# A SIMPSON CORRESPONDENCE FOR ABELIAN VARIETIES IN POSITIVE CHARACTERISTIC



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# The Stack of Higgs Bundles on X'

Denote by  $\mathbf{Higgs}' := \mathbf{Higgs}_r(X'/k)$  the stack of *Higgs bundles* of rank r on X'.

## Higgs bundles

A *Higgs bundle* of rank r on  $X'_T := X' \times_k T$  is a pair  $(E, \theta)$  where E is a locally free  $\mathcal{O}_{X'_T}$ -module of rank r, and

$$\theta: E \longrightarrow E \otimes_{\mathbb{O}_{X_{\tau}'}} \Omega^1_{X_{\tau}'/T}$$

is an  $\mathbb{O}_{X_T}$ -module morphism, such that the composition

$$E \xrightarrow{\theta} E \otimes \Omega^1_{X'_T/T} \xrightarrow{\theta \otimes \mathrm{id}} E \otimes \Omega^1_{X'_T/T} \otimes \Omega^1_{X'_T/T} \xrightarrow{\mathrm{id} \otimes \wedge} E \otimes \Omega^2_{X'_T/T}$$

is zero. Such a  $\theta$  is called a *Higgs field*. Equivalently, a Higgs bundle is an  $\left(\operatorname{Sym}^{\bullet}(\Omega^{1}_{X'_{T}/T})^{\vee}\right)$ -module, or even an  $\mathbb{O}_{\operatorname{T}^{*}(X'_{T}/T)}$ -module, that is locally free as an  $\mathbb{O}_{X'_{T}}$ -module.

# Characteristic Polynomial

Let  $\varphi : E \to E \otimes K$  be an  $\mathbb{O}_{X'}$ -module morphism, where E (resp. K) is an  $\mathbb{O}_{X'}$ -module locally free of rank r (resp. d). Then  $\varphi$  has a well defined characteristic polynomial

$$\chi_{\varphi}(t) = t^r - a_1 t^{r-1} + \dots + (-1)^r a_r$$
, with  $a_i \in \Gamma(X, \operatorname{Sym}^i K)$ .

This applies especially to any Higgs field  $\theta$  where  $K = \Omega^1_{X'_T/T}$ .

# **Spectral Cover**

Denote by  $\pi_T: T^*(X_T'/T) \to X_T'$  the projection. Let  $\lambda \in \Gamma(T^*(X_T'/T), \pi_T^*\Omega^1_{X_T'/T})$  be the *tautological* 1-*form*. Given a Higgs bundle  $(E, \theta)$ , the vanishing locus of the global section

$$\chi := \chi_{\theta}(\lambda) \in \Gamma\left(\mathrm{T}^*(X_T'/T), \mathrm{Sym}^r \Omega^1_{X_T'/T}\right)$$

defines a closed subscheme  $Z_{\chi}$  of  $T^*(X_T'/T)$ .

**Proposition**. (Cayley-Hamilton). The Higgs bundle  $(E, \theta)$  as above, viewed as an  $\mathfrak{O}_{T^*(X'_T/T)}$ -module, is supported on  $Z_{\chi}$ .

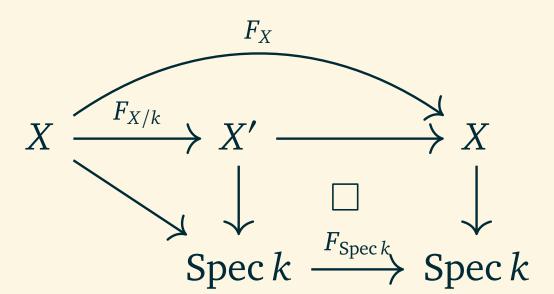
**Proposition.** The map  $Z_{\chi} \to X_T'$  is a finite map hence proper. However in general it is NOT flat.

The map  $Z_{\chi} \to X'_{T}$  is called the *spectral cover* of  $X'_{T}$ .

**Question**: Which polynomials can appear as the characteristic polynomial some Higgs field?

## Setup

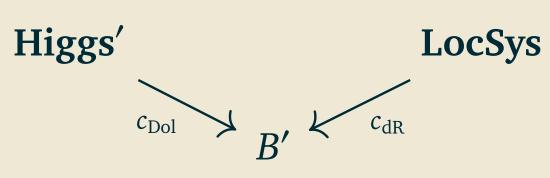
Let X/k be an d-dimensional  $abelian\ variety$  over an algebraically closed field k of characteristic p > 0. Assume T is an arbitrary object in the category (Sch/k). Denote by  $F_{X/k}$  the relative Frobenius morphism.



In this case,  $F_{X/k}$  is locally free of rank  $p^d$ . Moreover, for any T/k, the relative Frobenius  $F_{X_T/T}$  is the pullback of  $F_{X/k}$ .

## Main Theorem (working in progress not yet entirely written)

We have two stacks Higgs' and LocSys over the Hitchin base B':



There is an étale morphism  $U \rightarrow B$ , such that

 $\mathbf{Higgs'}|_{U} \simeq \mathbf{LocSys}|_{U}.$ 

#### **Idea of Proof**

The **Higgs**' side is the same as  $\mathfrak{G}$ -modules on the cotangent bundle of X' with certain support (the spectral cover); The **LocSys** side is the same as  $\mathfrak{D}_{X/k}$ -module on the cotangent bundle of X' with certain support. Noting that  $\mathfrak{D}_{X/k}$  is an Azumaya algebra, we can use the *Morita Theory* to establish the equivalence of both sides. So the aim is to look for splittings of the Azumaya algebra  $\mathfrak{D}_{X/k}$ .

### The Hitchin Base

$$B' := B_r(X'/k) := \operatorname{Spec}\left(\operatorname{Sym}_k^{\bullet}\left(\bigoplus_{i=1}^r \Gamma(X',\operatorname{Sym}^i \Omega^1_{X'/k})\right)\right).$$

# The maps $c_{dR}$ and $c_{Dol}$

 $c_{\mathrm{Dol}}:$  Higgs bundle  $(E,\theta)\longmapsto$  coefficients of  $\chi_{\theta}$   $c_{\mathrm{dR}}:$  Local system  $(E,\nabla)\longmapsto$  coefficients of  $(\chi_{\psi_{\nabla}'})^{1/p^d}$ 

## The Stack of Local System on X

Denote by LocSys := LocSys<sub>r</sub>(X/k) the stack of local systems of rank r on X.

## The Sheaf of Crystalline Differential Operators

Let  $D_{X/k}$  be the sheaf of crystalline differential operators, which is the "enveloping algebroid of the tangent Lie algebroid" of X: to an affine  $U \to X$ , it assigns the algebra that is generated over  $\mathfrak{G}_X(U)$  from the  $\mathfrak{G}_X(U)$ -module  $(\Omega^1_{X/k})^{\vee}(U)$  of k-derivations of  $\mathfrak{G}_X(U)$ , subject to the relations

$$v_1 \cdot v_2 - v_2 \cdot v_1 = [v_1, v_2], \quad \text{and} \quad v_1 \cdot f - f \cdot v_1 = v_1(f),$$

for all  $v_1, v_2 \in (\Omega^1_{X/k})^{\vee}(U)$  and  $f \in \mathcal{O}_X(U)$ .

**Theorem**. (Bezrukavnikov-Mirkovic-Rumynin). The direct image  $F_{X/k,*}D_{X/k}$  of  $D_{X/k}$  under the relative Frobenius  $F_{X/k}$  is an Azumaya algebra over its center  $Z(F_{X/k,*}D_{X/k})$  of rank  $p^{2d}$ . Moreover,

$$\Psi: \operatorname{Sym}^{\bullet}(\Omega^{1}_{X'/k})^{\vee} \xrightarrow{\sim} Z(F_{X/k,*}D_{X/k}).$$

In other words, it defines an *Azumaya algebra*  $\mathfrak{D}_{X/k}$  over the cotangent bundle  $T^*(X'/k)$ .

# Local Systems and D-modules

Denote by  $\tau: X_T \to X$  and  $f_T: X_T \to T$  the projections.

A *local system* of rank r is a (left)  $\tau^*(D_{X/k})$ -module that is locally free as an  $\mathbb{O}_{X_T}$ -module of rank r. This is equivalent to a pair  $(E, \nabla)$  consisting of a locally free  $\mathbb{O}_{X_T}$ -module of rank r and an *integrable* T-connection  $\nabla$ , i.e., an  $f_T^{-1}\mathbb{O}_T$ -module homomorphism

$$abla: E \longrightarrow E \otimes_{\mathbb{O}_{X_T}} \Omega^1_{X_T/T},$$

satisfying the Leibniz rule, whose curvature  $K_{\nabla}: E \to E \otimes \Omega^2_{X_T/T}$  is zero.

# The *p*-curvature

Let  $(E, \nabla)$  be a local system of rank r on  $X_T/T$ . Via the map  $\Psi$  in BMR's theorem, one obtains a Higgs field

$$\psi':=\psi'_
abla:F_{X_T/T,st}E\longrightarrow F_{X_T/T,st}E\otimes\Omega^1_{X_T'/T}$$

on the rank  $p^d r$  locally free sheaf  $F_{X/k,*}E$ . It is called the p-curvature of  $\nabla$ .

**Theorem**. (Laszlo-Pauly, Groechenig) The characteristic polynomial of  $\psi'$  is of the form  $(\chi')^{p^d}$ , where

$$\chi' = t^r - a_1 t^{r-1} + \cdots (-1)^r a_r$$
, and  $a_i \in \Gamma(X, \operatorname{Sym}^i \Omega^1_{X_T/T})$