Screw and Twist

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Agenda

- Screw (6D representation of rigid motion)
- Twist (6D representation of rigid motion velocity)

Rigid Transformation and $\mathbb{SE}(3)$

The Set of Rigid Transformations

- $\mathbb{SE}(3) := \left\{ T = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}, R \in \mathbb{SO}(3), t \in \mathbb{R}^3 \right\}$
- $\mathbb{SE}(3)$: “Special Euclidean Group”

- “Group”: closed under matrix multiplication and other conditions of group
- “Euclidean”: R and t
- “Special”: $\det(R) = 1$

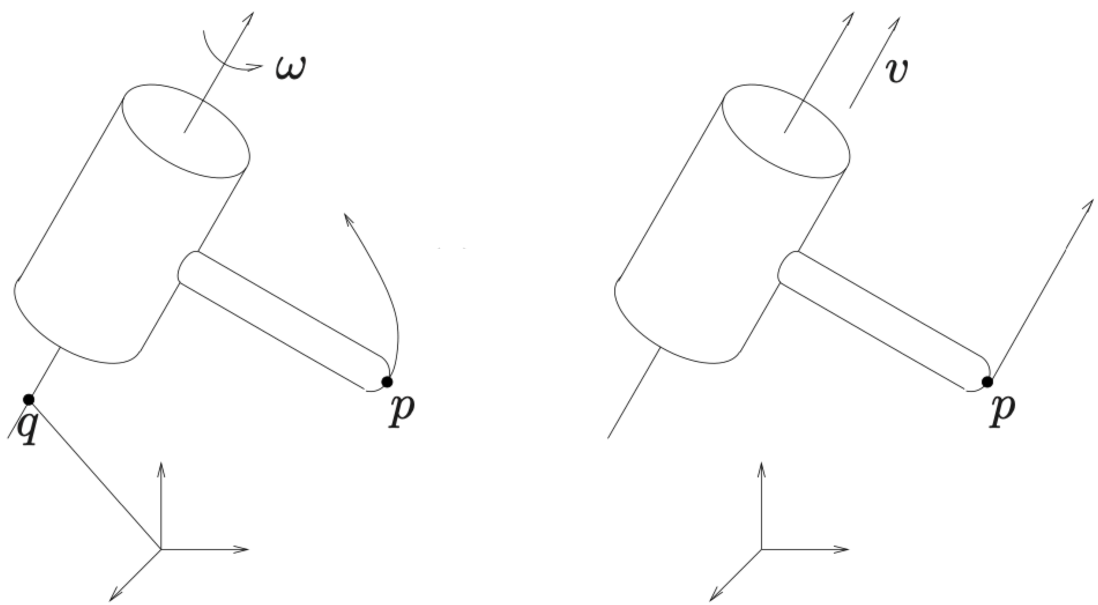
- 6 DoF

Conditions of Group: https://en.wikipedia.org/wiki/Group_(mathematics)#Definition

- Recall Euler's Theorem about $\mathbb{SO}(3)$:
    - Any rotation in $\mathbb{SO}(3)$ is equivalent to rotation about a fixed axis $\hat{\omega} \in \mathbb{R}^3$ through a positive angle θ
- Similar results for $\mathbb{SE}(3)$: Screw Parameterization

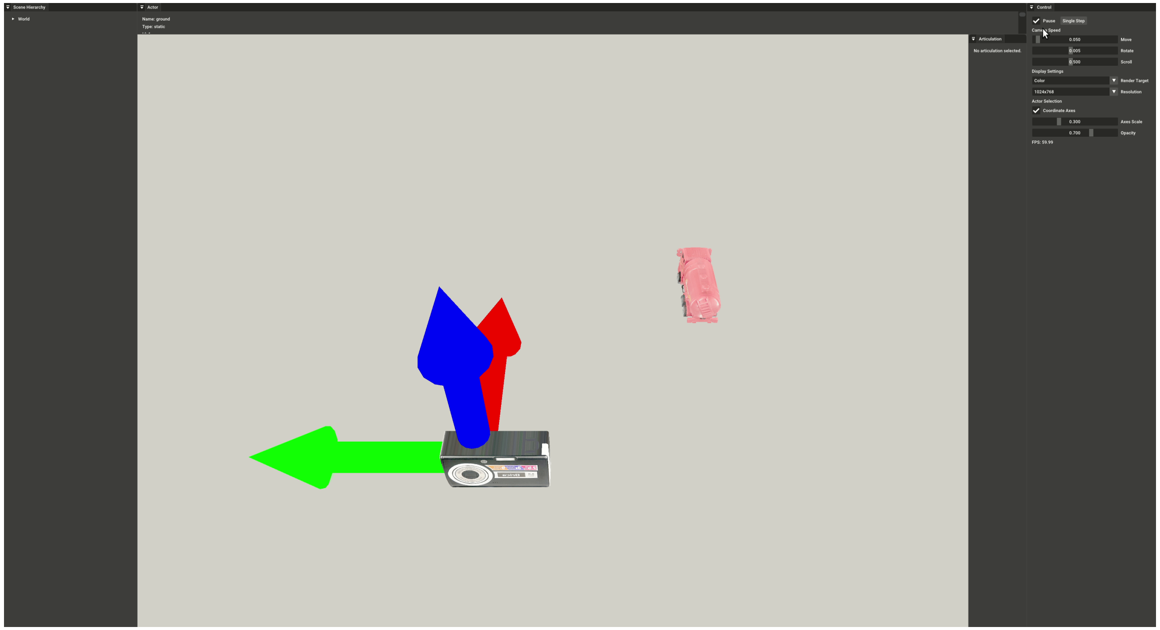
Screw Motion Theorem

- Any rigid body motion is equivalent to rotating about one axis while also translating along the axis
- **The axis may not pass the origin**



Screw Motion Theorem

- Any rigid body motion is equivalent to rotating about one axis while also translating along the axis
- Recall our question of “canonical” rigid transformation decomposition—by sharing rotation axis and translation direction, we identify the decomposition



Review: Lie algebra of $\mathbb{SO}(3)$

- Motion interpretation

  $\hat{\omega}$: motion direction

- Exponential coordinate

  $\vec{\theta} = \hat{\omega}\theta \in \mathbb{R}^3$ (rot vector)

- Exponential map

  $R = \exp([\hat{\omega}]\theta) \in \mathbb{SO}(3)$

- Tangent space at $R = I$

  $[\hat{\omega}]\theta \in \mathfrak{so}(3)$

Goal: The Lie Algebra of $\mathbb{SE}(3)$

- Motion interpretation

  $\hat{\omega}$: motion direction

- Exponential coordinate

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- Motion interpretation

  $\hat{\xi}$: 6D motion direction

- Exponential coordinate

  $\chi = \hat{\xi}\theta \in \mathbb{R}^6$ (screw)

- Exponential map

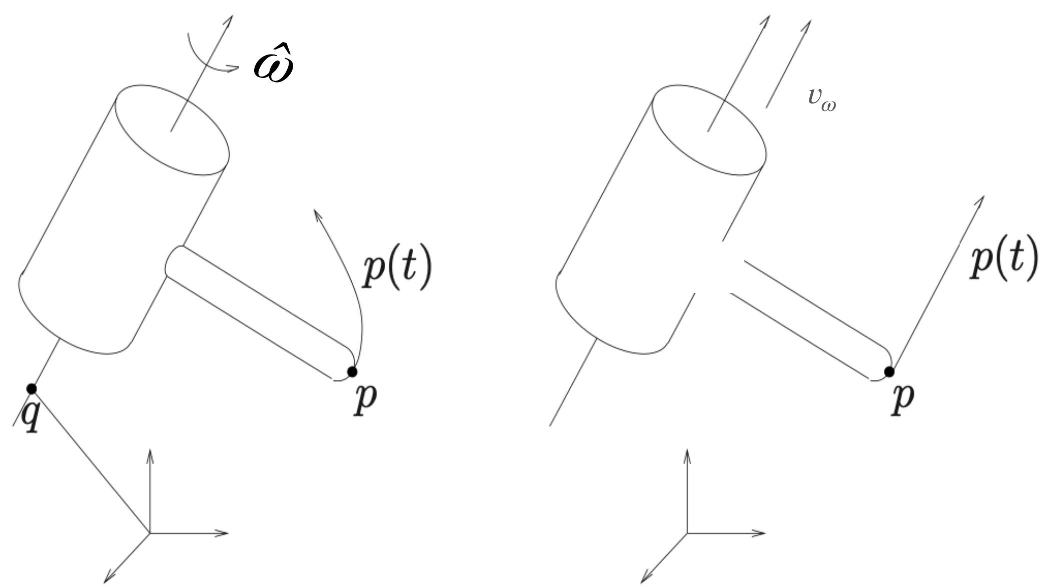
  $T = \exp([\hat{\xi}]\theta) \in \mathbb{SE}(3)$

- Tangent space at $T = I$

  $[\hat{\xi}]\theta \in \mathfrak{se}(3)$

An Imaged Motion for $T \in \mathbb{SE}(3)$

- Transforming by $T \iff$ **rotating** about one axis while also **translating** along the axis
- Assume an arbitrary point q on the axis, a **unit** vector $\hat{\omega}$ denoting axis, and the angle θ
- Assume the translation along $\hat{\omega}$ is d_ω



Screw Parameterization

- In $\mathbb{SO}(3)$, we have

$$\mathrm{Rot}(\hat{\omega}, \theta)x = (I + \theta[\hat{\omega}] + \frac{\theta^2}{2!}[\hat{\omega}]^2 + \frac{\theta^3}{3!}[\hat{\omega}]^3 + \cdots)x$$

- In $\mathbb{SE}(3)$, we have a similar result ($x \in \mathbb{R}^4$ by homogeneous coordinate):

$$\mathrm{Trans}(\hat{\omega}, \theta, q, d_\omega)x = (I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots)x$$

where $A = \begin{bmatrix} [\hat{\omega}\theta] & -[\hat{\omega}\theta]q + d_\omega \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times4}$

Screw Parameterization

$$\text{Trans}(\hat{\omega}, \theta, q, d_\omega)x = (I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots)x, \text{ where } A = \begin{bmatrix} [\hat{\omega}\theta] & -[\hat{\omega}\theta]q + d_\omega \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times4}$$

- Let $A = \begin{bmatrix} [\hat{\omega}] & d \\ 0 & 0 \end{bmatrix} \theta$, where $d = \dfrac{-[\hat{\omega}\theta]q + d_\omega}{\theta}$, then $T = \exp\left(\begin{bmatrix} [\hat{\omega}] & d \\ 0 & 0 \end{bmatrix} \theta \right)$
- The following rule introduces $\hat{\xi}$ so that $T = \exp\left(\begin{bmatrix} [\hat{\omega}] & d \\ 0 & 0 \end{bmatrix} \theta \right) \equiv e^{[\hat{\xi}]\theta}$:
    - $\hat{\xi} = \begin{bmatrix} d \\ \hat{\omega} \end{bmatrix} \in \mathbb{R}^6$ and $[\hat{\xi}]\theta = \begin{bmatrix} [\hat{\omega}] & d \\ 0 & 0 \end{bmatrix} \theta$

Screw Parameterization

- $\chi = \hat{\xi}\theta = \begin{bmatrix} d \\ \hat{\omega} \end{bmatrix} \theta = \begin{bmatrix} -[\hat{\omega}]q\theta + d_\omega \\ \hat{\omega}\theta \end{bmatrix}$ is called **screw,** or **exponential coordinate**
- Introducing the inverse function of $T = e^{[\chi]}$, $\chi = \log(T)$
- $\hat{\xi}$ is called **unit twist,** which describes **motion direction**

Compute $T \in \mathbb{SE}(3)$ from $\hat{\xi}\theta$

- Recall **Rodrigues Formula** for rotations:

$$e^{[\hat{\omega}]\theta} = I + [\hat{\omega}]\sin\theta + [\hat{\omega}]^2(1 - \cos\theta)$$

- Similarly, using Taylor's expansion definition of $\exp$,

$$e^{[\hat{\xi}]\theta} = I + \theta[\hat{\xi}] + (1 - \cos\theta)[\hat{\xi}]^2 + (\theta - \sin\theta)[\hat{\xi}]^3$$

- Further computation gives

$$e^{[\hat{\xi}\theta]} = \begin{bmatrix} e^{[\hat{\omega}\theta]} & (I - e^{[\hat{\omega}\theta]})(\hat{\omega} \times d) + \hat{\omega}\hat{\omega}^T d\theta \\ 0 & 1 \end{bmatrix}$$

Read by Yourself

Compute $\hat{\xi}\theta$ from $T \in \mathbb{SE}(3)$

- First, determine $\hat{\omega}\theta \in so(3)$ from the $SO(3)$ rotation
- The translation component of T is t, then d in $\hat{\xi}\theta = \begin{bmatrix} d \\ \hat{\omega} \end{bmatrix} \theta$ can be calculated as follow ($\theta \neq 0$):

$$d = (\frac{1}{\theta}I - \frac{1}{2}[\hat{\omega}] + (\frac{1}{\theta} - \frac{1}{2}\cot\frac{\theta}{2})[\hat{\omega}]^2)t$$

- $t \perp \hat{\omega} \iff \frac{1}{\theta}(I + [\hat{\omega}]^2)t = 0$, and there is no $\frac{1}{\theta}$ term in d

Read by Yourself

Summary

- **Exponential map**: $T = \mathrm{Trans}(\hat{\omega}, \theta, q, d_\omega) = e^{[\chi]}$
- **Screw**: $\chi = \begin{bmatrix} -[\hat{\omega}]q\theta + d_\omega \\ \hat{\omega}\theta \end{bmatrix}$ is the displacement of the 6D motion
- **Unit twist:** $\hat{\xi} = \begin{bmatrix} d \\ \hat{\omega} \end{bmatrix} \in \mathbb{R}^6$ so that $\chi = \hat{\xi}\theta$, the direction of the 6D motion

Libraries based on Screw Theory

- https://github.com/NxRLab/ModernRobotics/blob/master/packages/Python/modern_robotics/core.py
- https://petercorke.github.io/robotics-toolbox-python/intro.html#

Example of Screw Computation

Q: What is the screw $\chi = \hat{\xi}\theta$ given $T(\theta) = e^{[\hat{\xi}]\theta}$?

$$T(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha t) & -\sin(\alpha t) & \sin(\alpha t) \\ 0 & \sin(\alpha t) & \cos(\alpha t) & 1 - \cos(\alpha t) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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- Recall that given $R \in \mathbb{SO}(3)$, we can compute θ and $[\hat{\omega}]$

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- Recall that given $R \in \mathbb{SO}(3)$, we can compute θ and $[\hat{\omega}]$
- $\theta = \alpha t$, $[\hat{\omega}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, thus $\hat{\omega} = [1,0,0]^T$

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- $\theta = \alpha t$, $[\hat{\omega}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, $\hat{\omega} = [1,0,0]^T$
- Recall that $d = (\frac{1}{\theta}I - \frac{1}{2}[\hat{\omega}] + (\frac{1}{\theta} - \frac{1}{2}\cot\frac{\theta}{2})[\hat{\omega}]^2)t$
- With some calculation, we get $d = [0,1,0]^T$

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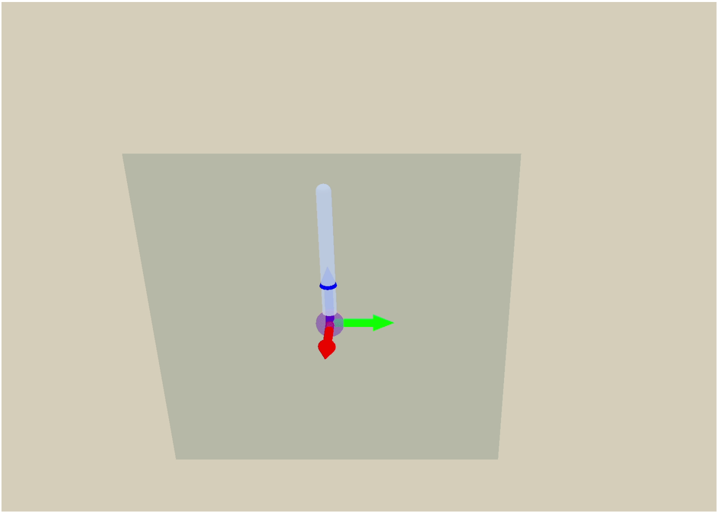
- With some calculation, we get $d = [0,1,0]^T$
- $\hat{\xi}\theta = \begin{bmatrix} d \\ \hat{\omega} \end{bmatrix} \theta = [0,1,0,1,0,0]^T \alpha t$, so $\chi = \hat{\xi}\theta = [0,\alpha t,0,\alpha t,0,0]^T$

Assume $T(\theta)$ describes the relative transformation of a body frame relative to spatial frame: $T^s_{s\to b}(\theta) \equiv T(\theta)$

$$T(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha t) & -\sin(\alpha t) & \sin(\alpha t) \\ 0 & \sin(\alpha t) & \cos(\alpha t) & 1-\cos(\alpha t) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- $\chi^s_{s\to b} = \hat{\xi}^s_{s\to b}\theta^s_{s\to b} = [0,\alpha t,0,\alpha t,0,0]^T$

$\chi^s_{s\to b}$ represents the linear transformation of rotating about a fixed axis



R: x-axis
G: y-axis
B: z-axis

Local Structure of $\mathbb{SE}(3)$

- Definition of Matrix Exponential:

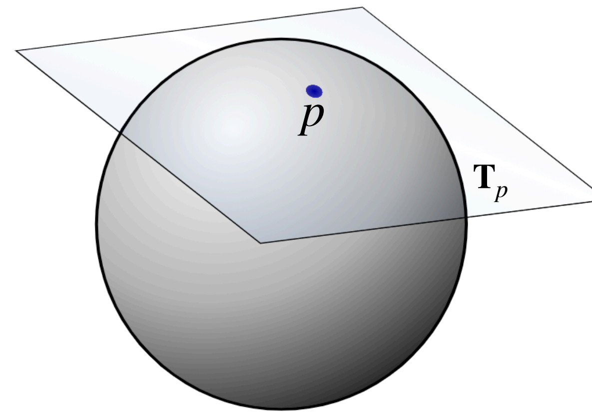
$$e^{[\hat{\xi}]\theta} = I + \theta[\hat{\xi}] + \frac{\theta^2}{2!}[\hat{\xi}]^2 + \frac{\theta^3}{3!}[\hat{\xi}]^3 + \cdots$$

- When $\theta \approx 0$, $e^{[\hat{\xi}]\theta} = I + \theta[\hat{\xi}] + o(\theta[\hat{\xi}])$
- $\forall T \in \mathbb{SE}(3), e^{\theta[\hat{\xi}]}T \approx T + \theta[\hat{\xi}]T$ when $\theta \approx 0$
    - Implies that $\mathbb{SE}(3)$ has a linear local structure (differentiable manifold)

Local Structure of $\mathbb{SE}(3)$

- By $e^{[\hat{\xi}]\theta} = I + \theta[\hat{\xi}] + o(\theta[\hat{\xi}])$ when $\theta \approx 0$,

$$e^{[\chi]} - I = [\chi] + o([\chi])$$



- Interpretation:
  - $[\chi]$ is a linear subspace of $\mathbb{R}^{4\times4}$
  - $e^{[\chi]} \to I$ as $[\chi] \to 0$
  - Any local movement in $\mathbb{SE}(3)$ around I, which is $e^{[\chi]} - I$, can be approximated by some small $[\chi]$
- The set of $[\chi]$ forms the tangent space of $\mathbb{SE}(3)$ at I

Lie algebra $\mathfrak{se}(3)$ of $\mathbb{SE}(3)$

- The set of $[\chi]$ forms the tangent space of $\mathbb{SE}(3)$ at I
  - Ex: What is the tangent space at any $T \in \mathbb{SE}(3)$?
- We give this set a name, the "Lie algebra of $\mathbb{SE}(3)$"
  - $\mathfrak{se}(3) := \left\{ \begin{bmatrix} S & t \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4} : S^T = -S \right\}$

The Lie algebra of $\mathbb{SE}(3)$

- Motion interpretation

  $\hat{\omega}$: motion direction

- Exponential coordinate

  $\vec{\theta} = \hat{\omega}\theta \in \mathbb{R}^3$

- Exponential map

  $R = \exp([\hat{\omega}]\theta) \in \mathbb{SO}(3)$

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- Tangent space at I

  $[\hat{\xi}]\theta \in \mathfrak{se}(3)$

Twist (6D Velocity Parameterization)

Setup

- Let us first parameterize the motion of a body frame by time:
  - An observer associated to $\mathscr{F}_o$ records the motion as $T^o_{s'\to b(t)}$, where the body frame is at $\mathscr{F}_{b(t)}$.

Twist

$$
\begin{aligned}
T^o_{s'\to b(t+\Delta t)} - T^o_{s'\to b(t)} &= T^o_{b(t)\to b(t+\Delta t)} T^o_{s'\to b(t)} - T^o_{s'\to b(t)} \\
&= e^{[\chi^o_{b(t)\to b(t+\Delta t)}]} T^o_{s'\to b(t)} - T^o_{s'\to b(t)} \\
&\approx [\chi^o_{b(t)\to b(t+\Delta t)}] T^o_{s'\to b(t)}
\end{aligned}
$$

- Divided by Δt and take the limit, we have

$$
\begin{aligned}
\dot{T}^o_{s'\to b(t)} &= \lim_{\Delta t\to 0} \left[\frac{\chi^o_{b(t)\to b(t+\Delta t)}}{\Delta t} \right] T^o_{s'\to b(t)} \\
&= [\xi^o_{b(t)}] T^o_{s'\to b(t)}
\end{aligned}
$$

- $\xi^o_{b(t)} := \lim_{\Delta t\to 0} \dfrac{\chi^o_{b(t)\to b(t+\Delta t)}}{\Delta t}$ is called "**twist**", the 6D instant velocity

Twist

- Twist: $\xi^o_{b(t)} := \lim_{\Delta t \to 0} \dfrac{\chi^o_{b(t)\to b(t+\Delta t)}}{\Delta t}$
- $[\xi^o_{b(t)}] = \dot{T}^o_{s'\to b(t)}(T^o_{s'\to b(t)})^{-1}$
- Note: $\xi^o_{b(t)} \neq \dot{\chi}^o_{s'\to b(t)}$ for general $\chi^o_{s\to b(t)}(t)$ (verify by yourself)

Linear Velocity from Twist

- The linear velocity of p^o caused by $T^o_{s'\to b(t)}$ at time t is

$$\mathbf{v}^o_p(t) = \lim_{\Delta t\to 0} \frac{T^o_{b(t)\to b(t+\Delta t)}p^o - p^o}{\Delta t} = \lim_{\Delta t\to 0} \frac{\exp([\chi^o_{b(t)\to b(t+\Delta t)}]) - I}{\Delta t} p^o$$

$$= \lim_{\Delta t\to 0} \frac{[\chi^o_{b(t)\to b(t+\Delta t)}]}{\Delta t} p^o = [\xi^o_{b(t)}]p^o$$

- Therefore, $\boxed{\mathbf{v}^o_p(t) = [\xi^o_{b(t)}]p^o}$

(Recall that, if a motion is a pure rotation, then $\mathbf{v}^o_p(t) = \omega^o_{b(t)} \times p^o$)

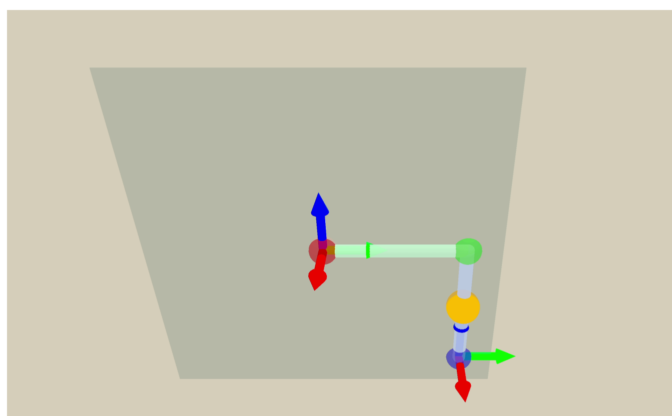
Example of Twist Computation

Example of Twist Computation

- Consider the example, but now an orange point is fixed to the end-effector frame (blue sphere)

- What is the **velocity of orange point at** $t = 0$? Given the pose of end effector frame as below:

$$T^s_{s\to b(t)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha t) & -\sin(\alpha t) & 1+\sin(\alpha t) \\ 0 & \sin(\alpha t) & \cos(\alpha t) & -\cos(\alpha t) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Example of Twist Computation

- The velocity of yellow point caused by the end-effector motion can be computed via twist
- Recall: $\mathbf{v}_p^s = [\xi_{b(t)}^s]p^s$

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- Recall: $\mathbf{v}_p^s = [\xi_{b(t)}^s]p^s$
- By $T_{s\to b(t)}^s = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha t) & -\sin(\alpha t) & 1+\sin(\alpha t) \\ 0 & \sin(\alpha t) & \cos(\alpha t) & -\cos(\alpha t) \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\dot{T}_{s\to b(t)}^s = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\sin(\alpha t) & -\cos(\alpha t) & \cos(\alpha t) \\ 0 & \cos(\alpha t) & -\sin(\alpha t) & \sin(\alpha t) \\ 0 & 0 & 0 & 0 \end{bmatrix}$,
- we have $[\xi_{s\to b(t)}^s] = \dot{T}_{s\to b(t)}^s (T_{s\to b(t)}^s)^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & \alpha & 0 & -\alpha \\ 0 & 0 & 0 & 0 \end{bmatrix}$

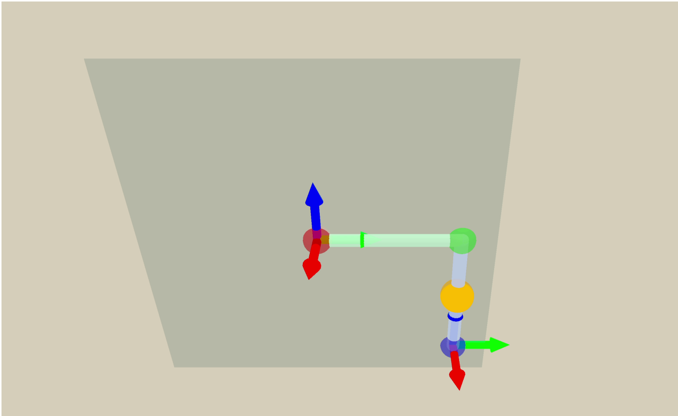
Example of Twist Computation

- The velocity of yellow point caused by the end-effector motion can be computed via twist

- Recall: $\mathbf{v}_p^s = [\xi^s_{b(t)}]p^s$

$$[\xi^s_{b(t)}] = \dot{T}^s_{s\to b(t)}(T^s_{s\to b(t)})^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & \alpha & 0 & -\alpha \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

At $t = 0, p^s = \begin{bmatrix} 0 \\ 1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$



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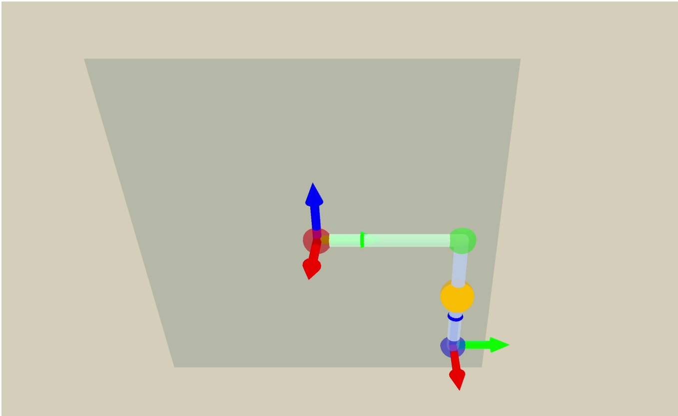
- Recall: $\mathbf{v}_p^s = [\xi_{b(t)}^s]p^s$

$$[\xi_{b(t)}^s] = \dot{T}_{s\to b(t)}^s (T_{s\to b(t)}^s)^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & \alpha & 0 & -\alpha \\ 0 & 0 & 0 & 0 \end{bmatrix}, p^s = \begin{bmatrix} 0 \\ 1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

$$\mathbf{v}_p^s = [\xi_{b(t)}^s]p^s = \begin{bmatrix} 0 \\ \frac{\alpha}{2} \\ 0 \\ 0 \end{bmatrix}$$

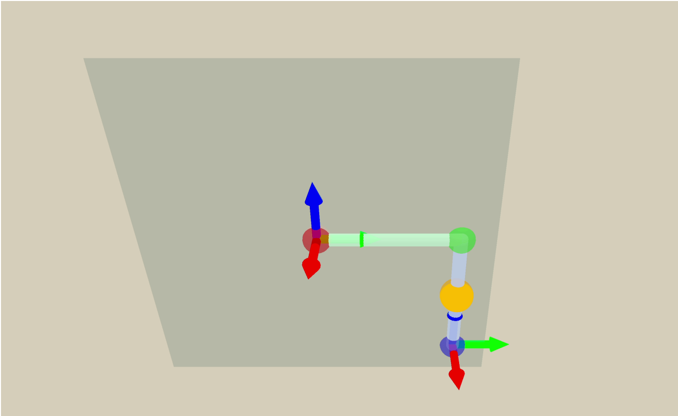


Example of Twist Computation

- We can verify this result by taking the derivative of $\frac{d}{dt}p^s(t)$

$$p^s(t) = \begin{bmatrix} 0 \\ 1 + \frac{1}{2}\sin(\alpha t) \\ -\frac{1}{2}\cos(\alpha t) \\ 1 \end{bmatrix}, \quad \frac{d}{dt}p^s(t) = \begin{bmatrix} 0 \\ \frac{\alpha}{2}\cos(\alpha t) \\ \frac{\alpha}{2}\sin(\alpha t) \\ 0 \end{bmatrix}$$

$$\mathbf{v}_p^s = [\xi_{b(t)}^s]p^s = \begin{bmatrix} 0 \\ \frac{\alpha}{2} \\ 0 \\ 0 \end{bmatrix} = \frac{d}{dt}p^s(t)\bigg|_{t=0}$$



Example of Twist Computation

- What is the body twist of the end effector?
- In the body frame of the end effector (blue sphere), the origin of the frame, which is the blue sphere, has a constant linear velocity, which is always $[0, \alpha, 0]$. The angular velocity is always $[\alpha, 0, 0]$.

So, $\xi_{b(t)}^{b(t)} = [0, \alpha, 0, \alpha, 0, 0]^T$