# Logic for CS

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## Notes for Ben's Course at UWaterloo[1]

*by*

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[1] https://www.youtube.com/playlist?list=PLPW2keNyw-utXOOzLR-Wp1poeE5LEtv3N

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# .1 Propositional Logic

## .1.1 Proof System

### Soundness, Consistency, and Maximality

**Definition 1** (consistent). *A set of w.f.f.$\Sigma$ is consistent if $\Sigma \nvdash \alpha$ or $\Sigma \nvdash \neg\alpha$ for all w.f.f.$\alpha$.*

**Claim 1.** *A set of w.f.f.$\Sigma$ is consistent iff $\exists\alpha, \Sigma \nvdash \alpha$.*

*Proof.* $\to$: pick any $\alpha$, either $\Sigma \nvdash \alpha$ or $\Sigma \vdash \alpha$, then there exists one that is not proved by $\Sigma$.

$\leftarrow$: assume that for some $beta, \Sigma \nvdash \beta$, if the first definition is violated, then, for some $\alpha$, $\Sigma \vdash \alpha$ and $\Sigma \vdash \neg\alpha$, then $\Sigma \vdash \Sigma \cup \{\alpha, \neg\alpha\} \vdash \{\alpha, \neg\alpha\}$. $\{\alpha, \neg\alpha\} \vdash \beta$ for every $\beta$, contradicting the assumption. □

**Corollary 1.** *If $\Sigma \subseteq \Sigma'$, then if $\Sigma$ is consistent, then so does $\Sigma'$.*

**Definition 2** (Soundness). *If $\vdash \alpha$, then $\alpha$ is a tautology.*

**Corollary 2.** *$\nvdash P$, since $P$ is a propositional variable, which is not a tautology.*

The universe $U$ (the set which includes everything) is the least consistent, since it contains everything. Therefore, for every $\alpha \in U, \neg\alpha \in U$, then $U$ can prove both.

$\vdash$ is about syntax while $\models$ is about semantics. $\vdash$ shows that though the machine doesn't know anything, it can reach some result by axioms and modus ponens. $\models$ shows that the result is true in the real world given by human being's assignment of truth values.

**Theorem 1.** *In a sound proof system, every satisfiable $\Sigma$ is consistent.*

*Proof.* *b.w.o.c*, assume the satisfiable $\Sigma$ is inconsistent. Then, for some $\alpha$, both $\Sigma \vdash \alpha$ and $\Sigma \vdash \neg\alpha$. If $\Sigma$ is satisfiable, then for some truth assignment $V$, $V$ satisfies all *w.f.f.*in $\Sigma$. By soundness, $\Sigma \models \alpha$ and $\Sigma \models \neg\alpha$. So for that assignment $V$, we get $V(\alpha) = T$ and $V(\neg\alpha) = T$, violating the truth table of negation. □

**Theorem 2** (Extended Soundness). *If $Sigma \vdash \alpha$, then $\Sigma \models \alpha$.*

**Corollary 3.** *If any set of w.f.f.is consistent, then, in particular* $\emptyset$*.*

**Definition 3** (Monotonicity). *$\forall \Sigma, \Sigma', \alpha$, if $\Sigma \vdash \alpha, \Sigma \subseteq \Sigma'$, then $\Sigma' \vdash \alpha$.*

## Consistency Continuation

**Theorem 3** (Deduction Theorem). *$\Sigma \cup \{\alpha\} \vdash \beta$ iff $\Sigma \vdash \alpha \to \beta$.*

**Theorem 4.** *Any consistent set of w.f.f.$\Sigma$ can be extended to $\Sigma$'s.t.$\Sigma' \supseteq \Sigma$ that is maximally consistent.*

*Proof.* We construct a sequence of set of *w.f.f.*. $\Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \ldots \subseteq \Sigma_i \subseteq \Sigma_{i+1}$ *s.t.*1) $\Sigma_0 = \Sigma$ and 2) For all $\Sigma_i$, $\Sigma_i$ is consistent, and 3) for fixed enumeration of all *w.f.f.*., $\alpha_1, \alpha_2, \ldots, \alpha_n$, for all $i$, either $\Sigma_i \vdash \alpha$ or $\Sigma_i \vdash \neg\alpha$.

Assume $\Sigma_i$ is defined and meets the requirements, let $\Sigma_{i+1}$ be $\Sigma_i$ if $\Sigma_i \vdash \neg\alpha_i$, otherwise $\Sigma_{i+1} = \Sigma_i \cup \{\alpha_i\}$.

Now we want to prove if $\Sigma_i$ meets the requirement, then so will $\Sigma_{i+1}$. Requirement 1) is trivial. Requirement 2) and 3) are simultaneously proved by showing $\Sigma_{i+1} \nvdash \neg\alpha_i$ if $\Sigma_{i+1} = \Sigma_i \cup \{\alpha_i\}$, which is shown by the following claim.

**Claim 2.** *$\forall \Sigma, \alpha$, if $\Sigma \cup \{\alpha\} \vdash \neg\alpha$, then $\Sigma \vdash \neg\alpha$.*

*Proof.* To prove this claim, it suffices to show $\vdash (\alpha \to \neg\alpha) \to \neg\alpha$ based on deduction theorem, which is a tautology. □

Now we show the $\Sigma' = \cup_{i=1}^{\infty} \Sigma_i$ is maximally consistent. To prove it, it suffices to show that $\forall \alpha_i, i \in N, \Sigma' \vdash \alpha$ or $\Sigma' \vdash \neg\alpha$. By requirement 2), $\Sigma_i \vdash \alpha_i$ or $\Sigma_i \vdash \neg\alpha_i$. Now we are going to show $\Sigma' \vdash \alpha_i$ or $\Sigma' \vdash \neg\alpha_i$.

*b.w.o.c*, suppose $\Sigma'$ is inconsistent. In this case, for some $\alpha$, $\Sigma' \vdash \alpha\, and\, \Sigma' \vdash \neg\alpha$. Let $\beta_1, \ldots \beta_k$ be the proof of $\alpha$ from $\Sigma'$ and $\gamma_1, \ldots, \gamma_l$ be the proof of $\neg\alpha$ from $\Sigma'$.

Each $\beta_i$ that is an assumption from $\Sigma'$ belongs to some $\Sigma_{m_i}$, and each $\gamma_i$ that is an assumption from $\Sigma'$ belongs to some $\Sigma_{m_j}$. Since both formal proofs of $\alpha$ and of $\neg\alpha$ are finite, there is some $i^*$ that is bigger than all of these $m_i$'s and $m_j$'s. Therefore, for each $beta_i$ or $gemma_j$ that are used as assumptions, $\beta_i, \gamma_j \in \Sigma_{i^*}$. Now by monotonicity, $\Sigma_{i^*} \vdash \alpha$ and $\Sigma_{i^*} \vdash \neg\alpha$, which is a contradiction to requirement 2).

$\square$

**Theorem 5.** *Every consistent $\Sigma$ is satisfiable.*

*Proof.* Let $\Sigma'$ be a maximally consistent set of *w.f.f.*.s *s.t.* $\Sigma \subseteq \Sigma'$. Define a truth assignment $V_{\Sigma'}$ as follows: $V_{\Sigma'}(P_i) = T$ iff $\Sigma' \vdash P_i$ where $P_i$ is a propositional variable. Now extend it into $V_{\Sigma'}(\alpha_i) = T$ iff $\Sigma' \vdash \alpha_i$.

$\square$