

1. Motivation and overview of the pro-étale for schemes.
(we ~~don't~~ won't use it, but the definitions are simpler)

Def $A \rightarrow B$ morphism of rings is weakly étale if both $A \rightarrow B$, $B \otimes B \rightarrow B$ are flat

$A \rightarrow B$ ind-étale if $B = \varprojlim B_i$, $A \rightarrow A_i$ étale

Prop. ~~if ind-étale~~ $A \rightarrow B$

- 1) ind-étale \Rightarrow weakly étale
- 2) weakly étale \Rightarrow cotangent map vanishes \Rightarrow formally étale
- 3) weakly ét. + finitely presented \Rightarrow étale

Thm $f: A \rightarrow B$ weakly étale

Then $\exists g: B \rightarrow C$ faithfully flat ind-étale: $g \circ f: A \rightarrow C$ ind-étale

Def $f: Y \rightarrow X$ of schemes weakly étale if f is flat and $\Delta_f: Y \rightarrow Y \times_X Y$ is flat

$X_{\text{proét}}$ = weakly étale schemes / X
 $\{X_i \rightarrow Y\}$ covers if it is a cover in fpqc topo.

e.g. x_1, \dots, x_n geom. pts on X

then $(\sqcup \text{Spec}(\mathcal{O}_{X, x_i}^{\text{sh}})) \sqcup (X \setminus \{x_1, \dots, x_n\}) \rightarrow X$ is cover in $X_{\text{proét}}$

but for $X = \mathbb{A}^1_{\mathbb{Z}}$, $x_1 = 0, x_2 = 1$ ~~not~~ if we take

$\sqcup_p \text{Spec}(\mathbb{Z}_p^{\text{sh}}) \rightarrow \text{Spec}(\mathbb{Z})$ not (not qc)

Q: Why weakly-étale? A: Local on the source and target

while being pro-étale is not even Zariski local on the source

(i.e. gluing of two $\varprojlim X_{1,i} \rightarrow X_1$, $\varprojlim X_{2,i} \rightarrow X_2$ might not have presentation as \varprojlim)

Advantage of $X_{\text{proét}}$ over $X_{\text{ét}}$:

$X_{\text{proét}}$ is replete (def. omitted, it has to do with some limits of surj. maps)

\Rightarrow Prop. ~~if $\{X_i\}$ is a directed system of schemes~~ $F_{n+2} \rightarrow F_{n+1} \rightarrow F_n \rightarrow \dots$ (like $(F, \mathbb{N}^{\text{op}} \rightarrow \text{Ab}(X_{\text{proét}}))$)
s.t. $\varprojlim F_n \rightarrow F_n$ surjective, then $\varprojlim F_n \cong \varprojlim F_n$

Motivation:

Outcome:

Recall $H^i(X_{\text{ét}}, \mathbb{Z}/\ell)$ or
but $H^i(X_{\text{ét}}, \mathbb{Q}_\ell) = \varprojlim H^i(X_{\text{ét}}, \mathbb{Z}/\ell^n) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$
artificial/ad hoc, but works fine

$D_c^b(X, \mathbb{Q}_\ell)$ also ad hoc (Deligne) and complicated

Let X be a V -stack and \mathcal{O}_X a constant sheaf on X . Then $H^i(X, \mathcal{O}_X) = H^i(X, \mathbb{Z})$

Natural map $v: X_{\text{proet}} \rightarrow X_{\text{et}}$

$\varprojlim F \in \text{Ab}(X_{\text{et}})$, $U = \varinjlim_{i \in I} U_i \in X_{\text{proet}}$ qcqs (ie. $|U|$ qcqs)

Then $\forall i: H^i(U, v^* F) = \varinjlim H^i(U_i, F)$

$\varprojlim \bullet \forall F \in \text{Ab}(X_{\text{et}})$, $F \rightarrow Rv_* v^* F$ is iso-

$\therefore f: X \rightarrow Y$ qcqs morphism. Then

$\forall F \in \text{Ab}(X_{\text{et}})$ $v^* Y_{\text{et}}^* F \rightarrow Rf_{\text{proet}*} v^* F$ is iso-

From now: X loc. noeth. adic / $\text{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ or (K, K^+) complete non-arch-field

Def. (uncompl.) sheaf $\mathcal{O}_X = v^* \mathcal{O}_{X_{\text{et}}}$, $\mathcal{O}_X^+ = v^* \mathcal{O}_{X_{\text{et}}}^+$

ii) $\hat{\mathcal{O}}_X^+ = \varprojlim \mathcal{O}_X^+ / p^n$, $\hat{\mathcal{O}}_X^+[1/p]$

(The sections of the above sheaves are only understood over so-called affinoid perfectoid)

Def. $U \in X_{\text{proet}}$

i) U is affinoid perfectoid if $U \cong \varinjlim_{i \in I} U_i$, $U_i = \text{Spa}(R_i, R_i^+)$, s.t. if $R^+ = \varinjlim R_i^+$ p -adic, $R = R^+[1/p]$,

then (R^+, R) is a perfectoid $(\mathbb{Q}_p, \mathbb{Z}/p)$ -algebra

ii) U is perfectoid if it has open cover by affinoid perfectoid

(this makes sense, as qcqs open subset of U affoid give rise to an object in X_{proet})

Prop. The set $U \in X_{\text{proet}}$ which are affinoid perfectoid forms a basis of the topology.

Sketch of pp for X -smooth:

• Fact: smoothness $\Rightarrow X$ admits locally an etale map to T^n , where $T^n = \text{Spa}(K\langle T_1^{\pm 1/p}, \dots, T_n^{\pm 1/p} \rangle, K^+ \langle T_1^{\pm 1/p}, \dots, T_n^{\pm 1/p} \rangle)$ \rightsquigarrow reduces to T^n

• For T^n consider $\hat{T}^n \in X_{\text{proet}}$

"perfectoid torus" $\rightarrow \hat{T}^n = \varprojlim_m \text{Spa}(K\langle T_1^{\pm 1/p^m}, \dots, T_n^{\pm 1/p^m} \rangle, K^+ \langle T_1^{\pm 1/p^m}, \dots, T_n^{\pm 1/p^m} \rangle)$ is affinoid perfectoid

Lm. & Thm. Let $U \in X_{\text{proet}}$ affinoid perfectoid, $U = \varinjlim_{i \in I} U_i$ as in the defn of aff. perf.

$U_i = \text{Spa}(R_i, R_i^+)$. Then $\mathcal{O}_X^+(U) = \varinjlim R_i^+$, $\mathcal{O}_X(U) = \varinjlim R_i$, $\hat{\mathcal{O}}_X^+(U) = R^+$, $\hat{\mathcal{O}}_X(U) = R$

$R_i^+ = \varinjlim R_i^{+p\text{-adic}}$
 $R = R^+[1/p]$

$\forall i > 0$ $H^i(U, \mathcal{O}_X) = H^i(U, \hat{\mathcal{O}}_X) = 0$

and $\forall i > 0$ $H^i(U, \mathcal{O}_X^+)$, $H^i(U, \hat{\mathcal{O}}_X^+)$ are almost zero, ie. killed by $\dots = v \circ c_{i,j}$

