GRASP Laboratory
General Robotics & Autonomous Systems and Processes

Mechatronic Modeling and Design with Applications in Robotics

Analytical Modeling (Part 2)

- A system is assumed to be at rest (zero initial conditions),
- A transfer function is defined by

$$G(s) := \frac{C(s)}{R(s)}$$

TF

Laplace transform of **system output**

Laplace transform of **system input**

$\frac{\text{output}}{\text{input}}$.

$s = \sigma + j\omega$.

σ Real, ω imaginary.

complex number.

frequency.

input-output function

Block diagram

$G(s) = C(s)P(s)$

Input. $R(s) \longrightarrow \boxed{G(s)} \longrightarrow C(s)$ output.

Transfer function of system

controller, plant.

$R(s) \longrightarrow \boxed{C(s)} \longrightarrow \boxed{P(s)} \longrightarrow C(s)$

PID. DC Motor

$\mathcal{L}T. \rightarrow t \rightarrow s.$

$\mathcal{L}T^{-1} - s \rightarrow t.$

in time domain

Note: input, system and output into three separate and distinct parts.

A general *n*th-order, linear, time-invariant differential equation:

output related

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1}\frac{d^{n-1}c(t)}{dt^{n-1}} + \cdots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1}\frac{d^{m-1}r(t)}{dt^{m-1}} + \cdots + b_0 r(t)$$

input related term

C(s) output R(s) input

where *c(t)* is the output, *r(t)* is the input.

$TF = \frac{\text{output}}{\text{input}}$

$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}\,dt$

Assume: zero initial conditions, and take the Laplace transform on both side:

$$(a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0)C(s) = (b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0)R(s)$$

output input output

output

$$T(s) = \frac{C(s)}{R(s)} = \frac{(b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0)}{(a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0)} \quad \Rightarrow \quad T(s) = \frac{C(s)}{R(s)}$$

TF. input input

Room temperature

inverse laplace transform

output input

$C(s) = R(s)G(s)$

$\mathcal{L}^{-1}\{C(s)\}$ system

Example

Find the transfer function represented by $\frac{dc(t)}{dt} + 2c(t) = r(t)$, and use the result to find the response $c(t)$ to a unit step input with zero initial conditions.

→ DE

output

$\frac{dc(t)}{dt} + 2c(t) = r(t)$ → Laplace transform $\mathcal{L}$.

$sC(s) + 2C(s) = R(s)$ → TF ⇒ $T(s)$: $(s+2)C(s) = R(s)$

s "t"

$T(s) = \frac{C(s)}{R(s)} = \frac{1}{s+2}$.

$C(s) = \frac{1}{s+2} R(s)$.

$R(s) \rightarrow \boxed{G(s)} \rightarrow C(s)$.

$\frac{1}{s+2}$

$C(s) = G(s) R(s) = \frac{1}{s+2} \cdot \frac{1}{s} = \frac{1}{s(s+2)}$

output

partial frational expansion → Laplace transform table.

$t \quad 0 \to \infty$

$\frac{1}{s(s+2)} = \frac{1}{s} \times \frac{1}{s+2} = \frac{K_1}{s} + \frac{K_2}{s+2} = \frac{K_1(s+2) + K_2 s}{s(s+2)}$

$\frac{1}{2} \times \frac{1}{s} - \frac{1}{2} \times \frac{1}{s+2}$

$K_1(s+2) + K_2 s = 1$

$K_1 \cdot s + 2K_1 + K_2 s = 1$

$(K_1 + K_2)s + 2K_1 = 1$

$K_1 + K_2 = 0. \quad 2K_1 = 1$

⇒ $K_1 = \frac{1}{2}$

$K_2 = -\frac{1}{2}$.

$\frac{1}{s(s+2)} = \frac{\frac{1}{2}}{s} + \frac{-\frac{1}{2}}{s+2}$.

t: ① ②.

One of the most important math tool in the course!

Definition:

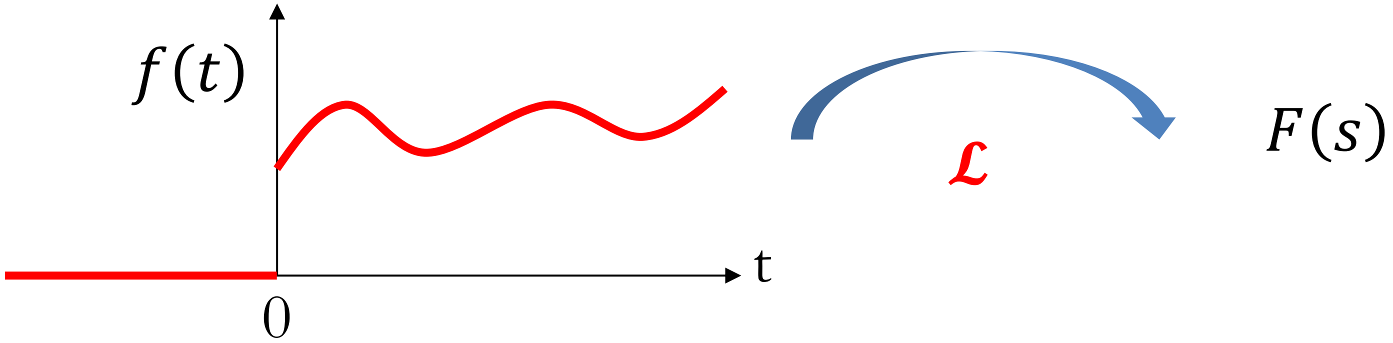
For a function $f(t)$ ($f(t) = 0 \; for \; t = 0$)

$s = \sigma + j\omega$.

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t) e^{-st} dt$$

(s: complex variable)

$F(s)$ is denoted as the Laplace transform of $f(t)$



Inverse Laplace Transform

Allow us to find $f(t)$ given $F(s)$:

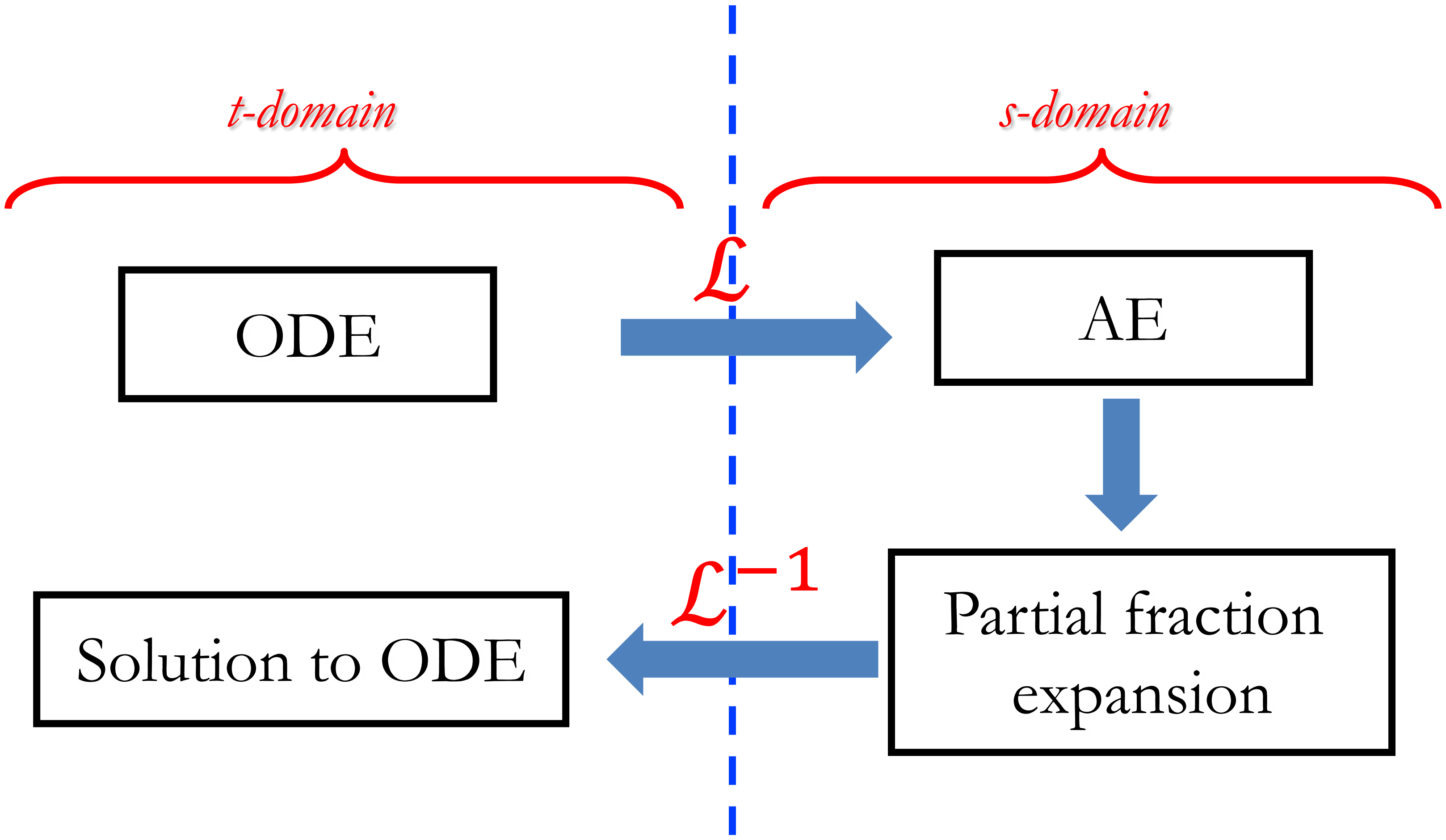
$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}\,ds = f(t)u(t)$$

where

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

An Advantage of Laplace Transform

Transform an ordinary differential equation (ODE) into an algebraic equation (AE).



Laplace Transform Table

No.	$f(t)$ time	$F(s)$ s domain
1	$\delta(t)$	1
2	$u(t)$	$\frac{1}{s}$
3	$\boxed{tu(t)}$	$\frac{1}{s^2}$
4	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5	$\boxed{e^{-at}u(t)}$	$\frac{1}{s+a}$
6	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

$\mathcal{L}$

$\mathcal{L}^{-1}$

$\frac{1}{s+2}$.

$\frac{1}{s+1}$

Laplace Transform Theorems (Properties)

Item no.	Theorem	Name
1.	$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)\, e^{-st} dt$	Definition
2.	$\mathscr{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathscr{L}\left[e^{-at} f(t)\right] = F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)] = e^{-sT} F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2 f}{dt^2}\right] = s^2 F(s) - sf(0-) - \dot{f}(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^n f}{dt^n}\right] = s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)\, d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s \to 0} sF(s)$	Final value theorem[1]
12.	$f(0+) = \lim_{s \to \infty} sF(s)$	Initial value theorem[2]

$\dot{y}(t)$

$sY(s)$

Partial-Fraction Expansion

Partial-Fraction Expansion: To convert the function to a sum of simpler terms.

E.g.,

$$F(s) = \frac{s^3 + 2s^2 + 6s + 7}{s^2 + s + 5}$$

Partial-Fraction Expansion

$$\Rightarrow \quad F(s) = s + 1 + \frac{2}{s^2 + s + 5}$$

Reminder:
Order of the numerator less than its denominator

$\mathcal{L}^{-1}$

$$\Rightarrow \quad f(t) = \mathcal{L}^{-1}\{s\} + \mathcal{L}^{-1}\{1\} + \mathcal{L}^{-1}\left\{\frac{2}{s^2 + s + 5}\right\}$$

3 Cases (Roots of the Denominator)

1. Real and Distinct

$$F(s) = \frac{2}{(s+1)(s+2)} \rightarrow \qquad F(s) = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)}$$

$$\frac{K_1(s+2) + K_2(s+1)}{(s+1)(s+2)}$$

$$K_1 s + 2K_1 + K_2 s + K_2 = 2$$

$K_1 \quad K_2$

$$\begin{cases} K_1 + K_2 = 0 \\ 2K_1 + K_2 = 2 \end{cases}$$

K_1

2. Real and Repeated

$$F(s) = \frac{2}{(s+1)(s+2)^2} \rightarrow \qquad F(s) = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)^2} + \frac{K_3}{(s+2)}$$

$-1 \quad -2 \quad -2$

$K_1 \quad K_2 \quad K_3$

3. Complex or Imaginary

$$F(s) = \frac{3}{s(s^2+2s+5)} \rightarrow \qquad F(s) = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2+2s+5}$$

$= 0$

roots.

$s^2 + 2s + 5 = 0$

$K_1 \quad K_2 \quad K_3$

Differentiation Theorem: $\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0)$; $\mathcal{L}\left\{\frac{d^2 f}{dt^2}\right\} = s^2F(s) - sf(0) - f'(0)$;

$\mathcal{L}\left\{\frac{d^n f}{dt^n}\right\} = s^nF(s) - \sum_{k-1}^{n} s^{n-k}\ f^{k-1}(0)$;

Example: Given the following differential equation, solve for $y(t)$ if all initial conditions are zeros.

$$\frac{d^2 y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 4y(t) = 4u(t)$$

input

$s^2 Y(s) + 2s Y(s) + 4Y(s) = 4U(s)$

$Y(s)(s^2+2s+4) = 4U(s)$

$Y(s) = \frac{4}{s^2+2s+4} U(s)$.

Inverse Laplace transform

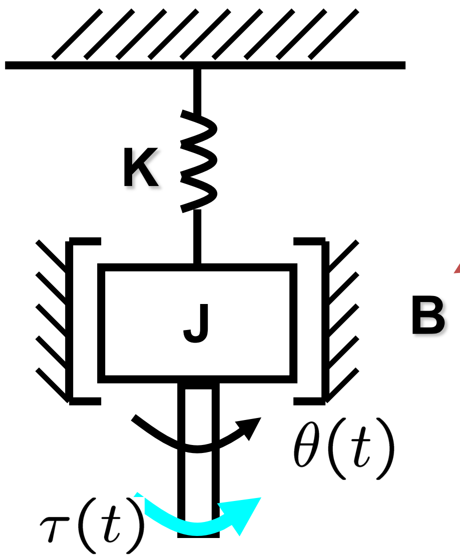
$s \to t$.

Example 1

$$\sum T = J\alpha = J\ddot{\theta}$$

$$\tau(t) - K\theta(t) - B\cdot\dot{\theta}(t) = J\ddot{\theta}(t)$$

$$J\ddot{\theta}(t) + B\cdot\dot{\theta}(t) + K\theta(t) = \tau(t)$$



Zero initial condition

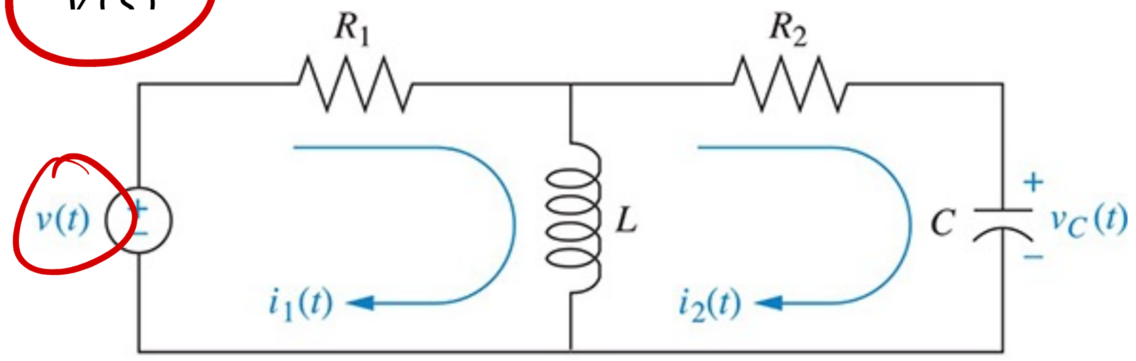
$$Js^2\Theta(s) + Bs\Theta(s) + K\Theta(s) = T(s)$$

Transfer Function $G(s) = \dfrac{\Theta(s)}{T(s)} = \dfrac{1}{Js^2 + Bs + K}$

Example 2

Given the network below, find the transfer function $\frac{I_2(s)}{V(s)}$.

$$T(s) = \frac{I_2(s)}{V(s)}.$$



Converting a TF to State Space

TF: input-output model.

$T(s) = \frac{\text{output}}{\text{input}}$. s domain

Assume the TF of a SISO system is as follows:

$$u \longrightarrow \boxed{G(s)} \longrightarrow y$$

$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du. \end{cases}$ time domain

$$G(s) = \frac{\overbrace{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}^{N}}{\underbrace{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}_{D}} \text{ where } m < n$$

order of D. higher order of N.

$n > m$

Then its state-space model can be written below:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \text{where } A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}_{n \times n},$$

6th order. $\boxed{s^6}$

6×6.

$$B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}_{n \times 1}, C = [b_0 \quad b_1 \quad \cdots \quad b_m \quad 0]_{1 \times n}, D = [0]$$

A B C D.

States.?

Example

$$G(s) = \frac{2s^2 + 5s + 3}{3s^3 + 7s^2 - 6s + 1} \quad \div 3.$$

input $U(s) \rightarrow \boxed{G(s)} \rightarrow Y(s)$ output

Please find its state-space model.

$$G(s) = \frac{\frac{2}{3}s^2 + \frac{5}{3}s + 1}{s^3 + \frac{7}{3}s^2 - 2s + \frac{1}{3}} \text{ (third-order system)}$$

Its state-space model: $\begin{cases} \dot{x} = Ax + Bu & \text{state equation} \\ y = Cx + Du & \text{output equation} \end{cases}$

TF - SS

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{3} & 2 & -\frac{7}{3} \end{bmatrix}_{3\times3}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{3\times1}, C = \begin{bmatrix} 1 & \frac{5}{3} & \frac{2}{3} \end{bmatrix}, D = [0]$$

Converting from State Space to TF

Assume the state-space model of a system is as follows:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

Take the Laplace Transform assuming zero initial conditions

$$\begin{cases} sX(s) = AX(s) + BU(s) & ① \\ Y(s) = CX(s) + DU(s) & ② \end{cases}$$

X(s)

Solving for X(s) in above equations

$X(s) = (sI - A)^{-1}BU(s)$ where I is the identity matrix

Substitute it to $y = Cx + Du$ ➔

$$Y(s) = [C(sI - A)^{-1}B + D]U(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

TF $G(s) = \frac{\text{output}}{\text{input}} = \frac{Y(s)}{U(s)}$

Example

A B

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} x + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + 0 \cdot u$$

C D = [0]

Please find its transfer function.

$$G(s) = C(sI - A)^{-1}B + D$$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \left(s \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} + [0]$$

✓

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s & -1 & 0 \\ 0 & 2 & -1 \\ 1 & 2 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} + [0]$$

$$\boxed{= \frac{10s^2+30s+20}{s^3+3s^2+2s+1}}$$

TF

SS model → TF

The End!!