

# **L3: Rotation**

Hao Su

# Agenda

- Multi-Link Rigid-Body Geometry
- Concepts of  $\mathbb{SO}(3)$  and  $\mathbb{SE}(3)$
- Angle-Axis Parameterization of Rotations
- Quaternions
- Local Structure of  $\mathbb{SO}(3)$

# **Multi-Link Rigid-Body Geometry**

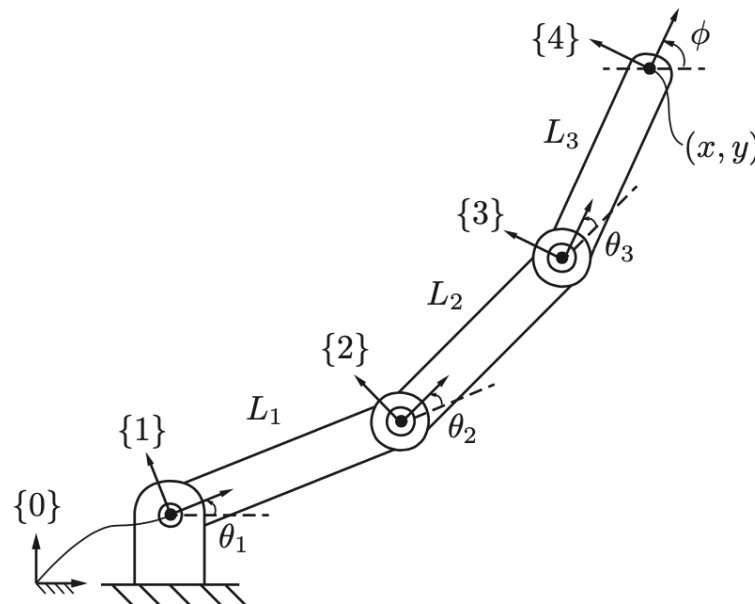
# Link and Joint

Link:

- **Links** are the rigid-body connected in sequence

Joint:

- **Joints** are the connectors between links. They determine the DoF of motion between adjacent links

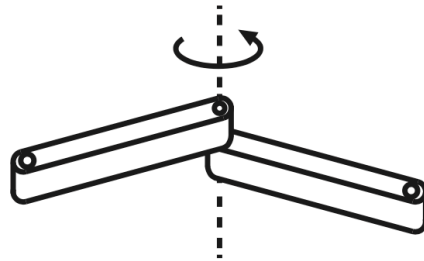


# Base Link and End-Effector Link

- Base link / root link:
  - The 0-th link of the robot
  - Regarded as the “fixed” reference
  - The spatial frame  $\mathcal{F}_s$  is attached to it
- End-effector link
  - The last link
  - e.g., the gripper
  - A frame  $\mathcal{F}_e$  is attached to it

# Two Common Joint Types

- Revolute/Hinge/Rotational joint



Revolute  
(R)

- Prismatic/Translational joint



Prismatic  
(P)

# Kinematics: The Basic Geometry

## Calculation Task

- Kinematics: Describing the motion of bodies (position and velocity)
- Kinematics **does not consider** how to achieve motion via force



# Kinematics Configuration

- Assuming frames are assigned to each link, we can parameterize **the pose of each joint**
  - Using the relative **angle** and **translation** between adjacent frames
- Two representations of the pose of the end-effector
  - **Joint space:** The space in which each coordinate is a vector of joint poses (**angles** around **joint axis**)
  - **Cartesian space:** The space of the rigid transformations of the end-effector by  $(R_{s \rightarrow e}, \mathbf{t}_{s \rightarrow e})$ , where  $\mathcal{F}_e$  is the end-effector frame

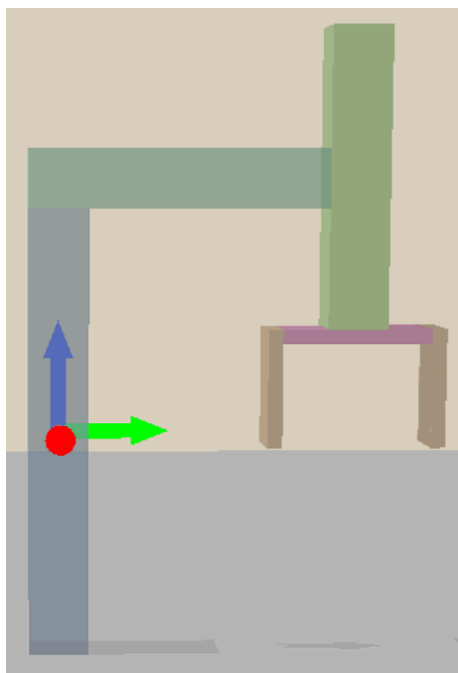


# Kinematics Equations

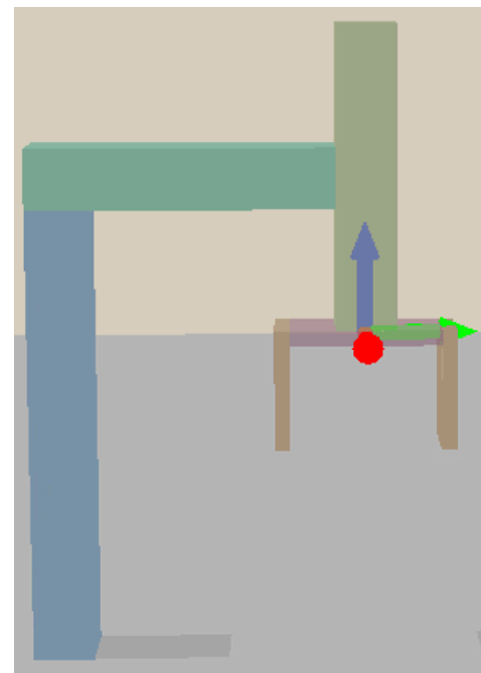
- Map the joint space coordinate  $\theta \in \mathbb{R}^n$  to a transformation matrix  $T$ :

$$T_{s \rightarrow e} = f(\theta)$$

- Calculated by composing transformations along the kinematic chain



base



end\_effector

$$T_{0 \rightarrow 3}^0 = T_{0 \rightarrow 1}^0 T_{1 \rightarrow 2}^1 T_{2 \rightarrow 3}^2 = \begin{bmatrix} \cos \theta_1 & -\sin \theta_1 & 0 & -\sin \theta_1(l_2 + l_3) \\ \sin \theta_1 & \cos \theta_1 & 0 & \cos \theta_1(l_2 + l_3) \\ 0 & 0 & 1 & l_1 - l_4 + \theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Inverse Kinematics

- Given the forward kinematics  $T_{s \rightarrow e}(\theta)$  and the target pose  $T_{target} = \mathbb{SE}(3)$ , find solutions  $\theta$  that satisfy  $T_{s \rightarrow e}(\theta) = T_{target}$

# How to Relate the Motion in Joint Space and Cartesian Space?

- Q1: If the robot moves by  $\Delta\theta$  in the joint space, how much will it move in the Cartesian space? (Forward Kinematics)
- Q2: If the robot would move the end-effector by  $\Delta x$  in the Cartesian space, how shall it change its joint poses? (Inverse Kinematic)
- Suppose we parameterize  $(R, t)$  by the **angles** around **axis**, we need to derive the differential map

- We will study the differentiability of rotation and rigid transformations

$\mathbb{SO}(3)$  **and**  $\mathbb{SE}(3)$

# $\mathbb{SO}(3)$ : The Space of Rotations

- $\mathbb{SO}(n) = \{R \in \mathbb{R}^{n \times n} : \det(R) = 1, RR^T = I\}$
- $\mathbb{SO}(n)$ : “Special Orthogonal Group”
- “Group”: roughly, closed under matrix multiplication
- “Orthogonal”:  $RR^T = I$
- “Special”:  $\det(R) = 1$
- $\mathbb{SO}(2)$ : 2D rotations, 1 DoF
- $\mathbb{SO}(3)$ : 3D rotations, 3 DoF

# $\mathbb{SE}(3)$ : The Space of Rigid Transformations

- $\mathbb{SE}(3) := \left\{ T = \begin{bmatrix} R & \mathbf{t} \\ 0 & 1 \end{bmatrix}, R \in \mathbb{SO}(3), \mathbf{t} \in \mathbb{R}^3 \right\}$
- $\mathbb{SE}(3)$ : “Special Euclidean Group”
- “Group”: roughly, closed under matrix multiplication
- “Euclidean”:  $R$  and  $\mathbf{t}$
- “Special”:  $\det(R) = 1$
- 6 DoF



- We need some theoretical understanding of  $\text{SO}(3)$  and  $\text{SE}(3)$ 
  - The topological structure
  - The parameterization
  - The differentiable properties

# **Angle-Axis Parameterization of Rotations**

# Euler's Theorem

- Any rotation is equivalent to a rotation about a fixed axis  $\hat{\omega} \in \mathbb{R}^3$  ( $\|\hat{\omega}\| = 1$ ) through a positive angle  $\theta$
- $\hat{\omega}$ : unit vector of rotation axis
- $\theta$ : angle of rotation
- $R \in \mathbb{SO}(3) := \text{Rot}(\hat{\omega}, \theta)$
- (In your mind, think  $R$  as a linear transformation)

**Given  $\hat{\omega}$  and  $\theta$ , what is  $R \in \text{SO}(3)$ ?**

# Skew-Symmetric Matrix

- $A$  is skew-symmetric  $A = -A^T$
- Skew-symmetric matrix operator:

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, [a] := \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

- Cross product can be a linear transformation
  - $a \times b = [a]b$

# Given $\hat{\omega}$ and $\theta$ , what is $R \in \text{SO}(3)$ ?

- We can show that, for any  $x \in \mathbb{R}^3$

$$\begin{aligned}\text{Rot}(\hat{\omega}, \theta)x &= x + (\sin \theta)\hat{\omega} \times x + (1 - \cos \theta)\hat{\omega} \times (\hat{\omega} \times x) \\ &= \{I + [\hat{\omega}]\sin \theta + [\hat{\omega}]^2(1 - \cos \theta)\}x\end{aligned}\quad (1)$$

# Given $\hat{\omega}$ and $\theta$ , what is $R \in \text{SO}(3)$ ?

- We can show that, for any  $x \in \mathbb{R}^3$   
$$\begin{aligned}\text{Rot}(\hat{\omega}, \theta)x &= x + (\sin \theta)\hat{\omega} \times x + (1 - \cos \theta)\hat{\omega} \times (\hat{\omega} \times x) \\ &= \{I + [\hat{\omega}]\sin \theta + [\hat{\omega}]^2(1 - \cos \theta)\}x \quad (1)\end{aligned}$$
- By Taylor's expansion of **sin**, **cos**,  $[\hat{\omega}]^3 = -[\hat{\omega}]$ , and above

$$\text{Rot}(\hat{\omega}, \theta)x = (I + \theta[\hat{\omega}] + \frac{\theta^2}{2!}[\hat{\omega}]^2 + \frac{\theta^3}{3!}[\hat{\omega}]^3 + \dots)x$$

# Given $\hat{\omega}$ and $\theta$ , what is $R \in \text{SO}(3)$ ?

- We can show that, for any  $x \in \mathbb{R}^3$   
$$\begin{aligned}\text{Rot}(\hat{\omega}, \theta)x &= x + (\sin \theta)\hat{\omega} \times x + (1 - \cos \theta)\hat{\omega} \times (\hat{\omega} \times x) \\ &= \{I + [\hat{\omega}]\sin \theta + [\hat{\omega}]^2(1 - \cos \theta)\}x\end{aligned}\quad (1)$$
- By Taylor's expansion of **sin**, **cos**,  $[\hat{\omega}]^3 = -[\hat{\omega}]$ , and above

$$\text{Rot}(\hat{\omega}, \theta)x = (I + \theta[\hat{\omega}] + \frac{\theta^2}{2!}[\hat{\omega}]^2 + \frac{\theta^3}{3!}[\hat{\omega}]^3 + \dots)x$$

- Recall Taylor's expansion of exponential,

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$$



# Given $\hat{\omega}$ and $\theta$ , what is $R \in \text{SO}(3)$ ?

- We can show that, for any  $x \in \mathbb{R}^3$   
$$\begin{aligned}\text{Rot}(\hat{\omega}, \theta)x &= x + (\sin \theta)\hat{\omega} \times x + (1 - \cos \theta)\hat{\omega} \times (\hat{\omega} \times x) \\ &= \{I + [\hat{\omega}]\sin \theta + [\hat{\omega}]^2(1 - \cos \theta)\}x\end{aligned}\quad (1)$$
- By Taylor's expansion of **sin**, **cos**,  $[\hat{\omega}]^3 = -[\hat{\omega}]$ , and above

$$\text{Rot}(\hat{\omega}, \theta)x = (I + \theta[\hat{\omega}] + \frac{\theta^2}{2!}[\hat{\omega}]^2 + \frac{\theta^3}{3!}[\hat{\omega}]^3 + \dots)x$$

- Recall Taylor's expansion of exponential,

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$$

- Formally, we have:

$$\text{Rot}(\hat{\omega}, \theta)x = e^{[\hat{\omega}]\theta}x, \forall x \in \mathbb{R}^3$$

Given  $\hat{\omega}$  and  $\theta$ , what is  $R \in \text{SO}(3)$ ?

- By  $\text{Rot}(\hat{\omega}, \theta)x = e^{[\hat{\omega}]\theta}x, \forall x \in \mathbb{R}^3$ ,

$$\text{Rot}(\hat{\omega}, \theta) \equiv e^{[\hat{\omega}]\theta}$$

- This is under such a **Definition of Matrix Exponential**:

$$e^{[\hat{\omega}]\theta} = I + \theta[\hat{\omega}] + \frac{\theta^2}{2!}[\hat{\omega}]^2 + \frac{\theta^3}{3!}[\hat{\omega}]^3 + \dots$$

- By (1) in the last slide, we obtain **Rodrigues** formula:

$$e^{[\hat{\omega}]\theta} = I + [\hat{\omega}]\sin \theta + [\hat{\omega}]^2(1 - \cos \theta)$$

Given  $\hat{\omega}$  and  $\theta$ , what is  $R \in \text{SO}(3)$ ?

- In the angle-axis representation of  $\text{Rot}(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta}$
- $\vec{\theta} = \hat{\omega}\theta$  is also called the **rotation vector**, or **exponential coordinate**

# Rodrigues Formula

- Definition of Matrix Exponential:

$$e^{[\hat{\omega}]\theta} = I + \theta[\hat{\omega}] + \frac{\theta^2}{2!}[\hat{\omega}]^2 + \frac{\theta^3}{3!}[\hat{\omega}]^3 + \dots$$

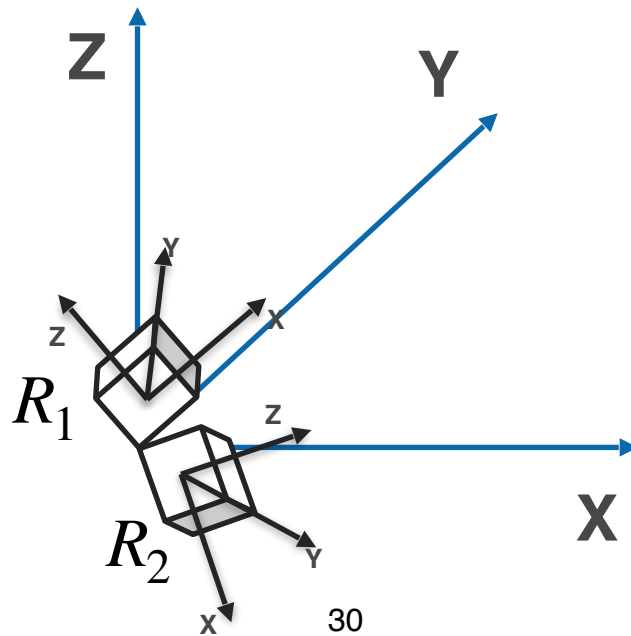
- Sum of infinite series? **Rodrigues Formula**
  - Can prove that  $[\hat{\omega}]^3 = -[\hat{\omega}]$
  - Then, use Taylor expansion of **sin** and **cos**
  - $e^{[\hat{\omega}]\theta} = I + [\hat{\omega}]\sin \theta + [\hat{\omega}]^2(1 - \cos \theta)$

# Given $R \in \text{SO}(3)$ , what is $\hat{\omega}$ and $\theta$ ?

- First question: Is there a **unique** parametrization?
  - No:
    1.  $(\hat{\omega}, \theta)$  and  $(-\hat{\omega}, -\theta)$  give the same rotation
    2. when  $R = I$ ,  $\theta = 0$  and  $\hat{\omega}$  can be arbitrary
    3.  $(\hat{\omega}, \pi)$  and  $(-\hat{\omega}, \pi)$  give the same rotation  
( $\text{tr}(R) = -1$ )
- If we restrict  $\theta \in (0, \pi)$ , a unique parameterization exists:
  - $\theta = \arccos \frac{1}{2}[\text{tr}(R) - 1], \quad [\hat{\omega}] = \frac{1}{2 \sin \theta}(R - R^T)$

# Distance between Rotations

- How to measure the distance between rotations  $(R_1, R_2)$ ?
- A natural view is to measure the (minimal) effort to rotate the body at  $R_1$  pose to  $R_2$  pose:  
 $\because (R_2 R_1^T) R_1 = R_2 \quad \therefore \text{dist}(R_1, R_2) = \theta(R_2 R_1^T) = \arccos \frac{1}{2} [\text{tr}(R_2 R_1^T) - 1]$



# Quaternion

Note: In this section,  $\vec{x} \in \mathbb{R}^3$  and  $q \in \mathbb{R}^4$

# Quaternion is a “Number”

- Recall the complex number  $a + b\mathbf{i}$
- Quaternion is a more generalized complex number:

$$q = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

- $w$  is the real part and  $\vec{v} = (x, y, z)$  is the imaginary part
- Imaginary:  $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$
- anti-commutative :  
 $\mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \mathbf{ki} = \mathbf{j} = -\mathbf{ik}$



# Properties of General Quaternions

- Vector form:  $q = (w, \vec{v})$
- Product:
  - For  $q_1 = (w_1, \vec{v}_1)$  and  $q_2 = (w_2, \vec{v}_2)$ ,  
$$q_1 q_2 = (w_1 w_2 - \vec{v}_1^T \vec{v}_2, w_1 \vec{v}_2 + w_2 \vec{v}_1 + \vec{v}_1 \times \vec{v}_2)$$
  - Not commutable (note that  $\vec{v}_1 \times \vec{v}_2 \neq \vec{v}_2 \times \vec{v}_1$ )
- Conjugate:  $q^* = (w, -\vec{v})$
- Norm:  $\|q\|^2 = w^2 + \vec{v}^T \vec{v} = qq^* = q^*q$
- Inverse:  $q^{-1} := \frac{q^*}{\|q\|^2}$

# Unit Quaternion as Rotation

- A **unit** quaternion  $\|q\| = 1$  can represent a rotation
  - Four numbers plus one constraint  $\rightarrow$  3 DoF
- Geometrically, the shell of a 4D sphere

# Build Rotation Quaternion

- Exponential coordinate  $\rightarrow$  Quaternion:  
 $q = [\cos(\theta/2), \sin(\theta/2)\hat{w}]$
- Quaternion is very close to angle-axis representation!
- Exponential coordinate  $\leftarrow$  Quaternion:

$$\theta = 2 \arccos(w), \quad \hat{w} = \begin{cases} \frac{1}{\sin(\theta/2)} \vec{v} & \theta \neq 0 \\ 0 & \theta = 0 \end{cases}$$

# Unit Quaternion as Rotation

- Rotate a vector  $\vec{x}$  by a quaternion  $q$ :
  1. Augment  $\vec{x}$  to  $x = (0, \vec{x})$
  2.  $x' = qxq^{-1}$
- Compose rotations by quaternion:
  - $(q_2(q_1xq_1^*)q_2^*)$ : first rotate by  $q_1$  and then by  $q_2$
  - Since  $(q_2(q_1xq_1^*)q_2^*) = (q_2q_1)x(q_1^*q_2^*)$ ,  
composing rotations is as simple as multiplying quaternions!

# Conversation between Quaternion and Rotation Matrix

- Rotation  $\leftarrow$  Quaternion

$$R(q) = E(q)G(q)^T$$

where  $E(q) = [-\vec{v}, wI + [\vec{v}]]$  and  
 $G(q) = [-\vec{v}, wI - [\vec{v}]]$

- Rotation  $\rightarrow$  Quaternion
  - Rotation  $\rightarrow$  Angle-Axis  $\rightarrow$  Quaternion

# Inspection

- Each rotation corresponds to two quaternions (“double-covering”)
- Need to normalize to unit length in networks. This normalization may cause big/small gradients in practice

# More about Quaternion

- Quaternion is computationally cheap:
  - Internal representation of Physical Engine and Robot
- Pay attention to convention  $(w, x, y, z)$  or  $(x, y, z, w)$ 
  - $(w, x, y, z)$ : SAPIEN, transforms3d, Eigen, blender, MuJoCo, V-Rep
  - $(x, y, z, w)$ : ROS, PhysX, PyBullet

# Summary of Quaternion

- Very useful and popular in practice
- 4D parameterization, compact and efficient to compute
- Good numerical properties in general



# Local Structure of $\mathbb{SO}(3)$

# Local Structure of $\mathbb{SO}(3)$

- Definition of Matrix Exponential:

$$e^{[\hat{\omega}]\theta} = I + \theta[\hat{\omega}] + \frac{\theta^2}{2!}[\hat{\omega}]^2 + \frac{\theta^3}{3!}[\hat{\omega}]^3 + \dots$$

- Note:

- $e^{A+B} = e^A e^B$  only when  $AB - BA = 0$

- When  $\theta \approx 0$ ,  $e^{[\hat{\omega}]\theta} = I + \theta[\hat{\omega}] + o(\theta[\hat{\omega}])$

# Local Structure of $\mathbb{SO}(3)$

- By  $e^{[\hat{\omega}]\theta} = I + \theta[\hat{\omega}] + o(\theta[\hat{\omega}])$  when  $\theta \approx 0$ ,

$$e^{[\vec{\theta}]} - I = [\vec{\theta}] + o([\vec{\theta}])$$

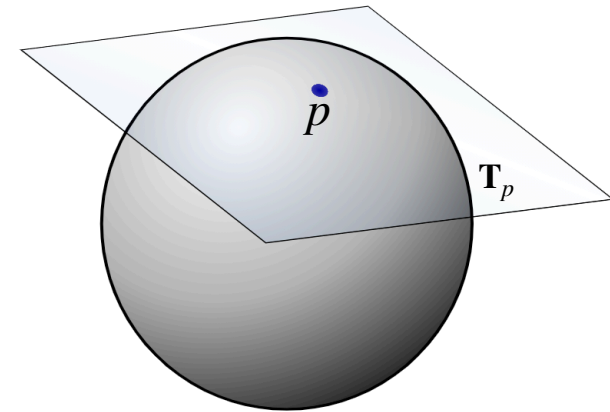
- Interpretation:

- $[\vec{\theta}]$  is a linear subspace of  $\mathbb{R}^{3 \times 3}$

- $e^{[\vec{\theta}]} \rightarrow I$  as  $[\vec{\theta}] \rightarrow 0$

- **Any** local movement in  $\mathbb{SO}(3)$  around  $I$ , which is  $\approx e^{[\vec{\theta}]} - I$ , can be approximated by  $[\vec{\theta}]$

- The set of  $[\vec{\theta}]$  forms the tangent space of  $\mathbb{SO}(3)$  at  $I$



# Lie algebra $\mathfrak{so}(3)$ of $\mathbb{SO}(3)$

- The set of  $[\vec{\theta}]$  is the tangent space of  $\mathbb{SO}(3)$  at  $R = I$ 
  - Ex: What is the tangent space at any  $R \in \mathbb{SO}(3)$ ?
    - $\because e^{[\vec{\theta}]} - I = [\vec{\theta}] + o([\vec{\theta}]), \therefore e^{[\vec{\theta}]}R - R = [\vec{\theta}]R + o([\vec{\theta}])$
    - i.e.,  $\forall R' \in \mathbb{SO}(3)$  near  $R$ ,  $\exists \vec{\theta} \in \mathbb{R}^3$  such that  $R' \approx R + [\vec{\theta}]R$
    - So the tangent space at  $R$  is  $\{SR : S \in \mathbb{R}^{3 \times 3}, S^T = -S\}$
- We give this set a name, the “Lie algebra of  $\mathbb{SO}(3)$ ”
  - $\mathfrak{so}(3) := \{S \in \mathbb{R}^{3 \times 3} : S^T = -S\}$

# Why called “algebra”?

- Introducing Lie bracket  $[A, B] = AB - BA$ , and the set of skew-symmetric matrices are closed under this binary operator
- Then, the set of skew-symmetric matrices form an “algebra”, because
  - The set is closed under Lie bracket
  - Left and right distributive law are satisfied under Lie bracket

$$\dot{R}$$

- Let us first parameterize the rotation of a body frame by time:
  - An observer associated to  $\mathcal{F}_o$  records the motion as  $R_{s' \rightarrow b(t)}^o$ , where the body frame is at  $\mathcal{F}_{b(t)}$ .

$$\dot{R}$$

$$\begin{aligned} R_{s' \rightarrow b(t+\Delta t)}^o - R_{s' \rightarrow b(t)}^o &= R_{b(t) \rightarrow b(t+\Delta t)}^o R_{s' \rightarrow b(t)}^o - R_{s' \rightarrow b(t)}^o \\ &= e^{[\chi_{b(t) \rightarrow b(t+\Delta t)}^o]} R_{s' \rightarrow b(t)}^o - R_{s' \rightarrow b(t)}^o \\ &\approx [\chi_{b(t) \rightarrow b(t+\Delta t)}^o] R_{s' \rightarrow b(t)}^o \end{aligned}$$

- Divided by  $\Delta t$  and take the limit, we have

$$\begin{aligned} \dot{R}_{s' \rightarrow b(t)}^o &= \lim_{\Delta t \rightarrow 0} \left[ \frac{\overrightarrow{\theta}_{b(t) \rightarrow b(t+\Delta t)}^o}{\Delta t} \right] R_{s' \rightarrow b(t)}^o = \left[ \lim_{\Delta t \rightarrow 0} \frac{\overrightarrow{\theta}_{b(t) \rightarrow b(t+\Delta t)}^o}{\Delta t} \right] R_{s' \rightarrow b(t)}^o \\ &= [\omega_{b(t)}^o] T_{s' \rightarrow b(t)}^o \end{aligned}$$

- $\omega_{b(t)}^o := \lim_{\Delta t \rightarrow 0} \frac{\overrightarrow{\theta}_{b(t) \rightarrow b(t+\Delta t)}^o}{\Delta t}$  is the instant angular velocity

# Basic Challenge in Parameterizing $\mathbb{SO}(3)$



# $\mathbb{SO}(3)$ : The Space of Rotations

- $\mathbb{SO}(n) = \{R \in \mathbb{R}^{n \times n} : \det(R) = 1, RR^T = I\}$
- $\mathbb{SO}(n)$ : “Special Orthogonal Group”
- “Group”: roughly, closed under matrix multiplication
- “Orthogonal”:  $RR^T = I$
- “Special”:  $\det(R) = 1$
- $\mathbb{SO}(2)$ : 2D rotations, 1 DoF
- $\mathbb{SO}(3)$ : 3D rotations, 3 DoF

# Parameterization

- Record  $R \in \text{SO}(3)$  by real numbers
- In other words, we look for mappings between  $\mathbb{R}^d$  and  $\text{SO}(3)$ :

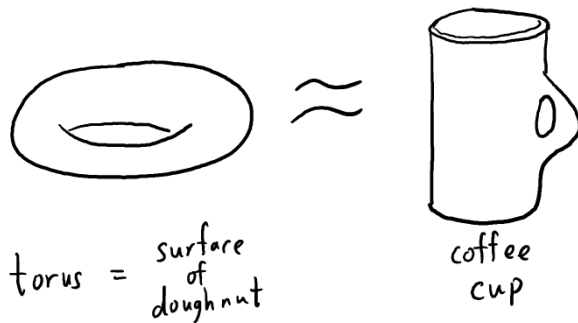
$$f(\theta) = R_\theta$$

# Prereq.: Topology

- Topology: Structural Properties of a Manifold

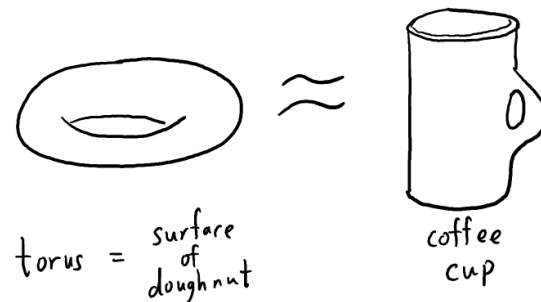
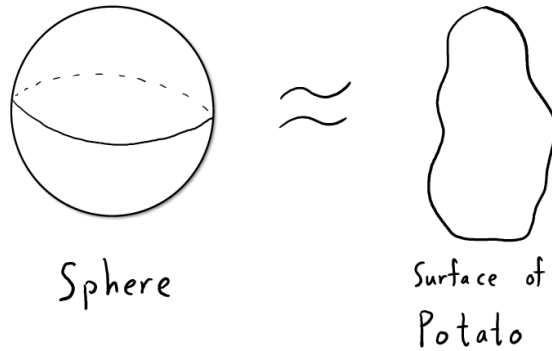


- Two surfaces  $M$  and  $N$  are *topologically equivalent* if there is a **differentiable bijection** between  $M$  and  $N$



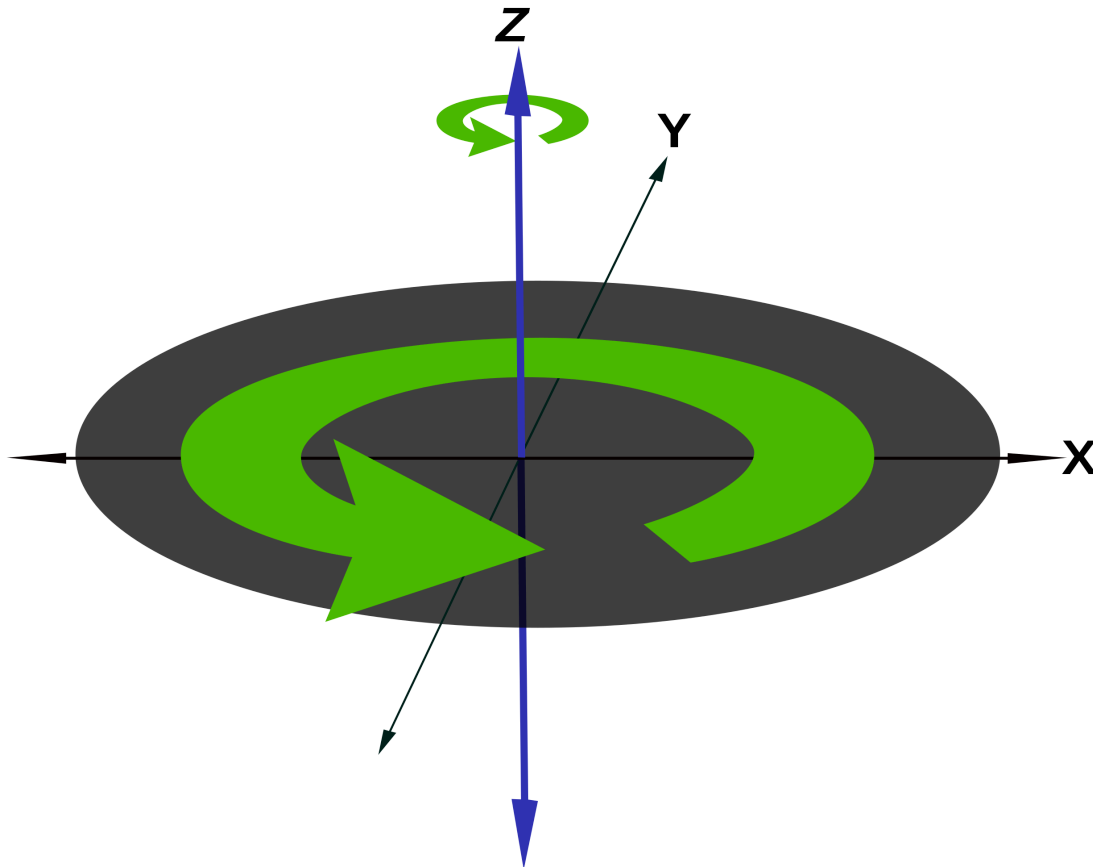
# Prereq: Topology

- More examples:



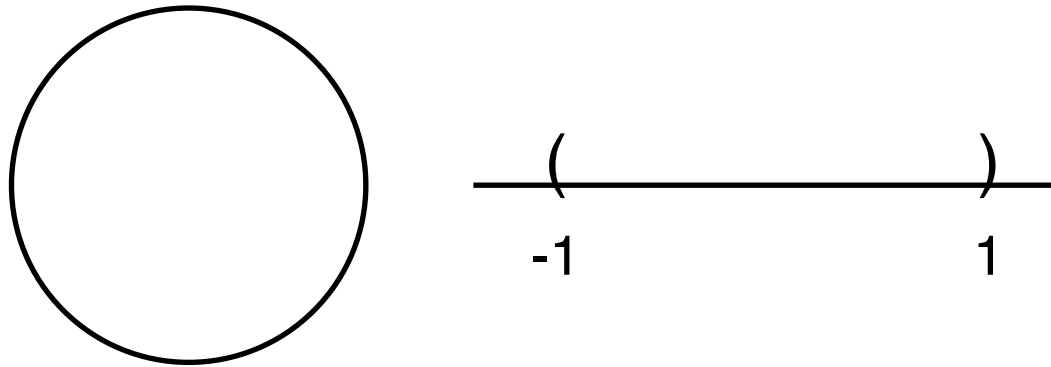
# Topology of $\mathbb{SO}(n)$

- The topology of  $\mathbb{SO}(2)$  is the same as a circle



# Topology of $\mathbb{S}^1(n)$

- Circles do not have the same topology as  $(-1,1)^n$   
 $\implies$  No differentiable bijections between  $\mathbb{S}^1(2)$  and  $(-1,1)^n$



- The topology of  $\mathbb{S}^1(3)$  is also different from  $(-1,1)^n$

# Parameterizing Rotations is Tricky

- Although  $\mathbb{SO}(3)$  only has 3 DoF, you cannot build a differentiable bijection between  $\mathbb{SO}(3)$  and any subset of  $\mathbb{R}^3$
- Even parameterizing  $\mathbb{SO}(3)$  by  $\mathbb{R}^d$  with  $d > 3$ ,
  - we cannot build differentiable bijections with  $(-1, 1)^d$
  - we have to either introduce constraints, or bear with singularities and the “multi-cover” issue
- The challenge brings a lot of trouble to optimization and learning

# Recent Progress on the Theoretical Understanding of Rotation Parameterization

- Revisiting the Continuity of Rotation Representations in Neural Networks, Xiang et al.



# Euler Angle is Very Intuitive



# Euler Angle to Rotation Matrix

- Rotation about principal axis is represented as:

$$R_x(\alpha) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{bmatrix}$$

$$R_y(\beta) := \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix}$$

$$R_z(\gamma) := \begin{bmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $R = R_z(\alpha)R_y(\beta)R_x(\gamma)$  for arbitrary rotation

# Inspection

- Euler Angle is not unique for some rotations.
- For example,

$$\begin{aligned} R_z(45^\circ)R_y(90^\circ)R_x(45^\circ) &= R_z(90^\circ)R_y(90^\circ)R_x(90^\circ) \\ &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \end{aligned}$$

# Inspection

- Gimbal lock:
  - $Df$  is rank-deficient at some  $\theta$
  - $\Rightarrow$  some movement in  $\mathbf{T}_R(\mathbb{SO}(3))$  cannot be achieved

# Inspection

- For example: When  $\beta = \pi/2$ ,

$$R = R_z(\alpha)R_y(\pi/2)R_x(\gamma)$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ \sin(\alpha + \gamma) & \cos(\alpha + \gamma) & 0 \\ -\cos(\alpha + \gamma) & \sin(\alpha + \gamma) & 0 \end{bmatrix}$$

since changing  $\alpha$  and  $\gamma$  has the same effects, a degree of freedom disappears!

# Summary

- Euler angle can parameterize every rotation and has good interpretability
- It is not a unique representation at some points
- There are some points where not every change in the target space (rotations) can be realized by a change in the source space (Euler angles)

# Summary of Rotation Representations

	Inverse?	Composing?	Any local movement in $SO(3)$ can be achieved by local movement in the domain?
Rotation Matrix	✓	✓	N/A
Euler Angle	Complicated	Complicated	No
Angle-axis	✓	Complicated	?
Quaternion	✓	✓	✓

? means no singularity with single exceptions

# Summary of Rotation Representations

- It is quite often that, we use
  - rotation matrices to define concepts
  - Euler angles to visualize rotations
  - angle-axis representation to visualize rotations and calculate derivatives
  - quaternion to write fast codes



# Resources

- In Python, you could use the transforms3d library
- For differentiable transformations, you can play with is “Kornia”, but use with cautious to its numerical properties
- “ceres” is a C++ library that is quite useful