Chapter 1

Example problem: 2D driven cavity flow in a quarter-circle domain with spatial adaptation.

In this example we shall demonstrate

- how easy it is to adapt the code for the solution of the driven cavity problem in a square domain, discussed in a `previous example` , to a different domain shape,
- how to apply body forces (e.g. gravity) in a Navier-Stokes problem,
- how to switch between the stress-divergence and the simplified forms of the incompressible Navier-Stokes equations.

1.1 The example problem

In this example we shall illustrate the solution of the steady 2D Navier-Stokes equations in a modified driven cavity problem: The fluid is contained in a quarter-circle domain and is subject to gravity which acts in the vertical direction. We solve the problem in two different formulations, using the stress-divergence and the simplified form of the Navier-Stokes equations, respectively, and by applying the gravitational body force via the gravity vector, $\mathbf{G}$, and via the body force function, $\mathbf{B}$, respectively.

Problem 1:

The 2D driven cavity problem in a quarter circle domain with gravity, using the stress-divergence form of the Navier-Stokes equations

Solve

$$Re\, u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{Re}{Fr} G_i + \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad (1)$$

and

$$\frac{\partial u_i}{\partial x_i} = 0,$$

in the quarter-circle domain $D = \{x_1 \geq 0, x_2 \geq 0 \text{ and } x_1^2 + x_2^2 \leq 1\}$, subject to the Dirichlet boundary conditions

$$\mathbf{u}|_{\partial D} = (0,0), \qquad (2)$$

on the curved and left boundaries; and

$$\mathbf{u}|_{\partial D} = (1,0), \qquad (3)$$

on the bottom boundary, $x_2 = 0$. Gravity acts vertically downwards so that $(G_1, G_2) = (0, -1)$.

When discussing the implementation of the Navier-Stokes equations in an earlier example , we mentioned that oomph-lib allows the incompressible Navier-Stokes equations to be solved in the simplified, rather than the (default) stress-divergence form. We will demonstrate the use of this feature by solving the following problem:

Problem 2:

The 2D driven cavity problem in a quarter circle domain with gravity, using the simplified form of the Navier-Stokes equations

Solve

$$Re\ u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + B_i + \frac{\partial^2 u_i}{\partial x_j^2}, \qquad (1)$$

and

$$\frac{\partial u_i}{\partial x_i} = 0,$$

in the quarter-circle domain $D = \{x_1 \geq 0, x_2 \geq 0 \text{ and } x_1^2 + x_2^2 \leq 1\}$, subject to the Dirichlet boundary conditions

$$\mathbf{u}|_{\partial D} = (0,0), \qquad (2)$$

on the curved and left boundaries; and

$$\mathbf{u}|_{\partial D} = (1,0), \qquad (3)$$

on the bottom boundary, $x_2 = 0$. To make this consistent with Problem 1, we define the body force function as $(B_1, B_2) = (0, -Re/Fr)$.

Note that in Problem 2, the gravitational body force is represented by the body force rather than the gravity vector.

1.1.1 Switching between the stress-divergence and the simplified forms of the Navier-Stokes equations

The two forms of the Navier-Stokes equations differ in the implementation of the viscous terms, which may be represented as

$$\frac{\partial^2 u_i}{\partial x_j^2} \quad \text{or} \quad \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

For an incompressible flow, $\partial u_i/\partial x_i = 0$, both forms are mathematically equivalent but the stress-divergence form is required for problems with free surfaces , or for problems in which traction boundary conditions are to be applied.

In order to be able do deal with both cases, oomph-lib's Navier-Stokes elements actually implement the viscous term as

$$\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \Gamma_i \frac{\partial u_j}{\partial x_i}\right).$$

By default the components of the vector Γ_i, are set to 1.0, so that the stress-divergence form is used. The components Γ_i are stored in the static data member

```
static Vector<double> NavierStokesEquations<DIM>::Gamma
```

of the NavierStokesEquations<DIM> class which forms the basis for all Navier-Stokes elements in oomph-lib. Its entries are initialised to 1.0. The user may over-write these assignments and thus re-define the values of Γ being used for a specific problem. [In principle, it is possible to use stress-divergence form for the first component of the momentum equations, and the simplified form for the second one, say. However, we do not believe that this is a particularly useful/desirable option and have certainly never used such (slightly bizarre) assignments in any of our own computations.]

1.1.2 Solution to problem 1

The figure below shows "carpet plots" of the velocity and pressure fields as well as a contour plot of the pressure distribution with superimposed streamlines for Problem 1 at a Reynolds number of $Re = 100$ and a ratio of Reynolds and Froude numbers (a measure of gravity on the viscous scale) of $Re/Fr = 100$. The velocity vanishes along the entire domain boundary, apart from the bottom boundary $(x_2 = 0)$ where the moving "lid" imposes a unit tangential velocity which drives a large vortex, centred at $(x_1, x_2) \approx (0.59, 0.22)$. The pressure singularities created by the velocity discontinuities at $(x_1, x_2) = (0, 0)$ and $(x_1, x_2) = (1, 0)$ are well resolved. The pressure plot shows that away from the singularities, the pressure decreases linearly with x_2, reflecting the effect of the gravitational body forces which acts in the negative x_2- direction.

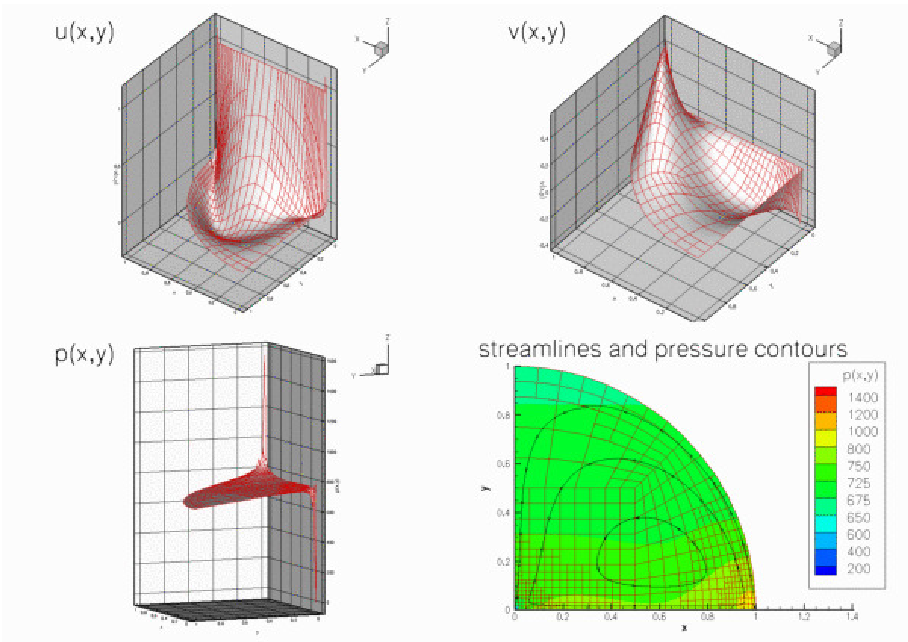

Figure 1.1 Plot of the velocity and pressure fields for problem 1 with Re=100 and Re/Fr=100, computed with adaptive Taylor-Hood elements.

1.1.3 Solution to problem 2

The next figure shows the computational results for Problem 2, obtained from a computation with adaptive Crouzeix-Raviart elements.

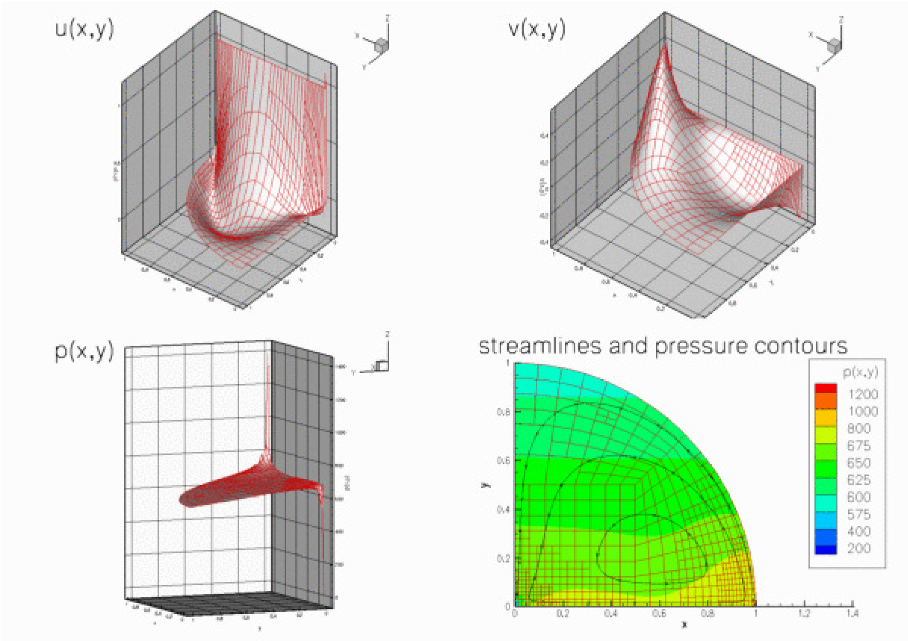

Figure 1.2 Plot of the velocity and pressure fields for problem 2 with Re=100 and Re/Fr=100, computed with adaptive Crouzeix-Raviart elements.

1.2 The code

We use a namespace `Global_Physical_Variables` to define the various parameters: The Reynolds number,

```
//==start_of_namespace==================================================
/// Namespace for physical parameters
//======================================================================
namespace Global_Physical_Variables
{
 /// Reynolds number
 double Re=100;
```

the gravity vector $\mathbf{G}$, and the ratio of Reynolds and Froude number, Re/Fr, which represents the ratio of gravitational and viscous forces,

```
 /// Reynolds/Froude number
 double Re_invFr=100;

 /// Gravity vector
 Vector<double> Gravity(2);
```

In Problem 2, gravity is introduced via the body force function $\mathbf{B}$ which we define such that Problems 1 and 2 are equivalent. (We use the gravity vector $\mathbf{G} = (0, -1)$ to specify the direction of gravity, while indicating it magnitude by Re/Fr.)

```
 /// Functional body force
 void body_force(const double& time, const Vector<double>& x,
                 Vector<double>& result)
 {
  result[0]=0.0;
  result[1]=-Re_invFr;
 }
```

Finally we define a body force function, which returns zero values, for use when solving Problem 1.

```
/// Zero functional body force
void zero_body_force(const double& time, const Vector<double>& x,
                     Vector<double>& result)
{
 result[0]=0.0;
 result[1]=0.0;
}

} // end_of_namespace
```

1.3 The driver code

First we create a DocInfo object to control the output, and set the maximum number of spatial adaptations to three.

```
//==start_of_main======================================================
/// Driver for QuarterCircleDrivenCavityProblem test problem
//=====================================================================
int main()
{

 // Set output directory and initialise count
 DocInfo doc_info;
 doc_info.set_directory("RESLT");

 // Set max. number of black-box adaptation
 unsigned max_adapt=3;
```

To solve problem 1 we define the direction of gravity, $\mathbf{G} = (0, -1)$, and set the entries in the NavierStokes←Equations<2>::Gamma vector to $(1, 1)$, so that the stress-divergence form of the equation is used [In fact, this step is not strictly necessary as it simply re-assigns the default values.]

```
 // Solve problem 1 with Taylor-Hood elements
 //--------------------------------------------
 {
  // Set up downwards-Gravity vector
  Global_Physical_Variables::Gravity[0] = 0.0;
  Global_Physical_Variables::Gravity[1] = -1.0;

  // Set up Gamma vector for stress-divergence form
  NavierStokesEquations<2>::Gamma[0]=1;
  NavierStokesEquations<2>::Gamma[1]=1;
```

Next we build problem 1 using Taylor-Hood elements and passing a function pointer to the zero_body_force(...) function (defined in the namespace Global_Physical_Variables) as the argument.

```
  // Build problem with Gravity vector in stress divergence form,
  // using zero body force function
  QuarterCircleDrivenCavityProblem<RefineableQTaylorHoodElement<2>
       >
   problem(&Global_Physical_Variables::zero_body_force);
```

Now problem 1 can be solved as in the previous example.

```
 // Solve the problem with automatic adaptation
 problem.newton_solve(max_adapt);

 // Step number
 doc_info.number()=0;

 // Output solution
 problem.doc_solution(doc_info);

} // end of problem 1
```

To solve problem 2 we set the entries in the `NavierStokesEquations<2>::Gamma` vector to zero (thus choosing the simplified version of the Navier-Stokes equations), define $\mathbf{G} = (0,0)$, and pass a function pointer to the `body_force`(...) function to the problem constructor.

```
// Solve problem 2 with Taylor Hood elements
//--------------------------------------------
{
 // Set up zero-Gravity vector
 Global_Physical_Variables::Gravity[0] = 0.0;
 Global_Physical_Variables::Gravity[1] = 0.0;

 // Set up Gamma vector for simplified form
 NavierStokesEquations<2>::Gamma[0]=0;
 NavierStokesEquations<2>::Gamma[1]=0;

 // Build problem with body force function and simplified form,
 // using body force function
 QuarterCircleDrivenCavityProblem<RefineableQTaylorHoodElement<2>
      >
  problem(&Global_Physical_Variables::body_force);
```

Problem 2 may then be solved as before.

```
 // Solve the problem with automatic adaptation
 problem.newton_solve(max_adapt);

 // Step number
 doc_info.number()=1;

 // Output solution
 problem.doc_solution(doc_info);

} // end of problem 2

} // end_of_main
```

1.4 The problem class

The problem class is very similar to that used in the `previous example` , with two exceptions:

- We pass a function pointer to the body force function $\mathbf{B}$ to the constructor and
- store the function pointer to the body force function in the problem's private member data.

```
//==start_of_problem_class============================================
/// Driven cavity problem in quarter circle domain, templated
/// by element type.
//====================================================================
template<class ELEMENT>
class QuarterCircleDrivenCavityProblem : public Problem
{

public:

 /// Constructor
 QuarterCircleDrivenCavityProblem(
  NavierStokesEquations<2>::NavierStokesBodyForceFctPt body_force_fct_pt);

 /// Destructor: Empty
 ~QuarterCircleDrivenCavityProblem() {}

 /// Update the after solve (empty)
 void actions_after_newton_solve() {}

 /// \short Update the problem specs before solve.
 /// (Re-)set velocity boundary conditions just to be on the safe side...
 void actions_before_newton_solve()
  {
  // Setup tangential flow along boundary 0:
  unsigned ibound=0;
  unsigned num_nod= mesh_pt()->nboundary_node(ibound);
  for (unsigned inod=0;inod<num_nod;inod++)
   {
    // Tangential flow
    unsigned i=0;
    mesh_pt()->boundary_node_pt(ibound,inod)->set_value(i,1.0);
    // No penetration
    i=1;
    mesh_pt()->boundary_node_pt(ibound,inod)->set_value(i,0.0);
   }

  // Overwrite with no flow along all other boundaries
  unsigned num_bound = mesh_pt()->nboundary();
  for(unsigned ibound=1;ibound<num_bound;ibound++)
   {
    unsigned num_nod= mesh_pt()->nboundary_node(ibound);
    for (unsigned inod=0;inod<num_nod;inod++)
     {
      for (unsigned i=0;i<2;i++)
       {
        mesh_pt()->boundary_node_pt(ibound,inod)->set_value(i,0.0);
       }
     }
   }
  } // end_of_actions_before_newton_solve


 /// After adaptation: Unpin pressure and pin redudant pressure dofs.
 void actions_after_adapt()
  {
   // Unpin all pressure dofs
   RefineableNavierStokesEquations<2>::
    unpin_all_pressure_dofs(mesh_pt()->element_pt());

   // Pin redundant pressure dofs
   RefineableNavierStokesEquations<2>::
    pin_redundant_nodal_pressures(mesh_pt()->element_pt());

   // Now pin the first pressure dof in the first element and set it to 0.0
   fix_pressure(0,0,0.0);
  } // end_of_actions_after_adapt

 /// Doc the solution
 void doc_solution(DocInfo& doc_info);

private:

 /// Pointer to body force function
 NavierStokesEquations<2>::NavierStokesBodyForceFctPt Body_force_fct_pt;

 /// Fix pressure in element e at pressure dof pdof and set to pvalue
 void fix_pressure(const unsigned &e, const unsigned &pdof,
                   const double &pvalue)
  {
   //Cast to proper element and fix pressure
   dynamic_cast<ELEMENT*>(mesh_pt()->element_pt(e))->
                          fix_pressure(pdof,pvalue);
  } // end_of_fix_pressure

}; // end_of_problem_class
```

1.5 The problem constructor

We store the function pointer to the body force function in the private data member `Body_force_fct_pt`.

```
//==start_of_constructor==================================================
/// Constructor for driven cavity problem in quarter circle domain
//========================================================================
template<class ELEMENT>
QuarterCircleDrivenCavityProblem<ELEMENT>::QuarterCircleDrivenCavityProblem
      (
 NavierStokesEquations<2>::NavierStokesBodyForceFctPt body_force_fct_pt) :
 Body_force_fct_pt(body_force_fct_pt)
```

As usual the first task is to create the mesh. We now use the `RefineableQuarterCircleSectorMesh<ELEMENT>`, which requires the creation of a `GeomObject` to describe geometry of the curved wall: We choose an ellipse with unit half axes (i.e. a unit circle).

```
{

 // Build geometric object that parametrises the curved boundary
 // of the domain

 // Half axes for ellipse
 double a_ellipse=1.0;
 double b_ellipse=1.0;

 // Setup elliptical ring
 GeomObject* Wall_pt=new Ellipse(a_ellipse,b_ellipse);

 // End points for wall
 double xi_lo=0.0;
 double xi_hi=2.0*atan(1.0);

 //Now create the mesh
 double fract_mid=0.5;
 Problem::mesh_pt() = new
  RefineableQuarterCircleSectorMesh<ELEMENT>(
   Wall_pt,xi_lo,fract_mid,xi_hi);
```

Next the error estimator is set, the boundary nodes are pinned and the Reynolds number is assigned, as before.

```
 // Set error estimator
 Z2ErrorEstimator* error_estimator_pt=new Z2ErrorEstimator;
 dynamic_cast<RefineableQuarterCircleSectorMesh<ELEMENT>*>(
  mesh_pt())->spatial_error_estimator_pt()=error_estimator_pt;

 // Set the boundary conditions for this problem: All nodes are
 // free by default -- just pin the ones that have Dirichlet conditions
 // here: All boundaries are Dirichlet boundaries.
 unsigned num_bound = mesh_pt()->nboundary();
 for(unsigned ibound=0;ibound<num_bound;ibound++)
  {
   unsigned num_nod= mesh_pt()->nboundary_node(ibound);
   for (unsigned inod=0;inod<num_nod;inod++)
    {
     // Loop over values (u and v velocities)
     for (unsigned i=0;i<2;i++)
      {
       mesh_pt()->boundary_node_pt(ibound,inod)->pin(i);
      }
    }
  } // end loop over boundaries

 //Find number of elements in mesh
 unsigned n_element = mesh_pt()->nelement();

 // Loop over the elements to set up element-specific
 // things that cannot be handled by constructor: Pass pointer to Reynolds
 // number
 for(unsigned e=0;e<n_element;e++)
  {
   // Upcast from GeneralisedElement to the present element
   ELEMENT* el_pt = dynamic_cast<ELEMENT*>(mesh_pt()->element_pt(e));

   //Set the Reynolds number, etc
   el_pt->re_pt() = &Global_Physical_Variables::Re;
```

Within this loop we also pass the pointers to Re/Fr, the gravity vector and the body-force function to the elements.

```
   //Set the Re/Fr
   el_pt->re_invfr_pt() = &Global_Physical_Variables::Re_invFr;
   //Set Gravity vector
   el_pt->g_pt() = &Global_Physical_Variables::Gravity;
   //set body force function
   el_pt->body_force_fct_pt() = Body_force_fct_pt;

  } // end loop over elements
```

The RefineableQuarterCircleSectorMesh<ELEMENT> contains only three elements and therefore provides a very coarse discretisation of the domain. We refine the mesh uniformly twice before pinning the redundant pressure degrees of freedom, pinning a single pressure degree of freedom, and assigning the equation numbers, as before.

```
 // Initial refinement level
 refine_uniformly();
 refine_uniformly();

 // Pin redudant pressure dofs
 RefineableNavierStokesEquations<2>::
  pin_redundant_nodal_pressures(mesh_pt()->element_pt());

 // Now pin the first pressure dof in the first element and set it to 0.0
 fix_pressure(0,0,0.0);

 // Setup equation numbering scheme
 cout <<"Number of equations: " << assign_eqn_numbers() << std::endl;

} // end_of_constructor
```

1.6 Post processing

The post processing function remains the same as in the previous examples .

```
//==start_of_doc_solution=================================================
/// Doc the solution
//========================================================================
template<class ELEMENT>
void QuarterCircleDrivenCavityProblem<ELEMENT>::doc_solution
      (DocInfo& doc_info)
{

 ofstream some_file;
 char filename[100];

 // Number of plot points
 unsigned npts=5;


 // Output solution
 sprintf(filename,"%s/soln%i.dat",doc_info.directory().c_str(),
         doc_info.number());
 some_file.open(filename);
 mesh_pt()->output(some_file,npts);
 some_file.close();

} // end_of_doc_solution
```

1.7 Comments and Exercises

1. Try making the curved boundary the driving wall [Hint: this requires a change in the wall velocities prescribed in `Problem::actions_before_newton_solve()`. The figure below shows what you should expect.]

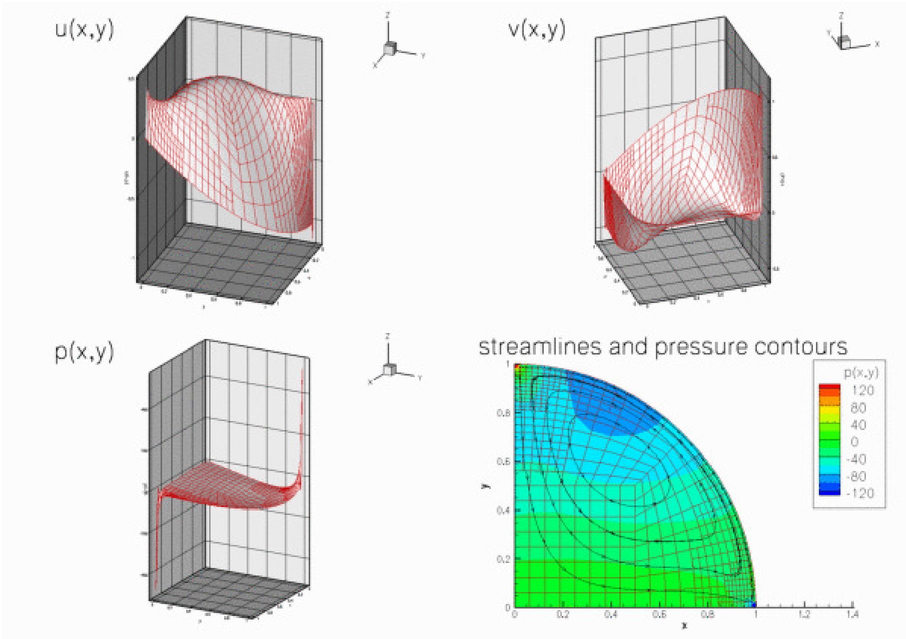

Figure 1.3 Plot of the velocity and pressure distribution for a circular driven cavity in which the flow is driven by the tangential motion of the curvilinear boundary.

1.8 Source files for this tutorial

- The source files for this tutorial are located in the directory:

  `demo_drivers/navier_stokes/circular_driven_cavity/`

- The driver code is:

  `demo_drivers/navier_stokes/circular_driven_cavity/circular_driven_cavity.cc`

1.9 PDF file

A `pdf version` of this document is available.