Formulation

We can always choose the coordinate system such that the start point is $(0,0)$ and the terminal be $(0,y)$

1 Notation

Denote the boundary of the each flow region be $f(x)$, which should be continuous and has corresponding subgradients. Suppose a path γ from $(x_0, f_0(x_0)) = (0,0)$ to $(x_n, f_n(x_n)) = (0,y)$ intersects the flow region boundaries at $\{(x_i, f_i)\}_{i=0}^n$ where $f_i = f_i(x_i)$ and denote the robot velocity in each region be $v = (v_x^i, v_y^i)$ and the flow velocity be $u = (u_x^i, u_y^i)$. Also the speed of robot has an upper bound, say V (i.e. $(v_x^i)^2 + (v_y^i)^2 \leq V^2$). And the time used in each region be t_i and the total time consumption is $T(\gamma) = \sum_{i=1}^n t_i(\gamma)$. The problem is $\min_\gamma T(\gamma)$

Property To achieve the optimality, in each region, the robot should move with maximum speed.
If not, say in region i, $(v_x^i)^2 + (v_y^i)^2 < V^2$, then we can let the speed be $(v_x^i + \epsilon, v_y^i + \eta)$ such that $(v_y^i - u_y^i)\epsilon = (v_x^i - u_x^i)\eta$. Since ϵ and η can be sufficiently small, the speed can always achieve the upper bound with the moving direction unchanged, which is because

$$\frac{v_x^i - u_x^i}{v_y^i - u_y^i} = \frac{v_x^i - u_x^i + \epsilon}{v_y^i - u_y^i + \eta}$$

Thus, we can always rewrite $(v_x^i, v_y^i) = (V\cos\theta_i, V\sin\theta_i)$

2 Formulation

The optimization is

$$\begin{aligned} \min_{x_1,\cdots,x_{n-1}} \ & \sum_{t=1}^n t_i(x_i) \\ \text{s.t.} \ & x_0 = x_n = 0 \\ & f_n = y \end{aligned}$$

within which, we take one specific term t_i and ignore the super- and sub-script, that is $t = t_i, v_x = v_x^i, x = x_i, f = f_i = f_i(x_i), x^- = x_{i-1}, f^- = f_{i-1} = f_{i-1}(x_{i-1})$ etc. We know that

$$\begin{aligned} & \frac{x - x^-}{f - f^-} = \frac{v_x - u_x}{v_y - u_y} \\ \Rightarrow & (x - x^-)(V\sin\theta - u_y) = (f - f^-)(V\cos\theta - u_x) \\ \Rightarrow & (f - f^-)\cos\theta - (x - x^-)\sin\theta = \frac{1}{V}(u_x(f - f^-) - u_y(x - x^-)) \\ \Rightarrow & \cos(\theta + \alpha) = \frac{u_x(f - f^-) - u_y(x - x^-)}{V\sqrt{(f - f^-)^2 + (x - x^-)^2}} \end{aligned}$$

where

$$\cos\alpha = \frac{f - f^-}{\sqrt{(f - f^-)^2 + (x - x^-)^2}}, \sin\alpha = \frac{x - x^-}{\sqrt{(f - f^-)^2 + (x - x^-)^2}}$$

Denote

$$C = \frac{u_x(f - f^-) - u_y(x - x^-)}{V\sqrt{(f - f^-)^2 + (x - x^-)^2}}$$

Then, $\theta = -\alpha \pm \arccos C$

$$\cos\theta = \cos(-\alpha \pm \arccos C) = C\cos\alpha \mp \sqrt{1-C^2}\sin\alpha$$
$$\sin\theta = \sin(-\alpha \pm \arccos C) = -C\sin\alpha \pm \sqrt{1-C^2}\cos\alpha$$

Then back to add the super- and sub-script, we have

$$C_i = \frac{u_x^i(f_i - f_{i-1}) - u_y^i(x_i - x_{i-1})}{V\sqrt{(f_i - f_{i-1})^2 + (x_i - x_{i-1})^2}} \tag{1}$$
$$\cos\alpha_i = \frac{f_i - f_{i-1}}{\sqrt{(f_i - f_{i-1})^2 + (x_i - x_{i-1})^2}} \tag{2}$$
$$\sin\alpha_i = \frac{x_i - x_{i-1}}{\sqrt{(f_i - f_{i-1})^2 + (x_i - x_{i-1})^2}} \tag{3}$$
$$\cos\theta_i = \cos(-\alpha_i \pm \arccos C_i) = C_i\cos\alpha_i \mp \sqrt{1-C_i^2}\sin\alpha_i \tag{4}$$
$$\sin\theta_i = \sin(-\alpha_i \pm \arccos C_i) = -C_i\sin\alpha_i \pm \sqrt{1-C_i^2}\cos\alpha_i \tag{5}$$

Combine (1)-(5), the final optimization is

$$\begin{aligned} \min_{x_1,\cdots,x_{n-1}} \; & \sum_{i=1}^{n} \frac{\sqrt{(x_i - x_{i-1})^2 + (f_i - f_{i-1})^2}}{\sqrt{(V\cos\theta_i - u_x^i)^2 + (V\sin\theta_i - u_y^i)^2}} \\ \text{s.t.} \; & x_0 = x_n = 0 \\ & f_n = y \end{aligned}$$