第十一周作业

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4. 设总体 X 服从标准正态分布, $X_1, \cdots, X_{16}$ 是来自总体 X 的简单随机样本, 写出下列统计量的分布:

(1) 样本均值 $\overline{X}$; (2) $\sum_{i=1}^{16} X_i^2$; (3) $\dfrac{3X_1}{\sqrt{\sum_{i=2}^{10} X_i^2}}$; (4) $\dfrac{X_1+X_2}{\sqrt{X_3^2+X_4^2}}$; (5) $\overline{X}-X_1$.

4. (1) $X \sim N(0,1)$　∴ $\sum_{i=1}^{16} X_i \sim N(0,16)$　∴ $\bar{X}=\dfrac{\sum_{i=1}^{16} X_i}{16} \sim N(0,\frac{1}{16})$

(2) $\sum_{i=1}^{16} X_i^2 \sim \chi^2(16)$

(3) $\dfrac{3X_1}{\sqrt{\sum_{i=2}^{10} X_i^2}} = \dfrac{X_1}{\sqrt{\frac{\sum_{i=2}^{10} X_i^2}{9}}} \sim t(9)$

(4) $X_1+X_2 \sim N(0,2) \Rightarrow \dfrac{X_1+X_2}{\sqrt{2}} \sim N(0,1)$

$X_3^2+X_4^2 \sim \chi^2(2)$

∴ $\dfrac{X_1+X_2}{\sqrt{X_3^2+X_4^2}} = \dfrac{\frac{X_1+X_2}{\sqrt{2}}}{\sqrt{\frac{X_3^2+X_4^2}{2}}} \sim t(2)$

(5) $\bar{X}-X_1 = \sum_{i=2}^{16} \dfrac{X_i}{16} - \dfrac{15}{16}X_1 \sim N(0, \frac{15}{16^2}+(\frac{15}{16})^2)$

即 $\bar{X}-X_1 \sim N(0,\frac{15}{16})$

7. 设总体 $X \sim N(\mu, \sigma^2), X_1, \cdots, X_9$ 是来自总体 X 的简单随机样本, $\overline{X}$ 是样本均值, S^2 是样本方差, 写出下列抽样分布:

(1) $\dfrac{3(\overline{X}-\mu)}{\sigma}$; (2) $\dfrac{3(\overline{X}-\mu)}{S}$; (3) $\dfrac{\sum_{i=1}^{9}(X_i-\overline{X})^2}{\sigma^2}$;

(4) $\dfrac{\sum_{i=1}^{9}(X_i-\mu)^2}{\sigma^2}$; (5) $\dfrac{9(\overline{X}-\mu)^2}{\sigma^2}$; (6) $\dfrac{9(\overline{X}-\mu)^2}{S^2}$;

(7) $\dfrac{2(X_1-X_2)^2}{(X_3-X_4)^2+(X_5-X_6)^2}$;

(8) $\dfrac{(X_1-Y_1)^2+(X_2-Y_1)^2+(X_3-Y_1)^2}{(X_4-Y_2)^2+(X_5-Y_2)^2+(X_6-Y_2)^2}$, 其中 $Y_1=\dfrac{X_1+X_2+X_3}{3}, Y_2=\dfrac{X_4+X_5+X_6}{3}$.

7. (1) $X \sim N(\mu,\sigma^2) \Rightarrow \bar{X} \sim N(\mu,\frac{\sigma^2}{n}) = N(\mu,\frac{\sigma^2}{9})$

(1) $\bar{X}-\mu \sim N(0,\frac{\sigma^2}{n}) \Rightarrow \dfrac{3(\bar{X}-\mu)}{\sigma} \sim N(0,1)$

(2) $\dfrac{S^2}{\sigma^2} = \dfrac{n}{n-1} \Rightarrow \dfrac{S}{\sigma} = \sqrt{\dfrac{n}{n-1}}$

∵ $\dfrac{3(\bar{X}-\mu)}{\sigma} \sim N(0,1)$　∴ $\dfrac{3(\bar{X}-\mu)}{S} \sim N(0,\frac{9(n-1)}{n^2}) = N(0,\frac{8}{9})$

(3) $X_i-\bar{X}=-\frac{\sum_{j\neq i}X_j}{9}+\frac{8X_i}{9}\sim N\left(0,8\times\frac{\sigma^2}{9^2}+\left(\frac{8}{9}\right)^2\cdot\sigma^2\right)$

$\therefore \frac{X_i-\bar{X}}{\sigma}\sim N\left(0,\frac{8}{9}\right) \Rightarrow \frac{X_i-\bar{X}}{S}\sim N(0,1)$ $\quad \cancel{\frac{\sigma^2}{S^2}=\frac{8}{9}}$

则 $\sum_{i=1}^{9}\left(\frac{X_i-\bar{X}}{S}\right)^2\sim\chi^2(9)$

又∵ $\frac{8S^2}{\sigma^2}\sim\chi^2(8)$

$\therefore \frac{\sum_{i=1}^{9}(X_i-\bar{X})^2}{\sigma^2}=\frac{\sum_{i=1}^{9}\left(\frac{X_i-\bar{X}}{S}\right)^2/9}{\frac{8S^2}{\sigma^2}/8}\sim F(9,8)$

(4) $\frac{X_i-\mu}{\sigma}\sim N(0,1)$

$\therefore \frac{\sum_{i=1}^{9}(X_i-\mu)^2}{\sigma^2}=\sum_{i=1}^{9}\left(\frac{X_i-\mu}{\sigma}\right)^2\sim\chi^2(9)$

(5) $\bar{X}-\mu\sim N\left(0,\frac{\sigma^2}{9}\right)$ $\quad\therefore \frac{\bar{X}-\mu}{\frac{\sigma}{3}}\sim N(0,1)$.

$\therefore \frac{9(\bar{X}-\mu)^2}{\sigma^2}\sim\chi^2(1)$

(6) $\frac{8S^2}{\sigma^2}\sim\chi^2(8)$

$\therefore \frac{9(\bar{X}-\mu)^2}{S^2}=\frac{\frac{9(\bar{X}-\mu)^2}{\sigma^2}/1}{\frac{8S^2}{\sigma^2}/8}\sim F(1,8)$

(7) $\frac{X_1-X_2}{\sqrt{2}\sigma}\sim N(0,1)$, $\frac{X_3-X_4}{\sqrt{2}\sigma}\sim N(0,1)$ $\frac{X_5-X_6}{\sqrt{2}\sigma}\sim N(0,1)$

$\therefore \frac{(X_1-X_2)^2}{2\sigma}\sim\chi^2(1)$. $\frac{(X_3-X_4)^2}{2\sigma^2}\sim\chi^2(1)$, $\frac{(X_5-X_6)^2}{2\sigma^2}\sim\chi^2(1)$

又∵ χ^2分布有可加性 $\quad\therefore \frac{(X_3-X_4)^2+(X_5-X_6)^2}{2\sigma^2}\sim\chi^2(2)$

则 $\frac{2(X_1-X_2)^2}{(X_3-X_4)^2+(X_5-X_6)^2}=\frac{\frac{X_1-X_2}{2\sigma^2}/1}{\frac{(X_3-X_4)^2+(X_5-X_6)^2}{2\sigma^2}/2}\sim F(1,2)$

(8) 易知 $\frac{X_i-Y_1}{\sigma}\sim N\left(0,\frac{2}{3}\right)$ $\quad i=1,2,3$.

$\frac{X_j-Y_2}{\sigma}\sim N\left(0,\frac{2}{3}\right)$ $\quad j=4,5,6$

$\therefore \frac{\sum_{i=1}^{3}(X_i-Y_1)^2}{\sum_{j=1}^{3}(X_j-Y_2)^2}=\frac{\frac{\sum_{i=1}^{3}(X_i-Y_1)^2}{\frac{2}{3}\sigma^2}/3}{\frac{\sum_{j=1}^{3}(X_j-Y_2)^2}{\frac{2}{3}\sigma^2}/3}\sim F(3,3)$

9. 设总体 $X\sim U(0,\theta),X_1,\cdots,X_5$ 是来自总体 X 的简单随机样本, $\overline{X}$ 是样本均值, S^2 是样本方差, 求 $E(\overline{X}),E(\overline{X}^2)$ 和 $E(S^2)$.

9. $E(\bar{X})=\frac{\sum_{i=1}^{5}E(X_i)}{5}=\frac{\theta}{2}$

$D(\bar{X})=E(\bar{X}^2)-[E(\bar{X})]^2=\frac{\sum_{i=1}^{5}D(X_i)}{25}=\frac{\frac{5}{12}\theta^2}{25}=\frac{\theta^2}{60}$

$\therefore E(\bar{X}^2)=\frac{\theta^2}{60}+\frac{\theta^2}{4}=\frac{4}{15}\theta^2$

$S^2=\frac{\sum_{i=1}^{5}(X_i-\bar{X})^2}{4}=\frac{5}{4}\sigma^2$

$\therefore E(S^2)=\frac{5}{4}E(\sigma^2)=\frac{5}{4}\cdot\frac{\theta^2}{60}=\frac{\theta^2}{48}$

10. 设总体 X 的密度函数

$$f(x)=\begin{cases}\lambda \mathrm{e}^{-\lambda x}, & x>0,\\ 0, & x\leqslant 0,\end{cases}$$

从总体中抽取容量是 10 的样本.

(1) 求样本均值的数学期望和方差;

(2) 记 $X_{(1)}=\min\{X_1,\cdots,X_{10}\}$, 求 $X_{(1)}$ 的数学期望和方差.

10.(1) 可见 $E(X)=\frac{1}{\lambda}$, $D(X)=\frac{1}{\lambda^2}$

故 $E(\bar{X})=E(X)=\frac{1}{\lambda}$, $D(\bar{X})=\frac{D(X)}{n}=\frac{1}{10\lambda^2}$

(2) $F(x)=\begin{cases}1-e^{-\lambda x}, & x>0\\ 0, & x\leq 0\end{cases}$

$\therefore \min\{X_1,\cdots,X_{10}\}$ 对应的. $G(x)=1-[1-F(x)]^{10}=\begin{cases}1-e^{-10\lambda x}, & x>0\\ 0, & x\leq 0\end{cases}$

$\therefore g(x)=\frac{d}{dx}G(x)=\begin{cases}10\lambda e^{-10\lambda x}, & x>0\\ 0, & x\leq 0\end{cases}$

则 $E(X_{(1)})=\frac{1}{10\lambda}$

$D(X_{(1)})=\frac{1}{100\lambda^2}$

11. 设 $X_1,\cdots,X_8$ 是来自标准正态总体的样本, $\overline{X}=\frac{1}{8}\sum_{i=1}^{8}X_i, S^2=\frac{1}{7}\sum_{i=1}^{8}(X_i-\overline{X})^2, X_9$ 是新增的样本, 试确定 $Y=\frac{2\sqrt{2}}{3}\frac{X_9-\overline{X}}{S}$ 的分布.

11. $X_9-\bar{X}=X_9-\frac{\sum_{i=1}^{8}X_i}{8}\sim N(0,1+\frac{8}{8^2})=N(0,\frac{9}{8})$

由于 $S^2=\frac{1}{7}\sum_{i=1}^{8}(X_i-\bar{X})^2$ $\therefore S^2=\frac{8}{7}\cdot\frac{\sum_{i=1}^{8}(X_i-\bar{X})^2}{8}$

由定理知 S ~~与 $\bar{X}$ 独立~~ $\frac{7S^2}{\sigma^2}\sim\chi^2(7)$. 其中 $\sigma^2=1$

则根据 $\frac{X_9-\bar{X}}{\frac{3}{2\sqrt{2}}}\sim N(0,1)$

$\frac{7S^2}{1}\sim\chi^2(7) \Rightarrow \frac{\frac{X_9-\bar{X}}{\frac{3}{2\sqrt{2}}}}{\sqrt{7S^2/7}}\sim t(7)$

即 $\frac{2\sqrt{2}}{3}\frac{(X_9-\bar{X})}{S}\sim t(7)$

12. 设总体 $X\sim N(\mu,\sigma^2), X_1,\cdots,X_5$ 和 $Y_1,\cdots,Y_9$ 是来自总体 X 的两个独立样本, $\overline{X}$ 和 $\overline{Y}$ 分别是两个样本的样本均值, S_1^2 和 S_2^2 分别是两个样本的样本方差.

(1) 若 $\frac{a(\overline{X}-\overline{Y})}{\sigma}\sim N(0,1)$, 求 a;

(2) 若 $\frac{b(\overline{X}-\overline{Y})}{\sqrt{S_1^2+2S_2^2}}\sim t(12)$, 求 b.

12.(1) $\bar{X}\sim N(\mu,\frac{\sigma^2}{5})$ $\bar{Y}\sim N(\mu,\frac{\sigma^2}{9})$

$\therefore \frac{\bar{X}-\bar{Y}}{\sigma}\sim N(0,\frac{14}{45})$ 则 $a=\sqrt{\frac{45}{14}}=\frac{3\sqrt{5}}{\sqrt{14}}=\frac{3}{14}\sqrt{70}$

(2) 由定理知 $\frac{\bar{X}-\bar{Y}-(\mu-\mu)}{\sqrt{\frac{1}{5}+\frac{1}{9}}\cdot\sqrt{\frac{4S_1^2+8S_2^2}{12}}}\sim t(12)$

即 $\frac{1}{\sqrt{\frac{14}{3\times 45}}}\cdot\frac{\bar{X}-\bar{Y}}{\sqrt{S_1^2+2S_2^2}}\sim t(12)$

则 $b=3\sqrt{\frac{15}{14}}=\frac{3}{14}\sqrt{210}$

13. 在两个等方差的正态总体中, 独立地各抽取一个容量为 7 的样本, 它们的样本方差分别为 S_1^2, S_2^2, 若 $P\left\{\max\left(\frac{S_1^2}{S_2^2}, \frac{S_2^2}{S_1^2}\right) > c\right\} = 0.05$, 求 c 的值.

13. $\frac{(7-1)S_1^2}{\sigma_1^2} \sim \chi^2(6)$ $\frac{(7-1)S_2^2}{\sigma_2^2} \sim \chi^2(6)$, 又 $\sigma_1^2 = \sigma_2^2$

$\therefore$ $\frac{S_1^2}{S_2^2} \sim F(6,6)$. 同理 $\frac{S_2^2}{S_1^2} \sim F(6,6)$

则 $\max\left\{\frac{S_1^2}{S_2^2}, \frac{S_2^2}{S_1^2}\right\} > c$ 的反事件为 $\frac{S_1^2}{S_2^2} \le c$ 且 $\frac{S_2^2}{S_1^2} \le c$

即 $P\{F \le c$ 且 $\frac{1}{F} \le c\} = 1-0.05 = 0.95$

$\Rightarrow P\{F \le c$ 且 $F \ge \frac{1}{c}\} = 0.95$

$\Rightarrow P\{F \le c\} - P\{F < \frac{1}{c}\} = 0.95$

$\Leftrightarrow P\{F \le c\} - P\{\frac{1}{F} > c\} = 0.95$

因 F 与 $\frac{1}{F}$ 同分布

$\therefore P\{F \le c\} - (1 - P\{F \le c\}) = 0.95$

$\Rightarrow P\{F \le c\} = 0.975$

查表可得 $F_{0.025}(6,6) = 5.82$

$\therefore c = 5.82$