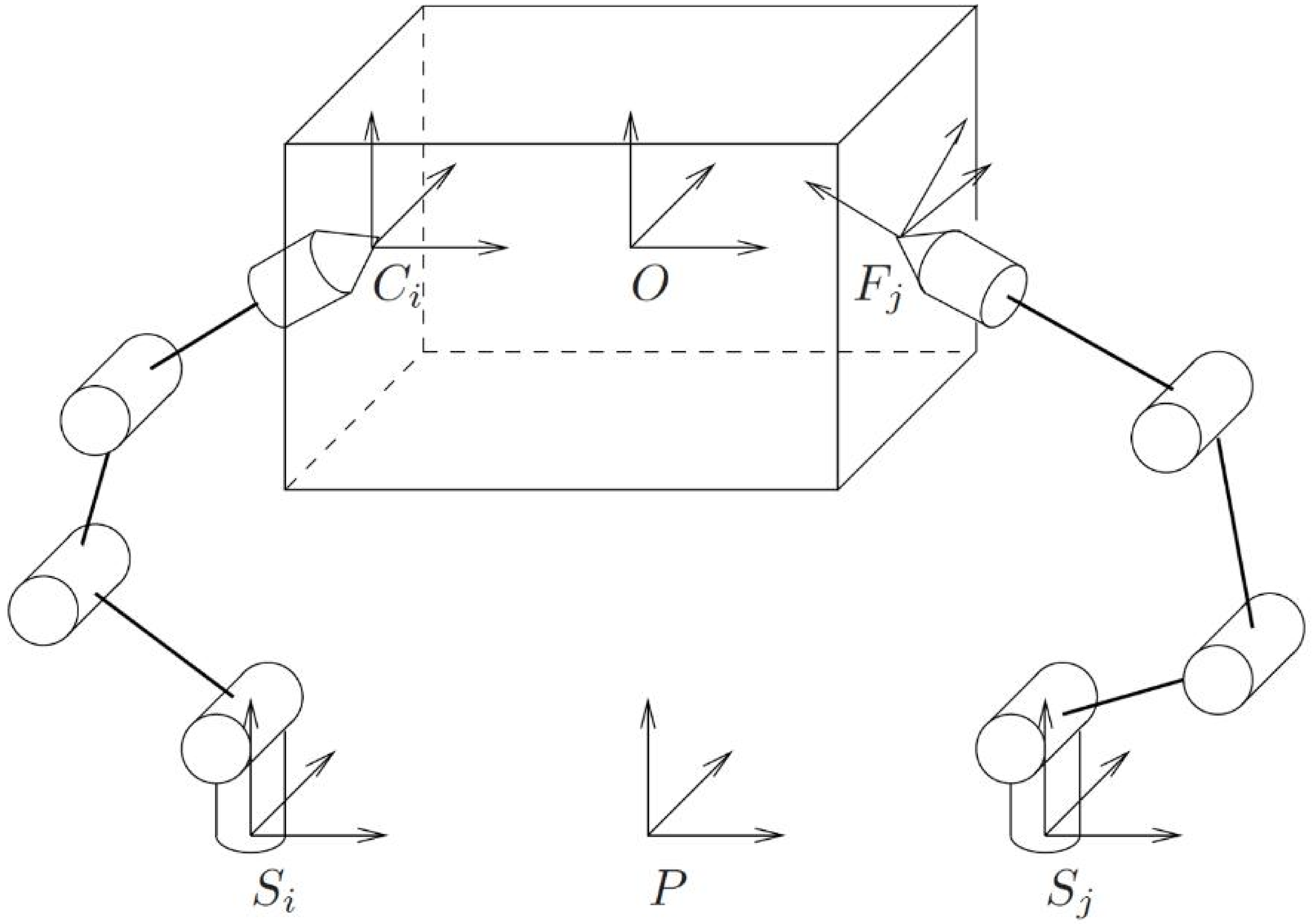

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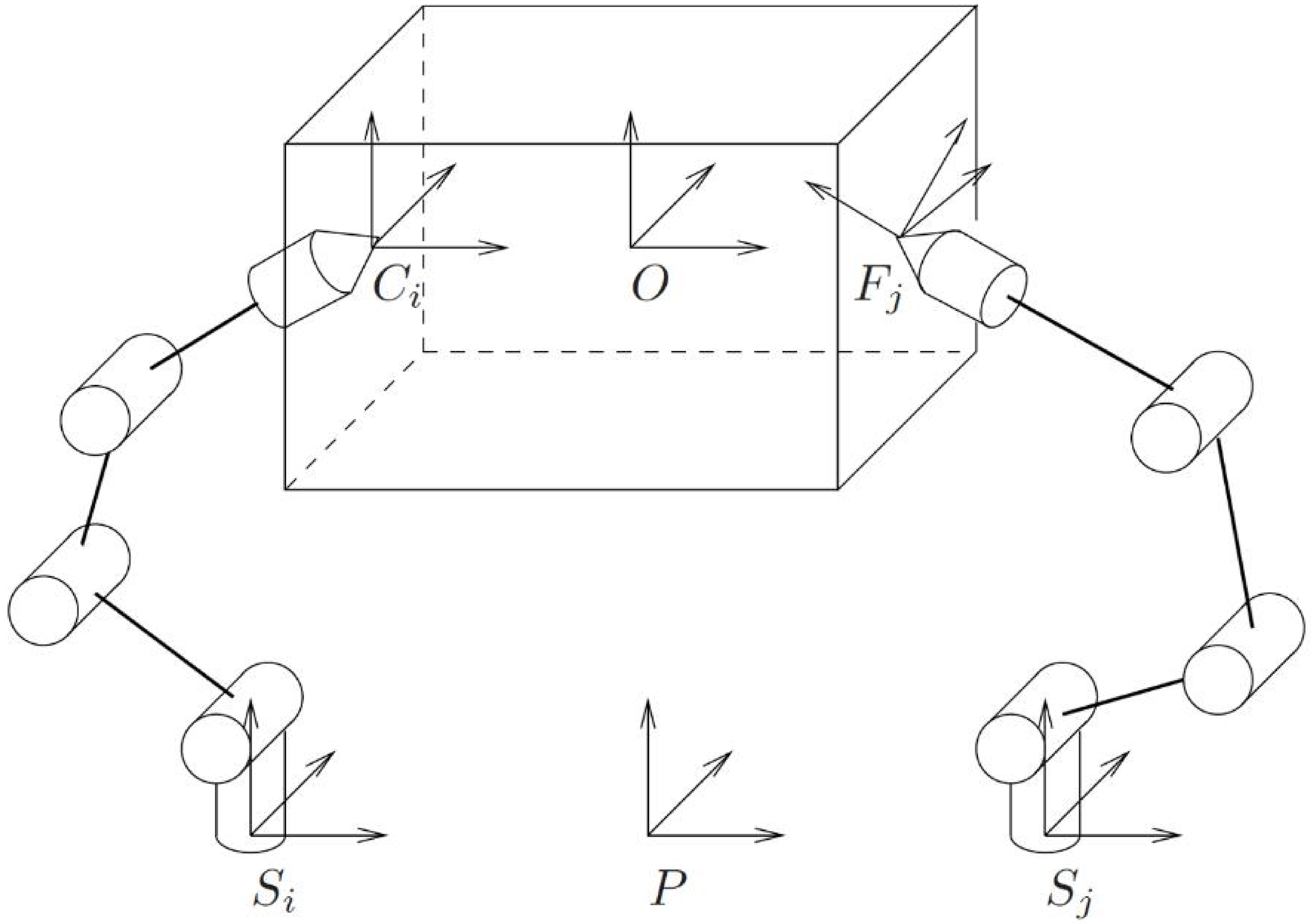

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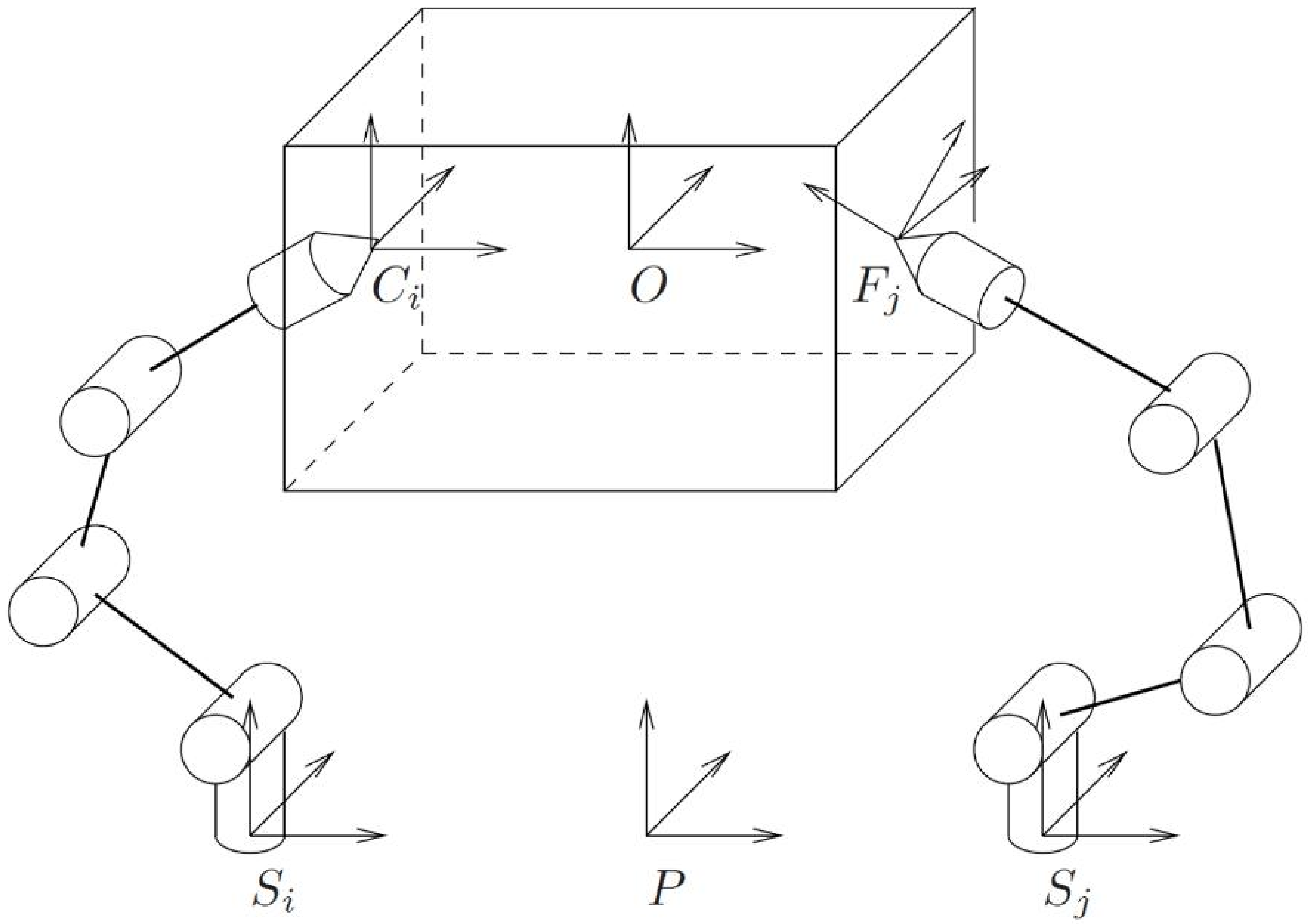

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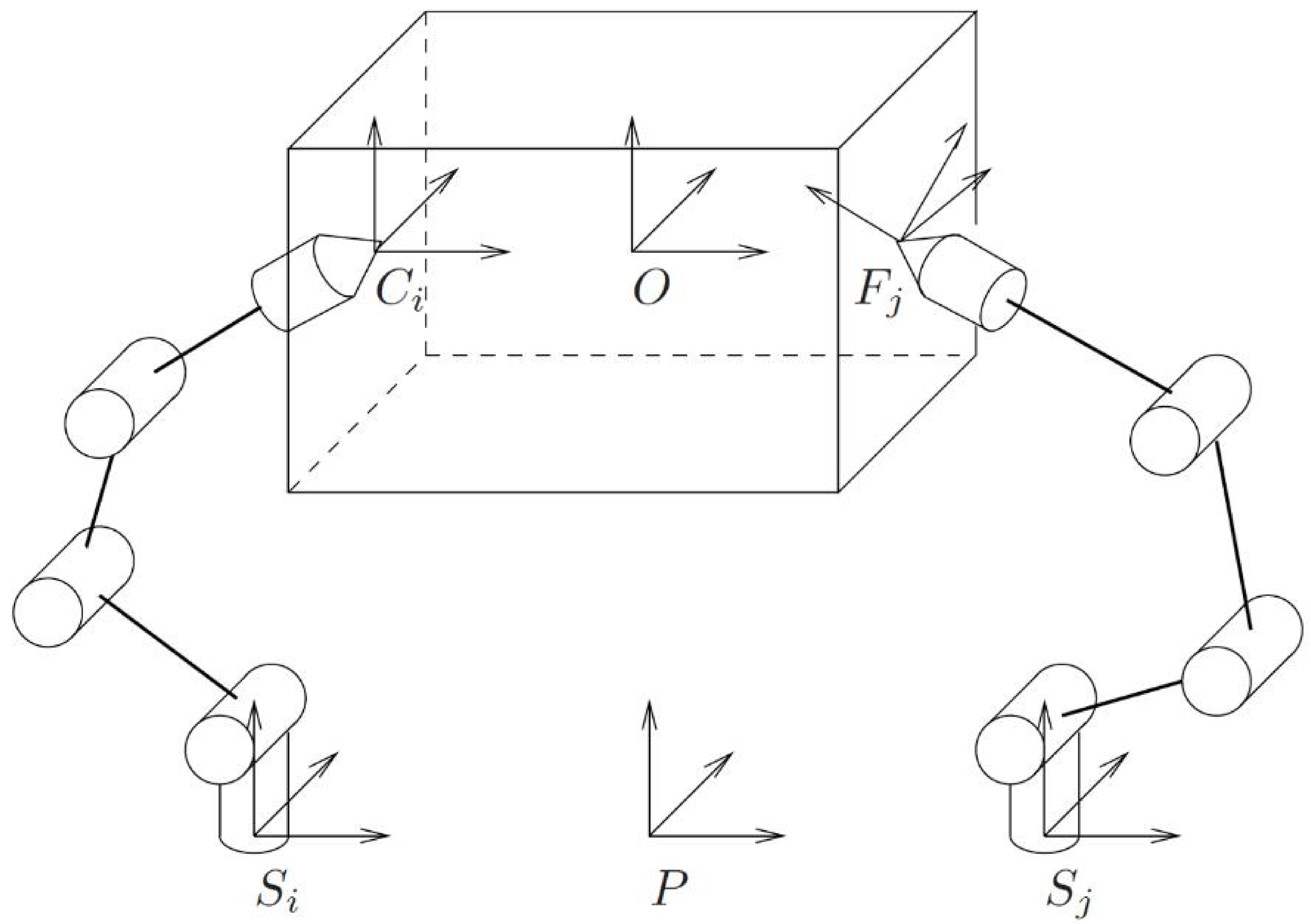

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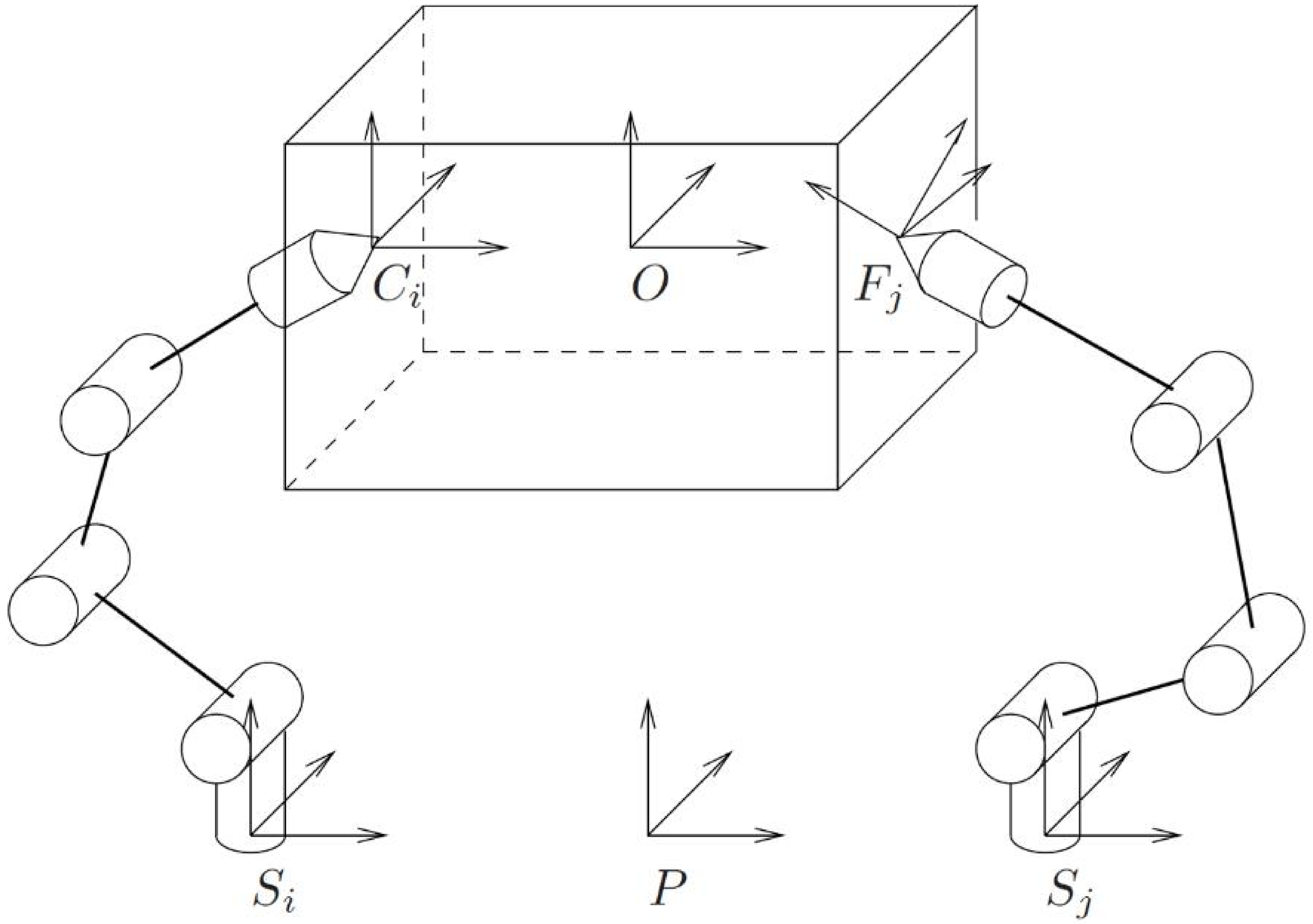
Q1: How to compute force at the tips from the torques at joints?



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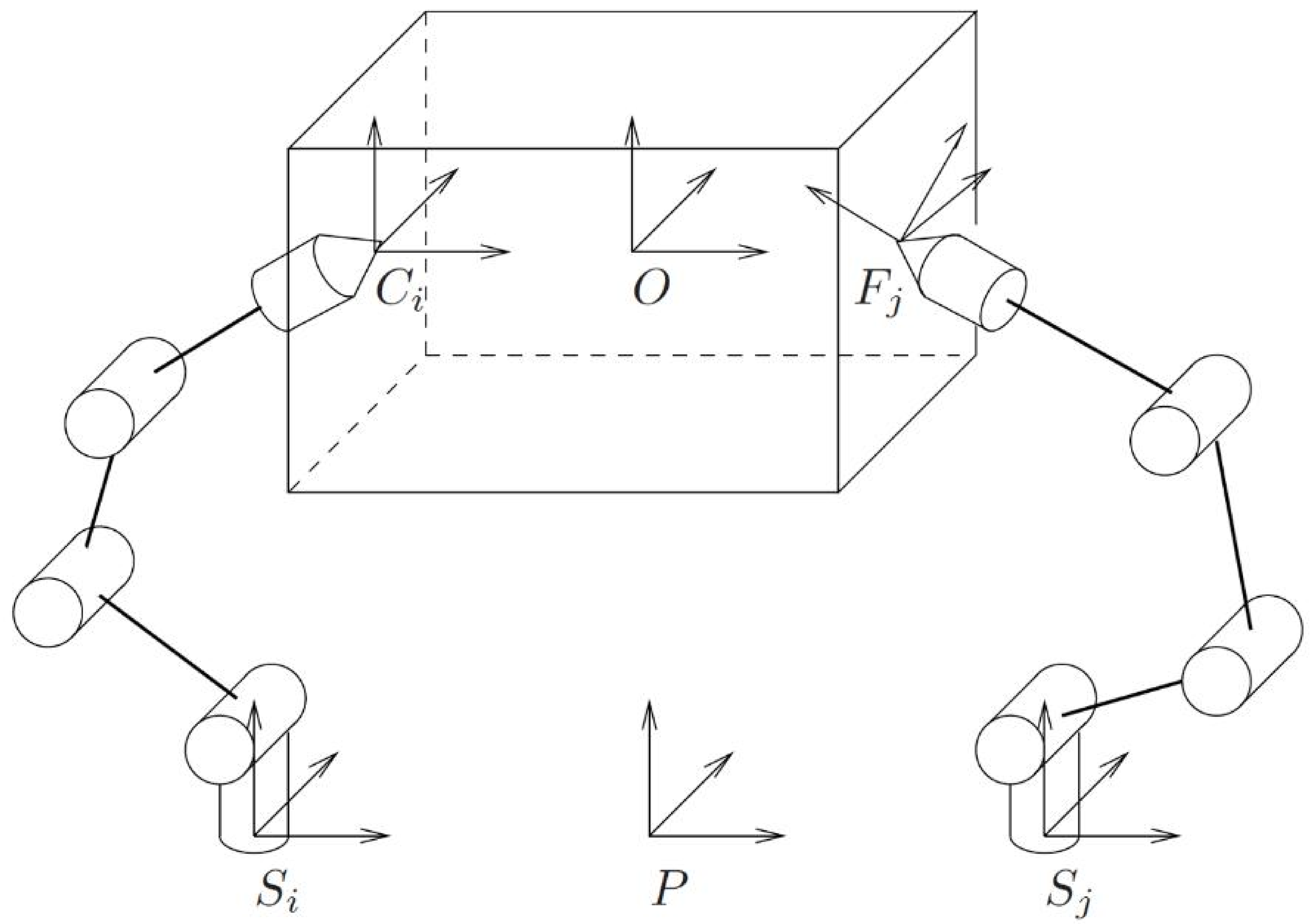
Q2: To keep the box static, what is the balance condition?



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Dynamics Example: Grasp

- Parameterization
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  - $\tau \in \mathbb{R}^n$: vector of joint forces/torques

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- Task
  - **Forward dynamics:** Determine acceleration $\ddot{\theta}$ given the state $(\theta, \dot{\theta})$ and the joint forces/torques

$$\ddot{\theta} = f(\tau; \theta, \dot{\theta})$$

  - **Inverse dynamics:** Finding torques/forces given state $\theta, \dot{\theta}$ and desired acceleration $\ddot{\theta}$

$$\tau = g(\ddot{\theta}; \theta, \dot{\theta})$$

Lagrangian vs. Newton-Euler Methods

- There are typically two ways to derive the equation of motion for an open-chain robot: Lagrangian method and Newton-Euler method

Lagrangian Formulation

- Energy-based method
- Dynamic equations in closed form
- Often used for study of dynamic properties and analysis of control methods

Newton-Euler Formulation

- Balance of forces/torques
- Dynamic equations numeric/recursive form
- Often used for numerical solution of forward/inverse dynamics

Lagrangian Method

Generalized Coordinates and Forces

- Consider k particles. Let $\boldsymbol{f}_i$ be the force acting on the ith particle, $\boldsymbol{p}_i$ be its position.

Generalized Coordinates and Forces

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- Now consider the case in which some particles are rigidly connected, imposing constraints on their positions

$$\alpha_j(p_1, \ldots, p_k) = 0, \quad j = 1, \ldots, n_c$$

  where n_c is the number of constraints

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  where n_c is the number of constraints
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Generalized Coordinates and Forces

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- We introduce $n := 3k - n_c$ independent variables q_i's, called the **generalized coordinates**

$$\begin{cases} \alpha_j(p_1, \ldots, p_k) = 0 \\ j = 1, \ldots, n_c \end{cases} \quad \Leftrightarrow \quad \begin{cases} p_i = \gamma_i(q_1, \ldots, q_n) \\ i = 1, \ldots, k \end{cases}$$

Generalized Coordinates and Forces

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- This idea of handling constraints can be extended to interconnected rigid bodies (kinematic chains).

Euler-Lagrange Equation

- Now let $q \in \mathbb{R}^n$ be the generalized coordinates and $\boldsymbol{f} \in \mathbb{R}^n$ be the generalized forces of some constrained dynamical system.
- **Lagrangian function:** $L(q, \dot{q}) = T(q, \dot{q}) - V(q)$
  - $T(q, \dot{q})$: kinetic energy of system
  - $V(q)$: potential energy
- **Euler-Lagrange Equations:**

$$\boldsymbol{f} = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}$$

Logic behind Concepts in Lagrangian Dynamics