A SHORT NOTE OF LINEAR ALGEBRA FOR FALL 2021

HAO SUN

This is an interesting example discussed throughout the course of linear algebra in the semester of Fall 2021. I would like to thank *Siqi Mai* (麦思淇) from *School of Biomedical Sciences and Engineering* in the class of 2021 for her notes, which help me to remember what I taught in this semester.

1. Warming Up Problem

We start with a problem.

Problem 1.1. Let A, B be two $n \times n$ matrices. Prove that

$$AB - BA \neq I,$$

where I is the identity matrix.

To solve this problem, we would like to introduce an important concept *the trace of a matrix* first.

Definition 1.2. Let $A = (a_{ij})$ be an $n \times n$ matrix. Define the *trace* of A to be sum of the entries on the diagonal, i.e.

$$\mathrm{tr}(A) := a_{11} + \cdots + a_{nn}.$$

Lemma 1.3. *We have the following properties about trace:*

(1) $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}]$(B),
(2) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

证明. We leave these two properties for the readers as exercises. $\square$

Now we can give a proof of the problem.

Proof of the Problem. Suppose that we have $AB - BA = I$. Taking the trace on both sides of the equation, we have

$$\mathrm{tr}(AB - BA) = \mathrm{tr}(I).$$

For the left hand side, the trace is zero

$$\mathrm{tr}(AB - BA) = \mathrm{tr}(AB) - \mathrm{tr}(BA) = 0,$$

and for the right hand side, the trace of I is n. This is a contradiction. $\square$

The difference $AB - BA$ is also called the *commutator of A and B*, and the corresponding binary operator is known as the *Lie bracket*. The problem tells us that for two arbitrary matrices, their commutators cannot be identity matrix. If we generalize this problem to linear transformations, the same argument holds:

Theorem 1.4. *Given two arbitrary linear transformations $T, S : V \to V$ for a finite dimensional vector space V, their commutator $T \circ S - S \circ T$ cannot be the identity map.*

Note that there is an important assumption in the statement that these linear transformations are of *finite dimensional* vector spaces. Now we can bring up a natural question: what happens when we consider infinite dimensional vector space?

2. Polynomials

Let $\mathbb{P}_n$ be the set of polynomials (with real coefficients), of which the degree are smaller than $n+1$. Let $\mathbb{P}$ be the set of *all* polynomials.

Lemma 2.1. *The sets $\mathbb{P}_n$ and $\mathbb{P}$ are vector spaces. Furthermore, the dimension of $\mathbb{P}_n$ is $n+1$, and the dimension of $\mathbb{P}$ is infinite.*

证明. We only prove the set $\mathbb{P}$ is a vector space, and the argument also holds for $\mathbb{P}_n$.

The zero polynomial $0(x) := 0$ is a special polynomial, and clearly, for any other polynomial $f(x)$, we have $f(x) + 0(x) = f(x)$. Therefore, we have a zero vector in $\mathbb{P}$, which is exactly the zero polynomial $0(x)$.

Let $f(x)$ and $g(x)$ be two polynomials, and let a, b be two real numbers. Clearly, $af(x)+bg(x)$ is also a polynomial. This actually means that given two elements in $\mathbb{P}$, all of their linear combinations are in $\mathbb{P}$.

This finishes the proof that $\mathbb{P}$ is a vector spaces.

To find the dimension, we only have to find a basis of $\mathbb{P}$. Actually, $\{1, x, x^2, \ldots, x^n, \ldots\}$ is a basis of $\mathbb{P}$, which includes infinitely many elements. Therefore, the dimension of $\mathbb{P}$ is infinite. $\square$

Now we consider two maps:

$$S : \mathbb{P} \to \mathbb{P}$$
$$T : \mathbb{P} \to \mathbb{P}$$

such that $S(f(x)) = xf(x)$ and $T(f(x)) = \frac{\partial f(x)}{\partial x}$.

Lemma 2.2. *Prove that T, S are linear transformations.*

证明. We only give the proof for S. Let $f(x)$ and $g(x)$ be two polynomials, and let a, b be two real numbers. We have

$$\begin{aligned}
S(af(x)+bg(x)) &= x(af(x)+bg(x)) \\
&= axf(x)+bxg(x) \\
&= aS(f(x))+bS(g(x)).
\end{aligned}$$

Therefore, S is a linear transformation. □

With respect to the above setup, we calculate the commutator of T and S:

$$\begin{aligned}
(T\circ S - S\circ T)(f(x)) &= T(S(f(x)) - S(T(f(x))) \\
&= T(x(f(x)) - S(\frac{\partial f(x)}{\partial x}) \\
&= f(x) + x\frac{\partial f(x)}{\partial x} - x\frac{\partial f(x)}{\partial x} \\
&= f(x).
\end{aligned}$$

The above calculation holds for an arbitrary polynomial $f(x)$. Therefore,

$$T\circ S - S\circ T = I,$$

where I is the identity map of $\mathbb{P}$.

Note that the definition of the linear transformation S and T can be defined similarly on $\mathbb{P}_n$. The differential operator T keeps the same, while S should send x^n to 0 (note that x^{n+1} is not in the set $\mathbb{P}_n$). With respect to these two operators on $\mathbb{P}_n$, it is easy to check that their commutator is not the identity map.

As a special case, we consider $n = 2$, and we take the basis $\{1, x, x^2\}$ of $\mathbb{P}_2$. With respect to this basis, the linear transformations S and T can be written as matrices:

$$S = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

As matrices, their commutator is

$$TS - ST = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$