

LOGIC FOR CS

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NOTES FOR BEN'S COURSE AT UWaterloo¹

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¹<https://www.youtube.com/playlist?list=PLPW2keNyw-utXOOzLR-Wp1poeE5LEtv3N>

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.I PROPOSITIONAL LOGIC

.I.I PROOF SYSTEM

Definition 1 (consistent). *A set of w.f.f. Σ is consistent if $\Sigma \not\vdash \alpha$ or $\Sigma \not\vdash \neg\alpha$ for all w.f.f. α .*

Claim 1. *A set of w.f.f. Σ is consistent iff $\exists\alpha, \Sigma \not\vdash \alpha$.*

Proof. \rightarrow : pick any α , either $\Sigma \not\vdash \alpha$ or $\Sigma \vdash \alpha$, then there exists one that is not proved by Σ .

\leftarrow : assume that for some beta, $\Sigma \not\vdash \beta$, if the first definition is violated, then, for some α , $\Sigma \vdash \alpha$ and $\Sigma \vdash \neg\alpha$, then $\Sigma \vdash \Sigma \cup \{\alpha, \neg\alpha\} \vdash \{\alpha, \neg\alpha\}. \{\alpha, \neg\alpha\} \vdash \beta$ for every β , contradicting the assumption. \square

Corollary 1. *If $\Sigma \subseteq \Sigma'$, then if Σ is consistent, then so does Σ' .*

Definition 2 (Soundness). *If $\vdash \alpha$, then α is a tautology.*

Corollary 2. $\not\vdash P$, since P is a propositional variable, which is not a tautology.

The universe U (the set which includes everything) is the least consistent, since it contains everything. Therefore, for every $\alpha \in U$, $\neg\alpha \in U$, U can prove both. It has the most unsteady stance:).

\vdash is about syntax while \models is about semantics. \vdash shows that though the machine doesn't know anything, it can reach some result by axioms and modus ponens. \models shows that the result is true in the real world given by human being's assignment of truth values.

Theorem 1. *In a sound proof system, every satisfiable Σ is consistent.*

Proof. b.w.o.c, assume the satisfiable Σ is inconsistent. Then, for some α , both $\Sigma \vdash \alpha$ and $\Sigma \vdash \neg\alpha$. If Σ is satisfiable, then for some truth assignment V , V satisfies all w.f.f. in Σ . By soundness, $\Sigma \models \alpha$ and $\Sigma \models \neg\alpha$. So for that assignment V , we get $V(\alpha) = T$ and $V(\neg\alpha) = T$, violating the truth table of negation. \square

The above theorem builds a bridge from semantic to syntax.

Theorem 2 (Extended Soundness). *If $\Sigma \vdash \alpha$, then $\Sigma \models \alpha$.*

Corollary 3. *If any set of w.f.f. is consistent, then, in particular \emptyset .*

Definition 3 (Maximally Consistent). *We say that Σ is maximally consistent if Σ is consistent, but, for every α , either $\Sigma \vdash \alpha$ or $\Sigma \cup \{\alpha\}$ is inconsistent.*

Simialr to the definition of maximally satisfiable, the system has the maximal knowledge, with which it can prove everything and rejects to embrace any unprovable things cause it will cause contradiction.

Example 1. Let $\Sigma \equiv \{P_1\}$ over the variables P_1, P_2, \dots

Claim 2. $\{P_1\}$ is consistent since it's satisfiable.

Claim 3. $\{P_1\}$ is not maximally consistent

Proof. It suffices to show that $\{P_1\} \not\models P_3$ and $\{P_1, P_3\}$ is consistent.

By soundness, it suffices to show that $\{P_1\} \not\models P_3$. Just find some truth assignment that $V \text{ s.t. } V(P_1) = T$ and $V(P_3) = F$. So $\{P_1\} \not\models P_3$.

Since $\{P_1, P_3\}$ is satisfiable, so it's consistent. \square

Lemma 1. *For every consistent Σ , there exists a maximally consistent $\Sigma' \supseteq \Sigma$.*

Proof. Let $\{\alpha_1, \dots, \alpha_i, \alpha_{i+1}, \dots\}$ is a set of w.f.f. over $\{P_1, P_2, \dots, P_i, \dots\}$ \square

Definition 4 (Monotonicity). $\forall \Sigma, \Sigma', \alpha$, if $\Sigma \vdash \alpha$, $\Sigma \subseteq \Sigma'$, then $\Sigma' \vdash \alpha$.

Theorem 3 (Deduction Theorem). $\Sigma \cup \{\alpha\} \vdash \beta$ iff $\Sigma \vdash \alpha \rightarrow \beta$.

Theorem 4. *Any consistent set of w.f.f. Σ can be extended to Σ' s.t. $\Sigma' \supseteq \Sigma$ that is maximally consistent.*

Proof. We construct a sequence of set of w.f.f. $\Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \dots \subseteq \Sigma_i \subseteq \Sigma_{i+1}$ s.t. 1) $\Sigma_0 = \Sigma$ and 2) For all Σ_i , Σ_i is consistent, and 3) for fixed enumeration of all w.f.f., $\alpha_1, \alpha_2, \dots, \alpha_n$, for all i , either $\Sigma_i \vdash \alpha$ or $\Sigma_i \vdash \neg\alpha$.

Assume Σ_i is defined and meets the requirements, let Σ_{i+1} be Σ_i if $\Sigma_i \vdash \neg\alpha_i$, otherwise $\Sigma_{i+1} = \Sigma_i \cup \{\alpha_i\}$.

Now we want to prove if Σ_i meets the requirement, then so will Σ_{i+1} . Requirement 1) is trivial. Requirement 2) and 3) are simultaneously proved by showing $\Sigma_{i+1} \not\vdash \neg\alpha_i$ if $\Sigma_{i+1} = \Sigma_i \cup \{\alpha_i\}$, which is shown by the following claim.

Claim 4. $\forall \Sigma, \alpha$, if $\Sigma \cup \{\alpha\} \vdash \neg\alpha$, then $\Sigma \vdash \neg\alpha$.

Proof. To prove this claim, it suffices to show $\vdash (\alpha \rightarrow \neg\alpha) \rightarrow \neg\alpha$ based on deduction theorem, which is a tautology. \square

Now we show the $\Sigma' = \bigcup_{i=1}^{\infty} \Sigma_i$ is maximally consistent. To prove it, it suffices to show that $\forall \alpha_i, i \in N, \Sigma' \vdash \alpha$ or $\Sigma' \vdash \neg\alpha$. By requirement 2), $\Sigma_i \vdash \alpha_i$ or $\Sigma_i \vdash \neg\alpha_i$. Now we are going to show $\Sigma' \vdash \alpha_i$ or $\Sigma' \vdash \neg\alpha_i$.

b.w.o.c., suppose Σ' is inconsistent. In this case, for some α , $\Sigma' \vdash \alpha$ and $\Sigma' \vdash \neg\alpha$. Let β_1, \dots, β_k be the proof of α from Σ' and $\gamma_1, \dots, \gamma_l$ be the proof of $\neg\alpha$ from Σ' .

Each β_i that is an assumption from Σ' belongs to some Σ_{m_i} , and each γ_j that is an assumption from Σ' belongs to some Σ_{m_j} . Since both formal proofs of α and of $\neg\alpha$ are finite, there is some i^* that is bigger than all of these m_i 's and m_j 's. Therefore, for each β_i or γ_j that are used as assumptions, $\beta_i, \gamma_j \in \Sigma_{i^*}$. Now by monotonicity, $\Sigma_{i^*} \vdash \alpha$ and $\Sigma_{i^*} \vdash \neg\alpha$, which is a contradiction to requirement 2). \square

Theorem 5. *In a sound proof system, every consistent Σ is satisfiable.*

Proof. Let Σ' be a maximally consistent set of w.f.f.s.s.t. $\Sigma \subseteq \Sigma'$. Define a truth assignment $V_{\Sigma'}$ as follows: $V_{\Sigma'}(P_i) = T$ iff $\Sigma' \vdash P_i$ where P_i is a propositional variable. Now extend it into $V_{\Sigma'}(\alpha_i) = T$ iff $\Sigma' \vdash \alpha_i$. \square

Proof. Σ is satisfiable by all-truth assignemnt, so it's consistent. To show it's maximally consistent, we need to show for every α , if $\Sigma \not\vdash \alpha$, then $\Sigma \cup \{\alpha\}$ is inconsistent. If $\Sigma \not\vdash \alpha$, then $\Sigma \not\vdash \alpha$, then $\Sigma \cup \{\alpha\}$ is unsatisfiable, so it's inconsistent. \square