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# MM Algorithm

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The MM algorithm is not an algorithm, but a **strategy** for constructing optimization algorithms.

An MM algorithm operates by creating a **surrogate function** that minorizes or majorizes the objective function. When the surrogate function is optimized, the objective function is driven uphill or downhill as needed.

In minimization MM stands for **Majorize–Minimize**, and in maximization MM stands for **Minorize–Maximize**.

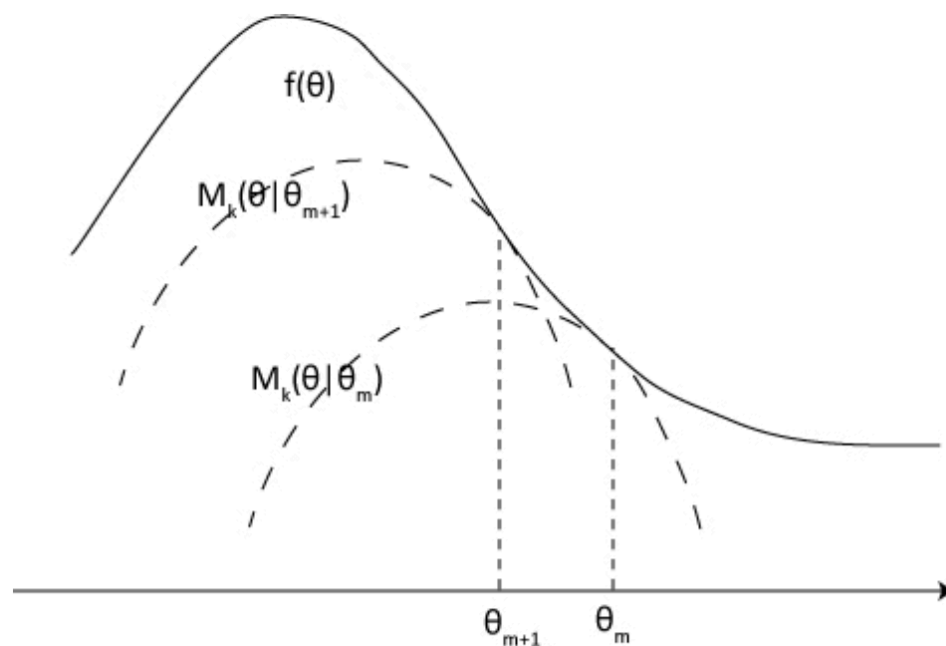
The **EM algorithm** can be thought as a special case of MM.

- We first focus on the **minimization** problem, in which MM = Majorize–Minimize.
- A function  $g(\theta|\theta^{(k)})$  is said to **majorize** the function  $f(\theta)$  at  $\theta^{(k)}$  provided

$$\begin{aligned} f(\theta) &\leq g(\theta|\theta^{(k)}) \text{ for all } \theta \\ f(\theta^{(k)}) &= g(\theta^{(k)}|\theta^{(k)}) \end{aligned}$$

- We choose a majorizing function  $g(\theta|\theta^{(k)})$  and **minimize** it (rather than minimizing  $f(\theta)$ ). Denote  $\theta^{(k+1)} = \arg \min_{\theta} g(\theta|\theta^{(k)})$ . Iterate until  $\theta^{(k)}$  converges.
- **Descent property:**  $f(\theta^{(k+1)}) \leq g(\theta^{(k+1)}|\theta^{(k)}) \leq g(\theta^{(k)}|\theta^{(k)}) = f(\theta^{(k)})$ .

- In a **maximization** problem, MM = Minorize–Maximize.
- To maximize  $f(\theta)$ , we **minorize** it by a surrogate function  $g(\theta|\theta^{(k)})$  and maximize  $g(\theta|\theta^{(k)})$  to produce the next iterate  $\theta^{(k+1)}$ .
- A function  $g(\theta|\theta^{(k)})$  is said to minorize the function  $f(\theta)$  at  $\theta^{(k)}$  provided that  $-g(\theta|\theta^{(k)})$  majorizes  $-f(\theta)$ .



One of the key criteria in judging majorizing or minorizing functions is their **ease of optimization**.

Successful MM algorithms in high-dimensional parameter spaces often rely on surrogate functions in which the individual parameter components are **separated**, i.e., for  $\theta = (\theta_1, \dots, \theta_p)$ ,

$$g(\theta \mid \theta^{(k)}) = \sum_{j=1}^p q_j(\theta_j),$$

where  $q_j(\cdot)$  are univariate functions.

Because the  $p$  univariate functions may be **optimized one by one**, this makes the surrogate function easier to optimize at each iteration.

- **Numerical stability:** warranted by the descent property
- **Simplicity:** substitute a simple optimization problem for a difficult optimization problem.
  - It can turn a non-differentiable problem into a smooth problem (Example 2).
  - It can separate the parameters of a problem (Example 3).
  - It can linearize an optimization problem (Example 3).
  - It can deal gracefully with equality and inequality constraints (Example 4).
  - It can generate an algorithm that avoids large matrix inversion (5).
- Iteration is the price we pay for simplifying the original problem.

- **(EM)** The E-step creates a surrogate function by identifying a complete-data log-likelihood function and evaluating it with respect to the observed data. The M-step maximizes the surrogate function. Every EM algorithm is an example of an MM algorithm.
- **(EM)** demands creativity in identifying the **missing data (complete data)** and technical skill in calculating an often complicated conditional expectation and then maximizing it analytically.
- **(MM)** pay attentions to the **convexity** of the objective function and **inequalities**.
- **(MM)** easier to understand and sometimes easier to apply than EM algorithms.

# Inequalities to construct majorizing/minorizing function — 7/22 —

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- **Property of convex function:**  $\kappa(\theta)$  is called convex if for any  $\theta_1, \theta_2, \lambda \in [0, 1]$

$$\kappa(\lambda\theta_1 + (1 - \lambda)\theta_2) \leq \lambda\kappa(\theta_1) + (1 - \lambda)\kappa(\theta_2)$$

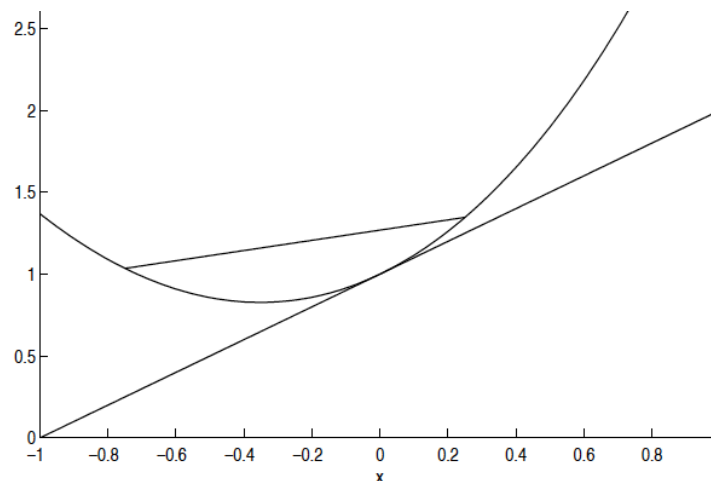
- **Jensen's Inequality:** For a convex function  $\kappa(x)$  and any random variable  $X$ ,

$$\kappa[E(X)] \leq E[\kappa(X)]$$

- **Supporting hyperplanes:** If  $\kappa(\cdot)$  is convex and differentiable, then

$$\kappa(\theta) \geq \kappa(\theta^{(k)}) + [\nabla\kappa(\theta^{(k)})]'(\theta - \theta^{(k)}),$$

and equality holds when  $\theta = \theta^{(k)}$ .





- **Arithmetic-Geometric Mean Inequality:** For nonnegative  $x_1, \dots, x_m$ ,

$$\sqrt[m]{\prod_{i=1}^m x_i} \leq \frac{1}{m} \sum_{i=1}^m x_i,$$

and the equality holds iff  $x_1 = x_2 = \dots = x_m$ .

*Proof by Jensen's inequality:*

Because negative logarithm is convex, we have

$$-\log \left( \frac{1}{m} \sum_{i=1}^m x_i \right) \leq \frac{1}{m} \sum_{i=1}^m -\log x_i = -\sum_{i=1}^m \log x_i^{1/m} = -\log \left( \prod_{i=1}^m x_i \right)^{1/m}$$

The monotonicity of  $-\log$  leads to the result.  $\square$

- **Cauchy-Schwartz Inequality:** For  $p$ -vectors  $x$  and  $y$ ,

$$x'y \leq \|x\| \cdot \|y\|,$$

where  $\|x\| = \sqrt{\sum_{i=1}^p x_i^2}$  is the norm of the vector.

- **Quadratic upper bound:** Suppose a convex function  $\kappa(\theta)$  is twice differentiable and have bounded curvature, we can find a positive definite matrix  $M$  such that  $M - \nabla^2 \kappa(\theta)$  is nonnegative definite. Then we can majorize  $\kappa(\theta)$  by a quadratic function with sufficient high curvature and tangent to  $\kappa(\theta)$  at  $\theta^{(k)}$ , i.e.,

$$\kappa(\theta) \leq \kappa(\theta^{(k)}) + [\nabla \kappa(\theta^{(k)})]' (\theta - \theta^{(k)}) + \frac{1}{2} (\theta - \theta^{(k)})' M (\theta - \theta^{(k)})$$

Note: flipping the above results, we can find a **quadratic lower bound** for a *concave* function, when  $M$  is *negative* definite and  $\nabla^2 \kappa(\theta) - M$  is nonnegative definite.

- By **Jensen's inequality** and the convexity of the function  $-\log(y)$ , we have for probability densities  $a(y)$  and  $b(y)$  that

$$-\log \left\{ \mathbb{E} \left[ \frac{a(Y)}{b(Y)} \right] \right\} \leq \mathbb{E} \left[ -\log \frac{a(Y)}{b(Y)} \right].$$

- If  $Y$  has the density  $b(y)$ , then  $\mathbb{E}[a(Y)/b(Y)] = 1$ . The left-hand side vanishes, and we obtain

$$\mathbb{E}[\log a(Y)] \leq \mathbb{E}[\log b(Y)],$$

which is sometimes known as the **information inequality** (Kullback-Leibler information).

- It is this inequality that guarantees that a minorizing function is constructed in the E-step of any EM algorithm, making every EM algorithm an MM algorithm.

- We have the **decomposition**

$$h^{(k)}(\theta) \equiv \mathbb{E}\{\log f(Y_{\text{obs}}, Y_{\text{mis}}|\theta)|Y_{\text{obs}}, \theta^{(k)}\} = \mathbb{E}\{\log c(Y_{\text{mis}}|Y_{\text{obs}}, \theta)|Y_{\text{obs}}, \theta^{(k)}\} + \log g(Y_{\text{obs}}|\theta)$$

- By the **information inequality**,

$$\mathbb{E}\{\log c(Y_{\text{mis}}|Y_{\text{obs}}, \theta)|Y_{\text{obs}}, \theta^{(k)}\} \leq \mathbb{E}\{\log c(Y_{\text{mis}}|Y_{\text{obs}}, \theta^{(k)})|Y_{\text{obs}}, \theta^{(k)}\}, \forall \theta$$

**Note:** here within the expectation operation,  $Y_{\text{mis}}|Y_{\text{obs}}, \theta^{(k)}$  is a random variable, with density function  $c(Y_{\text{mis}}|Y_{\text{obs}}, \theta^{(k)})$ .

- We obtain the **surrogate function** that minorizes the objective function

$$\log g(Y_{\text{obs}}|\theta) \geq h^{(k)}(\theta) - \mathbb{E}\{\log c(Y_{\text{mis}}|Y_{\text{obs}}, \theta^{(k)})|Y_{\text{obs}}, \theta^{(k)}\} \quad (1)$$

**Note:** The second term of (1) does not depend on  $\theta$ .

- Consider the sequence of numbers  $y_1, \dots, y_n$ . The sample median  $\theta$  minimizes the **non-differentiable objective function**

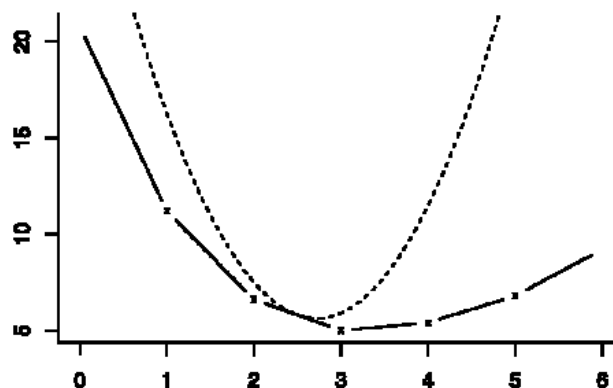
$$f(\theta) = \sum_{i=1}^n |y_i - \theta|.$$

- The **quadratic function**

$$h_i(\theta|\theta^{(k)}) = \frac{(y_i - \theta)^2}{2|y_i - \theta^{(k)}|} + \frac{1}{2}|y_i - \theta^{(k)}|$$

**majorizes**  $|y_i - \theta|$  at the point  $\theta^{(k)}$  (Arithmetic-Geometric Mean Inequality).

- Hence,  $g(\theta|\theta^{(k)}) = \sum_{i=1}^n h_i(\theta|\theta^{(k)})$  majorizes  $f(\theta)$ .



- We have following weighted sum of squares:

$$g(\theta|\theta^{(k)}) = \frac{1}{2} \sum_{i=1}^n \left[ \frac{(y_i - \theta)^2}{|y_i - \theta^{(k)}|} + |y_i - \theta^{(k)}| \right]$$

- The **minimum** of  $g(\theta|\theta^{(k)})$  occurs at

$$\theta^{(k+1)} = \frac{\sum_{i=1}^n w_i^{(k)} y_i}{\sum_{i=1}^n w_i^{(k)}}, \quad w_i^{(k)} = |y_i - \theta^{(k)}|^{-1}$$

- This algorithm works except when a weight  $w_i^{(k)} = \infty$ . It generalizes to **sample quantiles**, **least L1 regression** and **quantile regression**.

- Consider a sports league with  $n$  teams. Assign team  $i$  the skill level  $\theta_i$ , where  $\theta_1 = 1$  for identifiability. Bradley and Terry proposed the model

$$\Pr(i \text{ beats } j) = \frac{\theta_i}{\theta_i + \theta_j}.$$

- If  $b_{ij}$  is the number of times  $i$  beats  $j$ , then the likelihood of the data is

$$L(\boldsymbol{\theta}) = \prod_{i \neq j} \left( \frac{\theta_i}{\theta_i + \theta_j} \right)^{b_{ij}}.$$

We estimate  $\boldsymbol{\theta}$  by **maximizing**  $f(\boldsymbol{\theta}) = \ln L(\boldsymbol{\theta})$  and then rank the teams on the basis of the estimates.

- The log-likelihood is:  $f(\boldsymbol{\theta}) = \sum_{i \neq j} b_{ij} [\ln \theta_i - \ln(\theta_i + \theta_j)]$ .
- We need to **linearize** the term  $-\ln(\theta_i + \theta_j)$  to **separate parameters**.

- By the **supporting hyperplane property** ( $\kappa(\theta) \geq \kappa(\theta^{(k)}) + [\nabla \kappa(\theta^{(k)})]' (\theta - \theta^{(k)})$  when  $\kappa$  is convex) and the convexity of  $-\ln(\cdot)$ , we have

$$-\ln y \geq -\ln x - x^{-1}(y - x) = -\ln x - y/x + 1$$

- The inequality indicates that

$$-\ln(\theta_i + \theta_j) \geq -\ln(\theta_i^{(k)} + \theta_j^{(k)}) - \frac{\theta_i + \theta_j}{\theta_i^{(k)} + \theta_j^{(k)}} + 1$$

- Thus, the **minorizing** function is:

$$g(\theta|\theta^{(k)}) = \sum_{i \neq j} b_{ij} \left[ \ln \theta_i - \ln(\theta_i^{(k)} + \theta_j^{(k)}) - \frac{\theta_i + \theta_j}{\theta_i^{(k)} + \theta_j^{(k)}} + 1 \right].$$

- The parameters are now **separated**. We can easily find the optimal point

$$\theta_i^{(k+1)} = \frac{\sum_{i \neq j} b_{ij}}{\sum_{i \neq j} (b_{ij} + b_{ji}) / (\theta_i^{(k)} + \theta_j^{(k)})}.$$



- Consider the problem of **minimizing**  $f(\theta)$  subject to the **constraints**  $v_j(\theta) \geq 0$  for  $1 \leq j \leq q$ , where each  $v_j(\theta)$  is a concave, differentiable function.

- By the **supporting hyperplane property** and the convexity of  $-v_j(\theta)$ ,

$$v_j(\theta^{(k)}) - v_j(\theta) \geq -[\nabla v_j(\theta^{(k)})]'(\theta - \theta^{(k)}). \quad (2)$$

- Again, by the **supporting hyperplane property** and the convexity of  $-\ln(\cdot)$ , we have  $-\ln y + \ln x \geq -x^{-1}(y - x) \implies x(-\ln y + \ln x) \geq x - y$ . Then:

$$v_j(\theta^{(k)}) [-\ln v_j(\theta) + \ln v_j(\theta^{(k)})] \geq v_j(\theta^{(k)}) - v_j(\theta). \quad (3)$$

- By (2) and (3),

$$v_j(\theta^{(k)}) [-\ln v_j(\theta) + \ln v_j(\theta^{(k)})] + [\nabla v_j(\theta^{(k)})]'(\theta - \theta^{(k)}) \geq 0,$$

and the equality holds when  $\theta = \theta^{(k)}$ .

- Summing over  $j$  and multiplying by a positive tuning parameter  $\omega$ , we construct the **surrogate function** that majorizes  $f(\theta)$ ,

$$g(\theta|\theta^{(k)}) = f(\theta) + \omega \sum_{j=1}^q \left[ v_j(\theta^{(k)}) \ln \frac{v_j(\theta^{(k)})}{v_j(\theta)} + [\nabla v_j(\theta^{(k)})]' (\theta - \theta^{(k)}) \right] \geq f(\theta)$$

- **Note:**

- **Majorization gets rid of the inequality constraints.**
- The presence of  $\ln v_j(\theta)$  ensures  $v_j(\theta) \geq 0$ .
- An initial point  $\theta^{(0)}$  must be selected with all inequality constraints strictly satisfied. All iterates stay within the interior region but allows strict inequalities to become equalities in the limit.
- The minimization step of the MM algorithm can be carried out approximately by **Newton's method**.
- Where there are linear equality constraints  $A\theta = b$  in addition to the inequality constraints  $v_j(\theta) \geq 0$ , these should be enforced by introducing **Lagrange multipliers** during the minimization of  $g(\theta|\theta^{(k)})$ .

- We have an  $n \times 1$  vector  $Y$  of binary responses and an  $n \times p$  matrix  $X$  of predictors. The logistic regression model assumes that

$$\pi_i(\theta) \equiv \Pr(Y_i = 1) = \frac{\exp(\theta' x_i)}{1 + \exp(\theta' x_i)}.$$

Then the **log likelihood** is

$$l(\theta) \equiv \sum_{i=1}^n Y_i \theta' x_i - \sum_{i=1}^n \log \{1 + \exp(\theta' x_i)\}.$$

- The **Hessian** can be obtained by direct differentiation:

$$\nabla^2 l(\theta) = - \sum_{i=1}^n \pi_i(\theta) [1 - \pi_i(\theta)] x_i x_i'. \quad (4)$$

- Remember the definition of **quadratic lower bound**:

$$\kappa(\theta) \geq \kappa(\theta^{(k)}) + \left[ \nabla \kappa(\theta^{(k)}) \right]' (\theta - \theta^{(k)}) + \frac{1}{2} (\theta - \theta^{(k)})' M (\theta - \theta^{(k)})$$

where  $\kappa(\theta)$  is concave and twice differentiable, and  $M$  is a negative definite matrix.

- Since  $\pi_i(\theta) [1 - \pi_i(\theta)]$  is bounded above by 1/4, we may define the negative definite matrix  $M = -1/4 X' X$  such that  $\nabla^2 l(\theta) - M$  is nonnegative definite. Thus,

$$g(\theta|\theta^{(k)}) = l(\theta^{(k)}) + \left[ \nabla l(\theta^{(k)}) \right]' (\theta - \theta^{(k)}) + \frac{1}{2} (\theta - \theta^{(k)})' M (\theta - \theta^{(k)})$$

is a **quadratic lower bound** of  $l(\theta)$  (note:  $l(\theta)$  is convex).

- The MM algorithm proceeds by **maximizing**  $g(\theta|\theta^{(k)})$ , giving

$$\begin{aligned}\theta^{(k+1)} &= \theta^{(k)} - M^{-1} \nabla l(\theta^{(k)}) \\ &= \theta^{(k)} + 4(X'X)^{-1} X' [Y - \pi(\theta^{(k)})].\end{aligned}$$

- Computational advantage of the MM algorithm over Newton-Raphson
  - **MM**: invert  $X'X$  **only once**.
  - **NR**: invert the Hessian (4) **for every iteration**.

- **Convergence rate**

- NR: a quadratic rate  $\lim \|\theta^{(k+1)} - \hat{\theta}\| / \|\theta^{(k+1)} - \hat{\theta}\|^2 = c$  (constant)
- MM: a linear rate  $\lim \|\theta^{(k+1)} - \hat{\theta}\| / \|\theta^{(k+1)} - \hat{\theta}\| = c < 1$

- **Complexity of each iteration**

- NR: require evaluation and inversion of Hessian,  $O(p^3)$
- MM: separates parameters,  $O(p)$  or  $O(p^2)$

- **Stability of the algorithm**

- NR: behave poorly if started too far from an optimum point
- MM: guaranteed to increase/decrease the objective function at every iteration

In conclusion, well-designed MM algorithms tend to require more iterations but simpler iterations than Newton-Raphson; thus MM sometimes enjoy an advantage in computation speed and numerical stability.

- Quantile regression (Hunter and Lange, 2000)
- Survival analysis (Hunter and Lange, 2002)
- Paired and multiple comparisons (Hunter 2004)
- Variable selection (Hunter and Li, 2002)
- DNA sequence analysis (Sabatti and Lange, 2002)