A Simpson Correspondence for Abelian Varieties in Positive Characteristic

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The Stack of Higgs Bundles on X'

Denote by $\mathbf{Higgs}' := \mathbf{Higgs}_r(X'/k)$ the stack of *Higgs bundles* of rank r on X'.

Higgs bundles

A *Higgs bundle* of rank r on $X'_T := X' \times_k T$ is a pair (E, θ) where E is a locally free $\mathcal{O}_{X'_T}$-module of rank r, and

$$\theta : E \longrightarrow E \otimes_{\mathcal{O}_{X'_T}} \Omega^1_{X'_T/T}$$

is an $\mathcal{O}_{X_T}$-module morphism, such that the composition

$$E \xrightarrow{\ \theta\ } E \otimes \Omega^1_{X'_T/T} \xrightarrow{\ \theta\otimes \mathrm{id}\ } E \otimes \Omega^1_{X'_T/T} \otimes \Omega^1_{X'_T/T} \xrightarrow{\ \mathrm{id}\otimes\wedge\ } E \otimes \Omega^2_{X'_T/T}$$

is zero. Such a θ is called a *Higgs field*. Equivalently, a Higgs bundle is an $\left(\mathrm{Sym}^\bullet(\Omega^1_{X'_T/T})^\vee\right)$-module, or even an $\mathcal{O}_{\mathrm{T}^*(X'_T/T)}$-module, that is locally free as an $\mathcal{O}_{X'_T}$-module.

Characteristic Polynomial

Let $\varphi : E \to E \otimes K$ be an $\mathcal{O}_{X'}$-module morphism, where E (resp. K) is an $\mathcal{O}_{X'}$-module locally free of rank r (resp. d). Then φ has a well defined characteristic polynomial

$$\chi_\varphi(t) = t^r - a_1 t^{r-1} + \cdots + (-1)^r a_r, \quad \text{with} \quad a_i \in \Gamma(X, \mathrm{Sym}^i K).$$

This applies especially to any Higgs field θ where $K = \Omega^1_{X'_T/T}$.

Spectral Cover

Denote by $\pi_T : \mathrm{T}^*(X'_T/T) \to X'_T$ the projection. Let $\lambda \in \Gamma(\mathrm{T}^*(X'_T/T), \pi_T^*\Omega^1_{X'_T/T})$ be the *tautological 1-form*. Given a Higgs bundle (E, θ), the vanishing locus of the global section

$$\chi := \chi_\theta(\lambda) \in \Gamma\left(\mathrm{T}^*(X'_T/T), \mathrm{Sym}^r \Omega^1_{X'_T/T}\right)$$

defines a closed subscheme Z_χ of $\mathrm{T}^*(X'_T/T)$.

Proposition. (Cayley-Hamilton). The Higgs bundle (E, θ) as above, viewed as an $\mathcal{O}_{\mathrm{T}^*(X'_T/T)}$-module, is supported on Z_χ.

Proposition. The map $Z_\chi \to X'_T$ is a finite map hence proper. However in general it is NOT flat.

The map $Z_\chi \to X'_T$ is called the *spectral cover* of X'_T.

Question: Which polynomials can appear as the characteristic polynomial some Higgs field?

Setup

Let X/k be an d-dimensional *abelian variety* over an algebraically closed field k of characteristic $p > 0$. Assume T is an arbitrary object in the category (Sch/k). Denote by $F_{X/k}$ the relative Frobenius morphism.

$$\begin{tikzcd} X \arrow[r, "F_{X/k}"] \arrow[rr, bend left, "F_X"] \arrow[dr] & X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ & \mathrm{Spec}\, k \arrow[r, "F_{\mathrm{Spec}\, k}"] & \mathrm{Spec}\, k \end{tikzcd}$$

In this case, $F_{X/k}$ is locally free of rank p^d. Moreover, for any T/k, the relative Frobenius $F_{X_T/T}$ is the pullback of $F_{X/k}$.

Main Theorem

We have two stacks $\mathbf{Higgs}'$ and $\mathbf{LocSys}$ over the Hitchin base B':

$$\begin{tikzcd} \mathbf{Higgs}' \arrow[dr, "c_{\mathrm{Dol}}"'] & & \mathbf{LocSys} \arrow[dl, "c_{\mathrm{dR}}"] \\ & B' & \end{tikzcd}$$

There is an étale morphism $U \to B$, such that

$$\mathbf{Higgs}'|_U \simeq \mathbf{LocSys}|_U.$$

Idea of Proof

The $\mathbf{Higgs}'$ side is the same as $\mathcal{O}$-modules on the cotangent bundle of X' with certain support (the spectral cover); The $\mathbf{LocSys}$ side is the same as $\mathcal{D}_{X/k}$-module on the cotangent bundle of X' with certain support. Noting that $\mathcal{D}_{X/k}$ is an Azumaya algebra, we can use the *Morita Theory* to establish the equivalence of both sides. So the aim is to look for splittings of the Azumaya algebra $\mathcal{D}_{X/k}$.

The Hitchin Base

$$B' := B_r(X'/k) := \mathrm{Spec}\left(\mathrm{Sym}^\bullet_k\left(\bigoplus_{i=1}^r \Gamma(X', \mathrm{Sym}^i \Omega^1_{X'/k})\right)\right).$$

The maps c_{dR} and c_{Dol}

$$c_{\mathrm{Dol}} : \text{Higgs bundle } (E, \theta) \longmapsto \text{coefficients of } \chi_\theta$$

$$c_{\mathrm{dR}} : \text{Local system } (E, \nabla) \longmapsto \text{coefficients of } (\chi_{\psi'_\nabla})^{1/p^d}$$

The Stack of Local System on X

Denote by $\mathbf{LocSys} := \mathbf{LocSys}_r(X/k)$ the stack of local systems of rank r on X.

The Sheaf of Crystalline Differential Operators

Let $D_{X/k}$ be *the sheaf of crystalline differential operators*, which is the "*enveloping algebroid of the tangent Lie algebroid*" of X: to an affine $U \to X$, it assigns the algebra that is generated over $\mathcal{O}_X(U)$ from the $\mathcal{O}_X(U)$-module $(\Omega^1_{X/k})^\vee(U)$ of k-derivations of $\mathcal{O}_X(U)$, subject to the relations

$$v_1 \cdot v_2 - v_2 \cdot v_1 = [v_1, v_2], \quad \text{and} \quad v_1 \cdot f - f \cdot v_1 = v_1(f),$$

for all $v_1, v_2 \in (\Omega^1_{X/k})^\vee(U)$ and $f \in \mathcal{O}_X(U)$.

Theorem. (R. Bezrukavnikov, I. Mirkovic, and D. Rumynin). The direct image $F_{X/k,*}D_{X/k}$ of $D_{X/k}$ under the relative Frobenius $F_{X/k}$ is an Azumaya algebra over its center $Z(F_{X/k,*}D_{X/k})$ of rank p^{2d}. Moreover,

$$\Psi : \mathrm{Sym}^\bullet(\Omega^1_{X'/k})^\vee \xrightarrow{\sim} Z(F_{X/k,*}D_{X/k}).$$

In other words, it defines an *Azumaya algebra* $\mathcal{D}_{X/k}$ over the cotangent bundle $\mathrm{T}^*(X'/k)$.

Local Systems and D-modules

Denote by $\tau : X_T \to X$ and $f_T : X_T \to T$ the projections.

A *local system* of rank r is a (left) $\tau^*(D_{X/k})$-module that is locally free as an $\mathcal{O}_{X_T}$-module of rank r. This is equivalent to a pair (E, ∇) consisting of a locally free $\mathcal{O}_{X_T}$-module of rank r and an *integrable* T-connection ∇, i.e., an $f_T^{-1}\mathcal{O}_T$-module homomorphism

$$\nabla : E \longrightarrow E \otimes_{\mathcal{O}_{X_T}} \Omega^1_{X_T/T},$$

satisfying the Leibniz rule, whose curvature $K_\nabla : E \to E \otimes \Omega^2_{X_T/T}$ is zero.

The p-curvature

Let (E, ∇) be a local system of rank r on X_T/T. Via the map Ψ in BMR's theorem, one obtains a Higgs field

$$\psi' := \psi'_\nabla : F_{X_T/T,*}E \longrightarrow F_{X_T/T,*}E \otimes \Omega^1_{X'_T/T}$$

on the rank $p^d r$ locally free sheaf $F_{X/k,*}E$. It is called the p-curvature of ∇.

Theorem. (Laszlo-Pauly, Groechenig) The characteristic polynomial of ψ' is of the form $(\chi')^{p^d}$, where

$$\chi' = t^r - a_1 t^{r-1} + \cdots (-1)^r a_r, \quad \text{and} \quad a_i \in \Gamma(X, \mathrm{Sym}^i \Omega^1_{X_T/T})$$