

L5: Twist and Geometric Jacobian

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Agenda

- Interpretation and Computation of Twist
- Example of Twist Computation
- Change of Coordinates for Twists
- Jacobian of Kinematics Chain
- Inverse Kinematics

Review: Twist

$$\begin{aligned}
 T_{s' \rightarrow b(t+\Delta t)}^o - T_{s' \rightarrow b(t)}^o &= T_{b(t) \rightarrow b(t+\Delta t)}^o T_{s' \rightarrow b(t)}^o - T_{s' \rightarrow b(t)}^o \\
 &= e^{[\chi_{b(t) \rightarrow b(t+\Delta t)}^o]} T_{s' \rightarrow b(t)}^o - T_{s' \rightarrow b(t)}^o \\
 &\approx [\chi_{b(t) \rightarrow b(t+\Delta t)}^o] T_{s' \rightarrow b(t)}^o
 \end{aligned}$$

- Divided by Δt and take the limit, we have

$$\begin{aligned}
 \dot{T}_{s' \rightarrow b(t)}^o &= \lim_{\Delta t \rightarrow 0} \left[\frac{\chi_{b(t) \rightarrow b(t+\Delta t)}^o}{\Delta t} \right] T_{s' \rightarrow b(t)}^o \\
 &= [\xi_{b(t)}^o] T_{s' \rightarrow b(t)}^o
 \end{aligned}$$

- $\xi_{b(t)}^o := \lim_{\Delta t \rightarrow 0} \frac{\chi_{b(t) \rightarrow b(t+\Delta t)}^o}{\Delta t}$ is called “**twist**”, the 6D instant velocity

Review: Twist

- Twist: $\xi_{b(t)}^o := \lim_{\Delta t \rightarrow 0} \frac{\chi_{b(t) \rightarrow b(t+\Delta t)}^o}{\Delta t}$
- $[\xi_{b(t)}^o] = \dot{T}_{s' \rightarrow b(t)}^o (T_{s' \rightarrow b(t)}^o)^{-1}$
- Note: $\xi_{b(t)}^o \neq \dot{\chi}_{s' \rightarrow b(t)}^o$ for general $\chi_{s \rightarrow b(t)}^o(t)$ (verify by yourself)

Interpretation and Computation of Twist

- Let $\xi_{b(t)}^o = \begin{bmatrix} \nu_{b(t)}^o \\ \omega_{b(t)}^o \end{bmatrix} \in \mathbb{R}^6$, then $[\xi_{b(t)}^o] = \begin{bmatrix} [\omega_{b(t)}^o] & \nu_{b(t)}^o \\ 0 & 0 \end{bmatrix}$

Body Twist

- When $\mathcal{F}_o = \mathcal{F}_{b(t)}$, it is particularly easy to compute $\xi_{b(t)}^{b(t)}$
- Remarks
 - For body twist, when recording at time t , you should think it as ***first cloning the body frame and then record the movement using this cloned frame and keeping it static***
 - Body twist is “ego-centric” and sometimes simpler to specify for robotics. For example, if we take the gripper frame as the body frame. Using the body twist, we can express “move gripper forward” by a pure translation

Review: Linear Velocity from Twist

- The linear velocity of p^o caused by $T_{s' \rightarrow b(t)}^o$ at time t is

$$\begin{aligned}\mathbf{v}_p^o(t) &= \lim_{\Delta t \rightarrow 0} \frac{T_{b(t) \rightarrow b(t+\Delta t)}^o p^o - p^o}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\exp([\chi_{b(t) \rightarrow b(t+\Delta t)}^o]) - I}{\Delta t} p^o \\ &= \lim_{\Delta t \rightarrow 0} \frac{[\chi_{b(t) \rightarrow b(t+\Delta t)}^o]}{\Delta t} p^o = [\xi_{b(t)}^o] p^o\end{aligned}$$

- Therefore, $\mathbf{v}_p^o(t) = [\xi_{b(t)}^o] p^o$

(Recall that, if a motion is a pure rotation, then $\mathbf{v}_p^o(t) = \omega_{b(t)}^o \times p^o$)

Body Twist Computation

- By $\mathbf{v}_p^o(t) = [\xi_{b(t)}^o] p^o$ and $[\xi_{b(t)}^o] = \begin{bmatrix} [\omega_{b(t)}^o] & \nu_{b(t)}^o \\ 0 & 0 \end{bmatrix}$, let $p_{origin}^{b(t)} = [0,0,0,1]^T$, which is the origin of body frame

$$\mathbf{v}_{origin}^{b(t)}(t) = [\xi_{b(t)}^{b(t)}] p_{origin}^{b(t)} = \nu^{b(t)}$$

- Note that $\mathbf{v}_{origin}^{b(t)}(t)$ is the linear velocity of the origin of the body frame
- Therefore, $\xi_{b(t)}^{b(t)}$ is composed by the linear velocity of the origin and an angular velocity around the axis (may not pass the origin)
- In practice, we often write down the body twist first and then obtain the twist in other frames by change of coordinate

Compute $\xi_{b(t)}^o$ from Angle-Axis

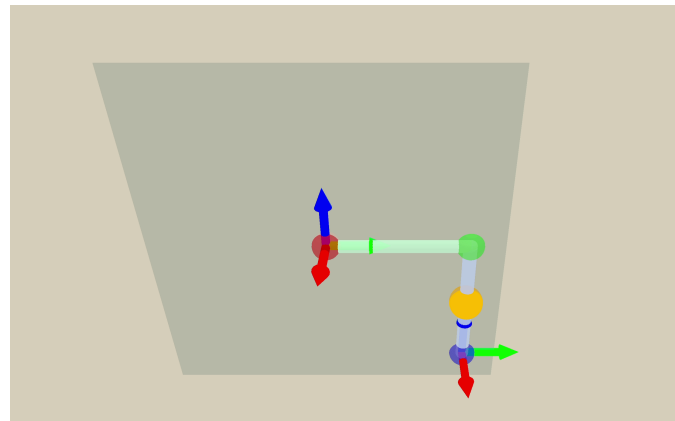
- Let $\xi_{b(t)}^o = \begin{bmatrix} \nu_{b(t)}^o \\ \omega_{b(t)}^o \end{bmatrix} \in \mathbb{R}^6$, then $[\xi_{b(t)}^o] = \begin{bmatrix} [\omega_{b(t)}^o] & \nu_{b(t)}^o \\ 0 & 0 \end{bmatrix}$
- The instant linear velocity can be decomposed into the rotation about the axis and translation along the axis
- Take a point $q^o \in \mathbb{R}^3$ on the axis,
 - By $\mathbf{v}_p^o(t) = [\xi_{b(t)}^o]p^o$, $\mathbf{v}_q^o = [\omega^o]q^o + \nu^o$
 - Since the only velocity of q^o is along $\hat{\omega}$, $\mathbf{v}_q^o = \mathbf{v}_\omega^o$
 - $\therefore \nu^o = -[\omega^o]q^o + \mathbf{v}_\omega^o$
- $\xi^o = \begin{bmatrix} -[\omega^o]q^o + \mathbf{v}_\omega^o \\ \omega^o \end{bmatrix}$ (all the symbols have $b(t)$ as the subscript)

Example of Twist Computation

Example of Twist Computation

- Consider the example (last lecture), but now an orange point is fixed to the end-effector frame (blue sphere)
- What is the **velocity of orange point at $t = 0$** ? Given the pose of end effector frame as below:

$$T_{s \rightarrow b(t)}^s = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha t) & -\sin(\alpha t) & 1 + \sin(\alpha t) \\ 0 & \sin(\alpha t) & \cos(\alpha t) & -\cos(\alpha t) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Example of Twist Computation

- The velocity of yellow point caused by the end-effector motion can be computed via twist
- Recall: $\mathbf{v}_p^s = [\xi_{b(t)}^s] p^s$

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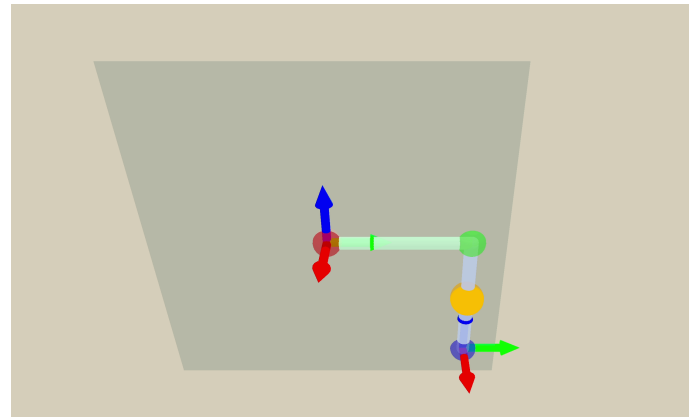
- we have $[\xi_{s \rightarrow b(t)}^s] = \dot{T}_{s \rightarrow b(t)}^s (T_{s \rightarrow b(t)}^s)^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & \alpha & 0 & -\alpha \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Example of Twist Computation

- The velocity of yellow point caused by the end-effector motion can be computed via twist
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$$[\xi_{b(t)}^s] = \dot{T}_{s \rightarrow b(t)}^s (T_{s \rightarrow b(t)}^s)^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & \alpha & 0 & -\alpha \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{At } t = 0, p^s = \begin{bmatrix} 0 \\ 1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

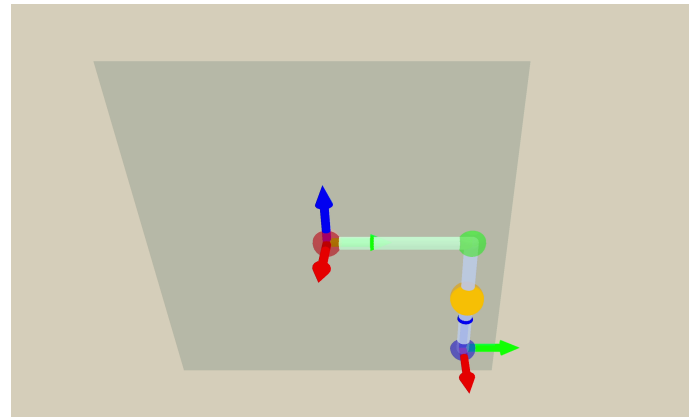


Example of Twist Computation

- The velocity of yellow point caused by the end-effector motion can be computed via twist
- Recall: $\mathbf{v}_p^s = [\xi_{b(t)}^s] p^s$

$$[\xi_{b(t)}^s] = \dot{T}_{s \rightarrow b(t)}^s (T_{s \rightarrow b(t)}^s)^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & \alpha & 0 & -\alpha \\ 0 & 0 & 0 & 0 \end{bmatrix}, p^s = \begin{bmatrix} 0 \\ 1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

$$\mathbf{v}_p^s = [\xi_{b(t)}^s] p^s = \begin{bmatrix} 0 \\ \frac{\alpha}{2} \\ 0 \\ 0 \end{bmatrix}$$

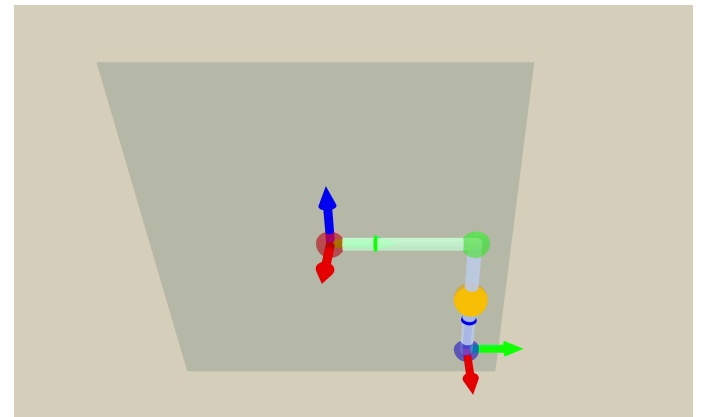


Example of Twist Computation

- We can verify this result by taking the derivative of $\frac{d}{dt} p^s(t)$

$$p^s(t) = \begin{bmatrix} 0 \\ 1 + \frac{1}{2} \sin(\alpha t) \\ -\frac{1}{2} \cos(\alpha t) \\ 1 \end{bmatrix}, \quad \frac{d}{dt} p^s(t) = \begin{bmatrix} 0 \\ \frac{\alpha}{2} \cos(\alpha t) \\ \frac{\alpha}{2} \sin(\alpha t) \\ 0 \end{bmatrix}$$

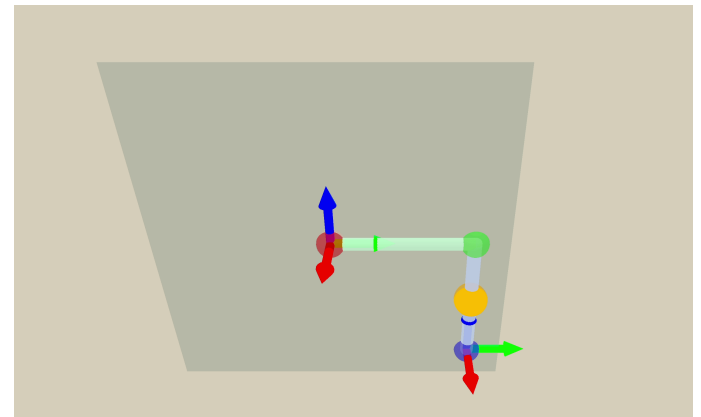
$$\mathbf{v}_p^s = [\xi_{b(t)}^s] p^s = \begin{bmatrix} 0 \\ \frac{\alpha}{2} \\ 0 \\ 0 \end{bmatrix} = \left. \frac{d}{dt} p^s(t) \right|_{t=0}$$



Example of Twist Computation

- What is the body twist of the end effector?
- In the body frame of the end effector (blue sphere), the origin of the frame, which is the blue sphere, has a constant linear velocity, which is always $[0, \alpha, 0]$. The angular velocity is always $[\alpha, 0, 0]$.

So, $\xi_{b(t)}^{b(t)} = [0, \alpha, 0, \alpha, 0, 0]^T$



Change of Coordinates for Twists

Review

- Recall that, the recordings by different observers are related by the similarity transformation:

$$T_{1 \rightarrow 2}^{s_1} = T_{s_1 \rightarrow s_2} T_{1 \rightarrow 2}^{s_2} (T_{s_1 \rightarrow s_2})^{-1}$$

Tricks in Recording Velocities

- If transformations could be recorded differently by observers, velocity should also be recorded differently

Relating 6D Velocities from Different Observers

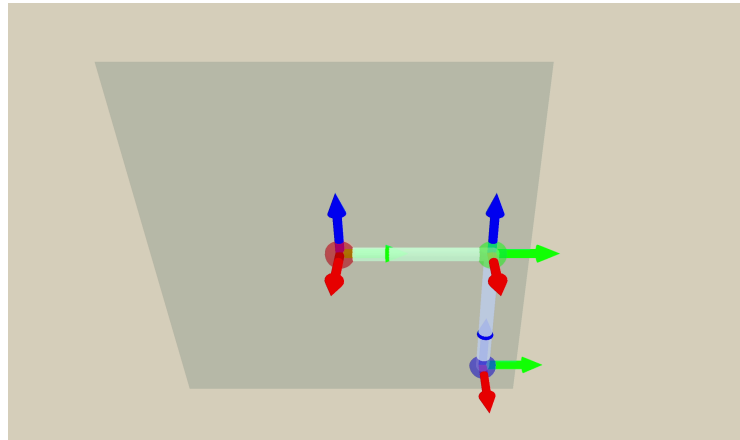
- Two observers record the same motion as $\xi_{b(t)}^{s_1}$ and $\xi_{b(t)}^{s_2}$
- **What is the relationship between $\xi_{b(t)}^{s_1}$ and $\xi_{b(t)}^{s_2}$?**

Example 1 of Change of Frame

- From our previous example, we know that the motion of the end-effector (blue sphere) can be recorded in the spatial frame (red sphere) as $\xi_{s \rightarrow b(t)}^s$:

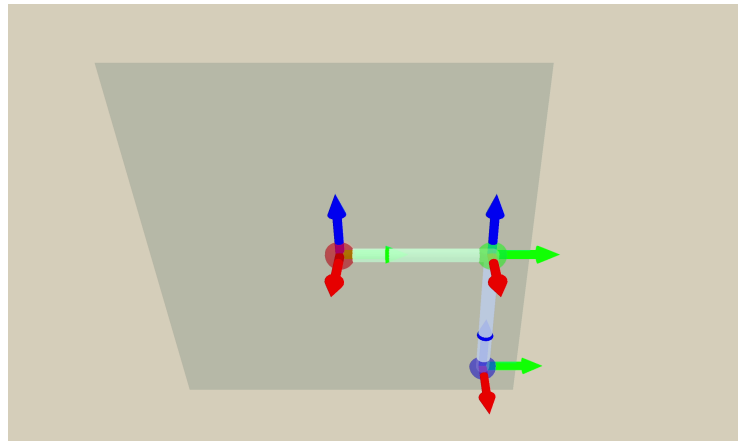
$$[\xi_{s \rightarrow b(t)}^s] = \dot{T}_{s \rightarrow b(t)}^s (T_{s \rightarrow b(t)}^s)^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & \alpha & 0 & -\alpha \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xi_{s \rightarrow b(t)}^s = [0, 0, -\alpha, \alpha, 0, 0]^T$$



Example 1 of Change of Frame

- From our previous example, we know that the motion of the end-effector (blue sphere) can be recorded in the spatial frame (red sphere) as $\xi_{s \rightarrow b(t)}^s$:
- Now we introduce a new frame \mathcal{F}_o , the frame of the green sphere. How can we record the same motion by \mathcal{F}_o as $\xi_{s \rightarrow b(t)}^o$?



Example 1 of Change of Frame

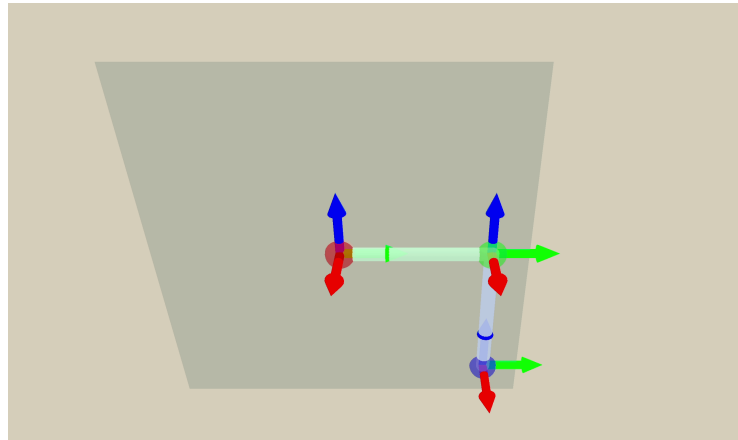
- From our previous example, we know that the motion of the end-effector (blue sphere) can be recorded in the spatial frame (red sphere) as $\xi_{s \rightarrow b(t)}^s$:
- By simple inspection, we can find end-effector is rotating about the x-axis of \mathcal{F}_o and the instant velocity along the axis is zero

$$\omega^o = [\alpha, 0, 0]^T$$

$$\hat{\omega}^o = [1, 0, 0]^T$$

$$q^o = [0, 0, 0]^T$$

$$\mathbf{v}_\omega^o = [0, 0, 0]^T$$



Example 1 of Change of Frame

- From our previous example, we know that the motion of the end-effector (blue sphere) can be recorded in the spatial frame (red sphere) as $\xi_{s \rightarrow b(t)}^s$:
- By simple inspection, we can find end-effector is rotating about the x-axis of \mathcal{F}_o and the instant velocity along the axis is zero

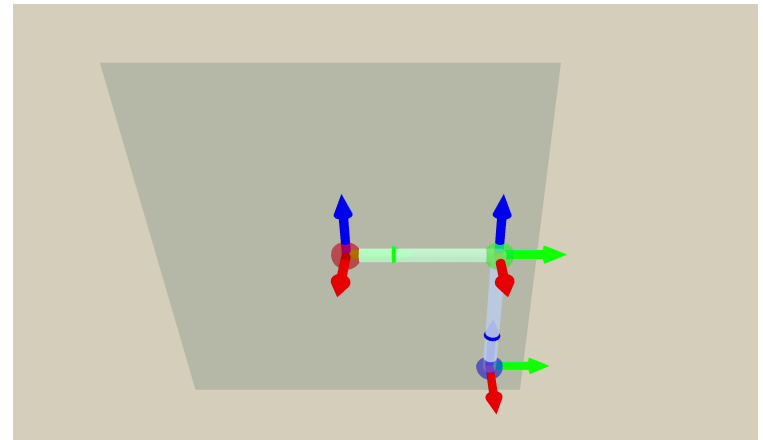
$$\omega^o = [\alpha, 0, 0]^T$$

$$\hat{\omega}^o = [1, 0, 0]^T$$

$$q^o = [0, 0, 0]^T$$

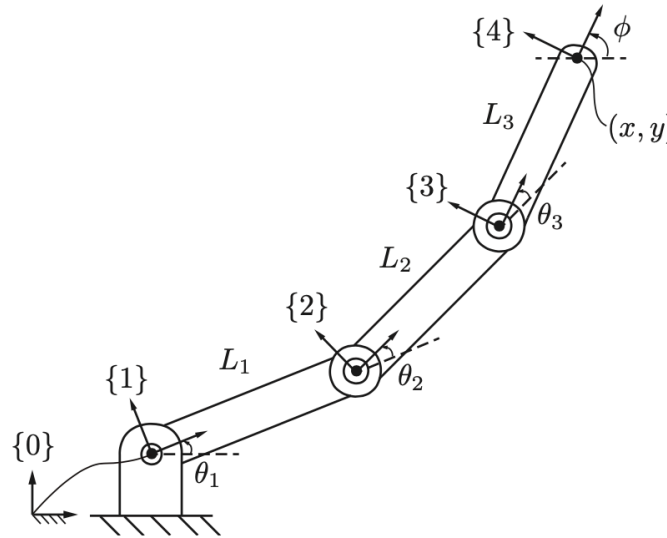
$$\mathbf{v}_\omega^o = [0, 0, 0]^T$$

- Recall: $\xi^o = \begin{bmatrix} -[\omega^o]q^o + \mathbf{v}_\omega^o \\ \omega^o \end{bmatrix}$
- Thus we have $\xi_{s \rightarrow b(t)}^o = [0, 0, 0, \alpha, 0, 0]^T$



Example 2 of Change of Frame

- For the 3-link robot arm



- Given $\xi_{L_3(t)}^3$, what is $\xi_{L_3(t)}^0$? Assume the transformation is $T_{L_0 \rightarrow L_3(t)}^0$ at time t .

Change of Frame by Similarity Transformation

- For two observers, one records by \mathcal{F}_{s_1} and the other by \mathcal{F}_{s_2} , then

- $\dot{T}_{s' \rightarrow b(t)}^{s_1} = [\xi_{b(t)}^{s_1}] T_{s' \rightarrow b(t)}^{s_1}$

- $\dot{T}_{s' \rightarrow b(t)}^{s_2} = [\xi_{b(t)}^{s_2}] T_{s' \rightarrow b(t)}^{s_2}$

Change of Frame by Similarity Transformation

$$[\xi_{b(t)}^{s_1}] = T_{s_1 \rightarrow s_2} [\xi_{b(t)}^{s_2}] T_{s_1 \rightarrow s_2}^{-1}$$

- When the observer's frame changes,
 - twist also conforms to the similarity transformation

Change of Frame by Similarity Transformation

- By $T_{s' \rightarrow b(t)}^{s_1} = T_{s_1 \rightarrow s_2} T_{s' \rightarrow b(t)}^{s_2} (T_{s_1 \rightarrow s_2})^{-1}$,

$$\begin{aligned}
 \dot{T}_{s' \rightarrow b(t)}^{s_1} &= T_{s_1 \rightarrow s_2} \dot{T}_{s' \rightarrow b(t)}^{s_2} (T_{s_1 \rightarrow s_2})^{-1} \Leftrightarrow [\xi_{b(t)}^{s_1}] T_{s' \rightarrow b(t)}^{s_1} = T_{s_1 \rightarrow s_2} [\xi_{b(t)}^{s_2}] T_{s' \rightarrow b(t)}^{s_2} (T_{s_1 \rightarrow s_2})^{-1} \\
 &\Leftrightarrow [\xi_{b(t)}^{s_1}] = T_{s_1 \rightarrow s_2} [\xi_{b(t)}^{s_2}] T_{s' \rightarrow b(t)}^{s_2} (T_{s_1 \rightarrow s_2})^{-1} (T_{s' \rightarrow b(t)}^{s_1})^{-1} \\
 &\Leftrightarrow [\xi_{b(t)}^{s_1}] = T_{s_1 \rightarrow s_2} [\xi_{b(t)}^{s_2}] T_{s' \rightarrow b(t)}^{s_2} \{ (T_{s_1 \rightarrow s_2})^{-1} (T_{s' \rightarrow b(t)}^{s_1})^{-1} T_{s_1 \rightarrow s_2} \} (T_{s_1 \rightarrow s_2})^{-1} \\
 &\Leftrightarrow [\xi_{b(t)}^{s_1}] T_{s_1 \rightarrow s_2} = T_{s_1 \rightarrow s_2} [\xi_{b(t)}^{s_2}] T_{s' \rightarrow b(t)}^b (T_{s' \rightarrow b(t)}^b)^{-1} (T_{s_1 \rightarrow s_2})^{-1} \\
 &\Leftrightarrow [\xi_{b(t)}^{s_1}] T_{s_1 \rightarrow s_2} = T_{s_1 \rightarrow s_2} [\xi_{b(t)}^{s_2}]
 \end{aligned}$$

$$[\xi_{b(t)}^{s_1}] = T_{s_1 \rightarrow s_2} [\xi_{b(t)}^{s_2}] T_{s_1 \rightarrow s_2}^{-1}$$

Adjoint Matrix

$$[\xi_{b(t)}^{s_1}] = T_{s_1 \rightarrow s_2} [\xi_{b(t)}^{s_2}] T_{s_1 \rightarrow s_2}^{-1}$$

- $\xi_{b(t)}^{s_1}$ is linear w.r.t. $\xi_{b(t)}^{s_2}$
- We introduce a matrix $[\text{Ad}_{T_{s_1 \rightarrow s_2}}] \in \mathbb{R}^{6 \times 6}$ to relate them:

$$\xi_{b(t)}^{s_1} = [\text{Ad}_{T_{s_1 \rightarrow s_2}}] \xi_{b(t)}^{s_2}$$

- Do computation based on the similarity transformation, and you can get

$$[\text{Ad}_{T_{s_1 \rightarrow s_2}}] = \begin{bmatrix} R_{s_1 \rightarrow s_2} & [\mathbf{t}_{s_1 \rightarrow s_2}] R_{s_1 \rightarrow s_2} \\ 0 & R_{s_1 \rightarrow s_2} \end{bmatrix}$$

Spatial Twist and Body Twist

- If we observe the motion of the body
 - _ from \mathcal{F}_s , the velocity is $\xi_{b(t)}^s$ (**spatial twist**)
 - _ from the moving object \mathcal{F}_b , the velocity is $\xi_{b(t)}^{b(t)}$ (**body twist**)

Spatial Twist and Body Twist

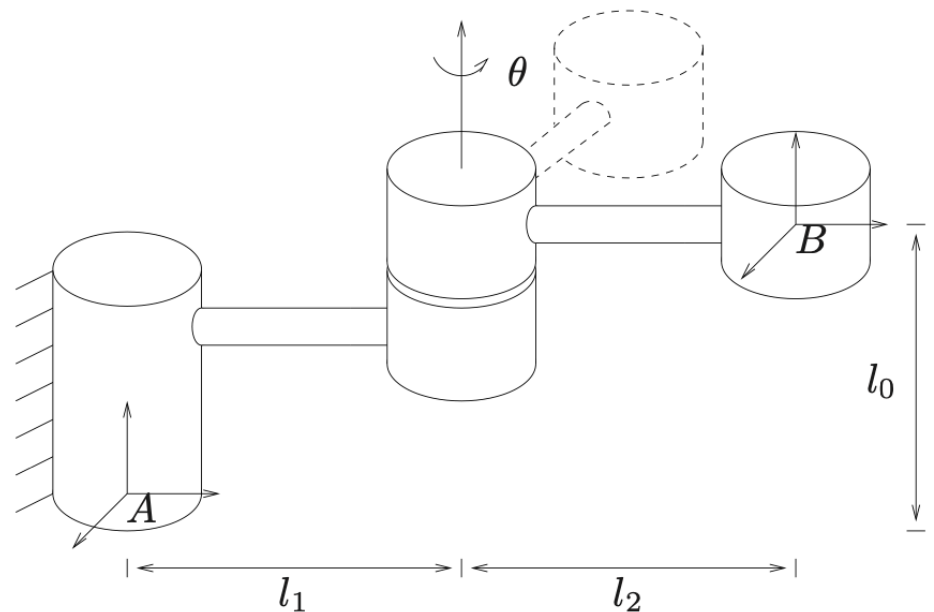
- By $\dot{T}_{s' \rightarrow b(t)}^s = [\xi_{b(t)}^s] T_{s' \rightarrow b(t)}$, $[\xi_{b(t)}^s] = \dot{T}_{s' \rightarrow b(t)}^s (T_{s' \rightarrow b(t)}^s)^{-1}$
- Note that we take $s' = s$ here
- Using the similarity transformation to change the frame, we have
 - $T_{s \rightarrow b(t)}^s [\xi_{b(t)}^{b(t)}] (T_{s \rightarrow b(t)})^{-1} = \dot{T}_{s \rightarrow b(t)}^s (T_{s \rightarrow b(t)}^s)^{-1}$
 - $\therefore [\xi_{b(t)}^{b(t)}] = (T_{s \rightarrow b(t)})^{-1} \dot{T}_{s \rightarrow b(t)}^s$

Example 3 of Change of Frame

- Given the motion of rigid-body

$$T_{A \rightarrow B(t)} = \begin{bmatrix} \cos \theta(t) & -\sin \theta(t) & 0 & -l_2 \sin \theta(t) \\ \sin \theta(t) & \cos \theta(t) & 0 & l_1 + l_2 \cos \theta(t) \\ 0 & 0 & 1 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- What is the spatial twist?
- What is the body twist?



Example 3 of Change of Frame

- Given the motion of rigid-body

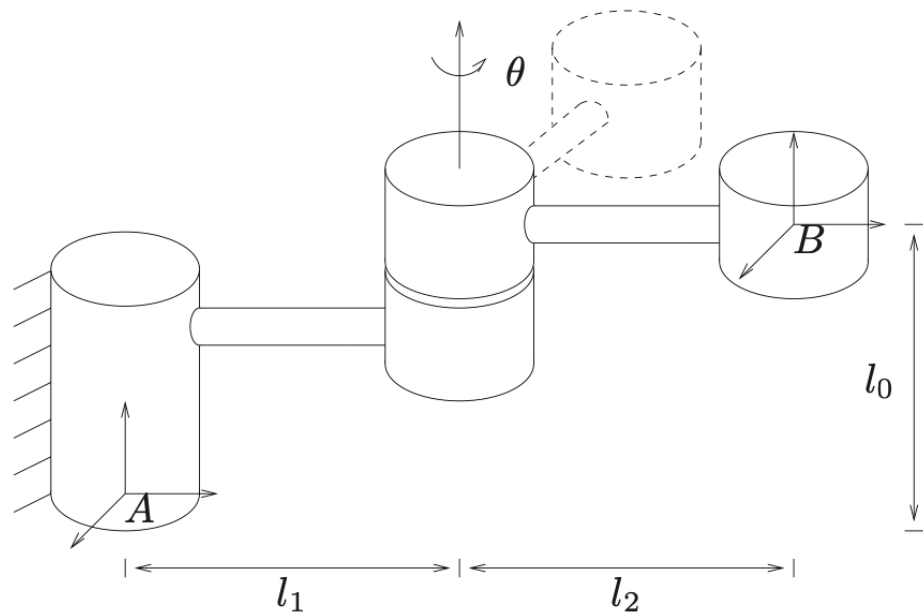
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$$\cdot [\xi_{B(t)}^A] = \dot{T}_{A \rightarrow B(t)} T_{A \rightarrow B(t)}^{-1}$$

$$\xi_{B(t)}^A = [l_1, 0, 0, 0, 0, 1]^T$$

$$\cdot [\xi_{B(t)}^B] = T_{A \rightarrow B(t)}^{-1} \dot{T}_{A \rightarrow B(t)}$$

$$\xi_{B(t)}^{B(t)} = [-l_2, 0, 0, 0, 0, 1]^T$$



```

import sympy as sp
from sympy import *

t = symbols("t")
l0 = symbols("l0")
l1 = symbols("l1")
l2 = symbols("l2")

T = Matrix(symarray('T', (4, 4)))
T[0, 0] = cos(t)
T[0, 1] = -sin(t)
T[0, 2] = 0
T[0, 3] = -l2 * sin(t)
T[1, 0] = sin(t)
T[1, 1] = cos(t)
T[1, 2] = 0
T[1, 3] = l1 + l2 * cos(t)
T[2, 0] = 0
T[2, 1] = 0
T[2, 2] = 1
T[2, 3] = l0
T[3, 0] = 0
T[3, 1] = 0
T[3, 2] = 0
T[3, 3] = 1

xi_s = sp.diff(T, t) @ sp.Inverse(T)
xi_s.simplify()

xi_b = sp.Inverse(T) @ sp.diff(T, t)
xi_b.simplify()

```

Example 3 of Change of Frame

• By $T_{A \rightarrow B(t)} = \begin{bmatrix} \cos \theta(t) & -\sin \theta(t) & 0 & -l_2 \sin \theta(t) \\ \sin \theta(t) & \cos \theta(t) & 0 & l_1 + l_2 \cos \theta(t) \\ 0 & 0 & 1 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, we have

$$R_{A \rightarrow B(t)} = \begin{bmatrix} \cos \theta(t) & -\sin \theta(t) & 0 \\ \sin \theta(t) & \cos \theta(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{t}_{A \rightarrow B(t)} = \begin{bmatrix} -l_2 \sin \theta(t) \\ l_1 + l_2 \cos \theta(t) \\ l_0 \end{bmatrix}.$$

• By $[\text{Ad}_{T_{A \rightarrow B(t)}}] = \begin{bmatrix} R_{A \rightarrow B(t)} & [\mathbf{t}_{A \rightarrow B(t)}] R_{A \rightarrow B(t)} \\ 0 & R_{A \rightarrow B(t)} \end{bmatrix}$,

$$[\text{Ad}_{T_{A \rightarrow B(t)}}] = \begin{bmatrix} \cos \theta(t) & -\sin \theta(t) & 0 & -l_0 \sin \theta(t) & -l_0 \cos \theta(t) & l_1 + l_2 \cos \theta(t) \\ \sin \theta(t) & \cos \theta(t) & 0 & l_0 \cos \theta(t) & -l_0 \sin \theta(t) & l_2 \sin \theta(t) \\ 0 & 0 & 1 & -l_1 \cos \theta(t) - l_2 & l_1 \sin \theta(t) & 0 \\ 0 & 0 & 0 & \cos \theta(t) & -\sin \theta(t) & 0 \\ 0 & 0 & 0 & \sin \theta(t) & \cos \theta(t) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 3 of Change of Frame

By $\xi_{A \rightarrow B(t)}^A = [l_1, 0, 0, 0, 0, 1]^T$

$\xi_{A \rightarrow B(t)}^B = [-l_2, 0, 0, 0, 0, 1]^T$

$$[\text{Ad}_{T_{s \rightarrow b}}] = \begin{bmatrix} \cos \theta(t) & -\sin \theta(t) & 0 & -l_0 \sin \theta(t) & -l_0 \cos \theta(t) & l_1 + l_2 \cos \theta(t) \\ \sin \theta(t) & \cos \theta(t) & 0 & l_0 \cos \theta(t) & -l_0 \sin \theta(t) & l_2 \sin \theta(t) \\ 0 & 0 & 1 & -l_1 \cos \theta(t) - l_2 & l_1 \sin \theta(t) & 0 \\ 0 & 0 & 0 & \cos \theta(t) & -\sin \theta(t) & 0 \\ 0 & 0 & 0 & \sin \theta(t) & \cos \theta(t) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

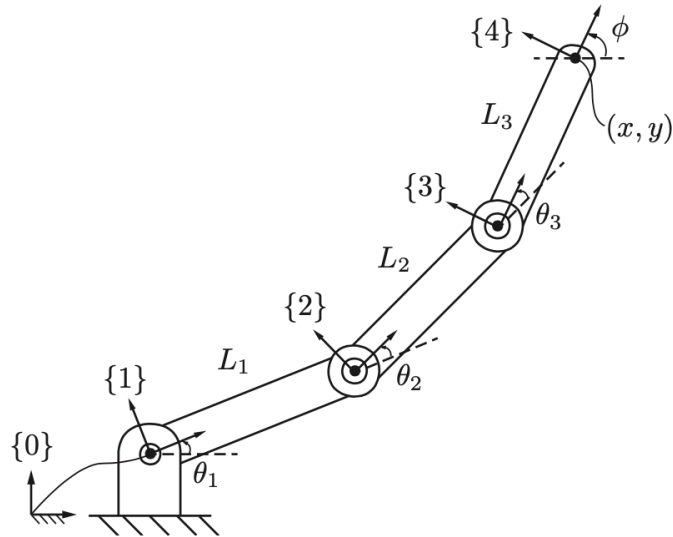
We can verify that $\xi_{A \rightarrow B(t)}^A = [\text{Ad}_{T_{s \rightarrow b}}] \xi_{A \rightarrow B(t)}^B$

Summary

- Twist ξ denotes the 6D motion velocity
- Relationship with \dot{T} : $\dot{T}_{s' \rightarrow b(t)}^o = [\xi_{b(t)}^o] T_{s' \rightarrow b(t)}^o$
- Change of frame:
 - $[\xi_{b(t)}^{s_1}] = T_{s_1 \rightarrow s_2} [\xi_{b(t)}^{s_2}] T_{s_1 \rightarrow s_2}^{-1}$
 - $\xi_{b(t)}^{s_1} = [Ad_{T_{s_1 \rightarrow s_2}}] \xi_{b(t)}^{s_2}$
- Spatial twist: $[\xi_{b(t)}^s] = \dot{T}_{s' \rightarrow b(t)}^s (T_{s' \rightarrow b(t)}^s)^{-1}$
- Body twist: $[\xi_{b(t)}^{b(t)}] = (T_{s \rightarrow b(t)})^{-1} \dot{T}_{s \rightarrow b(t)}^s$

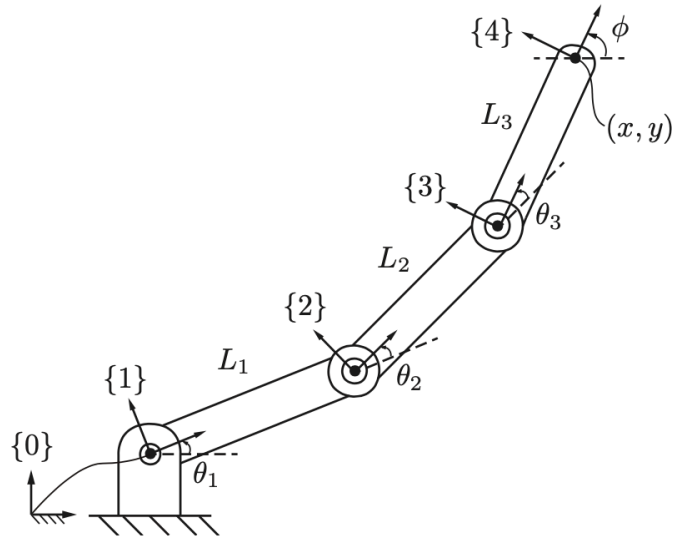
Jacobian of Kinematics Chain

Forward Kinematic Problem



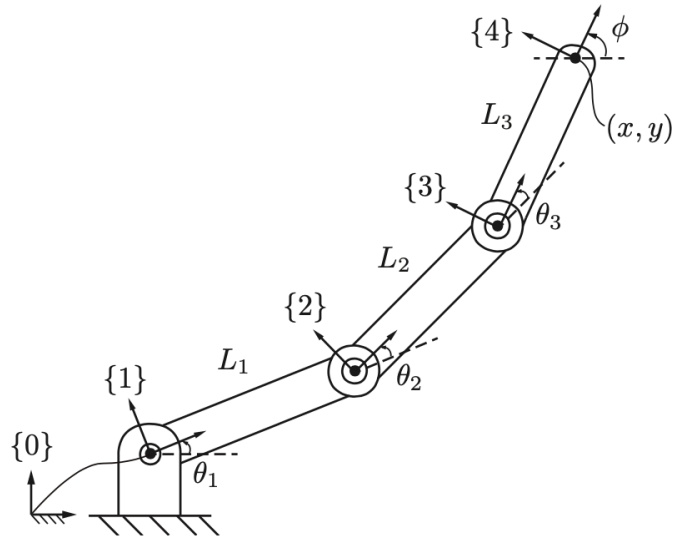
- Suppose that the arm moves
- How do I compute the velocity of the end-effector from the angular velocity of joints?

Spatial Frame Inverse Kinematics Problem



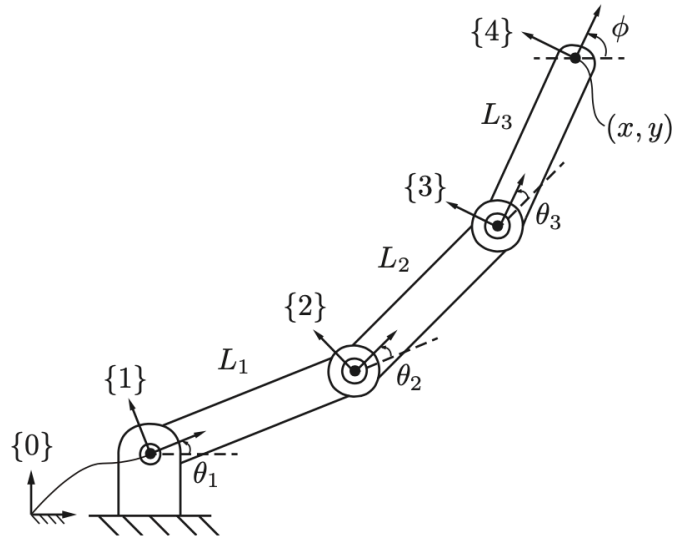
- If I specify the direction of the end-of-effector movement using the spatial frame, how can I change the joint angles?
- e.g. move to a pre-specified $T_{s \rightarrow e}^s$

Body Frame Inverse Kinematics Problem



- If I specify the direction of the end-of-effector movement using the body frame, how can I change the joint angles?
- e.g. move the end-effector forward along its link

Kinematic Equation



- We can solve the problems if we have $\xi_{e(t)} = f(\dot{\theta})$
- The language to describe the velocity of end-effector are
 - $\xi_{e(t)}^s$ for spatial frame query
 - $\xi_{e(t)}^{e(t)}$ for body frame query
- We will derive the f^s and $f^{e(t)}$

Spatial Geometric Jacobian

- Spatial Geometric Jacobian $J^s(\theta)$:

$$\xi_{e(t)}^s = J^s(\theta)\dot{\theta}$$

where $\theta \in \mathbb{R}^n$ (n joints), $J^s(\theta) \in \mathbb{R}^{6 \times n}$, and the i -th column of $J(\theta)$ is ${}^i\hat{\xi}_{e(t)}^s$, the twist when the movement is caused only by the i -th joint **while all other joints stay static**

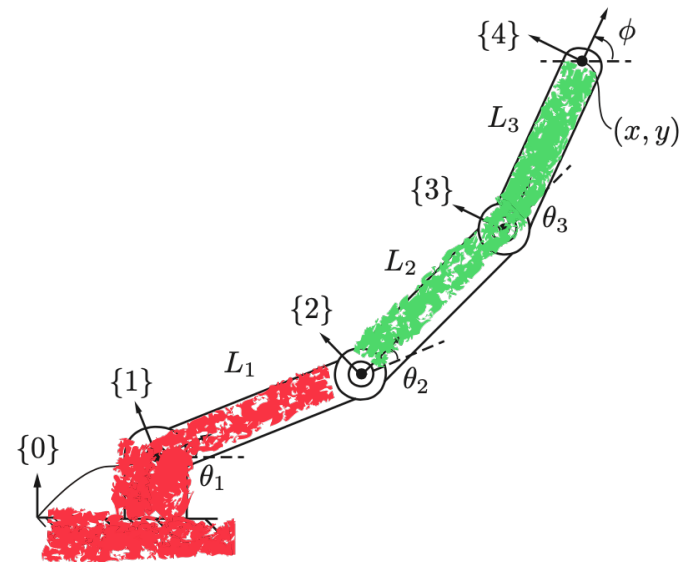
Spatial Geometric Jacobian

- Spatial Geometric Jacobian $J^s(\theta)$:

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where $\theta \in \mathbb{R}^n$ (n joints), $J^s(\theta) \in \mathbb{R}^{6 \times n}$, and the i -th column of $J(\theta)$ is ${}^i\hat{\xi}_{e(t)}^s$, the twist when the movement is caused only by the i -th joint **while all other joints stay static**

- For example, ${}^2\hat{\xi}_{e(t)}^s$ describes the motion of the green part, which is to revolute about Joint {2} (in this revolute joint, $\hat{\omega}^s$, q^s , and d^s are obvious).



Spatial Geometric Jacobian (Proof)

- First of all, $\dot{T}_{s' \rightarrow e(t)}^s = [\xi_{e(t)}^s] T_{s' \rightarrow e(t)}^s$
- Suppose that only θ_i can change at some $(\theta_1, \dots, \theta_n)$. Let ${}^i M_{s' \rightarrow e(t)}^s(\theta_i) := T_{s' \rightarrow e(t)}^s(\theta_1, \dots, \theta_n)$, then

$${}^i \dot{M}_{s' \rightarrow e(t)}^s = [{}^i \xi_{e(t)}^s] T_{s' \rightarrow e(t)}^s$$

- By total derivative,

$$\dot{T}_{s' \rightarrow e(t)}^s = \sum_i \frac{\partial T_{s' \rightarrow e(t)}^s}{\partial \theta_i} \dot{\theta}_i = \sum_i {}^i \dot{M}_{s' \rightarrow e(t)}^s = \sum_i [{}^i \xi_{e(t)}^s] T_{s' \rightarrow e(t)}^s$$

- Therefore, $[\xi_{e(t)}^s] = \sum_i [{}^i \xi_{e(t)}^s] = \sum_i [{}^i \hat{\xi}_{e(t)}^s] \dot{\theta}_i$

Body Geometric Jacobian

- The previous proof works for any recording frame. Simple substitution of $e(t)$ for s as the recording frame gives:
- Body Geometric Jacobian $J^{e(t)}(\theta)$:

$$\xi_{e(t)}^{e(t)} = J^{e(t)}(\theta)\dot{\theta}$$

where $J^{e(t)}(\theta) \in \mathbb{R}^{6 \times n}$, and the i -th column of $J(\theta)$ is ${}^i\hat{\xi}_{e(t)}^{e(t)}$, the twist when the movement is caused only by the i -th joint **while all other joints stay static**

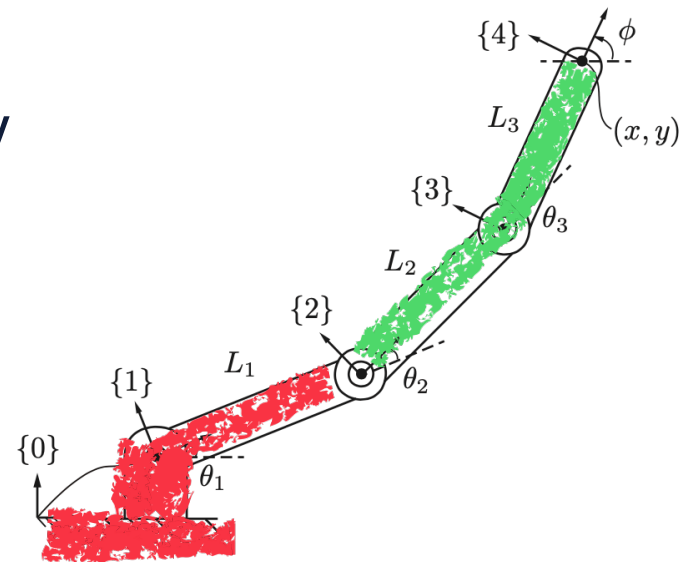
Body Geometric Jacobian

- Body Geometric Jacobian $J^{e(t)}(\theta)$:

$$\xi_{e(t)}^{e(t)} = J^{e(t)}(\theta) \dot{\theta}$$

where $J^{e(t)}(\theta) \in \mathbb{R}^{6 \times n}$, and the i -th column of $J(\theta)$ is $\hat{\xi}_{e(t)}^{e(t)}$, the twist when the movement is caused only by the i -th joint **while all other joints stay static**

- For example, $\hat{\xi}_{e(t)}^{e(t)}$ describes the motion of the green part observed by $\mathcal{F}_s = \mathcal{F}_{\{0\}}$, which is to revolute about Joint $\{2\}$
- For this revolute joint, $\hat{\omega}^{e(t)}$, $q^{e(t)}$, and $d^{e(t)}$ can be computed using $T_{\{2\} \rightarrow \{4\}}$.



Computation of Geometric Jacobian

- Just need to know ${}^i\hat{\xi}_o^{e(t)}$ for the recording frame \mathcal{F}_o
- When computing ${}^i\hat{\xi}_o^{e(t)}$, only the i -th joint can move
- Therefore, we can view as it as a single-joint problem, as our Example 1

Computation of Geometric Jacobian

- Method 1:
 - Figure out ${}^i\hat{\xi}_{e(t)}^o$ for each joint by first computing $\hat{\omega}^o$, q^o , and d^o (as in the robot arm example with red/green colors)
- Method 2:
 - ${}^i\hat{\xi}_{e(t)}^o = [\text{Ad}_{T_{o \rightarrow L_i}}] \hat{\xi}_{e(t)}^{L_i}$
 - Assume that the joint axis is aligned with the x-axis of \hat{L}_i
 - $\hat{\xi}_{e(t)}^{L_i} = [0,0,0,1,0,0]^T$ for prismatic joints
 - $\hat{\xi}_{e(t)}^{L_i} = [1,0,0,0,0,0]^T$ for revolute joints

Inverse Kinematics

Inverse Kinematics

- Position query
 - Given the forward kinematics $T_{s \rightarrow e}^s(\theta)$ and the target pose $T_{target} = \mathbb{SE}(3)$, find θ that satisfies $T_{s \rightarrow e}(\theta) = T_{target}$
- Velocity query
 - Given the twist of the end-effector, find the angular velocity that satisfies $\xi_{target} = J(\theta)\dot{\theta}$
- May have multiple solutions, a unique solution or no solution

Null Space of Jacobian

- Consider the velocity query IK task
- Recall that $\xi = J(\theta)\dot{\theta}$ for an n -joint kinematic chain, where J is a $6 \times n$ matrix
- When $n > 6$, the joint space is projected to a lower-dimensional space and J must exist a null space
- As a result, IK may have infinite solutions (a special solution + any vector in the null space of J)
- The null space adds flexibility to make motion plans

Analytical Solution

- Try to solve the equation $T_{target} = T(\theta)$ and get an analytical solution for θ

- e.g., solve θ_1 and θ_2 for

$$\begin{bmatrix} \cos \theta_1 & -\sin \theta_1 & 0 & -\sin \theta_1(l_2 + l_3) \\ \sin \theta_1 & \cos \theta_1 & 0 & \cos \theta_1(l_2 + l_3) \\ 0 & 0 & 1 & l_1 - l_4 + \theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = T_{target}$$

- For robots with more than 3-DoF, analytical solution can be very complex
 - e.g., for a 6-DoF robot, you will need several pages to write down the formula
- Some useful libraries: IKFast, IKBT

Numerical Solution

- Solving a nonlinear optimization problem
- Standard numerical optimization algorithms can be utilized, e.g. Newton-Raphson and Levenberg-Marquardt
- Numerical IK leverages the geometric Jacobian
$$\xi = J(\theta)\dot{\theta}$$

Levenberg–Marquardt Algorithm

- Error between the desired pose and the current one:

$$T_{err} = T_{target}T(\theta)^{-1} \in \mathbb{SE}(3)$$

- Calculate the corresponding screw:

$$\chi_{err} = \log(T_{err}) \in \mathfrak{se}(3)$$

- Recall that $\xi = J(\theta)\dot{\theta}$:

$$\xi\Delta t = J(\theta)\dot{\theta}\Delta t \Rightarrow \Delta\chi \approx J(\theta)\Delta\theta$$

Levenberg–Marquardt Algorithm

- In LM algorithm, we iteratively update θ
- In each iteration, we try to find a $\Delta\theta$ that minimizes:

$$S(\theta, \Delta\theta) = \|\chi_{err} - J(\theta)\Delta\theta\|^2 + \lambda\|\Delta\theta\|^2$$

- λ term stabilizes the optimization
- Closed-form solution:

$$(J^T J + \lambda I)\Delta\theta = J^T \chi_{err}$$

- Solve $\Delta\theta$ and then update θ by: $\theta \leftarrow \theta + \Delta\theta$

Levenberg–Marquardt Algorithm

$$(J^T J + \lambda I) \Delta \theta = J^T \chi_{err}$$

- Damping factor $\lambda \geq 0$ is adjusted at each iteration:
- If $S(\theta, \Delta \theta)$ is decreasing, a smaller λ (e.g., $\lambda \leftarrow 0.1\lambda$) can be used.
 - closer to the Gauss–Newton algorithm
- Otherwise, a larger λ (e.g., $\lambda \leftarrow 10\lambda$) can be used.
 - closer to the gradient-descent algorithm

Levenberg–Marquardt Algorithm

- LM algorithm may converge to a local minima, initial θ_0 is very important:
 - Sampling multiple θ_0 may boost the performance
- In most cases, θ comes with limit constraints:
 - $l[i] \leq \theta[i] \leq r[i]$
 - A joint can only translate (or rotate) within the limit
 - Invalid state rejection
 - Clipping during the optimization iterations

Kinematic Singularity

Question: is it always possible to move the end-effector to any direction $\hat{\xi}$ for a robot with $\text{DoF} \geq 6$

- **Kinematic singularity:**
 - A **robot configuration** where the robot's end-effector loses the ability to move in one direction instantaneously
- If $\text{rank}(J(\theta)) < 6$ at some θ , by $\Delta\xi = J(\theta)\Delta\theta$, $\Delta\xi$ can only be in a linear space with dimension $\text{rank}(J(\theta)) < 6$, losing its ability to move in some directions
- Note: Kinematic singularity does not mean that there exists a configuration that is not accessible (may get to the pose by some other motion trajectory)