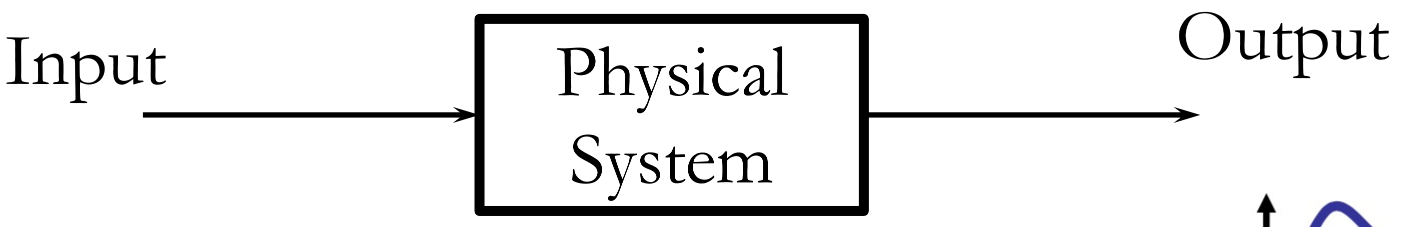
GRASP Laboratory
General Robotics & Autonomous Systems and Processes

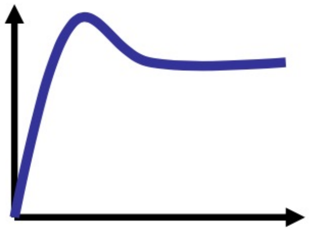
Mechatronic Modeling and Design with Applications in Robotics

Analytical Modeling (Part 1)

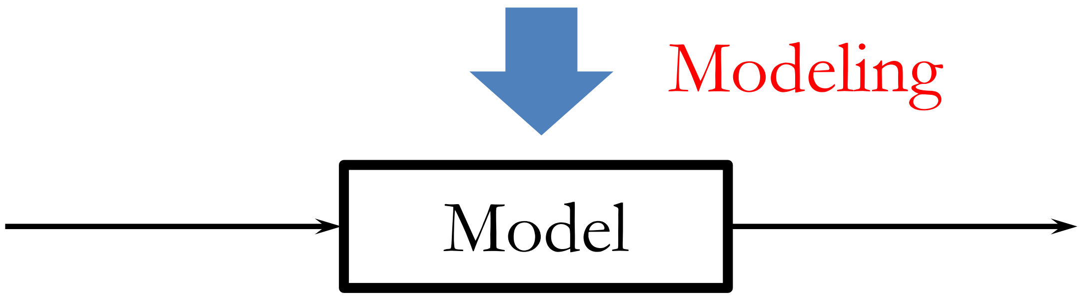
Mathematical Model

Representation of the input-output relationship of a physical system.





Acceleration	Car	Position
Voltage	Circuit	Current
Deposit	Bank Account	Balance

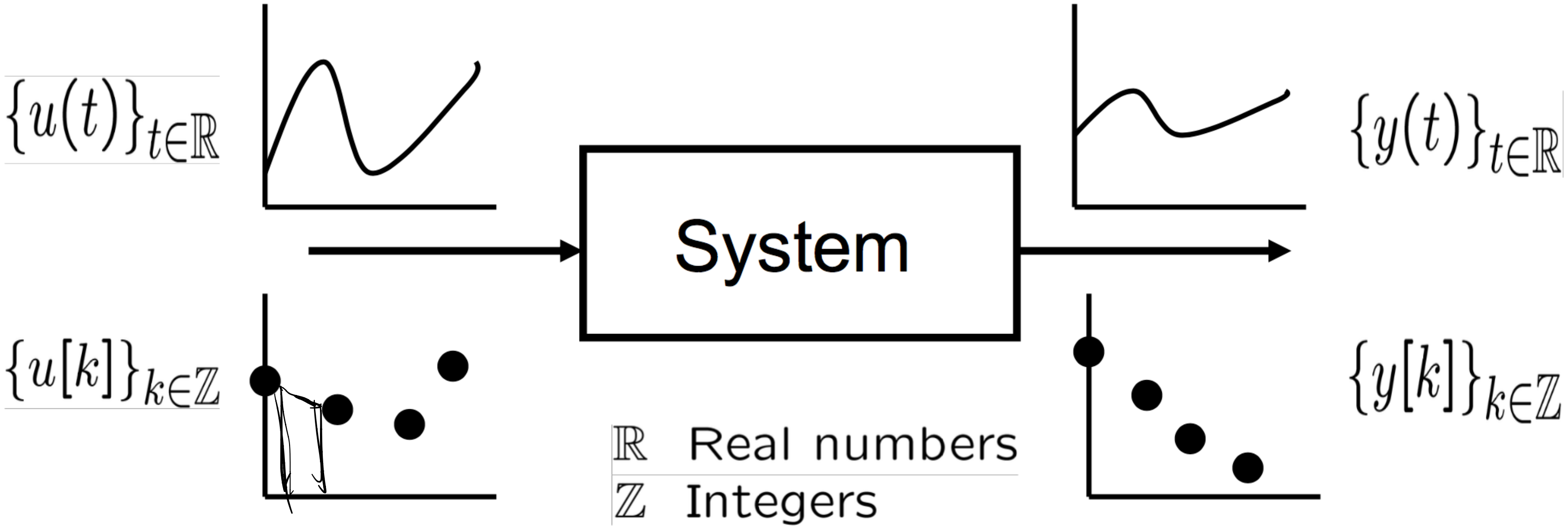


Model Classification

- Continuous-time and discrete-time
- Memoryless, causal and noncausal
- Lumped and distributed
- Time-invariant and time-varying
- Linear and nonlinear

Continuous-time and Discrete Time

Input/output vectors are continuous-time signals



MHz

- Discrete-time system
- Input/output vectors are discrete-time signals

Example

- Continuous-time system
  - Mass-spring-damper system

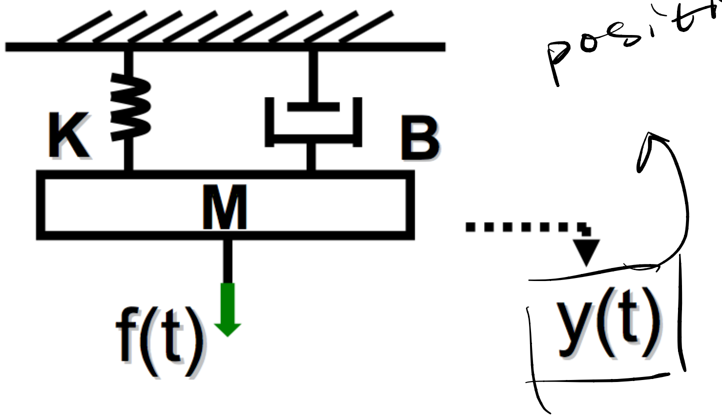
$$My''(t) = f(t) - By'(t) - Ky(t)$$

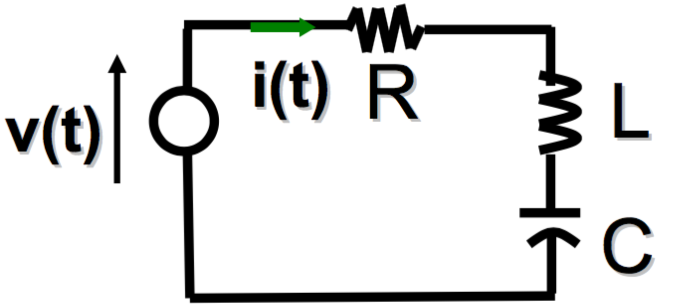
  - RLC circuit

$$v(t) = Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)dt$$

differential equation

position





- Discrete-time System
  - Digital computer
  - Daily balance of a bank account

$$y[k+1] = (1+a)y[k] + u[k]$$

difference equation

k step.

k: day 1, day 2, day 3.

y[k] : balance at k-th day
u[k] : deposit/withdrawal
a : interest rate

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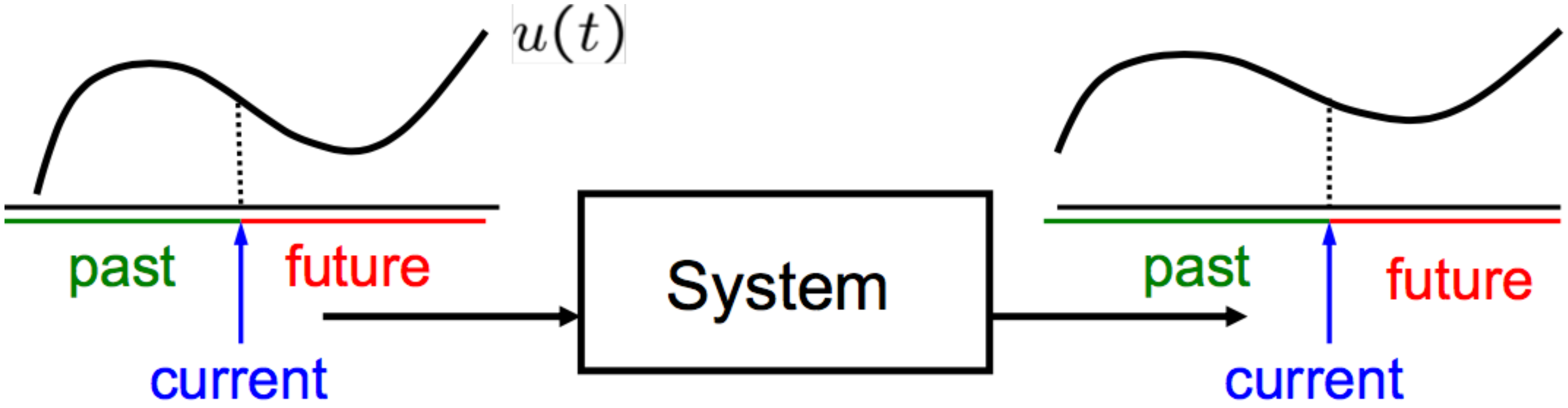
Memoryless, Causal and Noncausal

Memoryless system: Current output depends on ONLY current input.

Causal System: Current output depends on current and past input.

Noncausal system: Current output depends on future input.



Example

- Memoryless system
  - Spring: input *f(t)*, output *x(t)* ➔ $f(t) = kx(t)$
  - Resistor: input *v(t)*, output *i(t)* ➔ $v(t) = Ri(t)$
- Causal System
  - Input: acceleration; output: position of a car

Current position depends on not only current acceleration, but also all the past accelerations.

- **Noncausal System does not exist in real world; it exists only mathematically. (We only consider causal systems)**

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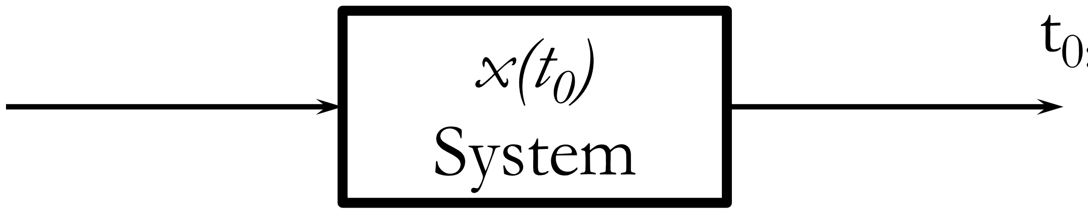
Lumped and Distributed

For a causal system,
(Current/future input)
(past input)

Current/Future output

To Memorize this info, we use a state vector $x(t_0)$

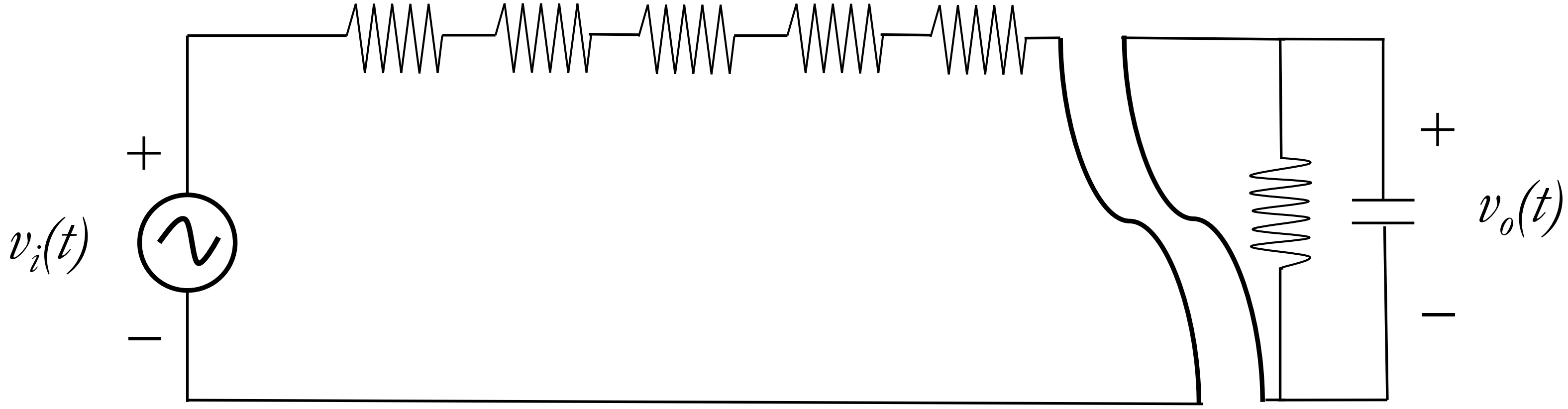


t_0: current time

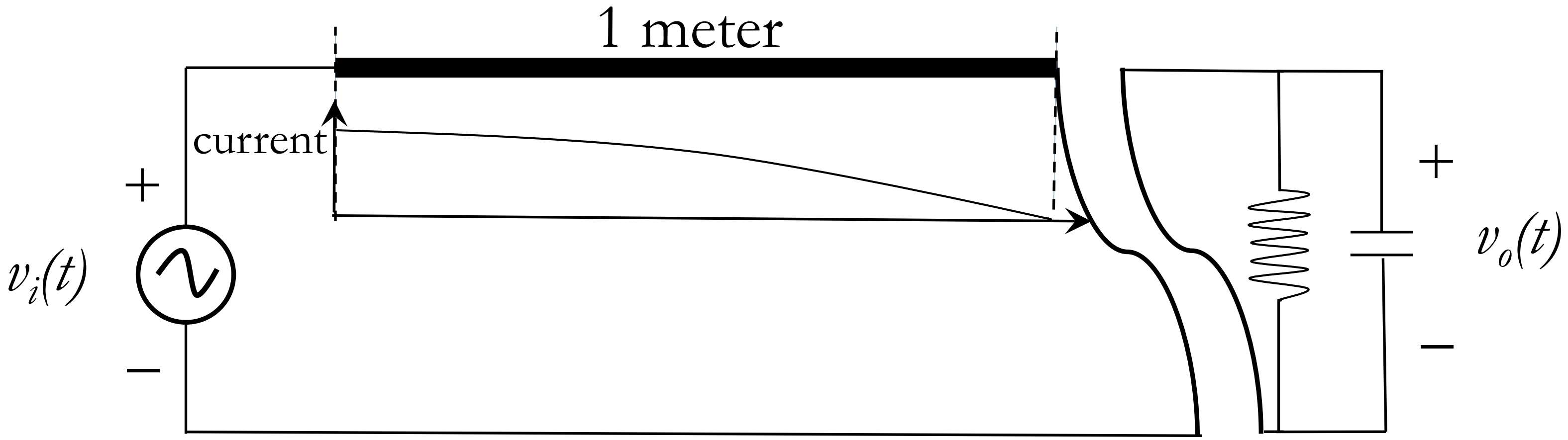
Lumped system: State vector is finite dimensional
Distributed system: State vector is infinite dimensional

Example

- Lumped System



- Distributed System



Model Classification

- Continuous-time and discrete-time
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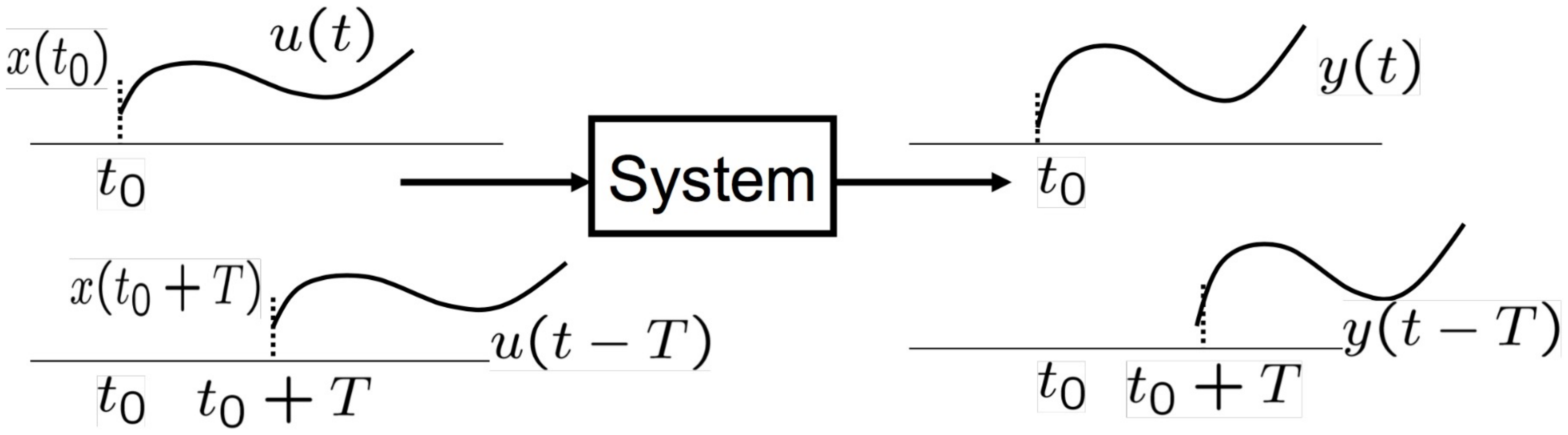
Time-invariant and Time-Varying

For a causal system, $\left.\begin{matrix} x(t_0) \\ u(t), t \geq t_0 \end{matrix}\right\} \Rightarrow y(t), t \geq t_0$

Time-invariant system: For any time shift T,

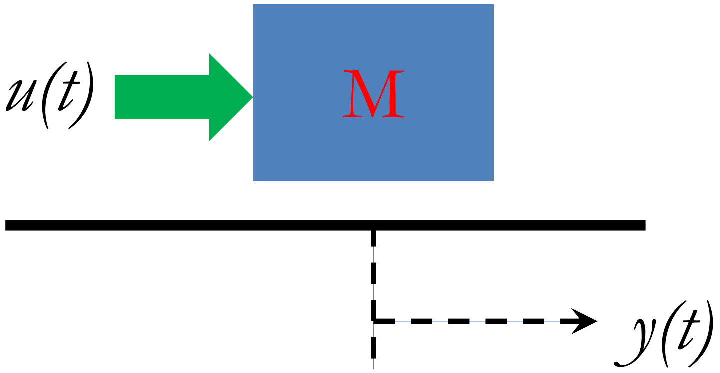
$$\left.\begin{matrix} x(t_0 + T) \\ u(t - T), t \geq t_0 + T \end{matrix}\right\} \Rightarrow y(t - T), t \geq t_0 + T$$

Time-varying system: Not time-invariant



Example

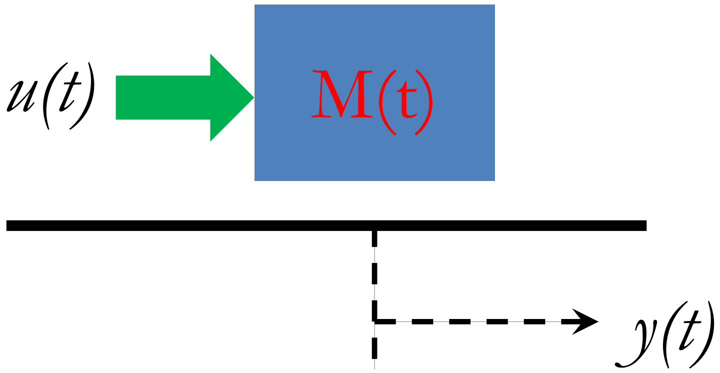
- Car, Rocket etc.



If we regard M to be constant (even though M changes very slowly), then this system is time-invariant.

$$My''(t) = u(t)$$

(Laplace applicable)



If we regard M to be Changing (due to fuel consumption), then this system is time-varying.

$$M(t)y''(t) = u(t)$$

(Laplace not applicable)

Model Classification

- Continuous-time and discrete-time
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Linear and Nonlinear

output $\curvearrowleft$ $y = f(x)$ $\nearrow$ input

For a causal system,

$$\left.\begin{matrix} x_i(t_0) \\ u_i(t), t \geq t_0 \end{matrix}\right\} \rightarrow y_i(t), t \geq t_0, i = 1, 2$$

$x = x_1$ $y_1 = f(x_1)$

$x = x_2$ $y_2 = f(x_2)$

$x = x_1 + x_2$

$f(x_1 + x_2) = y_1 + y_2$.

Linear system: A system satisfying superposition property

$$\left.\begin{matrix} \alpha_1 x_1(t_0) + \alpha_2 x_2(t_0) \\ \alpha_1 u_1 + \alpha_2 u_2(t), t \geq t_0 \end{matrix}\right\} \rightarrow \alpha_1 y_1(t) + \alpha_2 y_2(t),$$

$t \geq t_0 \; \forall \alpha_1, \alpha_2 \in \mathbb{R}$

Homogeniety

$y = f(x)$

$\alpha y = f(\alpha x)$

Nonlinear system: A system that does not satisfy superposition property.

$y = 5x$.

$x = x_1$ $y_1 = 5 \cdot x_1$

$x = x_2$ $y_2 = 5 \cdot x_2$

$x = x_1 + x_2$ $y = 5(x_1 + x_2)$

$= 5x_1 + 5x_2 = y_1 + y_2$

$y = 5x + 1$

$x = x_1$ $y_1 = 5x_1 + 1$

$x = x_2$ $y_2 = 5x_2 + 1$

$x = x_1 + x_2$ $y = 5(x_1 + x_2) + 1$

$= 5 \cdot x_1 + 5x_2 + \underline{1}$

$\neq y_1 + y_2 = 5x_1 + 5x_2 + 2$

Remarks

- All systems in real world are nonlinear.

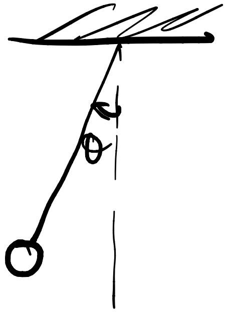
Linearization

Model $\neq$ perfect

$f(t) = Ky(t)$ ➔ This linear relation holds only for small *y(t)* and *f(t)* = Approximation

- However, linear approximation is often good enough for control purposes

- Linearization: approximation of a nonlinear system by linear system around some operating point



$mL^2\ddot{\theta}(t) = T(t) - mgL\sin\theta(t)$ $\quad mgL\theta(t)$

$\theta \to 0^\circ$

$\sin\theta \approx \theta$

$\sin\theta(t)$

↳ nonlinear term

Tayor series expension

State Space Model

Linear State-Space Models

$\begin{cases} \dot{x} = Ax + Bu & ① \\ y = Cx + Du & ② \end{cases}$

① → state equation

② → output equation

Continuous-time

$$\begin{cases} \dfrac{dx(t)}{dt} = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)x(t) + D(t)u(t) \end{cases}$$

$t \in \mathbb{R}$ (Real number)

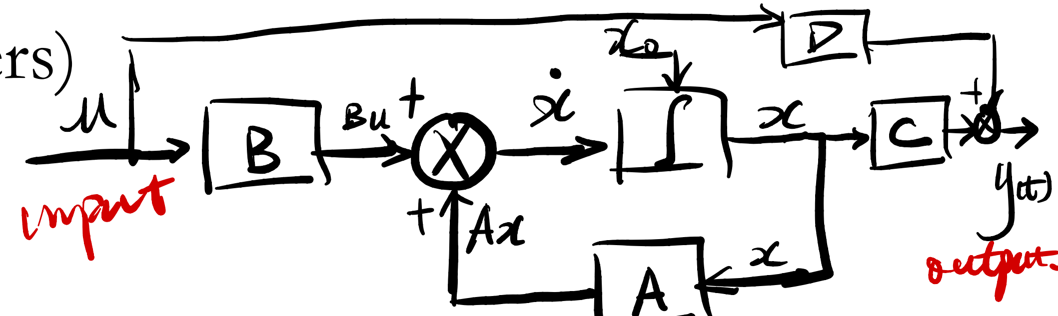
Discrete-time

$$\begin{cases} x[k+1] = A[k]x[k] + B[k]u[k] \\ y[k] = C[k]x[k] + D[k]u[k] \end{cases}$$

$k \in \mathbb{Z}$ (Integers)

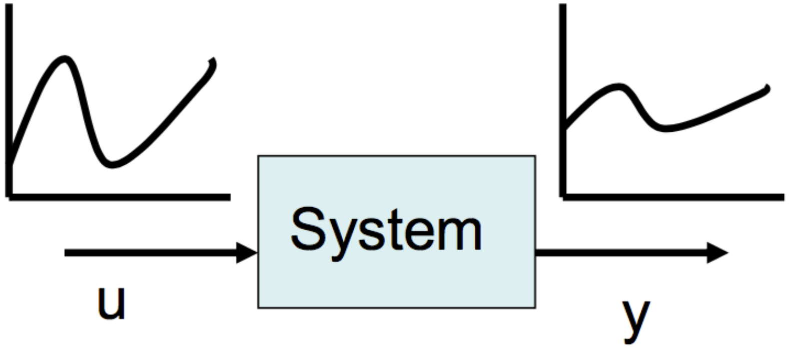
$\dot{x} = Ax + Bu$

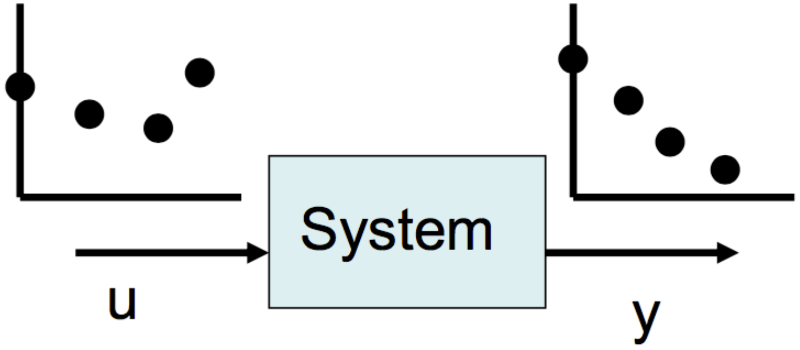
$y = Cx + Du.$



Block diagram

x: state vector
u: input vector
y: output vector





Remarks

- The first equation, called ***state equation***, is a first order ordinary differential (CT case) and difference (DT case) equation.
- The second equation, called ***output equation***, is an algebraic equation.
- Two equations are called ***state-space model***.
- If a system is ***time-invariant***, the matrices A, B, C, D are constant (independent of time).

  A, B, C, D,
- Pay attention to ***sizes of matrices and vectors***. They must by always compatible!

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du.$$

The State Space Model

n th order DE

Consider a general *n*th-order model of a dynamic system:

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + \underbrace{a_0 y(t)}_{\text{output}} = b_n\frac{d^n u(t)}{dt^n} + b_{n-1}\frac{d^{n-1}u(t)}{dt^{n-1}} + \cdots + b_1\frac{du(t)}{dt} + b_0 u(t)$$

→ input.

Assuming all initial conditions are all zeros.

n th order differential equation
to a 1st order differential equation

how to formulate the state equation

Goal: to derive a systematic procedure that transforms a differential equation of order *n* to a state space form representing a system of *n* first-order differential equations.

$\dot{x} = Ax + Bu$ 1st order DE.

Example

6th order differential equation.

Consider a dynamic system represented by the following differential equation:

$$y^{(6)} + 6y^{(5)} - 2y^{(4)} + y^{(2)} - 5y^{(1)} + 3y = 7u^{(3)} + u^{(1)} + 4u$$

where $y^{(i)}$ stands for the ith derivative: $y^{(i)} = d^i y/dt$. Find the state space model of the above system.

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

A B C D.

x is defined.

$\dot{x}$

u: input

y output

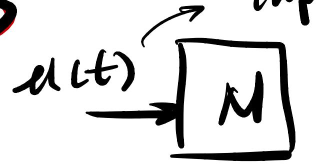
$$A_{6\times 6} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -3 & 5 & -1 & 0 & 2 & 6 \end{bmatrix} \quad B_{6\times 1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C_{1\times 6} = \begin{bmatrix} 4 & 1 & 0 & 7 & 0 & 0 \end{bmatrix} \qquad D = [0]$$

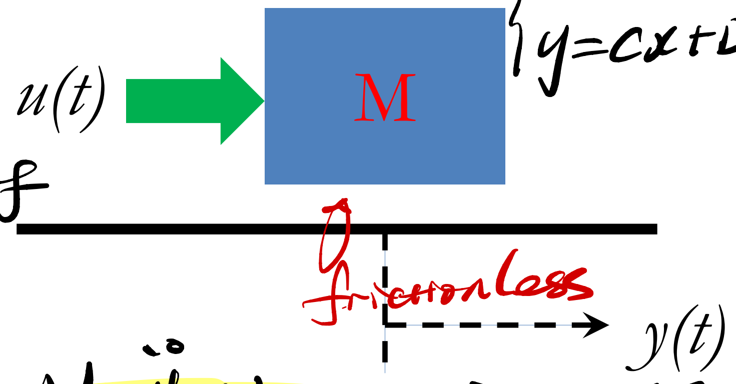
State-Space model

Example: Mass with a Driving Force

FBD



Newton's 2nd law of motion $\Sum F = ma$.



$\dot{x} = Ax + Bu$
$y = Cx + Du$

- By Newton's law, we have

$$M\ddot{y}(t) = u(t)$$

u: input force

y: output position

- Define state variables: $x_1(t) = y(t), x_2 = \dot{y}(t)$ $\quad u(t) = M \cdot \ddot{y}(t) \rightarrow \ddot{y} = \frac{1}{M}u(t)$

state $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u(t) \end{cases}$$

- Then,

$$\begin{cases} \dot{x}_1(t) = \dot{y}_1(t) = x_2(t) \\ \dot{x}_2(t) = \ddot{y}(t) = \frac{1}{M}u(t) \\ y(t) = x_1(t) \end{cases} \rightarrow \begin{cases} \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}_{2\times 2}$.

$B = \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}$

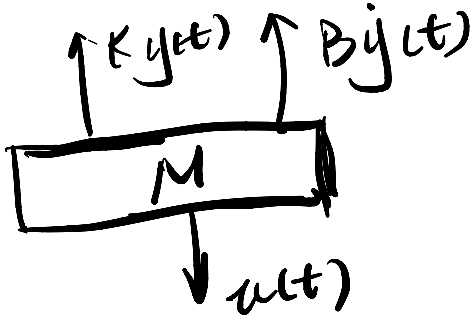
$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$ $\quad D = \begin{bmatrix} 0 \end{bmatrix}$

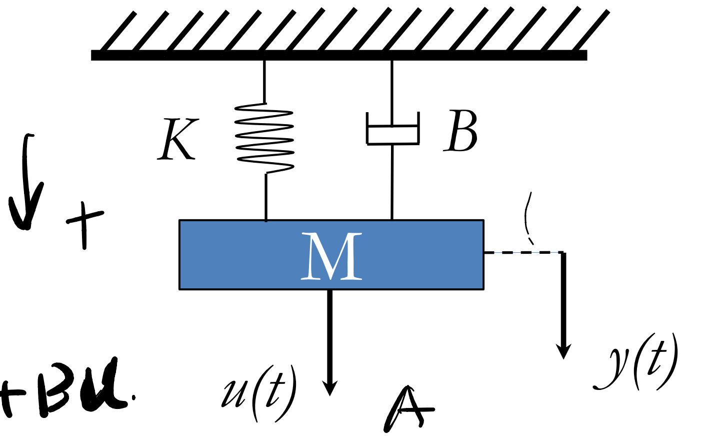
- By Newton's law

$$M\ddot{y}(t) = u(t) - B\dot{y}(t) - ky(t)$$

$$\ddot{y}(t) = \frac{1}{M}u(t) - \frac{B}{M}\dot{y}(t) - \frac{K}{M}y(t)$$

FBD.





- Define state variables

$$x_1(t) = y(t), x_2(t) = \dot{y}(t)$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} \qquad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix}$$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$\begin{cases} \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{B}{M} \end{bmatrix}}_{A} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}}_{B} u(t) \\ y(t) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{D} u(t) \end{cases}$$

$$\begin{cases} \dfrac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K/M & -B/M \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{B}{M} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

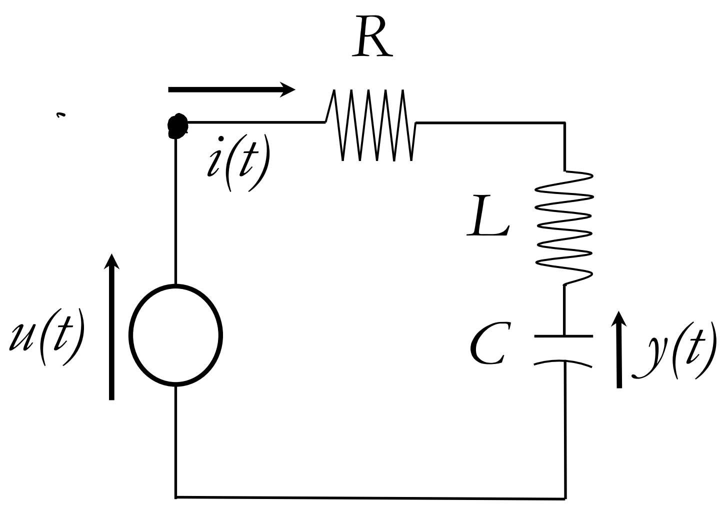
RLC Circuit

- $u(t)$: input voltage
- $y(t)$: output voltage
- By Kichhhoff's voltage law

current law (node)

(loop.)

Σ Voltage $= 0$



$$u(t) = Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(\tau)d\tau$$

*

Define State Variables (current for inductor, voltage for capacitor):

$$x_1(t) = i(t), x_2(t) = \frac{1}{C}\int i(\tau)d\tau$$

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du. \end{cases}$$

$$\begin{cases} \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -R/L & -1/L \\ 1/C & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$

2 additional questions in homework!

The End!!