

# LOGIC FOR CS



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NOTES FOR BEN'S COURSE AT UWaterloo<sup>1</sup>

*by*

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<sup>1</sup><https://www.youtube.com/playlist?list=PLPW2keNyw-utXOOzLR-Wp1poeE5LEtv3N>

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## .I PROPOSITIONAL LOGIC

### .I.I PROOF SYSTEM

**Definition 1** (consistent). *A set of w.f.f.  $\Sigma$  is consistent if  $\Sigma \vdash \alpha$  and  $\Sigma \vdash \neg\alpha$  does not hold together.*

Another definition of consistency is that  $\Sigma \not\vdash \perp$  where  $\perp$  is a contradiction<sup>a</sup>.

<sup>a</sup><https://planetmath.org/consistent>

**Claim 1.** *A set of w.f.f.  $\Sigma$  is consistent iff  $\exists\alpha, \Sigma \not\vdash \alpha$ .*

*Proof.*  $\rightarrow$ : pick any  $\alpha$ , either  $\Sigma \not\vdash \alpha$  or  $\Sigma \vdash \alpha$ , then there exists one that is not proved by  $\Sigma$ .

$\leftarrow$ : assume that for some  $\beta, \Sigma \not\vdash \beta$ , if the first definition is violated, then, for some  $\alpha, \Sigma \vdash \alpha$  and  $\Sigma \vdash \neg\alpha$ , then  $\Sigma \vdash \Sigma \cup \{\alpha, \neg\alpha\} \vdash \{\alpha, \neg\alpha\}. \{\alpha, \neg\alpha\} \vdash \beta$  for every  $\beta$ , contradicting the assumption.  $\square$

**Corollary 1.** *If  $\Sigma \subseteq \Sigma'$ , then if  $\Sigma$  is consistent, then so does  $\Sigma'$ .*

**Definition 2** (Soundness). *If  $\vdash \alpha$ , then  $\alpha$  is a tautology.*

*In another word, every theorem is a tautology.*

**Corollary 2.**  $\not\vdash P$ , since  $P$  is a propositional variable, which is not a tautology.

The universe  $U$  (the set which includes everything) is the least consistent, since it contains everything. Therefore, for every  $\alpha \in U, \neg\alpha \in U$ ,  $U$  can prove both. It has the most unsteady stance:).

$\vdash$  is about syntax while  $\models$  is about semantics.  $\vdash$  shows that though the machine doesn't know anything, it can reach some result by axioms and modus ponens.  $\models$  shows that the result is true in the real world given by human being's assignment of truth values.

**Theorem 1.** *In a sound proof system, every satisfiable  $\Sigma$  is consistent.*

*Proof.* *b.w.o.c.*, assume the satisfiable  $\Sigma$  is inconsistent. Then, for some  $\alpha$ , both  $\Sigma \vdash \alpha$  and  $\Sigma \vdash \neg\alpha$ . If  $\Sigma$  is satisfiable, then for some truth assignment  $V$ ,  $V$  satisfies all *w.f.f.* in  $\Sigma$ . By soundness,  $\Sigma \models \alpha$  and  $\Sigma \models \neg\alpha$ . So for that assignment  $V$ , we get  $V(\alpha) = T$  and  $V(\neg\alpha) = T$ , violating the truth table of negation.  $\square$

The above theorem builds a bridge from semantic to syntax.

**Theorem 2** (Extended Soundness). *If  $\Sigma \vdash \alpha$ , then  $\Sigma \models \alpha$ .*

**Corollary 3.** *If any set of *w.f.f.* is consistent, then, in particular  $\emptyset$ .*

**Definition 3** (Maximally Consistent). *We say that  $\Sigma$  is maximally consistent if  $\Sigma$  is consistent, but, for every  $\alpha$ , either  $\Sigma \vdash \alpha$  or  $\Sigma \cup \{\alpha\}$  is inconsistent.*

Consistency tells us that if  $\Sigma \vdash \alpha$ , then  $\Sigma \not\vdash \alpha$  doesn't hold. But if  $\Sigma \not\vdash \alpha$ , then we can't say  $\Sigma \vdash \alpha$  holds. Consistency only ensures no contradiction.

Maximally consistency tells us that if  $\Sigma \not\vdash \alpha$ , then  $\Sigma \vdash \alpha$  must hold. The maximality nature ensures that the negation of every nonprovable *w.f.f.* is provable.

**Example 1.** *Let  $\Sigma \equiv \{P_1\}$  over the variables  $P_1, P_2, \dots$*

**Claim 2.**  *$\{P_1\}$  is consistent since it's satisfiable.*

**Claim 3.**  *$\{P_1\}$  is not maximally consistent*

*Proof.* It suffices to show that  $\{P_1\} \not\vdash P_3$  and  $\{P_1, P_3\}$  is consistent.

By soundness, it suffices to show that  $\{P_1\} \not\models P_3$ . Just find some truth assignment that  $V \text{ s.t. } V(P_1) = T$  and  $V(P_3) = F$ . So  $\{P_1\} \not\models P_3$ .

Since  $\{P_1, P_3\}$  is satisfiable, so it's consistent.  $\square$

**Lemma 1.** *For every consistent  $\Sigma$ , there exists a maximally consistent  $\Sigma' \supseteq \Sigma$ .*

*Proof.* Let  $\{\alpha_1, \dots, \alpha_i, \alpha_{i+1}, \dots\}$  is a set of w.f.f. over  $\{P_1, P_2, \dots, P_i, \dots\}$   $\square$

**Definition 4** (Monotonicity).  $\forall \Sigma, \Sigma', \alpha$ , if  $\Sigma \vdash \alpha, \Sigma \subseteq \Sigma'$ , then  $\Sigma' \vdash \alpha$ .

**Theorem 3** (Deduction Theorem).  $\Sigma \cup \{\alpha\} \vdash \beta$  iff  $\Sigma \vdash \alpha \rightarrow \beta$ .

**Theorem 4.** *Any consistent set of w.f.f.  $\Sigma$  can be extended to  $\Sigma'$  s.t.  $\Sigma' \supseteq \Sigma$  that is maximally consistent.*

*Proof.* We construct a sequence of set of w.f.f.  $\Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \dots \subseteq \Sigma_i \subseteq \Sigma_{i+1}$  s.t. 1)  $\Sigma_0 = \Sigma$  and 2) For all  $\Sigma_i$ ,  $\Sigma_i$  is consistent, and 3) for fixed enumeration of all w.f.f.,  $\alpha_1, \alpha_2, \dots, \alpha_n$ , for all  $i$ , either  $\Sigma_i \vdash \alpha$  or  $\Sigma_i \vdash \neg \alpha$ .

Assume  $\Sigma_i$  is defined and meets the requirements, let  $\Sigma_{i+1}$  be  $\Sigma_i$  if  $\Sigma_i \vdash \neg \alpha_i$ , otherwise  $\Sigma_{i+1} = \Sigma_i \cup \{\alpha_i\}$ .

Now we want to prove if  $\Sigma_i$  meets the requirement, then so will  $\Sigma_{i+1}$ . Requirement 1) is trivial. Requirement 2) and 3) are simultaneously proved by showing  $\Sigma_{i+1} \not\vdash \neg \alpha_i$  if  $\Sigma_{i+1} = \Sigma_i \cup \{\alpha_i\}$ , which is shown by the following claim.

**Claim 4.**  $\forall \Sigma, \alpha$ , if  $\Sigma \cup \{\alpha\} \vdash \neg \alpha$ , then  $\Sigma \vdash \neg \alpha$ .

*Proof.* To prove this claim, it suffices to show  $\vdash (\alpha \rightarrow \neg \alpha) \rightarrow \neg \alpha$  based on deduction theorem, which is a tautology.  $\square$

Now we show the  $\Sigma' = \bigcup_{i=1}^{\infty} \Sigma_i$  is maximally consistent. To prove it, it suffices to show that  $\forall \alpha_i, i \in N, \Sigma' \vdash \alpha$  or  $\Sigma' \vdash \neg \alpha$ . By requirement 2),  $\Sigma_i \vdash \alpha_i$  or  $\Sigma_i \vdash \neg \alpha_i$ . Now we are going to show  $\Sigma' \vdash \alpha_i$  or  $\Sigma' \vdash \neg \alpha_i$ .

*b.w.o.c.*, suppose  $\Sigma'$  is inconsistent. In this case, for some  $\alpha$ ,  $\Sigma' \vdash \alpha$  and  $\Sigma' \vdash \neg\alpha$ . Let  $\beta_1, \dots, \beta_k$  be the proof of  $\alpha$  from  $\Sigma'$  and  $\gamma_1, \dots, \gamma_l$  be the proof of  $\neg\alpha$  from  $\Sigma'$ .

Each  $\beta_i$  that is an assumption from  $\Sigma'$  belongs to some  $\Sigma_{m_i}$ , and each  $\gamma_i$  that is an assumption from  $\Sigma'$  belongs to some  $\Sigma_{m_j}$ . Since both formal proofs of  $\alpha$  and of  $\neg\alpha$  are finite, there is some  $i^*$  that is bigger than all of these  $m_i$ 's and  $m_j$ 's. Therefore, for each *beta* <sub>$i$</sub>  or *gemma* <sub>$j$</sub>  that are used as assumptions,  $\beta_i, \gamma_j \in \Sigma_{i^*}$ . Now by monotonicity,  $\Sigma_{i^*} \vdash \alpha$  and  $\Sigma_{i^*} \vdash \neg\alpha$ , which is a contradiction to requirement 2). □

**Theorem 5.** *In a sound proof system, every consistent  $\Sigma$  is satisfiable.*

*Also called the completeness theorem.*

*Proof.* Let  $\Sigma'$  be a maximally consistent set of *w.f.f.s* s.t.  $\Sigma \subseteq \Sigma'$ . Define a truth assignment  $V_{\Sigma'}$  as follows:  $V_{\Sigma'}(P_i) = T$  iff  $\Sigma' \vdash P_i$  where  $P_i$  is a propositional variable. Now extend it into  $V_{\Sigma'}(\alpha_i) = T$  iff  $\Sigma' \vdash \alpha_i$ .

**Claim 5.** *For every formula  $\alpha$ ,  $\bar{V}_{\Sigma'}(\alpha) = T$  iff  $\Sigma' \vdash \alpha$ , where  $\bar{V}_{\Sigma'}$  is the extension of  $V_{\Sigma'}$  in the infinite set of *w.f.f.*, and  $\bar{V}_{\Sigma'}$  is the truth assignment that satisfies  $\Sigma'$ .*

*Proof.* By generalized induction on the set of all *w.f.f.*:  $I(\mathcal{P}, \{\rightarrow, \neg\})$  where  $\mathcal{P}$  is the set of propositional variables, and  $\{\rightarrow, \neg\}$  are the adequate connectives.  $I(\mathcal{P}, \{\rightarrow, \neg\})$  means that all *w.f.f.s* that are built from  $\mathcal{P}$  and  $\{\rightarrow, \neg\}$ .

**Induction base.**  $\alpha \in \mathcal{P}$ .

$\leftarrow$ :  $\bar{V}_{\Sigma'}(\alpha) = T$ , then  $\Sigma' \vdash \alpha$  by definition of  $V_{\Sigma'}$ .

$\rightarrow$ : Suppose  $\Sigma' \not\vdash \alpha$ , then  $\bar{V}_{\Sigma'}(\alpha) = F$  still by the definition.

Remember the relation between truth assignment and the proof symbol.  $\Sigma \vdash \alpha$  means that for every truth assignment  $V$  s.t.  $V(\Sigma) = T$ , then  $V(\alpha) = T$ , which can be simplified as  $V_\Sigma(\alpha) = T$ .



**Induction step.** Assume the claim holds for  $\alpha$  and for  $\beta$ , we need to show that for  $\neg\alpha$  and  $\alpha \rightarrow \beta$ .

$\neg\alpha$ :

$\leftarrow$ : If  $\Sigma' \vdash \neg\alpha$ , by the consistency,  $\Sigma' \not\vdash \alpha$ . So by the induction hypothesis,  $\bar{V}_{\Sigma'}(\alpha) = F$ , so by the truth table,  $\bar{V}_{\Sigma'}(\neg\alpha) = T$ .

$\rightarrow$ : Assume  $\Sigma' \not\vdash \neg\alpha$ , then by its maximality,  $\Sigma' \vdash \alpha$ . So by the induction hypothesis,  $\bar{V}_{\Sigma'}(\alpha) = T$ , so by the truth table,  $\bar{V}_{\Sigma'}(\neg\alpha) = F$ .

$\alpha \rightarrow \beta$ :

$\leftarrow$ :  $\Sigma' \not\vdash (\alpha \rightarrow \beta)$ , Since  $\vdash \beta \rightarrow (\alpha \rightarrow \beta)$ ,  $\Sigma' \not\vdash \beta$ . By the induction hypothesis,  $\bar{V}_{\Sigma'}(\beta) = F$ . Now we need to show that  $\bar{V}_{\Sigma'}(\alpha) = T$ . *b.w.o.c.*, assume  $\bar{V}_{\Sigma'}(\alpha) = F$ . By the induction hypothesis,  $\Sigma \not\vdash \alpha$ , and by the maximality of  $\Sigma'$ ,  $\Sigma' \vdash \neg\alpha$ . And by  $\vdash \neg\alpha \rightarrow (\alpha \rightarrow \beta)$ ,  $\Sigma' \vdash (\alpha \rightarrow \beta)$ , which contradicts  $\Sigma' \not\vdash (\alpha \rightarrow \beta)$ . Therefore,  $\bar{V}_{\Sigma'}(\alpha) = T$ , so this case is proved.

$\rightarrow$ :  $\Sigma' \vdash (\alpha \rightarrow \beta)$ . Assume  $\Sigma' \vdash \alpha$ , in which case  $\Sigma' \vdash \beta$  by modus ponens. Using the induction hypothesis,  $\bar{V}_{\Sigma'}(\alpha) = T$  and  $\bar{V}_{\Sigma'}(\beta) = T$ , so  $\bar{V}_{\Sigma'}(\alpha \rightarrow \beta) = T$ . Otherwise,  $\Sigma' \not\vdash \alpha$ , so by the induction hypothesis,  $\bar{V}_{\Sigma'}(\alpha) = F$ , so by truth table,  $\bar{V}_{\Sigma'}(\alpha \rightarrow \beta) = T$ .  $\square$

$\square$

*Proof.*  $\Sigma$  is satisfiable by all-truth assignemnt, so it's consistent. To show it's maximally consistent, we need to show for every  $\alpha$ , if  $\Sigma \not\vdash \alpha$ , then  $\Sigma \cup \{\alpha\}$  is inconsistent. If  $\Sigma \not\vdash \alpha$ , then  $\Sigma \not\vdash \alpha$ , then  $\Sigma \cup \{\alpha\}$  is unsatisfiable, so it's inconsistent.

Based on the definition of the truth table,  $V_{\Sigma} = T$ .

$\square$