On a une solution non nulle si $\det \begin{pmatrix} \cos(\gamma a) & \sin(\gamma a) \\ \cos(\gamma b) & \sin(\gamma b) \end{pmatrix} = 0$

$\Rightarrow \cos(\gamma a)\sin(\gamma b) - \cos(\gamma b)\sin(\gamma a) = \sin[\gamma(b-a)] = \sin[\gamma L] = 0$

d'où $\gamma_m L = m\pi \quad m \in \mathbb{N}^*$

$$\gamma_m = \frac{\omega_m}{C_L} = \frac{m\pi}{L} \quad \Rightarrow \quad \boxed{\omega_m = \frac{m\pi}{L} C_L = \frac{m\pi}{L}\sqrt{\frac{E}{\rho}}}$$

(CL1) : $C_m \cos(\gamma_m a) + D_m \sin(\gamma_m a) = 0$

$$C_m = -\frac{\sin(\gamma_m a)}{\cos(\gamma_m a)} D_m \qquad (*)$$

donc
$$g_m(x) = C_m \cos(\gamma_m x) + D_m \sin(\gamma_m x)$$
$$\underset{(*)}{=} D_m \left[\sin(\gamma_m x) - \frac{\sin(\gamma_m a)}{\cos(\gamma_m a)} \cos(\gamma_m x)\right]$$
$$= \frac{D_m}{\underbrace{\cos(\gamma_m a)}_{N_m}} \left[\underbrace{\cos(\gamma_m a)\sin(\gamma_m x) - \sin(\gamma_m a)\cos(\gamma_m x)}_{\sin[\gamma_m(x-a)]}\right]$$

par conséquent :

⊥té modes (normé)

$$\boxed{u(x,t) = \sum_{m=1}^{+\infty} \frac{1}{\alpha} \underbrace{N_m \sin[\gamma_m(x-a)]}_{X_m(x) \text{ modes propres}} f_m(t)}$$

avec $f_m(t) = \underline{A_m} \cos(\omega_m t) + \underline{B_m} \sin(\omega_m t)$

C. Initiales

6. Les fonctions propres $\chi_m(x)$ vérifient la relation d'orthogonalité

$$\int_a^b x^2 \chi_m(x)\chi_n(x)\,\mathrm{d}x = \delta_{mn}.$$

$$[x\chi_m(x)](x\chi_n(x)]$$

En déduire l'expression du coefficient de normalisation N_m.

$$\delta_{mn} = \begin{cases} 0 & m \neq n \quad \perp^{té} \\ 1 & m = n \quad \text{norme} \end{cases}$$

$$\int_a^b x^2 \chi_m^2(x)\,dx = 1$$

$$\int_a^b \cancel{x^2} \times \frac{1}{\cancel{x^2}} N_m^2 \sin^2[\gamma_m(x-a)]\,dx = 1$$

$$\rightarrow N_m^2 \int_a^b \underbrace{\sin^2[\gamma_m(x-a)]}_{\frac{1}{2}\left\{1-\cos[2\gamma_m(x-a)]\right\}}\,dx = 1$$

$$N_m^2 \times \left(\frac{b-a}{2}\right) = 1 \Rightarrow N_m^2 \frac{L}{2} = 1$$

$$\boxed{N_m = \sqrt{\frac{2}{L}}}$$

7. Déterminer la réponse de la poutre à une vibration longitudinale sinusoïdale du support :

$$u_s(t) = u_0 \sin(\Omega t).$$

$$\boxed{c_L^2 \frac{\partial^2}{\partial x^2}[x u(x,t)] - \frac{\partial^2}{\partial t^2}[x u(x,t)] + \cancel{\frac{f(x,t)}{\rho S_0 x}} = 0} \quad (3)$$

@ pas charg^t ext.

$u_{tot}(x,t) = u(x,t) + u_S(t)$
$\hookrightarrow$ déplacement du support

(3) $\Rightarrow$ $$c_L^2 \frac{\partial^2}{\partial x^2}\Big[x\, \underset{\substack{tot\\ \parallel \\ xu(x,t) + \cancel{x u_S(t)}}}{u}(x,t)\Big] - \frac{\partial^2}{\partial t^2}\Big[x\, \underbrace{u_{tot}(x,t)}_{\substack{\parallel \\ xu(x,t) + xu_S(t)}}\Big] = 0$$

$$c_L^2 \frac{\partial^2}{\partial x^2}\big[x\, u(x,t)\big] - \frac{\partial^2}{\partial t^2}\big[x\, u(x,t)\big] = + \overbrace{\frac{\partial^2 x u_S(t)}{\partial t^2}}^{-u_0 \Omega^2 \sin(\Omega t)}$$

$u_S(t) = u_0 \sin(\Omega t)$; $\ddot{u}_S(t) = -x u_0 \Omega^2 \sin(\Omega t)$

$$\boxed{c_L^2 \frac{\partial^2}{\partial x^2}\big[x\, u(x,t)\big] - \frac{\partial^2}{\partial t^2}\big[x\, u(x,t)\big] = -u_0 x \Omega^2 \sin(\Omega t)} \quad (5)$$

$u(x,t)$ solution décomposée sur les modes propres :

$$u(x,t) = \sum_{m=1}^{+\infty} X_m(x)\, \underbrace{Y_m(t)}_{\text{fonction inconnue de } t}$$

On remplace dans (5) :

$$c_L^2 \frac{\partial^2}{\partial x^2}\Big[x \sum_{m=1}^{+\infty} X_m(x)\, Y_m(t)\Big] - \frac{\partial^2}{\partial t^2}\Big[x \sum_{m=1}^{+\infty} X_m(x)\, Y_m(t)\Big] = -u_0 x \Omega^2 \sin(\Omega t)$$

$$c_L^2 \frac{\partial^2}{\partial x^2}\big[x X_m(x)\big] = -\omega_m^2\, x\, X_m(x)$$

$\underline{Rq}$: d'après (ii) $g''(x) + \frac{\omega^2}{c_L^2} g(x) = 0 \Rightarrow c_L^2 g''(x) = -\omega^2 g(x)$

si on prend $g(x) = x X_m(x)$

$$\Rightarrow \sum_{m=1}^{+\infty} \left\{ -\omega_m^2 x X_m(x) \Psi_m(t) - x X_m(x) \ddot{\Psi}_m(t) \right\} = -u_0 x \Omega^2 \sin(\Omega t)$$

$$\Rightarrow \sum_{m=1}^{+\infty} \left\{ \ddot{\Psi}_m(t) + \omega_m^2 \Psi_m(t) \right\} x X_m(x) = u_0 x \Omega^2 \sin(\Omega t)$$

$$\int_a^b (\ ) [x X_n(x)] dx \qquad \perp \overset{td}{=} \quad m = n$$

$$\int_a^b x^2 X_m(x) X_n(x) dx = \delta_{mn}$$

$$\int_a^b \left\{ \ddot{\Psi}_n(t) + \omega_n^2 \Psi_n(t) \right\} x^2 X_n^2(x)\, dx = \int_a^b u_0 x^2 \Omega^2 \sin(\Omega t) X_n(x)\, dx$$

$$\ddot{\Psi}_n(t) + \omega_n^2 \Psi_n(t) = u_0 \Omega^2 \int_a^b x^2 X_n(x) dx \ \sin(\Omega t)$$

avec $X_n(x) = \frac{1}{x} \sqrt{\frac{2}{L}} \sin[\gamma_n (x-a)]$

$$\int_a^b x^2 X_n(x) dx = \sqrt{\frac{2}{L}} \int_a^b \underset{\alpha}{x} \underset{\beta'}{\sin[\gamma_n (x-a)]} dx$$

$$= \sqrt{\frac{2}{L}} \left\{ \left[\frac{-x}{\gamma_n} \cos[\gamma_n(x-a)] \right]_a^b + \int_a^b \frac{\cos[\gamma_n(x-a)]}{\gamma_n} \, dx \right\}$$

$$= \sqrt{\frac{2}{L}} \left\{ -\frac{b}{\gamma_n} \cos[\gamma_n(\underbrace{b-a}_{L})] + \frac{a}{\gamma_n} + \frac{1}{\gamma_n^2} \left[\sin[\gamma_n(x-a)] \right]_a^b \right\}$$

$$= \sqrt{\frac{2}{L}} \left\{ \frac{1}{\gamma_n} [a - b\cos(\gamma_n L)] + \frac{1}{\gamma_n^2} [\sin(\gamma_n L) - 0] \right\}$$

or $\gamma_n L = n\pi$

$$\int_a^b x^2 \chi_n(x) \, dx = \sqrt{\frac{2}{L}} \left\{ \frac{L}{n\pi} [a - b \overbrace{\cos(n\pi)}^{(-1)^n}] + \frac{L^2}{n^2\pi^2} \overbrace{\sin(n\pi)}^{0} \right\}$$

$$\Rightarrow \ddot{\Psi}_n(t) + \omega_n^2 \Psi_n(t) = u_0 \Omega^2 \int_a^b x^2 \chi_n(x) \, dx \, \sin(\Omega t)$$

devient : $\boxed{\ddot{\Psi}_n(t) + \omega_n^2 \Psi_n(t) = I_n \sin(\Omega t)}$ (Δ)

avec $I_n = u_0 \Omega^2 \sqrt{\frac{2}{L}} \times \frac{L}{n\pi} [a - b(-1)^n]$

Solution de (Δ) est de la forme :

$$\Psi_n(t) = \Psi_n^{homog}(t) + \Psi_n^{(p)}(t)$$ → solution particulière

avec $\Psi_n^{homog}(t) = \alpha_n \cos(\omega_n t) + \beta_n \sin(\omega_n t)$

$$\varphi_n^{(p)}(t) = E_n \sin(\Omega t) \quad ; \quad \ddot{\varphi}_n^{(p)}(t) = -\Omega^2 E_n \sin(\Omega t)$$

$$(\Delta) \rightsquigarrow \quad (\omega_n^2 - \Omega^2)\, E_n \cancel{\sin(\Omega t)} = I_n \cancel{\sin(\Omega t)} \quad \forall t$$

$$E_n = \frac{I_n}{\omega_n^2 - \Omega^2}$$

$$\varphi_n(t) = \alpha_n \cos(\omega_n t) + \beta_n \sin(\omega_n t) + \frac{I_n}{\omega_n^2 - \Omega^2} \sin(\Omega t)$$

déterminés en utilisant les C. Initiales.