Hodge-Tate decomposition

Recall classical: let $K/\mathbb{Q}_p$ finite, & let $\mathbb{C}_p$ = completion of $\overline{K}$.
Let X/K be a sm. proper variety. Then, $\exists$ Galois equivariant decomposition

$$H^n(X_{\overline{K},\text{ét}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \simeq \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/K}) \otimes_K \mathbb{C}_p(-j)$$

where the isom. is functorial

Today $\boxed{\text{Thm}}$ Let $X/\mathbb{C}_p$ be a sm. proper rigid-analytic space. Then,

$\exists$ an E_2-spectral sequence

$$E_2^{ij}: H^i(X, \Omega^j_{X/C})(-j) \Rightarrow H^{i+j}(X_{\text{ét}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{C}_p$$

In particular, if X/K with $K/\mathbb{Q}_p$, all differentials are zero and we obtain the previous version.

Recall (Primitive comparison thm). If $X/\mathbb{C}_p$ is sm. proper, then the inclusion $\mathbb{C}_p \subset \widehat{\mathcal{O}}_X$ induces an isom.

Here we view X as an adic space

$$H^*(X_{\text{ét}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{C}_p \simeq H^*(X_{\text{proét}}, \widehat{\mathcal{O}}_X)$$

Hence, enough to prove $\qquad \Rightarrow H^{i+j}(X_{\text{proét}}, \widehat{\mathcal{O}}_X)$

Now, recall that étale morphisms are proétale, i.e. $\exists$ a canonical proj. map

$$\nu: X_{\text{proét}} \to X_{\text{ét}}$$

Then, $\boxed{\text{Thm}}$ follows from the following

Thm (local version) There is a canonical isom.

$$\Phi^1 : \Omega^1_{X/\mathbb{C}_p}(-1) \xrightarrow{\sim} R^1 \nu_* \widehat{\mathcal{O}}_X$$

and by taking products, we get

$$\Omega^j_{X/\mathbb{C}_p}(-j) \simeq R^j \nu_* \widehat{\mathcal{O}}_X$$

Rem. Taking Leray s.s. for ν + local version gives $\boxed{\text{Thm}}$

(*)

Lemma. $R^1 \nu_* \widehat{\mathcal{O}}_X$ is loc. free of rank n, and taking products we get

$$\Lambda^j R^1 \nu_* \widehat{\mathcal{O}}_X \simeq R^j \nu_* \widehat{\mathcal{O}}_X$$

Pf. Local statement $\rightsquigarrow$ assume X affinoid, and that $\exists\ X \to \mathbb{T}^n$ that factors as a composition of rational subsets and fin. ét. comp

Rem. Recall that we had the collorary:

Cor. The cotangent complex $L_{\widehat{\mathcal{O}}^+_X / \mathcal{O}_{\mathbb{C}_p}}$ vanishes mod. p on $X_{\text{proét}}$.

In particular, ~~the~~ its p-adic completion vanishes.

This means that there is no diff. geometric info on $(X_{\text{proét}}, \widehat{\mathcal{O}}_X)$.

But the thm. says that pushing forward $\widehat{\mathcal{O}}_X$ via ν down to $X_{\text{ét}}$, we get the differential forms on X.

It is enough to show:

1) The $\mathcal{O}_X(X)$-module $H^1(X_{proet}, \widehat{\mathcal{O}_X})$ is free of rank n

2) Taking cup products gives an isom. $\Lambda^i H^1(X_{proet}, \widehat{\mathcal{O}_X}) \simeq H^i(X_{proet}, \widehat{\mathcal{O}_X})$

3) Compatibility with étale localization on X.

First, one can reduce from X to $\mathbb{T}^n$ ~~via almost~~ using
- almost purity thm
- base change properties of gp cohom.

So enough to show $\mathbb{T}^n$. We sketch for $\mathbb{T}^1$ ($\mathbb{T}^n$ is similar).
"X"

Recall that $\mathbb{T}^1 := \mathrm{Spa}(\mathbb{C}_p\langle T^{\pm 1}\rangle, \mathcal{O}_{\mathbb{C}_p}\langle T^{\pm 1}\rangle)$

Here we have transition maps

$$X_{n+1} \xrightarrow{(-)^p} X_n \xrightarrow{(-)^p} X_{n-1} \to \cdots \to X$$

where $X_n \cong X$ for all n.

We write coord's $\quad X_n = \mathrm{Spa}(\mathbb{C}_p\langle T^{\pm \frac{1}{p^n}}\rangle, \mathcal{O}_{\mathbb{C}_p}\langle T^{\pm \frac{1}{p^n}}\rangle)$

We saw that $X_\infty := \{X_n\}_{n\in\mathbb{N}} \in X_{proet}$ is an affinoid perfectoid, with affinoid algebra $(\mathbb{C}_p\langle T^{\pm 1/p^\infty}\rangle, \mathcal{O}_{\mathbb{C}_p}\langle T^{\pm 1/p^\infty}\rangle)$.

~~Since each~~

Acyclicity thm $(H^i(X_\infty, \widehat{\mathcal{O}_X}) = 0 \quad i>0)$ implies

$$R\Gamma(X_{\infty,\, proet}, \widehat{\mathcal{O}_X}) \simeq \mathbb{C}_p\langle T^{\pm 1/p^\infty}\rangle$$

Now each $X_n \xrightarrow{(-)^p} X_{n-1}$ is a $\mu_p(\mathbb{C}_p)$-torsor. Hence,

$X_\infty \to X$ is a (pro-étale) $\mathbb{Z}_p(1)$-torsor

More explicitly, we have a (continuous) direct sum decomposition

$$C_p\langle T^{\pm 1/p^\infty}\rangle \simeq \widehat{\bigoplus_{i\in\mathbb{Z}[\frac{1}{p}]}} C_p\cdot T^i$$

which is equivariant for the $\mathbb{Z}_p(1)$-action. The action is as follows:

Let $\underline{\varepsilon} := (\varepsilon_n) \in \varprojlim \mu_{p^n}(C_p) =: \mathbb{Z}_p(1)$, then

$$T^{a/p^m} \longmapsto \varepsilon_m^a \cdot T^{a/p^m}$$ on every summand on RHS

Since $\mathbb{Z}_p(1)$ is a profinite gp, then

$$X_\infty \times_X X_\infty \simeq X_\infty \times \underline{\mathbb{Z}_p(1)}$$

top. constant sheaf on $X_{proét}$ given by $W \mapsto \mathrm{Map}_{conts}(|W|, \mathbb{Z}_p(1))$

top. space attached to $W \in X_{proét}$

~~This implies that $H^i($~~

Čech theory tells us that

$$H^i(X,\widehat{\mathcal{O}}_X) \simeq H^i\left(\widehat{\mathcal{O}}_X(X_\infty) \to \widehat{\mathcal{O}}_X(X_\infty\times_X X_\infty) \to \widehat{\mathcal{O}}_X(X_\infty\times_X X_\infty \times_X X_\infty) \to \cdots\right)$$

Hence, we get

$$H^i(X,\widehat{\mathcal{O}}_X) = H^i_{conts}(\mathbb{Z}_p(1), \widehat{\mathcal{O}}_X(X_\infty))$$

Continuous gp cohomology

In particular,

$$R\Gamma(X_{proét}, \widehat{\mathcal{O}}_X) \simeq R\Gamma_{conts}(\mathbb{Z}_p(1), C_p\langle T^{\pm 1/p^\infty}\rangle)$$

Now, if $\underline{\varepsilon} = (\varepsilon_n) \in \mathbb{Z}_p(1) = \varprojlim \mu_{p^n}(C_p)$,

continuous gp cohom. of pro-cyclic groups

$$R\Gamma(X_{proét}, \widehat{\mathcal{O}}_X) \simeq \widehat{\bigoplus_{i\in\mathbb{Z}[\frac{1}{p}]}}\left(C_p\cdot T^i \xrightarrow{T^i \mapsto (\underline{\varepsilon}^i - 1)\cdot T^i} C_p\cdot T^i\right)$$

if $i = \frac{a}{p^m}$, this is ε_m^a

In particular, for $i \in \mathbb{Z}$, we have $\varepsilon^i = 1$, so the differential is trivial on the summands indexed by $i \in \mathbb{Z}$.

If $i \in \mathbb{Z}[1/p] \setminus \mathbb{Z}$, then $\varepsilon^i - 1 \neq 0 \Rightarrow$ differential is an isom.

Hence, up to quasi-isom., we ignore the non-integral summands and get

$$R\Gamma(X_{\text{proét}}, \widehat{\mathcal{O}_X}) \simeq \widehat{\bigoplus_{i \in \mathbb{Z}}} \left(\mathbb{C}_p \cdot T^i \xrightarrow{0} \mathbb{C}_p \cdot T^i \right)$$

algebra $\longrightarrow$ · $H^1(X_{\text{proét}}, \widehat{\mathcal{O}}_X)$ is free of rank 1

· $H^i(X_{\text{proét}}, \widehat{\mathcal{O}}_X) \simeq \Lambda^i H^1(X_{\text{proét}}, \widehat{\mathcal{O}}_X)$

□

Now we want to give a global construction of the map

$$\boxed{\Phi^i : \Omega^i_{X/\mathbb{C}_p}(-i) \longrightarrow R^i \nu_* \widehat{\mathcal{O}_X}}$$ $\rightsquigarrow$ will be the isom.

First, choose a formal model $\mathfrak{X}/\mathcal{O}_{\mathbb{C}_p}$ of X, and write

$\mathfrak{X}_{\text{aff}}$:= categ. of affine opens in $\mathfrak{X}$ with "indiscrete topology" (so that all presheaves are sheaves)

Then, we obtain

$$(X_{\text{proét}}, \widehat{\mathcal{O}}_X) \xrightarrow{\nu} (X_{\text{ét}}, \mathcal{O}_X) \xrightarrow{\pi} (\mathfrak{X}_{\text{aff}}, \mathcal{O}_{\mathfrak{X}})$$

Write $\mu := \pi \circ \nu$.

~~Claim~~ We ~~show~~ can construct

$$\Phi^{1,1} : \Omega^1_{\mathfrak{X}/\mathcal{O}_{\mathbb{C}_p}} \longrightarrow R^1\mu_* \widehat{\mathcal{O}_X}(1)$$

where $\Omega^1_{\mathfrak{X}/\mathcal{O}_{\mathbb{C}_p}}$ is the sheaf of Kähler differentials on $\mathfrak{X}$:

- If $\mathfrak{X} = \mathrm{Spf}(R)$ $\longrightarrow$ $\mathcal{O}_{\mathbb{C}_p}$-alg. of fin. presentation

  then $\Omega^1_{\mathfrak{X}/\mathcal{O}_{\mathbb{C}_p}} :=$ coherent $\mathcal{O}_{\mathfrak{X}}$-sheaf associated to the continuous Kähler differentials on R.

  This is the p-adic completion of $\Omega^1_{R/\mathcal{O}_{\mathbb{C}_p}}$ in the alg. sense.

  Hence, the values of $\Omega^1_{\mathfrak{X}/\mathcal{O}_{\mathbb{C}_p}}$ on affines are p-adically complete

Once we have such a map, we get (adjoint formalism)

$$\begin{array}{c} \pi^*\Omega^1_{\mathfrak{X}/\mathcal{O}_{\mathbb{C}_p}} = \pi^{-1}\Omega^1_{\mathfrak{X}/\mathcal{O}_{\mathbb{C}_p}} \otimes_{\pi^{-1}\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_X \longrightarrow R^1\nu_*\widehat{\mathcal{O}_X}(1) \\ \wr\| \\ \Omega^1_{X/\mathbb{C}_p} \end{array}$$

and hence we obtain our desired map $\Phi^1 : \Omega^1_{X/\mathbb{C}_p} \to R^1\nu_*\widehat{\mathcal{O}}_X(1)$

From here, we get the Φ^i's by taking exterior products

Pf of claim:

Consider
$$\mathbb{Z}_p \hookrightarrow \mathcal{O}_{\mathbb{C}_p} \hookrightarrow \widehat{\mathcal{O}}^+_X$$

of sheaves of rings on $X_{\text{proét}}$. We get a canonical exact triangle

φ

$$\begin{array}{ccccc} L_{\mathcal{O}_{\mathbb{C}_p}/\mathbb{Z}_p} \otimes^L_{\mathcal{O}_{\mathbb{C}_p}} \widehat{\mathcal{O}}^+_X & \to & L_{\widehat{\mathcal{O}}^+_X/\mathbb{Z}_p} & \longrightarrow & L_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\mathbb{C}_p}} \\ & & & & \| \xrightarrow{\quad\quad} \text{previous cor} \\ & & & & 0 \bmod p \end{array}$$

Hence, after p-adic completion we have an isom.

$$\widehat{L_{\mathcal{O}_{\mathbb{C}_p}/\mathbb{Z}_p} \otimes_{\mathcal{O}_{\mathbb{C}_p}} \widehat{\mathcal{O}}_X^{+}} \simeq \widehat{L_{\widehat{\mathcal{O}}_X^{+}/\mathbb{Z}_p}}$$

<u>Facts</u>. 1) $\widehat{L_{\mathcal{O}_{\mathbb{C}_p}/\mathbb{Z}_p}} \simeq \Omega[1]$, where Ω is ~~the Tate module of~~ $\cancel{\Omega^1_{\mathcal{O}_{\mathbb{C}_p}/\mathbb{Q}_p}}$ a free $\mathcal{O}_{\mathbb{C}_p}$-module of rank 1 (indeed, $\Omega = \ker(\theta)/\ker(\theta)^2$,

where $\theta : A_{\mathrm{inf}}(\mathcal{O}_{\mathbb{C}_p}) \to \mathcal{O}_{\mathbb{C}_p}$ is the lift of the projection

$$\bar{\theta} : \mathcal{O}_{\mathbb{C}_p}^{\flat} = \lim_{\phi} \mathcal{O}_{\mathbb{C}_p}/p \longrightarrow \mathcal{O}_{\mathbb{C}_p}/p \;))$$

Galois equivariantly, it looks like $\mathcal{O}_{\mathbb{C}_p}(1)$ up to torsion.

Inverting p, we get

$$\mathrm{LHS}\left[\tfrac{1}{p}\right] \simeq \Omega \otimes_{\mathcal{O}_{\mathbb{C}_p}} \widehat{\mathcal{O}}_X^{+}[1]\left[\tfrac{1}{p}\right] \simeq \mathcal{O}_{\mathbb{C}_p}(1) \otimes_{\mathcal{O}_{\mathbb{C}_p}} \widehat{\mathcal{O}}_X[1] \simeq$$

$$\simeq \widehat{\mathcal{O}}_X(1)[1] \simeq \mathrm{RHS}\left[\tfrac{1}{p}\right]$$

Now, we pullback via μ:

$$\widehat{L_{\mathfrak{X}/\mathbb{Z}_p}} \longrightarrow R\mu_* \widehat{L_{\widehat{\mathcal{O}}_X^{+}/\mathbb{Z}_p}}$$

$$\longrightarrow R\mu_* \widehat{L_{\widehat{\mathcal{O}}_X^{+}/\mathbb{Z}_p}}\left[\tfrac{1}{p}\right] \simeq R\mu_* \widehat{\mathcal{O}}_X(1)[1]$$

Subclaim: $\mathcal{H}^0(\widehat{L_{\mathfrak{X}/\mathbb{Z}_p}}) \simeq \Omega^1_{\mathfrak{X}/\mathcal{O}_{C_p}}$

Therefore, passage to $\mathcal{H}^0$ yields our dessired

$$\Phi^{1,1} : \Omega^1_{\mathfrak{X}/\mathcal{O}_{C_p}} \longrightarrow R^1\mu_* \widehat{\mathcal{O}_X}(1)$$

Proof of subclaim:

$$\mathbb{Z}_p \to \mathcal{O}_{C_p} \longrightarrow \mathcal{O}_{\mathfrak{X}} \qquad \text{on } \mathfrak{X}_{aff}.$$

$$\Big\downarrow$$

$$L_{\mathcal{O}_{C_p}/\mathbb{Z}_p} \otimes_{\mathcal{O}_{C_p}} \mathcal{O}_{\mathfrak{X}} \to L_{\mathfrak{X}/\mathbb{Z}_p} \to L_{\mathfrak{X}/\mathcal{O}_C}$$

$$\Big\downarrow \text{(derived) p-adic completion} + \mathcal{H}^0$$

$$0 \to \mathcal{H}^0(\widehat{L_{\mathfrak{X}/\mathbb{Z}_p}}) \simeq \mathcal{H}^0(\widehat{L_{\mathfrak{X}/\mathcal{O}_{C_p}}})$$

$\Big\| \Rightarrow$ because $\mathfrak{X}$ is a top. fin. presented formal scheme over $\mathcal{O}_C$ (c.f. Gabber - Romero)

$$\Omega^1_{\mathfrak{X}/\mathcal{O}_{C_p}}$$

This finishes the subclaim and the claim.

So, we have $\Phi^i : \Omega^i_{X/C_p}(-i) \longrightarrow R^i\nu_* \widehat{\mathcal{O}_X}$

$$\Rightarrow$$

$$\oplus\Phi^i : \bigoplus_i \Lambda^i(\Omega^1_{X/C_p}(-1)) \longrightarrow \bigoplus_i R^i\nu_* \widehat{\mathcal{O}_X} \simeq \bigoplus_i \Lambda^i(R^1\nu_*\widehat{\mathcal{O}_X})$$

Left to show: this is an isom.

It is enough to show this for $i=1$.

Both sides are coherent sheaves of étal local nature on X.

Hence, we may assume $X = \mathbb{T}^n$ and go to global sections.

Enough to show

$$\Phi^1(X) : \Omega^1_{X/\mathbb{C}_p}(-1) \xrightarrow{\sim} H^1(X_{\text{proét}}, \widehat{\mathcal{O}}_X)$$

Both sides are rank n $\mathcal{O}_X(X)$-modules and compatible with products of adic spaces ($\mathbb{T}^n = \prod \mathbb{T}^1$) $\rightsquigarrow$ assume $n=1$.

Hence $X = \mathbb{T}^1 = \mathrm{Spa}\left(\mathbb{C}_p\langle T^{\pm 1}\rangle, \mathcal{O}_{\mathbb{C}_p}\langle T^{\pm 1}\rangle\right)$

$d\log(T) \in \Omega^1_{X/\mathbb{C}_p}$ is a generator.

$\rightarrow$ Enough to show that $\Phi^1(d\log(T))$ is a generator.