L4: Screw and Twist

Hao Su

Agenda

- Screw (6D representation of rigid motion)
- Twist (6D representation of rigid motion velocity)

Rigid Transformation and $\mathbb{SE}(3)$

The Set of Rigid Transformations

- $\mathbb{SE}(3) := \left\{ T = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}, R \in \mathbb{SO}(3), t \in \mathbb{R}^3 \right\}$
- $\mathbb{SE}(3)$: "Special Euclidean Group"

- "Group": closed under matrix multiplication and other conditions of group
- "Euclidean": R and t
- "Special": $\det(R) = 1$

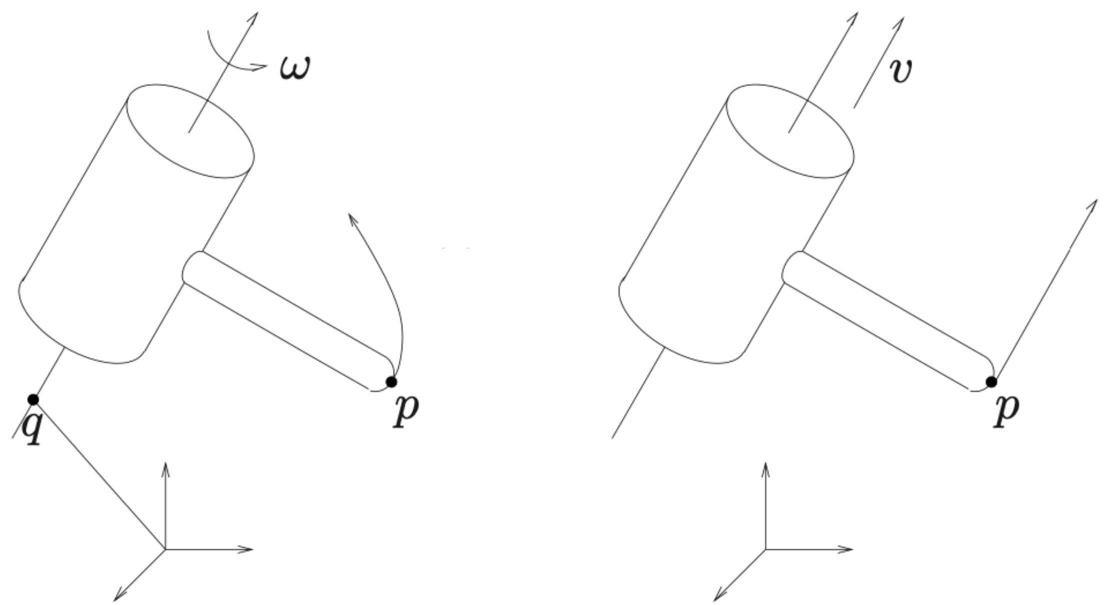
- 6 DoF

Conditions of Group: https://en.wikipedia.org/wiki/Group_(mathematics)#Definition

- Recall Euler's Theorem about $\mathbb{SO}(3)$:
  - Any rotation in $\mathbb{SO}(3)$ is equivalent to rotation about a fixed axis $\hat{\omega} \in \mathbb{R}^3$ through a positive angle θ
- Similar results for $\mathbb{SE}(3)$: Screw Parameterization
- (In your mind, think T as a linear transformation)

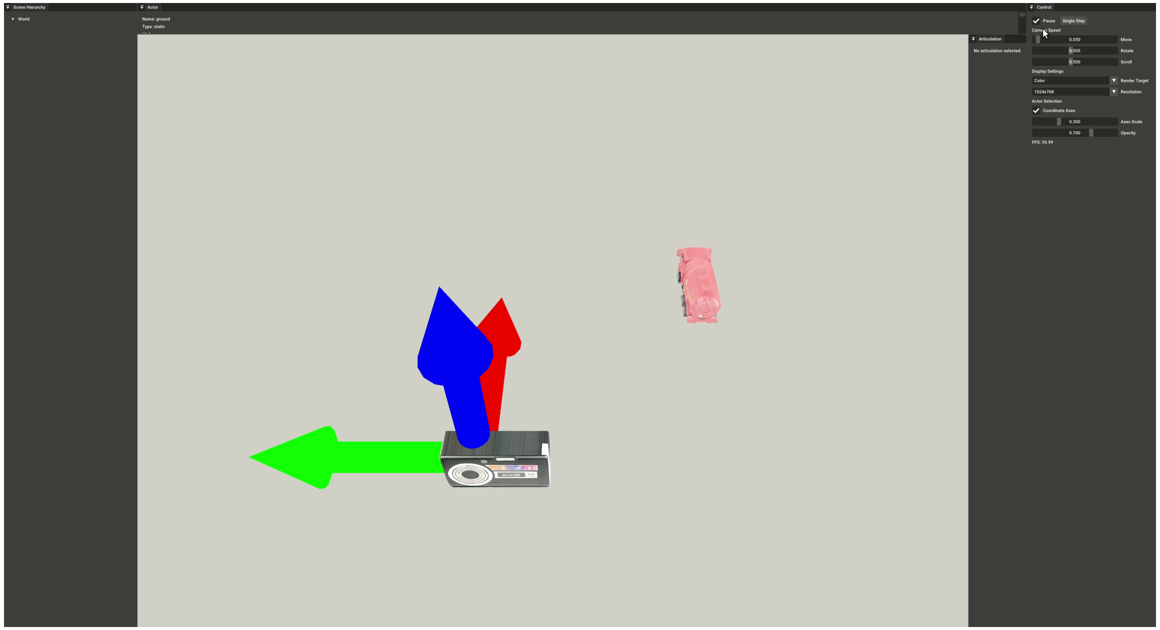
Screw Motion Theorem

- Any rigid body motion is equivalent to rotating about one axis while also translating along the axis
- **The axis may not pass the origin**



Screw Motion Theorem

- Any rigid body motion is equivalent to rotating about one axis while also translating along the axis
- Recall our question of “canonical” rigid transformation decomposition—by sharing rotation axis and translation direction, we identify the decomposition



Review: Lie algebra of $\mathbb{SO}(3)$

- Motion interpretation

$\hat{\omega}$: motion direction

- Exponential coordinate

$\overrightarrow{\theta} = \hat{\omega}\theta \in \mathbb{R}^3$ (rot vector)

- Exponential map

$R = \exp([\hat{\omega}]\theta) \in \mathbb{SO}(3)$

- Tangent space at $R = I$

$[\hat{\omega}]\theta \in \mathfrak{so}(3)$

Goal: The Lie Algebra of $\mathbb{SE}(3)$

- Motion interpretation

  $\hat{\omega}$: motion direction

- Exponential coordinate

  $\vec{\theta} = \hat{\omega}\theta \in \mathbb{R}^3$ (rot vector)

- Exponential map

  $R = \exp([\hat{\omega}]\theta) \in \mathbb{SO}(3)$

- Tangent space at $R = I$

  $[\hat{\omega}]\theta \in \mathfrak{so}(3)$

- Motion interpretation

  $\hat{\xi}$: 6D motion direction

- Exponential coordinate

  $\chi = \hat{\xi}\theta \in \mathbb{R}^6$ (screw)

- Exponential map

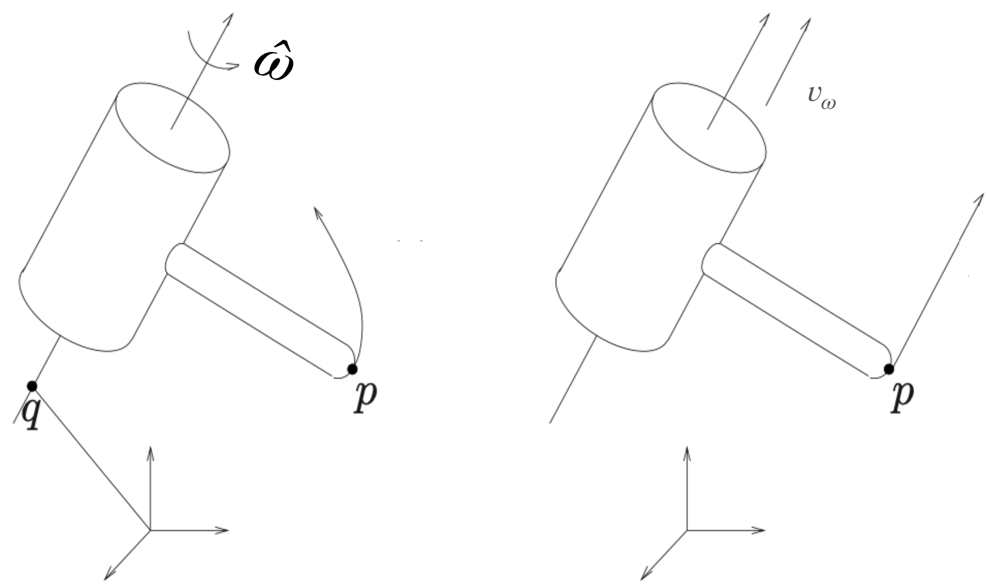
  $T = \exp([\hat{\xi}]\theta) \in \mathbb{SE}(3)$

- Tangent space at $T = I$

  $[\hat{\xi}]\theta \in \mathfrak{se}(3)$

An Imaged Motion for $T \in \mathbb{SE}(3)$

- Transforming by $T \iff$ **rotating** about one axis while also **translating** along the axis
- Assume an arbitrary point q on the axis, a **unit** vector $\hat{\omega}$ denoting axis, and the angle θ
- Assume the translation along $\hat{\omega}$ is d_ω



Screw Parameterization

- In $\mathbb{SO}(3)$, we have

$$\mathrm{Rot}(\hat{\omega}, \theta)x = (I + \theta[\hat{\omega}] + \frac{\theta^2}{2!}[\hat{\omega}]^2 + \frac{\theta^3}{3!}[\hat{\omega}]^3 + \cdots)x$$

- In $\mathbb{SE}(3)$, we have a similar result ($x \in \mathbb{R}^4$ by homogeneous coordinate):

$$\mathrm{Trans}(\hat{\omega}, \theta, q, d_\omega)x = (I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots)x$$

where $A = \begin{bmatrix} [\hat{\omega}\theta] & -[\hat{\omega}\theta]q + d_\omega \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times4}$

Screw Parameterization

$$\mathrm{Trans}(\hat{\omega}, \theta, q, d_\omega)x = (I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots)x, \text{ where } A = \begin{bmatrix} [\hat{\omega}\theta] & -[\hat{\omega}\theta]q + d_\omega \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times4}$$

- Similar to $\mathrm{Rot}(\hat{\omega}, \theta) \equiv e^{[\hat{\omega}]\theta}$, we have $\mathrm{Trans}(\hat{\omega}, \theta, q, d_\omega) = e^A$
- We try to align the form of $\mathrm{Rot}(\hat{\omega}, \theta)$ and $T = \mathrm{Trans}(\hat{\omega}, \theta, q, d_\omega)$:
  - Notice that the power of e is the product of a matrix that corresponds to motion direction and a scalar, we factor A similarly

Screw Parameterization

$$\mathrm{Trans}(\hat{\omega}, \theta, q, d_\omega)x = (I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots)x, \text{ where } A = \begin{bmatrix} [\hat{\omega}\theta] & -[\hat{\omega}\theta]q + d_\omega \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times4}$$

- Similar to $\mathrm{Rot}(\hat{\omega}, \theta) \equiv e^{[\hat{\omega}]\theta}$, we have $\mathrm{Trans}(\hat{\omega}, \theta, q, d_\omega) = e^A$
- We try to align the form of $\mathrm{Rot}(\hat{\omega}, \theta)$ and $T = \mathrm{Trans}(\hat{\omega}, \theta, q, d_\omega)$:
  - Notice that the power of e is the product of a matrix that corresponds to motion direction and a scalar, we factor A similarly
  - Let $A = \begin{bmatrix} [\hat{\omega}] & d \\ 0 & 0 \end{bmatrix} \theta$, where $d = \dfrac{-[\hat{\omega}\theta]q + d_\omega}{\theta}$, then $T = \exp\left(\begin{bmatrix} [\hat{\omega}] & d \\ 0 & 0 \end{bmatrix} \theta\right)$

Screw Parameterization

$$\mathrm{Trans}(\hat{\omega}, \theta, q, d_\omega)x = (I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots)x, \text{ where } A = \begin{bmatrix} [\hat{\omega}\theta] & -[\hat{\omega}\theta]q + d_\omega \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

- Similar to $\mathrm{Rot}(\hat{\omega}, \theta) \equiv e^{[\hat{\omega}]\theta}$, we have $\mathrm{Trans}(\hat{\omega}, \theta, q, d_\omega) = e^A$
- We try to align the form of $\mathrm{Rot}(\hat{\omega}, \theta)$ and $T = \mathrm{Trans}(\hat{\omega}, \theta, q, d_\omega)$:
  - Notice that the power of e is the product of a matrix that corresponds to motion direction and a scalar, we factor A similarly
  - Let $A = \begin{bmatrix} [\hat{\omega}] & d \\ 0 & 0 \end{bmatrix} \theta$, where $d = \dfrac{-[\hat{\omega}\theta]q + d_\omega}{\theta}$, then $T = \exp\left(\begin{bmatrix} [\hat{\omega}] & d \\ 0 & 0 \end{bmatrix} \theta\right)$
  - A special case:
    - **When the motion is translation-only, define $\hat{\omega} = 0$, $\theta = \|d_\omega\|$, and** $d = \dfrac{d_\omega}{\|d_\omega\|}$

Screw Parameterization

$$\text{Trans}(\hat{\omega}, \theta, q, d_\omega)x = (I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots)x, \text{ where } A = \begin{bmatrix} [\hat{\omega}\theta] & -[\hat{\omega}\theta]q + d_\omega \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

- Let $A = \begin{bmatrix} [\hat{\omega}] & d \\ 0 & 0 \end{bmatrix} \theta$, where $d = \dfrac{-[\hat{\omega}\theta]q + d_\omega}{\theta}$, then $T = \exp\left(\begin{bmatrix} [\hat{\omega}] & d \\ 0 & 0 \end{bmatrix} \theta\right)$
- The following rule introduces $\hat{\xi}$ so that $T = \exp\left(\begin{bmatrix} [\hat{\omega}] & d \\ 0 & 0 \end{bmatrix} \theta\right) \equiv e^{[\hat{\xi}]\theta}$:
  - $\hat{\xi} = \begin{bmatrix} d \\ \hat{\omega} \end{bmatrix} \in \mathbb{R}^6$ and $[\hat{\xi}]\theta = \begin{bmatrix} [\hat{\omega}] & d \\ 0 & 0 \end{bmatrix} \theta$

Screw Parameterization

- To sum up, $\mathrm{Trans}(\hat{\omega}, \theta, q, d_\omega) = e^{[\hat{\xi}]\theta}$, where
  - $\hat{\xi} = \begin{bmatrix} d \\ \hat{\omega} \end{bmatrix} \in \mathbb{R}^6$ ($\hat{\omega} = 0$ and $d = \dfrac{d_\omega}{\|d_\omega\|}$ if translation-only)
  - $d = \dfrac{-[\hat{\omega}\theta]q + d_\omega}{\theta}$,
  - $[\hat{\xi}]\theta = \begin{bmatrix} [\hat{\omega}] & d \\ 0 & 0 \end{bmatrix} \theta$
- $\chi = \hat{\xi}\theta = \begin{bmatrix} d \\ \hat{\omega} \end{bmatrix} \theta = \begin{bmatrix} -[\hat{\omega}]q\theta + d_\omega \\ \hat{\omega}\theta \end{bmatrix}$ is called **screw,** or **exponential coordinate**
- Introducing the inverse function of $T = e^{[\chi]}$, $\chi = \log(T)$
- $\hat{\xi}$ is called **unit twist,** which describes **motion direction**

Generate $T \in \mathbb{SE}(3)$ from $\hat{s}\theta$

- Recall **Rodrigues Formula** for rotations:

$$e^{[\hat{\omega}]\theta} = I + [\hat{\omega}]\sin\theta + [\hat{\omega}]^2(1 - \cos\theta)$$

- Similarly, using Taylor's expansion definition of $\exp$,

$$e^{[\hat{\xi}]\theta} = I + [\hat{\xi}] + \frac{1 - \cos\theta}{\theta^2}[\hat{\xi}]^2 + \frac{\theta - \sin\theta}{\theta^3}[\hat{\xi}]^3$$

- Further computation gives

$$e^{[\hat{\xi}\theta]} = \begin{bmatrix} e^{[\hat{\omega}\theta]} & (I - e^{[\hat{\omega}\theta]})(\hat{\omega} \times d) + \hat{\omega}\hat{\omega}^T d\theta \\ 0 & 1 \end{bmatrix}$$

Local Structure of $\mathbb{SE}(3)$

- Definition of Matrix Exponential:

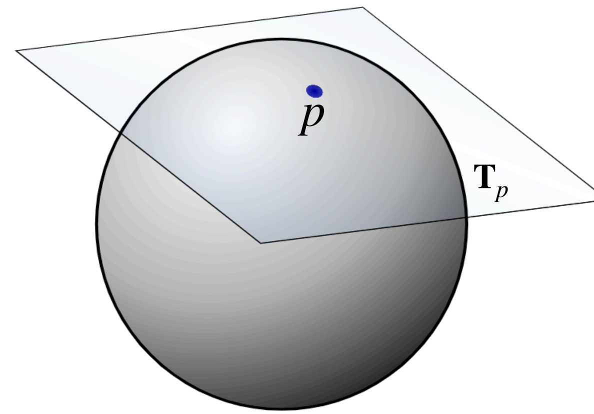
$$e^{[\hat{\xi}]\theta} = I + \theta[\hat{\xi}] + \frac{\theta^2}{2\,!}[\hat{\xi}]^2 + \frac{\theta^3}{3\,!}[\hat{\xi}]^3 + \cdots$$

- When $\theta \approx 0$, $e^{[\hat{\xi}]\theta} = I + \theta[\hat{\xi}] + o(\theta[\hat{\xi}])$
- $\forall T \in \mathbb{SE}(3), e^{\theta[\hat{\xi}]}T \approx T + \theta[\hat{\xi}]T$ when $\theta \approx 0$
  - Implies that $\mathbb{SE}(3)$ has a linear local structure (differentiable manifold)

Local Structure of $\mathbb{SE}(3)$

- By $e^{[\hat{\xi}]\theta} = I + \theta[\hat{\xi}] + o(\theta[\hat{\xi}])$ when $\theta \approx 0$,

$$e^{[\chi]} - I = [\chi] + o([\chi])$$



- Interpretation:
  - $[\chi]$ is a linear subspace of $\mathbb{R}^{4\times 4}$
  - $e^{[\chi]} \to I$ as $[\chi] \to 0$
  - Any local movement in $\mathbb{SE}(3)$ around I, which is $e^{[\chi]} - I$, can be approximated by some small $[\chi]$
- The set of $[\chi]$ forms the tangent space of $\mathbb{SE}(3)$ at I

Lie algebra $\mathfrak{se}(3)$ of $\mathbb{SE}(3)$

- The set of $[\chi]$ forms the tangent space of $\mathbb{SE}(3)$ at I
    - Ex: What is the tangent space at any $T \in \mathbb{SE}(3)$?
- We give this set a name, the "Lie algebra of $\mathbb{SE}(3)$"
    - $\mathfrak{se}(3) := \left\{ \begin{bmatrix} S & t \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4} : S^T = -S \right\}$

The Lie algebra of $\mathbb{SE}(3)$

- Motion interpretation

  $\hat{\omega}$: motion direction

- Exponential coordinate

  $\overrightarrow{\theta} = \hat{\omega}\theta \in \mathbb{R}^3$

- Exponential map

  $R = \exp([\hat{\omega}]\theta) \in \mathbb{SO}(3)$

- Tangent space at I

  $[\hat{\omega}]\theta \in \mathfrak{so}(3)$

- Motion interpretation

  $\hat{\xi}$: 6D motion direction

- Exponential coordinate

  $\chi = \hat{\xi}\theta \in \mathbb{R}^6$

- Exponential map

  $T = \exp([\hat{\xi}]\theta) \in \mathbb{SE}(3)$

- Tangent space at I

  $[\hat{\xi}]\theta \in \mathfrak{se}(3)$

Compute $T \in \mathbb{SE}(3)$ from $\hat{\xi}\theta$

- Recall **Rodrigues Formula** for rotations:

$$e^{[\hat{\omega}]\theta} = I + [\hat{\omega}]\sin\theta + [\hat{\omega}]^2(1 - \cos\theta)$$

- Similarly, using Taylor's expansion definition of $\exp$,

$$e^{[\hat{\xi}\theta]} = I + [\hat{\xi}] + \frac{1 - \cos\theta}{\theta^2}[\hat{\xi}]^2 + \frac{\theta - \sin\theta}{\theta^3}[\hat{\xi}]^3$$

- Further computation gives

$$e^{[\hat{\xi}\theta]} = \begin{bmatrix} e^{[\hat{\omega}\theta]} & (I - e^{[\hat{\omega}\theta]})(\hat{\omega} \times d) + \hat{\omega}\hat{\omega}^T d\theta \\ 0 & 1 \end{bmatrix}$$

Compute $\hat{\xi}\theta$ from $T \in \mathbb{SE}(3)$

- First, determine $\hat{\omega}\theta \in so(3)$ from the $SO(3)$ rotation
- The translation component of T is t, then d in $\hat{\xi}\theta = \begin{bmatrix} d \\ \hat{\omega} \end{bmatrix} \theta$ can be calculated as follow ($\theta \neq 0$):

$$d = (\frac{1}{\theta}I - \frac{1}{2}[\hat{\omega}] + (\frac{1}{\theta} - \frac{1}{2}\cot\frac{\theta}{2})[\hat{\omega}]^2)t$$

- $t \perp \hat{\omega} \iff \frac{1}{\theta}(I + [\hat{\omega}]^2)t = 0$, and there is no $\frac{1}{\theta}$ term in d

Read Motion Parameters from $\hat{\xi}\theta$

- Let us extract $\hat{\omega}, q, \theta, d_\omega$ from $\hat{\xi}\theta$
  - $\hat{\omega}$: we can directly read from $\hat{\xi}$
  - $q = [\hat{\omega}]^\dagger(\hat{\omega}\hat{\omega}^T - I)d$, where d can be read from $\hat{\xi}$
  - θ: we can directly read
  - $d_\omega = \hat{\omega}\hat{\omega}^T d\theta$

Summary

- **Exponential map**: $T = \mathrm{Trans}(\hat{\omega}, \theta, q, d_\omega) = e^{[s]}$
- **Screw**: $\chi = \begin{bmatrix} -[\hat{\omega}]q\theta + d_\omega \\ \hat{\omega}\theta \end{bmatrix}$ is the displacement of the 6D motion
- **Unit twist:** $\hat{\xi} = \begin{bmatrix} d \\ \hat{\omega} \end{bmatrix} \in \mathbb{R}^6$ so that $\chi = \hat{\xi}\theta$, the direction of the 6D motion

Example of Screw Computation

Q: What is the screw $\chi = \hat{\xi}\theta$ given $T(\theta) = e^{[\hat{\xi}]\theta}$?

$$T(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha t) & -\sin(\alpha t) & \sin(\alpha t) \\ 0 & \sin(\alpha t) & \cos(\alpha t) & 1 - \cos(\alpha t) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Q: What is the screw $\chi = \hat{\xi}\theta$ given $T(\theta) = e^{[\hat{\xi}]\theta}$?

$$T(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha t) & -\sin(\alpha t) & \sin(\alpha t) \\ 0 & \sin(\alpha t) & \cos(\alpha t) & 1 - \cos(\alpha t) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Recall that given $R \in \mathbb{SO}(3)$, we can compute θ and $[\hat{\omega}]$ (L3 P29)

Q: What is the screw $\chi = \hat{\xi}\theta$ given $T(\theta) = e^{[\hat{\xi}]\theta}$?

$$T(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha t) & -\sin(\alpha t) & \sin(\alpha t) \\ 0 & \sin(\alpha t) & \cos(\alpha t) & 1 - \cos(\alpha t) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Recall that given $R \in \mathbb{SO}(3)$, we can compute θ and $[\hat{\omega}]$ (L3 P29)
- $\theta = \alpha t$, $[\hat{\omega}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, thus $\hat{\omega} = [1,0,0]^T$

Q: What is the screw $\chi = \hat{\xi}\theta$ given $T(\theta) = e^{[\hat{\xi}]\theta}$?

$$T(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha t) & -\sin(\alpha t) & \sin(\alpha t) \\ 0 & \sin(\alpha t) & \cos(\alpha t) & 1 - \cos(\alpha t) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- $\theta = \alpha t$, $[\hat{\omega}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, $\hat{\omega} = [1,0,0]^T$
- Recall that $d = (\frac{1}{\theta}I - \frac{1}{2}[\hat{\omega}] + (\frac{1}{\theta} - \frac{1}{2}\cot\frac{\theta}{2})[\hat{\omega}]^2)t$
- With some calculation, we get $d = [0,1,0]^T$

Q: What is the screw $\chi = \hat{\xi}\theta$ given $T(\theta) = e^{[\hat{\xi}]\theta}$?

$$T(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha t) & -\sin(\alpha t) & \sin(\alpha t) \\ 0 & \sin(\alpha t) & \cos(\alpha t) & 1 - \cos(\alpha t) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

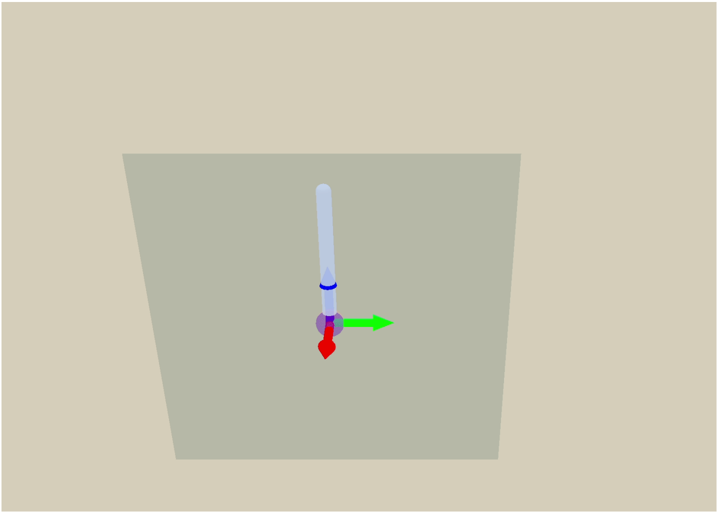
- $\theta = \alpha t$, $[\hat{\omega}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, $\hat{\omega} = [1,0,0]^T$
- With some calculation, we get $d = [0,1,0]^T$
- $\hat{\xi}\theta = \begin{bmatrix} d \\ \hat{\omega} \end{bmatrix} \theta = [0,1,0,1,0,0]^T \alpha t$, so $\chi = \hat{\xi}\theta = [0,\alpha t,0,\alpha t,0,0]^T$

Assume $T(\theta)$ describes the relative transformation of a body frame relative to spatial frame: $T^s_{s\to b}(\theta) \equiv T(\theta)$

$$T(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha t) & -\sin(\alpha t) & \sin(\alpha t) \\ 0 & \sin(\alpha t) & \cos(\alpha t) & 1-\cos(\alpha t) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- $\chi^s_{s\to b} = \hat{\xi}^s_{s\to b}\theta^s_{s\to b} = [0,\alpha t,0,\alpha t,0,0]^T$

$\chi^s_{s\to b}$ represents the linear transformation of rotating about a fixed axis



R: x-axis
G: y-axis
B: z-axis

- $\chi^s_{s\to b} = \hat{\xi}^s_{s\to b}\theta^s_{s\to b} = [0,\alpha t,0,\alpha t,0,0]^T$

$\chi^s_{s\to b}$ should represents the linear transformation of rotating about a fixed axis. Can we decode this information from $[0,\alpha t,0,\alpha t,0,0]^T$?

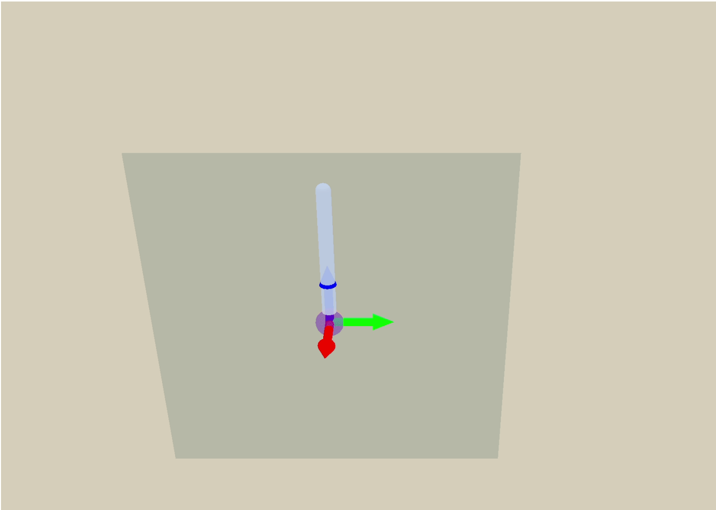
Q: What are $\hat{\omega}$, q, θ, d_ω for $T(\theta) = e^{[\hat{\xi}]\theta}$, where $\hat{\xi}\theta = [0,\alpha t,0,\alpha t,0,0]^T$?

- $\chi^s_{s\to b} = \hat{\xi}^s_{s\to b}\theta^s_{s\to b} = [0,\alpha t,0,\alpha t,0,0]^T$

$\chi^s_{s\to b}$ should represents the linear transformation of rotating about a fixed axis. Can we decode this information from $[0,\alpha t,0,\alpha t,0,0]^T$?

Q: What are $\hat{\omega}$, q, θ, d_ω for $T(\theta) = e^{[\hat{\xi}]\theta}$, where $\hat{\xi}\theta = [0,\alpha t,0,\alpha t,0,0]^T$?

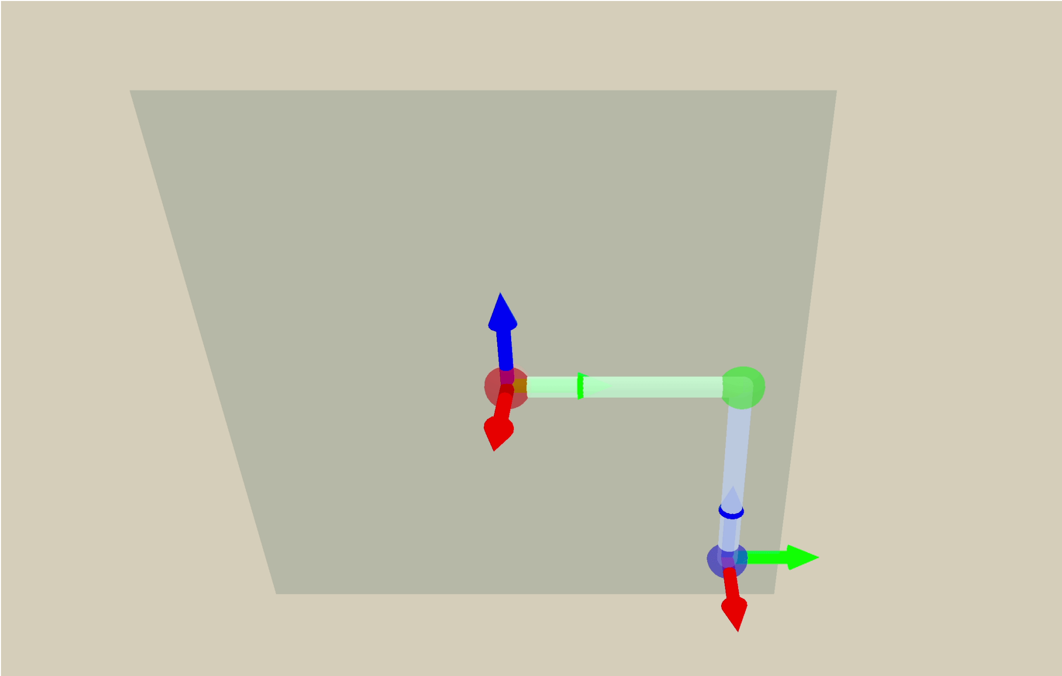
- Recall: $q = [\hat{\omega}]^\dagger(\hat{\omega}\hat{\omega}^T - I)d$
- With $\hat{\omega} = [1,0,0]^T$, $d = [0,1,0]^T$, we have $q = [0,0,1]^T$
- Recall: $d_\omega = \hat{\omega}\hat{\omega}^T d\theta$
- With $\theta = \alpha t$, we have $d_\omega = [0,0,0]^T$



Now, let's consider another case:

- A robot has two links (green stick and blue stick) connected by a revolute joint (green sphere). The end-effector (blue sphere) is connected to the end of the second link. The spatial frame is at the red sphere (static).

- What is the screw $\chi^s_{s\to e}(t)$ of the end-effector in the spatial frame?

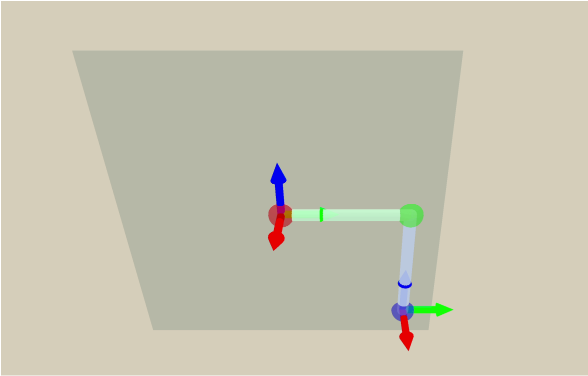


Read by Yourself

What is the screw $\chi^s_{s\to e}(t)$ of the end-effector in the spatial frame?

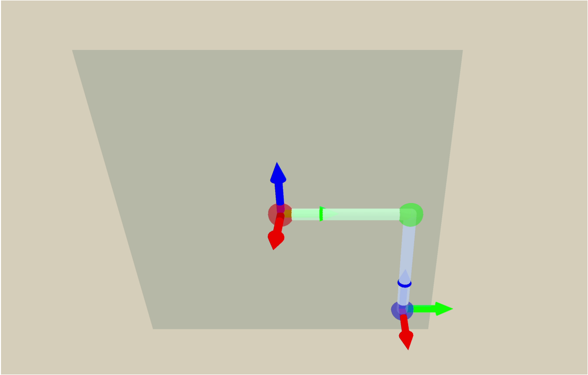
- Write down $T^s_{s\to e}$ by observation

$$T(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha t) & -\sin(\alpha t) & 1+\sin(\alpha t) \\ 0 & \sin(\alpha t) & \cos(\alpha t) & -\cos(\alpha t) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Read by Yourself

What is the screw $\chi^s_{s\to e}(t)$ of the end-effector in the spatial frame?



- Write down $T^s_{s\to e}$ by observation

$$T(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha t) & -\sin(\alpha t) & 1+\sin(\alpha t) \\ 0 & \sin(\alpha t) & \cos(\alpha t) & -\cos(\alpha t) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Similar as before, $\theta = \alpha t$, $\hat{\omega} = [1,0,0]^T$

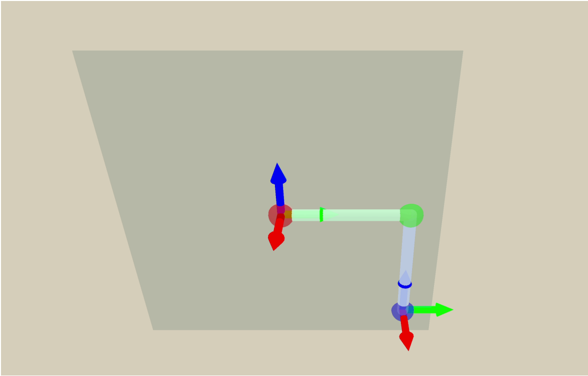
- $d = (\frac{1}{\theta}I - \frac{1}{2}[\hat{\omega}] + (\frac{1}{\theta} - \frac{1}{2}\cot\frac{\theta}{2})[\hat{\omega}]^2)t$

Read by Yourself

What is the screw $\chi^s_{s\to e}(t)$ of the end-effector in the spatial frame?

- Write down $T^s_{s\to e}$ by observation

$$T(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha t) & -\sin(\alpha t) & 1+\sin(\alpha t) \\ 0 & \sin(\alpha t) & \cos(\alpha t) & -\cos(\alpha t) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



- Similar as before, $\theta = \alpha t$, $\hat{\omega} = [1,0,0]^T$
- $d = (\frac{1}{\theta}I - \frac{1}{2}[\hat{\omega}] + (\frac{1}{\theta} - \frac{1}{2}\cot\frac{\theta}{2})[\hat{\omega}]^2)t$

$$d = \begin{bmatrix} 0 \\ \frac{(\sin(\theta)+1)\cot(\frac{\theta}{2}) - \cos(\theta)}{2} \\ -\frac{\sin(\theta)+\cos(\theta)\cot(\frac{\theta}{2})+1}{2} \end{bmatrix}$$, so complex!

Read by Yourself

What is the screw $\chi^s_{s\to e}(t)$ of the end-effector in the spatial frame?

- Screw $\chi^s_{s\to e}(t)$ is a function of time, since $\theta = \alpha t$

$$d = \begin{bmatrix} 0 \\ \frac{(\sin(\theta)+1)\cot(\frac{\theta}{2}) - \cos(\theta)}{2} \\ -\frac{\sin(\theta)+\cos(\theta)\cot(\frac{\theta}{2})+1}{2} \end{bmatrix}$$

- Even for a simple motion, the screw representation can be very complex
- Is there a better way of representing $T^s_{s\to e}(t)$ by screw?

Read by Yourself

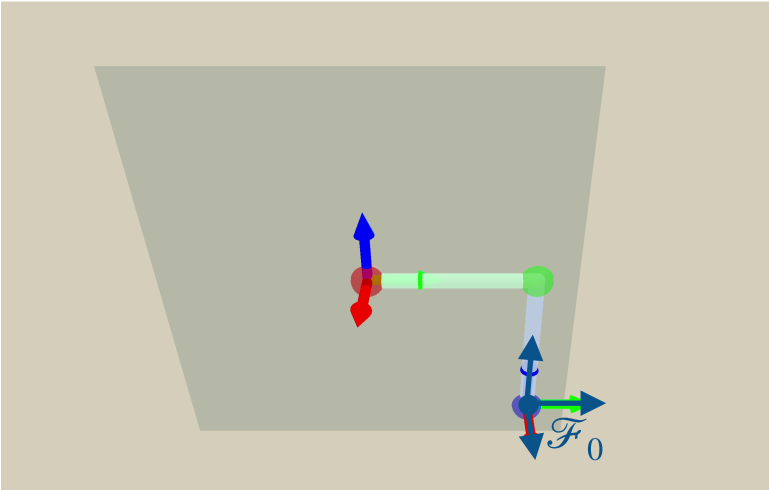
Is there a better way of representing $T^s_{s\to e}(t)$ by screw?

- Here we define a fixed auxiliary frame $\mathcal{F}_0$ and decompose $T^s_{s\to e}(t)$ into a composition of transformations

$$T^s_{s\to e}(t) = e^{[\chi^s_{s\to 0}]}e^{[\chi^0_{0\to e}]} \text{ (coord. trans. composition rule)}$$

- $\mathcal{F}_0 = \{p^s_0, (x^s_0, y^s_0, z^s_0)\}: p^s_0 = [0,1,-1]^T$ and (x^s_p, y^s_p, z^s_p) has the same direction as $\mathcal{F}_s$

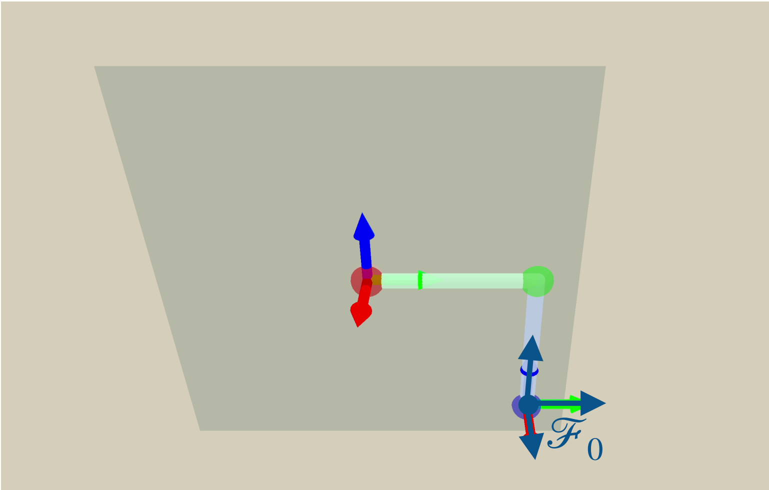
- Note that at $t=0$, $\mathcal{F}_0$ aligns with $\mathcal{F}_e$



Read by Yourself

Is there a better way of representing $T^s_{s\to e}(t)$ by screw?

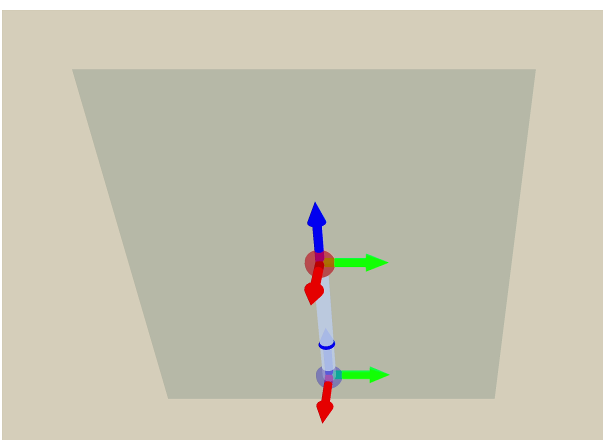
- $\mathcal{F}_0 = \{p^s_0, (x^s_0, y^s_0, z^s_0)\}: p^s_0 = [0,1,-1]^T$
- Note that at $t = 0$, $\mathcal{F}_0$ aligns with $\mathcal{F}_e$
- $$T^s_{s\to e}(t) = e^{[\chi^s_{s\to 0}]}e^{[\chi^0_{0\to e}]} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha t) & -\sin(\alpha t) & \sin(\alpha t) \\ 0 & \sin(\alpha t) & \cos(\alpha t) & 1-\cos(\alpha t) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

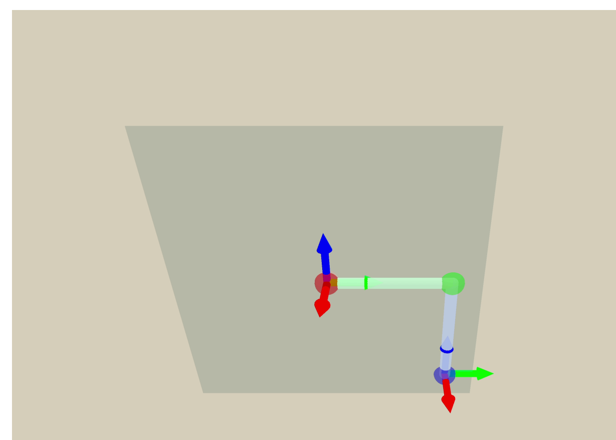


Read by Yourself

Is there a better way of representing $T^s_{s\to e}(t)$ by screw?

- $$T^s_{s\to e}(t) = e^{[\chi^s_{s\to 0}]}e^{[\chi^0_{0\to e}]} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha t) & -\sin(\alpha t) & \sin(\alpha t) \\ 0 & \sin(\alpha t) & \cos(\alpha t) & 1-\cos(\alpha t) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
- By simple inspection, $[\chi^s_{s\to 0}] = [0,1,-1,0,0,0]^T$
- As calculated in the previous example, $[\chi^0_{0\to e}] = [0,\alpha t,0,\alpha t,0,0]^T$

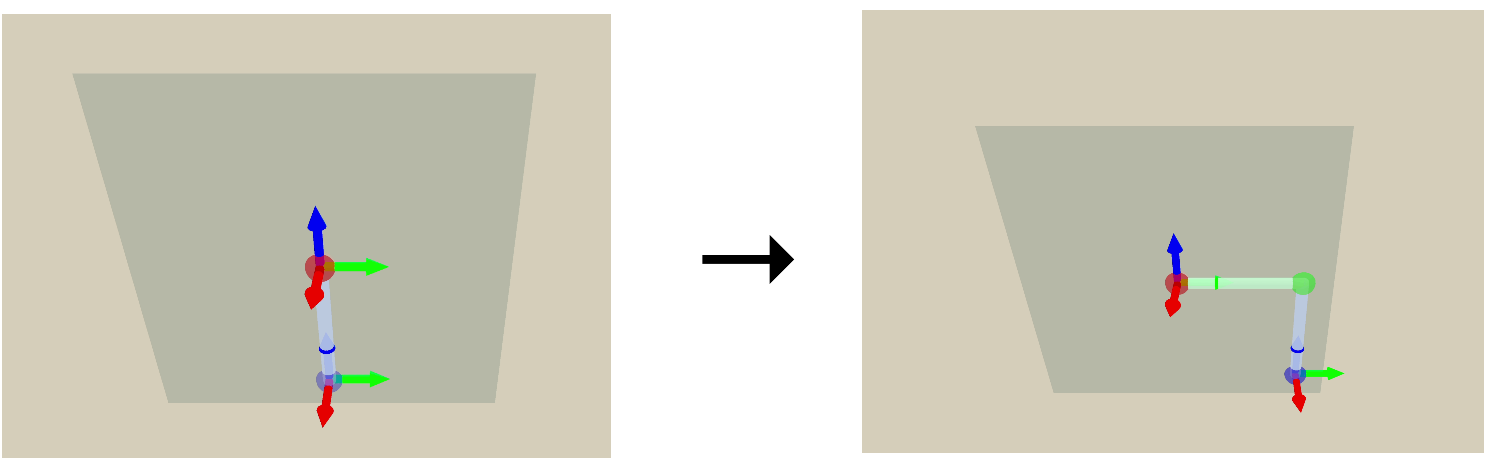




Read by Yourself

Is there a better way of representing $T^s_{s\to e}(t)$ by screw?

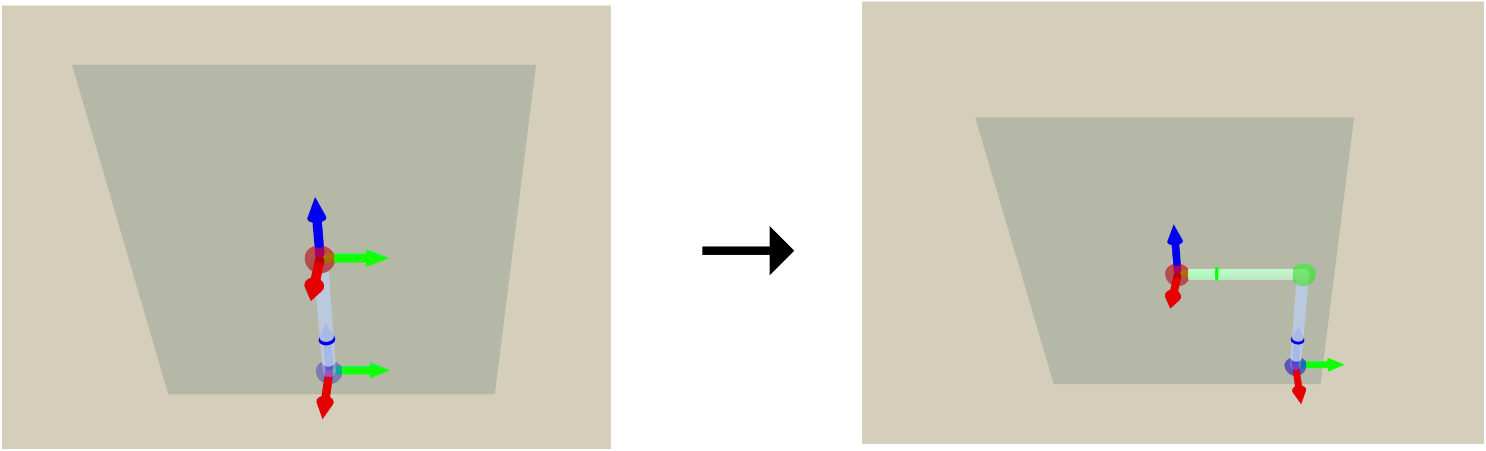
- Decomposing $T^s_{s\to e}(t)$ into two screw $e^{[\chi^s_{s\to 0}]}e^{[\chi^0_{0\to e}]}$ makes things easier!
- Why we select $\mathscr{F}_0 = \{p^s_0, (x^s_0, y^s_0, z^s_0)\}: p^s_0 = [0,1,-1]^T$ and (x^s_p, y^s_p, z^s_p) r ?



Read by Yourself

Is there a better way of representing $T^s_{s\to e}(t)$ by screw?

- Decomposing $T^s_{s\to e}(t)$ into two screw $e^{[\chi^s_{s\to 0}]}e^{[\chi^0_{0\to e}]}$ makes things easier
- Why we select $\mathscr{F}_0 = \{p^s_0, (x^s_0, y^s_0, z^s_0)\}$, where $p^s_0 = [0,1,-1]^T$ and (x^s_p, y^s_p, z^s_p) represent the same direction as $\mathscr{F}_s$?
- Observation: $\mathscr{F}_0$ aligns with $\mathscr{F}_e$ at $t = 0$



Read by Yourself

Is there a better way of representing $T^s_{s\to e}(t)$ by screw?

- Observation: $\mathcal{F}_0$ aligns with $\mathcal{F}_e$ at $t = 0$

$$T^s_{s\to e}(t) = e^{[\chi^s_{s\to 0}]}e^{[\chi^0_{0\to e}]} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha t) & -\sin(\alpha t) & \sin(\alpha t) \\ 0 & \sin(\alpha t) & \cos(\alpha t) & 1 - \cos(\alpha t) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$e^{[\chi^0_{0\to e(t)}]}$ is an identity matrix at $t = 0$

For motion of rotating about a fixed axis (common for revolute joint in real robot), screw will be very simple when it starts with an identity matrix

Read by Yourself

Libraries based on Screw Theory

- https://github.com/NxRLab/ModernRobotics/blob/master/packages/Python/modern_robotics/core.py
- https://petercorke.github.io/robotics-toolbox-python/intro.html#

Twist (6D Velocity Parameterization)

Setup

- Let us first parameterize the motion of a body frame by time:
  - An observer associated to $\mathcal{F}_o$ records the motion as $T^o_{s'\to b(t)}$, where the body frame is at $\mathcal{F}_{b(t)}$.

Twist

$$\begin{aligned} T^o_{s'\to b(t+\Delta t)} - T^o_{s'\to b(t)} &= T^o_{b(t)\to b(t+\Delta t)} T^o_{s'\to b(t)} - T^o_{s'\to b(t)} \\ &= e^{[\chi^o_{b(t)\to b(t+\Delta t)}]} T^o_{s'\to b(t)} - T^o_{s'\to b(t)} \\ &\approx [\chi^o_{b(t)\to b(t+\Delta t)}] T^o_{s'\to b(t)} \end{aligned}$$

- Divided by Δt and take the limit, we have

$$\begin{aligned} \dot{T}^o_{s'\to b(t)} &= \lim_{\Delta t\to 0} \left[\frac{\chi^o_{b(t)\to b(t+\Delta t)}}{\Delta t} \right] T^o_{s'\to b(t)} \\ &= [\xi^o_{b(t)}] T^o_{s'\to b(t)} \end{aligned}$$

- $\xi^o_{b(t)} := \lim_{\Delta t\to 0} \dfrac{\chi^o_{b(t)\to b(t+\Delta t)}}{\Delta t}$ is called "**twist**", the 6D instant velocity

Twist

- Twist: $\xi^{o}_{b(t)} := \lim\limits_{\Delta t \to 0} \dfrac{\chi^{o}_{b(t)\to b(t+\Delta t)}}{\Delta t}$
- $[\xi^{o}_{b(t)}] = \dot{T}^{o}_{s'\to b(t)}(T^{o}_{s'\to b(t)})^{-1}$
- Note: $\xi^{o}_{b(t)} \neq \dot{\chi}^{o}_{s'\to b(t)}$ for general $\chi^{o}_{s\to b(t)}(t)$ (verify by yourself)

Linear Velocity from Twist

- The linear velocity of p^o caused by $T^o_{s'\to b(t)}$ at time t is

$$\mathbf{v}^o_p(t) = \lim_{\Delta t \to 0} \frac{T^o_{b(t)\to b(t+\Delta t)} p^o - p^o}{\Delta t} = \lim_{\Delta t \to 0} \frac{\exp([\chi^o_{b(t)\to b(t+\Delta t)}]) - I}{\Delta t} p^o$$

$$= \lim_{\Delta t \to 0} \frac{[\chi^o_{b(t)\to b(t+\Delta t)}]}{\Delta t} p^o = [\xi^o_{b(t)}] p^o$$

- Therefore, $\boxed{\mathbf{v}^o_p(t) = [\xi^o_{b(t)}] p^o}$

(Recall that, if a motion is a pure rotation, then $\mathbf{v}^o_p(t) = \omega^o_{b(t)} \times p^o$)