# Natural Graph Wavelet Dictionaries: Methods and Applications

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# Outline


1. Background
2. Natural Organization of Graph Laplacian Eigenvectors
3. Natural Graph Wavelet Dictionaries
   - The VM-NGWP Dictionary
   - The LP-NGWP Dictionary
4. Approximation Experiment
5. Summary and Future Work

# Acknowledgment


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- ...

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# Motivation

**Wavelets**

- Have been quite successful on regular domains
- Have been extended to irregular domains $\Longrightarrow$ "2nd Generation Wavelets"

**For example:**

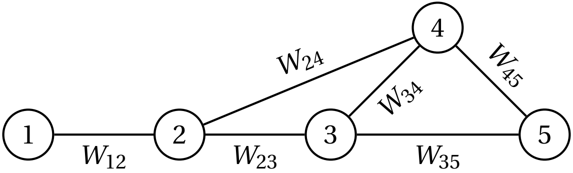
- Hammond, Vandergheynst, and Gribonval (2011): wavelets via spectral graph theory
- Coifman and Maggioni (2006): diffusion wavelets
  - Bremer *et al.* (2006): diffusion wavelet packets

**Key difficulty/issue:** The notion of frequency is *ill-defined* on graphs

# What is a graph?

Let $G$ be a *graph*.

- $V = V(G) = \{v_1, v_2, \ldots, v_N\}$ is the set of *nodes*, where $N := |V(G)|$. For simplicity, we usually use $i$ in place of $v_i$.
- $E = E(G) = \{e_1, e_2, \ldots, e_M\}$ is the set of *edges*, where $e_k = (i, j)$ represents an edge connecting adjacent nodes $i$ and $j$ for some $1 \le i, j \le N$, and $M := |E(G)|$.
- $W = W(G) \in \mathbb{R}^{N \times N}$ is the *edge weight matrix*, where $W_{ij}$ the edge weight between $i$ and $j$.



# How to define $W_{ij}$?

There are many ways to define $W_{ij}$.

- For *unweighted* graphs, we use

$$W_{ij} := \begin{cases} 1, & \text{if } i \sim j \text{ (i.e., } i \text{ and } j \text{ are adjacent),} \\ 0, & \text{otherwise.} \end{cases}$$

  This is often referred to as the *adjacency matrix*.

- For *weighted* graphs, $W_{ij}$ should indicate the *affinity* between nodes $i$ and $j$, e.g., if $v_i \sim v_j$, then

$$W_{ij} := 1/\operatorname{dist}(v_i, v_j),$$

  where $\operatorname{dist}(\cdot,\cdot)$ is a certain measure of dissimilarity, e.g., the Euclidean distance.

# Our Assumptions

In this talk, we assume that the graph is

- **finite.** $M, N < \infty$.
- **undirected.** Any $e_k \in E(G)$ does not specify a direction, which means that $W$ is symmetric.
- **connected.** Any two nodes $i, j \in V(G)$ are connected by a sequence of head-tail edges.
- **simple.** $G$ does not have any loops (an edge connecting a node to itself) or multiple edges (more than one edge connecting a pair of nodes).

The graph may be weighted or unweighted.

# Graph Laplacians

$$\begin{cases} D(G) := \operatorname{diag}(d_1, d_2, \ldots, d_N), & \text{the degree matrix, where } d_i := \sum_{j=1}^{N} W_{ij}, \\ L = L(G) := D(G) - W(G), & \text{the (unnormalized) Laplacian matrix.} \end{cases}$$

We have:

- sorted eigenvalues $0 = \lambda_0 < \lambda_1 \le \cdots \le \lambda_{N-1}$.
- associated eigenvectors $\boldsymbol{\phi}_0, \boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_{N-1}$.

The eigenvectors form a basis for $\mathbb{R}^N$. In particular:

- since $L$ is symmetric, the eigenvectors form an orthonormal basis.
- $\boldsymbol{\phi}_0 (= \mathbf{1}/N)$ is called the *DC vector*.
- $\boldsymbol{\phi}_1$ is called the *Fiedler vector*.

The *random-walk normalized Laplacian matrix* can be obtained by

$$L_{\mathrm{rw}}(G) := D(G)^{-1} L(G).$$

# Why Graph Laplacians?

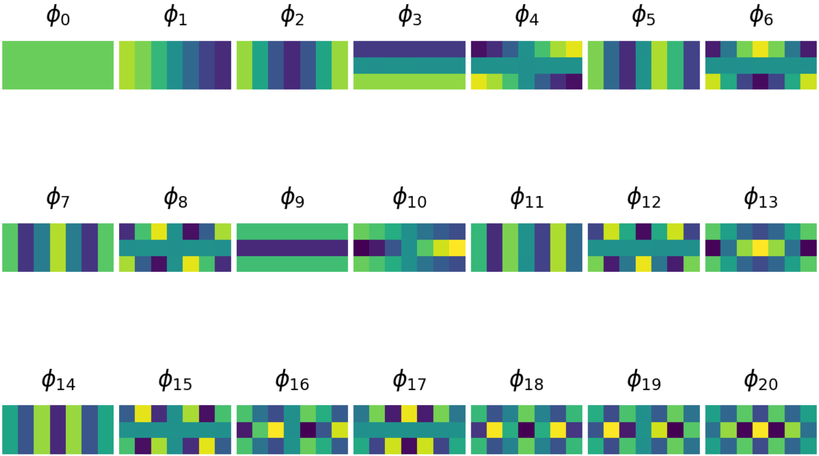
$$\underbrace{\begin{bmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix}}_{L(G)} = \underbrace{\begin{bmatrix} 1 & & & & & \\ & 2 & & & & \\ & & 2 & & & \\ & & & \ddots & & \\ & & & & 2 & \\ & & & & & 1 \end{bmatrix}}_{D(G)} - \underbrace{\begin{bmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 1 \\ & & & & 1 & 0 \end{bmatrix}}_{W(G)}$$

- The eigenvalues of $L$ are $\lambda_l = 2 - 2\cos(\pi l/N) = 4\sin^2(\pi l/2N)$, $l = 0 : N-1$.
- The corresponding eigenvectors are $\boldsymbol{\phi}_l(x) = a_{l;N}\cos(\pi l(x+\frac{1}{2})/N)$, $l, x = 0 : N-1$; $a_{l;N}$ is a const. s.t. $\|\boldsymbol{\phi}_l\|_2 = 1$.
- In 1D path $P_N$, $\lambda_l$ (eigenvalue) is a *monotonic* function w.r.t. the frequency $l$, and $\{\boldsymbol{\phi}_l\}_{l=0}^{N-1}$ are *the DCT Type II basis vectors*.
- So, people have viewed $\{\boldsymbol{\phi}_l\}_{l=0}^{N-1}$ and $\{\lambda_l\}_{l=0}^{N-1}$ as a generalization of the Fourier modes and their corresponding "frequencies" on general graphs, and consequently build the graph wavelets.

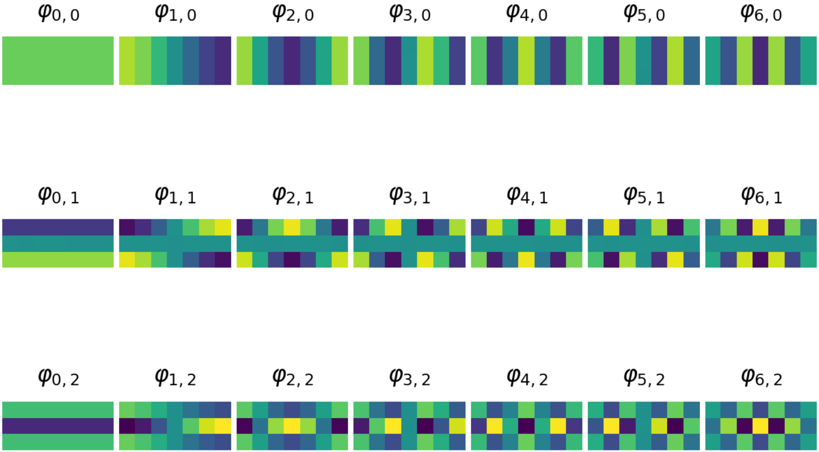
# Graph Wavelets

- The classical 1D wavelets are constructed by the *Littlewood-Paley theory*, i.e., clustering the Fourier modes into dyadic blocks based on their corresponding frequencies in the *dual domain*.
- Spectral Graph Wavelet Transform **(SGWT)** of Hammond et al. derived wavelets on a graph by viewing the eigenvalues as "frequencies", i.e., *organizing the graph Laplacian eigenvectors by the eigenvalue sequence*.
- For general graphs, however, this view point may lead to unexpected problems.

# $P_7 \times P_3$: Non-decreasing Eigenvalue Ordering



# $P_7 \times P_3$: Natural Frequency Ordering



## Problem

- The graph Laplacian eigenvectors of a general graph — even if it is ever so slightly more complicated than a path (or a cycle) — can behave in a much more complicated or unexpected manner than those of a path (or a cycle).
- So, it is impossible to tell how *the behaviors of the eigenvectors* change based solely on the eigenvalues.

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# Plan

- Given $G = (V, E, W)$ and its Laplacian eigenvectors $\{\boldsymbol{\phi}_l\}_{l=0:N-1}$.
- Our goal is to *organize the eigenvectors based on their "behaviors" on graphs*, e.g., their oriented oscillation pattern.
- Our plan is to define and compute quantitative similarity or difference between the eigenvectors.
- The usual $\ell^2$-distance doesn't work since $\left\|\boldsymbol{\phi}_i - \boldsymbol{\phi}_j\right\|_2 = \sqrt{2}\delta_{ij}$.
- We need to come up with some other *non-trivial metrics* to measure the behavioral difference between the eigenvectors.

# Metrics of Graph Laplacian Eigenvectors

- Saito considered using the *Ramified Optimal Transport* (ROT) distance between $\boldsymbol{\phi}_i^2$ and $\boldsymbol{\phi}_j^2$ on graphs to measure the difference.
- Cloninger and Steinerberger proposed a way to measure the similarity between $\boldsymbol{\phi}_i$ and $\boldsymbol{\phi}_j$ based on their *Hadamard* (HAD) product.
- Li and Saito invented another two promising distances.
  - The *Difference of Absolute Gradient* (DAG) distance
  - The *Time-Step Diffusion* (TSD) distance

# The DAG distance

- The idea of DAG is that we treat the absolute gradient of each eigenvector as its feature vector, i.e., $|\nabla_G \boldsymbol{\phi}_i| \in \mathbb{R}^M$.
- $|\nabla_G \boldsymbol{\phi}_l|(e) := |\phi_l(i) - \phi_l(j)|$ for $e = (i,j) \in E(G)$ and $l = 0 : N-1$.
- We use the *$\ell^2$-norm of the difference between the feature vectors* to quantify the distance between the eigenvectors.

$$
\begin{aligned}
d_{\mathrm{DAG}}(\boldsymbol{\phi}_i, \boldsymbol{\phi}_j)^2 &:= \langle |\nabla_G \boldsymbol{\phi}_i| - |\nabla_G \boldsymbol{\phi}_j|, |\nabla_G \boldsymbol{\phi}_i| - |\nabla_G \boldsymbol{\phi}_j| \rangle_E \\
&= \langle \nabla_G \boldsymbol{\phi}_i, \nabla_G \boldsymbol{\phi}_i \rangle_E + \langle \nabla_G \boldsymbol{\phi}_j, \nabla_G \boldsymbol{\phi}_j \rangle_E - 2\langle |\nabla_G \boldsymbol{\phi}_i|, |\nabla_G \boldsymbol{\phi}_j| \rangle_E \\
&= \lambda_i + \lambda_j - \sum_{x \in V} \sum_{y \sim x} |\phi_i(x) - \phi_i(y)| \cdot |\phi_j(x) - \phi_j(y)|
\end{aligned}
$$

  The last term can be viewed as a global average of absolute local correlation between the eigenvectors.

- The computational cost is *$O(M)$* for each $d_{\mathrm{DAG}}(\cdot,\cdot)$ evaluation provided that the eigenvectors have already been computed.

# Dual Graph

Given $G(V,E,W)$ and a metric of its Laplacian eigenvectors $d$ (e.g., $d = d_{\mathrm{DAG}}$), we build a *dual graph* $G^\star = G^\star(V^\star, E^\star, W^\star)$ by viewing the eigenvectors as its nodes, $V^\star = \{\boldsymbol{\phi}_0, \ldots, \boldsymbol{\phi}_{N-1}\}$, and the nontrivial affinity between eigenvector pairs as its edge weights,

$$W^\star_{ij} := 1/d(\boldsymbol{\phi}_{i-1}, \boldsymbol{\phi}_{j-1}), \quad i, j = 1 : N.$$

Using $G^\star$, which is a complete graph, for representing the graph spectral domain and studying relations between the eigenvectors is clearly more *natural* and effective than simply using the eigenvalue magnitudes.

# Visualization of the Arrangement of Eigenvectors

We assemble the *eigenvector distance matrix* $\Delta \in \mathbb{R}^{N \times N}$ by the mutual distance between the eigenvectors

$$\Delta_{ij} := d(\boldsymbol{\phi}_{i-1}, \boldsymbol{\phi}_{j-1}), \quad i, j = 1:N.$$

- $\Delta$ is symmetric and its diagonal entries are zeros.
- We input $\Delta$ and an embedding dimension $s$ (e.g., $s = 2$) to the *classical Multidimensional Scaling (MDS)* and get the coordinate matrix of classical scaling $X \in \mathbb{R}^{N \times s}$.
- The $(l+1)$-th row of $X$ represents the embedding of $\boldsymbol{\phi}_l$ in $\mathbb{R}^s$ $(l = 0 : N-1)$.

# An Example: $P_7 \times P_3$

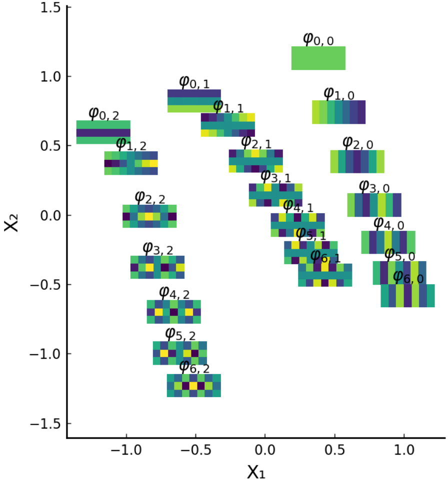


Figure: Embedding of the Laplacian eigenvectors of $P_7 \times P_3$ into $\mathbb{R}^2$ via $d_{\mathrm{DAG}}$ and the classical MDS algorithm.

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# Overview

Based on the dual graph $G^\star = G^\star(V^\star, E^\star, W^\star)$, we can construct various kinds of graph wavelet *dictionaries* (i.e., an overcomplete collection of wavelet vectors).

- Natural Graph Wavelet Packet (NGWP) dictionaries by
  - Varimax Rotations $\Longrightarrow$ the VM-NGWP dictionary.
  - Pair-Clustering $\Longrightarrow$ the PC-NGWP dictionary.
  - Lapped Orthogonal Projections $\Longrightarrow$ the LP-NGWP dictionary.
- Natural Graph Wavelet Frame (NGWF) and its reduced version.

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# Basic Steps to Generate the VM-NGWP Dictionary

1. *Bipartition the dual graph $G^\star$ recursively* via any method, e.g., spectral graph bipartition using the *Fiedler vectors*.
2. *Generate wavelet packet vectors* using the eigenvectors belonging to each subgraph of $G^\star$ that are *well localized on the primal domain $G$*.

# Hierarchical Bipartitioning of $G^\star$

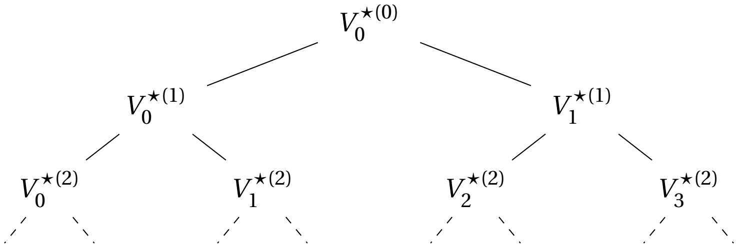


Figure: The hierarchical bipartition tree $\left\{V_k^{\star(j)}\right\}_{j=0:J;k=0:K^j-1}$ of the dual graph nodes $V^\star \equiv V_0^{\star(0)}$, which corresponds to the frequency domain bipartitioning used in the classical wavelet packet dictionary.

# Varimax Rotation

- Let $\Phi_k^{(j)} \in \mathbb{R}^{N \times N_k^j}$ be a matrix whose columns are the eigenvectors belonging to $V_k^{\star(j)}$.
- A varimax rotation is an orthogonal rotation, originally proposed by Kaiser and often used in *factor analysis* to maximize the variances of energy distribution of the input column vectors.
- It is equivalent to finding an orthogonal rotation that maximizes the overall *4th order moments*, i.e.,

$$\Psi_k^{(j)} := \Phi_k^{(j)} \cdot R_k^{(j)}, \quad \text{where } R_k^{(j)} = \arg \max_{R \in \mathrm{SO}(N_k^j)} \sum_{x=1}^{N} \sum_{y=1}^{N_k^j} \left[ \left( \Phi_k^{(j)} \cdot R \right)^4 \right]_{x,y}.$$

- The column vectors of obtained $\Psi_k^{(j)}$ are *more "localized" in the primal domain $G$* than those of $\Phi_k^{(j)}$.

# Example: Hierarchical Bipartition of $(P_7 \times P_3)^\star$

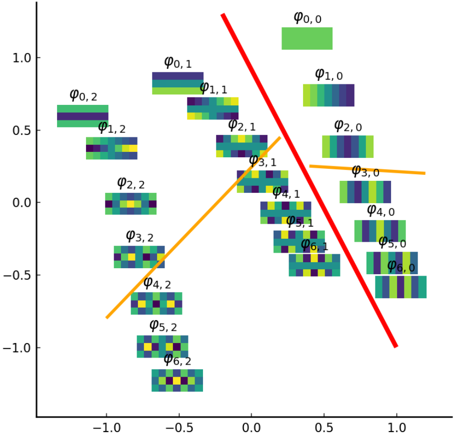


Figure: The result of the hierarchical bipartition algorithm applied to the dual geometry of $P_7 \times P_3$ via $d_{\mathrm{DAG}}$. The thick red line indicates the bipartition at $j = 1$ while the orange lines indicate those at $j = 2$.

# Example: Apply Varimax Rotation to Each Cluster

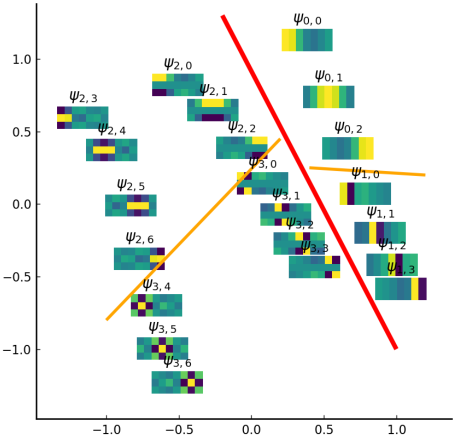


Figure: The VM-NGWP basis vectors of $P_7 \times P_3$ computed by the varimax rotations in the dual domain. Note that the column vectors of the basis matrix $\Psi_k^{(2)}$ are denoted as $\boldsymbol{\psi}_{k,l}$, $l = 0, 1, \ldots$, instead of $\boldsymbol{\psi}_{k,l}^{(2)}$ for simplicity.

# Example: The VM-NGWP Dictionary on $P_{512}$

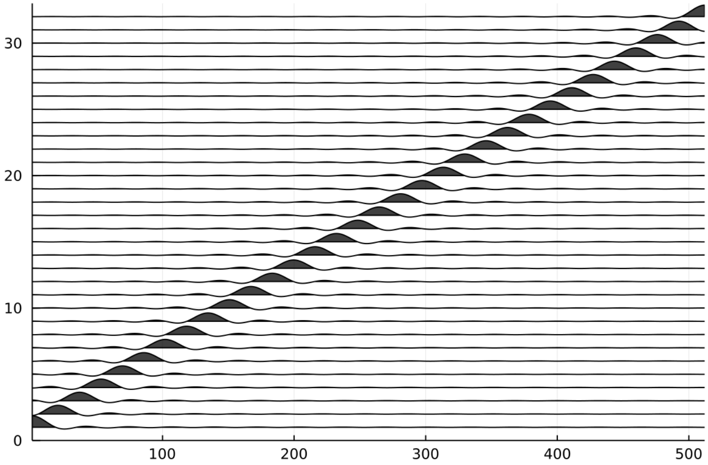


Figure: Father wavelet vectors $\Psi_0^{(4)}$

# Example: The VM-NGWP Dictionary on $P_{512}$

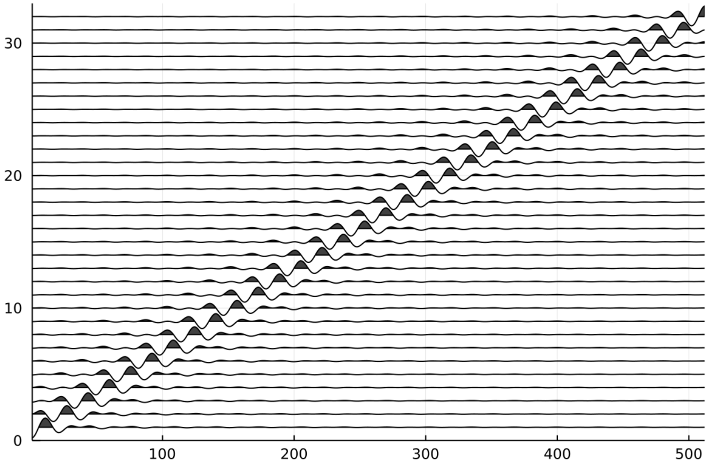


Figure: Mother wavelet vectors $\Psi_1^{(4)}$

# Example: The VM-NGWP Dictionary on $P_{512}$

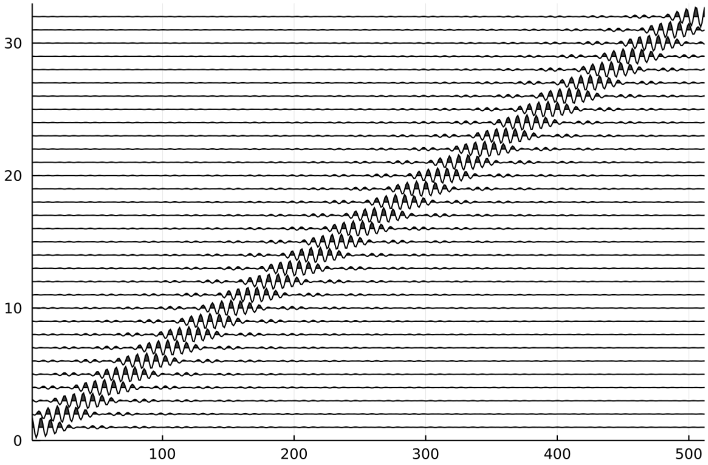


Figure: Wavelet packet vectors $\Psi_4^{(4)}$

# Remarks

- As we can see, our algorithm actually generates the classical *Shannon* wavelet packet dictionary when an input graph is the simple path $P_N$.
- The VM-NGWP dictionary can be viewed as a generalization of the *Shannon* wavelet packet dictionary.
- Can we generalize *other type* of wavelet packet dictionary to graph settings?

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# Soft Bipartition

- Previously, we used the sign of the Fiedler vectors, i.e., the *hard* way, to bipartition the dual graph recursively.
- These bipartitions naturally yield the *orthogonal splitting property*, i.e., the hard bipartition of $V^\star$ nicely yields two subspaces $\mathrm{span}\left(V_0^\star\right)$ and $\mathrm{span}\left(V_1^\star\right)$ such that

$$\mathrm{span}\left(V_0^\star\right) \bigoplus \mathrm{span}\left(V_1^\star\right) = \mathrm{span}\left(V^\star\right),$$
$$\mathrm{span}\left(V_0^\star\right) \perp \mathrm{span}\left(V_1^\star\right).$$

- A question is whether we can bipartition the dual graph in a *soft or lapped* way by allowing some spillovers across the cutoff boundary and still satisfy the orthogonal splitting property.
- If so, we can generalize the *Meyer* wavelet packet dictionary to the graph settings.

# Smooth Orthogonal Projector on $P_N$

- Let $V_0$, $V_1$ be the hard bipartitions of $P_N = (V, E)$.
- Coifman and Meyer introduced the *smooth orthogonal projector*

$$P_{V_k}\boldsymbol{f} := U^{\mathsf{T}} \chi_{V_k} U \boldsymbol{f}, \qquad k = 0, 1 \text{ and } \boldsymbol{f} \in \mathbb{R}^N.$$

It consists of three operators:

1) The *orthogonal folding operator* $U \in \mathbb{R}^{N \times N}$.
2) The *restriction operator* $\chi_{V_k} \in \mathbb{R}^{N \times N}$ is a diagonal matrix with

$$\left(\chi_{V_k}\right)_{x,x} = \begin{cases} 1, & \text{if } x \in V_k, \\ 0, & \text{otherwise.} \end{cases}$$

3) The *orthogonal unfolding operator* $U^{\mathsf{T}} \in \mathbb{R}^{N \times N}$.

- $P_{V_k}$ works as a smooth/soft version of $\chi_{V_k}$.

# Action Region on $P_N$

- The *action region* $(\beta-\eta, \beta+\eta) \subset (1, N)$ consists of the *cutoff boundary* $\beta$ and the *action region bandwidth* $\eta$.
- In particular, $(\beta-\eta, \beta)$ is the *negative action region* and $(\beta, \beta+\eta)$ is the *positive action region*.
- Denote $R^- := V \cap (\beta-\eta, \beta)$ and $R^+ := V \cap (\beta, \beta+\eta)$.
- Then, we can define the set of *reflection triples* $\left\{(v_i^-, v_i^+, r_i)\right\}_{i=1:N_p}$, where
  1) $v_i^-$ is the $i$-th closest node to $\beta$ in $R^-$,
  2) $v_i^+$ is the $i$-th closest node to $\beta$ in $R^+$,
  3) $v_i^-$, $v_i^+$ are a *reflection pair* about $\beta$ and their *reflection radius* $r_i := \beta - v_i^- = v_i^+ - \beta$,
  4) $N_p := \min\left(|R^-|, |R^+|\right)$.

  Note that $r_1 < r_2 < \cdots < r_{N_p} < \eta$.

# Orthogonal Folding Operator on $P_N$

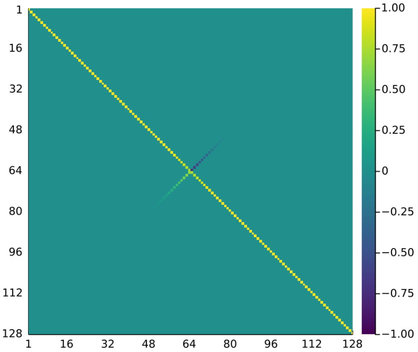
Then, the orthogonal folding operator $U := U(s, \beta, \eta) \in \mathbb{R}^{N \times N}$ associated with the action region $(\beta - \eta, \beta + \eta)$ and the set of reflection triples $\{(v_i^-, v_i^+, r_i)\}_{i=1:N_p}$ is defined by *modifying the identity matrix* $I_N$ as below

$$U_{v_i^-, v_i^-} := s\left(\frac{r_i}{\eta}\right), \qquad U_{v_i^-, v_i^+} := -s\left(-\frac{r_i}{\eta}\right),$$
$$U_{v_i^+, v_i^-} := s\left(-\frac{r_i}{\eta}\right), \qquad U_{v_i^+, v_i^+} := s\left(\frac{r_i}{\eta}\right).$$

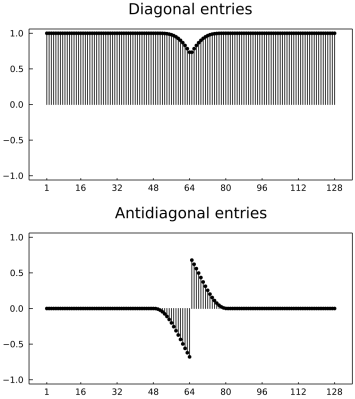
The function $s(t)$ above is the so-called *rising cutoff function*, which is a smooth version of the Heaviside step function

$$s(t) = \begin{cases} 0, & \text{if } t \le -1, \\ \sin[\frac{\pi}{4}(1 + \sin\frac{\pi}{2} t)], & \text{if } |t| < 1, \\ 1, & \text{if } t \ge 1. \end{cases}$$

# Example: Orthogonal Folding Operator on $P_{128}$



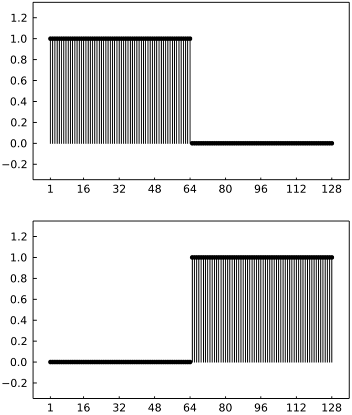

(a) Heatmap plot of $U$



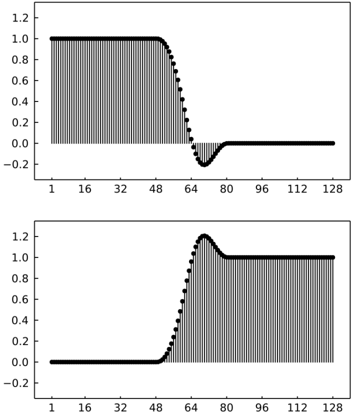

(b) Diagonal/Antidiagonal entries

Figure: The orthogonal folding operator $U$ on $P_{128}$ with $\beta = 64.5$ and $\eta = 16$.

# Example: Hard vs. Soft on $P_{128}$




(a) Hard bipartition



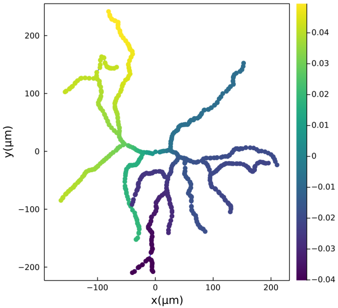

(b) Soft (Lapped) bipartition

Figure: Splitting $\boldsymbol{f} \equiv \mathbf{1} \in \mathbb{R}^{128}$ into $V_0 = 1:64$ and $V_1 = 65:128$, by the restriction operators (a), and by the smooth orthogonal projectors ($\beta = 64.5$, $\eta = 16$) (b).

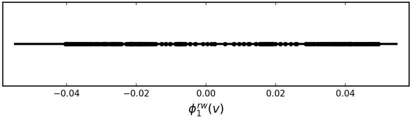
# To General Graphs

- Given $G(V,E,W)$, the Fiedler vector $\boldsymbol{\phi}_1^{\mathrm{rw}}$ of $L_{\mathrm{rw}}(G)$ provides *an embedding of the graph nodes into $\mathbb{R}$* such that the affinities between the nodes in $G$ are best preserved in $\mathbb{R}$.
- Naturally, we can set the action region $(\beta-\eta,\beta+\eta)$ to be $\left(-\epsilon\cdot\|\boldsymbol{\phi}_1^{\mathrm{rw}}\|_\infty, \epsilon\cdot\|\boldsymbol{\phi}_1^{\mathrm{rw}}\|_\infty\right)$ in the 1D embedding space, where $\beta=0$, $\eta=\epsilon\cdot\|\boldsymbol{\phi}_1^{\mathrm{rw}}\|_\infty$ , and $\epsilon\in(0,1)$ is the *relative action region bandwidth*.
- Denote $R_\epsilon^- := \left\{v^-\in V \,|\, \phi_1^{\mathrm{rw}}(v^-)\in\left(-\epsilon\cdot\|\boldsymbol{\phi}_1^{\mathrm{rw}}\|_\infty, 0\right]\right\}$ and $R_\epsilon^+ := \left\{v^+\in V \,|\, \phi_1^{\mathrm{rw}}(v^+)\in\left(0, \epsilon\cdot\|\boldsymbol{\phi}_1^{\mathrm{rw}}\|_\infty\right)\right\}$.
- Then, we can pair the nodes within $R_\epsilon^-$ and $R_\epsilon^+$, and define the reflection triples in a *similar manner* as in the $P_N$ case.
- Let us use an example to see how it works.

# Example: RGC #100 and Its 1D Embedding

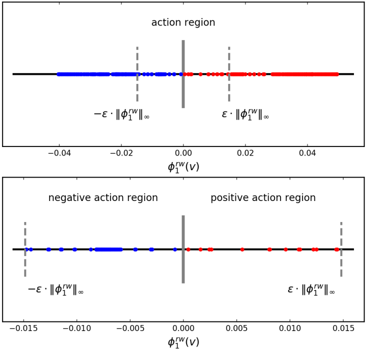



(a) $\boldsymbol{\phi}_1^{\mathrm{rw}}$ on the RGC #100

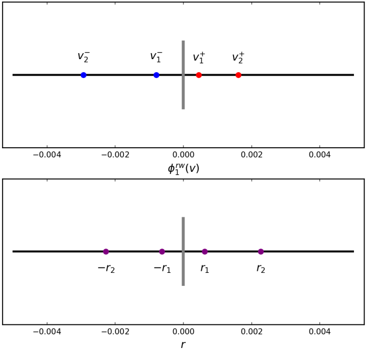



(b) 1D embedding

# Example: Reflection Triples of RGC #100




(a) The action region ($\epsilon = 0.3$)




(b) The (first two) reflection triples

Figure: Locating the positive and negative action regions on the 1D embedding of the RGC #100 (a); and finding the (first two) reflection triples, i.e., $(v_i^-, v_i^+, r_i)$ $(i = 1, 2)$, near the cutoff boundary $\beta = 0$ (b).

# Smooth Orthogonal Projector on General Graphs

- After we got the set of reflection triples on $G$, we can assemble the orthogonal folding operator by modifying the identity matrix $I_N$ in the *same way* as we did in the $P_N$ case.
- Consequently, we can generalize the smooth orthogonal projector to the graph settings.

# Example: Diagonal Entries of $U$

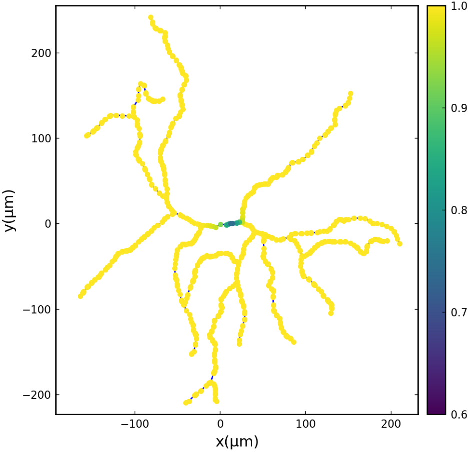



Figure: The diagonal entries of $U$ ($\epsilon = 0.3$) on RGC #100.

# Lapped NGWP

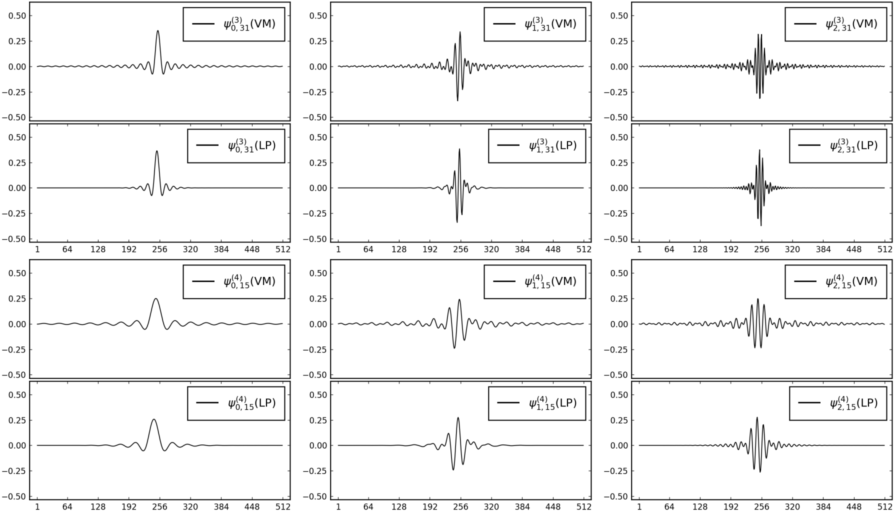
- *Applying the smooth orthogonal projector to the dual graph $G^\star$ recursively*, we can construct the LP-NGWP dictionary.
- Recall the construction of the VM-NGWP dictionary

$$\Phi_k^{(j)} = \Phi \chi_{V_k^{\star(j)}} I_k^{(j)}, \qquad \Psi_k^{(j)} = \mathrm{varimax}\left(\Phi_k^{(j)}\right),$$

  where $I_k^{(j)} \in \mathbb{R}^{N \times N_k^j}$ ($N_k^j := \left|V_k^{\star(j)}\right|$) is the *slicing operator* and it can be simply obtained by removing all the zero columns of the restriction operator $\chi_{V_k^{\star(j)}}$.
- Construct the LP-NGWP dictionary by *replacing the restriction operator with the smooth orthogonal operator*:

$$Y_k^{(j)} := \Phi P_{V_k^{\star(j)}} I_k^{(j)}, \qquad \Psi_k^{(j)} = \mathrm{varimax}\left(\mathrm{MGS}\left(Y_k^{(j)}\right)\right).$$

# Example: VM-NGWP vs. LP-NGWP ($\epsilon = 0.3$) on $P_{512}$



# Remarks

- As we can see, our algorithm actually generates the classical *Meyer* wavelet packet dictionary when an input graph is the simple path $P_N$.
- The LP-NGWP dictionary is a generalization of the *Meyer* wavelet packet dictionary whose basis vectors are *more localized* on primal domain $G$ compared to the ones of the VM-NGWP dictionary.

# Outline


1. Background
2. Natural Organization of Graph Laplacian Eigenvectors
3. Natural Graph Wavelet Dictionaries
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   - The LP-NGWP Dictionary
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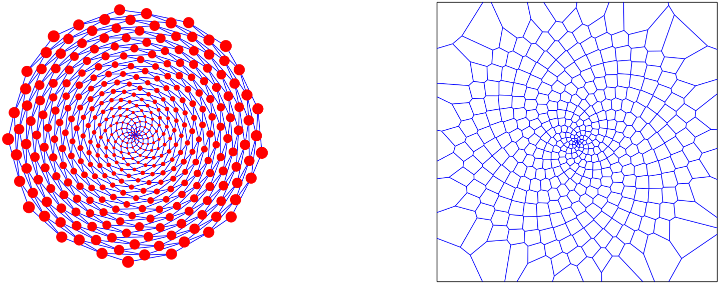
5. Summary and Future Work

# The Sunflower Graph



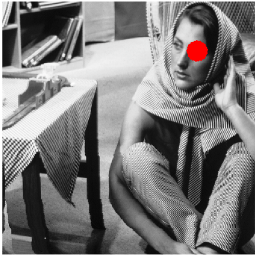
(a) Sunflower graph (b) Voronoi tessellation

Figure: (a) Sunflower graph ($N = 400$); node radii vary for visualization purpose; (b) its Voronoi tessellation.

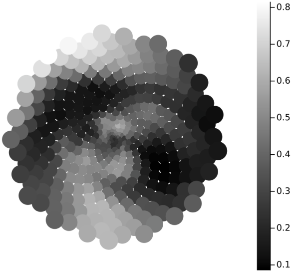
# Graph Signal Sampling Scheme

- The sunflower graph can be viewed as a simple model of the distribution of *photoreceptors in mammalian visual systems* due to cell generation and growth.
- Such a viewpoint motivates us the following sampling scheme:
  1) overlay the sunflower graph on a part of the standard Barbara image
  2) construct the Voronoi tessellation of the bounding square region with the nodes of the sunflower graph as its seeds
  3) compute the average pixel value within each Voronoi cell

# The Sunflower Barbara Eye Signal



(a) The sunflower graph overlaid on Barbara's left eye




(b) Barbara's left eye as an input graph signal

Figure: Barbara's left eye region sampled on the sunflower graph nodes (a) as a graph signal (b).

# Approximation Results

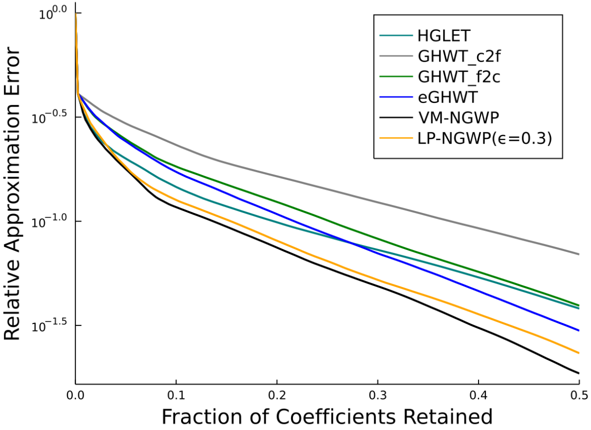
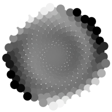
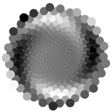



Figure: The relative $\ell^2$ approximation errors by various methods. The best basis algorithm (with $\ell^1$-norm cost) is used for these multiscale basis dictionaries.

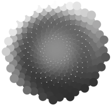
# Top 2-9 VM-NGWP Best Basis Vectors



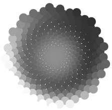
(a) $\boldsymbol{\psi}^{(11)}_{13,0} \equiv \boldsymbol{\phi}_{11}$



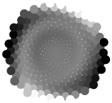
(b) $\boldsymbol{\psi}^{(11)}_{26,0} \equiv \boldsymbol{\phi}_{16}$



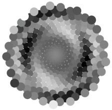
(c) $\boldsymbol{\psi}^{(5)}_{1,2}$



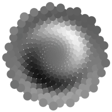
(d) $\boldsymbol{\psi}^{(5)}_{1,0}$



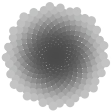
(e) $\boldsymbol{\psi}^{(11)}_{14,0} \equiv \boldsymbol{\phi}_{12}$



(f) $\boldsymbol{\psi}^{(7)}_{21,0}$

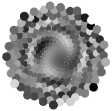


(g) $\boldsymbol{\psi}^{(7)}_{5,2}$

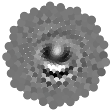


(h) $\boldsymbol{\psi}^{(11)}_{1,0} \equiv \boldsymbol{\phi}_{1}$

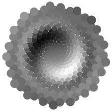
# Top 10-17 VM-NGWP Best Basis Vectors



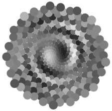
(a) $\boldsymbol{\psi}^{(11)}_{40,0} \equiv \boldsymbol{\phi}_{42}$



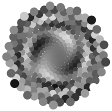
(b) $\boldsymbol{\psi}^{(3)}_{2,18}$



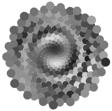
(c) $\boldsymbol{\psi}^{(6)}_{5,1}$



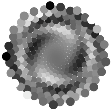
(d) $\boldsymbol{\psi}^{(6)}_{11,1}$



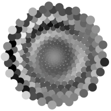
(e) $\boldsymbol{\psi}^{(11)}_{21,0} \equiv \boldsymbol{\phi}_{35}$



(f) $\boldsymbol{\psi}^{(6)}_{15,3}$



(g) $\boldsymbol{\psi}^{(11)}_{30,0} \equiv \boldsymbol{\phi}_{27}$



(h) $\boldsymbol{\psi}^{(6)}_{9,2}$

# Discussion

- The NGWP best bases performed the best.
- This Barbara's eye graph signal is not of piecewise-*constant* nature; rather, it is a *locally smooth* graph signal.
- The NGWP dictionaries containing smooth *and* localized basis vectors made a difference in performance compared to the HGLET best basis, the GHWT best basis and the eGHWT best basis.

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# Summary

- We introduced the DAG distance to measure the *behavioral difference* between the graph Laplacian eigenvectors.
- We constructed a *dual graph* $G^{\star}$ to organize the eigenvectors.
- We used the *classical MDS algorithm* to visualize such arrangements in low dimensional Euclidean space.
- We developed the VM-NGWP dictionary and the LP-NGWP dictionary.
- The VM-NGWP dictionary is a generalization of the classical *Shannon* wavelet packet dictionary to the graph settings.
- The LP-NGWP dictionary is a generalization of the classical *Meyer* wavelet packet dictionary to the graph settings.
- We demonstrated their potentials in graph signal approximations.

# Future Work

- Among all the metrics of graph Laplacian eigenvectors, explore which one is the best to use.
- Find fast algorithms to reduce the computational complexity of constructing the NGWP dictionaries.
- Study which $\epsilon$ is the best for the LP-NGWP dictionary in graph signal approximations.

# References


- https://github.com/UCD4IDS/MultiscaleGraphSignalTransforms.jl contains the code and useful information of this work.
- N. SAITO, *Local Fourier dictionary: a natural tool for data analysis*, in Wavelet Applications in Signal and Image Processing VII, M. A. Unser, A. Aldroubi, and A. F. Laine, eds., vol. 3813, International Society for Optics and Photonics, SPIE, 1999, pp. 610-624.
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- H. LI AND N. SAITO, *Metrics of graph Laplacian eigenvectors*, in Wavelets and Sparsity XVIII, Proc. SPIE 11138, D. Van De Ville, M. Papadakis, and Y.M. Lu, eds., 2019. Paper #111381K.
- A. CLONINGER, H. LI, AND N. SAITO, *Natural graph wavelet packet dictionaries*, J. Fourier Anal. Appl., 27 (2021).

# Thanks for your attention!
# Any questions?