INTEGRAL CALCULUS

NTU 108-1 Integral Calculus

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▸▸ Legal stuff

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- Links to the original files can be found at the text webpage

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Chapter 1

Integration

1.1▲ Integrals 積分

$$\delta x = \frac{b-a}{n} \quad x_i = a + i\delta x$$

$$\text{area of } A_i = f(x_i)\delta x \implies \sum_{i=1}^{n} = A_i = \sum_{i=1}^{n} f(x_i)\delta x$$

Definition1.1.1.

$$\text{area under } y = f(x) \text{ from } a \text{ to } b \begin{cases} = & \displaystyle\int_a^b f(x)dx \\ := & \displaystyle\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\delta x \end{cases}$$

▸▸ Techniques of Integrations

- Substitution Rule
  (change of variables)
- Trigonometric Integral ⋆
  $\int \sin^2 x \cos^3 x dx$
- Integration by Parts ⋆
  $\int \frac{P(x)}{Q(x)} dx$ where P, Q are polynomials

Example 1.1.2

$$y = f(x) = x^2 \quad a = 0, b = 1$$

$$\delta x = \frac{1-0}{n} = \frac{1}{n}$$

$$x_i = a + i \cdot \delta x = i \cdot \frac{1}{n} = \frac{i}{n}$$

$$\begin{aligned}
\sum_{i=1}^{n} f(x_i)\delta x &= \sum_{i=1}^{n} (\frac{i}{n})^2 \frac{1}{n} \\
&= \frac{1}{n^3} \sum_{i=1}^{n} i^2 \\
&= \frac{1}{n^3} \frac{n}{6} (n+1)(2n+1) \\
&= \frac{(n+1)(2n+1)}{6n^2}
\end{aligned}$$

let $n \to \infty$

$$\begin{aligned}
\sum_{i=1}^{\infty} f(x_i)\delta x &= \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} \\
&= \lim_{n\to\infty} \frac{2n^2+3n+1}{6n^2} \\
&= \lim_{n\to\infty} \frac{2+\frac{3}{n}+\frac{1}{n^2}}{6} \\
&= \frac{1}{3}
\end{aligned}$$

$$(\frac{1}{3} = \int_0^1 x^2 dx)$$

Example 1.1.2

1.2▲ Definite Integral 定積分

Definition1.2.1.

Let $x_i^* \in [x_{i-1}, x_i]$ be any point $\quad i = 1, \cdots, n$

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\delta x$$

if the limit exists

Properties of Definite Integral

- $\int_a^b f(x)dx = -\int_b^a f(x)dx \quad (\delta x = \frac{b-a}{n})$
- $\int_a^a f(x)dx = 0 \quad (\delta x = \frac{a-a}{n} = 0)$
- $\int_a^b cdx = c(b-a)$
- $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
- $\int_a^b cf(x)dx = c\int_a^b f(x)fd$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

1.3 Indefinite Integral 不定積分

Definition1.3.1.

$$\int f(x)dx = F(x) \text{ where } F'(x) = f(x)$$

Properties of Indefinite Integral

- $\int x^n dx = \frac{1}{n+1}x^{n+1} + c$
- $\int x^{-1}dx = lnx + c$
- $\int e^x dx = e^x + c$
- $\int a^x dx = \frac{1}{lna}a^x + c$
- $\int \sin x dx = -\cos x + c$
- $\int \cos x dx = \sin x + c$
- $\int \sec x^2 dx = \tan x + c$

- $\int \sec x \tan x dx = \sin x + c$
- $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$
- $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} + c$

Example 1.3.2

$$g(x) = \int_0^x \sqrt{1+t^2} dt$$

$$g'(x) = \frac{d}{dx} \int_0^x \sqrt{1+t^2} dt = \sqrt{1+x^2}$$

Example 1.3.2

Example 1.3.3

$$\begin{aligned} \frac{d}{dx} \int_1^{x^4} \sec t dt &= \frac{d}{d(x^4)} \int_1^{x^4} \sec t dt \cdot \frac{d(x^4)}{dx} \quad \text{by Chain Rule} \\ &= \frac{d}{du} \int_1^{u} \sec t dt (4x^3) \quad \text{let } x^4 = u \\ &= \sec u \cdot 4x^3 \quad \text{by F.T.C} \\ &= \sec(x^4) \cdot 4x^3 \end{aligned}$$

Example 1.3.3

1.4▲ Fundamental Theorem of Calculus 微積分基本定理

Theorem1.4.1.

If $f(x)$ is conti on $[a, b]$ and let $g(x) = \int_a^x f(t) dt (a \leqslant x \leqslant b)$. Then

(1) $g'(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x)$

(2) $\int_a^b f(x) dx = F(b) - F(a)$ where $F'(x) = f(x)$

Proof. F.T.C

We prove (1) first, let $h > 0$
To prove (1), let $x, x+h \in [a,b]$

$$\begin{aligned} g(x+h) - g(x) &= \int_a^{x+h} f(t)dt - \int_a^x f(t)dt \\ &= \int_x^{x+h} f(t)dt \end{aligned}$$

Use extreme value theorem on $f(x)$ for $[x, x+h]$

$$\max_{[x,x+h]} f(x) = f(u) \implies f(x) \leqslant f(u) \quad \forall x \in [x, x+h]$$

$$\min_{[x,x+h]} f(x) = f(v) \implies f(x) \geqslant f(v) \quad \forall x \in [x, x+h]$$

for some $u, v \in [x, x+h]$

$$\begin{aligned} \int_x^{x+h} f(v)dt &\leqslant \int_x^{x+h} f(t)dt \leqslant \int_x^{x+h} f(u)dt \\ hf(v) &\leqslant \int_x^{x+h} f(t)dt \leqslant hf(u) \\ f(v) &\leqslant \frac{1}{h}\int_x^{x+h} f(t)dt \leqslant f(u) \end{aligned}$$

$$\frac{g(x+h)-g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)dt$$

$$\lim_{h\to 0^+} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)dt$$

$$\lim_{h\to 0^+} f(v) = f(x)$$

$$\lim_{h\to 0^+} f(u) = f(x)$$

If we can show $\lim\limits_{h\to 0^-} \dfrac{g(x+h)-g(x)}{h} = f(x)$, then

$$g'(x) = f(x)$$

prove (2) from (1)

$$\begin{aligned} g'(x) &= f(x) = F'(x) \\ \implies & \frac{d}{dx}(g(x) - F(x)) = 0 \\ \implies & g(x) - F(x) = c \quad c \text{ is a const} \end{aligned}$$

$$g(x) = \int_a^x f(t)dt$$

$$g(a) = 0, g(b) = \int_a^b f(t)dt$$

$$\implies g(a) - F(a) = c$$
$$\implies F(a) = -c$$
$$\int_a^b f(t)dt = g(b) = F(b) + c = F(b) - F(a)$$

□

Example 1.4.2

$$\int_3^6 \frac{1}{x} dx$$

By F.T.C,

$$f(x) = \frac{1}{x} \implies F(x) = lnx + c$$
$$\int_3^6 \frac{1}{x} dx = (ln6 + \cancel{c}) - (ln3 + \cancel{c}) = ln6 - ln3 = ln2$$

Example 1.4.2

Example 1.4.3

$$\int_{-1}^3 \frac{1}{x^2} dx$$

$$f(x) = \frac{1}{x^2} \implies F'(x) = \frac{1}{x^2} = x^{-2} \implies F(x) = -x^{-1} + c$$
$$\int_{-1}^3 \frac{1}{x^2} dx = -\frac{1}{x}\Big|_{-1}^3 = (-\frac{1}{3}) - (1) = -\frac{4}{3} < 0$$

$f(x) = \frac{1}{x^2}$ is NOT defined at $x = 0$

Example 1.4.3

Example 1.4.4

Find $g'(x)$ where $g(x) = \int_{2x}^0 \frac{1}{1+t^3} dt$

Solution.

$$g(x) = -\int_0^{2x} \frac{1}{1+t^3} dt$$
$$g'(x) = -\frac{1}{1+(2x)^3} \cdot 2$$

Example 1.4.4

Joke1.4.5 (本書特有題).

殺鳥 $\implies$ 織田信長 讓鳥叫 $\implies$ 豐臣秀吉 等鳥叫 $\implies$ 德川家康

1.5▲ Substitution Rule

Definition1.5.1.

$$u = u(x) \implies \frac{du}{dx}u'(x)$$

$$\int f(u)du = \int (f(u(x))u'(x)dx$$

Proof. Substitution Rule (by Chain Rule + F.T.C)

let F satisfying $F' = f$

$$\frac{dF(u(x))}{dx} = F'(u(x))u'(x)$$

$$\int \frac{dF(u(x))}{dx} = \int F'(u(x))u'(x)dx$$

$$F(u(x)) + c = \int F'(u(x))u'(x)dx$$

$$\int F'(u)du + c = \int F'(u(x))u'(x)dx$$

$$\int f(u)du + c = \int f(u(x))u'(x)dx$$

□

Example 1.5.2

$$\int (\cos(x^2))dx$$

let $u = x^2$

$$\frac{du}{dx} = 2x \implies du = 2xdx \implies \frac{1}{2}du = xdx$$

$$\begin{aligned}\int(\cos u)x\,dx &= \frac{1}{2}\int \cos u\,du\\ &= \frac{1}{2}(\sin u + c)\\ &= \frac{1}{2}(\sin(x^2)+c)\\ &= \frac{1}{2}\sin(x^2)+c\end{aligned}$$

$$\begin{aligned}\int(\cos(x^2))dx &= \textstyle\int \cos(x^2)\frac{1}{2}d(x^2)\\ &= \frac{1}{2}\int\cos(x^2)d(x^2)\\ &= \frac{1}{2}\sin(x^2)+c\end{aligned}$$

Example 1.5.2

Example 1.5.3

$$\begin{aligned}\int\frac{x}{\sqrt{1-x^2}}dx &= \frac{1}{2}\int\frac{d(x^2)}{\sqrt{1-x^2}}\\ \text{let } u=x^2 &= \frac{1}{2}\int\frac{du}{\sqrt{1-u}}\\ &= -(1-u)^{\frac{1}{2}}+c\\ &= -(1-x^2)^{\frac{1}{2}}+c\end{aligned}$$

$$\begin{aligned}\int\frac{du}{\sqrt{1-u}} &= \int(1-u)^{-\frac{1}{2}}du\\ \text{let } 1-u=v &= \int v^{-\frac{1}{2}}(-dv)\\ &= \int v^{-\frac{1}{2}}dv\\ &= -(2v^{\frac{1}{2}}+c)\\ &= -2(1-u)^{\frac{1}{2}}+c\end{aligned}$$

$$\frac{dv}{du}=\frac{d}{du}(1-u)=-1$$

$$-dv=du$$

Example 1.5.3

Example 1.5.4

$$\begin{aligned}
\int \tan x dx &= \int \frac{\sin x}{\cos x} dx \\
&= -\int \frac{d(\cos x)}{\cos x} \\
\text{let } \cos x = u \quad &= -\int \frac{du}{u} \\
&= -ln u + c \\
&= -ln(\cos x) + c
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}(-ln(\cos x)) &= -\frac{-\sin x}{\cos x} \\
&= \tan x
\end{aligned}$$

Example 1.5.4

Example 1.5.5

Find $I = \int (2x+1)^{\frac{1}{2}} dx$

Solution. let $u = 2x+1$

$$\frac{du}{dx} = 2 \implies du = 2dx \implies dx = \frac{1}{2} du$$

$$\begin{aligned}
I &= \int u^{\frac{1}{2}} \frac{1}{2} du \\
&= \frac{1}{2} \int u^{\frac{1}{2}} du \\
&= \frac{1}{\cancel{2}} \frac{\cancel{2}}{3} u^{\frac{3}{2}} + c \\
&= \frac{1}{3} u^{\frac{3}{2}} + c \\
&= \frac{1}{3} (2x+1)^{\frac{3}{2}} + c
\end{aligned}$$

Example 1.5.5

Example 1.5.6

Find $I = \frac{lnx}{x} dx$

Solution. let $u = lnx$

$$\frac{du}{dx} = \frac{1}{x} \implies du = \frac{dx}{x}$$

$$\begin{aligned} I &= \int \frac{u}{x} dx \\ &= \int u \frac{dx}{x} \\ &= \int u du \\ &= \frac{1}{2} u^2 + c \\ &= \frac{1}{2} (lnx)^2 + c \end{aligned}$$

Example 1.5.6

Example 1.5.7

Find $I = \int_0^4 (2x+1)^{\frac{1}{2}} dx$

Solution. let $u = 2x + 1$

$$x = 0, u = 1$$

$$x = 4, u = 9$$

$$\begin{aligned} I &= \frac{1}{2} \int_{u=1}^{u=9} u^{\frac{1}{2}} du \\ &= \frac{1}{2} \frac{2}{3} u^{\frac{3}{2}} \Big|_{u=1}^{u=9} \\ &= \frac{1}{3} (9^{\frac{3}{2}} - 1^{\frac{3}{2}}) \\ &= \frac{1}{3} (27 - 1) \\ &= \frac{26}{3} \end{aligned}$$

Example 1.5.7

Example 1.5.8

Find $I = \int_1^e \frac{lnx}{x} dx$

Solution. let $u = lnx$

$$x = 1, u = 0$$

$$x = e, u = 1$$

$$\frac{du}{dx} = \frac{1}{x} \implies du = \frac{dx}{x} \implies dx = xdu = e^u du$$

$$\begin{aligned} I &= \int_{u=0}^{u=1} u du \\ &= \frac{1}{2}u^2\Big|_0^1 \\ &= \frac{1}{2}(1^2 - 0^2) \\ &= \frac{1}{2} \end{aligned}$$

Example 1.5.8

Joke1.5.9.

天才伽利略

1.6▲ Volume of Solids of Revolution 旋轉體體積

Notation1.6.1.

- Disk method (圓切法)
$$\int \pi(f(x))^2 dx$$
- Shell method (殼切法)
$$\int 2\pi x f(x) dx$$

Example 1.6.2

Find the volume of a sphere with radius r

Solution.

$$\begin{aligned} \text{volume} &= 2\int_0^r \pi y^2 dx \\ &= 2\pi \int_0^r y^2 dx \\ &= 2\pi \int_0^r (r^2 - x^2) dx \\ &= 2\pi(r^2 x - \frac{1}{3}x^3)\Big|_{x=0}^{x=r} \\ &= 2\pi(r^3 - \frac{1}{3}r^3 - 0) \\ &= 2\pi \cdot \frac{2}{3}r^3 \\ &= \frac{4}{3}\pi r^3 \end{aligned}$$

Example 1.6.2

Example 1.6.3

Find the volume of a right circular cone with height h and radius of base r

Solution.

$$\frac{y}{x} = \frac{r}{h} \implies y = \frac{r}{h} \cdot x$$

$$\begin{aligned}
\text{volume} &= \int_0^h \pi(\frac{r}{h}x)^2 dx \\
&= \pi \frac{r^2}{h^2} \int_0^h x^2 dx \\
&= \frac{\pi r^2}{h^2} \cdot \frac{1}{3} x^3 \Big|_{x=0}^{x=h} \\
&= \frac{\pi r^2}{\cancel{h^2}} \frac{1}{3} h^{\cancel{3}} \\
&= \frac{1}{3} \pi r^2 h
\end{aligned}$$

Example 1.6.3

Example 1.6.4

Find the volume of a pyramid whose base is a square with side L and where height is h

Solution.

$$\frac{y}{x} = \frac{\frac{L}{2}}{h} \implies y = \frac{L}{2h} x$$

$$\begin{aligned}
\text{volume} &= \int_0^h (2y)^2 dx \\
&= 4 \int_0^h (\frac{L}{2h} x)^2 dx \\
&= \cancel{4} \frac{L^2}{\cancel{4} h^2} \int_0^h x^2 dx \\
&= \frac{L^2}{\cancel{h^2}} \frac{1}{3} h^{\cancel{3}} \\
&= \frac{1}{3} L^2 h
\end{aligned}$$

Example 1.6.4

Example 1.6.5

Find the volume of a sphere with radius r

Solution.

$$x^2 + y^2 = r^2 \implies y = \sqrt{r^2 - x^2}$$

$$\begin{aligned} \text{volume} &= 2\int_0^r 2\pi x\sqrt{r^2 - x^2}dx \\ &= 4\pi \int_0^r x\sqrt{r^2 - x^2}dx \\ &= \frac{-4\pi}{3}(r^2 - x^2)^{\frac{3}{2}}\Big|_{x=0}^{x=r} \\ &= \frac{-4\pi}{3}(0 - r^3) \\ &= \frac{4}{3}\pi r^3 \end{aligned}$$

$$\begin{aligned} \int x\sqrt{r^2 - x^2}dx &= \int x(r^2 - x^2)^{\frac{1}{2}}dx \\ \text{let } x^2 = u &= \frac{1}{2}\int (r^2 - x^2)du \\ &= \frac{1}{\cancel{2}}(r^2 - u)^{\frac{3}{2}}\frac{\cancel{2}}{3}(-1) + c \\ &= \frac{-1}{3}(r^2 - u)^{\frac{3}{2}} + c \\ &= \frac{-1}{3}(r^2 - x^2)^{\frac{3}{2}} + c \end{aligned}$$

Example 1.6.5

1.7 Integration by Parts

Definition1.7.1.

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

$$\int \frac{d(f(x)(g(x))}{dx}dx = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$

$$f(x)g(x) = \int g(x)df(x) + \int f(x)dg(x)$$

let $f(x) = u, g(x) = v$

$$uv = \int vdu + \int udv$$

$$\int udv = uv - \int vdu$$

Notation1.7.2.

- $\int \text{poly} \cdot a^x dx$
- $\int \text{poly} \cdot \log_a x dx$
- $\int \text{poly} \cdot (\text{trigonometric fcn})\ dx$
- $\int \text{poly} \cdot \text{poly}\ dx$
- $\int a^x \cdot (\text{trigonometric fcn})\ dx$
- $\int \text{poly} \cdot \text{Inverse trigonometric fcn}\ dx$

Example 1.7.3

$$\begin{aligned}
\int lnx dx &= (lnx)x - \int x dlnx \quad \text{use I.B.P, let } u = lnx, v = x \\
&= xlnx - \int \cancel{x}\frac{1}{\cancel{x}}dx \\
&= xlnx - \int 1dx \\
&= xlnx - x + c
\end{aligned}$$

Example 1.7.3

Example 1.7.4

$$\begin{aligned}
\int xe^x dx &= x^2e^x - \int xd(xe^x) \\
&= x^2e^x - \int (xe^x + x^2e^x)dx \implies \text{fail but equality remains true} \\
\int xe^x dx &= \int \frac{1}{2}e^x d(x^2) \\
&= \frac{1}{2}(x^2e^x - \int x^2 de^x) \implies \text{fail} \\
\int xe^x dx &= \int xde^x \\
&= xe^x - \int x^0 e^x dx \quad \text{reduce the degree of } x \\
&= xe^x - e^x + c
\end{aligned}$$

Example 1.7.4

Example 1.7.5

$$\begin{aligned}\int x^2 lnx dx &= \frac{1}{3}\int lnx d(x^3)\\ &= \frac{1}{3}(x^3 lnx - \int x^3 d lnx)\\ &= \frac{1}{3}(x^3 lnx - \frac{1}{3}x^3) + c\end{aligned}$$

Example 1.7.5

Example 1.7.6

$$\begin{aligned}\int x\sin x dx &= \frac{1}{2}\int \sin x d(x^2)\\ &= \frac{1}{2}(x\sin x - \int x^2 d\sin x) \implies \text{fail}\\ \int x\sin x dx &= -\int x d\cos x\\ &= -(x\cos x - \int \cos x dx)\\ &= -x\cos x + \sin x + c\end{aligned}$$

Example 1.7.6

Example 1.7.7

$$\begin{aligned}\int x\sec^2 x dx &= \int x d\tan x\\ &= x\tan x - \int \tan x dx\\ &= x\tan x + ln\cos x + c\\ \int \tan x dx &= \int \frac{\sin x}{\cos x}dx\\ &= -\int \frac{d\cos x}{\cos x}\\ &= -\int \frac{du}{u}\\ &= -lnu + c\\ &= -ln\cos x + c\\ &= -ln\sec x + c\end{aligned}$$

Example 1.7.7

Example 1.7.8

$$\begin{aligned}\int x(x-1)^5dx &= \frac{1}{6}\int xd((x-1)^6)\\ &= \frac{1}{6}(x(x-1)^6-\int(x-1)^6dx)\\ &= \frac{1}{6}(x(x-1)^6-\frac{1}{7}(x-1)^7)+c\end{aligned}$$

Example 1.7.8

Example 1.7.9

$$\begin{aligned}\int e^x\sin xdx &= \int \sin xde^x\\ &= e^x\sin x-\int e^xd\sin x\\ &= e^x\sin x-\int e^x\cos xdx\\ &= e^x\sin x-\int\cos xde^x\\ &= e^x\sin x-(e^x\cos x-\int e^xd\cos x)\\ &= e^x\sin x-e^x\cos x-\int e^x\sin xdx\\ &= e^x(\sin x-\cos x)-\int e^x\sin xdx\\ &= \frac{1}{2}e^x(\sin x-\cos x)+c\end{aligned}$$

Example 1.7.9

Example 1.7.10

$$\begin{aligned}\int\tan^{-1}xdx &= x\tan^{-1}x-\int xd\tan^{-1}x\\ &= x\tan^{-1}x-\int\frac{x}{1+x^2}dx\\ &= x\tan^{-1}x-\frac{1}{2}ln(1+x^2)+c\end{aligned}$$

Example 1.7.10

Example 1.7.11

$$\begin{aligned}\int \sin^{-1} x dx &= x\sin^{-1} x - \int x d\sin^{-1} x \\ &= x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx \\ &= x\sin^{-1} x + (1-x^2)^{\frac{1}{2}} + c\end{aligned}$$

Example 1.7.11

Example 1.7.12

$$\begin{aligned}\int x\tan^{-1} x dx &= \frac{1}{2}\int \tan^{-1} x d(x^2) \\ &= \frac{1}{2}(x^2\tan^{-1} x - \int x^2 d\tan^{-1} x) \\ &= \frac{1}{2}(x^2\tan^{-1} x - \int \frac{x^2}{1+x^2} dx) \\ &= \frac{1}{2}(x^2\tan^{-1} x - \int (-1 + \frac{1}{1+x^2}) dx) \\ &= \frac{1}{2}(x^2\tan^{-1} x - x + \tan^{-1} x) + c\end{aligned}$$

Example 1.7.12

1.8▲ Trigonometric Integrals

Notation1.8.1.

- $\int \sin^m x \cos^n x dx$
- $\int \tan^m x \sec^n x dx$
- $\int \sin(mx)\cos(nx) dx$
- $\int \sin(mx)\sin(nx) dx$
- $\int \cos(mx)\cos(nx) dx$

Example 1.8.2

$$\begin{aligned}\int \cos^2 x dx &= \frac{1}{2}\int (1+\cos(2x) dx \\ &= \frac{1}{2}(x+\frac{1}{2}\sin(2x))+c\end{aligned}$$

Example 1.8.2

Example 1.8.3

$$\begin{aligned}\int \sin^5 x\cos^2 x dx &= \int \sin^4 x\cos^2 x\sin x dx \\ &= -\int (1-\cos^2 x)^2\cos^2 x d\cos x \\ &= -\int (1-u^2)^2 u^2 du \\ &= -\int (u^2-2u^4+u^6) du \\ &= -\frac{1}{3}u^3+\frac{2}{5}u^5-\frac{1}{7}u^7+c \\ &= -\frac{1}{3}\cos^3 x+\frac{2}{5}\cos^5 x-\frac{1}{7}\cos^7 x+c\end{aligned}$$

Example 1.8.3

Example 1.8.4

$$\begin{aligned}\int \sin^4 x\cos^3 x dx &= \int \sin^4 x\cos^2 x\cos x dx \\ &= \int u^4(1-u^2) du \\ &= \frac{1}{5}u^5-\frac{1}{7}u^7+c \\ &= \frac{1}{5}\sin^5 x-\frac{1}{7}\sin^7 x+c\end{aligned}$$

Example 1.8.4

Example 1.8.5

$$I=\int \sin^2 x\cos^4 x dx$$

$$\cos 2\theta = 2\cos^2\theta - 1 = 1-2\sin^2\theta$$

$$\begin{aligned} I &= \int \frac{1}{2}(1-\cos(2x))(\frac{1+\cos(2x)}{2})dx \\ &= \frac{1}{8}\int(1-\cos(2x))(1+2\cos(2x)+\cos^2 2x))dx \\ &= \frac{1}{16}\int(1-\cos(2x))(3+4\cos(2x)+\cos(4x))dx \\ &= \frac{1}{16}\int(3+4\cos(2x)+\cos(4x)-3\cos(2x)-4\cos^2(2x)-\cos(2x)\cos(4x))dx \end{aligned}$$

$$\begin{aligned} \int \cos(2x)\cos(4x)dx &= \frac{1}{2}\int(\cos(6x)+\cos(2x))dx \\ &= \frac{1}{2}(\frac{1}{6}\sin(6x)+\frac{1}{2}\sin(2x))+c \end{aligned}$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha+\beta)+\cos(\alpha-\beta))$$

Example 1.8.5

Example 1.8.6

$$\begin{aligned} \int \sec x dx &= \int \frac{\sec x(\sec x+\tan x)}{\sec x+\tan x}dx \\ &= \int \frac{(\sec^2 x+\sec x\tan x)dx}{\sec x+\tan x} \\ &= \int \frac{d(\tan x+\sec x)}{\tan x+\sec x} \\ &= ln \mid \sec x+\tan x \mid +c \end{aligned}$$

Example 1.8.6

Example 1.8.7

$$\begin{aligned} \int \tan x dx &= \int \frac{\sin x}{\cos x}dx \\ &= -ln \mid \cos x \mid +c \\ &= ln \mid \sec x \mid +c \end{aligned}$$

Example 1.8.7

Example 1.8.8

$$\begin{aligned}\int \tan^3 x dx &= \int \tan^2 x \tan x dx \\ &= \int (\sec^2 -1) \tan x dx \\ &= \int \tan x \sec^2 x dx - \int \tan x dx \\ &= \int \tan x d \tan x + ln \mid \cos x \mid \\ &= \frac{1}{2} \tan^2 x + ln \mid \cos x \mid +c\end{aligned}$$

$$y = \frac{ln(\frac{x}{m} - as)}{r^2}$$

$$e^{yr^2} = e^{ln(\frac{x}{m} - as)}$$

$$e^{yr^2} = \frac{x}{m} - as$$

$$m \cdot e^{yr^2} = x - mas$$

$$me^{rry} = x - mas$$

$$\int \sin^m x \cos^n x dx$$

Example 1.8.8

Notation1.8.9.

(1) either m or n is odd

$$\int \sin^5 x \cos^2 x dx = \int \sin^4 x \cos^2 x \sin x dx$$

(2) both m and n are even

$$\begin{aligned}\int \sin^4 x \cos^2 x dx &= \int \sin^4 x (1 - \sin^2 x) dx \\ &= \int (\sin^4 x - \sin^6 x) dx\end{aligned}$$

$$\begin{aligned}\int \sin^4 x dx &= \int (\frac{1 - \cos x}{2})^2 dx \\ &= \frac{1}{4} \int (1 - 2\cos(2x) + \cos^2(2x)) dx\end{aligned}$$

Example 1.8.10

$$\begin{aligned}\int \tan^6 x \sec^4 x dx &= \int \tan^6 x \sec^2 x \sec^2 dx \\ &= \int u^6(1+u^2)du \qquad \text{let } \tan x = u \\ &= \frac{1}{7}\tan^7 x + \frac{1}{9}\tan^9 x + c\end{aligned}$$

Example 1.8.10

Example 1.8.11

$$\begin{aligned}\int \tan^5 x \sec^7 x dx &= \int \tan^4 x \sec^6 x \tan x \sec x dx \\ &= \int u^6(u^2-1)^2 du \qquad \text{let } \sec x = u \\ &= \frac{1}{11}\sec^1 1x - \frac{2}{9}\sec^9 x + \frac{1}{7}\sec^7 x + c\end{aligned}$$

Example 1.8.11

Example 1.8.12

$$\begin{aligned}\int \tan^3 x dx &= \int \tan x \tan^2 x dx \\ &= \int \tan x(\sec^2 x - 1)dx \\ &= \int (\tan x \sec^2 x - \tan x)dx \\ &= \int \tan x d\tan x - \int \tan x dx \qquad = \frac{1}{2}\tan^2 x + c \\ &= \int \sec x d\sec x - \int \tan dx \qquad = \frac{1}{2}\sec^2 x + c = \frac{1}{2}(\tan^2 + 1) + c = \frac{1}{2}\tan^2 x + \frac{1}{2} + c\end{aligned}$$

Example 1.8.12

1.9 Trigonometric Substitution

Notation1.9.1.

- $\sqrt{a^2 - x^2} \implies$ let $x = a\sin\theta$
- $\sqrt{x^2 + a^2} \implies$ let $x = a\tan\theta$
- $\sqrt{x^2 - a^2} \implies$ let $x = a\sec\theta$

Example 1.9.2

$$\int \frac{\sqrt{9-x^2}}{x^2}dx = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}(\frac{x}{3}) + c$$

consider $\int \frac{\sqrt{1-x^2}}{x^2}dx$ first

let $x = \sin\theta, -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$

$$\sqrt{1-x^2} = \cos\theta$$

$$dx = \cos\theta d\theta$$

$$\begin{aligned}\int \frac{\sqrt{1-x^2}}{x^2}dx &= \int \frac{\cos\theta}{\sin^2\theta}\cos\theta d\theta \\ &= \int (\frac{\cos\theta}{\sin\theta})^2 d\theta \\ &= \int \cot^2\theta d\theta \\ &= \int (\csc^2\theta - 1)d\theta \\ &= -\cot\theta - \theta + c \\ &= -\frac{\sqrt{1-x^2}}{x} - \sin^{-1}x + c\end{aligned}$$

$$\sqrt{9-x^2} = 3\sqrt{1-\frac{x^2}{9}} = 3\sqrt{1-(\frac{x}{3})^2}$$

let $\frac{x}{3} = \sin\theta \implies x = 3\sin\theta$

Example 1.9.2

Example 1.9.3

$$\int \frac{dx}{x^4\sqrt{x^2+4}}$$

$$\sqrt{x^4+4} = 2\sqrt{\frac{x^2}{4}+1} = 2\sqrt{(\frac{x}{2})^2+1}$$

let $\frac{x}{2} = \tan\theta$

$$\sqrt{(\frac{x}{2})^2+1} = \sqrt{\tan^2\theta+1} = \sec\theta$$

$$dx = 2\sec^2\theta d\theta$$

$$\begin{aligned}
I &= \int \frac{2\sec^2\theta d\theta}{4\tan^2\theta 2\sec\theta} \\
&= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta} d\theta \\
&= \frac{1}{4}\int \frac{1}{\cos\theta}\frac{\cos^2\theta}{\sin^2\theta} d\theta \\
&= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta} d\theta \\
&= \frac{1}{4}\int \frac{du}{u^2} \qquad \text{let } \sin\theta = u \\
&= -\frac{1}{4}u^{-1} + c \\
&= -\frac{1}{4}\csc\theta + c \\
&= -\frac{1}{4}\frac{\sqrt{x^2+4}}{x} + c
\end{aligned}$$

Example 1.9.3

Example 1.9.4

$$\int \frac{dx}{\sqrt{x^2-16}}$$

$$\sec^2\theta - 1 = \tan^2\theta$$

$$\tan^2\theta + 1 = \sec^2\theta$$

let $\frac{x}{4} = \sec\theta$

$$x = 4\sec\theta$$

$$dx = 4\sec\theta\tan\theta d\theta$$

$$\begin{aligned}
\int \frac{4\sec\theta\tan\theta d\theta}{4\tan\theta} &= \int \sec\theta d\theta \\
&= (ln \mid \sec\theta + \tan\theta) + c \\
&= ln \mid \frac{x}{4} + \frac{\sqrt{x^2-16}}{4} \mid + c \\
&= ln \mid \frac{x+\sqrt{x^2-16}}{4} \mid + c \\
&= ln \mid x+\sqrt{x^2-16} \mid - ln 4 + c \\
&= ln \mid x+\sqrt{x^2-16} \mid + c
\end{aligned}$$

Example 1.9.4

Example 1.9.5

$$\int \frac{xdx}{\sqrt{3-2x-x^2}}$$

$$\begin{aligned} x-2x+3 &= -(x^2+2x)+3 \\ &= -(x+1)^2+4 \qquad \text{complete the square 配方} \\ &= -y^2+4 \end{aligned}$$

let $y = x+1$

$$dy = dx$$

$$\begin{aligned} \int \frac{(y-1)dy}{\sqrt{4-y^2}} &= \int (\frac{y}{\sqrt{4-y^2}} - \frac{1}{\sqrt{4-y^2}})dy \\ &= \int \frac{4\sin\theta\cos\theta d\theta}{2\cos\theta} - \int \frac{2\cos\theta}{2\cos\theta}d\theta \qquad \text{let } y = 2\sin\theta \implies dy = 2\cos\theta d\theta \\ &= 2\int \sin\theta d\theta - \int 1 d\theta \\ &= -2\cos\theta - \theta + c \\ &= -2\frac{\sqrt{4-y^2}}{2} - \sin^{-1}(\frac{y}{2}) + c \\ &= -\sqrt{-x^2-2x+3} - \sin^{-1}(\frac{x+1}{2}) + c \end{aligned}$$

Example 1.9.5

1.10▲ Improper Integrals 瑕積分

- Type I: infinite integral
  $\int_0^\infty xdx, \int_{-\infty}^0 \sin xdx$
- Type II: discontinuous integral 被積函數
  $\int_0^2 \frac{1}{x-1}dx \quad (\frac{1}{x-1}$ is not conti. of $x=1)$

Example 1.10.1

$$\begin{aligned} A(t) &= \int_1^t \frac{1}{x^2}dx \\ &= -\frac{1}{x}\Big|_{x=1}^{x=t} \\ &= -(\frac{1}{t}-1) \\ &= 1-\frac{1}{t} \end{aligned}$$

$$\begin{aligned} \lim_{t\to\infty} A(t) &= \lim_{t\to 0}(1-\frac{1}{t}) &= 1 \\ & \lim_{t\to\infty}(\int_1^t \frac{dx}{x^2}) &= 1 \\ & &:= \int_1^\infty \frac{dx}{x^2} \end{aligned}$$

Example 1.10.1

Definition1.10.2 (Integrals of type I).

- $\int_a^\infty f(x)dx := \lim_{t\to\infty}\int_a^t f(x)dx$

  If the limit exists, then we say $\int_a^\infty f(x)dx$ is convergent; otherwise, we say $\int_a^\infty f(x)dx$ is divergent.

- $\int_{-\infty}^a f(x)dx := \lim_{t\to-\infty}\int_t^a f(x)dx$

  If the limits exists, then we say $\int_{-\infty}^a f(x)dx$ is convergent; otherwise, we say $\int_{-\infty}^a f(x)dx$ is divergent.

- $\int_{-\infty}^\infty f(x)dx := \int_a^\infty f(x)dx + \int_{-\infty}^a f(x)dx$

  (1) Both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ converge $\implies \int_{-\infty}^\infty$ converges

  (2) Either $\int_a^\infty f(x)dx$ or $\int_{-\infty}^a f(x)dx$ diverges $\implies \int_{-\infty}^\infty$ diverges

Example 1.10.3

$$\begin{aligned}
\int_1^\infty \frac{1}{x}dx &:= \lim_{t\to\infty}\int_1^t \frac{1}{x}dx \\
&= \lim_{t\to\infty}(lnx\Big|_{x=1}^{x=t}) \\
&= \lim_{t\to\infty}(lnt - ln1) \\
&= \lim_{t\to\infty} lnt \\
&= \infty
\end{aligned}$$

Example 1.10.3

Example 1.10.4

$$\begin{aligned}
\int_\infty^\infty \frac{1}{1+x^2}dx &:= \int_0^\infty \frac{1}{1+x^2}dx + \int_\infty^0 \frac{1}{1+x^2}dx \\
&= \lim_{t\to\infty}\int_0^t \frac{1}{1+x^2}dx + \lim_{t\to\infty}\int_t^0 \frac{1}{1+x^2}dx \\
&= \lim_{t\to\infty}(\tan^{-1}x\Big|_{x=0}^{x=t}) + \lim_{t\to\infty}(\tan^{-1}x\Big|_{x=t}^{x=0}) \\
&= \lim_{t\to\infty}(\tan^{-1}t - \tan^{-1}0) + \lim_{t\to\infty}(\tan^{-1}0 - \tan^{-1}t) \\
&= \lim_{t\to\infty}\tan^{-1}t - \lim_{t\to\infty}\tan^{-1}t \\
&= \pi
\end{aligned}$$

Example 1.10.4

Example 1.10.5

$$\begin{aligned}
\int_1^\infty \frac{1}{x^p}dx \quad (p \neq 1) &:= \lim_{t\to\infty}\int_1^t \frac{1}{x^p}dx \\
&= \lim_{t\to\infty}(\frac{1}{1-p}x^{1-p}\Big|_{x=1}^{x=t}) \\
&= \lim_{t\to\infty}(\frac{1}{1-p}(t^{1-p}-1^{1-p})) \\
&= \lim_{t\to\infty}\frac{1}{1-p}\lim_{t\to\infty}(t^{1-p}-1) \\
&= \begin{cases} \dfrac{1}{p-1} & , \quad p>1 \\ \infty & , \quad p<1 \end{cases}
\end{aligned}$$

Example 1.10.5

Notation1.10.6.

- $\int_1^\infty \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1} & \text{(convergent)} \quad , \quad p > 1 \\ \infty & \text{(divergent)} \quad , \quad p \leqslant 1 \end{cases}$
- $\int_1^\infty \frac{dx}{x^2}, p = 2 > 1$
- $\int_1^\infty \frac{dx}{x^{\frac{1}{2}}}, p = \frac{1}{2} < 1$

Example 1.10.7 (Type II: Discontinuous Integral)

$$\begin{aligned} \int_2^5 \frac{dx}{\sqrt{x-2}} &= \lim_{t\to 2^+} \int_t^5 \frac{dx}{\sqrt{x-2}} \\ &= \lim_{t\to 2^+} (2(x-2)^{\frac{1}{2}}|_{x=2}^{x=5}) \\ &= \lim_{t\to 2^+} (2\sqrt{3} - 2(t-2)^{\frac{1}{2}}) \\ &= 2\sqrt{3} \end{aligned}$$

Example 1.10.7

1.11▲ Differential Equations

Definition1.11.1.

$$y'(t) = ky(t) \quad k : \text{const}$$

$$\begin{aligned} \frac{dy(t)}{dt} &= ky(t) \\ dy(t) &= ky(t)dt \\ \frac{dy(t)}{y(t)} &= kdt \\ \int \frac{dy}{y} &= k\int dt \\ lny(t) &= kt + c \\ e^{lny(t)} &= e^{kt+c} \\ y(t) &= e^c \cdot e^{kt} \\ y(t) &= c^* \cdot e^{kt} \end{aligned}$$

Example 1.11.2

$$\frac{dy}{dx} = \frac{x^2}{y^2} \quad y(0) = 2$$

$$\begin{aligned} \int y^2 dy &= \int x^2 dx \\ \frac{1}{3} y^3 &= \frac{1}{3} x^3 + c \end{aligned}$$

use $y(0) = 2$ to determine c

$$\frac{1}{3} 2^3 = \frac{1}{3} 0^3 + cc = \frac{8}{3}$$

The solution is $y^3 = x^3 + 8$

Example 1.11.2

Chapter 2

Applications of Integration

2.1▲ 1st-order linear ODE 一階線性常微分方程

Definition2.1.1 (Separable Equations).

$$\begin{aligned} \frac{dy}{dx} &= f(x)g(y) \\ \int \frac{dy}{g(y)} &= \int f(x)dx \end{aligned}$$

$$y'(x) + P(x)y(x) = Q(x) \quad (y \neq 0) \quad \text{where } P(x) \text{ and } Q(x) \text{ are given} \quad ---(\star)$$

Goal: solve $y(x)$
Idea: Integrating factor

$$(\star) \cdot I(x)$$

$$I(x)y'(x) + I(x)P(x)y(x) = I(x)Q(x)$$

Hope: $Iy' + IPy = (Iy)' \qquad ---(1)$
want $I(x)$ s.t. (1) is true

$$(1) \implies Iy' + IPy = I'y + Iy' \quad \text{product rule}$$

$$IP = I' = \frac{dI}{dx}$$

$$\frac{dI}{dx} = I(x)P(x)$$

$$\int \frac{dI}{I} = \int P(x)dx$$

$$\ln I \cancel{+c} = \int P(x)dx$$

$$I(x)\cancel{\cdot e^{c}} = e^{\int P(x)dx}$$

$$I(x) = \cancel{e^{-c}} e^{\int P(x)dx}$$

i.e: $I(x) = e^{\int P(x)dx}$ Integrating factor

$$? = Iy' + IPy = (Iy)'$$

$$e^{\int P(x)dx}y' + e^{\int P(x)dx}Py = (e^{\int P(x)dx}y)'$$

$$\frac{d}{dx}(e^{\int P(x)dx}y) = \frac{d}{dx}(e^{\int P(x)dx})y + e^{\int P(x)dx}y'$$

$$\frac{d}{dx}(e^{\int P(x)dx}) = e^{\int P(x)dx} \quad \text{chain rule}$$

$$\frac{d}{dx}(\int P(x)dx) = P(x) \quad \text{F.T.C}$$

(1) $\implies (Iy)' = IQ$, where $I(x) = e^{\int P(x)dx}$

$$\frac{d(Iy)}{dx} = I(x)Q(x)$$

$$\int d(Iy) = \int I(x)Q(x)dx$$

$$Iy\cancel{+c} = \int I(x)Q(x)dx$$

$$y(x) = \frac{1}{e^{\int P(x)dx}} \int (e^{\int P(x)dx})Q(x)dx$$

Example 2.1.2

Solve $y' + 3x^2y = 6x^2$ $\quad ---(2)$

Solution. Integrating factor

$$\begin{aligned} I(x) &= e^{\int P(x)dx} \\ &= e^{\int 3x^2dx} \\ &= e^{x^3+c} \\ &= \cancel{e^{c}} \cdot e^{x^3} \end{aligned}$$

$$(2)\cdot e^{x^3} \implies e^{x^3}y' + 3x^2ye^{x^3} = 6x^2e^{x^3}$$

$$\frac{d(e^{x^3})}{dx} = e^{x^3}y' = 6x^2e^{x^3}$$

$$\begin{aligned} \int d(e^{x^3}y) &= \int 6x^2e^{x^3}dx \\ &= \int 2e^{x^3}dx^3 \\ &= 2e^{y} + c \\ &= 2e^{x^3} + c \end{aligned}$$

$$e^{x^3}y = 2e^{x^3} + c$$
$$y = 2 + \frac{c}{e^{x^3}}$$

Example 2.1.2

Example 2.1.3

Solve Initial value problem (I.V.P) $\begin{cases} x^2y' + xy &= 1 & \text{ODE} \\ y(1) &= 2 & \text{Initial condition} \end{cases}$

Solution.

$$y' + \frac{1}{x}y = \frac{1}{x^2} \qquad ---(3)$$
$$I(x) = e^{\int \frac{1}{x}dx} = e^{lnx} = x$$
$$(3)\cdot I(x) \implies xy' + y = \frac{1}{x}$$
$$(xy)' = \frac{1}{x}$$
$$\frac{d(xy)}{dx} = \frac{1}{x}$$
$$\int d(xy) = \int \frac{1}{x}dx$$
$$xy = lnx + c$$

use $y(1) = 2$ to determine c

$$xy = lnx + c$$
$$x = 1, y = 2$$
$$1 \cdot 2 = ln1 + c$$
$$c = 2$$

The solution is $xy = lnx + 2$ or $y = \dfrac{lnx}{x} + \dfrac{2}{x}$

Example 2.1.3

Example 2.1.4

Solve $(\sec x)y' - y = \tan x e^{\cos x - \sin x} \qquad ---(4)$

Solution.

$$y' - \frac{y}{\sec x} = \sin x e^{\cos x - \sin x}$$
$$I(x) = e^{\int \cos x dx} = e^{\sin x}$$
$$\begin{aligned}(4)\cdot I(x) \implies e^{\sin x}y' + \cos x e^{\sin x}y &= \cancel{e^{\sin x}}\sin x e^{\cos x \cancel{-\sin x}} \\ &= \sin x e^{\cos x}\end{aligned}$$

$$(e^{\sin x}y)' = \sin x e^{\cos x}$$
$$\frac{d(e^{\sin x}y)}{dx} = \sin x e^{\cos x}$$
$$\int d(e^{\sin x}y) = \int \sin x e^{\cos x} dx$$
$$\begin{aligned} e^{\sin x}y &= -\int e^{\cos x} d\cos x \\ &= -e^{y} + c \\ &= -e^{\cos x} + c \end{aligned}$$
$$e^{\sin x}y = -e^{\cos x} + c$$

$y' + P(x)y = Q(x) \implies$ IF is $I(x) = e^{\int P(x)dx}$

Example 2.1.4

2.2▲ Arc Length 弧長

Definition2.2.1.

$$ds = \sqrt{(dx)^2 + (dy)^2}$$
$$\begin{aligned} \int_{x=a}^{x=b} ds &= \int_{x=a}^{x=b} \sqrt{(dx)^2 + (dy)^2} \\ &= \int_{x=a}^{x=b} \sqrt{\frac{dx^2}{dx^2} + \frac{dy^2}{dx^2}} dx \\ &= \int_a^b \sqrt{1 + (f'(x))^2} dx \end{aligned}$$

Given $x = g(y), c \leqslant y \leqslant d$, arc length $= \int_c^d \sqrt{1 + (g'(y))^2} dy$

Example 2.2.2

Find the arc length of $y = x^{\frac{3}{2}}$ from $(1,1)$ to $(4,8)$

Solution.

$$\begin{aligned} f(x) = x^{\frac{3}{2}} &\implies f'(x) = \frac{2}{3}x^{\frac{1}{2}} \\ &\implies 1 + (f'(x))^2 = 1 + \frac{4}{9}x \end{aligned}$$

$$\begin{aligned} \int \sqrt{1+x} dx &= \int (1+x)^{\frac{1}{2}} dx \\ &= \frac{2}{3}(1+x)^{\frac{3}{2}} + c \end{aligned}$$

$$
\begin{aligned}
\int_1^4 \sqrt{1+\frac{4}{9}x}dx &= \frac{4}{9}\frac{2}{3}(1+\frac{4}{9}x)^{\frac{3}{2}}\Big|_1^4 \\
&= \frac{8}{27}(10^{\frac{3}{2}} - \frac{13^{\frac{3}{2}}}{8}) \\
&= \frac{1}{27}(40^{\frac{3}{2}} - 13^{\frac{3}{2}})
\end{aligned}
$$

Example 2.2.2

Example 2.2.3

Find the arc length of $y = e^x$ from $(ln3, 3)$ to $(ln8, 8)$

Solution.

$$
\begin{aligned}
f(x) = e^x &\implies f'(x) = e^x \\
&\implies (f'(x))^2 = e^{2x}
\end{aligned}
$$

$$
\begin{aligned}
I &= \int \sqrt{1+e^{2x}}dx \\
\text{let } u &= \sqrt{1+e^{2x}} = (1+e^{2x})^{\frac{1}{2}} \\
du &= \cancel{\frac{1}{2}}(1+e^{2x})^{-\frac{1}{2}}\cancel{2}e^{2x}dx = \frac{u^2-1}{u}dx \\
I &= \int u(\frac{u}{u^2-1}du) \\
&= \int \frac{u^2}{u^2-1}du \\
&= \int (1+\frac{1}{u^2-1})du \\
&= u + \frac{1}{2}(ln(u-1) - ln(u+1)) + c
\end{aligned}
$$

$$
\begin{aligned}
\int_{ln3}^{ln8} \sqrt{1+e^{2x}}dx &= \sqrt{1+e^{2x}} + \frac{1}{2}(ln(\sqrt{1+e^{2x}}-1) - ln(\sqrt{1+e^{2x}}+1))\Big|_{ln3}^{ln8} \\
&= \sqrt{1+64} + \frac{1}{2}(ln(\sqrt{65}-1) - ln(\sqrt{65}+1)) \\
&\quad -(\sqrt{1+9} + \frac{1}{2}(ln(\sqrt{10}-1) - ln(\sqrt{10}+1))) \\
&= \sqrt{65} + \frac{1}{2}ln(\frac{\sqrt{65}-1}{\sqrt{65}+1}) - (\sqrt{10} + \frac{1}{2}ln(\frac{\sqrt{10}-1}{\sqrt{10}+1})) \\
&= \sqrt{65} - \sqrt{10} + \frac{1}{2}(ln(\frac{(\sqrt{65}-1)^2}{65-1}) - ln(\frac{(\sqrt{10}-1)^2}{10-1})) \\
&= \sqrt{65} - \sqrt{10} + \frac{1}{2}(ln(\frac{(\sqrt{65}-1)^2}{(\sqrt{10}-1)^2}) + ln(\frac{9}{64})) \\
&= \sqrt{65} - \sqrt{10} + ln(\frac{\sqrt{65}-1}{\sqrt{10}-1}) + ln\frac{3}{8} \\
&= 2 + ln3 - ln2
\end{aligned}
$$

Example 2.2.3

2.3 Calculus with Parametric Curve

Definition2.3.1 (Parametric Equations 參數式).

$$y = f(x)$$

$$\begin{cases} y = y(t) \\ x = x(t) \end{cases} \qquad t: \text{parameter}$$

Notation2.3.2.

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases} \qquad t: \text{parameter}$$

- Tangent

$$\text{slope} = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}$$

$$g'(t) = \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}f'(t) \quad \text{chain rule}$$

- Area

$$\int_a^b y(x)dx = \int_{t_1}^{t_2} y(f(t))\frac{dx}{dt}dt = \int_{t_1}^{t_2} y(f(t))f'(t)dt$$

- Arc Length

$$\int ds = \int \sqrt{(dx)^2 + (dy)^2} = \int \sqrt{\frac{(dx)^2}{(dt)^2} + \frac{(dy)^2}{(dt)^2}}dt = \int \sqrt{(f'(t))^2 + (g'(t))^2}dt$$

- Surface Area
  $y = y(x)$ around x-axis

$$\int 2\pi y(x)ds = \int 2\pi y(f(t)) \cdot \sqrt{(f'(t))^2 + (g'(t))^2}dt$$

- Cycloid 擺線

$$\overset{\frown}{PQ} = \overline{P_0Q}$$

$$x = \overline{P_0Q} - r\sin\theta = \overset{\frown}{PQ} - r\sin\theta = r\theta - r\sin\theta$$

$$y = r - r\cos\theta$$

$$\begin{cases} x(\theta) = r(\theta - \sin\theta) \\ y(\theta) = r(1 - \cos\theta) \end{cases}$$

Example 2.3.3

$$\begin{cases} x &= \cos t \\ y &= \sin t \end{cases}$$

$$\implies x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

circle centered at $(0,0)$ with radius 1

Example 2.3.3

Example 2.3.4

Find the slope of the tangent of Cycloid at $\theta = \dfrac{\pi}{3}$

Solution.

$$\begin{aligned} x(\theta) &= r(\theta - \sin\theta) \\ y(\theta) &= r(1 - \cos\theta) \end{aligned}$$

$$\begin{aligned} \text{slope} &= \frac{y'(\theta)}{x'(\theta)}\Big|_{\theta=\frac{\pi}{3}} \\ &= \frac{r\sin\theta}{r(1-\cos\theta)}\Big|_{\theta=\frac{\pi}{3}} \\ &= \frac{\frac{\sqrt{3}}{2}}{1-\frac{1}{2}} \\ &= \sqrt{3} \end{aligned}$$

Example 2.3.4

Example 2.3.5

$$\begin{aligned} \text{area } A &= \int_{\theta}^{2\pi} r(1-\cos\theta)\cdot r(1-\cos\theta)d\theta \\ &= r^2\int_{\theta}^{2\pi}(1-\cos\theta)^2 d\theta \\ &= r^2(\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{2}\theta)\Big|_0^{2\pi} \\ &= r^2(\frac{3}{\cancel{2}}\cdot\cancel{2}\pi) \\ &= 3\pi r^2 \end{aligned}$$

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\begin{aligned} \int(1-2\cos\theta+\cos^2\theta)d\theta &= \int\cos^2\theta d\theta \\ &= \int\frac{1}{2}(\cos 2\theta + 1)d\theta \end{aligned}$$

$$
\begin{aligned}
\text{arc length} &= \int_0^{2\pi} r\sqrt{(1-\cos\theta)^2+(\sin\theta)^2 d\theta} \\
&= r\int_0^{2\pi} \sqrt{2-2\cos\theta d\theta} \\
&= \sqrt{2}r\int_0^{2\pi} \sqrt{1-\cos\theta d\theta} \\
&= \sqrt{2}r\int_0^{2\pi} \sqrt{2\sin^2\frac{\theta}{2}d\theta} \\
&= 2r\int_0^{2\pi} \sqrt{\sin^2\frac{\theta}{2}d\theta} \\
&= 2r\int_0^{2\pi} \sin\frac{\theta}{2}d\theta \\
&= 2r(-2\cos\frac{\theta}{2})\Big|_0^{2\pi} \\
&= -4r((-1)-1) \\
&= 8r
\end{aligned}
$$

$$
\begin{aligned}
\cos 2\theta = 2\cos^2\theta - 1 &= 1-2\sin^2\theta \\
1-\cos 2\theta &= 2\sin^2\theta \\
1-\cos\theta &= 2\sin^2\frac{\theta}{2}
\end{aligned}
$$

$$
\begin{aligned}
\text{surface area} &= \int 2\pi(r(1-\cos\theta)(\sqrt{2}r\sqrt{1-\cos\theta})d\theta \\
&= 2\sqrt{2}\pi r^2\int(1-\cos\theta)^{\frac{1}{2}}d\theta
\end{aligned}
$$

$$
\begin{aligned}
\int\sin^3\theta d\theta &= \int\sin^2\theta\sin\theta d\theta \\
&= -\int(1-z^2)dz
\end{aligned}
$$

Example 2.3.5

Example 2.3.6

Find the surface area generated by rotating w.r.t x-axis

Solution.

$$\begin{aligned}
A &= \int_0^{2\pi} 2\pi y(\theta) ds \\
&= 2\pi\sqrt{2}r^2 \int_0^{2\pi} (1-\cos\theta)\sqrt{1-\cos\theta}d\theta \\
&= 2\pi\sqrt{2}r^2 \int_0^{2\pi} 2\sin^2\frac{\theta}{2}\sqrt{2}\sqrt{\sin^2\frac{\theta}{2}}d\theta \\
&= 8\pi r^2 \int_0^{2\pi} \sin^2\frac{\theta}{2}\sin\frac{\theta}{2}d\theta \\
&= -16\pi r^2 \int_0^{2\pi} (1-\cos^2\frac{\theta}{2})d\cos\frac{\theta}{2} \\
&= -16\pi r^2 \int_1^{-1} (1-z^2)dz \\
&= -16\pi r^2 (z-\frac{1}{3}z^3)\Big|_1^{-1} \\
&= -16\pi r^2(-1+\frac{1}{3}-(1-\frac{1}{3})) \\
&= -16\pi r^2(-\frac{4}{3}) \\
&= \frac{64}{3}\pi r^2
\end{aligned}$$

$$\begin{aligned}
ds &= \sqrt{(dx)^2+(dy)^2} \\
&= \sqrt{(\frac{dx}{d\theta})^2+(\frac{dy^2}{d\theta})^2} \\
&= \sqrt{2}r\sqrt{1-\cos\theta}d\theta
\end{aligned}$$

Example 2.3.6

2.4▲ Polar Coordinates 極坐標

Definition2.4.1.

$$\begin{cases} x &= r\cos\theta \quad ---(1) \\ y &= r\sin\theta \quad ---(2) \end{cases}$$

$$\frac{(2)}{(1)} = \frac{y}{x} = \tan\theta \implies \theta = \tan^{-1}\frac{y}{x}$$

$$(1)^2+(2)^2 = r^2+1 \implies r = \sqrt{x^2+y^2}$$

$$(r,\theta) = (\sqrt{x^2+y^2}, \tan^{-1}\frac{y}{x})$$

Notation2.4.2.

$$(1,\frac{\pi}{4}) = (1,\frac{\pi}{4}+2\pi) = (1,\frac{9}{4}\pi) = (-1,\frac{5}{4}\pi)$$

Expression of the same point by polar coordinate may not be unique.

▸▸ Symmetry 對稱

- x-axis
  $\theta \to -\theta$, the eqn is invariant
- y-axis
  $\theta \to \pi - \theta$, the eqn is invariant
- origin
  $r \to -r$, the eqn is invariant

Example 2.4.3

Plot the graph of $r = f(\theta) = 2\cos(2\theta)$

- $\theta \to -\theta$
$$r = 2\cos(2\theta) = 2\cos(2(-\theta))$$
  $\therefore$ the graph is symmetric w.r.t. x-axis
- $\theta \to \pi - \theta$
$$r = 2\cos(2\theta) = 2\cos(2(\pi - \theta))$$
$$\cos(2\pi - 2\theta) = \cos(2\pi)\cos(2\theta) + \sin(2\pi)\sin(2\theta) = \cos(2\theta)$$
  $\therefore$ the graph is symmetric w.r.t. y-axis
- $r \to -r$
$$-r = 2\cos(2\theta) \qquad r = -2\cos(2\theta)$$
$$\theta \to \theta + \pi$$
$$r = 2\cos(2\theta) = 2\cos(2(\theta+\pi)) = 2\cos(2\theta)\cos(2\pi) - \sin(2\theta)\sin(2\pi) = 2\cos(2\theta)$$
  $\therefore$ the graph is symmetric w.r.t. to $(0, 0)$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	2	$\sqrt{3}$	0	-1	-2

Example 2.4.3

Example 2.4.4

Find tangent of $r = 2\cos(2\theta)$ at $(1, \frac{\pi}{6})$

Solution.
$$\begin{cases} y &= r\sin\theta = 2\cos(2\theta)\cdot\sin\theta = y(\theta) \\ x &= r\cos\theta = 2\cos(2\theta)\cdot\cos\theta = x(\theta) \end{cases}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \left.\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right|_{\theta=\frac{\pi}{6}} = \frac{\cancel{2}(-2\sin(2\theta)\sin\theta + \cos(2\theta))\cdot\cos\theta}{\cancel{2}(-2\sin(2\theta)\cos\theta + \cos(2\theta))\cdot(-\sin\theta)}$$

Example 2.4.4

Example 2.4.5 (Cardioid 心臟線)

Plot the graph of $r = 1 + \sin\theta = f(\theta)$

- $\theta \to -\theta$

$$r = 1 + \sin(-\theta) = 1 - \sin\theta$$

  $\therefore$ the graph is NOT symmetric w.r.t. x-axis

- $\theta \to \pi - \theta$

$$r = 1 + \sin(\pi - \theta) = 1 + \sin\pi\cos\theta - \cos\pi\sin\theta = 1 + \sin\theta$$

  $\therefore$ the graph is symmetric w.r.t. y-axis

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{6}$	o	$\frac{\pi}{2}$	$\frac{\pi}{2}$
r	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2

Example 2.4.5

Example 2.4.6

Find the slope of the tangent of $r = 1 + \sin\theta$ at $\theta = \frac{\pi}{3}$

Solution.

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \left.\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right|_{\theta\frac{\pi}{3}} = \left.\frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta + (1+\sin\theta)(1-\sin\theta)}\right|_{\frac{\pi}{3}}$$

Example 2.4.6

▸▸ Area and Arc Length in Polar Coordinates

- Area

$$dA = \cancel{\pi} r^2 \frac{d\theta}{2\cancel{\pi}} = \frac{2}{1} r^2 d\theta$$

$$\int A = \frac{1}{2}\int_a^b r^2 d\theta$$

- Arc length

$$x = f(\theta)\cos\theta$$

$$y = f(\theta)\sin\theta$$

$$\begin{aligned}
(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2 &= (f'(\theta)\cos\theta + f(\theta)(-\sin\theta))^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2 \\
&= (f'(\theta))^2 + (f(\theta))^2
\end{aligned}$$

$$\begin{aligned}
ds\sqrt{(dx)^2 + (dy)^2} &= \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}d\theta \\
&= \sqrt{(f'(\theta))^2 + (f(\theta))^2}d\theta \\
&= \sqrt{(r')^2 + r^2}d\theta
\end{aligned}$$

Example 2.4.7

Find the length of $r = 1 + \sin\theta$

Solution.

$$r = f(\theta) = 1 + \sin\theta$$

$$f'(\theta) = \cos\theta$$

$$\begin{aligned}
(f(\theta))^2 + (f'(\theta))^2 &= (1 + \sin\theta)^2 + \cos^2\theta \\
&= 2 + 2\sin\theta
\end{aligned}$$

$$\begin{aligned}
\text{length} &= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{2 + 2\sin\theta}d\theta \\
&= 2\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 + \sin\theta}d\theta \\
&= 2\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} \\
&= 2\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{(\sin\frac{\theta}{2} + \cos\frac{\theta}{2})^2}d\theta \\
&= 2\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left|\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right|d\theta \\
&= 2\sqrt{2}(\int_{-\frac{\pi}{2}}^{0} \left|\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right|d\theta + \int_{0}^{\frac{\pi}{2}} \left|\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right|d\theta) \\
&= 2\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\frac{\theta}{2} + \cos\frac{\theta}{2}d\theta \\
&= 2\sqrt{2}(-2\cos\frac{\theta}{2} + 2\sin\frac{\theta}{2})\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= 2\sqrt{2}(\cancel{-2\frac{\sqrt{2}}{2}} + \cancel{2\frac{\sqrt{2}}{2}} - (-2\frac{\sqrt{2}}{2} - 2\frac{\sqrt{2}}{2})) \\
&= 2\sqrt{2} \cdot 2\sqrt{2} \\
&= 8
\end{aligned}$$

Example 2.4.7

Example 2.4.8 (Rose Curve 玫瑰線)

- Find the area of one loop of $r = 2\cos 2\theta$

$$\begin{aligned} A &= \int \frac{1}{2} r^2 d\theta \\ &= \cancel{2}\int_0^{\frac{\pi}{4}} \frac{1}{\cancel{2}}(2\cos 2\theta)^2 d\theta \\ &= 4\int_0^{\frac{\pi}{4}} \cos^2 2\theta d\theta \\ &= 2\int_0^{\frac{\pi}{4}} (0+\cos 4\theta) d\theta \\ &= 2(\theta + \frac{1}{4}\sin 4\theta)\Big|_0^{\frac{\pi}{4}} \\ &= 2(\frac{\pi}{4}) \\ &= \frac{\pi}{2} \end{aligned}$$

- Find the arc length of one loop of $r = 2\cos 2\theta$

Example 2.4.8

Example 2.4.9 (Cardioid)

- Plot the graph of $r = 1 - \cos\theta$

θ	0	π	$\frac{\pi}{2}$
r	0	2	1

- Find the area of $r = 1 - \cos\theta$

$$\begin{aligned} A &= \cancel{2}\int_0^{\pi} \frac{1}{\cancel{2}}(1-\cos\theta)^2 d\theta \\ &= \int_0^{\pi} (1 - 2\cos\theta + \cos^2\theta) d\theta \\ &= \frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\Big|_0^{\pi} \\ &= \frac{3}{2}\pi \end{aligned}$$

Example 2.4.9

Chapter 3

Sequence and Series

3.1▲ Sequence 數列

Definition3.1.1 $(f(n), n \in N)$**.**

- $a_n = \frac{1}{n} \quad (\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \cdots)$
- $a_n = n \quad (1, 2, 3, \cdots)$
- $a_n = (-1)^n \quad (-1, 1, -1, 1, \cdots)$

Q: Given a sequence (infinite) $\{a_n\}_{n=1}^{\infty}$. Is $\{a_n\}_{n=1}^{\infty}$ convergent or divergent?

- $a_n = \frac{1}{n} \quad (\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \cdots)$
  $\lim_{n\to\infty} a_n = 0$ convergent
- $a_n = n \quad (1, 2, 3, \cdots)$
  $\lim_{n\to\infty} a_n = \infty$ divergent
- $a_n = (-1)^n \quad (-1, 1, -1, 1, \cdots)$
  divergent

Theorem3.1.2.

If $\lim_{x\to\infty} a_x = L, x \in R$, then

$$\lim_{n\to\infty} a_n = L, n \in N$$

Example 3.1.3

$$a_n = \frac{lnn}{n}, n \in N$$

$$\lim_{x\to\infty} \frac{lnx}{x}, x \in R = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

$$\lim_{n\to\infty} \frac{lnn}{n} = 0, n \in N$$

Example 3.1.3

Example 3.1.4 (Geometric Sequence)

$\{r^n\}_{n=1}^{\infty}$ converges

$$\{r^n\}_{n=1}^{\infty} = \{r, r^2, r^3, \cdots\}$$

$$r = -1\{-1, 1, -1, 1, \cdots\}$$

$$-1 < r \leqslant 1$$

Example 3.1.4

Theorem3.1.5.

Assume

(1) $\{a_n\}_{n=1}^{\infty}$ is monotone 單調遞增或遞減
$a_n \leqslant a_{n+1}, \forall n = 0, 1, 2, \cdots$ or $a_n \geqslant a_{n+1}, \forall n = 0, 1, 2, \cdots$

(2) $\{a_n\}_{n=1}^{\infty}$ is bounded 有界
$\left|a_n\right| \leqslant M, \forall n = 1, 2, 3, \cdots$ for some $M > 0$

$\Longrightarrow$ $\{a_n\}_{n=1}^{\infty}$ converges

Example 3.1.6

$$x = \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}$$

$$\begin{aligned} x^2 &= 2 + \sqrt{2 + \sqrt{2 + \cdots}} \\ &= 2 + x \end{aligned}$$

$$x^2 - x - 2 = 0$$

$$(x+1)(x-2)=0$$

$$x=-1 \text{ or } 2$$

$$x=2$$

$$\begin{array}{lllll} a_1 & = & \sqrt{2} & & \\ a_2 & = & \sqrt{2+\sqrt{2}} & = & \sqrt{2+a_1} \\ a_3 & = & \sqrt{2+\sqrt{2+\sqrt{2}}} & = & \sqrt{2+a_2} \\ a_n & & & = & \sqrt{2+a_n} \end{array}$$

Consider $\{a_n\}_{n=1}^{\infty}$

- a_n monotone ($a_n \nearrow$ as $n \nearrow$)
- a_n bounded

Want to show $a_n \leqslant 2, \forall n \in N \quad ---(\star)$
Use Induction （數學歸納法）

- $n=1$

$$\left|a_1\right| = \left|\sqrt{2}\right| = \sqrt{2} \leqslant 2$$

  $\therefore n=1, (\star)$ is true

- If $n=k, (\star)$ is true $\implies n=k+1, (\star)$ is true

$$\left|a_{k+1}\right| = \left|\sqrt{2+a_k}\right| \leqslant \sqrt{2+2} = 2$$

$$\left|a_{k+1}\right| \leqslant 2$$

By Induction, $(\star)$ is true
By Thm, $\{a_n\}_{n=1}^{\infty}$ converges

Example 3.1.6

3.2▲ Series 級數

Definition3.2.1 (Series).

Summation of a sequence

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

Q: Convergence of a series?

$$\sum_{r=1}^{\infty} r^n = r + r^2 + r^3 + \cdots = \lim_{n\to\infty} \frac{r(1-r^n)}{1-r}$$

$$\text{convergence} \Leftrightarrow -1 < r < 1$$

Theorem3.2.2.

If $\sum_{n=1}^{\infty} a_n$ converges, then

$$\lim_{n\to\infty} a_n = 0$$

Notation3.2.3.

$\lim_{n\to\infty} a_n = 0$ does not imply $\sum_{n=1}^{\infty} a_n$ converges

Example 3.2.4

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

Assume $\sum_{n=1}^{\infty} \frac{1}{n} = S$ converges

$$\begin{array}{rcccccccccc}
S & = & 1 & + & \underbrace{(\frac{1}{2}+\frac{1}{3})}_{>\frac{1}{2}} & + & \underbrace{(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7})}_{>\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}} & + & \underbrace{(\frac{1}{8}+\frac{1}{9}+\cdots+\frac{1}{15})}_{>\frac{1}{16}+\frac{1}{16}+\cdot+\frac{1}{16}} & + \cdots \\
 & > & 1 & + & \frac{1}{2} & + & \frac{1}{2} & + & \frac{1}{2} & + \cdots \\
 & = & \infty & & & & & & &
\end{array}$$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$ but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

Example 3.2.4

3.3▲ Integral Test

Definition3.3.1.

Assume

(1) $a_n > 0$ for $n = m, m+1, \cdots$ (essentially positive)

(2) $a_n \searrow$

(3) a_n is conti. for $x \in [1, \infty)$

$$\Longrightarrow \sum_{n=1}^{\infty} a_n \text{ and } \int_1^{\infty} a_x dx \text{ both converge or diverge}$$

Example 3.3.2

Determine $\sum_{n=1}^{\infty} \frac{1}{n^p}, p > 0$ (P-series) converges or diverges

Solution.

(1) $\frac{1}{n^p} > 0 \quad \forall n = 1, 2, 3, \cdots$

(2) $a_n = \frac{1}{n^p} \searrow$

(3) $a_x = \frac{1}{x^p}$ conti for $x \geqslant 1$

By integral test $\Longrightarrow \sum_{n=1}^{\infty} \frac{1}{n^p}$ and $\int_1^{\infty} \frac{1}{x^p} dx$ both converges or diverges

Example 3.3.2

Example 3.3.3

Determine $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges or diverges by integral test

Solution.

(1) $a_n = \frac{1}{n^2+1} > 0 \quad \forall n = 1, 2, 3, \cdots$

(2) $a_n = \frac{1}{n^2+1} \searrow \qquad f(x) = \frac{1}{n^2+1} \Longrightarrow f'(x) = -(x^2+1)^{-2}(2x) < 0, x > 0$

(3) $a_x = \frac{1}{x^2+1}$ is conti. $x \geqslant 1$

By integral test, consider

$$\begin{aligned}\int_1^{\infty}\frac{1}{x^2+1}dx &= \lim_{t\to\infty}\left(\int_1^{\infty}\frac{1}{x^2+1}dx\right)\\ &= \lim_{t\to\infty}\left(\tan^{-1}x\Big|_{x=1}^{x=t}\right)\\ &= \lim_{t\to\infty}\left(\tan^{-1}t-\tan^{-1}1\right)\\ &= \lim_{t\to\infty}\tan^{-1}t-\lim_{t\to\infty}\tan^{-1}1\\ &= \frac{\pi}{2}-\frac{\pi}{4}\\ &= \frac{\pi}{4} \quad \text{converges}\end{aligned}$$

$\implies \sum_{n=1}^{\infty}\frac{1}{n^2+1}$ converges

$$\sum_{n=1}^{\infty}\frac{1}{n^2+1}\neq\int_1^{\infty}\frac{1}{x^2+1}dx$$

Compare with P-series

$$\sum_{n=1}^{\infty}\frac{1}{n^2+1}<\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6} \quad \text{converges}$$

$\implies \sum_{n=1}^{\infty}$ converges

Example 3.3.3

Example 3.3.4

Determine $\sum_{n=1}^{\infty}\frac{1}{n(lnn)^p}, p>0$ converges or diverges

Solution.

$$a_n=\frac{1}{n(lnn)^p}$$

(1) $a_n>0 \quad \forall n=2,3,\cdots$

(2) $a_n \searrow$

(3) $f(x)=\frac{1}{x(lnx)^p}$ is conti for $x\in[2,\infty)$

$$\begin{aligned}
\int_2^\infty \frac{1}{x(lnx)^p}dx &= \int_{x=2}^{x=\infty} \frac{dlnx}{(lnx)^p} \\
&= \int_{y=ln2}^{y=\infty} \frac{dy}{y^p} \\
&= \frac{1}{1-p} y^{1-p}\Big|_{ln2}^{\infty} \\
&= \left\{ \begin{array}{lcl} \displaystyle\int_{ln2}^{\infty} \frac{1}{y}dy &=& lny\Big|_{ln2}^{\infty} = \infty, p=1 \\ \displaystyle\int_{ln2}^{\infty} \frac{1}{y^p}dy &=& \displaystyle\frac{1}{1-p} y^{1-p}\Big|_{ln2}^{\infty} = \left\{ \begin{array}{ll} \text{converges} & , \ p>1 \\ \text{diverges} & , \ 0<p<1 \end{array} \right. \end{array} \right. \\
&= \left\{ \begin{array}{ll} \text{diverges} & , \ 0<p\leqslant 1 \\ \text{converges} & , \ p>1 \end{array} \right.
\end{aligned}$$

Example 3.3.4

Notation3.3.5.

$$\int_1^\infty \frac{1}{x^p}dx = \left\{ \begin{array}{ll} \text{diverges} & , \ 0<p\leqslant 1 \\ \text{converges} & , \ p>1 \end{array} \right.$$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \left\{ \begin{array}{ll} \text{diverges} & , \ 0<p\leqslant 1 \\ \text{converges} & , \ p>1 \end{array} \right.$$

$p=1$ Harmonic Series （調和級數）

3.4▲ Comparison Theorem

Theorem3.4.1 (Comparison Theorem).

- Subtraction 减法

$$a_n, b_n > 0 \quad a_n \leqslant b_n \quad n = 1, 2, \cdots$$

(1) $\sum_{n=1}^{\infty} b_n$ converges $\Longrightarrow \sum_{n=1}^{\infty} a_n$ converges

(2) $\sum_{n=1}^{\infty} a_n$ diverges $\Longrightarrow \sum_{n=1}^{\infty} b_n$ diverges

- Division 除法

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

(1) $c \neq 0$ and $c \neq \infty$

$\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converges or diverges

(2) $c = 0 \quad (\lim_{n\to\infty} \frac{a_n}{b_n} = 0)$

$\sum_{n=1}^{\infty} b_n$ converges $\Longrightarrow \sum_{n=1}^{\infty} a_n$ converges

$\sum_{n=1}^{\infty} a_n$ diverges $\Longrightarrow \sum_{n=1}^{\infty} b_n$ diverges

(3) $c = \infty$

$\sum_{n=1}^{\infty} a_n$ converges $\Longrightarrow \sum_{n=1}^{\infty} b_n$ converges

$\sum_{n=1}^{\infty} b_n$ diverges $\Longrightarrow \sum_{n=1}^{\infty} a_n$ diverges

Example 3.4.2

Determine $\sum_{n=1}^{\infty} \frac{1}{n^2 - n - 1}$ converges or diverges

Solution.

$$a_n = \frac{1}{n^2 - n - 1} \quad b_n = \frac{1}{n^2} \quad (\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges})$$

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n^2}{n^2-n-1}=1\neq 0$$

$$\frac{1}{n^2-n-1}\geqslant\frac{1}{n^2}\quad n=2,3,\cdots$$

$\Longrightarrow \sum_{n=1}^{\infty} a_n$ converges

Example 3.4.2

Example 3.4.3

Determine $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n(n+1)(n+2)}}$ converges or diverges

Solution.

$$a_n=\frac{1}{\sqrt{n(n+1)(n+2)}}\quad b_n=\frac{1}{n^{\frac{3}{2}}}\quad (\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}\text{ converges})$$

$$\lim_{n\to\infty}\frac{a_n}{b_n}=1\neq 0$$

$\Longrightarrow \sum_{n=1}^{\infty} a_n$ converges

Example 3.4.3

Example 3.4.4

Determine $\sum_{n=1}^{\infty}\sin(\frac{1}{n})$ converges or diverges

Solution.

$$a_n=\sin(\frac{1}{n})\quad b_n=\frac{1}{n}\quad (\sum_{n=1}^{\infty}\frac{1}{n}\text{ diverges})$$

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\sin(\frac{1}{n})}{\frac{1}{n}}=\lim_{n\to\infty}\frac{\sin m}{m}=1\neq 0$$

$\Longrightarrow \sum_{n=1}^{\infty} a_n$ diverges

Example 3.4.4

Example 3.4.5

Determine $\sum_{n=1}^{\infty}\frac{n+5}{\sqrt{n^7+n^2}}$ converges or diverges

Solution.

$$a_n = \frac{n+5}{\sqrt{n^7+n^2}} \quad b_n = \frac{1}{n^{\frac{4}{3}}} \quad (\sum_{n=1}^{\infty} \frac{1}{n^{\frac{4}{3}}} \text{ converges})$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1 \neq 0$$

$$\Longrightarrow \sum_{n=1}^{\infty} a_n \text{ converges}$$

Example 3.4.5

Example 3.4.6

Determine $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ converges or diverges

Solution.

$$a_n = \frac{1}{n^{1+\frac{1}{n}}} \quad b_n = \frac{1}{n^1} \quad (\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges})$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n^{\frac{1}{n}}} = 1 \neq 0$$

$$\lim_{x\to\infty} x^{\frac{1}{x}} \quad (\text{type } \infty)$$

$$y = x^{\frac{1}{x}} \Longrightarrow lny = \frac{1}{x} lnx$$

$$\lim_{x\to\infty} \frac{lnx}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

$$\lim_{x\to\infty} x^{\frac{1}{x}} = 1$$

$$\Longrightarrow \sum_{n=1}^{\infty} a_n \text{ diverges}$$

Example 3.4.6

Example 3.4.7

Determine $\sum_{n=1}^{\infty} \frac{n^3}{2^n}$ converges or diverges

Solution.

$$a_n = \frac{n^3}{2^n} = \frac{n^3}{(\frac{2}{1\cdot 1})^n (1\cdot 1)^n)} \quad b_n = \frac{1}{(\frac{2}{1\cdot 1})^n} (\sum_{n=1}^{\infty} \frac{1}{(\frac{2}{1\cdot 1})^n} \text{ converges})$$

geometric series 等比級數 with common ratio 公比 < 1

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n^2}{2^n ln2} = \frac{3}{ln2} \lim_{n\to\infty} \frac{n^2}{2^n} = \frac{3}{ln2} \lim_{n\to\infty} \frac{2n}{2^n ln2} = \frac{6}{(ln2)^2} \lim_{n\to\infty} \frac{1}{2^n} = 0$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3}{(1\cdot 1)^n} = 0$$

$\Longrightarrow \sum_{n=1}^{\infty} a_n$ converges

Example 3.4.7

3.5▲ Alternating Series 交錯級數

Definition3.5.1.

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

Assume

(1) $a_n > 0 \quad \forall n = 1, 2, 3, \cdots$

(2) $a_n \searrow$

$\Longrightarrow \sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges

Example 3.5.2

Determine $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges or diverges

Solution.

(1) $a_n = \frac{1}{n} > 0 \quad \forall n = 1, 2, \cdots$

(2) $a_n \searrow 0 \quad (a_n \searrow$ and $\lim_{n\to\infty} a_n = 0)$

$\Longrightarrow \sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges

Example 3.5.2

Proof.

$$S_{2n} \nearrow \text{ has a upper bound } a_1 \implies \lim_{n\to\infty} S_{2n} \text{ converges}$$
$$S^*_{2n+1} \searrow \text{ has a lower bound } 0 \implies \lim_{n\to\infty} S^*_{2n+1} \text{ converges}$$

Claim: $\lim\limits_{n\to\infty} S_{2n} = \lim\limits_{n\to\infty} S^*_{2n+1}$

$$\lim_{n\to\infty} S_{2n} - \lim_{n\to\infty} S^*_{2n+1} = \lim_{n\to\infty} (S_{2n} - S^*_{2n+1}) = 0$$
$$\lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} S^*_{2n+1}$$

$\implies \sum\limits_{n=1}^{\infty} (-1)^{n+1} a_n$ converges □

Example 3.5.3

Determine $\sum\limits_{n=1}^{\infty} (-1)^n \cos(\frac{\pi}{n})$ converges or diverges

Solution.

(1) $a_n = \cos(\frac{\pi}{n}) > 0 \quad \forall n = 3, 4, \cdots$

(2) $a_n \searrow \quad f(x) = \cos(\frac{\pi}{n}) \implies f'(x) =$

$$\lim_{n\to\infty} \cos(\frac{\pi}{n}) = 1 \neq 0$$

$\implies \sum\limits_{n=1}^{\infty} (-1)^n \cos(\frac{\pi}{n})$ diverges

Example 3.5.3

Example 3.5.4

Determine $\sum\limits_{n=1}^{\infty} (-1)^n \sin(\frac{\pi}{n})$ converges or diverges

Solution.

(1) $a_n = \sin(\frac{pi}{n}) > 0 \quad \forall n = 2, 3, \cdots$

(2) $a_n \searrow \quad f(x) = \sin(\frac{\pi}{x}) \implies f'(x) = \cos(\frac{\pi}{x}) \cdot (\frac{-\pi}{x^2}) < 0, x \geqslant 3$

$\implies \sum\limits_{n=1}^{\infty} (-1)^n \sin(\frac{\pi}{n})$ converges

Example 3.5.4

Example 3.5.5

Determine $\sum_{n=2}^{\infty}(-1)^n\frac{lnn}{n}$ converges or diverges?

Solution.

$$a_n = \frac{lnn}{n}$$

(1) $a_n = \frac{lnn}{n} > 0 \quad \forall n = 2, 3, \cdots$

(2) $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{lnn}{n} = \lim_{n\to\infty}\frac{\frac{1}{n}}{1} = 0$

let $f(x) = \frac{lnx}{x}$

$$f'(x) = \frac{x\frac{1}{x} - lnx}{x^2} = \frac{1}{x^2}(1 - lnx) \quad \text{Quotient Rule}$$

$\therefore$ As $x > e \implies f'(x) < 0 \quad (lne = 1)$
$\implies a_n \nearrow$ when $n > 3$
$\implies \sum_{n=2}^{\infty}(-1)^n\frac{lnn}{n}$ converges

Example 3.5.5

Example 3.5.6

Determine $\sum_{n=2}^{\infty}(-1)^n\frac{(lnn)^p}{n}, p > 0$ converges or diverges?

Solution.

$$a_n = \frac{(lnn)^p}{n}$$

(1) $a_n = \frac{(lnn)^p}{n} > 0 \quad \forall n = 2, 3, \cdots$

(2) $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{(lnn)^p}{n}$

$$\lim_{n\to\infty}\frac{p(lnn)^{p-1}\frac{1}{n}}{1} = p(\lim_{n\to\infty}\frac{(lnn)^{p-1}}{n}) = \begin{cases} 0 & , \quad p \leqslant 1 \\ \lim_{n\to\infty}\frac{(p-1)(lnn)^{p-2}\frac{1}{n}}{1} & , \quad p > 1 \end{cases}$$

$$p > 1: \quad p(p-1)\lim_{n\to\infty}\frac{ln^{p-2}}{n} = \begin{cases} 0 & , \quad 1 < p \leqslant 2 \\ \cdots & , \quad p > 2 \end{cases}$$

$$\lim_{n\to\infty}\frac{(lnn)^p}{n} = 0$$

let $f(x) = \dfrac{(lnx)^p}{x}$

$$f'(x) = \frac{\cancel{x}p(lnx)^{p-1}\cdot\frac{1}{\cancel{x}} - (lnx)^p \cdot 1}{x^2} = \frac{1}{x^2}(lnx)^{p-1}(p - lnx)$$

$\therefore$ As $p - lnx < 0 \quad (x > e^p) \implies f'(x) < 0$

$\implies \displaystyle\sum_{n=2}^{\infty}(-1)^n\frac{(lnn)^p}{n}$ converges

Example 3.5.6

3.6 Absolute Convergence

Definition3.6.1 (A.C and C.C).

(1) If $\displaystyle\sum_{n=1}^{\infty}\left|a_n\right|$ converges, then $\displaystyle\sum_{n=1}^{\infty}a_n$ is absolute convergent (A.C).

(2) If $\displaystyle\sum_{n=1}^{\infty}\left|a_n\right|$ diverges, but $\displaystyle\sum_{n=1}^{\infty}a_n$ converges, then $\displaystyle\sum_{n=1}^{\infty}a_n$ is conditionally convergent (C.C).

Notation3.6.2.

If $\displaystyle\sum_{n=1}^{\infty}a_n$ is A.C

$$\sum_{n=1}^{\infty}\left|a_n\right| < \infty \quad \text{by def}$$

$$\left|\sum_{n=1}^{\infty}a_n\right| \leqslant \sum_{n=1}^{\infty}\left|a_n\right| < \infty$$

$$-\sum_{n=1}^{\infty}\left|a_n\right| \leqslant \sum_{n=1}^{\infty}a_n \leqslant \sum_{n=1}^{\infty}\left|a_n\right|$$

$\implies \displaystyle\sum_{n=1}^{\infty}a_n$ converges

Example 3.6.3

Is $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2}$ is A.C?

Solution.

$$a_n = (-1)^n \frac{1}{n^2}$$

$$\sum_{n=1}^{\infty} \left|(-1)^n \frac{1}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ is convergent} \quad (\because p\text{-series with } p = 2)$$

$$\Longrightarrow \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2} \text{ is A.C)}$$

Example 3.6.3

Example 3.6.4

Is $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ is A.C?

Solution.

$$a_n = (-1)^n \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \left|(-1)^n \frac{1}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n} \text{ is divergent} \quad (\because p\text{-series with } p = 1)$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \text{ is alternating series}$$

$$a_n = \frac{1}{n}$$

(1) $a_n = \frac{1}{n} > 0 \quad \forall n = 1, 2, 3, \cdots$

(2) $a_n \searrow 0 \quad \lim_{n\to\infty} a_n = 0$

$$\Longrightarrow \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \text{ converges}$$

$$\Longrightarrow \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \text{ is C.C}$$

Example 3.6.4

3.7 Ratio and Root Tests

Definition3.7.1.

Consider $\sum_{n=1}^{\infty} a_n$

$$\begin{cases} \text{Ratio Test} & \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \geqslant 0 \\ \text{Root Test} & \lim_{n\to\infty}\left|a_n\right|^{\frac{1}{n}} = L \geqslant 0 \end{cases}$$

(1) $L < 1 \implies \sum_{n=1}^{\infty} a_n$ is A.C ($\implies \sum_{n=1}^{\infty} \left|a_n\right|$ converges)

(2) $L > 1 \implies \sum_{n=1}^{\infty} a_n$ diverges ($\implies \sum_{n=1}^{\infty} \left|a_n\right|$ diverges)

(3) $L = 1 \implies$ Inconclusive
Any conclusion cannot be drawn from this test (i.e the test fails)

Example 3.7.2

Does $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converge?

Solution.

$$a_n = \frac{n^2}{2^n}$$

- Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} = \lim_{n\to\infty}(\frac{(n+1)^2}{n}\frac{1}{2} = \frac{1}{2} < 1$$

$$\implies \sum_{n=1}^{\infty}\frac{n^2}{2^n} \text{ is A.C } (\implies \sum_{n=1}^{\infty}\frac{n^2}{2^n} \text{ converges})$$

- Root Test

$$\lim_{n\to\infty}\left|a_n\right|^{\frac{1}{n}} = \lim_{n\to\infty}(\frac{n^2}{2^n})^{\frac{1}{n}} = \lim_{n\to\infty}\frac{n^{\frac{2}{n}}}{2} = \frac{1}{2}(\lim_{n\to\infty} n^{\frac{2}{n}}) = \frac{1}{2} < 1$$

$$f(x) = x^{\frac{2}{x}} \quad lnf(x) = \frac{2}{x}lnx$$

$$2\lim_{n\to\infty}\frac{lnx}{x} = 2\lim_{n\to\infty}\frac{\frac{1}{x}}{1} = 0$$

$$\lim_{n\to\infty} x^{\frac{2}{x}} = 1$$

$$\implies \sum_{n=1}^{\infty} \frac{n^2}{2^n} \text{ is A.C}$$

Example 3.7.2

Example 3.7.3

Does $\sum_{n=1}^{\infty} (\frac{n^2+1}{2n^2+1})^n$ converge?

Solution.

$$a_n = (\frac{n^2+1}{2n^2+1})^n$$

- Root Test

$$\lim_{n\to\infty} ((\frac{n^2+1}{2n^2+1})^{\cancel{n}})^{\cancel{\frac{1}{n}}} = \lim_{n\to\infty} \frac{1n^2+1}{2n^2+1} = \frac{1}{2} < 1$$

$$\implies \sum_{n=1}^{\infty} (\frac{n^2+1}{2n^2+1}) \text{ is A.C}$$

Example 3.7.3

Example 3.7.4

Does $\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$ converges?

Solution.

$$a_n = \frac{(-3)^n}{n!}$$

- Ratio Test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(-3)^{n+1}}{(n+1)!}}{\frac{(-3)^n}{n!}}\right| = \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1$$

$$\implies \sum_{n\to\infty} \frac{(-3)^n}{n!} \text{ is A.C } (\implies \sum_{n=1}^{\infty} \frac{(-3)^n}{n!} \text{ converges})$$

Example 3.7.4

Example 3.7.5

Is $\sum_{n=1}^{\infty} (-1)^n \frac{2^n n!}{5\cdot 8\cdot 11\cdots(3n+2)}$ A.C?

Solution.

$$a_n = \frac{(-1)^n 2^n n!}{5 \cdot 8 \cdot 11 \cdots (3n+2)}$$

- Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{\cancel{(-1)^{n+1}}2^{n+1}(n+1)!}{\cancel{5\cdot8\cdot11\cdots(3n+2)}(3n+5)}}{\frac{\cancel{(-1)^n}2^n n!}{\cancel{5\cdot8\cdot11\cdots(3n+2)}}}\right| = \lim_{n\to\infty}\left|\frac{2(n+1)}{3n+5}\right| = \frac{2}{3} < 1$$

$$\implies \sum_{n=1}^{\infty} a_n \text{ is A.C} \quad \left(\text{i.e } \sum_{n=1}^{\infty}\left|a_n\right| < \infty\right)$$

Example 3.7.5

Example 3.7.6

Is $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n 5^n n!}{5\cdot 8\cdot 11\cdots(3n+2)}$ A.C?

Solution.

$$a_n = \frac{(-1)^n 5^n n!}{5 \cdot 8 \cdot 11 \cdots (3n+2)}$$

- Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{\cancel{(-1)^{n+1}}5^{n+1}(n+1)!}{\cancel{5\cdot8\cdot11\cdots(3n+2)}(3n+5)}}{\frac{\cancel{(-1)^n}5^n n!}{\cancel{5\cdot8\cdot11\cdots(3n+2)}}}\right| = \lim_{n\to\infty}\left|\frac{5(n+1)}{3n+5}\right| = \frac{5}{3} > 1$$

$$\implies \sum_{n=1}^{\infty} a_n \text{ diverges (NOT A.C)}$$

Example 3.7.6

Example 3.7.7

Is $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{(\tan^{-1} n)^n}$ A.C?

Solution.

$$a_n = \frac{(-1)^n}{(\tan^{-1} n)^n}$$

- Root Test

$$\lim_{n\to\infty}\left|a_n\right|^{\frac{1}{n}} = \lim_{n\to\infty}\left|\frac{1}{(\tan^{-1} n)^n}\right|^{\frac{1}{n}} = \lim_{n\to\infty}\frac{1}{|\tan^{-1} n|} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi} < 1$$

$$\implies \sum_{n=1}^{\infty} a_n \text{ is A.C}$$

Example 3.7.7

3.8▲ Power Series 冪級數

Definition3.8.1.

$$\sum_{n=1}^{\infty} c_n(x-a)^n = c_0(x-a)^0 + c_1(x-a)^1 + c_2(x-a)^2 + \cdots$$

$\sum_{n=1}^{\infty} c_n(x-a)^n$ is a power series about a (centered at a) or a power series in $(x-a)$

Notation3.8.2.

- "Power" means "n"
- Not a polynomial
- variable x in a series

Example 3.8.3

Find the interval of convergence (收斂區間) and the ratio of convergence of $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$

Solution.

$$b_n = \frac{(x-3)^n}{n} \qquad a = 3$$

$$c_n = \begin{cases} \frac{1}{n} & , \quad n = 1, 2, 3\cdots \\ 0 & , \quad n = 0 \end{cases}$$

Use Ratio Test

$$\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(x-3)^{n+1}}{n+1}}{\frac{(x-3)^n}{n}}\right| = \lim_{n\to\infty}\left|\frac{(x-3)n}{n+1}\right| = |x-3|\lim_{n\to\infty}\left|\frac{n}{n+1}\right| = |x-3| = L$$

- $|x-3| < 1 \implies \sum_{n=1}^{\infty} b_n$ is A.C
- $|x-3| > 1 \implies \sum_{n=1}^{\infty} b_n$ is divergent

- $|x-3| = 1 \implies$ Inconclusive

(1) $x = 2$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(\begin{array}{l} \text{alternating series} \\ k_n = \frac{1}{n} \searrow 0 \end{array} \right)$

$\implies \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges

(2) $x = 4$ $\displaystyle\sum_{n=1}^{\infty} \frac{1^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ (p-series with $p = 1$)

$\implies$ diverges

I.O.C is $2 \leqslant x < 4$ or $[2, 4)$ and R.O.C $= 1$

Example 3.8.3

Example 3.8.4 (Bessel Function)

Find the I.O.C and R.O.C of $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$

Solution.

$$b_n = \frac{(-1)^n (x-0)^{2n}}{2^{2n}(n!)^2}$$

Use Ratio Test

$$\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{\cancel{(-1)^{n+1}} x^{2n+2}}{2^{2n+2}((n+1)!)^2}}{\frac{\cancel{(-1)^n} x^{2n}}{x^{2n}(n!)^2}} \right| = \lim_{n\to\infty} \left| \frac{x^2}{4(n+1)^2} \right| = x^2 (\lim_{n\to\infty} \frac{1}{4(n+1)^2}) = 0 \quad \forall x \in R$$

I.O.C is $(-\infty, \infty)$ and R.O.C $= \infty$

Example 3.8.4

Example 3.8.5

Find the I.O.C of $\displaystyle\sum_{n=1}^{\infty} \frac{2^n (x-1)^n}{n}$

Solution.

$$b_n = \frac{2^n (x-1)^n}{n}$$

Use Ratio Test

$$\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{2^{n+1}(x-1)^{n+1}}{n+1}}{\frac{2^n (x-1)^n}{n}} \right| = \lim_{n\to\infty} \left| \frac{2(x-1)n}{n+1} \right| = |x-1| \lim_{n\to\infty} \frac{2n}{n+1} = 2|x-1|$$

- $2|x-1| < 1 \implies \displaystyle\sum_{n=1}^{\infty} b_n$ is A.C

- $2|x-1| > 1 \implies \sum_{n=1}^{\infty} b_n$ is divergent
- $2|x-1| = 1 \implies$ Inconclusive

(1) $x = \frac{1}{2} b_n = \frac{2^n(-\frac{1}{2})^n}{n} = \frac{(-1)^n}{n}$

$\implies \sum_{n=1}^{\infty} b_n$ converges

(2) $x = \frac{3}{2} b_n = \frac{2^n(\frac{1}{2})^n}{n} = \frac{1}{n}$

$\implies \sum_{n=1}^{\infty} b_n$ diverges

I.O.C is $\frac{1}{2} \leqslant x < \frac{3}{2}$

Example 3.8.5

Example 3.8.6

Find the I.O.C and R.O.C of $\sum_{n=1}^{\infty} n!x^n$

Solution.

$$b_n = n!x^n$$

Use Ratio Test

$$\lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty} \left|(n+1)x\right| = |x| \lim_{n\to\infty} (n+1)$$

I.O.C is $\{0\}$ and R.O.C $= 0$

Example 3.8.6

3.9▲ Representation of Functions as Power Series

Q: Does a function has a power series representation?

$$1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{as } |x| < 1$$

Example 3.9.1

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n \quad \text{as } |-x^2| < 1$$

- $\sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$
- $|-x^2| < 1 \implies x^2 < 1 \implies -1 < x < 1$ or $|x| < 1$

Example 3.9.1

Example 3.9.2

$$\frac{1}{2-x} = \frac{1}{2(1-\frac{x}{2})} = \frac{1}{2}\frac{1}{1-\frac{x}{2}} = \frac{1}{2}\sum_{n=0}^{\infty}(\frac{x}{2})^n \quad \text{as } \left|\frac{x}{2}\right| < 1$$

- $\frac{1}{2}\sum_{n=0}^{\infty}(\frac{x}{2})^n = \sum_{n=0}^{\infty} 2^{-n-1}x^n$
- $\left|\frac{x}{2}\right| < 1 \implies |x| < 2 \implies -2 < x < 2$

Example 3.9.2

Example 3.9.3

$$\begin{aligned}
\frac{x^3}{2+x^2} &= x^3(\frac{1}{x+x^2}) \\
&= x^3\frac{1}{2(1+\frac{x^2}{2})} \\
&= \frac{x^3}{2}\frac{1}{1-(-\frac{x^2}{2})} \\
&= \frac{x^3}{2}\sum_{n=0}^{\infty}(-\frac{x^2}{2})^n \\
&= \frac{x^3}{2}\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n}x^{2n} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^{2n+3}
\end{aligned}$$

Example 3.9.3

3.10▲ Term by Term Differentiation and Integration

Theorem3.10.1.

If $\sum_{n=0}^{\infty} c_n(x-a)^n$ has I.O.C $|x-a|<r$ and let $f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n$, then

(1) Differentiation

$$\begin{aligned} f'(x) &= \frac{d}{dx}(\sum_{n=0}^{\infty} c_n(x-a)^n \\ &= \sum_{n=0}^{\infty} \frac{d}{dx}(c_n(x-a)^n) \quad x\in(a-r,a+r) \\ &= \sum_{n=0}^{\infty} c_n\frac{d}{dx}((x-a)^n) \\ &= 0+c_1+2c_2(x-a)^1+\cdots \\ &= \sum_{n=1}^{\infty} c_n n(x-a)^{n-1} \end{aligned}$$

(2) Integration

$$\begin{aligned} \int f(x)dx &= \int \sum_{n=0}^{\infty} c_n(x-a)^n dx \\ &= \sum_{n=0}^{\infty}(\int c_n(x-a)^n dx) \\ &= \sum_{n=0}^{\infty}(c_n\frac{1}{n+1}(x-a)^{n+1})+C \end{aligned}$$

Example 3.10.2

Find p.s.r of $f(x)=ln(1-x)$

Solution.

$$f'(x)=\frac{-1}{1-x}=\sum_{n=0}^{\infty}-x^n \quad \text{as } |x|<1$$

$$\int f'(x)dx=\int(\sum_{n=0}^{\infty}-x^n)dx=-\sum_{n=0}^{\infty}(\int x^n dx)=-\sum_{n=0}^{\infty}\frac{1}{n+1}x^{n+1}+C \quad |x|<1$$

$$f(x)=-\sum_{n=0}^{\infty}\frac{1}{n+1}x^{n+1}+C \quad |x|<1$$

let $x = 0 \implies ln10 + C \implies C = 0$

$$ln(1-x) = \sum_{n=0}^{\infty} \frac{-1}{n+1} x^{n+1} \quad |x| < 1$$

Example 3.10.2

Example 3.10.3

Find p.s.r of $f(x) = ln(1-x)$

Solution.

$$\begin{aligned}
f'(x) = \frac{-1}{1-x} &= -\sum_{n=0}^{\infty} x^n && |x| < 1 \\
\int f'(x)dx &= \int -\sum_{n=0}^{\infty} x^n dx \\
&= -\sum_{n=0}^{\infty} (\int x^n dx) && |x| < 1 \\
&= -\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} && |x| < 1 \\
ln(1-x) + C &= -\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} && |x| < 1
\end{aligned}$$

let $x = 0$

$$ln1 + C = 0 \implies C = 0$$

$$ln(1-x) = \sum_{n=0}^{\infty} \frac{-1}{n+1} x^{n+1}$$

Example 3.10.3

Example 3.10.4

Find p.s.r of $f(x) = \tan^{-1} x$

Solution.

$$\begin{aligned}
f'(x) &= \frac{1}{1+x^2} \\
&= \frac{1}{1-1(-x^2)} \\
&= \sum_{n=0}^{\infty} (-x^2) && |-x^2| < 1 \\
&= \sum_{n=0}^{\infty} (-1)^n x^{2n} && |x| < 1
\end{aligned}$$

$$\begin{aligned}
\int f'(x)dx &= \int \sum_{n=0}^{\infty} (-1)^n x^{2n} dx \\
&= \sum_{n=0}^{\infty} \left(\int (-1)^n x^{2n} dx\right) \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1} + C \\
\tan^{-1} x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} + C
\end{aligned}$$

let $x = 0$

$$\tan^{-1} 0 = 0 + C \implies C = 0$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

let $x = 1$

$$\tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{d}{dx}(1-x)^{-1} = (1-x)^{-2}$$

Example 3.10.4

Example 3.10.5

Find p.s.r of $f(x) = \dfrac{1}{(1-x)^2}$

Solution.

$$\begin{aligned}
\int f(x)dx &= \int (1-x)^{-2} dx \\
&= \frac{1}{1-x} + C \\
&= \sum_{n=0}^{\infty} x^n + C \qquad |x| < 1
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}\int f(x)dx &= \frac{d}{dx}\sum_{n=0}^{\infty} x^n + C \qquad |x| < 1 \\
&= \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) + \frac{d}{dx}C \\
&= \sum_{n=0}^{\infty}\left(\frac{d}{dx}x^n\right) \\
&= \sum_{n=1}^{\infty} n x^{n-1} \\
\frac{1}{(1-x)^2} &= \sum_{n=1}^{\infty} n x^{n-1} \qquad |x| < 1
\end{aligned}$$

Example 3.10.5

Example 3.10.6

Does $\sum_{n=1}^{\infty}(-1)^n(\sqrt{x+1}-\sqrt{n})$ converge?

Solution.

$$a_n = \sqrt{n+1}-\sqrt{n} < 0 \quad n = 1, 2, \cdots$$

- $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n}) = \lim_{n\to\infty}\frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{(\sqrt{n+1}+\sqrt{n})} = 0$
- $a_n = \dfrac{1}{\sqrt{n+1}+\sqrt{n}} \searrow$

$\Longrightarrow \sum_{n=1}^{\infty}(-1)^n a_n$ converges

Example 3.10.6

Example 3.10.7

Find p.s.r of $\int \frac{dx}{1+x^7}$

Solution.

$$\begin{aligned}
f(x) &= \int \frac{dx}{1+x^7} \\
f'(x) &= \frac{d}{dx}\int \frac{dx}{1+x} \\
&= \frac{1}{1+x^7} \\
&= \frac{1}{1-(-x^7)} \\
&= \sum_{n=0}^{\infty}(-x^7)^n \qquad |-x^7| < 1 \\
&= \sum_{n=0}^{\infty}(-1)^n x^{7n} \qquad |x| < 1
\end{aligned}$$

$$\begin{aligned}
\int f'(x)dx &= \int \sum_{n=0}^{\infty}(-1)^n x^{7n} \qquad |x| < 1 \\
f(x) &= \sum_{n=0}^{\infty}(-1)^n \int x^{7n}dx + C \qquad |x| < 1 \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{7n+1}x^{7n+1} + C \qquad |x| < 1
\end{aligned}$$

Example 3.10.7

3.11 Taylor and Maclaurin Series

Theorem3.11.1.

Assume $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ has p.s.r, then

$$c_n = \frac{f^{(n)}(a)}{n!}$$

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots$$

(1) let $x = a \implies c_0 = f(a) = \dfrac{f(a)}{0!}$

(2) $f'(x) = 0 + c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$
let $x = a \implies c_1 = f'(a) = \dfrac{f'(a)}{1!}$

(3) $f''(x) = 0 + 2c_2 + 3 \cdot 2c_3(x-a) + 4 \cdot 3c_4(x-a) + \cdots$
let $x = a \implies c_2 = \dfrac{1}{2} f''(a) = \dfrac{f''(a)}{2!}$

(4) $f'''(x) = 0 + 3 \cdot 2c_3 + 4 \cdot 3 \cdot 2c_4(x-a) + \cdots$
let $x = a \implies c_3 = \dfrac{f'''(a)}{3!}$

$\vdots$

$c_n = \dfrac{f^{(n)}(a)}{n!}$

Definition3.11.2.

(1) $\sum_{n=0}^{\infty} c_n (x-a)^n$, whose $c_n \dfrac{f^{(n)}(a)}{n!}$, is called Taylor series of $f(x)$ about a
(Given $f(x)$ and $a \implies$ Taylor series of $f(x)$ of a)

(2) $\sum_{n=0}^{\infty} c_n x^n$ with $c_n = \dfrac{f^{(n)}(0)}{n!}$ is called Maclaurin series of $f(x)$

Example 3.11.3

Find Maclaurin series of $f(x) = e^x$

Solution.

$$\begin{aligned} f(x) &= e^x \\ f'(x) &= e^x \\ &\vdots \\ f^{(n)}(x) &= e^x \\ f^{(n)}(0) &= 1 \end{aligned}$$

$$c_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!}$$

$\therefore$ Maclaurin series of $f(x) = e^x$ is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

Example 3.11.3

Example 3.11.4

Find Maclaurin series of $f(x) = \sin x$

Solution.

$$\begin{array}{rcrcrcr} f(x) &=& \sin x &\Longrightarrow& f(0) &=& 0 \\ f'(x) &=& \cos x &\Longrightarrow& f'(0) &=& 1 \\ f''(x) &=& -\sin x &\Longrightarrow& f''(0) &=& 0 \\ f'''(x) &=& -\cos x &\Longrightarrow& f'''(0) &=& -1 \\ f^{(4)}(x) &=& \sin x &\Longrightarrow& f^{(4)}(0) &=& 0 \end{array}$$

$$c_n = \frac{(-1)^n}{(2n+1)!}$$

$\therefore$ Maclaurin series of $f(x) = \sin x$ is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots (= \sin x)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

(1) $\sin(-x) = -\sin x$

(2) $\lim\limits_{n\to\infty} \dfrac{\sin x}{x} = 1$

Example 3.11.4

Example 3.11.5

Find Maclaurin series of $f(x) = \cos x$

Solution.

$$\begin{array}{rcrcrcr} f(x) &=& \cos x &\Longrightarrow& f(0) &=& 1 \\ f'(x) &=& -\sin x &\Longrightarrow& f'(0) &=& 0 \\ f''(x) &=& -\cos x &\Longrightarrow& f''(0) &=& -1 \\ f'''(x) &=& \sin x &\Longrightarrow& f'''(0) &=& 0 \\ f^{(4)}(x) &=& \cos x &\Longrightarrow& f^{(4)}(0) &=& 1 \end{array}$$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} &= \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n}{(2n)!}x^{2n} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} \\ &= \cos x \end{aligned}$$

Q: What is the I.O.C of $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$?

$$a_n = \frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

Use Ratio Test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{\cancel{(-1)^{n+1}}}{(2n+3)!}x^{2n+3}}{\frac{\cancel{(-1)^{n}}}{(2n+1)!}x^{2n+1}}\right| = \lim_{n\to\infty} \left|\frac{x^2}{(2n+2)(2n+3)}\right| = x^2(\lim_{n\to\infty} \frac{1}{(2n+2)(2n+3)}) = 0$$

Example 3.11.5

Lemma3.11.6.

$$\lim_{n\to\infty} (n!)^{\frac{1}{n}} = \infty$$

Example 3.11.7

Is $\sum_{n=0}^{\infty} \frac{(-3)^n}{n!}$ A.C?

Solution. Use Root Test

- $a_n = \dfrac{(-3)^n}{n!}$
- $\lim\limits_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim\limits_{n\to\infty} \dfrac{3}{(n!)^{\frac{1}{n}}} = 0 < 1$

$\implies \sum\limits_{n=0}^{\infty}$ is A.C

Example 3.11.7

Example 3.11.8

Find I.O.C of $\sum\limits_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{x^{2n}(n!)^2}$

Solution. Use Root Test

- $a_n = \dfrac{(-1)^n x^{2n}}{x^{2n}(n!)^2}$
- $\lim\limits_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim\limits_{n\to\infty} \dfrac{x^2}{4((n!)^{\frac{1}{n}})^2} = \dfrac{x^2}{4} \lim\limits_{n\to\infty} \dfrac{1}{((n!)^{\frac{1}{n}})^2} = 0 \quad \forall x \in R < 1$

$\implies$ I.O.C is R or $(-\infty, \infty)$

Example 3.11.8

Example 3.11.9

Find I.O.C of $\sum\limits_{n=0}^{\infty} n! x^n$

Solution. Use Root Test

- $a_n = n! x^n$
- $\lim\limits_{n\to\infty} |x|(n!)^{\frac{1}{n}} = (\lim\limits_{n\to\infty} (n!)^{\frac{1}{n}})|x| = \begin{cases} 0 < 1 & , \quad x = 0 \\ \infty > 1 & , \quad x \neq 0 \end{cases}$

$\implies$ I.O.C is 0

Example 3.11.9

Example 3.11.10

Find M.S of $f(x) = \begin{cases} e^{-\frac{1}{x^2}} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$

Solution.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$
\begin{aligned}
f(0) &= 0 \\
f'(0) &= \lim_{h\to\infty} \frac{f(0+h)-f(0)}{h} \\
&= \lim_{h\to\infty} \frac{e^{-\frac{1}{h^2}}}{h} \\
&= \lim_{m\to\infty} \frac{e^{-m^2}}{\frac{1}{m}} \\
&= \lim_{m\to\infty} \frac{m}{e^{m^2}} \\
&\overset{L}{=} \lim_{m\to\infty} \frac{1}{2me^{m^2}} \\
&= 0 \\
f''(0) &= \lim_{h\to\infty} \frac{f'(h)-f'(0)}{h} \\
&= \lim_{h\to\infty} \frac{e^{-\frac{1}{h^2}}}{2h^4} \\
&= 0 \\
&\vdots \\
f^{(n)}(0) &= 0 \quad n = 1, 2, \cdots
\end{aligned}
$$

M.S of $f(x)$ is $\sum_{n=0}^{\infty} \frac{0}{n!} x^n = 0$ but $f(x) \neq 0$

Example 3.11.10

Notation3.11.11.

	限制	equal to Taylor series
11.9	Yes	一定
11.10	No	不一定

Example 3.11.12

$$
\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots) - 1 - x}{x^2} = \frac{1}{2}
$$

Example 3.11.12

Example 3.11.13

Find the first non-zero terms in the Maclaurin series of $f(x)$

Solution.

(a) $f(x) = \tan x$

$$\begin{aligned}
\tan x &= \frac{\sin x}{\cos x} \\
\sin x &= x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots \qquad x \in R \\
\cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots \qquad x \in R
\end{aligned}$$

$$\begin{array}{r|cccccccc}
 & x & + & \frac{x^3}{3} & + & \frac{2}{15}x^5 & & & \\
\hline
1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots & x & - & \frac{x^3}{6} & + & \frac{x^5}{120} & - & \cdots & \\
 & x & - & \frac{x^3}{2} & + & \frac{x^5}{24} & & & \\
\hline
 & & & \frac{x^3}{3} & - & \frac{x^5}{30} & & & \\
 & & & \frac{x^3}{3} & - & \frac{x^5}{6} & + & \frac{x^7}{72} & \\
\hline
 & & & & & \frac{x}{15}x^5 & & &
\end{array}$$

(b) $f(x) = e^x \sin x$

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots \qquad x \in R \\
\sin x &= x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots \qquad x \in R \\
e^x \sin x &= (1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots) \cdot (x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots) \\
&= x + x^2 + x^3(-\frac{1}{6} + \frac{1}{2}) \\
&= x + x^2 + \frac{x^3}{3} + \cdots \qquad x \in R
\end{aligned}$$

Example 3.11.13

Example 3.11.14

Find the Taylor series for $f(x) = e^x$ at $a = 2$

Solution.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n = e^2(\sum_{n=0}^{\infty} \frac{1}{n!}(x-2)^n)$$

Find I.O.C

$$\lim_{n\to\infty} (n!)^{\frac{1}{n}} = \infty$$

Use Root Test

$$\begin{aligned}
\lim_{n\to\infty} \frac{|x-2|}{(n!)^{\frac{1}{n}}} &= |x-2| \lim_{n\to\infty} \frac{1}{(n!)^{\frac{1}{n}}} \\
&= |x-2| \cdot 0 \\
&= 0 \quad \forall x \in R
\end{aligned}$$

$\Longrightarrow$ I.O.C is R or $(-\infty, \infty)$

Example 3.11.14

Example 3.11.15

Use the Maclaurin series for $f(x) = \sin x$ to find the Maclaurin series for $g(x) = \cos x$

Solution.

$$\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad x \in R \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad x \in R \\
\int \sin x &= \frac{d}{dx} (\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}) \\
&= \sum_{n=0}^{\infty} (\frac{(-1)^n}{(2n+1)!} (2n+1) x^2 n) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \quad x \in R
\end{aligned}$$

Example 3.11.15

Definition3.11.16.

$Tn(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k$: the n-th degree of Taylor polynomial of f at a

3.12▲ Taylor Inequality

Definition3.12.1.

If

$$|f^{(n+1)}(x)| \leqslant M \text{ for } |x-a| \leqslant d$$

Then

$$|f(x) - Tn(x)| \leqslant \frac{M}{(n+1)!}(x-a)^{n+1} \text{ for } |x-a| \leqslant d$$

Where

$$Tn(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$Rn(x) = f(x) - Tn(x)$: Remainder to of Taylor series of f at a

Example 3.12.2

Show that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, x \in R$

Solution.

- I.O.C is R
  Use Ratio Test
  $$a_n = \frac{x^n}{n!}$$
  $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = |x|\lim_{n\to\infty}\frac{1}{n+1} = 0 < 1 \quad \forall x \in R$$
- $f(x) = e^x \implies f^{(n)}(x) = e^x \quad \forall x = 0, 1, 2, \cdots$
  When $|x| \leqslant d$
  $$|e^x| \leqslant e^d = M$$
  $$M = e^d, a = 0$$
  By Taylor Inequality,
  $$\left|f(x) - \sum_{k=0}^{\infty}\frac{x^k}{k!}\right| \leqslant \frac{e^d}{(n+1)!}x^{n+1}$$
  Use the fact $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ conveys $\forall x \in R$
  $$\lim_{n\to\infty} a_n = 0 \quad \forall x \in R$$
  $$\lim_{n\to\infty}\frac{x^n}{n!} = 0$$

$$\lim_{n\to\infty}\left|f(x)-\sum_{k=0}^{\infty}\frac{x^k}{k!}\right|=0$$

$$\sum_{k=0}^{\infty}\frac{x^k}{k!}=f(x)=e^x$$

Example 3.12.2

Example 3.12.3

Show that $(\infty)^{\frac{1}{\infty}}=\infty^0$

Solution.

$$\lim_{n\to\infty}(n!)^{\frac{1}{n}}=\infty$$

$$e^x=1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots \quad x\in R$$

let $x=n$

$$\begin{aligned}
e^x &\geqslant \frac{x^n}{n!} \quad x>0\\
e^n &\geqslant \frac{n^n}{n!}\\
(e^n)^{\frac{1}{n}} &\geqslant \frac{(n^n)^{\frac{1}{n}}}{(n!)^{\frac{1}{n}}}\\
e &\geqslant \frac{n}{(n^n)^{\frac{1}{n}}}\\
e(n!)^{\frac{1}{n}} &\geqslant n\\
e\lim_{n\to\infty}(n!)^{\frac{1}{n}} &\geqslant \lim_{n\to\infty} n\\
\lim_{n\to\infty}(n!)^{\frac{1}{n}} &\geqslant \frac{\lim_{n\to\infty} n}{e}=\infty
\end{aligned}$$

Example 3.12.3

▸▸ Show Taylor Inequality

Idea F.T.C

$$
\begin{aligned}
f(x) &= f(a)+\int_a^x f'(t)dt \\
&= f(a)+\int_a^x f'(t)d(t-x) \\
&\overset{I.B.P}{=} f(a)+f'(t)(t-x)\Big|_{t=a}^{t=x}-\int_a^x (t-x)f''(t)dt \\
&= f(a)-f'(a)(a-x)-\int_a^x \frac{1}{2}f''(t)d((t-x)^2) \\
&\overset{I.B.P}{=} f(a)-f'(a)(a-x)-\frac{1}{2}(f''(t)(t-x)^2\Big|_{t=a}^{t=x}-\int_a^x (t-x)^2 f'''(t)dt) \\
&= f(a)+f'(a)(x-a)+\frac{1}{2}(1-f''(a)(a-x)^2)+\frac{1}{2}\int_a^x (t-x)^2 f'''(t)dt \\
&= f(a)+f'(a)(x-a)+\frac{1}{2}f''(a)(x-a)^2+\frac{1}{3!}\int_a^x f'''(t)d((t-x)^3) \\
&\overset{I.B.P}{=} f(a)+f'(a)(x-a)+\frac{1}{2}f''(a)(x-a)^2+\frac{1}{3!}f'''(a)(x-a)^3\Big|_{t=a}^{t=x}-\int_a^x (t-x)^3 f^{(4)}(t)dt \\
&= f(a)+f'(a)(x-a)+\frac{1}{2}f''(a)(x-a)^2+\frac{1}{3!}f'''(a)(x-a)^3-\frac{1}{3!}\int_a^x (t-x)^3 f^{(4)}(t)dt \\
&\vdots \\
f(x) &= f(a)+f'(a)(x-a)+\frac{1}{2!}f''(a)(x-a)^2+\cdots+\frac{1}{n!}f^{(n)}(a)(x-a)^n \\
&\quad \underline{+(-1)^n\frac{1}{n!}\int_a^x (t-x)^n f^{(t+1)}(t)dt} \quad Rn(x) \\
&= \sum_{k=0}^{\infty}\frac{f^{(n)}(a)}{k!}(x-a)^k+\frac{(-1)^n}{n!}\int_a^x (t-x)^n f^{(n+1)}(t)dt \\
&= Tn(x)+Rn(x)
\end{aligned}
$$

Proof. Taylor Inequality

$$
\begin{aligned}
Rn(x) &= \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt \\
|Rn(x)| &\leqslant \frac{1}{n!}\int_a^x |x-t||f^{(n+1)}(t)|dt \\
&\leqslant \frac{1}{n!}\int_a^x |x-t|^n M dt \\
&= \frac{n}{n!}\frac{(x-a)^{n+1}}{n+1} \\
&= \frac{M}{(n+1)!}(x-a)^{n+1} \\
&\leqslant \frac{M}{(n+1)!}|x-a|^{n+1}
\end{aligned}
$$

$$
\begin{aligned}
\int_a^x |x-t|^n dt &= \int_a^x (x-t)^n dt \\
&= \frac{-1}{n+1}(x-t)^{n+1}\Big|_{t=a}^{t=x} \\
&= \frac{-1}{n+1}(0-(x-a)^{n+1}) \\
&= \frac{1}{n+1}(x-a)^{n+1}
\end{aligned}
$$

□

Example 3.12.4

(a) Approximate $f(x) = x^{\frac{1}{3}}$ by a Taylor polynomial of degree 2 at $a = 8$

Solution.

$$T_2(x) = \sum_{k=0}^{2} \frac{f^{(k)}(2)}{k!}(x-a)^k$$

$$\begin{aligned} f(x) &= x^{\frac{1}{3}} & f(8) &= 2 \\ f'(x) &= \frac{1}{3}x^{-\frac{2}{3}} & f'(8) &= \frac{1}{12} \\ f''(x) &= -\frac{2}{9}x^{-\frac{5}{3}} & f''(8) &= -\frac{2}{9}\cdot\frac{1}{32} = \frac{-1}{144} \end{aligned}$$

$$T_2(x) = f(8) + f'(8)(x-8) + \frac{f''(8)}{2}(x-8)^2 = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$$

$T_2(x)$ is a quadratic (二次) fcn.

(b) How accurate is the approximation when $7 \leqslant x \leqslant 9$?

Solution.

$$n = 2, a = 8, d = 1$$

$$f'''(x) = \frac{10}{27}\cdot\frac{1}{x^{\frac{8}{3}}}$$

$$|f'''(x)| = \left|\frac{10}{27}\cdot\frac{1}{x^{\frac{8}{3}}}\right| \leqslant \frac{10}{27}\cdot\frac{1}{7^{\frac{8}{3}}} = M \text{ when } |x-8| \leqslant 1$$

$$|R_2(x)| \leqslant \frac{M}{3!}|x-8|^3 \leqslant \frac{1}{3!}\cdot\frac{10}{27}\cdot\frac{1}{7^{\frac{8}{3}}}\cdot 1^3 \doteqdot 0.000344$$

$$|x-8| \leqslant 1$$

Example 3.12.4

3.13 Review

- Divergence Test

  $\displaystyle\sum_{n=0}^{\infty} a_n$ converges $\implies \displaystyle\lim_{n\to\infty} a_n = 0$

  ($\displaystyle\lim_{n\to\infty} a_n \neq 0$ or doesn't exist $\implies \displaystyle\sum_{n=0}^{\infty}$ diverges)

- Alternating Series Test

$$\sum_{n=0}^{\infty}(-1)^n a_n \quad a_n > 0$$

  (1) $\displaystyle\lim_{n\to\infty} a_n = 0$

  (2) $a_n \searrow$ in n

- Integral test

  (1) $a_n > 0 \quad \forall n = 1, 2, \cdots$

  (2) $a_n \searrow$ in a

  $$\sum_{n=1}^{\infty} a_n \neq \int_1^{\infty} a_n dn$$
  Both converge or diverge

- Comparison Test

  $$\sum a_n \quad \sum b_n \qquad a_n, b_n > 0$$

  – 減法

    (1) $a_n \geqslant b_n \quad \sum a_n$ converges $\implies \sum b_n$ converges

    (2) $a_n \geqslant b_n \quad \sum b_n$ diverges
    $\implies \sum a_n$ diverges

  – 除法 $\lim_{n\to\infty} \dfrac{a_n}{b_n} = L$

    (1) $L = 0 \quad \sum b_n$ converges $\implies \sum a_n$ converges

    (2) $L = \infty \quad \lim_{n\to\infty} \dfrac{b_n}{a_n} = 0$

    (3) $L \neq 0$ and $L \neq \infty \quad (0 < L < \infty) \implies \sum a_n$ and $\sum b_n$ both converge or diverge

- Ratio and Root Test

  – Ratio Test

  $$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

    (1) $L < 1 \quad \sum a_n$ is A.C $\implies \sum a_n$ converges (i.e $\sum |a_n|$ converges)

    (2) $L > 1 \quad \sum a_n$ diverges

    (3) $L = 1$ Inconclusive
    $\sum_{n=1}^{\infty} \dfrac{1}{n}$ diverges $\implies L = 1$
    $\sum_{n=1}^{\infty} \dfrac{1}{n^2}$ converges $\implies L = 1$

  – Root test

  $$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L$$

    (1) $L < 1 \quad \sum a_n$ is A.C $\implies \sum a_n$ converges (i.e $\sum |a_n|$ converges)

    (2) $L > 1 \quad \sum a_n$ diverges

    (3) $L = 1$ Inconclusive

- Power Series: Representation
  $\sum_{n=0}^{\infty} c_n(x-a)^n$ depends on x
  I.O.C is $|x-a| \leqslant R$
  R.O.C is R
  no negative power

  (1) $\dfrac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n \quad |x| < 1$

  (2) $\int f'(x)dx = f(x) + C$
  $\int \sum (c_n(x-a)^n)dx$
  $\sum (\int c_n(x-a)^n dx)$

- Taylor Series of f at a $\sum_{n=0}^{\infty} \dfrac{f^{(n)}(a)}{n!}$
  when $a = 0 \implies$ Maclaurin series

  (1) $e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} \quad x \in R$

  (2) $\sin x = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)!} x^{2n+1} \quad x \in R$

  (3) $\cos x = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n)!} x^{2n} \quad x \in R$

Chapter 4

Partial Derivatives

4.1 Partial Derivatives

Definition4.1.1.

$$f(x) \quad x \in R \quad \text{single variable}$$
$$f(x,y) \quad x,y \in R \quad \text{multi variables}$$

Example 4.1.2

$$z = f(x,y) = \sqrt{9 - x^2 - y^2}$$

- Domain

$$9 - x^2 - y^2 \geqslant 0$$
$$x^2 + y^2 \leqslant 9$$

$\implies$ Domain is $\{(x,y)|x^2 + y^2 \leqslant 9\}$

- Range

$$0 \leqslant x^2 + y^2 \leqslant 9$$
$$0 \leqslant 9 - x^2 - y^2 \leqslant 9$$
$$0 \leqslant \sqrt{9 - x^2 - y^2} \leqslant 3$$

$\implies$ Range is $0 \leqslant z \leqslant 3$

- Graph

$$z = \sqrt{9 - x^2 - y^2}$$
$$z^2 = 9 - x^2 - y^2$$

$$x^2 + y^2 + z^2 = 9$$
$$(x-0)^2 + (y-0)^2 + (z-0)^2 = 3^2$$
$$\Longrightarrow \text{上半球面 (sphere)} \quad \text{center} = (0,0) \quad \text{radius} = 3$$

- Level Curve（等高線）

$$z = \sqrt{9 - x^2 - y^2}$$
$$k = 9 - x^2 - y^2$$
$$x^2 + y^2 = 9 - k$$
$$0 \leqslant k \leqslant 9$$
$$k = 0 \quad x^2 + y^2 = 3^2$$
$$k = 1 \quad x^2 + y^2 = (2\sqrt{2})^2$$

Example 4.1.2

4.2▲ Limit and Continuity

- 1D $\Longrightarrow \lim\limits_{x \to a} f(x)$
- 2D $\Longrightarrow \lim\limits_{(x,y) \to (a,b)} f(x,y)$

Definition4.2.1.

$\lim\limits_{A \to B} f(A) = L$ if $\forall \epsilon > 0 \quad \exists \delta > 0$
s.t. if $d(A,B) < \delta$, then

$$|f(A) - L| < \epsilon$$

Example 4.2.2

Does $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$ exists? homogeneous 齊次

Solution.
Along the line $y = mx$ m: const

$$\lim_{(x,m)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x \to 0} \frac{\cancel{x^2}(1-m^)}{\cancel{x^2}(1+m^2)} = \frac{1-m^2}{1+m^2} \quad \text{depend on } m$$

$\Longrightarrow$ The limit doesn't exist

Example 4.2.2

Example 4.2.3

Does $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4}$ exists?

Solution.

$$y = mx^k \quad k: \text{ const}$$
$$\frac{x\cdot m^2x^{2k}}{x^2+m^4x^{4k}} = \frac{m^2x^{2k+1}}{x^2+mx^4k}$$
$$2k+1 = 2 = 4k$$
$$k = \frac{1}{2}$$

Along $y = mx^{\frac{1}{2}}$

$$\lim_{(x,mx^{\frac{1}{2}})\to(0,0)} \frac{x\cdot m^2x}{x^2+m^4x^2} = \lim_{x\to 0}\frac{\cancel{x^2}m^2}{\cancel{x^2}(1+m^4)} = \frac{m^2}{1+m^4} \quad \text{depend on } m$$

$\implies$ The limit doesn't exist

Example 4.2.3

Example 4.2.4

Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^2+y^2}$

Solution. 1

$$\left|\frac{x^2y}{x^2+y^2}\right| = \left|\frac{x^2}{x^2+y^2}\right|\left|y\right| \leqslant 1|y| = |y|$$
$$\lim_{(x,y)\to(0,0)} 0 \leqslant \lim_{(x,y)\to(0,0)}\left|\frac{x^2y}{x^2+y^2}\right| \leqslant \lim_{(x,y)\to(0,0)} |y|$$

By Squeeze Theorem

$$\lim_{(x,y)\to(0,0)}\left|\frac{x^2y}{x^2+y^2}\right| = 0 = \lim_{(x,y)\to(0,0)}\frac{x^2y}{x^2+y^2}$$

Solution. 2

Use polar coordinates $(x,y)\to(r,\theta)$

$$x = r\cos\theta \quad y = r\sin\theta$$
$$\lim_{(x,y)\to(0,0)}\frac{x^2y}{x^2+y^2} = \lim_{r\to 0}\frac{r^3\cos^2\theta\sin\theta}{r^2} = 0$$

Example 4.2.4

Example 4.2.5

Does $\lim_{(x,y)\to(0,0)} \dfrac{x^2+\sin^2 y}{2x^2+y^2}$ exists?

Solution. Key: $\lim_{y\to 0} \dfrac{\sin y}{y} = 1 \quad y = mx$

Along $y = mx$

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{x^2+(\frac{\sin y}{mx})^2(mx)^2}{2x^2+y^2} &= \lim_{x\to 0} \frac{\cancel{x^2}+m^2\cancel{x^2}(\frac{\sin(mx)}{mx})^2}{2\cancel{x^2}+m^2\cancel{x^2}} \\
&= \frac{1}{1+m^2}\lim_{x\to 0}(1+m^2(\frac{\sin(mx)}{mx})^2) \\
&= \frac{1}{1+m^2}(1+m^2\cdot 1)
\end{aligned}$$

$\implies$ The limit doesn't exist

Example 4.2.5

Notation4.2.6.

- $f(x)$ is continue at a if $\lim_{x\to a} f(x) = \underline{f(a)}$
- $f(x,y)$ is continue at (x_0,y_0) if $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \underline{f(x_0,y_0)}$

Example 4.2.7

Find a such that $f(x,y) = \begin{cases} \dfrac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} & , \quad (x,y)\neq(0,0) \\ a & , \quad (x,y)=(0,0) \end{cases}$ is conti. at $(0,0)$

Solution. 1

We need $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$

$$\begin{aligned}
\text{LHS} &= \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}\cdot\frac{\sqrt{x^2+y^2+1}+1}{\sqrt{x^2+y^2+1}+1} \\
&= \lim_{(x,y)\to(0,0)} \frac{\cancel{(x^2+y^2)}(\sqrt{x^2+y^2+1}+1)}{\cancel{x^2+y^2}} \\
&= 2 = a = \text{RHS}
\end{aligned}$$

$\implies a = 2$

Solution. 2

Use $x = r\cos\theta \quad y = r\sin\theta$

$$\begin{aligned}
\text{LHS} &= \lim_{r\to 0} \frac{r^2}{\sqrt{r^2+1}-1} \\
&\vdots \\
&= 2
\end{aligned}$$

Example 4.2.7

4.3 First Partial Derivatives

Recall: $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

- $\frac{\partial}{\partial x}(x,y)\Big|_{(a,b)} = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$
- $\frac{\partial}{\partial y}f(x,y)\Big|_{(a,b)} = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}$

Notation4.3.1.

$$\frac{\partial}{\partial x}f(x,y) = \frac{\partial f(x,y)}{\partial x} = f_x(x,y) = \partial x f(x,y)$$

Example 4.3.2

$f(x,y) = x^2 + xy + 2y^2$. Find f_x and f_y

Solution.

$$\begin{aligned} f_x &= \frac{\partial}{\partial x}f(x,y) \\ &= \frac{\partial}{\partial x}(x^2+xy+2y^2) \\ &= 2x+y+0 \end{aligned}$$

$$\begin{aligned} f_y &= \frac{\partial}{\partial y}f(x,y) \\ &= \frac{\partial}{\partial y}(x^2+xy+2y^2) \\ &= 0+x+4y \end{aligned}$$

Example 4.3.2

4.4 Second Partial Derivatives

- $(f_x)_x = f_{xx} = \frac{\partial}{\partial x}(\frac{\partial f(x,y)}{\partial x}) = \frac{\partial^2 f(x,y)}{\partial x^2}$
- $(f_x)_y = f_{xy} = \frac{\partial}{\partial y}(\frac{\partial f(x,y)}{\partial x}) = \frac{\partial^2 f(x,y)}{\partial y \partial x}$

- $(f_y)_x = f_{yx} = \frac{\partial}{\partial x}(\frac{\partial f(x,y)}{\partial y}) = \frac{\partial^2 f(x,y)}{\partial x \partial y}$
- $(f_y)_y = f_{yy} = \frac{\partial}{\partial y}(\frac{\partial f(x,y)}{\partial y}) = \frac{\partial^2 f(x,y)}{\partial y^2}$

Example 4.4.1

$$f(x,y) = x^2 + xy + 2y^2$$

$$\begin{array}{ll} f_{xx} = (f_x)_x = 2 & f_{yx} = (f_y)_x = 1 \\ f_{xy} = (f_x)_y = 1 & f_{yy} = (f_y)_y = 4 \end{array}$$

Example 4.4.1

Example 4.4.2

$u(x,y) = e^x \sin y$. Find $u_{xx} + u_{yy}$.

Solution.

$$u = e^x \sin y$$
$$u_x = e^x \sin y$$
$$u_{xx} = (u_x)_x = e^x \sin y$$
$$u_y = e^x \cos y$$
$$u_{yy} = (u_y)_y = e^x(-\sin y)$$

$\Longrightarrow u_{xx} + u_{yy} = 0$ Laplace Equation

Example 4.4.2

Theorem4.4.3 (Clairaut Theorem)**.**

If f_{xy} and f_{yx} are conti in D, then

$$f_{xy} = f_{yx} \text{ in } D$$

Example 4.4.4

If you are told ∃ a fcn. s.t. $\begin{cases} f_x &= 2x+3y \\ f_y &= 4x-y \end{cases}$. Believe it or not?

Solution.

$$(f(x,y) \in c^2 \quad \text{兩次微分後連續}$$

$$\begin{cases} f_x &= 2x+3y &\implies f_{xy} = (f_x)_y = 3 \\ f_y &= 4x-y &\implies f_{yx} = (f_y)_x = 4 \end{cases}$$

$\implies$ I don't believe it.

Example 4.4.4

Example 4.4.5

If $u(x,t) = u(x-ct) \quad c:$ const. Find $u_{tt} - c^2 u_{xx}$

Solution.
Let $z = x - ct$

$$u_x = u_z \cdot z_x = u_z \cdot 1 = u_z$$

$$u_{xx} = (u_z)_z \cdot z_x = u_{zz} \cdot 1 = u_{zz}$$

$$u_t = u_z \cdot z_t = u_z \cdot (-c) = -cu_z$$

$$u_{tt} = (-cu_z)_z) \cdot z_t = -cu_{zz}(-c) = c^2 u_z z$$

$$u_{tt} - c^2 u_{xx} = c^2 u_{zz} - c^2 u_{zz} = 0$$

Example 4.4.5

Notation4.4.6.

- Laplace Equation
  $u(x,y) = e^x \sin y$ satisfies $u_{xx} + uyy = 0$
- Wave Equation
  $u(x,t) = u(x-ct)$ solves $u_{tt} - c^2 u_{xx} = 0$

Example 4.4.7

$$u(x,t) = \sin(x-ct) \overset{c=1}{=} \sin(x-t)$$

$$\begin{array}{ll} t=0 &, \sin x \\ t=1 &, \sin(x-1) \\ t=2 &, \sin(x-2) \end{array}$$

Example 4.4.7

4.5▲ Partial Derivatives by Implicit Differentiation

Example 4.5.1

If $xyz = \cos(x+y+y)$. Find $\frac{\partial z}{\partial x}(= z_x), \frac{\partial z}{\partial y}(= z_y)$

Solution.

$$z = z(x,y)$$

$$xyz(x,y) = \cos(x+y+z(x,y)) \quad \text{Product Rule}$$

$$y(z + xz_x) = -\sin(x+y+z(x,y)) \cdot (1+0+z_x) \quad \text{Chain Rule}$$

$$yz + xyz_x = -\sin(x+y+z') - z_x \sin(x+y+z)$$

$$z_x = \frac{-yz - \sin(x+y+z)}{xy + \sin(x+y+z)}$$

$$z_y = \frac{-xz - \sin(x+y+z)}{xy + \sin(x+y+z)}$$

Example 4.5.1

Example 4.5.2

$f(x,y) = (\frac{1}{3}(x^3+y^3))(e^{-\sin y})$. Find $f_x(1,0) = \frac{\partial f}{\partial x}\Big|_{(1,0)}$

Solution. 1

$$f_x(x,y) = e^{\sin y} \cdot x^2$$

$$f_x(1,0) = e^{-0} \cdot 1^2 = 1$$

Solution. 2

$$f(x,0) = \frac{1}{3}x^3$$

$$f_x(x,o) = x^2$$

$$f_x(1,0) = 1^2 = 1$$

Example 4.5.2

Example 4.5.3

$$f(x,y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2+y^2} & , \quad (x,y) \neq (0,0) \\ 0 & , \quad (x,y) = (0,0) \end{cases}$$

(1) Find f_x and f_y when $(x,y) \neq (0,0)$

Solution.

$$
\begin{aligned}
f_x &= \frac{(x^2+y^2)(3x^2y-y^3)-(x^3y-xy^3)(2x)}{(x^2+y^2)^2} \quad \text{Quotient Rule} \\
&= \frac{x^4+4x^2y^2-y^4}{(x^2+y^2)^2} \\
f_y &= \frac{(x^2+y^2)(x^3-3xy^2)-(x^3y-xy^3)(2y)}{(x^2+y^2)^2} \\
&= \frac{x(x^4-4x^2y^2-y^4)}{(x^2+y^2)^2}
\end{aligned}
$$

(2) Find $f_x(0,0)$ and $f_y(0,0)$

Solution.

$$
\begin{aligned}
f_x(0,0) &:= \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} \\
&= \lim_{h\to 0} \frac{0-0}{h} \\
&= 0 \\
f_y(0,0) &:= \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} \\
&= 0
\end{aligned}
$$

(3) Find $f_{xy}(0,0)$ and $f_{yx}(0,0)$

Solution.

$$
\begin{aligned}
(f_x)_y(0,0) &= g_y(0,0) \\
&= \lim_{h\to 0} \frac{g(o,h)-g(0,0)}{h} \\
&= \lim_{h\to 0} \frac{f_x(0,h)-f_x(0,0)}{h} \\
&= \lim_{h\to 0} \frac{\frac{h(-\cancel{h^4})}{\cancel{h^4}}-f_x(0,0)}{h} \\
&= \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}} \\
&= -1 \\
(f_y)_x(0,0) &= J_x(0,0) \\
&= \lim_{h\to 0} \frac{J(h,o)-J(0,0)}{h} \\
&= \lim_{h\to 0} \frac{f_y(h,o)-f_y(0,0)}{h} \\
&= \lim_{h\to 0} \frac{\frac{h^{\cancel{5}}}{\cancel{h^4}}-f_y(0,0)}{h} \\
&= \lim_{h\to 0} \frac{h}{h} \\
&= 1
\end{aligned}
$$

Example 4.5.3

4.6 Tangent Planes

Notation4.6.1.

$$z = f(x, y) \quad p = (x_0, y_0, z_0)$$

Q: Find the tangent plane of $z = f(x, y)$ at P

$c_1 : z = f(x_0, y)$
T_1: tangent line of c_1 at x_0
$c_2 : z = f(x, y_0)$
T_x: tangent line of c_2 at y_0

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

If $c \neq 0$

$$\frac{a}{c}(x - x_0) + \frac{b}{c}(y - y_0) + (z - z_0) = 0$$

Normal Vector $= (A, B, 1)$

$$A(x - x_0) + B(y - y_0) + (z - z_0) = 0$$

- $x = x_0$
  $T_1 : B(y - y_0) + (z - z_0) = 0$
  $\implies B = -\dfrac{z - z_0}{y - y_0} = -f_y(x_0, y_0)$
- $y = y_0$
  $T_2 : A(x - x_0) + (z - z_0) = 0$
  $\implies A = -\dfrac{z - z_0}{x - x_0} = -f_x(x_0, y_0)$

$$(A, B, 1) = (-f_x(x_0, y_0), -f_y(x_0, y_0), 1)$$

Tangent plane

$$z - f_x(x, y) = 0$$
$$(-f_x, -f_y, 1)|_{(x_0, y_0)}$$
$$F = z - f(x, y)$$
$$(F_x, F_y, F_z)|_{(x_0, y_0)}$$

Example 4.6.2

$z = f(x,y) = 2x^2 + y^2$. Find the tangent plane of $z = f(x,y)$ at $(1,1,3)$

Solution.
Normal vector is $(-f_x, -f_y, 1)|_{(1,1)} = (-4x, -2y, 1)|_{(1,1)} = (-4, -2, 1)$
Tangent plane

$$-4(x-1) - 2(y-1) + (z-3) = 0$$

$$-4x - 2y + z + 3 = 0$$

Example 4.6.2

4.7 ▲ Linear Approximation

Notation4.7.1.

$$z = f(x,y) \quad (x_0, y_0, z_0)$$

The tangent plane of $z = f(x,y)$ at (x_0, y_0, z_0) is

$$-f_x(x-x_0) - f_y(y-y_0) + (z-z_0) = 0$$

$$z = z_0 + f_x(x-x_0) + f_y(y-y_0)$$
$$f(x,y) = f(x_0,y_0) + f_x(x-x_0) + f_y(y-y_0)$$

Example 4.7.2

$f(x,y) = x^2 + y^2 \quad (x_0, y_0) = (3,4)$. Approximate $f(301, 399)$

Solution.

$$\begin{aligned} f(x,y) &\approx f(3,4) + f_x|_{(3,4)}(x-3) + f_y|_{(3,4)}(y-4) \\ &= 25 + 6(x-3) + 8(y-4) \\ &= -25 + 6x + 8y \end{aligned}$$

$\Longrightarrow \; f(301, 399) \doteqdot 25$

Example 4.7.2

4.8▲ Chain Rule

Definition4.8.1.

$$\frac{d}{dx}(f(g(x)) = \frac{d}{d(g(x))}f(g(x)) \cdot g'(x)$$

Example 4.8.2

$f(x, y) = x^2 + xy + y^2$ where $x = \cos t, y = \sin t$. Find $\dfrac{d}{dt}f(x(t), y(t))$.

Solution. 1

$$\begin{aligned} f(x(t), y(t)) &= \cos^2 t + \cos t \sin t + \sin^2 t \\ &= 1 + \cos t \sin t \\ \frac{d}{dt}f(x(t), y(t)) &= -\sin^2 t + \cos^2 t \end{aligned}$$

Solution. 2

$$\begin{aligned} \frac{d}{dx}f(x(t), y(t)) &= f_x \cdot x't + f_y \cdot y't \\ &= (2x + y)(-\sin t) + (x + 2y)(\cos t) \\ &= (2\cos t + \sin t)(-\sin t) + (\cos t + 2\sin t)(\cos t) \\ &= \cos t \sin t(-2 + 2) - \sin^2 t + \cos^2 t \\ &= -\sin^2 t + \cos^2 t \end{aligned}$$

Example 4.8.2

Notation4.8.3.

If $f(x(t, s), y(t, s))$

$$\begin{aligned} \frac{\partial f}{\partial t} &= f_x \cdot x_t + f_y \cdot y_t \\ \frac{\partial f}{\partial s} &= f_s x_s + f_y y_s \end{aligned}$$

Example 4.8.4

$$(x, y) \implies (r, \theta)$$

$$\begin{cases} x = r\cos\theta = x(r, \theta) \\ y = r\sin\theta = y(r, \theta) \end{cases}$$

Example 4.8.4

Example 4.8.5

$f(x,y) = z = f(x(r,s), y(r,s)), \left\{ \begin{array}{lllll} x & = & r^2+s^2 & = & x(r,s) \\ y & = & 2rs & = & y(r,s) \end{array} \right.$. Find z_{rr}

Solution.

$$\begin{array}{rcl} z_r & = & z_x \cdot \cot x_r + z_y \cdot y_r \\ & = & z_x(2r) + z_y(2s) \\ & = & 2(rx_r + sz_y) \end{array}$$

$$\begin{array}{rcl} z_{rr} & = & (z_r)_r \\ & = & 2(rz_r + sz_y)_r \\ & = & 2(z_x + 2r^2 z_{xx} + 2rsz_{xy} + 2s^2 z_{yy} + 2rsz_{yx}) \end{array}$$

$$\begin{array}{rcl} (rz_x)_r & = & 1z_x + r(z_x)_r \\ & = & z_x + r(g_x \cdot x_r + g_y \cdot y_r) \\ & = & z_x + r(z_{xx}(2r) + z_{xy}(2s)) \\ & = & z_x + 2r(rz_{xx} + sz_{xy}) \end{array}$$

$$\begin{array}{rcl} (sz_y)_r & = & s(z_y)_r \\ & = & s(z_{yx}(2r) + z_{yy}(2s)) \end{array}$$

Example 4.8.5

Example 4.8.6

$u(x,y) = u(x(r,\theta), y(r,\theta))$. Let $\left\{ \begin{array}{lll} x & = & r\cos\theta \\ y & = & r\sin\theta \end{array} \right.$. Show $u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$

Solution.

$$
\begin{aligned}
u_r &= u_x \cdot x_r + u_y \cdot y_r \\
&= u_x \cos\theta + u_y \sin\theta
\end{aligned}
$$

$$
\begin{aligned}
u_\theta &= u_x \cdot x_\theta + u_y \cdot y_\theta \\
&= u_x(-r\sin\theta) + u_y(r\cos\theta)
\end{aligned}
$$

$$
\begin{aligned}
u_{rr} &= (u_r)_r \\
&= (\cos\theta u_x + \sin\theta u_y)_r \\
&= \cos\theta(u_x)_r + \sin\theta(u_y)_r \\
&= \cos\theta((u_x)_x \cdot x_r + (u_x)_y \cdot y_r) + \sin\theta((u_y)_x \cdot x_r + (u_y)_y \cdot y_r) \\
&= u_{xx}(\cos^2\theta) + u_{yy}(\sin^2\theta) + u_{xy}(2\cos\theta\sin\theta)
\end{aligned}
$$

$$
\begin{aligned}
u_{\theta\theta} &= (u_\theta)_\theta \\
&= r(-\sin\theta u_x + \cos\theta u_y)_\theta \\
&= r(-\cos\theta u_x - \sin\theta(u_x)_\theta - \sin\theta u_y + \cos\theta(u_y)_\theta \\
&= r(-\cos\theta u_x - \sin\theta(u_{xx} \cdot x_\theta + u_{xy} \cdot y_\theta) - \sin\theta u_y + \cos\theta(u_{yx} \cdot x_\theta + u_{yy} \cdot y_\theta)) \\
&= u_{xx}(r^2\sin^2\theta) + u_{yy}(r^2\cos^2\theta) + u_{xy}(2\cos\theta\sin\theta)
\end{aligned}
$$

$$
\begin{aligned}
\text{RHS} &= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \\
&= u_{xx}(\cos^2\theta) + u_{yy}(\sin^2\theta) + u_{xy}(2\cos\theta\sin\theta) + \frac{\cos\theta}{r}u_x + \frac{\sin\theta}{r}u_y \\
&\quad + u_{xx}(\sin^2\theta) + u_{yy}(\cos^2\theta) + u_{xy}(-2\sin\theta\cos\theta) + \frac{1}{r}(-\cos\theta u_x - \sin\theta u_y) \\
&= u_{xx} + u_{yy} \\
&= \text{LHS}
\end{aligned}
$$

Example 4.8.6