Pro-étale top. for adic spaces — Marcin Lara

1. Motivation and overview of the pro-étale for schemes.
(we ~~don't~~ won't use it, but the definitions are simpler)

Def $A \to B$ morphism of rings is <u>weakly étale</u> if both $A \to B$, $B \otimes B \to B$ are flat

$A \to B$ <u>ind-étale</u> if $B = \varinjlim_i B_i$, $A \to A_i$ étale

Prop. ~~1) ind-étale~~ $A \to B$
1) ind-étale $\Rightarrow$ weakly étale
3) weakly ét. + finitely presented $\Rightarrow$ étale
2) weakly étale $\Rightarrow$ cotangent cpx vanishes $\Rightarrow$ formally étale

Thm $f: A \to B$ weakly étale
Then $\exists g: B \to C$ faith. flat ind-étale : $g \circ f: A \to C$ ind-étale

Def $f: Y \to X$ of schemes <u>weakly-étale</u> if f is flat and $\Delta_f: Y \to Y \times_X Y$ is flat

$X_{proét}$ = weakly étale schemes $/X$
$\{Y_i \to Y\}$ covers if it is a cover in fpqc topo.

e.g. $x_1, \dots, x_n$ closed geom. pts on X

then $\left(\bigsqcup_i \operatorname{Spec}(\mathcal{O}^{sh}_{X,x_i})\right) \sqcup (X \setminus \{x_1, \dots, x_n\}) \to X$ cover in $X_{proét}$

but ~~for $x_1, x_2, x_3, \dots$ not if we take~~

$\bigsqcup_p \operatorname{Spec}(\mathbb{Z}_p^{sh}) \to \operatorname{Spec}(\mathbb{Z})$ <u>not</u> (not <u>qc</u>)

Q: Why weakly-étale? A: Local on the source an target while being pro-étale is not even Zariski local on the source
(i.e. gluing of two $\varprojlim X_{1,i} \to X_1$, $\varprojlim X_{2,j} \to X_2$ ... might not have presentation as $\varprojlim$)

Advantage of $X_{proét}$ over $X_{ét}$:

$X_{proét}$ is <u>replete</u> (def. omitted, it has to do with surjectivity of limits of surj. maps

$\Rightarrow$ **Prop.** ~~If~~ ... $F_{n+2} \to F_{n+1} \to F_n \to \dots$ ($F: \mathbb{N}^{op} \to Ab(X_{proét})$)
s.t. $\forall_n F_{n+1} \twoheadrightarrow F_n$ surjective, then $\varprojlim_n F_n \simeq R\varprojlim_n F_n$

~~Motivation~~:
Outcome: Recall $H^i(X_{ét}, \mathbb{Z}/\ell)$ ok
but $H^i(X_{ét}, \overline{\mathbb{Q}}_\ell) = \varprojlim H^i(X_{ét}, \mathbb{Z}/\ell^n\mathbb{Z}) \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell$
artificial / ad hoc, but works fine

$D^b_c(X, \overline{\mathbb{Q}}_\ell)$ also ad hoc (Deligne) and complicated

[illegible]

Rmk: "following work well for: - locally noeth - perfectoid."

→ locally noeth. adic sp. / K = locally Spa(A,A^+), A - strongly Noeth Tate k-alg

2. Étale and proétale for adic spaces

~~(A,A^+) 2 strongly noetherian Tate k-algebra~~

Def i) $(R,R^+)\to(S,S^+)$ of affinoid k-alg. is called ~~ét~~ finite étale if S/R f.ét. with the induced top., and $S^+ =$ int. cl. $(R^+\subset S)$

ii) $X\to Y$ adic sp. is finite étale if
∃ cover of Y by open affinoids $V\subset Y$ s.t. $f^{-1}(V)$ is open affinoid and
$$(\mathcal{O}_Y(V),\mathcal{O}_Y^+(V))\to(\mathcal{O}_X(U),\mathcal{O}_X^+(U)) \text{ is f.ét.}$$

iii) $f: X\to Y$ étale if $\forall_{x\in X}\ \exists_{x\in} U\subset X$ open, $f(x)\in V\subset Y$ open, W

$$\begin{tikzcd} U \arrow[r, hook, "\text{open embedding}"] \arrow[dr, "f|_U"'] & W \arrow[d, "\text{f. et.}"] \\ & V \end{tikzcd}$$

Prop $X\to Y\leftarrow Z$ X, Y, Z loc. noeth., $X\to Y$ locally of f.t.
$\Rightarrow X\times_Y Z$ exists as an adic space

"Recall" **Prop.** $X\to Y\leftarrow Z$ perfectoid spaces (over K), then
$X\times_Y Z$ exists as adic sp. and is perfectoid

not always true for general adic spaces, ~~but it works for locally of finite type~~

Lm.
$$\begin{tikzcd} & Z \arrow[d] \\ X \arrow[r, "\text{f.ét/ét}"] & Y \end{tikzcd}$$
Z ← perfectoid, X adic, Y adic, /K

$\Rightarrow$ $X\times_Y Z$ exists in adic
$X\times_Y Z$ is perfectoid
$X\times_Y Z\to Z$ f.ét / ét.

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pro-X_{et} := cat of pro-objects assoc. to X_{et}
= objects as functors $F: I\to X_{et}$, i.e. "$\varprojlim_{i\in I} U_i$" $U_i\in X_{et}$
(I small, cofiltered index cat)

morphisms $\mathrm{Hom}(``\varprojlim_{i\in I} U_i", ``\varprojlim_{j\in J} V_j") = \varprojlim_{j\in J}\varinjlim_{i\in I}\mathrm{Hom}(U_i,V_j)$

Obs Cofiltered limits exist in pro-X_{et} (combining double lim into single one)

associate top sp. $U = ``\varprojlim_{i\in I} U_i"$, [illegible]
$|U| = \varprojlim_{i\in I}|U_i|$

Def $U, V\in$ pro-X_{et}
$U\to V$ ét / f.ét if [illegible] $\exists\, U_0\to V_0$ ét / f.ét, $V\to V_0$: $U = U_0\times_{V_0} V$

$U\in$ pro-X_{et} is pro-étale over X if U is isomorphic in pro-X_{et} to "$\varprojlim_{i\in I} U_i$" with $\forall_{i,j\gg 0}\ U_i\to U_j$ finite étale surjective.

$X_{proét}$ = full subcat of $\{U\in \text{pro-}X_{et} \mid U \text{ is pro-étale over } X\}$

coverings: $\{f_i: U_i\to U\}$
[illegible] s.t. one can write $U_i\to U$ as $U_i = \varprojlim_{\mu<\lambda} U_\mu$
$U_\mu\in X_{proét}$, λ - some ordinal

Natural map $\nu: X_{proét} \to X_{ét}$

<u>Lm</u> $\mathcal{F} \in Ab(X_{ét})$, $U = $ "$\varprojlim_{i\in I} U_i$" $\in X_{proét}$ qcqs (i.e. $|U|$ qcqs)

Then $\forall i$ $H^i(U, \nu^*\mathcal{F}) = \varinjlim H^i(U_j, \mathcal{F})$

<u>Lm</u> • $\forall \mathcal{F} \in Ab(X_{ét})$, $\mathcal{F} \to R\nu_*\nu^*\mathcal{F}$ is iso.

• $f: X \to Y$ qcqs morphism. Then $\forall \mathcal{F} \in Ab(X_{ét})$ $\nu_Y^* Rf_{ét*}\mathcal{F} \to Rf_{proét*}\nu_X^*\mathcal{F}$ is iso.

From now: X loc. noeth. adic / $Spa(\mathbb{Q}_p, \mathbb{Z}_p)$ or (K, K^{+}) ← complete non-arch-field

<u>Def</u>: i) (uncompl.) struct. sheaf $\mathcal{O}_X = \nu^*\mathcal{O}_{X_{ét}}$, $\mathcal{O}_X^+ = \nu^*\mathcal{O}^+_{X_{ét}}$

ii) $\hat{\mathcal{O}}_X^+ = \varprojlim \mathcal{O}_X^+/p^n$, $\hat{\mathcal{O}}_X^+[\frac{1}{p}]$

(The sections of the above sheaves are easily understood over so-called affinoid perfectoids)

<u>Def</u>. $U \in X_{proét}$

i) U is <u>affinoid perfectoid</u> if $U \simeq$ "$\varprojlim_{i\in I} U_i$", $U_i = Spa(R_i, R_i^+)$,

s.t. if $R^+ = \widehat{\varinjlim R_i^+}^{p\text{-adic}}$, $R = R^+[p^{-1}]$,

then (R^+, R) is a perfectoid $(\mathbb{Q}_p, \mathbb{Z}_p)$-algebra

ii) U is <u>perfectoid</u> if it has open cover by affinoid perfectoid

(this makes sense, as qcqs open subset of $|U|$ would give rise to an object in $X_{proét}$)

<u>Prop</u> The set $U \in X_{proét}$ which are affinoid perfectoid forms a basis of the topology.

Sketch of pf for X-smooth:

• Fact: smoothness ⇒ X admits locally an étale map to $\mathbb{T}^n$, where $\mathbb{T}^n = Spa(K\langle T_1^{\pm1}, \dots, T_n^{\pm1}\rangle, K^+\langle T_1^{\pm1}, \dots, T_n^{\pm1}\rangle)$

⇝ reduces problem to $\mathbb{T}^n$

• For $\mathbb{T}^n$ consider $\tilde{\mathbb{T}}^n \in \mathbb{T}^n_{proét}$

"perfectoid torus" → $\tilde{\mathbb{T}}^n = $ "$\varprojlim_m Spa(K\langle T_1^{\pm 1/p^m}, \dots, T_n^{1/p^m}\rangle, K^+\langle T_1^{\pm 1/p^m}, \dots, T_n^{\pm 1/p^m}\rangle)$"

is affinoid perfectoid

Lm. & Thm. Let $U \in X_{proét}$ affinoid perfectoid, $U = $ "$\varprojlim_{i\in I} U_i$" ← as in the defn of aff. perf.

$U_i = Spa(R_i, R_i^+)$, $R^+ = \widehat{\varinjlim R_i^+}^{p\text{-adic}}$, $R = R^+[p^{-1}]$. Then $\mathcal{O}_X^+(U) = \varinjlim R_i^+$, $\mathcal{O}_X(U) = \varinjlim R_i$, $\hat{\mathcal{O}}_X^+(U) = R^+$, $\hat{\mathcal{O}}_X(U) = R$

& $\forall i > 0$ $H^i(U, \mathcal{O}_X) = H^i(U, \hat{\mathcal{O}}_X) = 0$

and $\forall i > 0$ $H^i(U, \mathcal{O}_X^+)$, $H^i(U, \hat{\mathcal{O}}_X^+)$ are <u>almost zero</u>, i.e. killed by