DIFFERENTIAL CALCULUS

NTU 108-1 Differential Calculus

Li-Chang Hung

This document was typeset on Monday 2nd December, 2019.

▸▸ Legal stuff

- Edited by Hao-Yu Chien under the framework by Joel Feldman, Andrew Rechnitzer and Elyse Yeager.
- Links to the original files can be found at the text webpage

Contents

Chapter 1

Functions

1.1 ▲ Function 函數

Definition 1.1.1.

x (D: domain 定義域) $\to f(x)$ (R: range 值域)

- one to one
- many to one
- one to many

Example 1.1.2

$$f(x) = \sqrt{x-1}, x \in R$$

domain is $\{x \in R \mid x \geqslant 1\} = \{x \in R : x \geqslant 1\} = [1, \infty)$
range is $\{y \in R \mid y \geqslant 0\} = [0, \infty)$
$\therefore$ non-function

Example 1.1.2

1.2 ▲ Even and Odd Functions 偶函數與奇函數

Definition 1.2.1.

- $f(x)$ is an even function if $f(x) = f(-x)$
  (i.e graph of $f(x)$ is symmetric with respect to y-axis)
- $f(x)$ is an odd function if $f(x) = -f(x)$

Joke 1.2.2.

$9x - 7i > 3(3x - 7u)$
$9x - 7i > 9x - 21u$
$-7i > -21u$
$i < 3u$
I love u

1.3 ▲ Polynomial 多項式

Definition 1.3.1.

$f(x) = a_n x^n + a_{n-1} x^{n+1} + \cdots + a_0 (a_n \neq 0)$

- Degree (次數): n

Example 1.3.2

$$f(x) = x^3 + 23x + 17$$

Example 1.3.2

1.4 ▲ Rational Function 有理函數

Definition 1.4.1.

$f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials.

1.5 ▲ Exponential Function 指數函數

Definition 1.5.1.

$f(x) = a^x$

- base (底數): a
- power (指數): x

Chapter 2

Limits

2.1 ▲ Limit 極限

Definition 2.1.1 (Working definition).

$\lim_{x \to a} f(x) = L$

- As x approaches a, $f(x)$ approaches L.

$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \ (0 < \theta < \frac{\pi}{2})$

	$\triangle$ OBA	$\subset$	sector (扇形) OBA	$\subset$	$\triangle$ OB'A
area	$\frac{1}{2} \cdot 1 \sin\theta$	$\leqslant$	$\pi \cdot 1^2 \frac{\theta}{2\pi}$	$\leqslant$	$\frac{1}{2} \cdot 1 \tan\theta$
$\Longrightarrow$	$\sin\theta$	$\leqslant$	θ	$\leqslant$	$\tan\theta$
$\Longrightarrow$	1	$\leqslant$	$\frac{\theta}{\sin\theta}$	$\leqslant$	$\frac{1}{\cos\theta}$
$\Longrightarrow$	θ	$\leqslant$	$\frac{\sin\theta}{\theta}$	$\leqslant$	1
$\Longrightarrow$	$\lim_{\theta \to 0^+} \cos\theta$	$\leqslant$	$\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta}$	$\leqslant$	$\lim_{\theta \to 0^+} 1$

$\Longrightarrow \lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1$

Example 2.1.2

Find $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\cos x}{x^2} &= \lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)} \\
&= \lim_{x\to 0}\frac{\sin^2 x}{x^2(1+\cos x)} \\
&= \lim_{x\to 0}(\frac{\sin x}{x})^2\cdot\lim_{x\to 0}\frac{1}{1+\cos x} \\
&= (\lim_{x\to 0}\frac{\sin x}{x})^2\cdot\frac{1}{2} \\
&= 1\cdot\frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

or

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\cos x}{x^2} &= \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2} \\
&= \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{4(\frac{x}{2})^2} \\
&= \frac{1}{2}\lim_{x\to 0}\frac{\sin^2\frac{x}{2}}{(\frac{x}{2})^2} \\
&= \frac{1}{2}\cdot 1 \\
&= \frac{1}{2}
\end{aligned}$$

Example 2.1.2

2.2 ▲ Ways to Find Limits

- Direct substitution

  $\lim_{x\to a} f(x) = f(a)$

- Factorization 因式分解

- Rationalization 有理化

- Squeeze Theorem 夾擠定理

Theorem 2.2.1.

If

$$f(x) \leqslant g(x) \leqslant h(x) \text{near} x = a$$

and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$

Example 2.2.2 $\left(\text{type } \frac{0}{0} \star\right)$

$$\lim_{x \to 0} \frac{(x+3)^2 - 9}{x} = \lim_{x \to 0} \frac{x^2 + 6x + 9 - 9}{x} = \lim_{x \to 0} \frac{x(x+6)}{x} = 6$$

Example 2.2.2

Example 2.2.3 $\left(\text{type } \frac{0}{0} \star\right)$

$$\lim_{x \to 0} \frac{\sqrt{x^2 - 9} - 3}{x^2} = \lim_{x \to 0} \frac{(\sqrt{x^2+9} - 3)(\sqrt{x^2+9} + 3)}{x^2(\sqrt{x^2 - 9} + 3)} = \lim_{x \to 0} \frac{1}{\sqrt{x^2+9} + 3} = \frac{1}{6}$$

Example 2.2.3

Example 2.2.4 $\left(\text{type } \frac{0}{0} \star\star\right)$

$$\begin{aligned}
\lim_{x \to 2} \frac{\sqrt{x^3 + x^2 - 8} - 2}{x - 2} &= \lim_{x \to 2} \frac{(\sqrt{x^3 + x^2 - 8} - 2)(\sqrt{x^3 + x^2 - 8} + 2)}{(x-2)(\sqrt{x^3 + x^2 - 8} + 2)} \\
&= \lim_{x \to 2} \frac{x^3 + x^2 - 12}{(x-2)(\sqrt{x^3 + x^2 - 8} + 2)} \\
&= \lim_{x \to 2} \frac{(x-2)(x^2 + 3x + 6)}{(x-2)(\sqrt{x^3 + x^2 - 8} + 2)} \\
&= \frac{4 + 6 + 6}{4} \\
&= 4
\end{aligned}$$

or

$$\lim_{x\to 2}\frac{\sqrt{x^3+x^2-8}-2}{x-2}=\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}=f'(2)$$
$$f(x)=x^3+x^2-8$$
$$f'(x)=\frac{1}{2}(x^3+x^2-8)^{-\frac{1}{2}}(3x^2+2x)$$
$$\implies f'(2)=\frac{1}{2}(\frac{1}{2})(12-4)=4$$

Example 2.2.4

Example 2.2.5 (type $\frac{0}{0}$ ⋆⋆)

$$\begin{aligned}\lim_{x\to 2}\frac{\sqrt{1+\sqrt{2+x}}-\sqrt{3}}{x-2} &= \lim_{x\to 2}\frac{(\sqrt{1+\sqrt{2+x}}-\sqrt{3})(\sqrt{1+\sqrt{2+x}}+\sqrt{3})}{(x-2)(\sqrt{1+\sqrt{2+x}}+\sqrt{3})}\\ &= \lim_{x\to 2}\frac{1+\sqrt{2+x}-3}{(x-2)(\sqrt{1+\sqrt{2+x}}+\sqrt{3})}\\ &= \lim_{x\to 2}\frac{(\sqrt{2+x}-2)(\sqrt{2+x}+2)}{(x-2)(\sqrt{1+\sqrt{2+x}}+\sqrt{3})(\sqrt{2+x}+2)}\\ &= \lim_{x\to 2}\frac{2+x-4}{(x-2)(\sqrt{1+\sqrt{2+x}}+\sqrt{3})(\sqrt{2+x}+2)}\\ &= \frac{1}{2\sqrt{3}\cdot 4}\\ &= \frac{1}{8\sqrt{3}}\end{aligned}$$

Example 2.2.5

Example 2.2.6 (type $\frac{0}{0}$ ⋆)

$$\begin{aligned}\lim_{x\to 0}\frac{\tan x-\sin x}{x^3} &= \lim_{x\to 0}\frac{\frac{\sin x}{\cos x}-\sin x}{x^3}\\ &= \lim_{x\to 0}\frac{\sin x(1-\cos x)(1+\cos x)}{x^3\cos x(1+\cos x)}\\ &= \lim_{x\to 0}\frac{\sin^3 x}{x^3}c\frac{1}{\cos x(1+\cos x)}\\ &= \lim_{x\to 0}\frac{\sin^3 x}{x^3}\cdot\lim_{x\to 0}\frac{1}{\cos x(1+\cos x)}\\ &= \frac{1}{2}\end{aligned}$$

Example 2.2.6

Example 2.2.7 (type $\frac{0}{0}$ $\star$)

$$\begin{aligned}
\lim_{x\to 0}\frac{\cos x-1}{\sin(x\sin x)} &= \lim_{x\to 0}\frac{(\cos x-1)(\cos x+1)}{\sin(x\sin x)(\cos x+1)}\\
&= -\lim_{x\to 0}\frac{sim^2 x}{\sin(x\sin x)}\cdot\frac{1}{\cos x+1}\\
&= -\lim_{x\to 0}\frac{x\sin x}{\sin(x\sin x)}\cdot\frac{\sin^2 x}{x\sin x}\cdot\frac{1}{\cos x+1}\\
&= -\lim_{x\to 0}\frac{x\sin x}{\sin(x\sin x)}\cdot\lim_{x\to 0}\frac{\sin^2 x}{x\sin x}\cdot\lim_{x\to 0}\frac{1}{\cos x+1}\\
&= -\frac{1}{2}
\end{aligned}$$

Example 2.2.7

Example 2.2.8

$$\lim_{x\to 0} x^2\sin(\frac{1}{x})$$

$$-1 \leqslant \sin x \leqslant 1, \forall x \in R$$

$$\left|x^2\sin(\frac{1}{x})\right| \leqslant\mid x^2\cdot 1\mid = x^2, x\in R$$

$$\implies -x^2 \leqslant x^2\sin(\frac{1}{x}) \leqslant x^2, x\in R$$

$$\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0$$

By Squeeze Theorem, we have $\lim_{x\to 0} x^2\sin(\frac{1}{x}) = 0$

Example 2.2.8

2.3 ▲ ϵ - δ language

Definition 2.3.1.

$\lim_{x\to a} f(x) = L$

- $\forall\epsilon > 0$, $\exists\delta$ $(=\delta(\epsilon))$ s.t. if $\mid x-a\mid < \delta$ then $\mid f(x)-L\mid < \epsilon$

Example 2.3.2

Prove $\lim_{x\to 3}(x+2) = 5$ using ϵ - δ language

Want to prove:
$\forall\epsilon > 0, \exists\delta = \delta(\epsilon)$ s.t. if $| x - 3 | < \delta$ then $| (x + 2) - 5 | < \epsilon$
Experiment:
When $\epsilon = 0.1, \exists\delta$ s.t. if $| x - 3 | < \delta$ then $| x - 3 | < 0.1$
If $\delta = 0.1$ ✓
If $\delta = 0.2$ ×
If $\delta = 0.05$ ✓
$\forall\epsilon > 0, \exists\delta = \epsilon$ s.t. if $| x - 3 | < \delta$ then $| x - 3 | < \epsilon$

Example 2.3.2

Example 2.3.3

Prove $\lim\limits_{x\to 5} x^2 = 25$ using ϵ - δ language

Want to show:
$\forall\epsilon > 0, \exists\delta = \delta(\epsilon)$ s.t. if $| x - 5 | < \delta$ then $| x^2 - 25 | < \epsilon$
When $| x - 5 | < 1$:
$-1 < x - 5 < 1 \Rightarrow 9 < x + 5 < 11$
$| x^2 - 25 | = | (x - 5)(x + 5) | < | x - 5 | \cdot 11$ hope $< \epsilon$
hope $| x - 5 | < \dfrac{\epsilon}{11}$
take $\delta = \min(1, \dfrac{\epsilon}{11})$
Verification:
$\forall\epsilon > 0, \exists\delta = \min(1, \dfrac{\epsilon}{11})$
check
$| x - 5 | < \min(1, \dfrac{\epsilon}{11}) \leqslant \dfrac{\epsilon}{11}$
$\Longrightarrow | x^2 - 25 | < \epsilon$
$| x^2 - 25 | = | x - 5 | \cdot | x + 5 | \leqslant \dfrac{\epsilon}{11} \cdot 11 = \epsilon$

Example 2.3.3

Joke 2.3.4.

九九乘法表 (table) $\Rightarrow$ 地板算

2.4 ▲ Limit Law

Assume $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$ exist, then:

- $\lim\limits_{x\to a}(f(x) + g(x)) = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$
- $\lim\limits_{x\to a}(f(x) - g(x)) = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x)$

- $\lim_{x\to a}(f(x)\cdot g(x)) = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)$

- $\lim_{x\to a}(\frac{f(x)}{g(x)}) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

Remarks:

- $\lim_{x\to a}(c\cdot f(x)) = \lim_{x\to a} c\cdot \lim_{x\to a} f(x) = c\cdot \lim_{x\to a} f(x)$

- $\lim_{x\to a}(f(x))^n = \lim_{x\to a}(f(x))^{n-1}\cdot \lim_{x\to a} f(x) = (\lim_{x\to a} f(x))^n$

- $\lim_{x\to 0} x^2 \sin(\frac{1}{x}) \neq (\lim_{x\to 0} x^2)\cdot(\lim_{x\to 0}\sin(\frac{1}{x}))$

Example 2.4.1

Assume $\lim_{x\to 0}\frac{f(x)}{x^2} = 5$

(a)Find $\lim_{x\to 0} f(x)$

$$\lim_{x\to 0}(\frac{f(x)}{x^2}\cdot x^2) = \lim_{x\to 0} f(x)$$

$$(\lim_{x\to 0}\frac{f(x)}{x^2})\cdot(\lim_{x\to 0} x^2) = 5\cdot 0 = 0$$

(b)Find $\lim_{x\to 0}\frac{f(x)}{x}$

$$\lim_{x\to 0}(\frac{f(x)}{x^2}\cdot x) = \lim_{x\to 0}\frac{f(x)}{x}$$

$$(\lim_{x\to 0}\frac{f(x)}{x^2})\cdot(\lim_{x\to 0} x) = 5\cdot 0 = 0$$

Example 2.4.1

2.5 ▲ Continuity

Definition 2.5.1.

$f(x)$ is conti. at $x = a$ if

- $f(x)$ is defined at $x = a$
  $f(a)$ makes sense
- $\lim_{x \to a} f(x)$ exists
  $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$
- $\lim_{x \to a} f(x) = f(a)$
  $\forall \epsilon > 0, \exists \delta = \delta(\epsilon) > 0$ s.t if $| x - a | < \delta \Rightarrow | f(x) - f(a) | < \epsilon$

Joke 2.5.2.

原子小金剛
撞牆穿牆

Example 2.5.3 (Removable Discontinuity)

$$f(x) = \frac{x^2 - x - 2}{x - 2}$$

How would you define $f(2)$ in order to make $f(x)$ is conti. at $x = 2$?

Sol:

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{(x+1)(x-2)}{x-2} = 3$$

Example 2.5.3

Example 2.5.4

$$f(x) = \begin{cases} cx^2 + 2x & , \quad x < 2 \\ x^3 - cx & , \quad x > 2 \end{cases}$$

For what value of the const c is the fcn. f conti. on $(-\infty, \infty)$?

$$\begin{aligned} \lim_{x \to 2^+} f(x) &= \lim_{x \to 2^+} (x^3 - cx) &= 8 - 2c \\ \lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-} (cx^2 + 2x) &= 4c + 4 \end{aligned}$$

$$\begin{aligned} 8-2c &= 4c+4 \\ 4 &= 6c \\ c &= \frac{2}{3} \end{aligned}$$

Example 2.5.4

2.6 ▲ Intermediate Value Theorem

Theorem 2.6.1.

Suppose that $f(x)$ is conti. $[a,b]$.

- If $f(a) < f(b)$ and $\forall k \in R$ with $f(a) < k < f(b)$, then $\exists c \in (a,b)$ s.t $f(c) = k$
- If $f(x)$ is conti. on $[a,b]$ and $\exists N \in R$ s.t $f(b) < N < f(a)$, then $\exists c \in (a,b)$ s.t $f(c) = N$
- If $N = 0$ in I.V.T, $f(b) < 0 < f(a)$, $\exists c \in (a,b)$ s.t $f(c) = 0$

The proof relies on " Least Upper Bound Axiom"

Definition 2.6.2.

If $S \in R$,

- b is called an upper bound of S if $\forall x \in S \implies \leqslant b$
- b is called a least upper bound if
  - b is an upper bound of S
  - b is less than or equal to every other upper bound of S.

Lemma 2.6.3.

Let $M, N \in R$.
If $M > N - \epsilon \quad \forall \epsilon > 0$, then $M > N$

pf:
Suppose that If $M < N$, $\exists \epsilon > 0$ s.t $M + \epsilon < N$
$\implies$ $\underline{M < N - \epsilon}$ **a contradiction** $\implies M \geqslant N$

Proof. Intermediate Value Theorem
Let $A = \{x \in [a,b] \quad f(x) \leqslant k\}$

- $A \neq \phi$ (empty set)
  $(\because f(a) < k \implies a \in A)$
- A is bounded above

By Least Upper Bound axiom $\implies$ A has an l.u.b c
Denote sup $A = c$
want to show: $f(c) = k$

Claim: $c \in [a,b]$

- b is an upper bound of $S \implies c \leqslant b$
  $\left\{ \begin{array}{c} c \text{ is an upper bound of } S \\ f(a) < k \implies a \in A \end{array} \right. \implies a \leqslant c \implies a \leqslant c \leqslant b$

$f(x)$ is conti. at $x = c \Leftrightarrow \forall \epsilon > 0, \exists \delta > 0$ s.t if $\mid x - c \mid < \delta$
$f(c) - \epsilon < f(x) < f(c) + \epsilon - (\star)$

Claim: $f(c) \leqslant k$
pf:
$c = \sup A \implies c - \delta$ is not an upper bound of A
$\therefore \exists x_1 \in A \implies f(x_1 \leqslant k)$ s.t $c - \delta < x_1 \leqslant c \implies c - x_1 < \delta$
$(\star) \implies f(c) < f(x) + \epsilon \leqslant k + \epsilon \implies f(c) \leqslant k$

Claim: $c < b$
pf:
If $c = b$, $\underline{k < f(b) = f(c) \leqslant k}$ **a contradiction**

Claim: $f(c) \geqslant k$
pf:
$c < b \implies \exists x_2$ s.t $c < x + 2 < b$ and $x_2 - c < \delta$
$(\star) \implies f(x_2 < f(c) + \epsilon$
$x_2 < c \implies x_2 \notin A \implies f(x_2) > k \implies k < f(x_2) < f(c) + \epsilon \implies f(c) > k - \epsilon$
(Lemma) $\implies f(c) \geqslant k$ $\square$

Example 2.6.4

Show that there is a root of the eqn. $\sin x = x^2 - x$ in $(1,2)$

pf:
let $f(x) = x^2 - x - \sin x$ conti.

$$\begin{array}{rcccccc} f(1) &=& 1 - 1 - \sin 1 &=& -\sin 1 &<& 0 \\ f(2) &=& 4 - 2 - \sin 2 &=& 2 - \sin 2 &>& 0 \end{array}$$

By I.V.T, $\exists c \in (1,2)$ s.t $f(c)=0$

Example 2.6.4

Example 2.6.5

Prove that $\cos x = x^3$ has at least one real root.

pf:
let $f(x) = \cos x - x^3$
observe:

$$\begin{aligned} \lim_{x\to\infty} f(x) &= -\infty \\ \lim_{x\to\infty} f(x) &= \infty \end{aligned}$$

$$\begin{array}{rclcl} f(100) &=& \cos 100 - 100^3 &<& 0 \\ f(-100) &=& \cos -100 + 100)^3 &>& 0 \end{array}$$

By I.V.T, $\exists c \in (-100,100)$ s.t $f(c)=0$

Example 2.6.5

Example 2.6.6

Show that $f(x) = \begin{cases} x^4 \sin\frac{1}{x} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$ is conti. on $(-\infty,\infty)$

pf:
We need to show $\lim\limits_{x\to 0} f(x) = f(0)$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}(x^4 \sin\frac{1}{x})$$

$$-1 \leqslant \sin\frac{1}{x} \leqslant 1$$

$$\implies -x^4 \leqslant x^4 \sin\frac{1}{x} \leqslant x^4$$

$$\lim_{x\to 0} -x^4 = \lim_{x\to 0} x^4 = 0$$

By Sqeeze thm, $\lim\limits_{x\to 0} x^4 \sin\frac{1}{x} = 0 \implies \lim\limits_{x\to 0} f(x) = 0 = f(0)$

Example 2.6.6

Example 2.6.7

Assume $f(x)$ is conti. on $[-1,1]$. Show that $\exists c \in (-1,1)$ s.t $f(c) = \dfrac{c}{1-c^2}$ (i.e: $x=c$ is a root of $f(x) = \dfrac{x}{1-x^2}$)

pf:

let $g(x) = 1 - x^2 \cdot f(x) - x$
$g(1) = (1-1)f(1) - 1 < 0$
$g(-1) = (1-1)f(-1) + 1 > 0$
$g(x)$ is conti. on $[-1,1]$
By I.V.T, $\exists c \in (-1,1)$ s.t $g(c) = 0$

$f(x)$ is conti. at $x = a \Leftrightarrow \lim_{x \to a} f(x) = f(a)$

Example 2.6.7

2.7 ▲ Velocity

Definition 2.7.1.

- Average velocity $(t = a \to t = a+h) = \dfrac{f(a+h) - f(a)}{a+h-a}$
- Instantaneous velocity $= \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h} := f'(a)$

Chapter 3

Derivatives

3.1 ▲ Derivatives 導（函）數

Definition 3.1.1.

- The derivative of $f(x)$ at $x = a$ is given by

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} := f'(a)$$

- The derivative of $f(x)$ is given by

$$\lim_{y\to x}\frac{f(y)-f(x)}{y-x} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} := f'(x)$$

Example 3.1.2

Find the derivative of $f(x) = x^2 - 2x + 3$ at $x = 2$ by definition

pf:

$$f'(2) = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{(2+h)^2-2(2+h)+3-3}{h} = \lim_{h\to 0}\frac{h(h+2)}{h} = 2$$

Example 3.1.2

Example 3.1.3

Let $f(x) = x^3 - x$. Find the derivative of $f(x)$

Sol:

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{(x+h)^3-(x+h)-x^3+x}{h} \\ &= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3-h}{h} \\ &= \lim_{h\to 0}\frac{h(3x^2+3xh+h^2-1)}{h} \\ &= 3x^2-1 \end{aligned}$$

Example 3.1.3

Notation 3.1.4.

- $f'(x) = y'(x) = \dfrac{df(x)}{dx} = \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}}{\mathrm{d}x}f(x)$
- $f'(a) = y'(a) = \dfrac{\mathrm{d}f(x)}{\mathrm{d}x}\Big| x = a = \dfrac{\mathrm{d}y}{\mathrm{d}x}\Big| x = a \quad (\neq \dfrac{\mathrm{d}}{\mathrm{d}x}f(a) = 0)$

Joke 3.1.5.

活生生血淋淋的栗子

3.2 ▲ Differentiability

Definition 3.2.1.

$f(x)$ is differentiable at $x = a$ if $f'(a)$ exists (i.e. $\lim_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ exists)

Example 3.2.2

Discuss differentiability of $f(x) = |x|$ at $x = 0$

Sol:

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{\mid h\mid -0}{h}$$

$$\begin{aligned} \lim_{h\to 0^+}\frac{|h|}{h} &= \lim_{h\to 0^+}\frac{h}{h} &= 1 \\ \lim_{h\to 0^-}\frac{|h|}{h} &= \lim_{h\to 0^-}\frac{-h}{h} &= -1 \end{aligned}$$

$\Longrightarrow \lim_{h\to 0} \frac{|h|}{h}$ doesn't exist
$\Longrightarrow f'(0)$ doesn't exists
$\Longrightarrow f(x)$ is not differentiable at $x = 0$ but $f(x)$ is conti. at $x = 0$

Example 3.2.2

Theorem 3.2.3 (Differentiability $\Longrightarrow$ Continuity)**.**

If $f(x)$ is differentiable at $x = a$, $f(x)$ is continuous at $x = a$ (i.e: $\lim_{x\to a} f(x) = f(a)$)

pf:
$f(x)$ is diff. at $x = a$
i.e $f'(a) = \lim_{h\to} \frac{f(a+h) - f(a)}{h} = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$ exists

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a}(x - a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x\to a}(x - a) \Longrightarrow \lim_{x\to a}(f(x) - f(a)) = 0$$

$$\begin{aligned} &\Longrightarrow && \lim_{x\to a}(f(x) - f(a) + f(a)) = \lim_{x\to a} f(x)) \\ &= && \lim_{x\to a}(f(x) - f(a)) + \lim_{x\to a} f(a)) \\ &= && 0 + f(a) \\ &= && f(a) \end{aligned}$$

3.3 ▲ Differentiation Rule

If $f'(x)$ and $g'(x)$ exist, and c is any consts.

- $\frac{dc}{dx} = 0$

$$f(x) = c \Longrightarrow f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = 0$$

- $\frac{\mathrm{d}}{dx}(cf(x)) = cf'(x)$

$$\frac{d}{dx}(cf(x)) = \lim_{h\to 0} \frac{cf(x+h) - cf(x)}{h} = c \cdot f'(x)$$

- $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$

$$\begin{aligned} \frac{d}{dx}(f(x) + g(x)) &= \lim_{h\to 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h} \\ &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= f'(x) \pm g'(x) \end{aligned}$$

- $\frac{d}{dx}(f(x)-g(x)) = f'(x)-g'(x)$

$$\begin{aligned}
\frac{d}{dx}(f(x)-g(x)) &= \lim_{h\to 0}\frac{(f(x+h)-g(x+h))-(f(x)-g(x))}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} - \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
&= f'(x)-g'(x)
\end{aligned}$$

- $\frac{d}{dx}(f(x)\cdot g(x)) = f'(x)g(x)+f(x)g'(x)$ (Product Rule)

$$\begin{aligned}
\frac{d}{dx}(f(x)\cdot g(x)) &= \lim_{h\to 0}\frac{f(x+h)\cdot g(x+h)-f(x)\cdot g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)\cdot g(x+h)-f(x)\cdot g(x+h)+f(x)\cdot g(x+h)-f(x)\cdot g(x)}{h} \\
&= \lim_{h\to 0}\frac{g(x+h)(f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{f(x)(g(x+h)-g(x)}{h} \\
&= \lim_{h\to 0} g(x+h)\cdot \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} f(x)\cdot \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
&= g(x)f'(x)+f(x)g'(x) \\
&= f'(x)g(x)+f(x)g'(x)
\end{aligned}$$

- $\frac{d}{dx}(\frac{f(x)}{g(x)}) = \frac{g(x)f'(x)-g'(x)f(x)}{(g(x))^2}$ (Quotient Rule)

$$\frac{d}{dx}(\frac{f(x)}{g(x)}) = \frac{d}{dx}(f(x)\frac{1}{g(x)}) = f'(x)\frac{1}{g(x)} + f(x)\frac{d}{dx}(\frac{1}{g(x)})$$

$$\begin{aligned}
\frac{d}{dx}(\frac{1}{g(x)}) &= \lim_{h\to 0}\frac{\frac{1}{g(x+h)}-\frac{1}{g(x)}}{h} \\
&= \lim_{h\to 0}\frac{g(x)-g(x+h)}{h\cdot g(x+h)g(x)} \\
&= \lim_{h\to 0}\frac{-g(x+h)+g(x)}{h}\cdot \lim_{h\to 0}\frac{1}{g(x+h)g(x)} \\
&= \lim_{h\to 0}\frac{-g(x+h)+g(x)}{h}\cdot \frac{1}{g(x)}\lim_{h\to 0}\frac{1}{g(x+h)} \\
&= -g'(x)\cdot\frac{1}{g(x)}\frac{1}{g(x)} \\
&= \frac{-g'(x)}{(g(x))^2}
\end{aligned}$$

$$\frac{d}{dx}(\frac{f(x)}{g(x)}) = \frac{f'(x)}{g(x)} + f(x)\frac{-g'(x)}{(g(x))^2} = \frac{f'(x)g(x)-f(x(g'(x)}{(g(x))^2}$$

Joke 3.3.1.

熱氣球 $\Rightarrow$ 數學沒用

3.4 ▲ Derivatives of Trigonometric Functions

- $\frac{d}{dx}\sin x = \cos x$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x(\cos h-1)^2}{h}+\lim_{h\to 0}(\cos x\cdot\frac{\sin h}{h}) \\
&= \sin x\lim_{h\to 0}\frac{(\cos h-1)(\cos h+1)}{h\cdot\cos x+1}+\lim_{h\to 0}(\cos x\cdot\frac{\sin h}{h}) \\
&= \sin x\lim_{h\to 0}\frac{-\sin^2 h}{h^2}\cdot\frac{h}{\cos h+1}+\lim_{h\to 0}(\cos x\cdot\frac{\sin h}{h}) \\
&= \sin x\cdot -1\cdot 0+\lim_{h\to 0}(\cos x\cdot\frac{\sin h}{h}) \\
&= 0+\cos x \\
&= \cos x
\end{aligned}$$

- $\frac{d}{dx}\cos x = -\sin x$

$$\begin{aligned}
\frac{d}{dx}\cos x &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} \\
&= \lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h} \\
&= \lim_{h\to 0}(\cos\frac{\cos h-1}{h})-\lim_{h\to 0}(\sin\frac{\sin h}{h}) \\
&= \cos x\cdot 0-\sin x \\
&= -\sin x
\end{aligned}$$

$$\lim_{h\to 0}\frac{\cos h-1}{h}=\lim_{h\to 0}\frac{(\cos h-1)(\cos h+1)}{h(\cos +1)}=\lim_{h\to 0}\frac{\cos^2 h-1}{h(\cos h+1)}=0$$

- $\frac{d}{dx}\tan x = \sec^2 x$

$$\begin{aligned}
\frac{d}{dx}\tan x &= \frac{d}{dx}(\frac{\sin x}{\cos x}) \\
&= \frac{\cos x\sin x-\sin x\cos x}{\cos^2 x} \qquad \text{(Quotient Rule)} \\
&= \frac{\cos^2 x+\sin^2 x}{\cos^2 x} \\
&= \sec^2 x
\end{aligned}$$

- $\frac{d}{dx}\sec x = \sec x \tan x$

$$\begin{aligned}\frac{d}{dx}\sec x &= \frac{d}{dx}(\frac{1}{\cos x})\\ &= \frac{\cos x \cdot 0 - 1(\cos x)}{\cos^2 x} \quad \text{(Quotient Rule)}\\ &= \frac{\sin x}{\cos^2 x}\\ &= \frac{\sin x}{\cos x}\sec x\\ &= \sec x \tan x\end{aligned}$$

- $\frac{d}{dx}\csc x = -\csc x \cot x$
- $\frac{d}{dx}\cot x = -\csc^2 x$

3.5 ▲ Derivatives of Polynomials

- $\frac{d}{dx}(x^n) = n \cdot x^{n-1}, n \in N$

$$\begin{aligned}\lim_{y\to x}\frac{f(y)-f(x)}{y-x} &= \lim_{y\to x}\frac{y^n - x^n}{y-x}\\ &= \lim_{y\to x}\frac{(y-x)(y^{n-1}+xy^{n-1}+\cdots+x^{n-1})}{y-x}\\ &= x^{n-1}+x\cdot x^{n-2}+\cdots+x^{n-1}\\ &= n\cdot x^{n-1}\end{aligned}$$

Example 3.5.1

$$\frac{d}{dx}(x^2) = 2x^{2-1} = 2x$$

Example 3.5.1

Example 3.5.2

$$\frac{d}{dx}(x^3) = 3x^{3-1} = 3x^2$$

Example 3.5.2

Notation 3.5.3.

$$\begin{aligned} f'(x) &= \frac{df(x)}{dx} \\ f''(x) &= \frac{d}{dx}(\frac{df(x)}{dx}) = \frac{d^2f(x)}{dx^2} \quad (\text{not } \frac{d^2f(x)}{d^2x^2}) \\ f^{(n)}(x) &= \frac{d^nf(x)}{dx^n} \end{aligned}$$

Example 3.5.4

$$\begin{aligned} \frac{d^2}{dx^2}x^2 &= \frac{d}{dx}(\frac{d}{dx}(x^2)) \\ &= \frac{d}{dx}(2x) \\ &= 2\frac{d}{dx}x \\ &= 2(1x^{1-1}) \\ &= 2 \end{aligned}$$

Example 3.5.4

3.6 ▲ Derivatives of e^x

- $(e^x)' = \frac{d}{dx}e^x = e^x \quad (e \doteqdot 2.718281828459045\cdots)$

  $f(x) = a^x \quad (a > 0 \text{ const.})$

  $$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} \\ &= a^x\lim_{h\to 0}\frac{a^h-1}{h} \end{aligned}$$

  $f_h(a) = \frac{a^h-1}{h} \quad (h \text{ is fixed})$

  - $f(a)$ is conti.
  - $f(a) \nearrow \quad$ (increasing in a)

  When $a = 2$,

  $$\lim_{h\to 0}\frac{2^h-1}{h} = \lim_{h\to 0}f_h(2) \doteqdot 0.69 < 1$$

When $a = 3$,

$$\lim_{h\to 0}\frac{3^h - 1}{h} = \lim_{h\to 0} f_h(3) \doteqdot 1.10 > 1$$

By I.V.T, $\exists a_0 \in (2,3)$ s.t $\lim\limits_{h\to 0} f_h(a_0) = 1$

$\implies f'(x) = {a_0}^x \cdot \lim\limits_{h\to 0}\dfrac{{a_0}^h - 1}{h} = {a_0}^x$

i.e: $\dfrac{d}{dx}({a_0}^x) = {a_0}^x \quad (a_0 = e)$

Joke 3.6.1.

躍（一ㄠ丶）躍（一ㄠ丶）欲試

▸▸ e Defined by Limit

let $f(x) = lna$

$$f'(x) = \frac{1}{x}$$

$$\begin{aligned}
1 = f'(x) &= \lim_{h\to 0}\frac{f(1-h) - f(1)}{h} \\
&= \lim_{h\to 0}\frac{ln(1+h)}{h} \\
&= \lim_{h\to 0} ln((1+h)^{\frac{1}{h}}) \\
&= ln(\lim_{h\to 0}(1+h)^{\frac{1}{h}})
\end{aligned}$$

$$\begin{aligned}
lnA &= 1 \\
A &= e
\end{aligned}$$

$$\therefore e = \lim_{h\to 0}(1+h)^{\frac{1}{h}}$$

$$h = \frac{1}{k} \implies e = \lim_{k\to\infty}(1 + \frac{1}{k})^k$$

Example 3.6.2

$$(x^6)' = 6x^{6-1} = 6x^5$$

Example 3.6.2

Example 3.6.3

$$(\frac{1}{x^2})' = (x^{-2})' = -2x^{-2-1} = -2^{-3} = \frac{-2}{x^{-3}}$$

Example 3.6.3

Example 3.6.4

$$(\sqrt[3]{x^2})' = (x^{\frac{2}{3}})' = \frac{2}{3}x^{\frac{2}{3}-1} = \frac{2}{3}x^{-\frac{1}{3}}$$

Example 3.6.4

Example 3.6.5

$$(x^9 + 12x^5 - 4x^4 + 10x^3 - 6x + 5)' = 8x^7 + 60x^4 - 16x^3 + 30x^2 - 6$$

Example 3.6.5

Example 3.6.6

Find the tangent line to the curve $y = x\sqrt{x}$ at $(1, 1)$

sol:
let $f(x) = x\sqrt{x} = x^{\frac{3}{2}}$

$$\begin{aligned} f'(x) &= \frac{3}{2}x^{\frac{1}{2}} \\ f'(1) &= \frac{3}{2} \end{aligned}$$

tangent line : $\frac{y-1}{x-1} = \frac{3}{2}$

Example 3.6.6

Example 3.6.7

Find the points at the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal

sol:
let $f(x) = x^4 - 6x^2 + 4$

$$f'(x) = 4x^3 - 12x$$

let $f'(x) = 0$ i.e: solve $4x^3 - 12^x = 0$

$$x(x^2 - 3) = 0$$

$$x = 0, \pm\sqrt{3}$$

Example 3.6.7

Example 3.6.8 (Product Rule)

$$f(x) = x^2 \sin x$$

$$\begin{aligned} f'(x) &= (x^2)' \sin x + x^2 (\sin x)' \\ &= 2x \sin x + x^2 \cos x \end{aligned}$$

Example 3.6.8

Example 3.6.9 (Quotient Rule)

$$f(x) = \frac{\sec x}{1 + \tan x}$$

$$\begin{aligned} f'(x) &= \frac{(1+\tan x)(\sec x)' - (\sec x)(1+\tan x)'}{(1+\tan x)^2} \\ &= \frac{\sec x \tan x + \sec x \tan^2 x - \sec^3 x}{(1+\tan x)^2} \\ &= \frac{\sec x(\tan x - 1)}{(1+\tan x)^2} \end{aligned}$$

$$\tan^2 x + 1 = \sec^2 x$$

$$\sec x \tan x + \sec x(\cancel{\sec^2 x} - 1) - \cancel{\sec^3 x} = \sec x(\tan x - 1)$$

Example 3.6.9

Example 3.6.10

(i) $f(x) = x \cdot e^x$

$$\begin{aligned} f'(x) &= (x)'e^x + x(e^x)' \\ &= 1e^x + xe^x \\ &= e^x(1+x) \end{aligned}$$

(ii) Find $f^n(x)$

$$\begin{aligned} f &= xe^x \\ f' &= e^x(1+x) \\ f'' &= e^x(1+x) + e^x \cdot 1 \\ &= e^x(2+x) \\ f''' &= e^2(2+x) + e^x \cdot 1 \\ &= e^x(3+x) \end{aligned}$$

guess: $f^n(x) = e^x(n+x)$
prove it by <u>induction</u>

$$\begin{aligned} \frac{d}{dx}1 &= 0 \\ \frac{d}{dx}e^x &= e^x \\ (e^x)'' &= e^x \\ (e^x)''' &= e^x \end{aligned}$$

Example 3.6.10

Example 3.6.11 (Quotient Rule)

$$f(x) = \frac{x^2+x-2}{x^3+6}$$

$$\begin{aligned} f'(x) &= \frac{(x^3+6)(2x+1)-(x^2+x-2)(3x^2)}{(x^3+6)^2} \\ &= \frac{2x^4+x^3+12x+6-3x^4-3x^3+6x^2}{(x^3+6)^2} \\ &= \frac{-x^4-2x^3+6x^2+12x+6}{(x^3+6)^2} \end{aligned}$$

Example 3.6.11

Example 3.6.12

Find an equation of the tangent line to the curve $y = \dfrac{e^x}{1+x^2}$ at $(1, \dfrac{e}{2})$

sol:
let $f(x) = \dfrac{e^x}{1+x^2}$

$$\begin{aligned} f'(x) &= \frac{(1+x^2)(e^x)-e^x(2x)}{(1+x^2)^2} \\ &= \frac{e^x(x^2-2x+1)}{1+x^2)^2} \\ f'(1) &= \frac{e^x(1-2+1)}{4} = 0 \end{aligned}$$

$\implies$ eqn. of the tangent line is $y = \dfrac{e}{2}$

Example 3.6.12

Notation 3.6.13.

$$e^x = y$$
$$\log_e e^x = \log_e y$$
$$x = \log_e y = lny = Lny$$

Example 3.6.14

$f(x) = \cos x.$ Find $f^{(27)}(x)$

$$\begin{array}{rcl} f'(x) & = & -\sin x \\ f''(x) & = & -\cos x \\ f'''(x) & = & \sin x \\ f''''(x) & = & \cos x \end{array}$$

$$\Longrightarrow \quad f^{(27)}(x) = \sin x$$

Example 3.6.14

3.7 ▲ Chain Rule 連鎖律

Definition 3.7.1.

Let $h(x) = f(g(x))$
If f and g are differentiable

$$\begin{array}{rcl} h'(x) & = & f'(g(x)) \cdot g'(x) \\ & = & f'(u)|_{u=g(x)} \cdot g'(x) \\ & = & \dfrac{df(g(x))}{dg(x)} \cdot \dfrac{dg(x)}{dx} \end{array}$$

Example 3.7.2

$$\frac{d}{dx}(\sin 2x)$$

$$\begin{array}{rcl} h(x) & = & \sin(2x) \\ f(x) & = & \sin x \Longrightarrow f'(x) = \cos x \\ g(x) & = & 2x \Longrightarrow g'(x) = 2 \end{array}$$

$$f(g(x)) = f(2x) = \sin(2x)$$

$$\begin{aligned}\frac{d\sin(2x)}{dx} &= \frac{d\sin(2x)}{d(2x)}\cdot\frac{d(2x)}{dx}\\ &= \frac{d\sin y}{dy}\cdot 2\\ &= 2\cos y\\ &= 2\cos(2x)\end{aligned}$$

Example 3.7.2

Example 3.7.3

$$\frac{de^{2x}}{dx}$$

$$\begin{aligned}\frac{de^{2x}}{dx} &= \frac{de^{2x}}{d(2x)}\cdot\frac{d(2x)}{dx}\\ &= e^{y}\cdot 2\\ &= e^{2x}\cdot 2\end{aligned}$$

$$f(x)=a^x(a>0) \implies f'(x)=?$$

If $a=e=2.718281828459045$

$$(e^x)'=e^x$$

If $a\neq e$

$$\begin{aligned}\frac{d}{dx}(a^x) &= ((e^{lna})^x)'=(e^{lna\cdot x})'\\ &= \frac{d(e^{(lna)x})}{d((lna)x)}\cdot\frac{d((lna)x)}{dx}\\ &= e^{y}\cdot lna\\ &= e^{(lna)x}\cdot lna\\ &= a^x\cdot lna\\ \frac{da^x}{dx} &= lna\cdot a^x\end{aligned}$$

when $a=e$

$$\frac{de^x}{dx}=lne\cdot e^x=1e^x=e^x$$

Example 3.7.3

Example 3.7.4

$$\frac{d\tan(\sin x)}{dx}$$

$$\begin{aligned}\frac{d\tan(\sin x)}{dx} &= \frac{d\tan(\sin x)}{d\sin x}\cdot\frac{d\sin x}{dx}\\ &= \sec^2(\sin x)\cos x\end{aligned}$$

Example 3.7.4

Proof. Chain Rule

Let $\epsilon_1 = \dfrac{g(x)-g(a)}{x-a} - g'(a)$ $(\epsilon_1 = \epsilon_1(x))$

$$\begin{aligned}\lim_{x\to a}\epsilon_1 &= \lim_{x\to a}\left(\frac{g(x)-g(a)}{x-a} - g'(a)\right)\\ &= \lim_{x\to a}\left(\frac{g(x)-g(a)}{x-a}\right) - \lim_{x\to a} g'(a)\\ &= g'(a) - g'(a)\end{aligned}$$

$\therefore \epsilon_1 \to 0$ as $x \to a$

$$g(x) - g(a) = (g'(a)+\epsilon_1)(x-a) \qquad ---(1)$$

Similarity:

Let $y = g(x)$, $b = g(a)$ $\qquad ---(3)$

$$\epsilon_2 = \frac{f(y)-f(b)}{y-b} - f'(b) \quad (\epsilon_2 = \epsilon_2(y))$$

$\therefore \epsilon_2 \to 0$ as $y \to b$

$$f(y) - f(b) = (f'(b)+\epsilon_2)(y-b) \qquad ---(2)$$

$$\begin{aligned}(3) &\implies f(g(x) - f(g(a)) = (f'(g(a))+\epsilon_2)(g(x)-g(a))\\ (1) &\implies (f'(g(a))+\epsilon_2)(g'(a)+\epsilon_1)(x-a)\end{aligned}$$

$$\lim_{x\to a}\frac{f(g(x))-f(g(a))}{x-a} = \lim_{x\to a}(f'(g(a)+\epsilon_2)(g'(a)+\epsilon_1)$$

$$\begin{aligned}\frac{d}{dx}(f(g(x))|_{x=a} &= \lim_{x\to a}(f'(g(a))+\epsilon_2)\lim_{x\to a}(g'(a)+\epsilon_1)\\ &= (f(g(a))+0)\cdot(g'(a)+0)\\ &= f'(g(a))\cdot g'(a)\end{aligned}$$

$$(1): g(x) \approx g(a) + g'(a)(x-a)$$

$$(2): g(y) \approx g(b) + g'(b)(y-b)$$

$$f(g(x)) - f(g(a)) \approx f'(g(a))(g(x)-g(a)) \approx f'(g(a))g'(a)(x-a)$$

□

3.8 ▲ Linear Approximation (Linearization)

$$g(x) \approx g(a) + g'(a)(x-a) \quad \text{(As } x \text{ is close to } a\text{)}$$
$$g'(a) = \frac{g(x)-g(a)}{x-a}$$
$$f(x) \approx f(a) + f'(a)(x-a)$$
$$g(x) \approx g(a) + g'(a)(x-a)$$
$$f(x)g(x) \approx f(a)g(a) + f(a)g'(a) + f'(a)g(a) + f'(a)g'(a)(x-a)^2$$
$$\frac{f(x)g(x)-f(a)g(a)}{x-a} \approx f(a)g(a) + f(a)g'(a) + f'(a)g'(a)(x-a)$$
$$\frac{d}{dx}(f(x)g(x)) \approx f'(a)g(a) + f(a)g'(a) \quad \text{product rule}$$
$$\frac{d}{dx}(x^a) = ax^{a-1}, a \in R$$

Example 3.8.1

$f(x) = a^x$. Find $f'(x)$

$$a^x = (e^{lna})^x = e^{(lna)x}$$
$$\frac{d}{dx}(a^x) = a^x lna \quad a > 0$$

Example 3.8.1

Example 3.8.2

Find linearization of $f(x) = (x+3)^{\frac{1}{2}}$ at $a = 1$.

$$\begin{aligned} f(x) &\approx f(a) + f'(a)(x-a) \\ f'(x) &= \frac{1}{2}(x+3)^{\frac{1}{2}} \\ f'(1) &= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \\ f(1) &= 2 \end{aligned}$$

when $x \approx 1$

$$f(x) \approx 2 + \frac{1}{4}(x-1) = \frac{1}{4}x + \frac{7}{4}$$

- Find $\sqrt{398} < 4 = 2$

$$f(0.98) \approx \frac{1}{4} \cdot 0.98 + \frac{7}{4} = 1.995$$

- Find $\sqrt{405} > 4 = 2$

$$f(1.05) \approx \frac{1}{4} \cdot 1.05 + \frac{7}{4} = 2.0125$$

Example 3.8.2

Example 3.8.3

Find the linearization of $f(\theta) = \sin\theta$ at $a = 0$

$$\begin{aligned} f(\theta) &\approx f(a) + f'(a)(\theta - a) \\ &= 0 + 1(\theta - 0) \\ &= \theta \end{aligned}$$

$\therefore \sin\theta \approx \theta$ as $\theta \approx 0$

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

Example 3.8.3

3.9 ▲ Implicit Differentiation

Example 3.9.1

Find an eqn. of the tangent line to $x^2 + y^2 = 25$ at $(3, 4)$

$$4x^2 + y^2 = 25$$

$$\begin{aligned} y(x) &= +\sqrt{25 - x^2} = (25 - x^2)^{\frac{1}{2}} \\ y'(x) &= \frac{d(z^{\frac{1}{2}})}{dz}\frac{dz}{dx} \\ &= \frac{1}{2}z^{-\frac{1}{2}}(-2x) \\ &= -xz^{-\frac{1}{2}} \\ &= \frac{-x}{\sqrt{25 - x^2}} \\ y'(3) &= \frac{-3}{4} \end{aligned}$$

$$x^2 + (y(x))^2 = 25 \qquad ---(1)$$

$$\frac{d}{dx}(1) \implies \frac{d}{dx}(x^2 + (y(x))^2) = \frac{d}{dx}(25)$$

$$\cancel{2}x + \cancel{2}y(x)y'(x) = 0$$

$$y'(x) = -\frac{x}{y}$$

Example 3.9.1

Notation 3.9.2.

Folium of Descartes: $x^3 + y^3 = 6xy$

Example 3.9.3

(1) Find $y'(x)$ if $x^3 + y^3 = 6xy$

$$x^3 + (y(x))^3 = 6x \cdot y(x) \qquad ---(\star)$$

$$\frac{d}{dx}(\star) \implies 3x^2 + 3y(x)^2 y'(x) = 6\frac{d}{dx}(x \cdot y(x))$$
$$= 6(1 \cdot y(x) + x \cdot y'(x))$$

$$3x^2 + 3y^2y' = 6(y + xy')$$

$$y' = \frac{2y - x^2}{y^2 - 2x}$$

(2) Find the tangent line to the folium of Descartes at $(3,3)$

$$y'(3) = \frac{2y - x^2}{y^2 - 2x}\Big|_{(x,y)=(3,3)} = \frac{-3}{3} = -1$$

tangent line:

$$\frac{y-3}{x-3} = -1$$

Example 3.9.3

Example 3.9.4

Find y' if $\sin(x + y) = y^2 \cos x$

$$\sin(x + y(x)) = (y(x))^2 \cos x \qquad ---(2)$$

$$\frac{d}{dx}(2) \implies \cos(x + y(x)) \cdot (1 + y'(x)) = (2y(x) \cdot y'(x)) \cos x + (y(x))^2(-\sin x)$$

$$\cos(x + y)(1 + y') = 2(\cos x)yy' - (\sin x)y^2$$

$$y' = \frac{y^2 \sin x + \cos(x + y)}{2y \cos x - \cos(x + y)}$$

Example 3.9.4

Example 3.9.5

Find $f'(x)$ if $f(x) = \log_a x \quad (a > 0 \text{ const.})$

$$\begin{aligned} y(x) = \log_a x = a^{y(x)} &= x \qquad ---(3) \\ \frac{d}{dx}(3) \implies \frac{d}{dx}(a^{y(x)}) &= \frac{d}{dx}x \\ \frac{d(a^{y(x)})}{dy(x)}\frac{dy(x)}{dx} &= 1 \\ y'(x) &= \frac{1}{lna}a^{-y(x)} \\ &= \frac{1}{lna}\frac{1}{x} \end{aligned}$$

Example 3.9.5

- $\frac{d}{dx}(x^a) = ax^{a-1}$, $a \in R$
- $\frac{d}{dx}(a^x) = a^x lna$, $a > 0$
  when $a = e \implies (e^x)' = e^x$
- $\frac{d}{dx}\log_a x = \frac{1}{lna}\frac{1}{x}$, $a > 0$
  when $a = e \implies (lnx)' = \frac{1}{x}$

3.10 ▲ Derivatives of Inverse Trigonometric Functions

$y = \sin^{-1} x = \arcsin x$
If $\sin y = \sin(\sin^{-1} x) = x, x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

- $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

let $y(x) = \sin^{-1} x$

$$\begin{aligned} \sin(y(x)) &= x & ---(1) \\ \frac{d}{dx}(1) \implies \cos(y(x)) \cdot y'(x) &= 1 \\ y'(x) &= \frac{1}{\cos(y(x))} \\ &= \frac{1}{\sqrt{1-x^2}} \end{aligned}$$

- $\dfrac{d}{dx}(\cos^{-1} x) = -\dfrac{1}{\sqrt{1-x^2}}$
- $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$

  let $y(x) = \tan^{-1} x$

$$\begin{aligned} \tan(y(x)) &= x & ---(2) \\ \frac{d}{dx}(2) \implies \sec^2 y(x) \cdot y'(x) &= 1 \\ y'(x) &= \frac{1}{\sec^2 y(x)} \\ &= \cos^2 y(x) = \frac{1}{1+x^2} \end{aligned}$$

- $\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{x\sqrt{x^2-1}}$

  let $y(x) = \sec^{-1} x$

$$\begin{aligned} \frac{d}{dx}(y(x)) &= x & ---(3) \\ \frac{d}{dx}(3) \implies \sec y(x) \tan y(x) \cdot y'(x) &= 1 \\ y'(x) &= \frac{1}{\sec y(x) \tan y(x)} \\ &= \frac{1}{x}\frac{1}{\sqrt{x^2-1}} = \frac{1}{x\sqrt{x^2-1}} \end{aligned}$$

- $\dfrac{d}{dx}(\csc^{-1} x) = -\dfrac{1}{x\sqrt{x^2-1}}$
- $\dfrac{d}{dx}(\cot^{-1} x) = -\dfrac{1}{1+x^2}$

Joke 3.10.1.

記 $\Longrightarrow$ 不要記

Example 3.10.2

$$\frac{d}{dx}(f(x)^{g(x)})$$

let $y(x) = f(x)^{g(x)}$

$$\begin{aligned}
\text{take } ln \Longrightarrow \quad lny(x) = ln(f(x)^{g(x)}) &= (g(x))(lnf(x)) && ---(4) \\
\frac{d}{dx}(4) \Longrightarrow \quad \frac{1}{y(x)}y'(x) &= \frac{d}{dx}(g(x)lnf(x)) \\
&= g'(x)lnf(x) + g(x)\frac{f'(x)}{f(x)} && \text{(product rule)} \\
y'(x) &= y(x)(g'(x) \cdot lnf(x) + \frac{g(x)}{f(x)} \cdot f'(x)) \\
\frac{d}{dx}(f(x)^{g(x)}) &= f(x)^{g(x)}(g'(x) \cdot lnf(x) + \frac{g(x)}{f(x)} \cdot f'(x))
\end{aligned}$$

Example 3.10.2

Example 3.10.3

$f(x) = x^{\sqrt{x}}$. Find $f'(x)$

$$\begin{aligned}
\text{take } ln \Longrightarrow \quad lnf(x) = lnx^{\sqrt{x}} &= \sqrt{x}lnx && ---(5) \\
\frac{d}{dx}(5) \Longrightarrow \quad \tfrac{1}{f(x)}f'(x) &= \frac{1}{2}x^{-\frac{1}{2}}lnx + \sqrt{x}\frac{1}{x} \\
f'(x) &= f(x)x^{-\frac{1}{2}}(\frac{1}{2}lnx + 1) \\
&= \frac{x^{\sqrt{x}}}{2\sqrt{x}}(lnx + 2)
\end{aligned}$$

Example 3.10.3

Example 3.10.4

$f(x) = \dfrac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$. Find $f'(x)$

$$\begin{aligned}
\text{take } ln \implies \quad lnf(x) &= \frac{3}{4}lnx + \frac{1}{2}ln(x^2+1) - 5ln(3x+2) \qquad ---(6) \\
\frac{d}{dx}(6) \implies \quad \frac{f'(x)}{f(x)} &= \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+1} - 5\cdot\frac{3}{3x+2} \\
f'(x) &= \frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+1} - 5\cdot\frac{3}{3x+2}\right)
\end{aligned}$$

Example 3.10.4

Joke 3.10.5.

解剖霸王龍 $\implies$ 西方做無聊的事

Notation 3.10.6.

- $\dfrac{d}{dx}(a^b) = 0 \qquad a, b \in R$ consts
- $\dfrac{d}{dx}(f(x)^b) = b \cdot f(x)^{-1} \cdot f'(x) \qquad b$: const

$$\begin{aligned}
\frac{d}{dx}(f(x)^b) &= \frac{df(x)}{df(x)} \cdot \frac{df(x)}{dx} \qquad \text{(chain rule)} \\
&= b \cdot y^{b-1} \cdot f'(x) \\
&= b \cdot f(x)^{b-1} \cdot f'(x)
\end{aligned}$$

- $\dfrac{d}{dx}(a^{g(x)}) = a^{g(x)} \cdot lna \cdot g'(x) \qquad a > 0$ const
- $\dfrac{d}{dx}(f(x)^{g(x)})$ take ln in y, and differentiate

$$\begin{aligned}
\frac{d}{dx}lny(x) &= \frac{dlny(x)}{dy(x)} \cdot \frac{dy(x)}{dx} \\
&= \frac{1}{y(x)}y'(x) \\
&= \frac{y'(x)}{y(x)}
\end{aligned}$$

Joke 3.10.7 (Anagram).

Tom Hanks = monk hats
Mel Gibson = big lemons

3.11 ▲ Exponential Growth and Decay

t: time
$y(t)$: population
under ideal condition, assume:

$$\begin{aligned} y'(t) &= \frac{dy}{dt} \propto y(t) \\ \text{differential eqn.} \implies y'(t) &= k \cdot y(t) \quad k: \text{ const} \qquad ---(\star) \\ y(t) &= c \cdot e^k t \end{aligned}$$

Q: solve $(\star)$

$$k = 1$$
$$y'(t) = y(t) \qquad --(1)$$

$$\begin{aligned} y(t) &= e^t && \text{solves (1)} \\ y(t) &= 2e^t && \text{solves (1)} \\ y(t) &= c \cdot e^t && \text{solves (1)} \quad c\text{: any const} \end{aligned}$$

Q: solve c

$$\text{initial value problem} \begin{cases} y'(t) = k \cdot y(t) & t > 0 \\ y(0) = y_0 & y_0\text{: const, initial condition (I.C)} \end{cases}$$

$$\begin{aligned} y(t) &= c \cdot e^{kt} \\ y(0) &= c \cdot 1 = c \\ c &= y_0 \end{aligned}$$

$\therefore$ The *sol* of I.VP. is $y(t) = y_0 \cdot e^{kt}$
$t > 0 \implies \lim_{t \to \infty} y(t) = \infty$

- verify $y(t) = y_0 \cdot e^{kt}$ solves $(\star)$

$$y'(t) = y_0 \frac{d}{dt}(e^{kt}) = \frac{de^{kt}}{dkt} \cdot \frac{dkt}{dt} \cdot y_0 = e^{kt} \cdot k \cdot y_0 = k(y_0 e^{kt})$$

- verify $y(0) = y_0$ is satisfied

$$\begin{aligned} y(t) &= y_0 \cdot e^{kt} \\ y(0) &= y_0 \cdot e^0 = y_0 \end{aligned}$$

▸▸ Compound Interest

\$ 1000 (A_0) invested 6% (r) per year (t)

$$
\begin{aligned}
\text{annual 3 years} &\implies 1000(1+0.06)^4 \doteqdot 1191.02 \\
\text{semi-annual 3 years} &\implies 1000(1+0.03)^6 \doteqdot 1194.05 \\
\text{quarterly 3 years} &\implies 1000(1+0.015)^{12} \doteqdot 1196.68 \\
\text{daily 3 years} &\implies 1000(1+\frac{0.06}{365})^{3.365} \doteqdot 1197.20
\end{aligned}
$$

$$
\begin{aligned}
\lim_{n\to\infty} A_0(1+\frac{r}{n})^{nt} &= \lim_{n\to\infty} A_0(1+\frac{1}{\frac{n}{r}})^{\frac{n}{r}\cdot rt} \\
\text{無時不刻 3 years} \implies \quad &= \lim_{n\to\infty} A_0((1+\frac{1}{k})^k)^{rt} \\
&= A_0 \lim_{n\to\infty}((1+\frac{1}{k})^k)^{rt} \\
&= A_0 e^{rt}
\end{aligned}
$$

▸▸ Radioactive Decay

Decay rate of radioactive $\propto$ remaining mass ($= m(t)$)

$$
\begin{aligned}
\frac{dm(t)}{mt} &\propto m(t) \\
\frac{dm(t)}{dt} &= k \cdot m(t) \quad k < 0 \text{ const}
\end{aligned}
$$

$$m(t) = m(0)\cdot e^{kt}$$

Example 3.11.1

The half-life of Ra (鐳) is 1590 years.

$$\frac{1}{2}\cancel{m(0)} = \cancel{m(0)}e^{k\cdot 1590}$$

$$-ln2 = ln\frac{1}{2} = 1590k$$

$$k = \frac{-ln2}{1590}$$

Q: $m(0) = 100(\text{mg}) \implies m(1000) = ?$

$$m(1000) = m(0)\cdot e^{\frac{-ln2}{1590}\cdot 1000} \doteqdot 65 \text{ (mg)}$$

Q: $m(t) = 30(\text{mg}) \implies t = ?$

$$30 = 100e$$

$$ln30 = ln100 + (\frac{-ln2}{1590})t \implies t = 2762 \text{ (yr)}$$

Example 3.11.1

Example 3.11.2

$$
\begin{aligned}
(x(t))^2 + (y(t))^2 &= 5^2 \quad ---(1) \\
\frac{d}{dt}(1) \implies \cancel{2}x(t)x'(t) + \cancel{2}y(t)y'(t) &= 0 \\
\frac{x(t)}{y(t)} &= -\frac{y(t)}{x(t)} \\
y'(t) &= -\frac{3}{4}
\end{aligned}
$$

Example 3.11.2

Chapter 4

Applications of Derivatives

4.1 Maximum and Minimum

Example 4.1.1

- $f(x) = \cos x$
  $x \in R$ has infinity many local max and local min
  abs max $= 1$
  abs min$= -1$
- $f(x) = x^2$
  local max: none
  local min: $x = 0 \implies$ abs min
  $f(x) = x^2 \geqslant 0$
- $f(x) = x^3$
  local max: none
  local min: none

Example 4.1.1

4.2 Extreme Value Theorem

Theorem 4.2.1.

Assume $f(x)$ is conti. on $[a, b]$, then
$f(x)$ has a abs max $f(C)$
$f(x)$ has a abs min $f(D)$
for some $C, D \in [a, b]$

4.3 ▲ Fermat's Theorem

Theorem 4.3.1.

Assume

(1) $f(x)$ has a local max or local min at c

(2) $f'(c)$ exists (f is differentiable at $x = c$)

Then

$$f'(c) = 0$$

Notation 4.3.2.

$$\begin{cases} f'(c) \neq 0 \\ f'(c)\text{exists} \end{cases} \implies f(c) \text{ is neither a local max nor a local min}$$

Proof. Fermat's Theorem

(1) If f has a local max at c
$\implies f(c+h) \leqslant f(c)$ if $|\,h\,|$ is small enough

(2) If $f'(c)$ exists
$$\implies f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h}$$

$f(c+h) - f(c) \leqslant 0$ if $|\,h\,|$ is small enough

- if $h > 0$
$$\begin{aligned} &\implies \frac{f(c+h)-f(c)}{h} \leqslant 0 \\ &\implies \lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} \leqslant 0 \\ &\implies \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \leqslant 0 \\ &\implies f'(c) \leqslant 0 \end{aligned}$$

- if $h < 0$
$$\begin{aligned} &\implies \frac{f(c+h)-f(c)}{h} \geqslant 0 \\ &\implies \lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} \geqslant 0 \\ &\implies \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \geqslant 0 \\ &\implies f'(c) \geqslant 0 \end{aligned}$$

$f'(c) \geqslant 0$ and $f'(c) \leqslant 0 \implies f'(c) = 0$ $\square$

Remark 4.3.3.

(1) Reverse of Fermat's Thm is false
$f(x) = x^3 \implies f'(x) = 3x^2 \quad f'(0) = 0$
$x = 0$ is neither a local max nor a local min

(2) $f(x) = \mid h \mid \implies f(x) > 0 \quad f(0) = 0$
$f'(0)$ doesn't exist
0 is a local min (abs min)

4.4 ▲ Critical Point (Critical Number)

Definition 4.4.1.

- If $f'(c)$ doesn't exist or $f''(c) = 0$, then c is called a critical point of f
- If f has a local max or a local min at c, then c is a critical point of f

Example 4.4.2

Find critical point of $f(x) = x^{\frac{3}{5}}(4 - x)$

$$\begin{aligned} f'(x) &= \frac{3}{5}x^{-\frac{2}{5}}(4 - x) + x^{\frac{3}{5}} \cdot -1 \\ &= \frac{1}{5}x^{-\frac{2}{5}}(12 - 3x - 5x) \\ &= \frac{1}{5}x^{-\frac{2}{5}}(12 - 8x) \\ &= \frac{1}{5} \cdot \frac{12 - 8x}{x^{\frac{2}{5}}} \end{aligned}$$

let $f(x)' = 0$

$$x = \frac{3}{2}$$

Find x s.t. $f'(x)$ doesn't exist $\implies$ $f'(0)$ doesn't exist

Example 4.4.2

4.5 ▲ Rolle's Theorem

Theorem 4.5.1.

Assume

(H1) $f(x)$ is conti. in $[a, b]$

(H2) $f(x)$ is differentiable in (a, b)

(H3) $f(a) = f(b)$

Then $\exists c \in (a, b)$ s.t $f'(c) = 0$

Remark 4.5.2. 等高兩點間必有波峰或波谷

Proof. Rolle's Theorem
Without loss of generality, we may assume

$$f(a) = f(b) = 0$$

Otherwise let $g(x) = f(x) - f(a) = f(x) - f(b)$, then

$$g(a) = 0 \quad \text{and} \quad g(b) = 0$$

3 cases:

(1) $f(x) = 0 \quad \forall x \in (a, b)$
$\implies f'(x) = 0 \quad \forall x \in (a, b)$

(2) $\exists x \in (a, b)$ s.t. $f(x) > 0$
Extreme Value Thm, (H1) $\implies$ f has a local max at $c \in (a, b)$
Fermat's Thm, (H2) $\implies$ $f'(c) = 0$

(3) $\exists x \in (a, b)$ s.t $f(x) < 0$
Extreme Value Thm, (H1) $\implies$ f has a local min at $c \in (a, b)$
Fermat's Thm, (H2) $\implies$ $f'(c) = 0$

□

Example 4.5.3

Prove $x^3 + x - 1 = 0$ has one real root

let $f(x) = x^3 + x - 1$

$$\begin{cases} f(1) = 1 > 0 \\ f(-1) = -3 < 0 \end{cases}$$

By I.V.T, $\exists c \in (-1,1)$ s.t. $f(c)=0$
Assume x_1 and x_2 are two roots of $f(x)=0$

$$f(x_1)=f(x_2)=0$$

$$\begin{cases} \exists k \in (x_1,x_2) \text{ s.t.} f(k)=0 \\ f'(x)=3x^2+1 \geqslant 1 > 0 \quad \forall x \in R \end{cases}$$

$\Longrightarrow$ contradiction

Example 4.5.3

4.6 ▲ Mean Value Theorem

Theorem 4.6.1.

Assume

(H1) $f(x)$ is conti. in $[a,b]$

(H2) $f(x)$ is differentiable in (a,b)

Then $\exists c \in (a,b)$ s.t. $f'(c)=\dfrac{f(a)-f(b)}{a-b}$

Remark 4.6.2. When $f(a)=f(b)$ in M.V.T, M.V.T becomes Rolle's Thm

Proof. Mean Value Theorem
Let $h(x)=f(a)+\dfrac{f(a)-f(b)}{a-b}(x-a)$
Let $g(x)=f(x)-h(x)$

$$\begin{cases} g(a) &= f(a)-h(a)=0 \\ g(b) &= f(b)-h(b)=0 \end{cases}$$

$$\frac{h(x)-f(a)}{x-a}=\frac{f(a)-f(b)}{a-b}$$

$g(x)=f(x)-h(x)$ is conti. on (a,b) and diff. on (a,b)
By Rolle's Thm, $\exists c \in (a,b)$ s.t. $g'(c)=0$

$$\begin{aligned} h'(x) &= \frac{f(a)-f(b)}{a-b} \\ g'(x) &= f'(x)-h'(x) \\ g'(c) &= f'(c)-\frac{f(a)-f(b)}{a-b} \\ f'(c) &= \frac{f(a)-f(b)}{a-b} \end{aligned}$$

□

Example 4.6.3

$f(x) = \sin\sqrt{x+1}$

- Find $f'(x)$

$$\begin{aligned} f(x) &= \sin(x+1)^{\frac{1}{2}} \\ f'(x) &= \cos(\sqrt{x+1}) \cdot \frac{1}{2}(x+1)^{-\frac{1}{2}} \cdot 1 \\ &= \frac{\cos\sqrt{x+1}}{2\sqrt{x+1}} \end{aligned}$$

- $\lim\limits_{x\to 0} \dfrac{\sin\sqrt{x+1} - \sin 1}{x-0} = \lim\limits_{x\to 0} \dfrac{f(x)-f(0)}{x-0} = f'(0) = \dfrac{\cos 1}{2}$

- Prove $\sin\sqrt{x+1} < \frac{1}{2}x + \sin 1$ for $x > 0$

  By M.V.T, $\exists c \in (0,x)$ s.t. $f'(c) = \dfrac{f(x)-f(0)}{x-0} = \dfrac{\sin\sqrt{x+1} - \sin 1}{x}$

$$f'(c) = \frac{\cos\sqrt{c+1}}{2\sqrt{c+1}} \leqslant \frac{1}{2\sqrt{x+1}} < \frac{1}{2\sqrt{0+1}} = \frac{1}{2}$$

Example 4.6.3

Joke 4.6.4.

你想不到 8

4.7 ▲ L'Hospital's Rule

Theorem 4.7.1.

Consider $\lim_{x \to a} \frac{f(x)}{g(x)}$

- type $\frac{0}{0}$
  $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$
- type $\frac{\infty}{\infty}$
  $\lim_{x \to a} f(x) = \pm\infty$
  $\lim_{x \to a} g(x) = \pm\infty$

Assume

(1) f and g are differentiable

(2) $g'(x) \neq 0$ on an open interval containing a (except possibility at a.)

Then for type $\frac{0}{0}$ or $\frac{\infty}{\infty}$,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

▸▸ Baby L'Hospital's Rule

Theorem 4.7.2.

Assume

(1) $f(a) = g(a) = 0$

(2) f' and g' are conti.

(3) $g'(a) \neq 0$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Proof. Baby L'Hospital's Rule

$$
\begin{aligned}
\lim_{x\to a}\frac{f(x)}{g(x)} &= \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}\\
&= \lim_{x\to a}\frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}\\
&= \frac{\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}{\lim_{x\to a}\frac{g(x)-g(a)}{x-a}} && \text{(limit law)}\\
&= \frac{f'(a)}{g'(a)}\\
&= \frac{\lim_{x\to a} f'(x)}{\lim_{x\to a} g'(x)}\\
&= \lim_{x\to a}\frac{f'(x)}{g'(x)}
\end{aligned}
$$

□

Example 4.7.3

$$
\begin{aligned}
&\lim_{x\to 1}\frac{lnx}{x-1} \quad (\text{type } \frac{0}{0})\\
&= \lim_{x\to 1}\frac{\frac{1}{x}}{1}\\
&= 1
\end{aligned}
$$

Example 4.7.3

Example 4.7.4

$$
\begin{aligned}
&\lim_{x\to\infty}\frac{e^x}{x^2} \quad (\text{type } \frac{\infty}{\infty})\\
&= \lim_{x\to\infty}\frac{e^x}{2x} \quad (\text{type } \frac{\infty}{\infty})\\
&= \lim_{x\to\infty}\frac{e^x}{2} \quad (\text{type } \frac{\infty}{2})\\
&= \infty
\end{aligned}
$$

$$e^x >> x^2$$

Example 4.7.4

Example 4.7.5

$$\begin{aligned}
&\lim_{x\to 0}\frac{\tan x - x}{x^3} && (\text{type } \frac{0}{0})\\
&= \lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2} && (\text{type } \frac{0}{0})\\
&= \lim_{x\to 0}\frac{2\sec^2 x\tan x}{6x} && (\text{type } \frac{0}{0})\\
&= \frac{1}{3}\lim_{x\to 0}\frac{2\sec x\sec x\tan x + \sec^4 x}{1} && (\text{type } \frac{1}{1})\\
&= \frac{1}{3}
\end{aligned}$$

Example 4.7.5

Example 4.7.6

$$\begin{aligned}
&\lim_{x\to 0^+} xlnx \qquad (\text{type } 0\infty)\\
&= \lim_{x\to 0^+}\frac{lnx}{\frac{1}{x}}\\
&= \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}\\
&= \lim_{x\to 0^+}(-x)\\
&= 0
\end{aligned}$$

Example 4.7.6

Example 4.7.7

$$\begin{aligned}
&displaystyle \lim_{x\to\frac{\pi}{2}^-}(\sec x - \tan x) && (\text{type } \infty - \infty)\\
&= \lim_{x\to\frac{\pi}{2}^-}(\frac{1}{\cos x} - \frac{\sin x}{\cos x})\\
&= \lim_{x\to\frac{\pi}{2}^-}\frac{1-\sin x}{\cos x} && (\text{type } \frac{0}{0})\\
&= \lim_{x\to\frac{\pi}{2}^-}\frac{-\cos x}{-\sin x} && (\text{type } \frac{0}{-1})\\
&= 0
\end{aligned}$$

Example 4.7.7

Example 4.7.8

$$\lim_{x\to 0^+}(1+\sin(4x))^{\cot x} \quad (\text{type } f(x)^{g(x)})$$

$$\begin{aligned} y(x) &= (1+\sin(4x))^{\cot x} \\ ln y(x) &= \cot x \cdot ln(1+\sin(4x)) \end{aligned}$$

Find $\lim_{x\to 0^+}(ln y(x))$

$$\begin{aligned} \lim_{x\to 0^+}(ln y(x)) &= \lim_{x\to 0^+}\cot x ln(1+\sin(4x)) && (\text{type } \infty 0) \\ &= \lim_{x\to 0^+}\frac{ln(1+\sin(4x))}{\tan x} && (\text{ type } \frac{0}{0}) \\ &= \lim_{x\to 0^+}\frac{\frac{4\cos(4x)}{1+\sin(4x)}}{\sec^2 x} && (\text{type } \frac{\frac{4}{1+0}}{1}) \\ &= 4 \end{aligned}$$

$$\begin{aligned} \lim_{x\to 0^+}(ln y(x)) &= 4 \\ ln(\lim_{x\to 0^+} y(x)) &= \lim_{x\to 0^+} y(x) = e^4) \end{aligned}$$

Example 4.7.8

Joke 4.7.9.

導遊口才很好真不是 guide

4.8 ▲ Inflection Point (反曲點)

Definition 4.8.1.

A point c is called a inflection point of f if on c, the curve $y = f(x)$ changes from convex ($f'' > 0$, $\cup$) to concave ($f'' < 0$, $\cap$) or concave to convex. That is, if c is an inflection point then $f''(c) = 0$.

- If $f' > 0 \implies f \nearrow$
  $f' < 0 \implies f \swarrow$
  $\nearrow$ then $\searrow \implies \cap$ (concave downward 凹口向下)
  $\searrow$ then $\nearrow \implies \cup$ (concave upward 凹口向上/凸口向下)
- If $f'' > 0 \implies (f')' > 0 \quad \cup$
  $f'' < 0 \implies (f')' < 0 \quad \cap$

▸▸ 1st derivative test

$f'(c) = 0$
$f'(c^-) : +, \quad f'(c^+) : -, \quad f'(c) = 0 : \text{local max} \quad \cap$
$f'(c^-) : -, \quad f'(c^+) : +, \quad f'(c) = 0 : \text{local min} \quad \cup$

▸▸ 2nd derivative test

$f'(c) = 0$
$f''(c) : +, \quad f'(c) = 0 : \text{local min} \quad \cup$
$f''(c) : -, \quad f'(c) = 0 : \text{local max} \quad \cap$

4.9 ▲ Optimization Problems

$$\begin{cases} \text{objective fcn} \\ \text{condition(s)} \end{cases} \Longrightarrow \text{max or min}$$

Example 4.9.1

Find the point on the parabola $y^2 = 2x$ which closest to $(1, 4)$

$$\begin{cases} d(x,y) &= \sqrt{(x-1)^2 + (y-4)^2} \\ x &= \dfrac{y^2}{2} \end{cases}$$

Q: Minimize $d(x, y)$ under the condition $y^2 = 2x$
let

$$\begin{aligned} f(x,y) &= (x-1)^2 + (y-4)^2 \\ &= (\frac{1}{2}y^2 - 1)^2 + (y-4)^2 \\ &= \frac{1}{4}y^4 - \cancel{y^2} + 1 + \cancel{y^2} - 8y + 16 \\ &= \frac{1}{4}y^4 - 8y + 17 \\ &= g(y) \end{aligned}$$

Find critical point of $g(y)$
let $g'(y) = 0$

$$g'(y) = y^3 - 8 = 0$$
$$(y-2)(y^2 + 2y + 4) = 0$$
$$y = 2$$

- 1st derivative test

$f'(c^-)$	$f'(c)$	$f'(c^+)$		
$-$	0	$+$	$\cup$	local min
$+$	0	$-$	$\cap$	local max

$$\begin{aligned} g'(y) &= y^3 - 8 \\ g'(2) &= 0 \\ g'(2^-) &< 0 \\ g'(2^+) &> 0 \end{aligned}$$

$\implies$ local min

- 2nd derivative test

$$\begin{array}{cccc} f'(c) & f''(c) & & \\ 0 & + & \cup & \text{local min} \\ 0 & - & \cap & \text{local max} \end{array}$$

$$\begin{aligned} g''(y) &= 3y^2 \\ g'(2) &= 12 > 0 \end{aligned}$$

$\implies$ local min

Example 4.9.1

Example 4.9.2

Find the area of the largest rectangle that can be inscribed (內接) in a semi-circle of radius r

$$\begin{cases} A(x,y) &= 2xy \\ r^2 &= x^2 + y^2 \end{cases}$$

Q: Max $A(x,y)$

$$A(x,y) = 2xy = 2x(r^2 - x^2)^{\frac{1}{2}} := f(x)$$

$$f'(x) = 2(r^2 - x^2)^{\frac{1}{2}} + x(-x)(r^2 - x^2)^{\frac{1}{2}} = \frac{2}{(r^2 - x^2)^{\frac{1}{2}}}((r^2 - x^2) - x^2)$$

critical point
let $f'(x) = 0 \implies r^2 - 2x^2 \implies x = \frac{r}{\sqrt{2}}$
when $f'(x)$ doesn't exist $\implies x = r \implies y = 0$

- 1st derivative test

$$\begin{aligned} f'(x) &= \frac{2(r^2 - x^2)}{(r^2 - x^2)^{\frac{1}{2}}} \\ f'(\frac{r}{\sqrt{2}}^-) &> 0 \\ f'(\frac{r}{\sqrt{2}}^+) &< 0 \end{aligned}$$

$\implies$ local max

Example 4.9.2

4.10 ▲ Antiderivatives（反導函數）

Definition 4.10.1.

$F(x)$ is "an" antiderivative of $f(x)$ if $F'(x) = f(x)$

Notation 4.10.2.

Why"an" not "the"?

If $F'(x) = f(x)$
$\implies (F(x) + c)' = f(x) \qquad c$: const. interp of x
$\implies F(x) + c$ is also an antiderivative of $f(x)$

Example 4.10.3

$$\begin{array}{rcl} f(x) & = & \sin x \\ F(x) & = & -\cos x + c \end{array}$$

Example 4.10.3

Example 4.10.4

$$\begin{array}{rcl} f(x) & = & \frac{1}{x} \\ F(x) & = & lnx + c \end{array}$$

Example 4.10.4

Example 4.10.5

$$\begin{array}{rcl} f(x) & = & x^n \\ F(x) & = & \frac{1}{n+1}x^{n+1} + c \end{array}$$

Example 4.10.5