GRASP Laboratory
General Robotics & Autonomous Systems and Processes

Mechatronic Modeling and Design with Applications in Robotics

Linear Graph Example

Sketching a Linear Graph

1. Identify the energy storage elements, energy dissipation elements, and source elements in the system (1-port elements, each represented by 1 branch)
2. Identify any multi-port elements (e.g., transformers)
3. Identify the terminals of each element (i.e., action points and reference points)
4. Recognize how the elements are interconnected (series or parallel and to what elements?) and sketch a schematic diagram (e.g., circuit diagram)
5. Starting from a convenient node point (typically, ground reference) draw a branch (typically, for a source), link it to another branch through a node, and so on, to form a loop
6. Repeat Step 5 until the entire system is completed (i.e., all the elements in the system are included and connected)

Compatibility (Loop) Equations

Kirchhoff voltage law.

Loop: Closed path formed by two or more branches

Loop Equation (Compatibility Equation): Sum of across variables in a loop is zero.

Sign Convention:

ccw + positive direction

1. Go in counter-clockwise direction of loop
2. In direction of branch arrow (except in a T-source) ➔ Across variable drops ➔ Positive

Note: Loop equation (compatibility) ➔ across variable remains the same (i.e., unique) at any given point in the loop at a given time (E.g., a mass and spring connected to the same point must have the same velocity ➔ point is intact; does not break or snap; system is compatible).

Number of "Primary" Loops

"Minimal" set of loops from which any other loop can be determined.

Primary loop set is an "independent" set. It will generate all the independent loop equations.

Note: Loops closed by broken-line (inertia) branches should be included in counting primary loops.

Mechanical Example

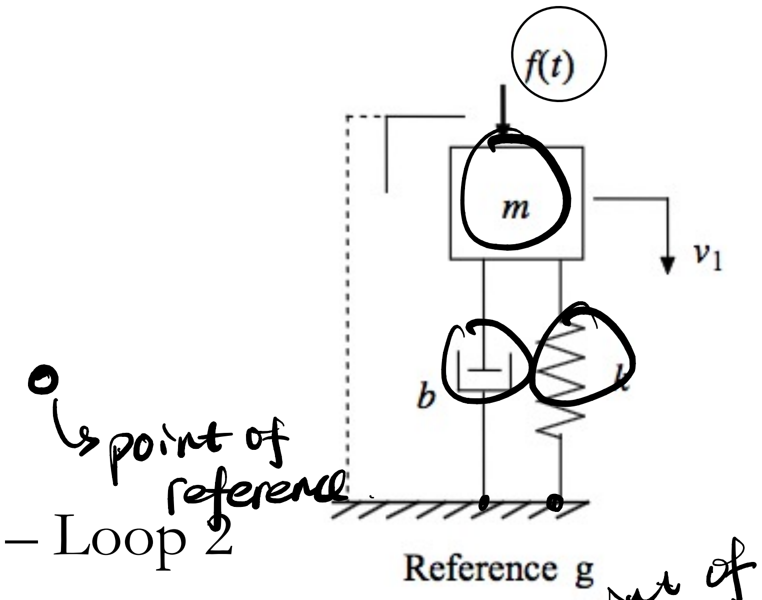
Three Primary Loops; Force source

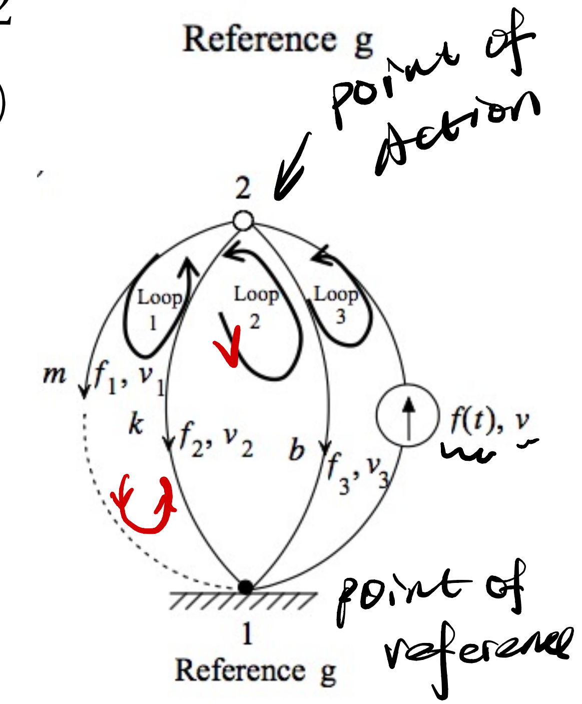
e.g., Loop 1: *m-k*; loop 2: *k-b*; Loop 3: *b-f*

Other Choices: (*b-k*, *m-b* and *m-f*) or (*b-k*, *m-b* and *f-k*), etc.

Note: *m-k* loop= (*m-b* loop) - (*b-k* loop); or Loop 1=(*m-b* loop) – Loop 2

f-m loop = (*f-b* loop)+(*b-m* loop); or *f-m* loop=Loop 3 + (*b-m* loop)





Verify these relations using loop equations

Loop 1 Equation: $v_1 - v_2 = 0$; Loop 2 Equation: $v_2 - v_3 = 0$

Loop 3 Equation: $v_3 - v = 0$; *m-b* Loop Equation: $v_1 - v_3 = 0$

f-m Loop Equation: $v_1 - v = 0$

Continuity (Node) Equations

Kirchhoff current law.

Node: Point where two or more branches meet

Node Equation: Sum of all through variables at node = 0
"What goes in must come out" ➔ continuity of through variables at a node

current through variable

Mechanical Systems: Force balance or equilibrium equation; Newton's third law; etc.
Electrical Systems: Current balance; Kirchoff's current law; conservation of charge; etc.
Hydraulic Systems: Conservation of matter
Thermal Systems: Conservation of energy

Sign Convention: Into node is positive
Previous Example: Two nodes. Corresponding node equations are identical:

✓ **Node 2 Equation:** $-f_1-f_2-f_3+f = 0$ same equation one of two equations.
✓ **Node 1 Equation:** $f_1+f_2+f_3-f = 0$

Required number of node equations = Number of nodes - 1

2 Nodes. 2 of 3.
6 Nodes 5 of 6.

Electrical Example

Three primary loops; two nodes (1 primary node); current source.

point of action

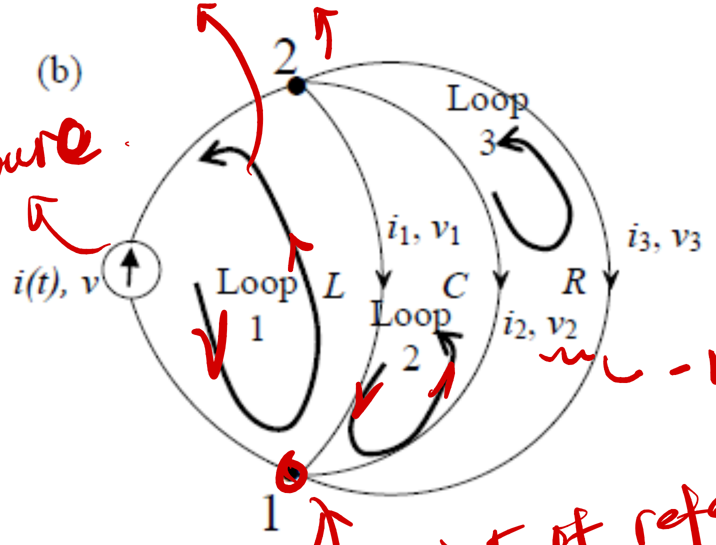
power source - current

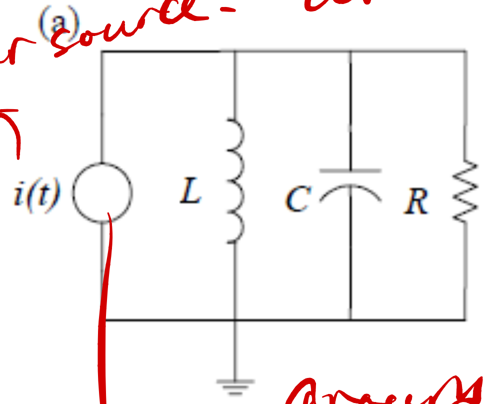
Possible choice of 3 primary loops:

1 Loop 1 (L-v) Equation: $-v_1 + v = 0$

2 Loop 2 (C-L) Equation: $-v_2 + v_1 = 0$

3 Loop 3 (R-C) Equation: $-v_3 + v_2 = 0$





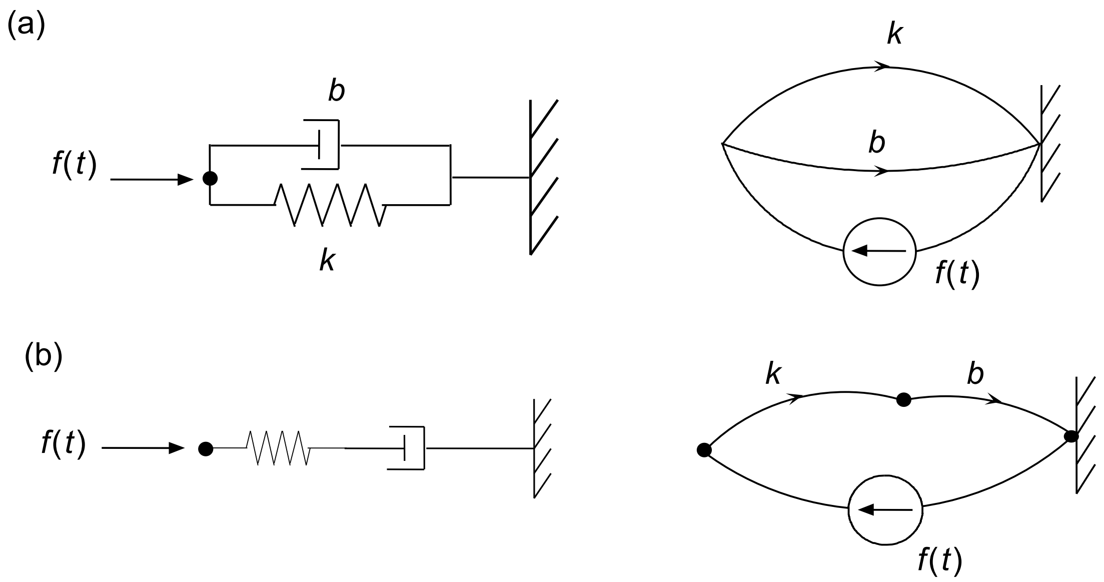
Node equations:

4 Node 2 Equation: $i - i_1 - i_2 - i_3 = 0$

Node 1 Equation: $-i + i_1 + i_2 + i_3 = 0$ (Sign-reversed Node 2 equation)

Series and Parallel Connections

Series System	Parallel System
Through variables are the same Across variables are not the same (they add algebraically)	Across variables are the same Through variables are not the same (they add algebraically)



Spring (k)-damper (b) systems with a force source and their linear graphs:
(a) Elements in parallel (has 2 loops); (b) Elements in series (has 1 loop).

State Models from Linear Graphs

Sign Convention

- **Power flows into action point and out of reference point. This direction is shown by the branch arrow. Exception: In a source element power flows out of the action point.**
- **Through variable (*f*), across variable (*v*), and power flow (*fv*) are positive in the same direction at action point. At reference point *v* is positive in the same direction given by linear-graph arrow, but *f* is taken positive in the opposite direction.**
- **In writing node equations: Flow into a node is positive**
- **In writing loop equations: Loop direction is counterclockwise. A potential (A-variable) "drop" is positive (same direction as branch arrow. Exception: In a T-source arrow is in the direction in which A-variable increases)**

***Note*: Once the sign convention is established, the actual values of the variables can be positive or negative depending on their actual direction.**

Steps in Obtaining a State Model

A-Type across variable

T-Type → through

1. Choose state variables (across variables for independent A-type elements and through variables for independent T-type elements)
2. Write constitutive equations for independent energy storage elements ➔ state-space shell
3. Do similarly for remaining elements (dependent energy storage elements, dissipation elements, transformation elements—two port, etc.)
4. Write compatibility equations for primary loops   loop equations.
5. Write continuity equations for primary nodes (total nodes – 1)
6. In the state space shell, retain state and input variables only. Eliminate all other variables using loop and node equations and extra constitutive equations

$\dot{x} = Ax + Bu$

$y = Cx + Du$

u : input

y : output

A B C D

x ? State

→ check. whether modeling is possible.

General Observation

#sources = s; # branches = b ➔ Total # unknown variables = $2b - s$

constitutive equations = $b - s$; # primary loops = ℓ ➔ # loop equations = ℓ

nodes = n ➔ # node equations = $n - 1$

Total # equations = $(b\text{-}s) + \ell + (n\text{-}1) = b + \ell + n\text{-}s\text{-}1$ → step 1 → 5

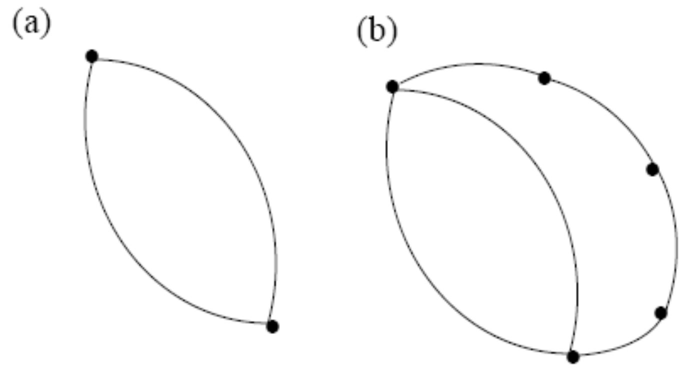
We must have: Unknowns = Equations ➔ $2b - s = b + \ell + n\text{-}s\text{-}1$ ➔ $\ell = b\text{-}n+1$

1. Constitutive eqs.
2. loop equations
3. Node equation.

Topological Result

\# sources = s; # branches = b; # nodes = n

Proof of $\ell = b\text{-}n+1$ (i) **for a Linear Graph (Proof by Induction)**



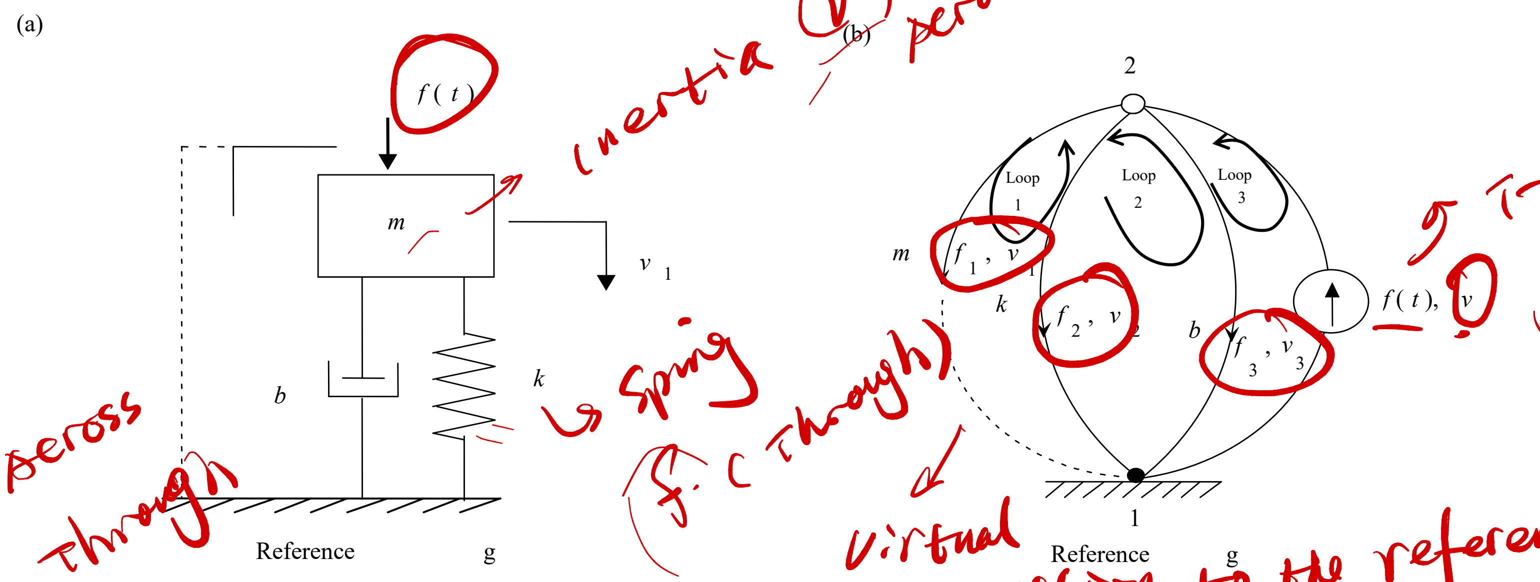
\# of unknowns
= # of equation

Step 1: Start with Graph (a): $\ell = 1$, $b = 2$, $n = 2$ ← (i) is satisfied.
Add new loop by using m nodes and $m + 1$ branches, as in Graph (b)
$\ell \to 2$; $n \to 2+m$; $b \to 2+m+1$ → (i) is still satisfied
Note: $m = 0$ is a special case

Step 2: Start with a general linear graph having ℓ loops, b ranches, and n nodes that satisfies (i); Add a new loop by using m nodes and $m + 1$ branches
$\ell \to \ell+1$; $n \to n+m$; $b \to b+m+1$ → (i) is still satisfied
→ By induction, (i) is true in general.

Example 1

Example 1



v: across
f: through

→ T-type source.
→ f(t) known.

(6 equations 7 unknown)

branches b = 4; # nodes n = 2; # sources s = 1; # primary loops l = 3

unknowns = $v_1, f_1, v_2, f_2, v_3, f_3, v$ = 7 (*Note*: $f(t)$, the input variable, is known)

constitutive equations (one each for m, k, b) = $b - s$ = 3 4 − 1 = 3.

node equations = $n - 1$ = 1

loop equations = 3 (Note: 3 primary loops) 3 + 1 + 3 = 7 equations

check!!!

➔

Total # equations = constitutive eqns + node eqns + loop eqns = 3 + 1 + 3 = 7.

➔ system is solvable (7 unknowns and 7 equations).

Example 1 (Cont'd)

5 steps: State space model

A, B, C, D.

States: x

~~Sta~~ Across Through

Step 1. State Variables: Velocity v_1 of mass m and force f_2 of spring k ➔ $x_1 = v_1$; $x_2 = f_2$

Input variable = applied forcing function (force source) $f(t)$

Step 2. Constitutive equations for m and k ➔ **State-space Shell (Model Skeleton):**

✓ Newton's 2nd law for m: $\dot{v}_1 = (1/m) f_1$ (i)

✓ Hooke's law for k: $\dot{f}_2 = k v_2$ (ii)

Step 3. Remaining Constitutive Eqn (for damper): $f_3 = b v_3$ (iii)

Step 4. Node and Loop Equations:

Node eqn (for Node 2): $f - f_1 - f_2 - f_3 = 0$ (iv)

Loop eqn for loop 1: $v_1 - v_2 = 0$ (v)

Loop eqn for loop 2: $v_2 - v_3 = 0$ (vi)

Loop eqn for loop 3: $v_3 - v = 0$ (not needed since v is not needed) (vii)

Step 5: Eliminate Auxiliary Variables:

Eliminate f_1 and v_2 in (i) and (ii).

From (v): $v_2 = v_1$

From (iv) and (iii): $f_1 = -f_2 - b v_3 + f$ ➔ $f_1 = -f_2 - b v_1 + f$ (from (vi) and (v))

Substituting these into the state-space shell (equations (i) and (ii)) we get the state model

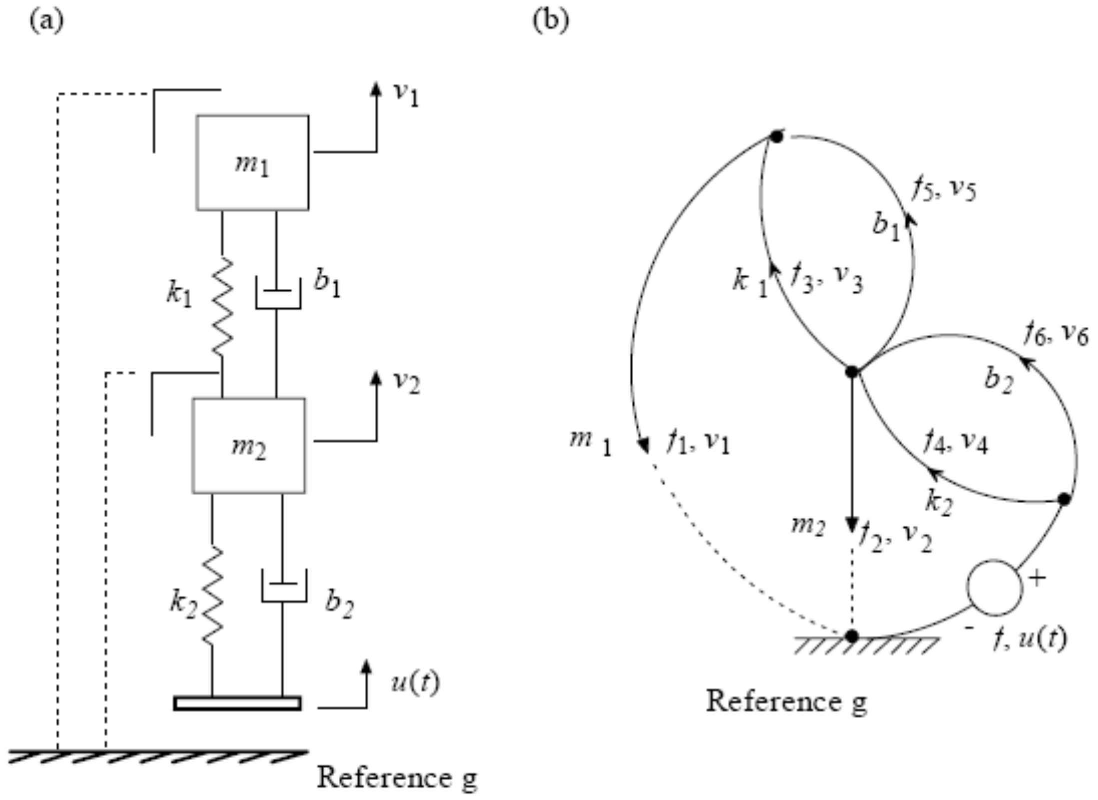
$$\begin{cases} \dot{v}_1 = -\frac{b}{m} v_1 - \frac{1}{m} f_2 + \frac{1}{m} f \\ \dot{f}_2 = k v_1 \end{cases} \Rightarrow A = \begin{bmatrix} -b/m & -1/m \\ k & 0 \end{bmatrix}; \; B = \begin{bmatrix} 1/m \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \dot{v}_1 \\ \dot{f}_2 \end{bmatrix} = \begin{bmatrix} -b/m & -1/m \\ k & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ f_2 \end{bmatrix} + \begin{bmatrix} 1/m \\ 0 \end{bmatrix} f(t)$$

State vector $x = [x_1 \quad x_2]^T = [v_1 \quad f_2]^T$ and input vector $u = f(t)$ ➔ 2nd order system

Note: It would have been easier if both loops contained state variable v_1

Example 2



Main System (m_2, k_2, b_2); Dynamic Absorber (m_1, k_1, b_1); Velocity Source $u(t)$

 # branches = b = 7; # nodes = n = 4; # sources = s = 1

 # independent loops = l = 4; # unknowns = $2b - s = 13$

 # constitutive eqns = $b - s = 6$; # node eqns = $n - 1 = 3$; # loop eqns = 4

Check: # Unknowns = $2b - s = 13$; # Eqns = $(b - s) + (n - 1) + l = 6+3+4 = 13$

Example 2 (Cont'd)

Step 1. **Four independent energy storage elements (m_1, m_2, k_1, k_2) ➔**
State variables $\boldsymbol{x} = [x_1, \quad x_2 \quad x_3 \quad x_4]^T = [v_1, \quad v_2 \quad f_3 \quad f_4]^T$; **Input variable = $u(t)$.**

Step 2. **Skeleton State Equations (Model Shell):**

Newton's 2nd law for mass m_1: $\dot{v}_1 = \dfrac{1}{m_1} f_1$; **Newton's 2nd law for mass m_2:** $\dot{v}_2 = \dfrac{1}{m_2} f_2$

Hooke's law for spring k_1: $\dot{f}_3 = k_1 v_3$; **Hooke's law for spring k_2:** $\dot{f}_4 = k_2 v_4$

Step 3. **Remaining Constitutive Equations:**

For damper b_1: $f_5 = b_1 v_5$; **For damper b_2:** $f_6 = b_2 v_6$

Step 4. **Node Equations:** $-f_1 + f_3 + f_5 = 0; \quad -f_3 - f_5 - f_2 + f_4 + f_6 = 0; \quad -f_4 - f_6 + f = 0$

Loop equations: $v_1 - v_2 + v_3 = 0; \quad v_2 - u + v_4 = 0; \quad -v_4 + v_6 = 0; \quad -v_3 + v_5 = 0$

Example 2 (Cont'd)

Eliminate auxiliary variables ➔ State Equations:

$$\dot{v}_1 = -(b_1/m_1)v_1 + (b_1/m_1)v_2 + (1/m_1)f_3$$

$$\dot{v}_2 = (b_1/m_2)v_1 - \left[(b_1+b_2)/m_2\right]v_2 - (1/m_2)f_3 + (1/m_2)f_4 + (b_2/m_2)u(t)$$

$$\dot{f}_3 = -k_1v_1 + k_1v_2$$

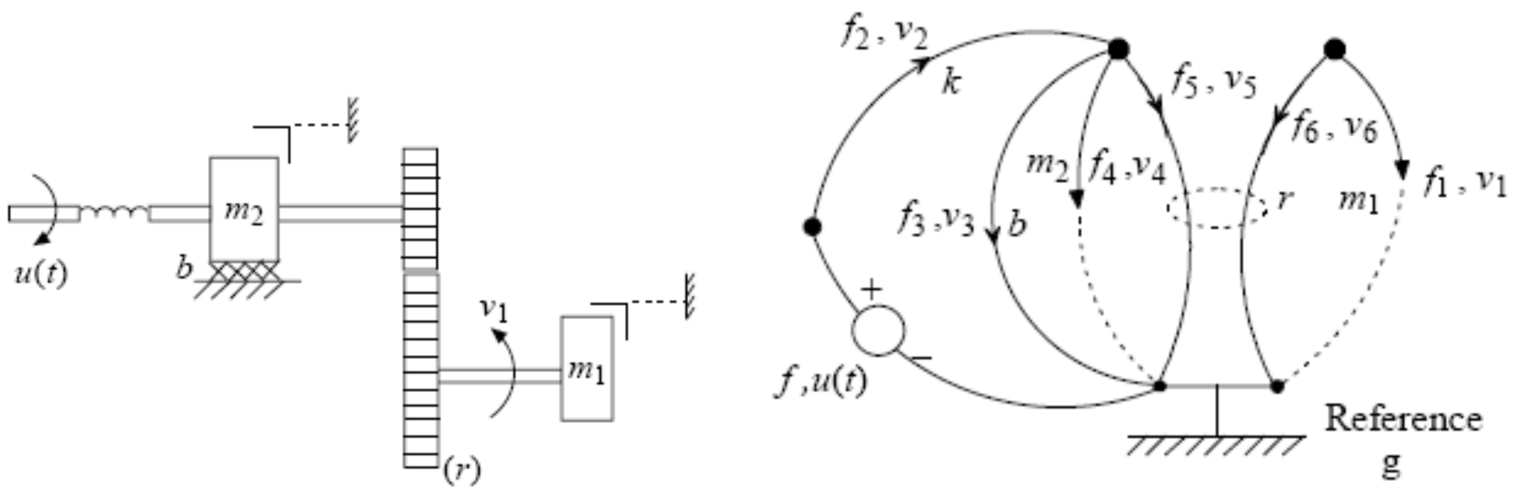
$$\dot{f}_4 = -k_2v_2 + k_2u(t)$$

System matrix $A = \begin{bmatrix} -b_1/m_1 & b_1/m_1 & 1/m_1 & 0 \\ b_1/m_2 & -(b_1+b_2)/m_1 & -1/m_2 & 1/m_2 \\ -k_1 & k_1 & 0 & 0 \\ 0 & -k_2 & 0 & 0 \end{bmatrix}$

Input distribution matrix $B = \begin{bmatrix} 0 \\ b_2/m_2 \\ 0 \\ k_2 \end{bmatrix}$

***Note*: Not necessary to write the node equation containing the through variable of the A-type source, because that through variable is not needed**

Example 3



Rotary-Motion System with Velocity Source and Gear Transmission; Linear Graph

Step 1. **Two inertial elements m_1 and m_2 are not independent ➔ With spring there are only two independent energy storage elements. State Variables:** $[x_1 \quad x_2]^T = [v_1 \quad f_2]^T$

Step 2. **Constitutive equations for m_1 and k:** $\dot{v}_1 = \frac{1}{m_1} f_1$; $\quad \dot{f}_2 = k v_2$

Step 3. **Remaining Constitutive Equations:**

For damper: $f_3 = b v_3$; **For "dependent" inertia m_2:** $\dot{v}_4 = \frac{1}{m_2} f_4$

For transformer (pair of meshed gear wheels): $v_6 = r v_5$; $f_6 = -\frac{1}{r} f_5$

Example 3 (Cont'd)

Step 4: Node equations: $-f_6 - f_1 = 0;\ f - f_2 = 0;\ f_2 - f_3 - f_4 - f_5 = ($

Loop equations: $v_6 - v_1 = 0;\ v_3 - v_4 = 0;\ v_4 - v_5 = 0;\ -v_2 + u(t) - v_3 =$

Step 5: Equations from steps 3 and 4 → Auxiliary variables:

$$f_1 = \frac{1}{r}\left[f_2 - \frac{b}{r}v_1 - \frac{m_2}{r}\dot{v}_1\right];\quad v_2 = -\frac{1}{r}v_1 + u(t)$$

→ **State equations:**

$$\dot{v}_1 = -\left[\frac{b}{\left(m_1 r^2 + m_2\right)}\right]v_1 + \left[\frac{r}{\left(m_1 r^2 + m_2\right)}\right]f_2$$

$$\dot{f}_2 = -\frac{k}{r}v_1 + ku(t)$$

→

$$\boldsymbol{A} = \begin{bmatrix} -b/m & r/m \\ -k/r & 0 \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 0 \\ k \end{bmatrix}$$

$m = m_1 r^2 + m_2$ = ***equivalent inertia*** **of** $\boldsymbol{m_1}$ **and** $\boldsymbol{m_2}$ **expressed at** $\boldsymbol{m_2}$

Example 3 (Cont'd)

Note **1:**

We could have defined an equivalent inertia expressed at m_1 as:

$$m' = m_1 + \frac{m_2}{r^2}$$

Note **2:**

If it is not immediately clear if two energy storage elements are dependent, one approach would be to introduce two different state variables for them. At the end it will be found that the two variables are not independent, and that one of them can be eliminated.

A more systematic approach uses graph trees (see my extra notes; outside the scope of the course)

Note **3: There is no need to write a loop equation involving the dependent across variable of a T-type source, because that dependent variable is not something that we are asked to determine (typically)**

Note **4: Similarly, there is no need to write a node equation involving the dependent through variable of an A-type source**

The End!!