Automatique Linéaire I

TD 1 : Etude des systèmes linéaires invariants

Exercice 1 :

$$\boxed{m\ddot{x}(t) + c\dot{x}(t) + k\,x(t) = e(t)}$$

→ Le système est linéaire invariant d'ordre 2.

2- Fonction de Transfert $F(p)$

$$TL\left[m\ddot{x}(t)+c\dot{x}(t)+k\,x(t)\right]=TL\left[e(t)\right]$$

$$m\,TL\left[\ddot{x}(t)\right]+c\,TL\left[\dot{x}(t)\right]+k\,TL\left[x(t)\right]=TL\left[e(t)\right]$$

$$\begin{cases} TL\left[\ddot{x}(t)\right]=p^2\,TL\left[x(t)\right]-p\,x(0)-\dot{x}(0) \\ TL\left[\dot{x}(t)\right]=p\,TL\left[x(t)\right]-x(0) \end{cases}$$

En considerant les condition initiales $x(0)=0$ et $\dot{x}(0)=0$
et en notant $TL\left[x(t)\right]=X(p)$ et $TL\left[e(t)\right]=E(p)$.

$$m\,p^2X(p)+c\,p\,X(p)+k\,X(p)=E(p)$$

$$\boxed{F(p)=\frac{X(p)}{E(p)}=\frac{1}{mp^2+cp+k}}$$

3- Deduction de la fonction de Transfert à partir de la réponse impulsionnelle.

$$F(p)=TL\left[h(t)\right]$$

$$=TL\left[\frac{1}{\sqrt{c^2-4km}}\left(e^{p_1t}-e^{p_2t}\right)\right]$$

$$=\frac{1}{\sqrt{c^2-4km}}\cdot\left(TL\left[e^{p_1t}\right]-TL\left[e^{p_2t}\right]\right)$$

Rappel : $TL\left[e^{at}\right] = \dfrac{1}{p-a}$

Ainsi

$$F(p) = \frac{1}{\sqrt{c^2-4km}}\left[\frac{1}{p-p_1} - \frac{1}{p-p_2}\right]$$

$$F(p) = \frac{1}{\sqrt{c^2-4km}}\left[\frac{p_1-p_2}{(p-p_1)(p-p_2)}\right]$$

$$p_1 - p_2 = -\frac{c}{2m} + \frac{1}{2m}\sqrt{c^2-4km} + \frac{c}{2m} + \frac{1}{2m}\sqrt{c^2-4km}.$$

$$p_1 - p_2 = \frac{1}{m}\sqrt{c^2-4km}$$

$$(p-p_1)(p-p_2) = p^2 - (p_1+p_2)p + p_1 p_2$$

$$= p^2 - \left(-\frac{c}{m}\right)p + k/m.$$

Ainsi

$$F(p) = \frac{1}{\sqrt{c^2-4km}}\;\frac{1/m\sqrt{c^2-4km}}{p^2 + c/m\,p + k/m} = \frac{1/m}{p^2 + c/m\,p + k/m}$$

$$\boxed{F(p) = \frac{1}{mp^2 + cp + k}}$$

4- Pôles, zéros et gain statique de $F(p)$

<u>Pôles</u>: solutions de l'équation $mp^2+cp+k=0$.

$\Delta = c^2-4km$

Sachant que $c^2>4km$, alors $\Delta>0$.

Les pôles sont ainsi $\boxed{\begin{cases} p_1 = \dfrac{-c-\sqrt{c^2-4km}}{2m} \\ p_2 = \dfrac{-c+\sqrt{c^2-4km}}{2m} \end{cases}}$

<u>Zéros</u>: $\boxed{\text{pas de zéros}}$

<u>Gain statique</u>: $GS = F(0)$

$$\boxed{GS = \frac{1}{k}}$$

5- Diagramme de Bode asymptotique

$$F(p) = \frac{1/m}{(p-p_1)(p-p_2)} \quad \text{avec } p_1 > p_2$$

$$= \frac{\frac{1}{m p_1 p_2}}{\left(1-\frac{p}{p_1}\right)\left(1-\frac{p}{p_2}\right)}$$

$p = j\omega$

$$F(\omega) = \frac{\frac{1}{m\omega_1\omega_2}}{\left(1+j\frac{\omega}{\omega_1}\right)\left(1+j\frac{\omega}{\omega_2}\right)} \quad \text{avec } \begin{cases} \omega_1 = -p_1 \\ \omega_2 = -p_2 \end{cases} \quad \omega_2 > \omega_1$$

$$F(\omega) = \underbrace{\frac{\frac{1}{m\omega_1\omega_2}}{\left(1+j\frac{\omega}{\omega_1}\right)}}_{F_1(\omega)} \cdot \underbrace{\frac{1}{1+j\frac{\omega}{\omega_2}}}_{F_2(\omega)}$$

<u>$F_1(\omega)$</u> : gain

$$|F_1(\omega)|_{db} = 20\log_{10}\left[|F_1(\omega)|\right] = 20\log_{10}\left[\frac{\frac{1}{m\omega_1\omega_2}}{\sqrt{1+\left(\frac{\omega}{\omega_1}\right)^2}}\right]$$

$$|F_1(\omega)|_{db} = 20\log_{10}\left[\frac{1}{m\omega_1\omega_2}\right] - 10\log_{10}\left[1+\left(\frac{\omega}{\omega_1}\right)^2\right]$$

- $\omega \ll \omega_1$: $|F_1(\omega)|_{db} \to 20\log_{10}\left[\frac{1}{m\omega_1\omega_2}\right]$
- $\omega \gg \omega_1$: $|F_1(\omega)|_{db} \longrightarrow 20\log_{10}\left[\frac{1}{m\omega_1\omega_2}\right] - 20\log_{10}\left[\frac{\omega}{\omega_1}\right]$

Phase: $\varphi[F_1(\omega)] = -\arctg\left[\frac{\omega}{\omega_1}\right]$

- $\omega \ll \omega_1$: $\varphi[F_1(\omega)] \to 0$
- $\omega \gg \omega_1$: $\varphi[F_1(\omega)] \to -\pi/2$

$F_2(\omega)$ gain

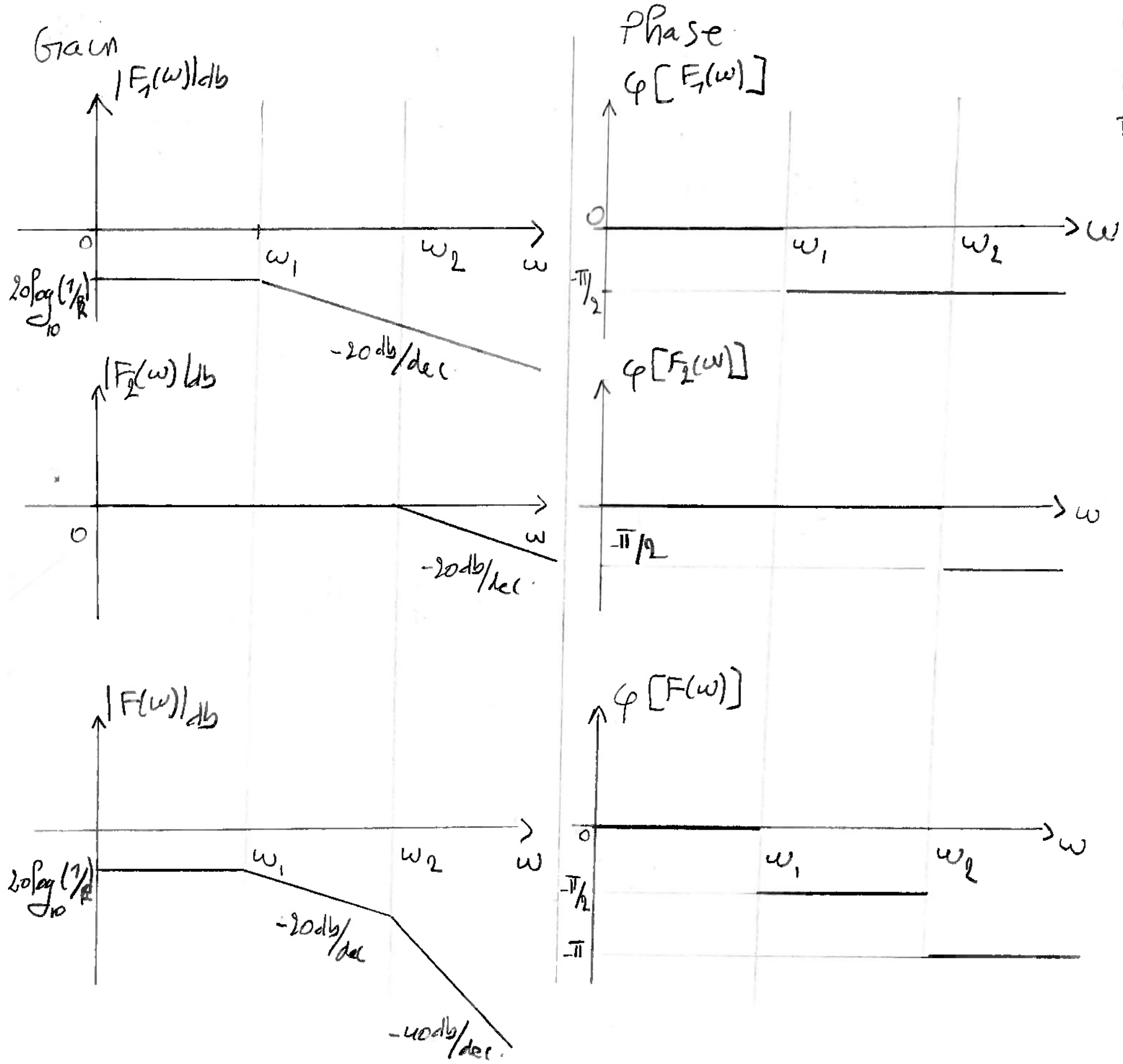
$$|F_2(\omega)|_{db} = 20\log_{10}\left[|F_2(\omega)|\right] = 20\log_{10}\left[\frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_2}\right)^2}}\right]$$

$$|F_2(\omega)|_{db} = -10\log_{10}\left[1+\left(\frac{\omega}{\omega_2}\right)^2\right]$$

- $\omega \ll \omega_2$: $|F_2(\omega)|_{db} \to 0$
- $\omega \gg \omega_2$: $|F_2(\omega)|_{db} \to -20\log_{10}\left(\frac{\omega}{\omega_2}\right)$

Phase $\varphi[F_2(\omega)] = -\arctg\left[\frac{\omega}{\omega_2}\right]$

- $\omega \ll \omega_2$: $\varphi[F_2(\omega)] \to 0$
- $\omega \gg \omega_2$: $\varphi[F_2(\omega)] \to -\pi/2$



$$\frac{1}{m\omega_1\omega_2} = \frac{1}{k}$$

$20\log_{10}(1/k) < 0$ car $k > \underline{1}$

Exercice 2 : Tracés dans le plan de Bode

1. $G_1(p) = \dfrac{K}{p^2(1+\tau p)}$ avec $0 < K < 1$ et $0 < \tau < 1$

$G_1(p) = \dfrac{K}{p^2\left(1 - \dfrac{p}{p_1}\right)}$ avec $p_1 = -1/\tau$

$p = j\omega$

$G_1(\omega) = \underbrace{\dfrac{-1}{(j\omega)^2}}_{G_{11}} \cdot \underbrace{\dfrac{K}{1 + j\dfrac{\omega}{\omega_1}}}_{G_{12}}$ avec $\omega_1 = -p_1 = 1/\tau$

G_{11} gain

$|G_{11}(\omega)|_{db} = 20\log_{10}\left|\dfrac{1}{\omega^2}\right| = -20\log_{10}(\omega^2) = \underbrace{-40\log_{10}(\omega)}_{-40db/dec}$

phase : $\varphi[G_{11}(\omega)] = -\arg(j\omega)^2 = -2\,\mathrm{arctg}[\omega]$

$\omega \to \infty$: $\varphi[G_{11}(\omega)] \to -2 \cdot \pi/2 = -\pi$

G_{12} gain

$|G_{12}(\omega)|_{db} = 20\log_{10}(K) - 10\log_{10}\left[1 + \left(\dfrac{\omega}{\omega_1}\right)^2\right]$

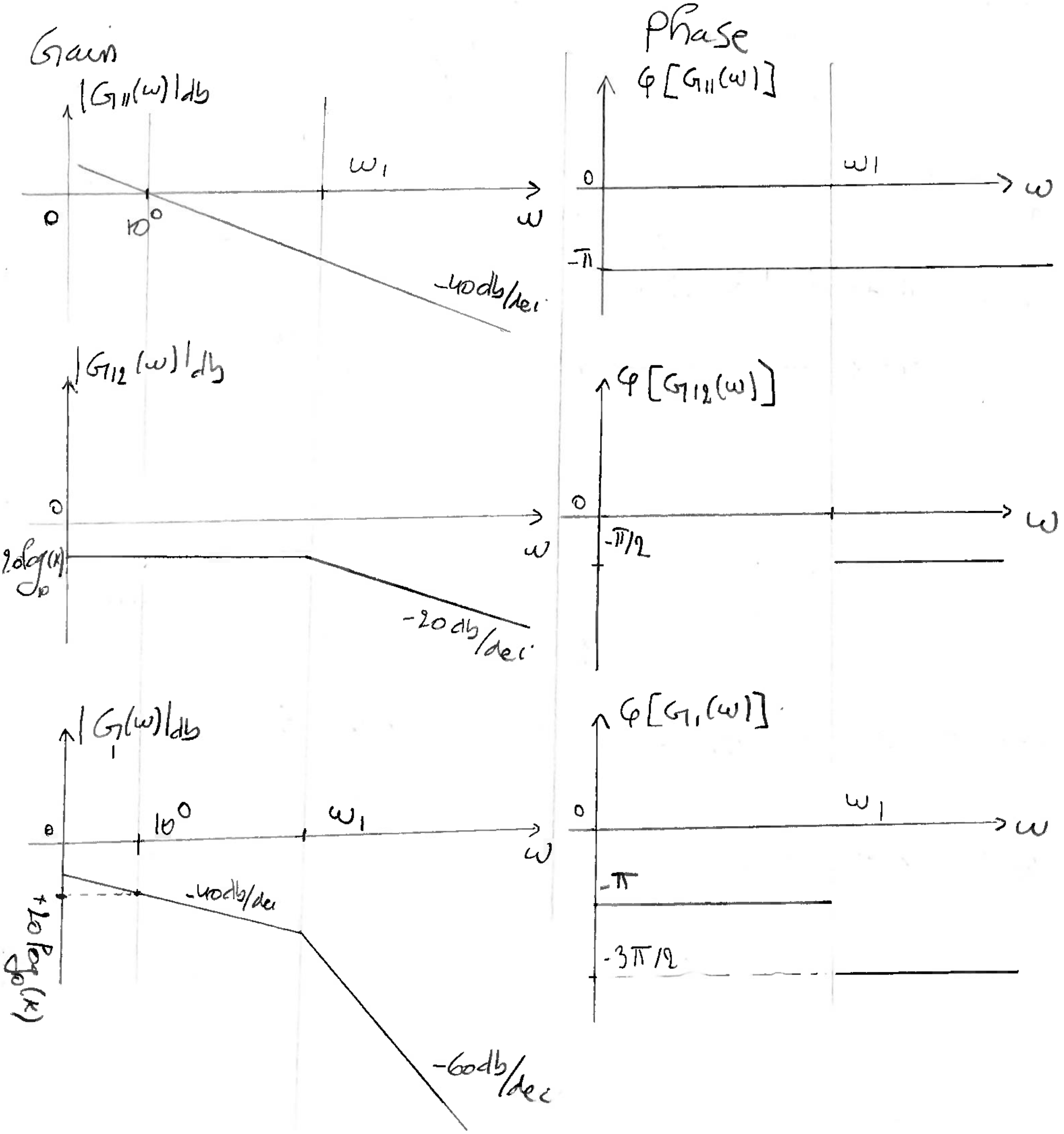
$\omega \ll \omega_1$: $|G_{12}(\omega)|_{db} \to 20\log_{10}(K)$

$\omega \gg \omega_1$: $|G_{12}(\omega)|_{db} \to 20\log_{10}(K) - 20\log_{10}\left(\dfrac{\omega}{\omega_1}\right)$

Phase $\varphi[G_{12}(\omega)] = -\operatorname{arctg}\left[\frac{\omega}{\omega_1}\right]$.

$\omega \ll \omega_1$: $\varphi[G_{12}(\omega)] \to 0$

$\omega \gg \omega_2$: $\varphi[G_{12}(\omega)] \to -\pi/2$



avec $0<K<1$ et $\propto 2<1$

2. $G_2(p) = \dfrac{K\,p\,(1+\tau_1 p)}{(1+\tau_2 p)(1+\tau_3 p)}$ $\quad 0 < K < 1$

$\tau_1 > \tau_2 > \tau_3 > 0$

$$G_2(p) = \frac{K\,p\left(1-\frac{p}{p_1}\right)}{\left(1-\frac{p}{p_2}\right)\left(1-\frac{p}{p_3}\right)} \quad \text{avec} \quad \begin{cases} p_1 = -1/\tau_1 \\ p_2 = -1/\tau_2 \\ p_3 = -1/\tau_3 \end{cases}$$

$p = j\omega$

$$G_2(\omega) = \frac{K\,j\omega\left(1+j\frac{\omega}{\omega_1}\right)}{\left(1+j\frac{\omega}{\omega_2}\right)\left(1+j\frac{\omega}{\omega_3}\right)} \quad \text{avec} \quad \begin{cases} \omega_1 = -p_1,\ \omega_2 = -p_2,\ \omega_3 = -p_3 \\ \omega_1 < \omega_2 < \omega_3 \end{cases}$$

$$G_2(\omega) = \underbrace{K\,j\omega}_{G_{20}}\ \underbrace{\left(1+j\frac{\omega}{\omega_1}\right)}_{G_{21}}\ \underbrace{\frac{1}{\left(1+j\frac{\omega}{\omega_2}\right)}}_{G_{22}}\ \underbrace{\frac{1}{\left(1+j\frac{\omega}{\omega_3}\right)}}_{G_{23}}$$

Attention G_{20} et G_{21} ne sont pas causales. Elles seront traitées indépendamment uniquement pour le tracé des asymptotes du diagramme de Bode de $G_2(\omega)$.

G_{20}: gain

$$|G_{20}(\omega)|_{db} = 20\log_{10}(K) + \underbrace{20\log_{10}(\omega)}_{+20\,db/dec}$$

Phase

$$\varphi\left[G_{20}(\omega)\right] = \arg\left[j\omega\right] = \pi/2.$$

G_{21}: gain

$$|G_{21}(\omega)|_{db} = 10\log_{10}\left(1+\left(\frac{\omega}{\omega_1}\right)^2\right)$$

$\omega \ll \omega_1$: $|G_{21}(\omega)|_{db} \to 0$

$\omega \gg \omega_1$: $|G_{21}(\omega)|_{db} \to \underbrace{+20\log_{10}(\omega/\omega_1)}_{+20\,db/de}$

Phase:

$$\varphi\left[G_{21}(\omega)\right] = +\arctg\left[\frac{\omega}{\omega_1}\right].$$

$\omega \ll \omega_1$: $\varphi\left[G_{21}(\omega)\right] \to 0$

$\omega \gg \omega_1$: $\varphi\left[G_{21}(\omega)\right] \to +\pi/2.$

G_{2i} $(i = 2,3)$ Gain

$$|G_{2i}(\omega)|_{db} = -10\log_{10}\left[1+\left(\frac{\omega}{\omega_i}\right)^2\right]$$

$\omega \ll \omega_1$: $|G_{2_1}(\omega)|_{db} \to 0$

$\omega \gg \omega_i$ $|G_{2_i}(\omega)|_{db} \to -20\log_{10}\left(\frac{\omega}{\omega_i}\right)$.

Phase : $\varphi[G_{2_i}(\omega)] = -\operatorname{arctg}\left(\frac{\omega}{\omega_i}\right)$

$\omega \ll \omega_i$: $\varphi[G_{2_i}(\omega)] \to 0$

$\omega \gg \omega_i$: $\varphi[G_{2_i}(\omega)] \to -\pi/2$.