Chapter 5 Counting

Discrete Mathematics and Its Applications
Zhejiang University/CS/Course/Xiaogang Jin
E-Mail: xiaogangj@cise.zju.edu.cn

5.1 The Basics of Counting

□ BASIC COUNTING PRINCIPLE

THE SUM RULE: Suppose that the tasks $T_1, T_2, \cdots, T_m$ can be down in $n_1, n_2, \cdots, n_m$ ways respectively and no two of these tasks can be down at the same time. The the number of ways to do one of these tasks is $n_1 + n_2 + \cdots + n_m$.

Example: Suppose that either a member of the mathematics faculty or a student who is a mathematics major is chosen as representative to a university committee. How many different choices are there for this representative if there are 37 members of mathematics faculty and 83 mathematics majors?

Example: A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. How many possible projects are there to choose from?

The sum rule can be phrased in terms of sets as follow:

If $A_1, A_2, \cdots, A_m$ are disjoint sets, then the number of elements in the union of these sets is the sum of the number of elements in them.

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$$

(Task T_i is choose an element form A_i for $i = 1, 2, \cdots, m$.)

THE PRODUCT RULE : Suppose that a procedure can be performed in k successive steps, step 1 can be done in n_1 ways; step 2 can be done in n_2 ways; $\cdots$; step k can be done in n_k ways. Then the number of different ways that the procedure can be performed is the product $n_1 \cdot n_2 \cdot \cdots \cdot n_k$.

Example: The chairs of an auditorium are to be labeled with a letter and a positive integer not exceeding 100. What is the largest number of chairs that can be labelled differently? $(26 \cdot 100 = 2600)$

Example: Counting Functions

1) How many functions are there from a set with m elements to one with n elements?

2) How many one to one functions are there from a set with m elements to one with n elements?

Solution: 1) n^m.

2) When $m > n$, there are no one-to-one functions.

When $m \leq n$, there are $n(n-1)\cdots(n-m+1)$ one-to-one functions.

Example: Counting Subsets of Finite Set

1) Use the product rule to show that the number of different subset of set S is $2^{|S|}$.

2) How many binary relations are there from a set with m elements to one with n elements?

Solution: 1) Let S be a finite set. List the elements of S in arbitrary order.

– there is a bijection φ between subsets of S and bit strings of length $|S|$.

– By product rule, there are $2^{|S|}$ bit strings of length $|S|$.

Hence, $|P(S)| = 2^{|S|}$.

2) $2^{m \times n}$.

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The product rule can be phrased in terms of sets as follow:

If $A_1, A_2, \cdots, A_m$ are finite sets, then the number of elements in the cartesian product of these sets is the product of the number of elements in each set.

$$|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_m|.$$

(The task of choosing an element in the Cartesian product $|A_1 \times A_2 \times \cdots \times A_m|$ is done by choosing an element in A_1, an element in A_2, $\cdots$, and an element in A_m.)

□ MORE COMPLEX COUNTING PROBLEMS

Example: In version of the computer language BASIC, the name of a variable is a string of one or two alphanumeric characters, where uppercase and lowercase letters are not distinguished.(An alphanumeric character is either one of the 26 English letters or one of the 10 digits.) Moreover, a variable name must begin with a letter and must be different from the five strings of two characters that are reserved for programming use. How many different variable names are there in this version of BASIC?

Solution: V : the number of different variable names in this version of BASIC

– V_1 : the number of these that are one character long $V_1 = 26$

– V_2 : the number of these that are two characters long $V_2 = 26 \times 36 - 5$

Then by the sum rule, $V = V_1 + V_2 = 26 + 931 = 957$.

Example: Each user on a computer system has a password, which is six to eight character long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Solution: P : the total number of possible password

P_i : the number of of possible password of length i, respectively $(i = 6, 7, 8)$

- $P_6 = 36^6 - 26^6 = 1,867,866,560.$
- $P_7 = 36^7 - 26^7 = 70,332,353,920.$
- $P_8 = 36^8 - 26^8 = 2,612,2822,842,880.$

Then by the sum rule, $P = P_6 + P_7 + P_8 = 2,684,483,063,360.$

5.2 The Pigeonhole Principle

□ **The Pigeonhole Principle:** If $k+1$ or more objects are placed into k boxes then there is at least one box containing two or more of the objects.

– Dirichlet Drawer Principle

Example: Among any group of 367 people, there must be at least two with the same birthday, because there are 366 possible birthdays.

□ **The Generalized Pigeonhole Principle**: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Proof: Suppose that none of the contains more than $\lceil N/k \rceil - 1$ objects.

Then the total number of objects is at most

$$\boxed{\lceil N/k \rceil < (N/k) + 1}$$

$$k(\lceil N/k \rceil - 1) < k(((N/k) + 1) - 1) = N$$

There are a total of N objects. – **Contradiction!**

Some Elegant Applications of Pigeonhole Principle

Example: Show that among any $n+1$ positive integers not exceeding $2n$ there must be an integer that divides one of other integers.

Solution: Write each of the $n+1$ integers $a_1, a_2, \cdots, a_{n+1}$ as a power of 2 times an odd integer.

$$\boxed{a_j = 2^k q_j\colon\ k \in \mathbb{Z}^+ \wedge (q_j \text{ odd}) \text{ for } j = 1, \cdots, n+1}$$

$\boxed{\text{Pigeonhole: } \{q_j\}_{j=1}^{n}}$ $\leftarrow q_j \leq 2n$ and odd

$\boxed{\text{Pigeon: } \{a_j\}_{j=1}^{n+1}}$ $\Rightarrow \exists\ q_i = q_j$

Then, $a_i = a^{k_i} q_i$ and $a_j = a^{k_j} q_j$. It follows that if $k_i < k_j$, then a_i divides a_j.

Definition: Suppose that $a_1, \cdots, a_N$ is a sequence of real number.

- A subsequence of this sequence is a sequence of the form $a_{i_1}, \cdots, a_{i_m}$, where $1 \le i_1 < \cdots < i_m \le N$. Hence, a subsequence is a sequence obtained from the original sequence by including some terms of the original sequence in their original order.
- A sequence is called strictly increasing if each term is larger than the one that precedes it, and it is called strictly decreasing if each term is smaller than the one that precedes it.

Theorem: Every sequence of n^2+1 distinct real numbers contains a subsequence of length $n+1$ that is either strictly increasing or strictly decreasing.

Example: The sequence 8, 11, 9, 1, 4, 6, 12, 10, 5, 7 contain 10 terms. Note that $10 = 3^2 + 1$. There are four increasing subsequences of length four, namely, 1, 4, 6, 12; 1, 4, 6, 10; 1, 4, 6, 7; 1, 4, 5, 7. There is also a decreasing subsequences of length four, namely, 11, 9, 6, 5.

Proof: Let $a_1, a_2, \cdots, a_{n^2+1}$ be a sequence of n^2+1 distinct real numbers.

$$\boxed{a_k \leftrightarrow (i_k, d_k)}$$

i_k — the length of the <u>longest</u> increasing subsequence starting at a_k
d_k — the length of the <u>longest</u> decreasing subsequence starting at a_k

Suppose that there are no increasing or decreasing subsequence of length $n+1$.

$$\boxed{1 \le i_k, d_k \le n \qquad (k = 1, \cdots, n^2+1)}$$

Hence, there are n^2 possible ordered pair for (i_k, d_k).

Pigeon: $\{a_j\}_{j=1}^{n^2+1}$

Pigeonhole: $\{(i_k, d_k) : 1 \leq i_k, d_k \leq n\}$ $\leftarrow n^2$

$\Rightarrow$ There exist terms a_s and a_t, with $s < t$ such that $i_s = i_t$ and $d_s = d_t$.

Impossible!

Because the terms of the sequence are distinct, either $a_s < a_t$ or $a_s > a_t$.

i) If $a_s < a_t$, then , since $i_s = i_t$, an increasing subsequence of length $i_t + 1$ can be built starting at a_s, by taking a_s followed by an increasing subsequence of length i_t beginning at a_t. —**contradiction**

ii) If $a_s > a_t$, it can be shown similarly that d_s must be greater than d_t.

Example: Let $x_1, x_2, \cdots, x_n$ be a sequence of integers, then there are some successive integers in the sequence such that their sum can be divided by n.

Solution: Let $A_i = \sum_{k=1}^{i} x_k$.

1) If $\exists i$, $n \mid A_i$, the proposition is true.

2) If $n \nmid A_i, i = 1, 2, \cdots, n$:

$\boxed{\text{Pigeon: } \{A_i\}_{i=1}^{n}}$ $\boxed{\text{Pigeonhole: } [1]_n, \cdots, [n-1]_n}$

$\Rightarrow \exists i < j$, such that $A_i \equiv A_j (\mod n)$

Hence, $n \mid (A_j - A_i)$ and $A_j - A_i = x_{i+1} + x_{i+2} + \cdots + x_j$.

E-Mail: xiaogangj@cise.zju.edu.cn

Example: Every sequence $a_1, a_2, \cdots, a_N$ $(N = 2^n)$consisted of n distinct negative integers contains a successive subsequence such that their product is a perfect square.

Solution: Let $x_1, x_2, \cdots, x_n$ be n negative integers and

$$A_i = \prod_{k=1}^{i} a_k \text{ and } A_i = {x_1}^{\alpha_{i1}} {x_2}^{\alpha_{i2}} \cdots {x_n}^{\alpha_{in}}$$

where $\alpha_{ik} \geq 0$.

1) If $\exists i$, A_i be a perfect square, the proposition is true.

2) If $A_i, i = 1, 2, \cdots, n$ are not perfect square:

$$A_i \leftrightarrow (\alpha_{i1}, \cdots, \alpha_{in}) \leftrightarrow (\alpha_{i1}(\text{mod } 2), \cdots, \alpha_{in}(\text{mod } 2))$$

$\boxed{\text{Pigeon: } \{A_i\}_{i=1}^{2^n}}$

$2^n - 1$

$\downarrow$

$\boxed{\text{Pigeonhole: } \{0,1\} \times \{0,1\} \times \cdots \times \{0,1\} - (0,0,\cdots,0)}$

$\Rightarrow \exists i < j$, such that

$$\frac{A_j}{A_i} = {x_1}^{2k_1}{x_2}^{2k_2}\cdots{x_n}^{2k_n} = ({x_1}^{k_1}{x_2}^{k_2}\cdots{x_n}^{k_n})^2$$

Hence, $\frac{A_j}{A_i} = a_{i+1}a_{i+2}\cdots a_j$ is a perfect square.

Example: During a month with 30 days a baseball team plays at least 1 game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play eaxactly 14 games.

Solution: Let a_j be the number of games played on or before the jth day of the month.

$$a_1 < a_2 < \cdots < a_{30} \qquad 1 \le a_j \le 45$$

$$a_1 + 14, a_2 + 14, \cdots, a_{30} + 14 \qquad 15 \le a_j + 14 \le 59$$

Pigeon: $\{a_j\}_{j=1}^{30}$, $\{a_j + 14\}_{j=1}^{30}$

Pigeonhole: $1 \sim 59$

$$\Rightarrow \exists\ a_i = a_j + 14$$

Example: (Ramsey Theory) Assume that in a group of six people, each pair of individuals consists of two friends or two enemies. Show that there are either three mutual friends or three mutual enemies in the group.

Solution: Let A be one of the six people. Of the five other people in the group, there are either three
three or more who are enemies of

This follows from the generalized pi
objects are divided into two sets, or
elements..

In former case, suppose that B, C
of these three individuals are frie
group of three mutual friends. ot
three mutual enemies. The proof
three or more enemies of A, proc

Homework 5:

6^{th} edition	5^{th} edition
P344 1,3,4,8	P310 1,3,4,8
12	12
14	14
20	18
28	26
36	34
44	42

5.3 Permutations and Combinations

□ PERMUTATIONS

Definition: Given a set of distinct objects $X = \{x_1, \cdots, x_n\}$

- a **permutation** of X is an ordered arrangement of x_1, $\cdots$, x_n
- a r-**permutation**, where $r \leq n$ is an ordering of a subset of r-elements of X.
- The number of r-permutations of a set of distinct elements is denoted by $P(n, r)$

Theorem: $P(n,r) = n(n-1)\cdots(n-r+1) = n!/(n-r)!$
In particular, note that $p(n,n) = n!$

Proof: Select element 1 in n ways, 2 in $n-1$ ways, $\cdots$, r in $n-r+1$ ways multiply these using 1st product Principle.

Example: There are $3! = 6$ permutations of three elements a, b, c:

$$abc \quad bac \quad cab \quad acb \quad bca \quad cba$$

COMBINATIONS

Definition: Let $X = \{x_1, x_2, \cdots, x_n\}$ be a set containing n distinct elements.

• an r-**combination** of X is an unordered selection of r elements of X.

• the number of r-combinations of a set of n distinct elements is denoted by $C(n, r)$.

Theorem: $C(n, r) = n!/(n-r)!r! = P(n, r)/r!$

Proof: The product principle says that $P(n, r)$ is product of $C(n, r)$ and number of orderings of r elements, namely $r!$.

□ BINOMIAL COEFFICIENTS

Theorem: (Binomial theorem) If a and b are real numbers and n is a positive integer, then

$$\begin{aligned}(a+b)^n =& C(n,0)a^n b^0 + C(n,1)a^{n-1}b^1 + \cdots \\ &+ C(n,n-1)a^1 b^{n-1} + C(n,n)a^0 b^n\end{aligned}$$

Proof: $(a+b)^n = \underbrace{(a+b)(a+b)\cdot\cdots\cdot(a+b)}_{n \text{ factors}}$

A term of form $a^{n-k}b^k$ arises from choosing a from $n-k$ factors and b from k factors. This can be done in $C(n,k)$ number of ways since this counts the number of ways of selecting k things from n items.

Pascal's Triangle

$$(a+b)^3 = C(3,0)a^3 + C(3,1)a^2b + C(3,2)ab^2 + C(3,3)b^3$$

If we put $a = b = 1$ we obtain $2^3 = 1 + 3 + 3 + 1$

Pascal's triangle

1		
1 1		Border consists of 1's
1 2 1		Other values is sum of
1 3 3 1		two numbers above it
1 4 6 4 1		
1 5 10 10 5 1	⇐terms for 2^3	

Theorem: PASCAL'S IDENTITY Let n and k be positive integers with $n \geq k$. Then

$$C(n+1,k) = C(n,k) + C(n,k-1)$$

Theorem: Let n be a positive integer. Then

$$\sum_{k=0}^{n} C(n,k) = 2^n$$

Theorem: VANDERMONDE'S IDENTITY Let m, n and r be nonnegative integers with r not exceeding either m or n. Then

$$C(m+n,r) = \sum_{k=0}^{r} C(m,r-k)C(n,k)$$

5.4 Generalized Permutations & Combinations

□ PERMUTATIONS WITH REPETITION

Theorem: The number of r-permutations of a set of n objects with repetition allowed is n^r.

Proof: There are n ways to select an element of the set for each of the r positions in the r-permutation when repetition is allowed, since for each choice all n objects are available. Hence, by the product rule there are n^r r-permutation when repetition is allowed.

Remark: r different objects , n different boxes and every box can contain more than one objects.

□ COMBINATIONS WITH REPETITION

Example: How many ways are there to select four pieces of fruit from a bowl containing apples, oranges, and pears if the order in which the pieces are selected not matter, only the type of fruit and not the individual piece matters, and there are at least four pieces of each type of fruit in the bowl?

Solution: To solve this problem we list all the ways possible to select the fruit. There are 15 ways:

4 apples	4 oranges	4 pears
3 apples, 1 orange	3 apples, 1 pear	3 oranges, 1 apple
3 oranges, 1 pear	3 pears, 1 apple	3 pears, 1 orange
2 oranges, 2 pears	2 pears, 2 apples	2 pears, 2 oranges
2 apples, 1 orange, 1 pear		2 oranges, 1 apple, 1 pear
2 pears, 1 orange, 1 apple		

We could have	**apple**	**orange**	**pear**
	XXX |	X |	
or we could have	**apple**	**orange**	**pear**
	XX |	X |	X

The solution is the number of 4-combination with repetition allowed from a three element set, $\{apple, orange, pear\}$.

Theorem: There are $C(n+r-1, r)$ r-combinations from a set with n elements when repetition of elements is allowed.

Proof: Each r-combination of a set with n elements when repetition is allowed can be represented by a list of $n-1$ bars and r stars.

The $n-1$ bars are used to mark off n different cells, with the ith cell containing a star of each time the ith element of the set occurs in the combination.

For instance, a 6-combination of a set with four elements is represented with three bars and six stars. Here

$$** \mid * \mid\mid * * *$$

represents the combination containing exactly two of the first element, one of the second element, none of the third element, and three of the fourth element of the set.

As we have seen, each different list contain $n-1$ bars and r stars corresponds to an r-combination of the set with n elements when repetition is allowed.

The number of such list is $C(n+r-1, r)$, since each list corresponds to a choice of the r position to place the r stars from the $n-1+r$ positions that contain r stars and $n-1$ bars.

Example: Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen? Assume that only the type of cookie, and not the individual cookies or the order in which they are chosen, matters.

Solution: The number of ways to choose six cookies is the number of 6-combinations of a set with four elements. From Theorem this equals $C(4+6-1,6) = C(9,6) = 84$.

Example: How many solutions in nonnegative integers are there to the equation $x_1 + x_2 + x_3 + x_4 = 29$? How many of these satisfy $x_1 > 0, x_2 > 1, x_3 > 2, x_4 \geq 0$?

Solution: (a) Note that a solution corresponds to a way of selecting 29 items from a set with four elements, so that x_1 items of type one, x_2 items of type two, x_3 items of type three, and x_4 items of type four are chosen. Hence, the number of solutions is equal to the number of 29-combinations with repetition allowed from a set with four elements. From Theorem it follows that there are

$$C(4 + 29 - 1, 29) = C(32, 29) = C(32, 3) = 4960$$

solutions

(b) $C(4 + 23 - 1, 23) = C(26, 23) = C(26, 3) = 2600$

Example: How many ways to place $2t+1$ indistinguishable balls into 3 distinguishable boxes such that the total number of balls in arbitrary two boxes is larger than the third one?

Solution:

$$C(3+2t+1-1, 2t+1) - 3C(3+t-1, t) = C(2t+3, 2) - 3C(t+2, 2)$$

□ DISTRIBUTING OBJECTS INTO BOXES

Example: How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards?

Solution: We note that the first player can be dealt 5 cards in $C(52,5)$ ways. The second player can be dealt 5 cards in $C(47,5)$ ways. The third player can be dealt 5 cards in $C(42,5)$ ways. Finally, the fourth player can be dealt 5 cards in $C(37,5)$ ways.

Hence, by the product rule, the total number of ways to deal 5 cards each

$$C(52,5)C(47,5)C(42,5)C(37,5) = \frac{52!}{47!5!} \cdot \frac{47!}{42!5!} \cdot \frac{42!}{37!5!} \cdot \frac{37!}{32!5!}$$

$$= \frac{52!}{5!5!5!5!32!}$$

Theorem: The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed in to box i, $i = 1, 2, \cdots, k$ equals

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

5.5 Generating Permutations and Combinations

□ GENERATING PERMUTATIONS

- We will describe one algorithm that is based on the **lexicographic ordering** of the set of permutations of $\{1, 2, \cdots, n\}$.

- The permutation $a_1 a_2 \cdots a_n$ **precedes** the permutation of $b_1 b_2 \cdots b_n$, if for some k, with $1 \le k \le n$, $a_1 = b_1, \cdots,$ $a_{k-1} = b_{k-1}$, and $a_k < b_k$.

For example, the permutation 23415 of the set $\{1, 2, 3, 4, 5\}$ precedes the permutation 23514.

Main idea:

A procedure that constructs the next permutation in lexicographic order following a given permutation $a_1a_2\cdots a_n$.

1) Find the integer a_j and a_{j+1} with $a_j < a_{j+1}$ and

$$a_{j+1} > a_{j+2} > \cdots > a_n$$

i.e. the last pair of adjacent integers in the permutation where the first integer in the pair is smaller than the second.

2) the next largest permutation in lexicographic order is obtain by putting the jth position the least integer among $a_j, a_{j+1}, \cdots$, and a_n, that is greater than a_j and listing in increasing order the rest of integers $a_j, a_{j+1}, \cdots$, and a_n in positions $j+1$ to n.

Example: What is the next largest permutation in lexicographic order after 362541.

Solution: 364125.

Example: Generate the permutation of the integers $1, 2, 3, 4$ in lexicographic order.

Solution: $1234 \rightarrow 1243 \rightarrow 1324 \rightarrow 1342 \rightarrow 1423 \rightarrow 1432$
$\rightarrow 2134 \rightarrow 2143 \rightarrow 2314 \rightarrow 2341 \rightarrow 2413 \rightarrow 2431$
$\rightarrow 3124 \rightarrow 3142 \rightarrow 3214 \rightarrow 3241 \rightarrow 3412 \rightarrow 3421$
$\rightarrow 4123 \rightarrow 4132 \rightarrow 4213 \rightarrow 4231 \rightarrow 4312 \rightarrow 4321$

Zhejiang University/CS/Course/Xiaogang Jin
E-Mail: xiaogangj@cise.zju.edu.cn

ALGORITHM **Generating the Next Larg**
Lexicographic Order.
procedure $nextpermutation(a_1 a_2 \cdots a_n :$ perm
not equal to $n\ n-1\ \cdots\ 2\ 1$
$j := n - 1$
while $a_j > a_{j+1}$
$j := j - 1$
$\{\ j$ is the largest subscript with $a_j < a_{j+1}\}$
$k := n$
while $a_j > a_k$
  $k := k - 1$
$\{\ a_k$ is the smallest integer greater than a_j to
interchange a_k and a_j
$r := n$
$s := j + 1$
while $r > s$
begin
interchange a_r and a_s
  $r := r - 1$
  $s := s + 1$
end
{this puts the tail end of the permutation afte

Homework 6:

6^{th} edition	5^{th} edition
P353 6 10 36 37-38 42	P319 6 10 36 37-38 42
P361 26 28 32	P325 26 28 32
P369 11 24	P334 10 24
P379 6 10 14 16 32	P342 6 10 14 16 42