GRASP Laboratory
General Robotics & Autonomous Systems and Processes

Mechatronic Modeling and Design with Applications in Robotics

Analytical Modeling (Part 2)

Transfer Function

- A system is assumed to be at rest (zero initial conditions),
- A transfer function is defined by

$s = \sigma + j\omega$. Real, imaginary, complex number, frequency.

system Transfer function

$$G(s) := \frac{C(s)}{R(s)}$$

Laplace transform of **system output**

Laplace transform of **system input**

TF of output / TF of input. $\frac{\text{Output}}{\text{Input}}$ s domain

Block diagram

$R(s) \longrightarrow \boxed{G(s)} \longrightarrow C(s)$

output TF

$C(s) = G(s)\,R(s)$.

$G(s) = \frac{C(s)}{R(s)}$

Input TF

controller, Plant

$R(s) \rightarrow \boxed{C(s)} \rightarrow \boxed{P(s)} \rightarrow C(s)$

$G(s)$

PID. DC Motor

Transfer Function

$\mathcal{L}[f(t)]$ ~~t~~ $\to s$

Note: input, system and output into three separate and distinct parts.

A general *n*th-order, linear, time-invariant differential equation:

$\mathcal{L}^{-1}[F(s)]$ $s \to t$.

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1}\frac{d^{n-1}c(t)}{dt^{n-1}} + \cdots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1}\frac{d^{m-1}r(t)}{dt^{m-1}} + \cdots + b_0 r(t)$$

where *c(t)* is the output, *r(t)* is the input.

$TF = \frac{\text{output}}{\text{input}}$.

$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$.

Assume: zero initial conditions, and take the Laplace transform on both side:

$$(a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0)\boxed{C(s)} = (b_m s^m + a_{m-1}s^{m-1} + \cdots + a_0)\boxed{R(s)}$$

output

input . output

$$\Rightarrow \quad T(s) = \frac{C(s)}{R(s)} = \frac{(b_m s^m + a_{m-1}s^{m-1} + \cdots + a_0)}{(a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0)} \quad \Rightarrow \quad \boxed{G(s)} = \frac{C(s)}{R(s)}$$

TF

TF

system TF

system

input

$f(t) = \mathcal{L}^{-1}\{C(s)\}$

$t: 0 \to \infty$

$$\boxed{C(s)} = R(s)G(s)$$

Example

Find the transfer function represented by $\frac{dc(t)}{dt} + 2c(t) = r(t)$, and use the result to find the response $c(t)$ to a unit step input with zero initial conditions.

$t: 0 \to \infty$

$\frac{dc(t)}{dt} + 2c(t) = r(t) \Rightarrow$ Laplace Transform

system TF

Analytical Model.

$\frac{C(s)}{R(s)}$

$s\,C(s) + 2C(s) = R(s) \rightarrow$ TF

$TF: G(s) = \frac{C(s)}{R(s)} = \frac{1}{(s+2)}$

$(s+2)\,C(s) = R(s)$

Input $R(s) \rightarrow$ system $\boxed{G(s)} \rightarrow C(s)$ output

$C(s) = G(s)\,R(s) = \frac{1}{(s+2)} \cdot \frac{1}{s} = \frac{1}{s(s+2)}$

partial Fractional expansion.

$\frac{1}{s(s+2)} = \frac{1}{s} \times \frac{1}{s+2} = \frac{K_1}{s} + \frac{K_2}{s+2} = \frac{K_1(s+2) + K_2 s}{s(s+2)}$, $K_1(s+2)+K_2 s = 1$

$\Rightarrow K_1(s+2) + K_2 s = 1$

$K_1 s + 2K_1 + K_2 s = 1$

$\begin{cases} K_1 + K_2 = 0 \\ 2K_1 = 1 \end{cases} \Rightarrow K_1 = \frac{1}{2}$

$K_2 = -\frac{1}{2}$

$C(s) = \frac{\frac{1}{2}}{s} - \frac{\frac{1}{2}}{s+2} \Rightarrow c(t) = \mathcal{L}^{-1}\left\{\frac{\frac{1}{2}}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{-\frac{1}{2}}{s+2}\right\}$

Laplace Transform

One of the most important math tool in the course!

Definition:

For a function $f(t)$ ($f(t) = 0 \; for \; t = 0$)

$s = \sigma + j\omega$

Real

complex

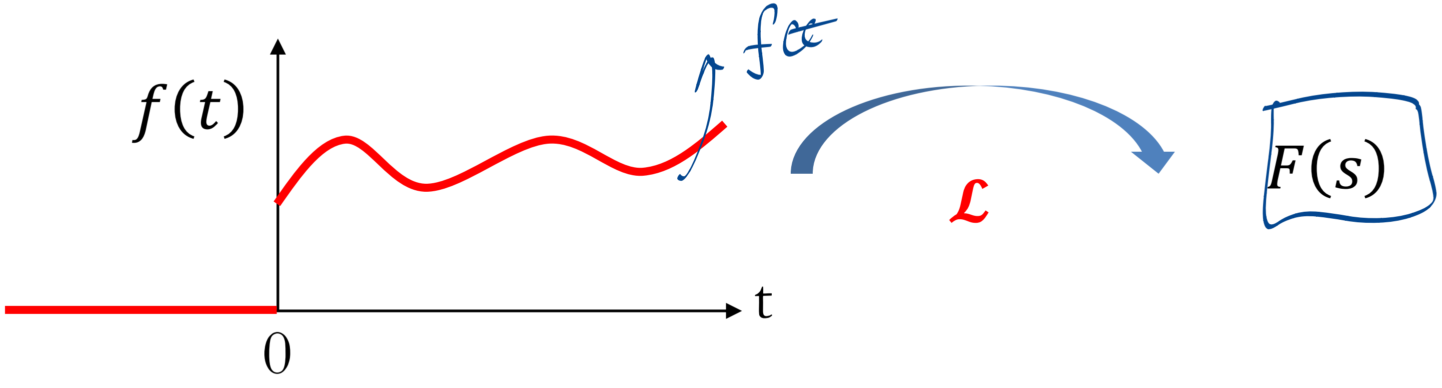
frequency

f(t)

$$\boxed{F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t) e^{-st} dt}$$

(s: complex variable)

$F(s)$ is denoted as the Laplace transform of $f(t)$



Inverse Laplace Transform

Allow us to find $f(t)$ given $F(s)$:

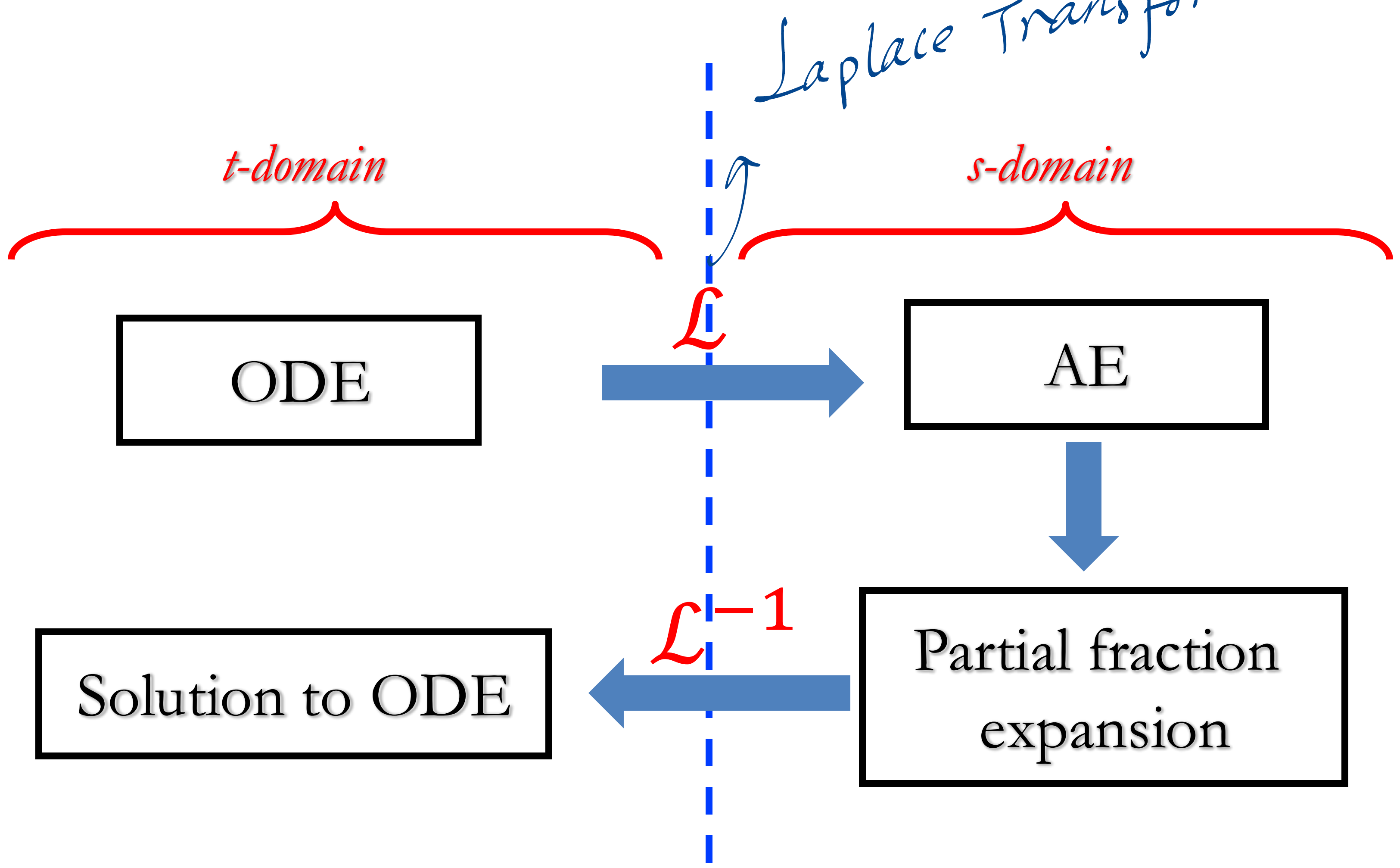
$$\mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} \boxed{F(s)} e^{st}\,ds = \boxed{f(t)}u(t)$$

where

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

An Advantage of Laplace Transform

Transform an ordinary differential equation (ODE) into an algebraic equation (AE).



Laplace Transform Table

No.	$\boldsymbol{f(t)}$ time domain	$\boldsymbol{F(s)}$ s domain
1	$\delta(t)$	1
2	$u(t)$	$\frac{1}{s}$
3	$tu(t)$ $\xrightarrow{\mathcal{L}}$	$\frac{1}{s^2}$
4	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5	$e^{-at}u(t)$ $\xleftarrow{\mathcal{L}^{-1}}$	$\frac{1}{s+a}$
6	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

Laplace Transform Theorems (Properties)

Item no.	Theorem	Name
1.	$\mathscr{L}\left[f(t)\right] = F(s) = \int_{0-}^{\infty} f(t)\, e^{-st} dt$	Definition
2.	$\mathscr{L}\left[kf(t)\right] = kF(s)$	Linearity theorem
3.	$\mathscr{L}\left[f_1(t) + f_2(t)\right] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathscr{L}\left[e^{-at} f(t)\right] = F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}\left[f(t-T)\right] = e^{-sT} F(s)$	Time shift theorem
6.	$\mathscr{L}\left[f(at)\right] = \frac{1}{a} F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2 f}{dt^2}\right] = s^2 F(s) - sf(0-) - f(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^n f}{dt^n}\right] = s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)\, d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s \to 0} sF(s)$	Final value theorem[1]
12.	$f(0+) = \lim_{s \to \infty} sF(s)$	Initial value theorem[2]

Partial-Fraction Expansion

Partial-Fraction Expansion: To convert the function to a sum of simpler terms.

E.g.,

$$F(s) = \frac{s^3 + 2s^2 + 6s + 7}{s^2 + s + 5}$$

s domain

Inverse Laplace Transform

Partial-Fraction Expansion

$$F(s) = s + 1 + \frac{2}{s^2 + s + 5}$$

Reminder:
Order of the numerator less than its denominator

$\mathcal{L}^{-1}$

$$f(t) = \mathcal{L}^{-1}\{s\} + \mathcal{L}^{-1}\{1\} + \mathcal{L}^{-1}\left\{\frac{2}{s^2 + s + 5}\right\}$$

3 Cases (Roots of the Denominator)

1. Real and Distinct

$$F(s) = \boxed{\frac{2}{(s+1)(s+2)}} \rightarrow \quad F(s) = \boxed{\frac{K_1}{(s+1)}} + \boxed{\frac{K_2}{(s+2)}} = \frac{K_1(s+2) + K_2(s+1)}{(s+1)(s+2)}$$

1 2

$K_1(s+2) + K_2(s+1) = 2.$

$K_1 s + 2K_1 + K_2 s + K_2 = 2.$

$(K_1 + K_2)s + 2K_1 + K_2 = 2$

$$\begin{cases} K_1 + K_2 = 0 \\ 2K_1 + K_2 = 2. \end{cases}$$

2. Real and Repeated

$$F(s) = \frac{2}{(s+1)(s+2)^2} \rightarrow \quad F(s) = \boxed{\frac{K_1}{(s+1)}} + \boxed{\frac{K_2}{(s+2)^2}} + \boxed{\frac{K_3}{(s+2)}}$$

1 2 3

3. Complex or Imaginary

$$F(s) = \frac{3}{s(s^2+2s+5)} \rightarrow \quad F(s) = \boxed{\frac{K_1}{s}} + \frac{K_2 s + K_3}{s^2+2s+5}$$

K_1, K_2, K_3

Solution to ODEs via Laplace Transform

Differentiation Theorem: $\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0)$; $\mathcal{L}\left\{\frac{d^2f}{dt^2}\right\} = s^2F(s) - sf(0) - f'(0)$;

$\mathcal{L}\left\{\frac{d^nf}{dt^n}\right\} = s^nF(s) - \sum_{k-1}^{n} s^{n-k}\ f^{k-1}(0)$;

Example: Given the following differential equation, solve for $y(t)$ if all initial conditions are zeros.

output. input $y(t)$?

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 4y(t) = 4u(t)$$

$s^2Y(s) + 2Y(s)s + 4Y(s) = 4U(s)$

$Y(s)(s^2 + 2 + 4) = 4U(s)$

$Y(s) = \boxed{\frac{4}{s^2+2+4}U(s)}$

Inverse Laplace Transform

$y(t)$ ✓.

Laplace Transform in s domain (output.).

$s \to t$.

Example 1

$$\Sigma \tau = J\alpha = J\ddot{\theta}$$

$$\tau(t) - K\theta(t) - B\dot{\theta}(t) = J\ddot{\theta}(t)$$

$$J\ddot{\theta}(t) + B\dot{\theta}(t) + K\theta(t) = \tau(t)$$

zero initial conditions:

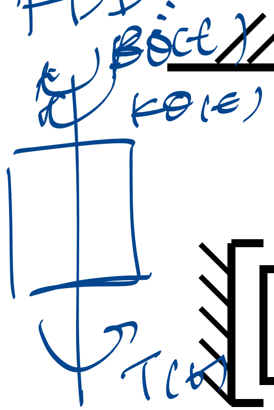
$$Js^2\Theta(s) + Bs\Theta(s) + K\Theta(s) = T(s)$$

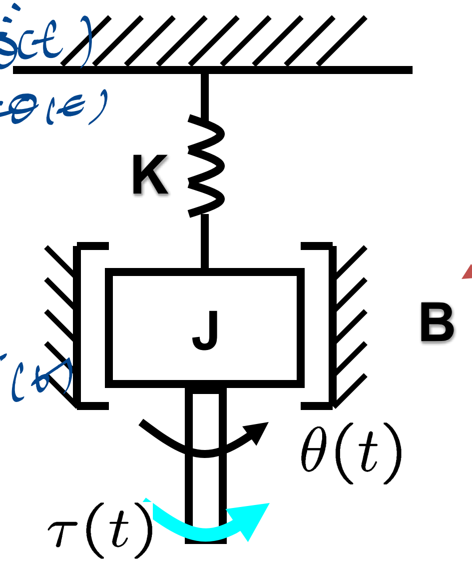
Transfer Function of the system

$$\frac{\text{output}}{\text{input}}$$

$$\underbrace{G(s) = \frac{\Theta(s)}{T(s)} = \frac{1}{Js^2 + Bs + K}}$$

FBD





friction between bob and air

Example 2

Find the transfer function relating the capacitor voltage, $V_C(s)$, to the input voltage, $V(s)$

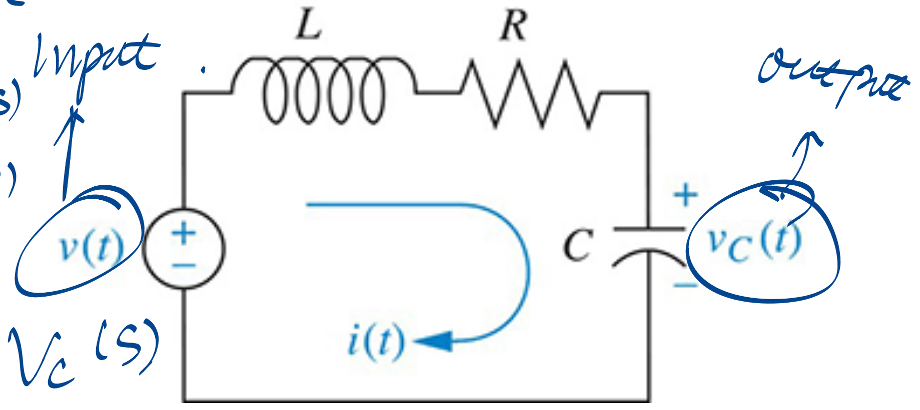
TF: $G(s) = \dfrac{V_c(s)}{V(s)}$

$V_c(t) = \frac{1}{C}\int_0^t i(\tau)\,d\tau$

$V_c(s) = \frac{1}{C}\,\frac{1}{s}\cdot I(s)$ input

$V_c(s) = \frac{1}{Cs}\cdot I(s)$

$I(s) = Cs\,V_c(s)$



$v(t) - V_L - V_R - V_c = 0$

$v(t) = L\frac{di}{dt} + i\cdot R + V_c(t)$

output

$V(s) = Ls\,I(s) + I(s)\,R + V_c(s)$

input

$V(s) = Ls\,Cs\,V_c(s) + Cs\,V_c(s)\,R + V_c(s)$.

$G(s) = \dfrac{V_c(s)}{V(s)} = \dfrac{1}{LCs^2 + RCs + 1}$

→ output

→ input

Converting a TF to State Space

input / output model in s. "t"

Assume the TF of a SISO system is as follows:

$$u \longrightarrow \boxed{G(s)} \longrightarrow y$$

$$G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \quad \text{where } m<n$$

Then its state-space model can be written below:

$$\boxed{\begin{cases}\dot{x} = Ax + Bu \\ y = Cx + Du\end{cases}} \quad \text{where } A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}_{n\times n},$$

state equation

output equation

$n=6$. 6×6. 36

A B C D

$$B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}_{n\times 1}, \quad C = [b_0 \quad b_1 \quad \cdots \quad b_m \quad 0]_{1\times n}, \quad D = [0]$$

6×1

1×6

Example

$$G(s) = \frac{2s^2 + 5s + 3}{\boxed{3}s^3 + 7s^2 - 6s + 1}$$

$$U(s) \xrightarrow{\text{input}} \boxed{G(s)} \xrightarrow{\text{output}} Y(s)$$

Please find its state-space model.

$$G(s) = \frac{\frac{2}{3}s^2 + \frac{5}{3}s + 1}{s^3 + \frac{7}{3}s^2 - 2s + \frac{1}{3}}$$

$$G(s) = \frac{\frac{2}{3}s^2+\frac{5}{3}s+1}{s^3+\frac{7}{3}s^2-2s+\frac{1}{3}}$$ (third-order system)

A, $\boxed{B}$, C D

Its state-space model: $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{3\times1} \qquad D = [0].$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{3} & 2 & -\frac{7}{3} \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & \frac{5}{3} & \frac{2}{3} \end{bmatrix}, D = [0]$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{3} & 2 & -\frac{7}{3} \end{bmatrix}_{3\times3}.$$

$$C = \begin{bmatrix} 1 & \frac{5}{3} & \frac{2}{3} \end{bmatrix}_{1\times3}$$

$TF \rightarrow SS$.

Converting from State Space to TF

Assume the state-space model of a system is as follows:

$$\boxed{\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}}$$

$f(t)$ $\mathcal{L}\{f(t)\} = F(s)$.

Take the Laplace Transform assuming zero initial conditions

input

$$\begin{cases} sX(s) = AX(s) + BU(s) \\ Y(s) = CX(s) + DU(s) \end{cases}$$

output

$G(s) = \frac{\text{output}}{\text{input}}$

Solving for X(s) in above equations

Identity Matrix.

$\boxed{X(s)} = (sI - A)^{-1}BU(s)$ where I is the identity matrix

Substitute it to $y = Cx + Du$ ➔

$$Y(s) = [C(sI - A)^{-1}B + D]U(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = C(s\boxed{I} - \boxed{A})^{-1}B + D$$

Example

$$\left\{\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} x + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + 0 \cdot u\end{aligned}\right.$$

$$\left\{\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du.\end{aligned}\right.$$

$A \quad B \quad C \quad D$

Please find its transfer function.

$$G(s) = C(sI - A)^{-1}B + D$$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \left(s \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} + [0]$$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s & -1 & 0 \\ 0 & 2 & -1 \\ 1 & 2 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} + [0]$$

$$\boxed{= \frac{10s^2+30s+20}{s^3+3s^2+2s+1}} = TF$$

$G(s)$.

SS model $\rightarrow$ TF.

s domain.
input/output

Analytical Models.
$\left\{\begin{array}{l} \text{TF} \\ \text{SS} \end{array}\right.$

$t.$ $\left\{\begin{array}{l} \text{state equation} \\ \text{output equation} \end{array}\right.$

The End!!