A ADDITIONAL PROOFS

Theorem 1. *The MEW and Least Expected Regret Winner are equivalent.*

Proof. Let $\boldsymbol{P}^{\mathsf{M}}$ denote a general voting profile with $\Omega(\boldsymbol{P}^{\mathsf{M}}) = \{\boldsymbol{P}_1, \ldots, \boldsymbol{P}_z\}$. The expected regret of a candidate w can be rewritten as follows.

$$\begin{aligned}
&\mathbb{E}(\texttt{Regret}(w, \boldsymbol{P}^{\mathsf{M}})) \\
&= \sum_{i=1}^{z} \texttt{Regret}(c, \boldsymbol{P}_i) \cdot \Pr(\boldsymbol{P}_i \mid \boldsymbol{P}^{\mathsf{M}}) \\
&= \sum_{i=1}^{z} \left(\max_{c \in C} s(c, \boldsymbol{P}_i) - s(w, \boldsymbol{P}_i) \right) \cdot \Pr(\boldsymbol{P}_i \mid \boldsymbol{P}^{\mathsf{M}}) \\
&= \sum_{i=1}^{z} \max_{c \in C} s(c, \boldsymbol{P}_i) \cdot \Pr(\boldsymbol{P}_i \mid \boldsymbol{P}^{\mathsf{M}}) - \mathbb{E}(s(w, \boldsymbol{P}^{\mathsf{M}}))
\end{aligned}$$

The first term $\sum_{i=1}^{z} \max_{c \in C} s(c, \boldsymbol{P}_i) \cdot \Pr(\boldsymbol{P}_i \mid \boldsymbol{P}^{\mathsf{M}})$ is a constant value, when $\boldsymbol{P}^{\mathsf{M}}$ and the voting rule are fixed. Thus, $\mathbb{E}(\texttt{Regret}(w, \boldsymbol{P}^{\mathsf{M}}))$ is minimized by maximizing $\mathbb{E}(s(w, \boldsymbol{P}^{\mathsf{M}}))$, the expected score of the candidate w. □

Theorem 2. *The MEW and Meta-Election Winner are equivalent.*

Proof. Let $\boldsymbol{P}^{\mathsf{M}}$ denote a general voting profile with $\Omega(\boldsymbol{P}^{\mathsf{M}}) = \{\boldsymbol{P}_1, \ldots, \boldsymbol{P}_z\}$, and $\boldsymbol{P}_{meta} = (\boldsymbol{P}_1, \ldots, \boldsymbol{P}_z)$ denote the large meta profile where rankings in $\boldsymbol{P}_i$ are weighted by $\Pr(\boldsymbol{P}_i \mid \boldsymbol{P}^{\mathsf{M}})$. According the definition of the Meta-Election Winner, $s(w, \boldsymbol{P}_{meta}) = \max_{c \in C} s(c, \boldsymbol{P}_{meta})$. As a result, for any candidate c,

$$\mathbb{E}(s(c, \boldsymbol{P}^{\mathsf{M}})) = \sum_{\boldsymbol{P} \in \Omega(\boldsymbol{P}^{\mathsf{M}})} s(c, \boldsymbol{P}) \cdot \Pr(\boldsymbol{P} \mid \boldsymbol{P}^{\mathsf{M}}) = s(c, \boldsymbol{P}_{meta})$$

Her expected score in $\boldsymbol{P}^{\mathsf{M}}$ are precisely her score in $\boldsymbol{P}_{meta}$. The two winner definitions are optimizing the same metric. □

Theorem 3. *The FCP is #P-complete.*

Proof. First, we prove its membership in #P. The FCP is the counting version the following decision problem: given a partial order $\boldsymbol{\nu}$, an item c, and an integer j, determine whether $\boldsymbol{\nu}$ has a linear extension $\boldsymbol{\tau} \in \Omega(\boldsymbol{\nu})$ where c is ranked at j. This decision problem is obviously in NP, meaning that the FCP is in #P.

Then, we prove that the FCP is #P-hard by reduction. Recall that counting $|\Omega(\boldsymbol{\nu})|$, the number of linear extensions of a partial order $\boldsymbol{\nu}$, is #P-complete [2]. This problem can be reduced to the FCP by $|\Omega(\boldsymbol{\nu})| = \sum_{j=1}^{m} N(c@j \mid \boldsymbol{\nu})$.

In conclusion, the FCP is #P-complete. □

Lemma 1. *If the ranking model* M *is a partial order* $\boldsymbol{\nu}$ *of* m *items that represents a uniform distribution of* $\Omega(\boldsymbol{\nu})$*, the REP-t is* $FP^{\#P}$*-complete.*

Proof. First, we prove that the REP-t is in $\mathrm{FP}^{\#P}$. Recall that $\Omega(\boldsymbol{\nu})$ is the linear extensions of a partial order $\boldsymbol{\nu}$, and $N(c@1|\boldsymbol{\nu})$ is the number of linear extensions in $\Omega(\boldsymbol{\nu})$ where candidate c is at rank 1. Then $\Pr(c@1|\boldsymbol{\nu}) = N(c@1|\boldsymbol{\nu})/|\Omega(\boldsymbol{\nu})|$. Consider that counting $N(c@1 \mid \boldsymbol{\nu})$ is in #P (Theorem 3) and counting $|\Omega(\boldsymbol{\nu})|$ is #P-complete [2], so $\Pr(c@1|\boldsymbol{\nu})$ is in $\mathrm{FP}^{\#P}$.

In the rest of this proof, we prove that the REP-t is #P-hard by reduction from the #P-complete problem of counting $|\Omega(\boldsymbol{\nu})|$.

Let c^* denote an item that has no parent in $\boldsymbol{\nu}$. Let $\boldsymbol{\nu}_{-c^*}$ denote the partial order of $\boldsymbol{\nu}$ with item c^* removed. If we are interested in the probability that c^* is placed at rank 1, we can write $\Pr(c^*@1|\boldsymbol{\nu}) = N(c^*@1 \mid \boldsymbol{\nu})/|\Omega(\boldsymbol{\nu})|$. The item c^* has been fixed at rank 1, so any placement of the rest items will definitely satisfy any relative order involving c^*. That is to say, the placement of the rest items just need to satisfy $\boldsymbol{\nu}_{-c^*}$, which leads to $N(c^*@1 \mid \boldsymbol{\nu}) = |\Omega(\boldsymbol{\nu}_{-c^*})|$.

For example, let $\boldsymbol{\nu}' = \{c_1 \succ c_4, c_2 \succ c_4, c_3 \succ c_4\}$. Then $N(c_1@1 \mid \boldsymbol{\nu}') = |\Omega(\boldsymbol{\nu}'_{-c_1})| = |\Omega(\{c_2 \succ c_4, c_3 \succ c_4\})|$.

Then we re-write $\Pr(c^*@1|\boldsymbol{\nu}) = N(c^*@1|\boldsymbol{\nu})/|\Omega(\boldsymbol{\nu})| = |\Omega(\boldsymbol{\nu}_{-c^*})|/|\Omega(\boldsymbol{\nu})|$. The oracle for $\Pr(c^*@1|\boldsymbol{\nu})$ manages to reduce the size of the counting problem from $|\Omega(\boldsymbol{\nu})|$ to $|\Omega(\boldsymbol{\nu}_{-c^*})|$. This oracle should be as hard as counting $|\Omega(\boldsymbol{\nu})|$. Thus calculating $\Pr(c^*@1|\boldsymbol{\nu})$ is $\mathrm{FP}^{\#P}$-hard.

In conclusion, the REP-t is $\mathrm{FP}^{\#P}$-complete. □

Lemma 2. *If the ranking model* M *is a partial order* $\boldsymbol{\nu}$ *of* m *items that represents a uniform distribution of* $\Omega(\boldsymbol{\nu})$*, the REP-b is* $FP^{\#P}$*-complete.*

Proof. This proof adopts the same approach as the proof of Lemma 1.

Let m be the number of items in the ranking model M. For the membership proof that the REP-b is in $\mathrm{FP}^{\#P}$, let $N(c@m|\boldsymbol{\nu})$ denote the number of linear extensions in $\Omega(\boldsymbol{\nu})$ where candidate c is at the bottom rank m. Then $\Pr(c@m|\boldsymbol{\nu}) = N(c@m|\boldsymbol{\nu})/|\Omega(\boldsymbol{\nu})|$. Consider that counting $N(c@m|\boldsymbol{\nu})$ is in #P (Theorem 3) and counting $|\Omega(\boldsymbol{\nu})|$ is #P-complete [2], so $\Pr(c@m \mid \boldsymbol{\nu})$ is in $\mathrm{FP}^{\#P}$.

In the proof of Lemma 1, item c^* is an item with no parent in the partial order $\boldsymbol{\nu}$. In the current proof, item c^* is set to be an item with no child in $\boldsymbol{\nu}$. The $\boldsymbol{\nu}_{-c^*}$ still denotes the partial order of $\boldsymbol{\nu}$ but with item c^* removed. Then the probability that item c^* at the bottom rank m is $\Pr(c^*@m \mid \boldsymbol{\nu}) = N(c^*@m \mid \boldsymbol{\nu})/|\Omega(\boldsymbol{\nu})| = |\Omega(\boldsymbol{\nu}_{-c^*})|/|\Omega(\boldsymbol{\nu})|$. The oracle for $\Pr(c^*@m \mid \boldsymbol{\nu})$ manages to reduce the size of the counting problem again from $|\Omega(\boldsymbol{\nu})|$ to $|\Omega(\boldsymbol{\nu}_{-c^*})|$. Thus, this oracle is #P-hard, and calculating $\Pr(c^*@m \mid \boldsymbol{\nu})$ is $\mathrm{FP}^{\#P}$-hard.

In conclusion, the REP-b is $\mathrm{FP}^{\#P}$-complete. □

Theorem 4. *If the ranking model* M *is a partial order* $\boldsymbol{\nu}$ *that represents a uniform distribution of* $\Omega(\boldsymbol{\nu})$*, the Rank Estimation Problem is* $FP^{\#P}$*-complete.*

Proof. First, we prove that the REP is in $\mathrm{FP}^{\#P}$. Recall that $\Omega(\boldsymbol{\nu})$ is the linear extensions of a partial order $\boldsymbol{\nu}$, and $N(c@j|\boldsymbol{\nu})$ is the number of linear extensions in $\Omega(\boldsymbol{\nu})$ where candidate c is at rank j. Then $\Pr(c@j|\boldsymbol{\nu}) = N(c@j|\boldsymbol{\nu})/|\Omega(\boldsymbol{\nu})|$. Consider that counting $N(c@j|\boldsymbol{\nu})$ is #P-complete (Theorem 3) and counting $|\Omega(\boldsymbol{\nu})|$ is #P-complete [2] as well. So $\Pr(c@j|\boldsymbol{\nu})$ is in $\mathrm{FP}^{\#P}$.

Lemma 1 demonstrates that REP-t, a special case of REP, is $\mathrm{FP}^{\#P}$-hard. Thus REP is #P-hard as well.

In conclusion, REP is $\mathrm{FP}^{\#P}$-complete. □

Theorem 5. *Given a general voting profile $\boldsymbol{P}^{\mathsf{M}}$ and a positional scoring rule $\boldsymbol{r}_m$, the ESC problem can be reduced to the REP.*

Proof. Recall that the MEW w maximizes the expected score, i.e.,

$$s(w, \boldsymbol{P}^{\mathsf{M}}) = \max_{c \in C} \mathbb{E}(s(c, \boldsymbol{P}^{\mathsf{M}}))$$

The voting profile $\boldsymbol{P}^{\mathsf{M}}$ contains n ranking distributions $\{\mathsf{M}_1, \dots, \mathsf{M}_n\}$, so

$$\mathbb{E}(s(c, \boldsymbol{P}^{\mathsf{M}})) = \sum_{i=1}^{n} \mathbb{E}(s(c, \mathsf{M}_i))$$

where $\mathbb{E}(s(c, \mathsf{M}_i))$ is the expected score of c from voter v_i.

$$\mathbb{E}(s(c, \mathsf{M}_i)) = \sum_{j=1}^{m} Pr(c@j \mid \mathsf{M}_i) \cdot \boldsymbol{r}_m(j)$$

where $c@j$ denotes candidate c at rank j, and $\boldsymbol{r}_m(j)$ is the score of rank j.

Let $\mathbb{T}$ denote the complexity of calculating $Pr(c@j|\mathsf{M}_i)$. The original MEW problem can be solved by calculating $Pr(c@j|\mathsf{M}_i)$ for all m candidates, m ranks and n voters, which leads to the complexity of $O(n \cdot m^2 \cdot \mathbb{T})$. □

Theorem 6. *The REP for rank k is equivalent to the ESC problem over either one or both of the $(k-1)$-approval and k-approval rules.*

Proof. The ESC problem has be reduced to the REP (Theorem 5). This proof will focus on the other direction, i.e., reducing the REP to the ESC problem.

Let $\Pr(c@j \mid \mathsf{M})$ denote the probability of placing candidate c at rank j over a ranking distribution M. Let $\boldsymbol{P}^{\mathsf{M}}$ denote a single-voter profile consisting of only this ranking distribution M.

When $k = 1$, the REP can be reduced to solving the ESC problem under plurality or 1-approval rule.

$$\Pr(c@1 \mid \mathsf{M}) = \mathbb{E}(s(c \mid \boldsymbol{P}^{\mathsf{M}}, \text{1-approval}))$$

When $k = m$, the REP can be reduced to solving the ESC problem under veto or $(m-1)$-approval rule.

$$\Pr(c@m \mid \mathsf{M}) = 1 - \mathbb{E}(s(c \mid \boldsymbol{P}^{\mathsf{M}}, (m-1)\text{-approval}))$$

When $2 \leq k \leq m$, the REP can be reduced to solving the ESC problem twice under k-approval and $(k-1)$-approval rules.

$$\begin{aligned}\Pr(c@k \mid \mathsf{M}) = {} & \mathbb{E}(s(c \mid \boldsymbol{P}^{\mathsf{M}}, k\text{-approval})) \\ & - \mathbb{E}(s(c \mid \boldsymbol{P}^{\mathsf{M}}, (k-1)\text{-approval}))\end{aligned}$$

□

Theorem 7. *Given a partial voting profile $\boldsymbol{P}^{PO}$, a distinguished candidate c, and plurality rule $\boldsymbol{r}_m$, the ESC problem of calculating $\mathbb{E}(s(c \mid \boldsymbol{P}^{PO}, \boldsymbol{r}_m))$ is $FP^{\#P}$-complete.*

Proof. Firstly, we prove the membership of the ESC problem as a $\text{FP}^{\#P}$ problem. Consider that the REP is $\text{FP}^{\#P}$-complete over partial orders (Theorem 4), and the ESC problem can be reduced to the REP (Theorem 5) So the ESC problem is in $\text{FP}^{\#P}$ for partial voting profiles.

Secondly, we prove that the ESC problem is $\text{FP}^{\#P}$-hard, even for plurality rule, by reduction from the REP-t that is $\text{FP}^{\#P}$-hard (Lemma 1).

Let $\boldsymbol{\nu}$ denote the partial order of the REP-t problem. Recall that the REP-t problem aims to calculate $\Pr(c@1 \mid \boldsymbol{\nu})$ for a given item c. Let $\boldsymbol{P}^{\boldsymbol{\nu}}$ denote a voting profile consisting of just this partial order $\boldsymbol{\nu}$. The answer to the REP-t problem is the same as the answer to the corresponding ESC problem, i.e., $\Pr(c@1 \mid \boldsymbol{\nu}) = \mathbb{E}(s(c \mid \boldsymbol{P}^{\boldsymbol{\nu}}, \text{plurality}))$. So the ESC problem is $\text{FP}^{\#P}$-hard, even for plurality voting rule.

In conclusion, the ESC problem is $\text{FP}^{\#P}$-complete, under plurality rule. □

Theorem 8. *Given a partial voting profile $\boldsymbol{P}^{PO}$, a distinguished candidate c, and veto rule $\boldsymbol{r}_m$, the ESC problem of calculating $\mathbb{E}(s(c \mid \boldsymbol{P}^{PO}, \boldsymbol{r}_m))$ is $FP^{\#P}$-complete.*

Proof. This proof adopts the same approach as the proof of Theorem 7.

Firstly, the membership proof that the ESC is in $\text{FP}^{\#P}$ is based on the conclusions that the REP is $\text{FP}^{\#P}$-complete over partial orders (Theorem 4), and that the ESC can be reduced to the REP (Theorem 5) So the ESC is in $\text{FP}^{\#P}$ for partial voting profiles.

Secondly, we prove that the ESC is $\text{FP}^{\#P}$-hard, under veto voting rule, by reduction from the REP-b that is $\text{FP}^{\#P}$-hard (Lemma 2).

Let $\boldsymbol{\nu}$ denote the partial order of the REP-b problem. Recall that the REP-b problem aims to calculate $\Pr(c@m \mid \boldsymbol{\nu})$ for a given item c. Let $\boldsymbol{P}^{\boldsymbol{\nu}}$ denote a voting profile consisting of just this partial order $\boldsymbol{\nu}$. The answer to the ESC indirectly solves the REP-b, i.e., $\Pr(c@m \mid \boldsymbol{\nu}) = 1 - \mathbb{E}(s(c \mid \boldsymbol{P}^{\boldsymbol{\nu}}, \text{veto}))$. So the ESC problem is $\text{FP}^{\#P}$-hard under veto rule.

In conclusion, the ESC is $\text{FP}^{\#P}$-complete, under veto rule. □

Theorem 9. *Given a partial voting profile $\boldsymbol{P}^{PO}$, a distinguished candidate c, and k-approval rule $\boldsymbol{r}_m$, the ESC problem of calculating $\mathbb{E}(s(c \mid \boldsymbol{P}^{PO}, \boldsymbol{r}_m))$ is $FP^{\#P}$-complete.*

Proof. First the proof that the Expected Score Computation (ESC) is in $\text{FP}^{\#P}$ is the same as the proof of Theorem 7. Now we prove that the ESC problem is $\text{FP}^{\#P}$-hard, under k-approval rule $\boldsymbol{r}_m$, by reduction from the REP-t problem that is $\text{FP}^{\#P}$-hard (Lemma 1).

Let $\boldsymbol{\nu}$ denote the partial order of the REP-t problem. Recall that the REP-t problem aims to calculate $\Pr(c@1 \mid \boldsymbol{\nu})$ for a given item c. Let $\boldsymbol{\nu}_+$ denote a new partial order by inserting $(k-1)$ ordered items $d_1 \succ \dots \succ d_{k-1}$ into $\boldsymbol{\nu}$ such that item d_{k-1} is preferred to every item in $\boldsymbol{\nu}$. Such placement of items $\{d_1, \dots, d_{k-1}\}$ is to guarantee that all linear extensions of $\boldsymbol{\nu}_+$ start with $d_1 \succ \dots \succ d_{k-1}$ and these linear extensions will be precisely the linear extensions of $\boldsymbol{\nu}$ after removing $\{d_1, \dots, d_{k-1}\}$.

Let $\boldsymbol{P}^{\boldsymbol{\nu}_+}$ denote a voting profile consisting of just this partial order $\boldsymbol{\nu}_+$. The answer to the ESC problem for item c is $\mathbb{E}(s(c \mid \boldsymbol{P}^{\boldsymbol{\nu}_+}, k\text{-approval}))$. Since there is only one partial order $\boldsymbol{\nu}_+$ in the voting profile, $\mathbb{E}(s(c \mid \boldsymbol{P}^{\boldsymbol{\nu}_+}, k\text{-approval})) = \sum_{j=1}^{k} \Pr(c@j \mid \boldsymbol{\nu}_+)$. Recall that the first $(k-1)$ items in any linear extension of $\boldsymbol{\nu}_+$ starts with $d_1 \succ \dots \succ d_{k-1}$, so $\forall 1 \leq j \leq (k-1), \Pr(c@j \mid \boldsymbol{\nu}_+) = 0$, which leads to $\mathbb{E}(s(c \mid \boldsymbol{P}^{\boldsymbol{\nu}_+}, k\text{-approval})) = \Pr(c@k \mid \boldsymbol{\nu}_+)$. Since $\boldsymbol{\nu}_+$ is constructed by inserting $(k-1)$ items before items in $\boldsymbol{\nu}$, $\Pr(c@k \mid \boldsymbol{\nu}_+) = \Pr(c@1 \mid \boldsymbol{\nu})$. So $\mathbb{E}(s(c \mid \boldsymbol{P}^{\boldsymbol{\nu}_+}, k\text{-approval})) = \Pr(c@1 \mid \boldsymbol{\nu})$.

Algorithm 5 REP solver for rRSM

Input: Item c, rank k, $\mathsf{rRSM}(\boldsymbol{\sigma}, \Pi)$
Output: $\Pr(c@k \mid \boldsymbol{\sigma}, \Pi)$

1: $\alpha_0 := |\{\sigma_i | \sigma_i \succ_{\boldsymbol{\sigma}} c\}|$, $\beta_0 := |\{\sigma_i \mid c \succ_{\boldsymbol{\sigma}} \sigma_i\}|$
2: $\mathcal{P}_0 := \{\langle \alpha_0, \beta_0 \rangle\}$ and $q_0(\langle \alpha_0, \beta_0 \rangle) := 1$
3: **for** $i = 1, \ldots, (k-1)$ **do**
4: $\quad \mathcal{P}_i := \{\}$
5: $\quad$ **for** $\langle \alpha, \beta \rangle \in \mathcal{P}_{i-1}$ **do**
6: $\quad\quad$ **if** $\alpha > 0$ **then**
7: $\quad\quad\quad$ Generate a new state $\langle \alpha', \beta' \rangle = \langle \alpha - 1, \beta \rangle$.
8: $\quad\quad\quad$ **if** $\langle \alpha', \beta' \rangle \notin \mathcal{P}_i$ **then**
9: $\quad\quad\quad\quad \mathcal{P}_i.add(\langle \alpha', \beta' \rangle)$
10: $\quad\quad\quad\quad q_i(\langle \alpha', \beta' \rangle) := 0$
11: $\quad\quad\quad$ **end if**
12: $\quad\quad\quad q_i(\langle \alpha', \beta' \rangle) \mathrel{+}= q_{i-1}(\langle \alpha, \beta \rangle) \cdot \sum_{j=1}^{\alpha} \Pi(i, j)$
13: $\quad\quad$ **end if**
14: $\quad\quad$ **if** $\beta > 0$ **then**
15: $\quad\quad\quad$ Generate a new state $\langle \alpha', \beta' \rangle = \langle \alpha, \beta - 1 \rangle$.
16: $\quad\quad\quad$ **if** $\langle \alpha', \beta' \rangle \notin \mathcal{P}_i$ **then**
17: $\quad\quad\quad\quad \mathcal{P}_i.add(\langle \alpha', \beta' \rangle)$
18: $\quad\quad\quad\quad q_i(\langle \alpha', \beta' \rangle) := 0$
19: $\quad\quad\quad$ **end if**
20: $\quad\quad\quad q_i(\langle \alpha', \beta' \rangle) \mathrel{+}= q_{i-1}(\langle \alpha, \beta \rangle) \cdot \sum_{j=\alpha+2}^{\alpha+1+\beta} \Pi(i, j)$
21: $\quad\quad$ **end if**
22: $\quad$ **end for**
23: **end for**
24: **return** $\sum_{\langle \alpha, \beta \rangle \in \mathcal{P}_{k-1}} q_{k-1}(\langle \alpha, \beta \rangle) \cdot \Pi(k, \alpha + 1)$

The answer to the REP-t problem has been reduced to the ESC problem. So the ESC problem is $\mathsf{FP}^{\#P}$-hard, under k-approval rule.

In conclusion, the ESC problem is $\mathsf{FP}^{\#P}$-complete, under k-approval rule. □

B TRACTABILITY OVER RSM VOTING PROFILES

RSM [4] denoted by $\mathsf{RSM}(\boldsymbol{\sigma}, \Pi, p)$ is another generalization of the Mallows. It is parameterized by a reference ranking $\boldsymbol{\sigma}$, a probability function Π where $\Pi(i, j)$ is the probability of the j^{th} item selected at step i, and a probability function $p : \{1, ..., m-1\} \to [0, 1]$ where $p(i)$ is the probability that the i^{th} selected item preferred to the remaining items. In contrast to the RIM that randomizes the item insertion position, the RSM randomized the item insertion order. In this paper, we use RSM as a ranking model, i.e., $p \equiv 1$ such that it only outputs rankings. This ranking version is named rRSM and denoted by $\mathsf{rRSM}(\boldsymbol{\sigma}, \Pi)$.

Example 6. $\mathsf{rRSM}(\boldsymbol{\sigma}, \Pi)$ *with* $\boldsymbol{\sigma} = \langle a, b, c \rangle$ *generates* $\boldsymbol{\tau}=\langle c, a, b \rangle$ *as follows. Initialize* $\boldsymbol{\tau}_0=\langle\rangle$. *When* $i = 1$, $\boldsymbol{\tau}_1=\langle c \rangle$ *by selecting* c *with probability* $\Pi(1, 3)$, *which making the remaining* $\boldsymbol{\sigma} = \langle a, b \rangle$. *When* $i = 2$, $\boldsymbol{\tau}_2=\langle c, a \rangle$ *by selecting* a *with probability* $\Pi(2, 1)$, *which making the remaining* $\boldsymbol{\sigma} = \langle b \rangle$. *When* $i = 3$, $\boldsymbol{\tau}=\langle c, a, b \rangle$ *by selecting* b *with probability* $\Pi(3, 1)$. *Overall,* $\Pr(\boldsymbol{\tau} \mid \boldsymbol{\sigma}, \Pi)=\Pi(1, 3) \cdot \Pi(2, 1) \cdot \Pi(3, 1)$.

Theorem 16. *Given a positional scoring rule* $\boldsymbol{r}_m$, *a RSM voting profile* $\boldsymbol{P}^{\mathsf{rRSM}} = (\mathsf{rRSM}_1, \ldots, \mathsf{rRSM}_n)$, *and candidate* w, *determining* $w \in MEW(\boldsymbol{r}_m, \boldsymbol{P}^{\mathsf{rRSM}})$ *is in* $O(nm^4)$.

Proof. Given any $\mathsf{rRSM} \in \boldsymbol{P}^{\mathsf{rRSM}}$, candidate c, and rank j, the $Pr(c@j \mid \mathsf{rRSM})$ is computed by Algorithm 5 in a fashion that is similar to Algorithm 3. This is also a Dynamic Programming (DP) approach. The states are in the form of $\langle \alpha, \beta \rangle$, where α is the number of items before c, and β is that after c in the remaining $\boldsymbol{\sigma}$. For state $\langle \alpha, \beta \rangle$, there are $(\alpha + 1 + \beta)$ items in the remaining $\boldsymbol{\sigma}$. Algorithm 5 only runs up to $i = (k-1)$ (in line 3), since item c must be selected at step k and the rest steps do not change the rank of c any more. Each step i generates at most $(i + 1)$ states, corresponding to $[0, \ldots, i]$ items are selected from items before c in the original $\boldsymbol{\sigma}$. The complexity of Algorithm 5 is bounded by $O(m^2)$. It takes $O(nm^4)$ to obtain the expected scores of all candidates and to determine the MEW.

□

Example 7. *Let* $\mathsf{rRSM}(\boldsymbol{\sigma}, \Pi)$ *denote a RSM where* $\boldsymbol{\sigma} = \langle \sigma_1, \sigma_2, \sigma_3, \sigma_4 \rangle$, *and* $\Pi = [[0.1, 0.3, 0.4, 0.2], [0.2, 0.5, 0.3], [0.3, 0.7], [1]]$. *Assume we are interested in* $\Pr(\sigma_2@3 \mid \boldsymbol{\sigma}, \Pi)$, *the probability of* $\mathsf{rRSM}(\boldsymbol{\sigma}, \Pi)$ *placing* σ_2 *at rank* 3.

- *Before running RSM, there is* $\alpha_0 = 1$ *item before* σ_2 *and* $\beta_0 = 2$ *items after* σ_2 *in* $\boldsymbol{\sigma}$. *So the initial state is* $\langle \alpha_0, \beta_0 \rangle = \langle 1, 2 \rangle$, *and* $q_0(\langle 1, 2 \rangle) = 1$.
- *At step* $i = 1$, *the selected item can be either from* $\{\sigma_1\}$ *or* $\{\sigma_3, \sigma_4\}$. *So two new states are generated here.*
  - *The* σ_1 *is selected with probability* $\Pi(1, 1) = 0.1$, *which generates a new state* $\langle 0, 2 \rangle$, *and* $q_1(\langle 0, 2 \rangle) = q_0(\langle 1, 2 \rangle) \cdot \Pi(1, 1) = 0.1$.
  - *An item* $\sigma \in \{\sigma_3, \sigma_4\}$ *is selected with probability* $\Pi(1, 3) + \Pi(1, 4) = 0.6$, *which generates a new state* $\langle 1, 1 \rangle$, *and* $q_1(\langle 1, 1 \rangle) = q_0(\langle 1, 2 \rangle) \cdot 0.6 = 0.6$.

  So $\mathcal{P}_1 = \{\langle 0, 2 \rangle, \langle 1, 1 \rangle\}$, $q_1 = \{\langle 0, 2 \rangle \mapsto 0.1, \langle 1, 1 \rangle \mapsto 0.6\}$.
- *At step* $i = 2$, *iterate states in* $\mathcal{P}_1$.
  - *For state* $\langle 0, 2 \rangle$, *the selected item must be from the last two items in remaining reference ranking. A new state* $\langle 0, 1 \rangle$ *is generated with probability* $\Pi(2, 2) + \Pi(2, 3) = 0.8$.
  - *For state* $\langle 1, 1 \rangle$, *the selected item is either the first or last item in remaining reference ranking. A new state* $\langle 0, 1 \rangle$ *is generated with probability* $\Pi(2, 1) = 0.1$, *and another state* $\langle 1, 0 \rangle$ *is generated with probability* $\Pi(2, 3) = 0.3$.

  So $\mathcal{P}_2 = \{\langle 0, 1 \rangle, \langle 1, 0 \rangle\}$ *and*

  □ $q_2(\langle 0, 1 \rangle) = q_1(\langle 0, 2 \rangle) \cdot 0.8 + q_1(\langle 1, 1 \rangle) \cdot 0.1 = 0.1 \cdot 0.8 + 0.6 \cdot 0.1 = 0.14$

  □ $q_2(\langle 1, 0 \rangle) = q_1(\langle 1, 1 \rangle) \cdot 0.3 = 0.6 \cdot 0.3 = 0.18$
- *At step* $i = 3$, *item* σ_2 *must be selected to meet the requirement. For each state* $\langle \alpha, \beta \rangle \in \mathcal{P}_2$, *the rank of* σ_2 *is* $(\alpha+1)$ *in the corresponding remaining ranking. So* $\Pr(\sigma_2@3 | \boldsymbol{\sigma}, \Pi) = q_2(\langle 0, 1 \rangle) \cdot \Pi(3, 1) + q_2(\langle 1, 0 \rangle) \cdot \Pi(3, 2) = 0.14 \cdot 0.3 + 0.18 \cdot 0.7 = 0.168$.