A SIMPSON CORRESPONDENCE FOR ABELIAN VARIETIES IN POSITIVE CHARACTERISTIC



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The Stack of Higgs Bundles on X'

Denote by $\mathbf{Higgs}' := \mathbf{Higgs}_r(X'/k)$ the stack of *Higgs bundles* of rank r on X'.

Higgs bundles

A *Higgs bundle* of rank r on $X'_T := X' \times_k T$ is a pair (E, θ) where E is a locally free $\mathfrak{O}_{X'_T}$ -module of rank r, and

$$\theta: E \longrightarrow E \otimes_{\mathbb{O}_{X_{\tau}'}} \Omega^1_{X_{\tau}'/T}$$

is an \mathbb{O}_{X_T} -module morphism, such that the composition

$$E \xrightarrow{\theta} E \otimes \Omega^1_{X'_T/T} \xrightarrow{\theta \otimes \mathrm{id}} E \otimes \Omega^1_{X'_T/T} \otimes \Omega^1_{X'_T/T} \xrightarrow{\mathrm{id} \otimes \wedge} E \otimes \Omega^2_{X'_T/T}$$

is zero. Such a θ is called a *Higgs field*. Equivalently, a Higgs bundle is an $\left(\operatorname{Sym}^{\bullet}(\Omega^{1}_{X'_{T}/T})^{\vee}\right)$ -module, or even an $\mathbb{O}_{\operatorname{T}^{*}(X'_{T}/T)}$ -module, that is locally free as an $\mathbb{O}_{X'_{T}}$ -module.

Characteristic Polynomial

Let $\varphi : E \to E \otimes K$ be an $\mathbb{O}_{X'}$ -module morphism, where E (resp. K) is an $\mathbb{O}_{X'}$ -module locally free of rank r (resp. d). Then φ has a well defined characteristic polynomial

$$\chi_{\varphi}(t) = t^r - a_1 t^{r-1} + \dots + (-1)^r a_r$$
, with $a_i \in \Gamma(X, \operatorname{Sym}^i K)$.

This applies especially to any Higgs field θ where $K = \Omega^1_{X'_T/T}$.

Spectral Cover

Denote by $\pi_T : T^*(X_T'/T) \to X_T'$ the projection. Let $\lambda \in \Gamma(T^*(X_T'/T), \pi_T^*\Omega^1_{X_T'/T})$ be the *tautological* 1-*form*. Given a Higgs bundle (E, θ) , the vanishing locus of the global section

$$\chi := \chi_{\theta}(\lambda) \in \Gamma\left(\mathrm{T}^*(X_T'/T), \operatorname{Sym}^r\Omega^1_{X_T'/T}\right)$$

defines a closed subscheme Z_{χ} of $T^*(X_T'/T)$.

Proposition. (Cayley-Hamilton). The Higgs bundle (E, θ) as above, viewed as an $\mathcal{O}_{T^*(X'_T/T)}$ -module, is supported on Z_{χ} .

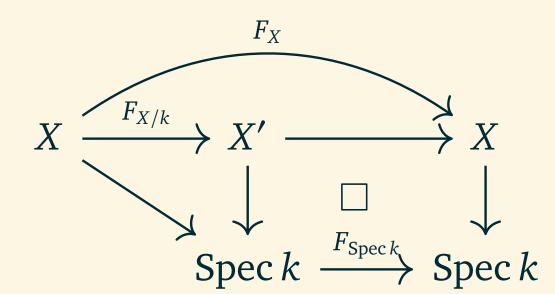
Proposition. The map $Z_{\chi} \to X_T'$ is a finite map hence proper. However in general it is NOT flat.

The map $Z_{\chi} \to X_T'$ is called the *spectral cover* of X_T' .

Question: Which polynomials can appear as the characteristic polynomial some Higgs field?

Setup

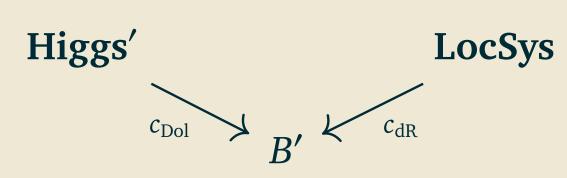
Let X/k be an d-dimensional $abelian\ variety$ over an algebraically closed field k of characteristic p > 0. Assume T is an arbitrary object in the category (Sch/k). Denote by $F_{X/k}$ the relative Frobenius morphism.



In this case, $F_{X/k}$ is locally free of rank p^d . Moreover, for any T/k, the relative Frobenius $F_{X_T/T}$ is the pullback of $F_{X/k}$.

Main Theorem

We have two stacks Higgs' and LocSys over the Hitchin base B':



There is an étale morphism $U \rightarrow B$, such that

 $\mathbf{Higgs'}|_{U} \simeq \mathbf{LocSys}|_{U}.$

Idea of Proof

The **Higgs**' side is the same as \mathfrak{G} -modules on the cotangent bundle of X' with certain support (the spectral cover); The **LocSys** side is the same as $\mathfrak{D}_{X/k}$ -module on the cotangent bundle of X' with certain support. Noting that $\mathfrak{D}_{X/k}$ is an Azumaya algebra, we can use the *Morita Theory* to establish the equivalence of both sides. So the aim is to look for splittings of the Azumaya algebra $\mathfrak{D}_{X/k}$.

The Hitchin Base

$$B' := B_r(X'/k) := \operatorname{Spec}\left(\operatorname{Sym}_k^{\bullet}\left(\bigoplus_{i=1}^r \Gamma(X',\operatorname{Sym}^i\Omega^1_{X'/k})\right)\right).$$

The maps c_{dR} and c_{Dol}

 $c_{\mathrm{Dol}}:$ Higgs bundle $(E,\theta)\longmapsto$ coefficients of χ_{θ} $c_{\mathrm{dR}}:$ Local system $(E,\nabla)\longmapsto$ coefficients of $(\chi_{\psi_{\nabla}'})^{1/p^d}$

The Stack of Local System on X

Denote by LocSys := LocSys_r(X/k) the stack of local systems of rank r on X.

The Sheaf of Crystalline Differential Operators

Let $D_{X/k}$ be the sheaf of crystalline differential operators, which is the "enveloping algebroid of the tangent Lie algebroid" of X: to an affine $U \to X$, it assigns the algebra that is generated over $\mathcal{O}_X(U)$ from the $\mathcal{O}_X(U)$ -module $(\Omega^1_{X/k})^{\vee}(U)$ of k-derivations of $\mathcal{O}_X(U)$, subject to the relations

$$v_1 \cdot v_2 - v_2 \cdot v_1 = [v_1, v_2], \text{ and } v_1 \cdot f - f \cdot v_1 = v_1(f),$$

for all $v_1, v_2 \in (\Omega^1_{X/k})^{\vee}(U)$ and $f \in \mathcal{O}_X(U)$.

Theorem. (R. Bezrukavnikov, I. Mirkovic, and D. Rumynin). The direct image $F_{X/k,*}D_{X/k}$ of $D_{X/k}$ under the relative Frobenius $F_{X/k}$ is an Azumaya algebra over its center $Z(F_{X/k,*}D_{X/k})$ of rank p^{2d} . Moreover,

$$\Psi: \operatorname{Sym}^{\bullet}(\Omega^{1}_{X'/k})^{\vee} \xrightarrow{\sim} Z(F_{X/k,*}D_{X/k}).$$

In other words, it defines an *Azumaya algebra* $\mathfrak{D}_{X/k}$ over the cotangent bundle $T^*(X'/k)$.

Local Systems and *D*-modules

Denote by $\tau: X_T \to X$ and $f_T: X_T \to T$ the projections.

A *local system* of rank r is a (left) $\tau^*(D_{X/k})$ -module that is locally free as an \mathbb{O}_{X_T} -module of rank r. This is equivalent to a pair (E, ∇) consisting of a locally free \mathbb{O}_{X_T} -module of rank r and an *integrable* T-connection ∇ , i.e., an $f_T^{-1}\mathbb{O}_T$ -module homomorphism

$$abla: E \longrightarrow E \otimes_{\mathbb{O}_{X_T}} \Omega^1_{X_T/T},$$

satisfying the Leibniz rule, whose curvature $K_{\nabla}: E \to E \otimes \Omega^2_{X_T/T}$ is zero.

The *p*-curvature

Let (E, ∇) be a local system of rank r on X_T/T . Via the map Ψ in BMR's theorem, one obtains a Higgs field

$$\psi':=\psi'_
abla:F_{X_T/T,st}E\longrightarrow F_{X_T/T,st}E\otimes\Omega^1_{X_T'/T}$$

on the rank $p^d r$ locally free sheaf $F_{X/k,*}E$. It is called the p-curvature of ∇ .

Theorem. (Laszlo-Pauly, Groechenig) The characteristic polynomial of ψ' is of the form $(\chi')^{p^d}$, where

$$\chi' = t^r - a_1 t^{r-1} + \cdots (-1)^r a_r$$
, and $a_i \in \Gamma(X, \operatorname{Sym}^i \Omega^1_{X_T/T})$