AM1: Array Comms and Processing

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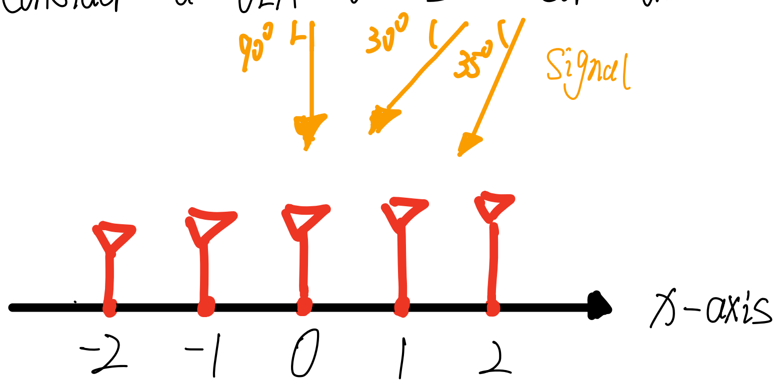
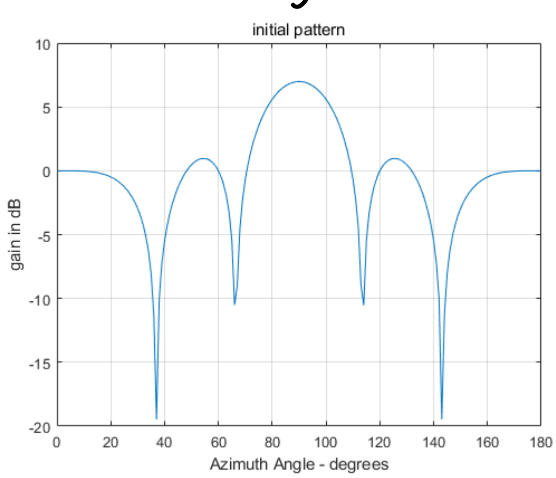
0. Array geometry

Consider a ULA of 5 sensor distributed along x-axis in the 3D real space:



$$\text{array} = \begin{bmatrix} -2 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \qquad \text{direction of Signal} = \begin{bmatrix} 30, & 0 \\ 35, & 0 \\ 90, & 0 \end{bmatrix}$$

1. Pattern of the given array



There are no beamforming.

all w is equal to 1.

2. Theoretical Convariance Matrix Formation:

$$\underset{N\times 1}{x(t)} \triangleq \underset{N\times M}{S(p)} \cdot \underset{M\times M}{m(t)} + \underset{N\times 1}{n(t)}$$

$\begin{cases} N: \text{attennas num} \\ M: \text{co-channel message source num} \end{cases}$

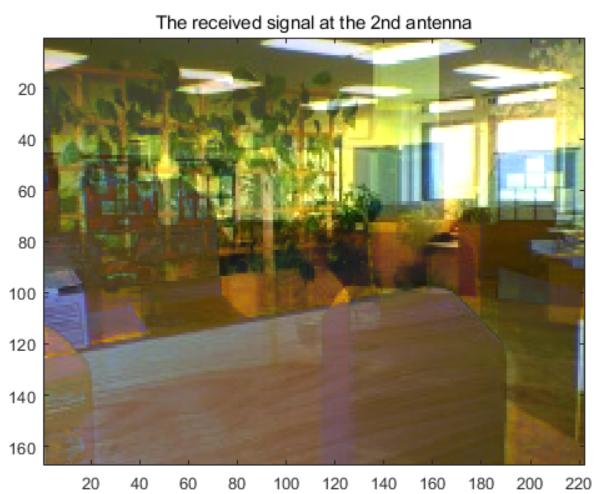
$$R_{xx} \triangleq \mathcal{E}\{x(t)x(t)^H\}$$
$$= \mathcal{E}\{(S\underline{m}(t)+n(t))\cdot(S\underline{m}(t)+n(t))^H\}$$
$$= \mathcal{E}\{S\underline{m}(t)\underline{m}(t)^H S^H + n(t)n(t)^H + S\underline{m}(t)\cdot n(t)^H + \underline{n}(t)\underline{m}(t)^H S^H\}$$
$$= S\,\mathcal{E}\{\underline{m}(t)\underline{m}(t)^H\}S^H + \mathcal{E}\{n(t)n(t)^H\} + \underbrace{S\,\mathcal{E}\{m(t)n(t)^H\}}_{=0} + \underbrace{\mathcal{E}\{n(t)m(t)^H\}S^H}_{=0.}$$
$$= SR_{mm}S^H + R_{nn}$$
$$= SR_{mm}S^H + \sigma_n^2 I_n$$

Define following in matlab:

```
S = spv(array,directions); % array manifold vectors

sigma2 = 0.0001; % noise power
Rmm = eye(3);
Rxx_theoretical = S*Rmm*S' + sigma2*eye(5,5); % Theoretical covariance matirx
```

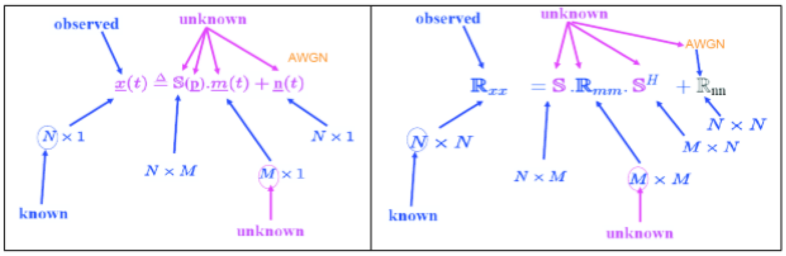
3. Practical Covariance Martix



After loading the data, we can display the received image at 2nd antenna.

4 & 5 Detection Problem.

Summary - General Problem Formulation



- Condition: $M < N$
- Estimate $M, \underline{p} = [p_1, p_2, ..., p_M]^T, \mathbb{R}_{mm}, \sigma_n^2$, etc
  where p_i is a parameter of interest associated with the i^{th} source.

Actually, we only know the system param N. Other info or param are needed to be estimated.

For Detection Problem, we want to estimate the num of message source M

Remember that:

$$R_{xx} \triangleq \mathcal{E}\{\underline{x}(t)\underline{x}(t)^H\} = \underbrace{S R_{mm} S^H}_{R_{signal}} + \underbrace{R_{nn}}_{\sigma_n^2 I_N} \qquad M < N$$

Using eigen-decomposition of R_{xx}, we have:

$$R_{xx} = E D E^H$$

Also, $R_{xx} = R_{signal} + R_{noise}$

$$= E\underline{\Lambda}E^H + \sigma_n^2 I_N = E\underline{\Lambda}E^H + \sigma_n^2 \underbrace{EE^H}_{I_N}$$

$$= E\underbrace{(\underline{\Lambda} + \sigma_n^2 I_N)}_{D}E^H$$

$$\underline{\Lambda} = \begin{bmatrix} \lambda_1 & & & & & & 0 \\ & \lambda_2 & & & & & \\ & & \lambda_3 & & & & \\ & & & \ddots \lambda_m & & & \\ & & & & 0 & & \\ & & & & & 0 \ddots & \\ 0 & & & & & & 0 \end{bmatrix} \Rightarrow D = \begin{bmatrix} \lambda_1+\sigma_n^2 & & & & & \\ & \lambda_2+\sigma_n^2 & & & & \\ & & \ddots & & & \\ & & & \lambda_m+\sigma_n^2 & & \\ & & & & \sigma_n^2 & \\ & & & & & \sigma_n^2 \cdots \sigma_n \end{bmatrix}$$

$\Rightarrow eig_{min}\{R_{xx}\} = \sigma_n^2$ the multiplicity of D is $N-$

Therefore, theoretically, the number of emitting source M can be determined by the eigenvalues of the covariance matrix R_{xx}:

$$M = N - (\text{multiplicity of } \text{eig}_{min}\{R_{xx}\})$$

eig_theoretical | eig

5x1 double

	1	2
1	1.0000e-04	
2	1.0000e-04	
3	0.0953	
4	4.9681	
5	9.9369	

$\} \to \sigma_n^2$

In Matlab simulation, we can find $M = 5-2 = \underline{3}$.

However, in practice

$$\sigma_1^2 \neq \sigma_2^2 \neq \sigma_3^2 \neq \cdots \neq \sigma_m^2 \cdots \neq \sigma_N$$

This is also can verified in Matlab:

eig_au

5x1 double

	1
1	0.0100
2	0.0101
3	173.3260
4	1.0720e+04
5	4.3818e+05
6	
7	

$\} \sigma_n$

eig_im

5x1 double

	1
1	-9.4922e-11
2	7.0614e-11
3	190.8705
4	2.0391e+04
5	2.1784e+05
6	
7	

$\} \sigma_n$

We can find $\sigma_4^2 \approx \sigma_5^2$ but they are not exact equality.

In this case, we introduce AIC and MDL (In step 13)

6&7 Estimation Problem – Conventional Approach

The WIENER – HOPF Beamformer:

$$\underline{w} = c \cdot R_{xx}^{-1} \underline{S}_{\text{desired signal}}$$

$\hookrightarrow$ SPV $(90°)$

MAX the SNIR at array O/P.

~~DOA estimation not required~~

Proof:

This is the Minimum Variance Beamformer that solve the following optimisation problem:

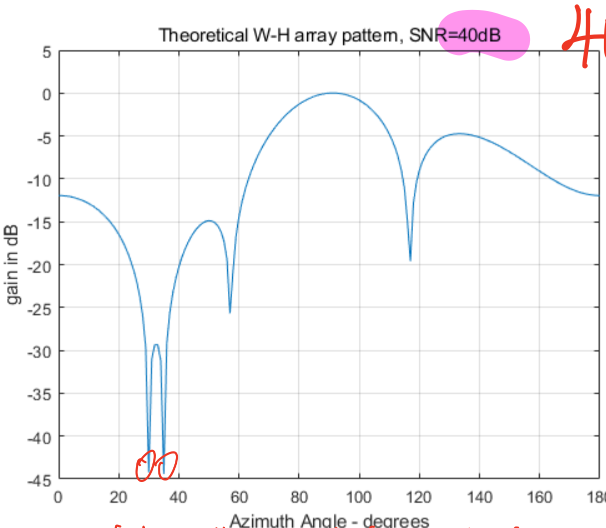
$$\min_{\underline{w}} \left(\underline{w}^H R_{xx} \underline{w} \right)$$
$$\text{s.t. } \underline{w}^H S(\theta) = 1$$

where, the objective function aims to minimise the effort of the desired signal plus noise. However the constraint $\underline{w}^H S(\theta) = 1$ prevents the gain reduction in the direction of desired signal.

$$\Rightarrow \underline{w} = c \cdot R_{xx}^{-1} \underline{S}_d$$

where c = a constant scalar chosen such as $\underline{w}^H S(\theta) = 1$

40dB



two nulls at interference directions.

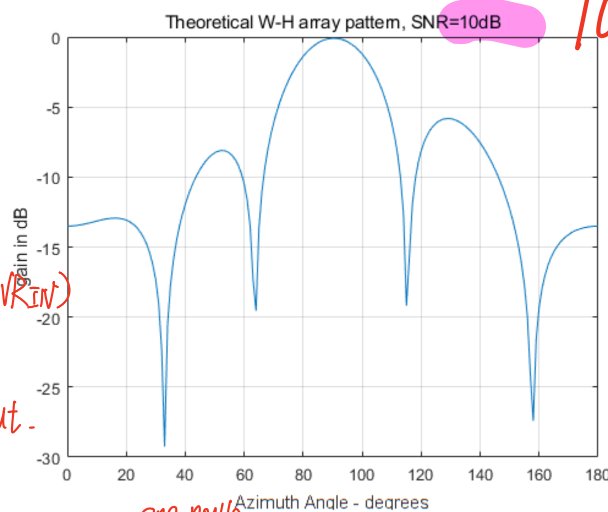
compare

$WH = f(SNR_{IN})$

$\Downarrow$

Max SNRout.

10dB



one nulls.

Here are the result of W-H beamformer in Matlab with two level SNR.

SNR = 40 dB (high) , SNR = 10 dB (low)

we find that:

When the SNR is low (e.g 10 dB), we don't have nulls at two unkown inference directions. However, we still have the maximum SNIR achieveable but the beamformer allows some interference to pass through to the output in order to reduce the strong noise and overall to maximise the SNI/ at the output of the beamformer

8. Conclusions of WH beamformer

☆ Hence, the performance of the WH is the function of the SNR. if the SNR is high, we can observe the directions of the interference as nulls in the array pattern but not when it low.

9&10 Estimation Problem: Music Algorithm

It estimate the intersection $\mathcal{L}[E_n]$ and the array manifold.

The project $\underline{S}(p)$ on the subspace $\mathcal{L}[E_n]$ is given by:

$$\underline{z}(p) = P_{E_n} \cdot \underline{S}(p)$$

The norm-squared of $\underline{z}(p)$ can be written as:

$$\mathcal{E}(p) = \underline{z}(p)^H z(p) = S(p)^H \underbrace{P_{E_n}^H P_{E_n}}_{P_{E_n}} \underline{S}(p)$$

$$= \underline{S}(p)^H P_{E_n} \underline{S}(p)$$

$$= \underline{S}(p)^H E_n \underbrace{(E_n^H E_n)}_{I_{N \times N}}^{-1} E_n^H \underline{S}(p)$$

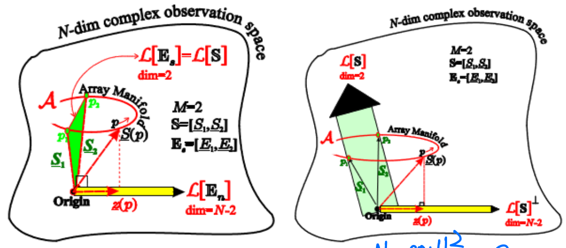
$$= \underline{S}(p)^H E_n E_n^H \underline{S}(p)$$

The magnitude of vector $\underline{z}(p) = 0$

This will give us the green point p_1, p_2, p_3 belong to the green. $\mathcal{L}(S)$ subspace.

$$\Rightarrow \boxed{\mathcal{E}(p) = 0 \quad \text{iff} \begin{cases} p = p_1 \\ \text{or} \\ p = p_2 \end{cases}}$$

- Example:



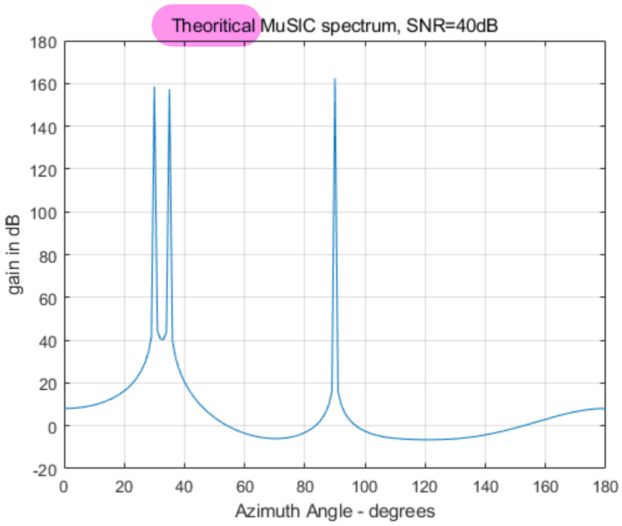
$\|z(p)\|^2 = 0.$

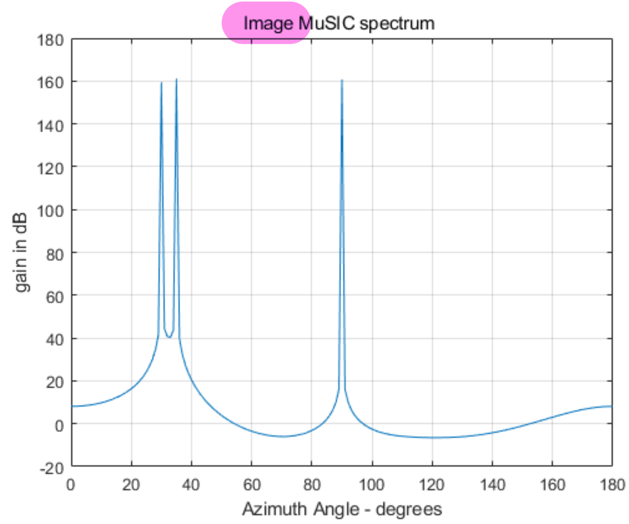
- Note:

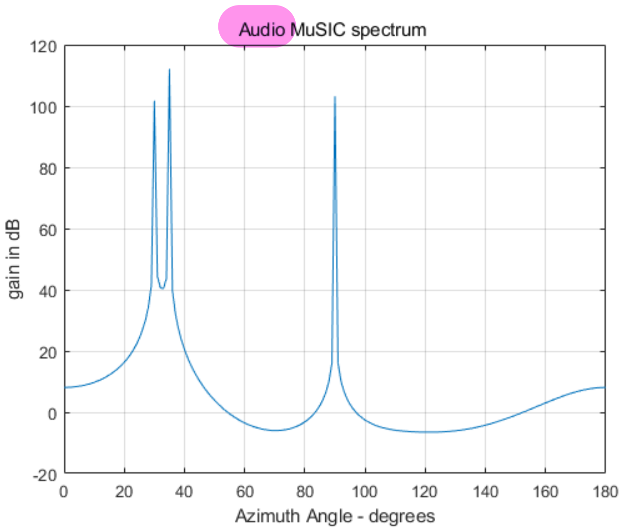
$$\underline{S}_1, \underline{S}_2, \underline{E}_1, \underline{E}_2 \perp \mathcal{L}[\mathbb{E}_n]$$
$$\underline{S}_1, \underline{S}_2, \underline{E}_1, \underline{E}_2 \in \mathcal{L}[\mathbb{E}_s]$$

$\mathcal{L}[\mathbb{E}_s]$ is a plane which intersects the array manifold in 2 points $\underline{S}_1, \underline{S}_2$

$$\Rightarrow [p_1, p_2, \ldots, p_M]^T = \arg\min_{p} \mathcal{E}(p)$$







Three of the figure shows the inverse of the MUSic algorithm cost function (Maximisation rather than minisation)

The peak of point is estimated DOAS.

— $(30^\circ, 0^\circ)$, $(35^\circ, 0^\circ)$, $(90^\circ, 0^\circ)$

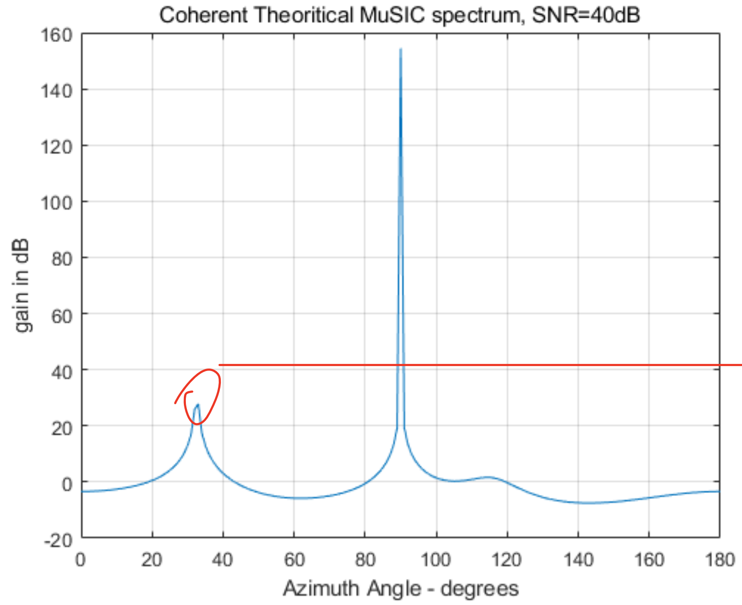
11. Multipath — Coherent Sources

if all source message are fully uncorrelated and the power is 1

$$R_{mm} = eye(3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

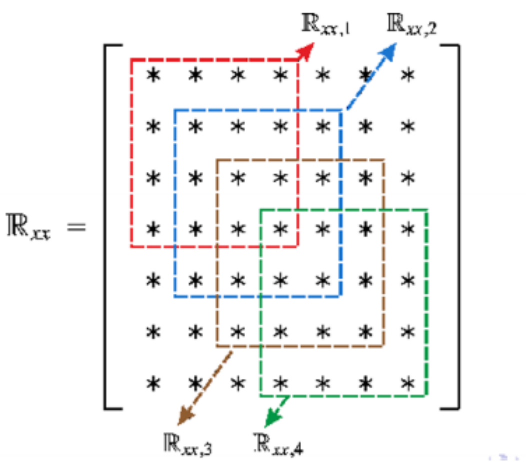
if they are coherent (i.e. $30^\circ, 35^\circ$ are coherent):

$$R_{mm} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



The MUSIC fails to estimate the two coherent signal at 30° and 35°.

This problem can be combatted by using a pre-processor (before using MUSIC) known as: "Spatial Smoothing"



Parition the array to sub-array and calculate the average of all these subarray covariance:

$$R_{xx\_smooth} = \frac{1}{N}\sum_{S=1}^{N} R_S$$

where, $N = L - K + 1$

$R_S = R_{xx}(S:S+K-1,\ S:S+K-1)$

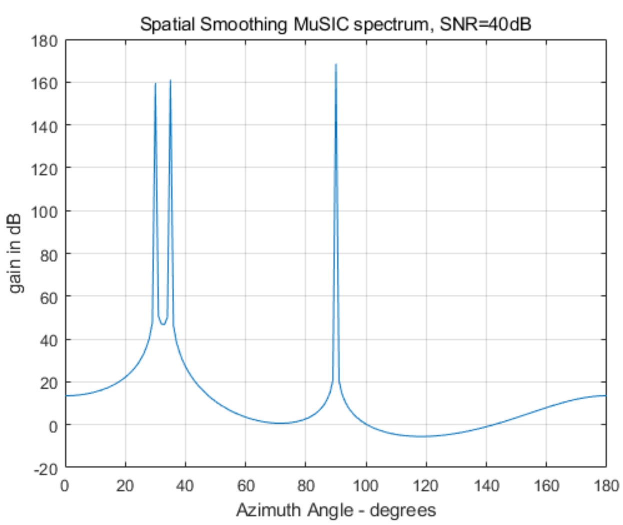
L is the elements in ULA.

K is the subarray length

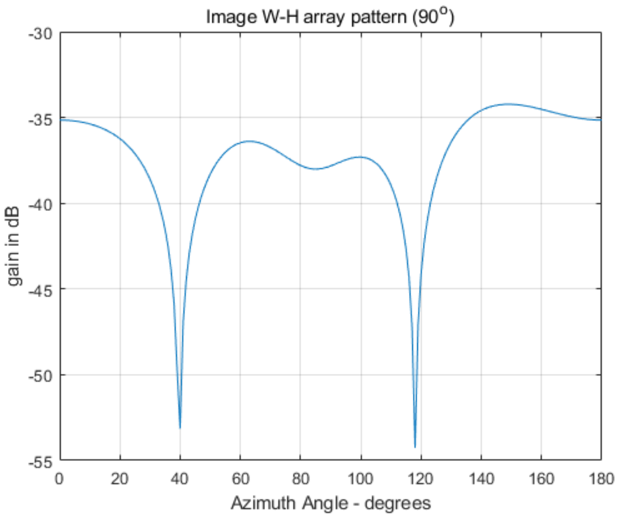
However, the rank of R_{xx_smooth} is $K < N$

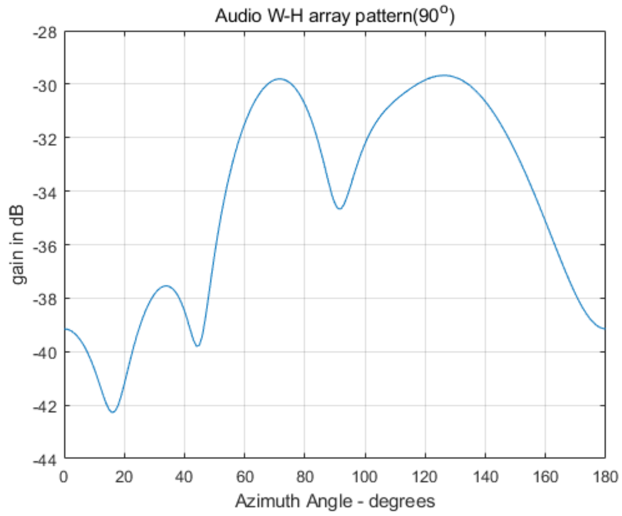
it reduces the aperture of arrays.

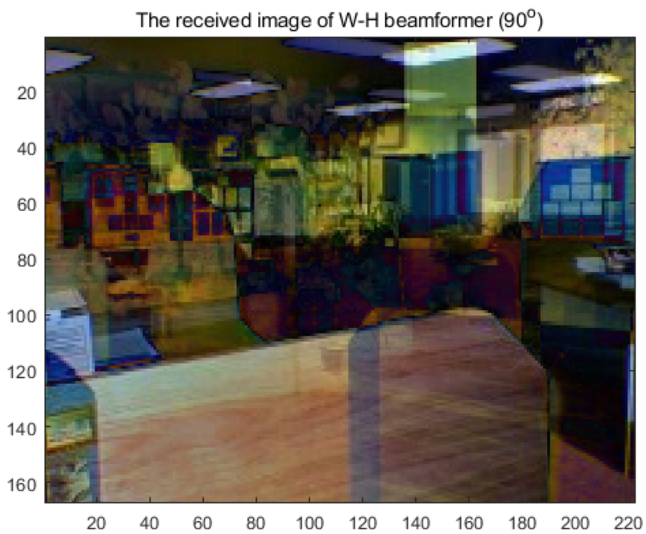
Then, we can use MUSIC again, results shown in LHS



12. Estimations & Reception problem — Supperresolution Techniques:







Using W-H beamformer, the array pattern of image and audio shows it is not good enough at desired direction 90°.

And the received image at 90° shows a lot of interference.

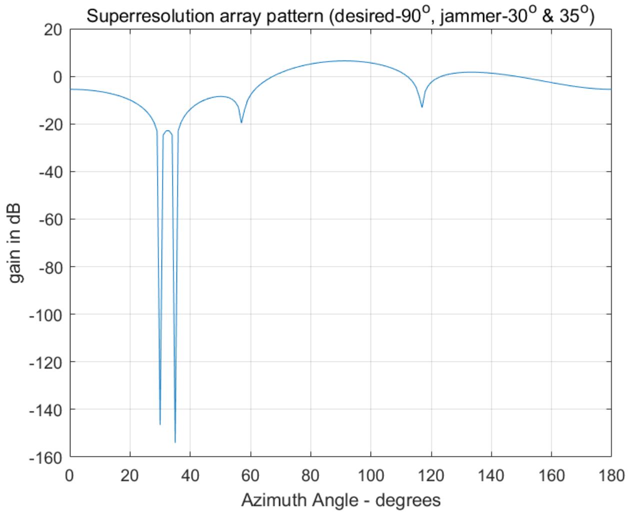
12.3 Superresolution Beamformer based on DOA estimation

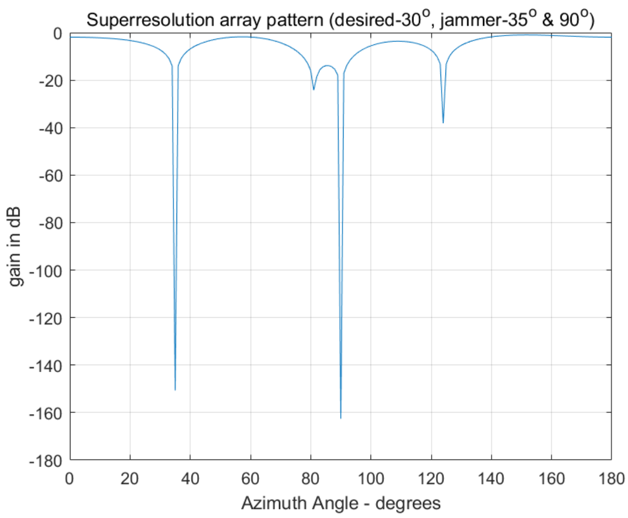
$$\underline{w} = P_{SJ}^{\perp} \underline{S}_d$$

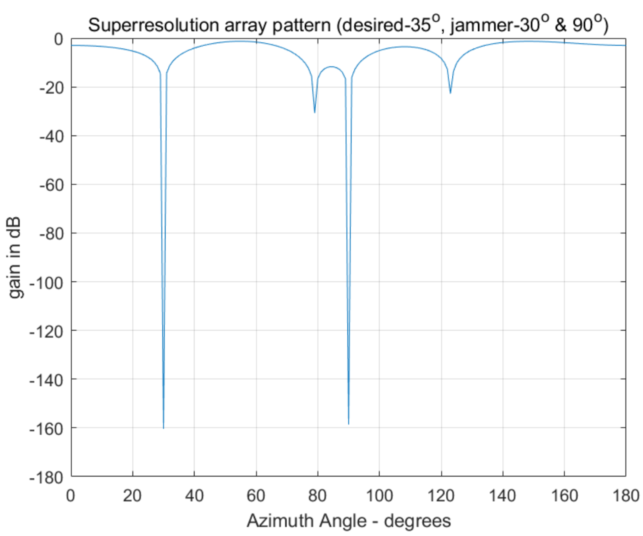
where $S = [\underline{S}_d, \underline{S}_J$

$$P_{S_J}^{\perp} = I_N - S_J (S_J^H S_J)^{-1} S_J^H$$

The Pattern of the arrays and result of image shown below:







13. Practical Detection Criterion

a. AIC :

$$\min_{k} \{ AIC(k) \}$$

where

$$AIC(k) \triangleq \underbrace{-2 \ln \left(\max_{k} LF^{(k)} \right)}_{\text{log- likelihood}} + \underbrace{2k}_{\text{bias correction.}}$$

b. MDL

$$\min_{k} \{ MDL(k) \}$$

where

$$MDL(k) \triangleq -\ln (\max LF^{(k)}) + \frac{1}{2} k \ln L$$

C. LF

$$LF^{(k)} = -L \cdot \ln (\det \{ R_{xx}^{(k)} \}) - Tr \{ R_{xx}^{(k)} \}^{-1} \cdot \hat{R}_{xx}$$

$$= \ln \left(\frac{ \prod_{l=k+1}^{N} d_l^{1/(N-k)} }{ \frac{1}{N-k} \sum_{l=k+1}^{N} d_l } \right)^{(N-k)L}$$

where $k \in (0, 1, \ldots, N-1)$

and $d_l =$ the l-th eigenvalue of $\hat{R}_{xx}$

the ratio :

$$\frac{\text{geometric mean of the smallest } N-k \text{ eigenvalue of } \hat{R}_{xx}}{\text{arithmetic mean of the smallest } N-k \text{ eigenvalue of } \hat{R}_{xx}}$$

$$\Rightarrow \begin{cases} AIC(k) = -2\ln\left(\dfrac{\prod_{l=k+1}^{N} d_l^{1/(N-k)}}{\frac{1}{N-k}\sum_{l=k+1}^{N} d_l}\right)^{(N-k)L} + 2k(2N-k) \\ MDL(k) = -\ln\left(\dfrac{\prod_{l=k+1}^{N} d_l^{1/(N-k)}}{\frac{1}{N-k}\sum_{l=k+1}^{N} d_l}\right)^{(N-k)L} + \frac{1}{2}k(2N-k)\ln L \end{cases}$$

Matlab result:

MIN

AIC

1x5 double

	1	2	3	4	5
1	4.0390e+...	2.9147e+...	125.7277	42.1580	48.0000
2					
3					

MDL

1x5 double

	1	2	3	4	5
1	2.0195e+...	1.4732e+...	91.0355	58.0543	66.2575
2					
3					

$$\therefore \; M\_AIC = 4-1 = 3$$
$$M\_MDL = 4-1 = 3$$