Linear programming I

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Short introduction

- **Linear programming** (LP) or **linear optimization** is a set of optimization algorithms.
- Short history: began in 1947 when George Dantzig devised the simplex method.
- Goal: Optimize **linear** objective functions, subject to (s.t.) a set of **linear** constraints (equality or inequality).
- Widely used in various industries for production planning, investment assignment, transportation problem, traffic control, etc. The LP class is usually offered by Industrial Engineering department.
- Applications in statistics: regularized regression with L_1 penalty (e.g., LASSO), quantile regression, support vector machine, etc.

A simple motivating example

Consider a hypothetical company that manufactures two products A and B.

1. Each item of product A requires 1 hour of labor and 2 units of materials, and yields a profit of 1 dollar.
2. Each item of product B requires 2 hours of labor and 1 unit of materials, and yields a profit of 1 dollar.
3. The company has 100 hours of labor and 100 units of materials available.

Question: what's the company's optimal strategy for production?

The problem can be formulated as following optimization problem. Denote the amount of production for product A and B by x_1 and x_2:

$$
\begin{aligned}
max\ z &= x_1 + x_2, \\
s.t.\ x_1 + 2x_2 &\leq 100 \\
2x_1 + x_2 &\leq 100 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

Graphical representation

The small, 2-variable problem can be represented in a 2D graph.
The isoprofit line (any point on the line generate the same profit) is perpendicular to the gradient of objective function. The problem can be solved by sliding the isoprofit line.

Properties of the optimal solution

Optimal solution might not exist (e.g., the objective function is unbounded in the solution space).

But if the optimal solution exists:

- Interior point: NO.
- Corner point: Yes. The corner points are often referred to as "extreme points".
- Edge point: Yes, only when the edge is parallel to the isoprofit line. In this case, every point on that edge has the same objective function value, including the corner point of that edge.
- **Key result**: If an LP has an optimal solution, it must have a extreme point optimal solution (can be rigorously proved using the **General Representation Theorem**). This greatly reduce our search of optimal solutions: only need to check a finite number of extreme points.

The augmented LP

The inequality constrains can be converted into equality constrains by adding non-negative "slack" variables. For example,

$$x_1 + 2x_2 \leq 100 \Longleftrightarrow x_1 + 2x_2 + x_3 = 100, x_3 \geq 0.$$

Here x_3 is called a "slack" variable.

The original problem can be expressed as the following slack form:

$$\begin{aligned} max\ z &= x_1 + x_2, \\ s.t.\ x_1 + 2x_2 + x_3 &= 100 \\ 2x_1 + x_2 + x_4 &= 100 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

This augmentation is necessary for **simplex** algorithm.

LP in matrix notation

The augmented **standard form** of LP problem, expressed in matrix notation is:

$$max\ z = c\mathbf{x}$$
$$s.t.\ A\mathbf{x} = b$$
$$\mathbf{x} \geq 0$$

It is required that RHS $b \geq 0$. Here $\mathbf{x}$ is augmented, e.g., including the original and slack variables. For our simple example, we have:

$$\mathbf{x} = [x_1, x_2, x_3, x_4]^T$$
$$c = [1, 1, 0, 0]$$
$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$
$$b = [100, 100]^T$$

Finding extreme points

We know the problem has an optimal solution at the extreme point if the optimal solution exists $\Longrightarrow$ we can try all extreme points and find the best one.

Assume there are m constraints an n unknowns (so A is of dimension $m \times n$). In our case $m = 2$ and $n = 4$.

- It is required that $m < n$ (why?)
- Assume $rank(A) = m$, e.g., rows of A are independent, or there's no redundant constraints.
- An extreme point is the intersection of n linearly independent hyperplanes. The constraints $A\mathbf{x} = b$ provide m such hyperplanes. How about the remaining $n - m$ of them?

Solution: first set $n - m$ of the unknowns to zero. Essentially this gets rid of $n - m$ variables and leaves us with m unknowns and m equations, which can be solved.

Terminology: The $n - m$ variables set to zero are called *nonbasic variables* (NBV) and denoted by $\mathbf{x}_N$. The rest m variables are *basic variables* (BV), denoted by $\mathbf{x}_B$.

Let B be the columns of A that are associated with the basic variables, and N be the columns associated with nonbasic variables (B is square and has full rank):

$$A\mathbf{x} = b \Longrightarrow [B, N]\begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{bmatrix} = b \Longrightarrow \mathbf{x}_B = B^{-1}b$$

So the point $\mathbf{x}_N = 0, \mathbf{x}_B = B^{-1}b$ is an extreme point.

Bad news: there are too many of them! With 20 variables and 10 constraints, there are $\binom{20}{10} = 184,756$ extreme points.

Given the LP problem

$$max\ z = c\mathbf{x}, \quad s.t.\ A\mathbf{x} = b, \quad \mathbf{x} \geq 0$$

First characterize the solution at a extreme point:

$$B\mathbf{x}_B + N\mathbf{x}_N = b \Longrightarrow \mathbf{x}_B = B^{-1}b - B^{-1}N\mathbf{x}_N \Longrightarrow \mathbf{x}_B = B^{-1}b - \sum_{j \in R} B^{-1}a_j x_j$$

Here R is the set of nonbasic variables, and a_j's are columns of A.

Substitute this expression to the objective function,

$$\begin{aligned} z = c\mathbf{x} &= c_B\mathbf{x}_B + c_N\mathbf{x}_N \\ &= c_B\left[B^{-1}b - \sum_{j \in R} B^{-1}a_j x_j\right] + \sum_{j \in R} c_j x_j \\ &= c_B B^{-1}b - \sum_{j \in R}(c_B B^{-1}a_j - c_j)x_j \end{aligned}$$

The current extreme point is optimal if $c_B B^{-1}a_j - c_j \geq 0$ for $\forall j$ (why?).

Iterative method:

- No need to enumerate all extreme points.
- Never go to extreme point which yield smaller objective function, e.g., the objective function always increases during iterations.

Consider our simple example with:

$$c = [1, 1, 0, 0], A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}, b = [100, 100]^T$$

First assume we choose x_1 and x_3 as basic variables, then

$$c_B = [1, 0], c_N = [1, 0], B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}, N = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, R = \{2, 4\}$$

Solving $x_B = B^{-1}b = [50, 50]^T$. Are we optimal?

Check $c_B B^{-1} a_j - c_j \equiv w_j$ for $j \in R$.

$$w_2 = [1,0] \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 1 = -0.5$$

$$w_4 = [1,0] \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} - 0 = 0.5$$

- $w_2 < 0$ so we know the solution is not optimal $\Longrightarrow$ increasing x_2 will increase the objective function z.
- Since we will increase x_2, it will no longer be a nonbasic variable (will not be zero). It is referred to as **entering basic variable**.
- But when we increase x_2, it will change the values of other variables. How much can we increase x_2?

Now go back to look at $x_B = B^{-1}b - \sum_{j \in R} B^{-1}a_j x_j$. To simplify the notations, define $y_j = B^{-1}a_j$ for $j \in R$, and $\bar{b} = B^{-1}b$.

$$y_2 = B^{-1}a_2 = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix}$$

$$y_4 = B^{-1}a_4 = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}$$

$$\bar{b} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 100 \\ 100 \end{bmatrix} = \begin{bmatrix} 50 \\ 50 \end{bmatrix}$$

Plug these back to the expessions for x_B:

$$\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 50 \\ 50 \end{bmatrix} - \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix} x_2 - \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix} x_4$$

Holding $x_4 = 0$ and increasing x_2, x_1 or x_3 will eventually become negative.

How much can we increase $x_2 \Longrightarrow$ increasing x_2, which basic variable (x_1 and x_3) will hit zero first?

The basic variable that hits zero first is called the **leaving basic variable**. Given index of entering basic variable k, we find index for the leaving basic variable, denoted by l, as following:

$$l = \operatorname*{argmin}_i \left\{ \frac{\bar{b}_i}{y_{ki}}, 1 \le i \le m, y_{ki} > 0 \right\}$$

For our example, $k = 2$, $\bar{b}_1/y_{21} = 50/0.5 = 100$, $\bar{b}_2/y_{22} = 50/1.5 = 33.3$. So the **second** basic variable is leaving, which is x_3.

Be careful of the indexing here (especially in programming). It is the second of the current basic variable that is leaving, not x_2!

Next iteration, with x_1 and x_2 as basic variables. We will have

$$c_B = [1,1], c_N = [0,0], B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, R = \{3,4\}$$

We will get $x_B = [33.3, 33.3]^T$, and

$$w_3 = [1,1]\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 0 = 0.33$$

$$w_4 = [1,1]\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} - 0 = 0.33$$

Both w's are positive so we are at optimal solution now!

- It only considers extreme points.
- It moves as far as it can until the movement is blocked by another constraint.
- It's greedy: find the edge that optimize the objective function the fastest and move along that edge.

To make it formal, the **steps for Simplex method** are:

1. Randomly choose a set of basic variables (often use the slack variables because it's easy).
2. **Find entering basic variable**. Define the set of nonbasic variable to be R. For each $j \in R$, compute $C_B B^{-1} a_j - c_j \equiv \omega_j$. If all $\omega_j \geq 0$, the current solution is optimal. Otherwise find $k = argmin_{j \in R} \omega_j$, the kth nonbasic variable will be EBV.
3. **Find leaving basic variable**. Obtain $l = argmin_i \left\{ \frac{\bar{b}_i}{y_{ki}}, 1 \leq i \leq m, y_{ki} > 0 \right\}$. The lth current basic variable will be LBV.
4. Iterate steps 2 and 3.

Tableau implementation of the Simplex method (SKIP!!)

The LP problem: $max\ z = c\mathbf{x},\ \ s.t.\ A\mathbf{x} = b,\ \ \mathbf{x} \geq 0$. With BV and NBV and corresponding matrices, we can reorganize them as:

$$
\begin{aligned}
z &= c_B B^{-1} b - (c_B B^{-1} N - c_N)\mathbf{x}_N. \\
\mathbf{x}_B &= B^{-1} b - B^{-1} N \mathbf{x}_N.
\end{aligned}
$$

This is to express the objective function and BVs in terms of NBVs and constants. We do this because:

- Expressing z in terms of NBVs and constants so that we will know whether the current solution is optimal, and how to choose EBV.
- Expressing BVs in terms of NBVs and constants in order to choose LBV.

This can be organized into following tableau form:

	z	x_B	x_N	RHS
z	1	0	$c_B B^{-1} N - c_N$	$c_B B^{-1} b$
x_B	0	I	$B^{-1} N$	$B^{-1} b$

The LP problem is:

$$\begin{aligned} max \quad & 50x_1 + 30x_2 + 40x_3 \\ s.t. \quad & 2x_1 + 3x_2 + 5x_3 \leq 100 \\ & 5x_1 + 2x_2 + 4x_3 \leq 80 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Step 1: Add slack variables (x_4, x_5) to get equality constraints.
Step 2: Choose initial basic variables - the easiest is to choose the slack variables as BV, then put into tableau (verify):

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	-50	-30	-40	0	0	0
x_4	0	2	3	5	1	0	100
x_5	0	5	2	4	0	1	80

Choose x_1 to be EBV (why?).

Step 3: Now to find LBV, need to compute ratios of the RHS column to the EBV column, get $x_4 : 100/2$, $x_5 : 80/5$. We need to take the BV with minimum of these ratios, e.g., x_5, as the LBV.

Step 4: Update the tableau: replace x_5 by x_1 in BV. We must update the table so that x_1 has coefficient of 1 in row 2 and coefficient 0 in every other row. This step is call **pivoting**.

To do so, the new row 3 = old row 3 divided by 5:

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1						
x_4	0						
x_1	0	1	2/5	4/5	0	1/5	16

To eliminate the coefficient of x_1 in row 1 and row 2, we perform row operations:

- new row 1 $=$ old row 1 $+ 50 \times$ new row 3.
- new row 2 $=$ old row 2 – $2 \times$ new row 3.

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	-10	0	0	10	800
x_4	0	0	11/5	17/5	1	-2/5	68
x_1	0	1	2/5	4/5	0	1/5	16

Step 5: Now x_2 will be EBV (because it has negative coefficient). By calculating ratios, x_4 is LBV (verify!). Another step of pivoting gives:

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	0	$15\frac{5}{11}$	$4\frac{6}{11}$	$8\frac{2}{11}$	$1109\frac{1}{11}$
x_2	0	0	1	17/11	5/11	-2/11	$30\frac{10}{11}$
x_1	0	1	0	2/11	-2/11	3/11	$3\frac{7}{11}$

Now all coefficients for NBV are positive, meaning we have reached the optimal solution.

The RHS column gives the optimal objective function value, as well as the values of BVs at the optimal solution.

Handling $\geq$ constraints

So far we assume all constraints are $\leq$ with non-negative RHS. What is some constraints are $\geq$? For example, the following LP problem:

$$\begin{aligned} max \quad & -x_1 - 3x_2 \\ s.t. \quad & x_1 + 2x_2 \geq 1 \\ & 5x_1 + x_2 \leq 10 \\ & x_1, x_2 \geq 0 \end{aligned}$$

For the $\geq$ constraint we can subtract a slack variable (“surplus variable”):

$$\begin{aligned} max \quad & -x_1 - 3x_2 \\ s.t. \quad & x_1 + 2x_2 - x_3 = 1 \\ & 5x_1 + x_2 + x_4 = 10 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

The problem now is that we cannot use the slack variables as initial basic variable anymore because they are not feasible (violate the non-negative constraint).

Use artificial variables

When it is difficult to find a basic feasible solution for the original problem, we create an artificial problem that we know is feasible. For the example, we add x_5 and get:

$$\begin{aligned} max \quad & -x_1 - 3x_2 \\ s.t. \quad & x_1 + 2x_2 - x_3 + x_5 = 1 \\ & 5x_1 + x_2 + x_4 = 10 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

x_5 is called an “artificial variable” because it’s neither a decision nor a slack variable. It is now easy to obtain initial basic variables (x_5 and x_4).

Problem: A feasible solution to this problem might not be feasible to the original problem. For example, we could have x_1, x_2, x_3 equals 0 and $x_5 = 1$. This violate the original constraint $x_1 + 2x_2 \geq 1$. This is caused by non-zero artificial variable in the results!

Solution: Force the artificial variable (x_5) to be 0. There are two methods.

Two phase method

The problem can be solved in two steps. First we create another LP problem of minimizing x_5 subject to the same constraints. This is called the **Phase I** problem.

$$
\begin{aligned}
max \quad & x_0 = -x_5 \\
s.t. \quad & x_1 + 2x_2 - x_3 + x_5 = 1 \\
& 5x_1 + x_2 + x_4 = 10 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}
$$

This can be solved using the Simplex method.

- If the optimal objective function value is 0, we have a feasible solution.
- The optimal point of the Phase I problem provides a set of initial basis for the original problem. We can then eliminate the artificial variables and use the optimal point of the Phase I problem as initial value to solve the original problem.

Put into tableau, get

	x_0	x_1	x_2	x_3	x_4	x_5	RHS
x_0	1	0	0	0	0	1	0
x_5	0	1	2	-1	0	1	1
x_4	0	5	1	0	1	0	10

First eliminate the coefficient for artificial variables: new Row 1=Old row 1 - Old row 2):

	x_0	x_1	x_2	x_3	x_4	x_5	RHS
x_0	1	-1	-2	1	0	0	-1
x_5	0	1	2	-1	0	1	1
x_4	0	5	1	0	1	0	10

Pivot out x_5, and pivot in x_2, get:

	x_0	x_1	x_2	x_3	x_4	x_5	RHS
x_0	1	0	0	0	0	1	0
x_2	0	1/2	1	-1/2	0	1/2	1/2
x_4	0	9/2	0	1/2	1	-1/2	19/2

Now x_0 is optimized at $x_5 = 0$. We can eliminate x_5 and use $x_2 = 1/2$, $x_4 = 19/2$ as initial solution for the original problem (this is the **Phase II** problem).

The tableau for the initial problem is:

	z	x_1	x_2	x_3	x_4	RHS
z	1	1	3	0	0	0
x_2	0	1/2	1	-1/2	0	1/2
x_4	0	9/2	0	1/2	1	19/2

This is not a valid tableau!! (why?)

Need to adjust and get:

	z	x_1	x_2	x_3	x_4	RHS
z	1	-1/2	0	3/2	0	-3/2
x_2	0	1/2	1	-1/2	0	1/2
x_4	0	9/2	0	1/2	1	19/2

Continue to finish!

Another technique is to modify the objective function to include artificial variable, but with a big penalty:

$$\begin{aligned}
max \quad & -x_1 - 3x_2 - Mx_5 \\
s.t. \quad & x_1 + 2x_2 - x_3 + x_5 = 1 \\
& 5x_1 + x_2 + x_4 = 10 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}$$

M is assumed to be huge that it dominates the objective function.

- Using usual Simplex, all artificial variables will be first pivot out and become NBV because the using of M.
- Once the artificial variables are out, we obtain a set of initial basis for the original problem. We can then eliminate the artificial variables and solve the original problem the usual way.

Step for solving a general LP

To solve a general LP with $\leq$, $\geq$ or $=$ constraints:

1. Make all right hand size ≥ 0.
2. Add slack variables for $\leq$ constraints, and surplus variable for $\geq$ constraints.
3. Add artificial variables for $\geq$ or $=$ constraints.
4. Use slack and artificial variables as initial basic variables.
5. Set up Phase I problem or use big-M method to pivot out all artificial variables (e.g., make all artificial variables nonbasic variables).
6. Use the optimal solution from Phase I or big-M as initial solution, eliminate artificial variables from the problem and finish the original problem.

LP solver software

There are a large number of LP solver software both commercial or freely available. See Wikipedia page of "linear programming" for a list.

- In `R`, Simplex method is implemented as `simplex` function in `boot` package.
- In `Matlab`, the `optimiztion` toolbox contains `linprog` function.
- IBM ILOG `CPLEX` is commercial optimization package written in C. It is very powerful and highly efficient for solving large scale LP problems.

```
simplex                 package:boot                R Documentation

Simplex Method for Linear Programming Problems

Description:

  This function will optimize the linear function a%*%x subject to
  the constraints A1%*%x <= b1, A2%*%x >= b2, A3%*%x = b3 and
  x >= 0.  Either maximization or minimization is possible but the
  default is minimization.

Usage:

 simplex(a, A1 = NULL, b1 = NULL, A2 = NULL, b2 = NULL, A3 = NULL,
         b3 = NULL, maxi = FALSE, n.iter = n + 2 * m, eps = 1e-10)
```

Simplex Example in R

```
> library(boot)
> a = c(50, 30, 40)
> A1 = matrix(c(2,3,5,5,2,4), nrow=2, byrow=TRUE)
> b1 = c(100, 80)
> simplex(-a, A1, b1)

Linear Programming Results

Call : simplex(a = -a, A1 = A1, b1 = b1)

Minimization Problem with Objective Function Coefficients
 x1  x2  x3
-50 -30 -40

Optimal solution has the following values
       x1         x2         x3
 3.636364 30.909091  0.000000
The optimal value of the objective  function is -1109.09090909091.
```

LP problem for resource allocation

For a typical maximizing LP problem like the following (with 3 variables and 2 constraints):

$$\begin{aligned} max \quad & c_1x_1 + c_2x_2 + c_3x_3 \\ s.t. \quad & a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \le b_1 \\ & a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \le b_2 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

Economical interpretation of the problem:

- x_j: unit of production of product $j, j = 1, 2, 3$. Unknown to be obtained.
- c_j: profit per unit of product $j, j = 1, 2, 3$.
- a_{ij}: unit of material i ($i = 1, 2$) required to produce 1 unit of product j.
- b_i: unit of available material $i, i = 1, 2$.

The goal is to maximize the profit, subject to the material constraints.

Now assume a buyer consider to buy our entire inventory of materials but not sure how to price the materials, but s/he knows that we will only do the business if selling the materials yields higher return than producing the product.

Buyer's business strategy: producing one unit **less** of product j will save us:

- a_{1j} unit of material 1, and a_{2j} unit of material 2.

Buyer want to compute the unit prices of materials to **minimize** his/her cost, subject to the constraints that we will do business (that we will not make less money). Assume the unit price for the materials are y_1 and y_2, the buyer will face the following optimization problem (called **Resource valuation problem**):

$$
\begin{aligned}
min \quad & b_1 y_1 + b_2 y_2 \\
s.t. \quad & a_{11} y_1 + a_{21} y_2 \geq c_1 \\
& a_{12} y_1 + a_{22} y_2 \geq c_2 \\
& a_{13} y_1 + a_{23} y_2 \geq c_3 \\
& y_1, y_2 \geq 0
\end{aligned}
$$

The buyer's LP problem is called the "**dual**" problem of the original problem, which is called the "**primal** problem".

In matrix notation, if the **primal** LP problem is:

$$\begin{aligned} max \quad & cx \\ s.t. \quad & Ax \le b, x \ge 0 \end{aligned}$$

The corresponding dual problem is:

$$\begin{aligned} min \quad & b^T y \\ s.t. \quad & A^T y \ge c^T, y \ge 0 \end{aligned}$$

Or to express in the canonical form (a maximization problem with $\le$ constraints):

$$\begin{aligned} max \quad & -b^T y \\ s.t. \quad & -A^T y \le -c^T, y \ge 0 \end{aligned}$$

Dual is the "negative transpose" of the primal. It's easy to see, the dual of the dual problem is the primal problem.

Duality in non-canonical form

What if the primal problem doesn't fit into the canonical form (e.g., with $\geq$ or $=$ constraints, unrestricted variable, etc.)? The general rules of converting are:

- The variable types of the dual problem is determined by the constraints types of the primal:

Primal (max) constraints	Dual (min) variable
$\leq$	≥ 0
$\geq$	≤ 0
$=$	unrestricted

- The constraints types of the dual problem is determined by the variable types of the primal:

Primal (max) variable	Dual (min) constraints
≥ 0	$\geq$
≤ 0	$\leq$
unrestricted	$=$

If the primal problem is:

$$
\begin{aligned}
\mathit{max} \quad & 20x_1 + 10x_2 + 50x_3 \\
\mathit{s.t.} \quad & 3x_1 + x_2 + 9x_3 \leq 10 \\
& 7x_1 + 2x_2 + 3x_3 = 8 \\
& 6x_1 + x_2 + 10x_3 \geq 1 \\
& x_1 \geq 0, x_2 \ \mathit{unrestricted}, x_3 \leq 0
\end{aligned}
$$

The dual problem is:

$$
\begin{aligned}
\mathit{min} \quad & 10y_1 + 8y_2 + y_3 \\
\mathit{s.t.} \quad & 3y_1 + 7y_2 + 6y_3 \geq 20 \\
& y_1 + 2y_2 + y_3 = 10 \\
& 9y_1 + 3y_2 + 10y_3 \leq 50 \\
& y_1 \geq 0, y_2 \ \mathit{unrestricted}, y_3 \leq 0
\end{aligned}
$$

Weak duality Theorem: *the objective function value of the primal problem (max) at any feasible solution is always less than or equal to the objective function value of the dual problem (min) at any feasible solution.*

So if $(x_1, \ldots, x_n)$ is a feasible solution for the primal problem, and $(y_1, \ldots, y_m)$ is a feasible solution for the dual problem, then $\sum_j c_j x_j \le \sum_i b_i y_i$.

Proof:

$$\begin{aligned}
\sum_j c_j x_j &\le \sum_j \left(\sum_i y_i a_{ij} \right) x_j \\
&= \sum_{ij} y_i a_{ij} x_j \\
&= \sum_i \left(\sum_j a_{ij} x_j \right) y_i \\
&\le \sum_i b_i y_i.
\end{aligned}$$

In other words, we will only do business if selling the material makes us more money.

Strong duality

- So we know now that at feasible solutions for both, the objective function of the dual problem is always greater or equal.
- A question is that if there is a difference between the largest primal value and the smallest dual value? Such difference is called the "**Duality gap**".
- The answer is provided by the following theorem.

Strong Duality Theorem: If the primal problem has an optimal solution, then the dual also has an optimal solution and there is no duality gap.

Economic interpretation: at optimality, the resource allocation and resource valuation problems give the same objective function values. In other words, in the ideal economic situation, using the materials or selling the materials give the same profit.

Sketch of the proof (SKIP!)

Recall for the LP problem in standard form: $max\ z = cx, s.t.\ Ax \leq b, x \geq 0$. Let x^* be an optimal solution. Let B be the columns of A associated with the BV, and R is the set of columns associated with the NBV. We have $c_B B^{-1} a_j - c_j \geq 0$ for $\forall j \in R$.

- Define $y^* \equiv (c_B B^{-1})^T$, we will have $y^{*T} A \geq c$.
- Furthermore, $y^{*T} \geq 0$ (why?).

Thus y^* is a feasible solution for the dual problem.

How about optimality? We know $y^{*T} b \geq y^{*T} Ax \geq cx$. Want to prove $y^{*T} b = cx^*$. This requires two steps:

1. $y^{*T} b = y^{*T} Ax^* \Leftrightarrow y^{*T}(b - Ax^*) = 0$
2. $y^* Ax^* = cx^* \Leftrightarrow (y^{*T} A - c)x^* = 0$

This can be seen from the optimal tableau:

	z	x_B	x_N	RHS
z	1	0	$c_B B^{-1} N - c_N$	$c_B B^{-1} b$
x_B	0	I	$B^{-1} N$	$B^{-1} b$

An optimal solution $x^* = [x_B^*, 0]^T$, $A = [B, N]$, $c = [c_B, c_N]$:

1. becomes $c_B B^{-1} b - c_B B^{-1}[B, N][x_B^*, 0]^T = c_B B^{-1} b - c_B x_B^* = 0$
2. becomes $(c_B B^{-1}[B, N] - [c_B, c_N])[x_B^*, 0]^T = [0, c_B B^{-1} N - c_N][x_B^*, 0]^T = 0$

□

Review

- Linear programming (LP) is an constrained optimization method, where both the objective function and constraints must be linear.
- If an LP has an optimal solution, it must have an extreme point optimal solution. This great reduces the search space.
- Simplex is a method to search for an optimal solution. It goes through the extreme points and guarantees the increase of objective function at every step. It has better performance than exhaustive search the extreme points.
- Primal–dual property of the LP.
- Weak and strong duality.