
Lecture 1: Random number generation, permutation test, and the bootstrap

August 17, 2020

- Statistical simulation (Monte Carlo) is an important part of statistical method research.
- The statistical theories/methods are all based on assumptions. So most theorems state something like “if the data follow these models/assumptions, then ...”.
- The theories can hardly be verified in real world data because (1) the real data never satisfy the assumption; and (2) the underlying truth is unknown (no “gold standard”).
- In simulation, data are “created” in a well controlled environment (model assumptions) and all truth are known. So the claim in the theorem can be verified.

- Random number generator is the basis of statistical simulation. It serves to generate random numbers from predefined statistical distributions.
- Traditional methods (flip a coin or dice) work, but can't scale up.
- Computational methods are available to generate “pseudorandom” numbers.

The random number generation often starts from generating $\text{uniform}(0,1)$. The most common method: “**Linear congruential generator**”:

$$X_{n+1} = (aX_n + c) \bmod m$$

Here, a , c , and m are predefined numbers:

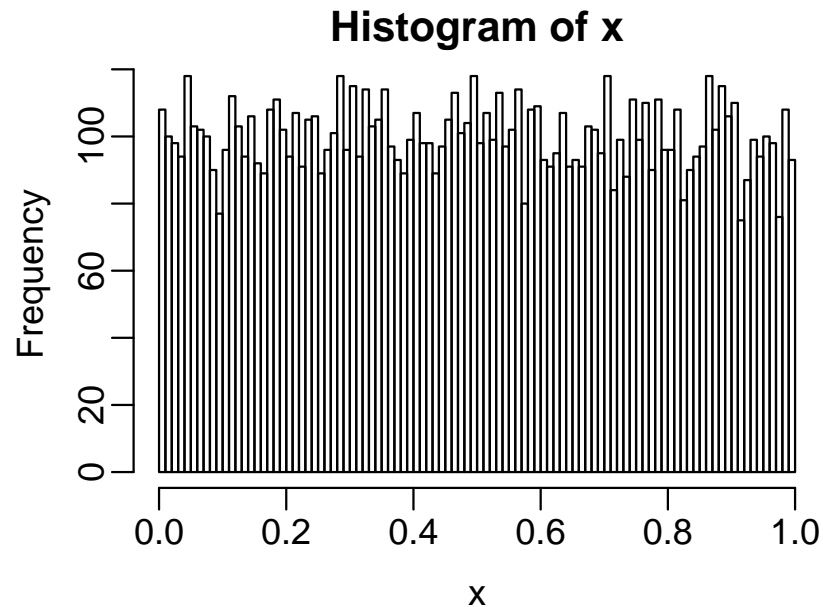
- X_0 : random number “seed”.
- a : multiplier, 1103515245 in `glibc`.
- c : increment, 12345 in `glibc`.
- m : modulus, for example, 2^{32} or 2^{64} .

$U_n = X_n/m$ is distributed as $\text{Uniform}(0,1)$.

Linear congruential generator

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```
a = 1103515245; c = 12345; m = 2^32
n = 10000
x = numeric(n)
x[1] = 1
for( i in 2:n) {
    x[i] = (a*x[i-1] + c) %% m
}
x = x/m
hist(x, 100)
```



A few remarks about Linear congruential generator:

- The numbers generated will be exactly the same using the same seed.
- Want cycle of generator (number of steps before it begins repeating) to be large.
- Don't generate more than $m/1000$ numbers.

RNG in R:

- `set.seed` is the function to specify random seed.
- Read the help for `.Random.seed` for more description about random number generation in R.
- `runif` is used to generate $\text{uniform}(0,1)$ r.v.
- My recommendation: always set and save random number seed during simulation, so that the simulation results can be reproduced.

When the distribution **has a cumulative distribution function (cdf)** F , the r.v. can be obtained by inverting the cdf (“inversion sampling”). This is based on the theory that the cdf is distributed as Uniform (0,1):

Algorithm: Assume F is the cdf of distribution \mathcal{D} . Given $u \sim \text{unif}(0, 1)$, find a unique real number x such that $F(x) = u$. Then $x \sim \mathcal{D}$.

Example: exponential distribution. When $x \sim \text{exp}(\lambda)$, the cdf is: $F(x) = 1 - \exp(-\lambda x)$. The inversion of cdf is: $F^{-1}(u) = -\log(1 - u)/\lambda$. Then to generate exponential r.v., do:

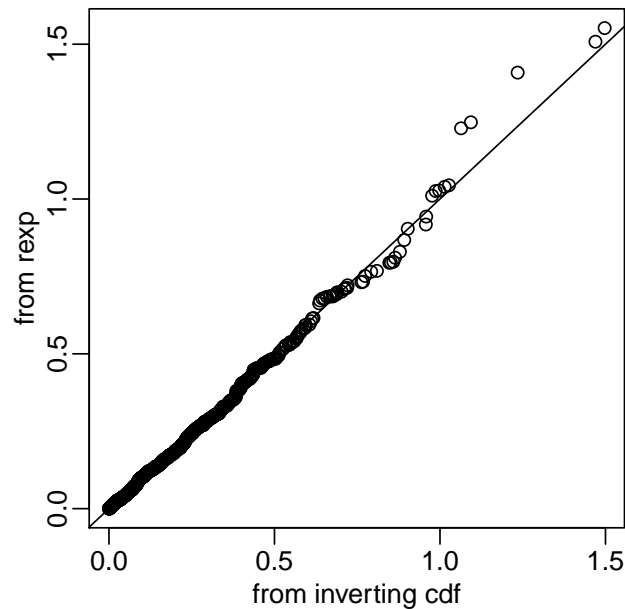
- Generate $\text{uniform}(0,1)$, r.v., denote by u .
- Calculate $x = -\log(1 - u)/\lambda$.

When the inverted cdf is unavailable, one has to rely on other methods such as **acceptance-rejection**. This will be covered later in MCMC classes.

Example: simulate exponential r.v.

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```
lambda=5  
u = runif(1000)  
x = -log(1-u) / lambda  
## generate from R's function  
x2 = rexp(1000, lambda)  
## compare  
qqplot(x, x2, xlab="from inverting cdf", ylab="from rexp")  
abline(0,1)
```



Difficulty: Generating random vectors is more difficult, because we need to consider the correlation structure.

Solution: Generate **independent** r.v.'s, then apply some kind of transformation.

Example: simulate from multivariate normal distribution $MVN(\mu, \Sigma)$

Let \mathbf{Z} be a p -vector of independent $N(0, 1)$ r.v.'s, Given $p \times p$ matrix \mathbf{D} ,

$$\text{var}(\mathbf{D}^T \mathbf{Z}) = \mathbf{D}^T \text{var}(\mathbf{Z}) \mathbf{D} = \mathbf{D}^T \mathbf{D}$$

The simulation steps are:

1. Perform **Cholesky decomposition** on Σ to find \mathbf{D} : $\Sigma = \mathbf{D}^T \mathbf{D}$.
2. Simulate $\mathbf{Z} = (z_1, \dots, z_p)' \sim \text{iid } N(0, 1)$
3. Apply transformation $\mathbf{X} = \mathbf{D}^T \mathbf{Z} + \mu$.

R function `mvrnorm` available in MASS package.

Generating multivariate random vector from other distributions are usually harder.

Recommended book: ***Multivariate Statistical Simulation: A Guide to Selecting and Generating Continuous Multivariate Distributions.***


```
## specify mean and variance/covariance matrix
mu = c(0,1)
Sigma = matrix(c(1.7, 0.5, 0.5, 0.8), nrow=2)

## Cholesky decomposition
D = chol(Sigma)

## generate 500 Z's.
Z = matrix(rnorm(1000), nrow=2)
## transform
X = t(D) %*% Z + mu

## check the means X
> rowMeans(X)
[1] -0.08976896  0.95802769

## check the variance/covariance matrix of X
> cov(t(X))
      [,1]      [,2]
[1,] 1.7392114 0.5609027
[2,] 0.5609027 0.7380548
```

- In statistical inference, it is important to know the distribution of some statistics under null hypothesis (H_0), so that quantities like p-values can be derived.
- The null distribution is available theoretically in some cases. For example, assume $X_i \sim N(\mu, \sigma^2), i = 1, \dots, n$. Under $H_0 : \mu = 0$, we have $\bar{X} \sim N(0, \sigma^2/n)$. Then H_0 can be tested by comparing \bar{X} with $N(0, \sigma^2/n)$.
- When null distribution cannot be obtained, it is useful to use **permutation test** to “create” a null distribution from data.

The basic procedure of permutation test for H_0

- Permute data under H_0 for a number of times. Each time recompute the test statistics. The test statistics obtained from the permuted data form the null distribution.
- Compare the observed test statistics with the null distribution to obtain statistical significance.

Assume there are two sets of independent normal r.v.'s with the same known variance and different means: $X_i \sim N(\mu_1, \sigma^2)$, $Y_i \sim N(\mu_2, \sigma^2)$. We wish to test $H_0 : \mu_1 = \mu_2$.

Define test statistics: $t = \bar{X} - \bar{Y}$. We know under null, we have $t \sim N(0, 2\sigma^2/n)$ (assuming same sample size n in both groups). Using permutation test, we do:

1. Pool X and Y together, denote the pooled vector by Z .
2. Randomly shuffle Z . For each shuffling, take the first n items as X (denote as X^*) and the next n items as Y (denote as Y^*).
3. Compute $t^* = \bar{X}^* - \bar{Y}^*$.
4. Repeat steps 2 and 3 for a number of times. The result t^* 's form the null distribution of t .
5. To compute p-values, calculate $Pr(|t^*| > |t|)$.

NOTE: the random shuffling is based on H_0 , that X and Y are iid distributed.

```
> x=rnorm(100, 0, 1)
> y=rnorm(100, 0.5, 1)
> t.test(x,y)
```

Welch Two Sample t-test

data: x and y

t = -1.9751, df = 197.962, p-value = 0.04965

```
> nsims=50000
> t.obs = mean(x) - mean(y)
> t.perm = rep(0, nsims)
> for(i in 1:nsims) {
+   tmp = sample(c(x,y))
+   t.perm[i] = mean(tmp[1:100]) - mean(tmp[101:200])
+ }
> mean(abs(t.obs) < abs(t.perm))
[1] 0.04814
```

- Under linear regression setting (without intercept) $y_i = \beta x_i + \epsilon_i$. We want to test the coefficient: $H_0 : \beta = 0$.
- Observed data are (x_i, y_i) pairs.
- Use ordinary least square estimator for β , denote as $\hat{\beta}(\mathbf{x}, \mathbf{y})$.

The permutation test steps are:

1. Keep y_i unchanged, permute (change the orders of) x_i to obtain a vector, denoted as x_i^* .
2. Obtain estimate under the permuted data: $\hat{\beta}^*(\mathbf{x}^*, \mathbf{y})$
3. Repeat steps 1 and 2. $\hat{\beta}^*$ form the null distribution for $\hat{\beta}$.
4. P-value = $Pr(|\hat{\beta}^*| > |\hat{\beta}|)$.

NOTE: the random shuffling of x_i is based on the H_0 , that is there is no association between x and y .

```
> x = rnorm(100); y = 0.2 * x + rnorm(100)
> summary(lm(y~x-1))
```

Coefficients:

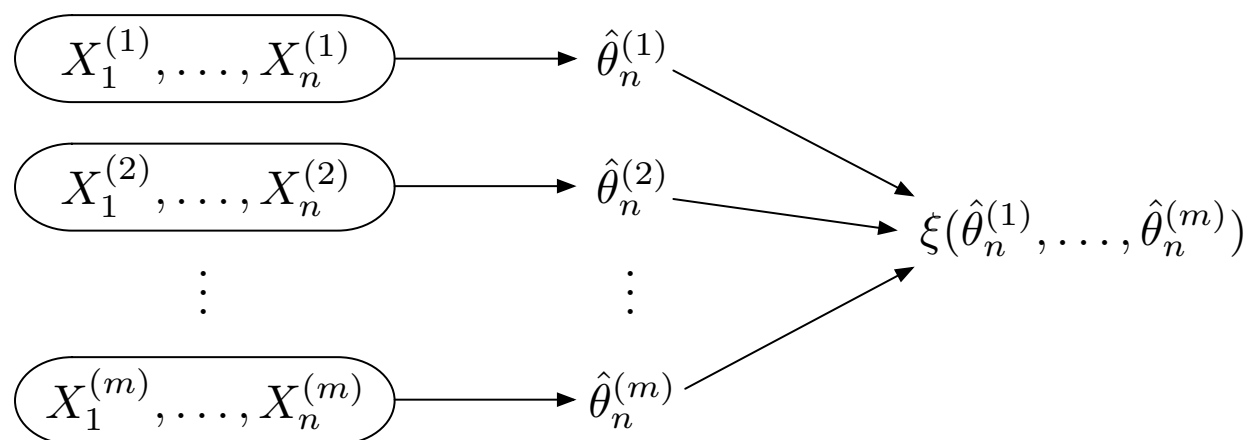
	Estimate	Std. Error	t value	Pr(> t)
x	0.1502	0.1050	1.431	0.156

```
> nsims=5000
> beta.obs = coef(lm(y~x-1))
> beta.perm = rep(0, nsims)
> for(i in 1:nsims) {
+   xstar = sample(x)
+   beta.perm[i] = coef(lm(y~xstar-1))
+ }
> mean(abs(beta.obs) < abs(beta.perm))
[1] 0.157
```

- “Bootstrap” is a simple procedure to estimate the sampling distribution (such as mean, variance, confidence interval, etc.) of some statistics.
- Developed by Brad Efron (see Efron (1979) AOS), extending the “jackknife” algorithm.
- The basic idea is to resample the observed data **with replacement** and create a distribution of the statistics.
- Show good performances compared with jackknife.
- Computationally intensive, but algorithmically easy.

- Observe data $\mathbf{x} = \{x_1, \dots, x_n\}$.
- Parameter of interest is θ , for example, $\theta = E[X]$.
- Let $\hat{\theta}(\mathbf{x})$ be an estimator for θ (such as the MLE). Note $\hat{\theta}$ is a random variable.
- We want to obtain some quantity from $\hat{\theta}$, denoted as $\xi(\hat{\theta})$, for example, the distributional properties of $\hat{\theta}$: its mean, variance, quantiles, etc.

Ideally, we would need to observe a number of independent datasets, compute $\hat{\theta}$ from each of them, and then compute the $\xi(\hat{\theta})$.



Assume $x_i \sim \text{iid } f(\theta)$, where f is known.

The **parametric bootstrap** procedure involves repeating following steps for N times. At the k^{th} time, do:

1. Simulate \mathbf{x}_i^* iid from $f(\theta)$.
2. Compute $\hat{\theta}_i(\mathbf{x}_i^*)$.

Then ξ can be calculated from $\hat{\theta}_i(\mathbf{x}_i^*)$.

Problem setup is the same as in parametric bootstrap, except that the distribution f is unknown. In this case, since \mathbf{x} cannot be generated from a known parametric distribution, they will be drawn from the observed data.

The **non-parametric bootstrap** procedure involves repeating following steps for N times. Assume the observed data has n data points. At the k^{th} time, do:

1. Draw \mathbf{x}_i^* from the observed data \mathbf{x} . Note that \mathbf{x}_i^* must have the same length as \mathbf{x} , and the drawing is sampling **with replacement**.
2. Compute $\hat{\theta}_i(\mathbf{x}_i^*)$.

Then the ξ can be calculated from $\hat{\theta}_i(\mathbf{x}_i^*)$.

So the only difference between parametric and non-parametric bootstrap is the way to generate data:

- In parametric bootstrap: simulate from parametric distribution.
- In non-parametric bootstrap: sample with replacement from observed data.

Problem setup:

- Under linear regression setting (again we omit the intercept to simplify the problem): $y_i = \beta x_i + \epsilon_i$.
- We wish to study the property of OLS estimator, denoted by $\hat{\beta}(\mathbf{x}, \mathbf{y})$.

Parametric bootstrap is based on assumption that $\epsilon_i \sim N(0, \sigma^2)$. Steps are:

1. Obtain $\hat{\beta}(\mathbf{x}, \mathbf{y})$ from observed data.
2. Sample $\epsilon_i^* \sim N(0, \sigma^2)$.
3. Create new \mathbf{y} : $y_i^* = \hat{\beta} x_i + \epsilon_i^*$.
4. Estimate the coefficient based on new data: $\hat{\beta}^*(\mathbf{x}, \mathbf{y}^*)$

Repeat steps 2–4 for many times, then the properties of OLS estimator (such as mean/variance) can be estimated from $\hat{\beta}^*(\mathbf{x}^*, \mathbf{y})$.

Non-parametric bootstrap doesn't require the distributional assumption on ϵ_i . The residuals are resampled from the observed values.

1. Obtain $\hat{\beta}(\mathbf{x}, \mathbf{y})$ from observed data.
2. Compute the observed residuals: $\hat{\epsilon}_i = y_i - \hat{\beta}x_i$.
3. Sample ϵ_i^* by drawing from $\{\hat{\epsilon}_i\}$ with replacement.
4. Create new \mathbf{y} : $y_i^* = \hat{\beta}x_i + \epsilon_i^*$.
5. Estimate the coefficient based on new data: $\hat{\beta}^*(\mathbf{x}, \mathbf{y}^*)$

We will estimate the 95% confidence interval for regression coefficient.

Generate data and compute theoretical value:

```
> x = rnorm(100)
> y = 0.5 * x + rnorm(100)
> fit = lm(y~x-1)
> confint(fit)
      2.5 %      97.5 %
x 0.4002036 0.7638744
```

Parametric bootstrap - sample residual from normal distribution:

```
> nsims = 1000
> beta.obs = coef(fit)
> beta.boot = rep(0, nsims)
> for(i in 1:nsims) {
+   eps.star = rnorm(100)
+   y.star = beta.obs * x + eps.star
+   beta.boot[i] = coef(lm(y.star~x-1))
+ }
> quantile(beta.boot, c(0.025, 0.975))
      2.5%      97.5%
0.4098746 0.7595921
```

Non-parametric bootstrap - sample residual from observed values:

```
> eps.obs = y - beta.obs*x
> for(i in 1:nsims) {
+   eps.star = sample(eps.obs, replace=TRUE)
+   y.star = beta.obs *x + eps.star
+   beta.boot[i] = coef(lm(y.star~x-1))
+ }
> quantile(beta.boot, c(0.025, 0.975))
      2.5%      97.5%
0.4011628 0.7690787
```

In big data set, bootstrap poses significant computational challenge, since the bootstrapped data must have the same length as the original data.

The “ b out of n bootstrap”

Bickel et al. (1997) *Resampling fewer than n observations: Gains, losses, and remedies for losses*, **Statistica Sinica**:

1. Repeatedly subsample b data points **with replacement** from the original data (of size n), and then compute $\hat{\theta}_b$ from the subsample.
2. Compute ξ from $\hat{\theta}_b$'s.
3. Analytically correct the results using prior knowledge of the convergence rate of $\hat{\theta}_b$.

“**Bag of Little Bootstrap**” (BLB) approach, from Kleiner et al. (2014) *A scalable bootstrap for massive data*. **JRSSB**:

1. Subsample s subsets from the original data (of size n), each with size b .
2. For each subsample, do:
 - (a) Repeatedly sample n points with replacement from the subsample (up-sampling), and compute $\hat{\theta}_n^*$ on each resample.
 - (b) Compute an estimate of ξ based on $\hat{\theta}_n^*$'s, denote as ξ_s .
3. Average ξ_s 's as the final estimate of ξ .

Advantages

- More automatic than the “ b out of n bootstrap” approach.
- Since $b \ll n$, the size- n subsamples are highly repetitive. For example each resample can be represented as a vector of counts from an n -trial uniform multinomial distribution over b objects. This leads to less memory usage and faster computing.
- Highly parallelizable.

- Random number generation:
 - Linear congruential generator for generating Uniform(0,1) r.v.;
 - Inverting cdf to generate r.v. from other distributions;
 - simulate random vectors from MVN.
- Permutation test. The key is to shuffle data under null hypothesis, then recompute test statistics and form the null distribution.
- Bootstrap algorithm. Include parametric (draw from parametric distribution) or non-parametric (draw from observed data with replacement).
- Smart approaches for big data bootstrap.
- After class: review slides, play with the R codes.