The decalage functor

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1 Definition and examples

Fix a ring A and $f \in A$: regular (= nonzerodivisor).

Definition. (cf. [1, Definition 5.1]) For any complex $K^\bullet$ of f-torsionfree A-modules, define a subcomplex $\eta_f(K^\bullet) \subset K^\bullet[\frac{1}{f}]$ by

$$\eta_f(K^\bullet)^i \overset{\text{def}}{=} \{\alpha \in f^i K^i \,|\, d\alpha \in f^{i+1} K^{i+1}\}.$$

Lemma. (cf. [1, Lemma 5.2]) For any complex $K^\bullet$ of f-torsionfree A-modules and any integer $i \in \mathbb{Z}$, there exists a natural isomorphism

$$H^i(\eta_f K^\bullet) \simeq H^i(K^\bullet)/H^i(K^\bullet)[f] \simeq f H^i(K^\bullet).$$

Proof. The second isomorphism is obvious. Let us check the first one. Since $K^\bullet$ is f-torsionfree, we have a bijection

$$Z^i(K^\bullet) \xrightarrow{\simeq} Z^i(\eta_f K^\bullet)\,;\, \alpha \mapsto f^i\alpha.$$

We shall show that this bijection induces an isomorphism

$$H^i(K^\bullet)/H^i(K^\bullet)[f] \xrightarrow{\simeq} H^i(\eta_f K^\bullet).$$

Indeed, for any $\alpha \in Z^i(K^\bullet)$ with $[\alpha]$ image in $H^i(K^\bullet)$, we have

$$\begin{aligned}
[f^i\alpha] = 0 \text{ in } H^i(\eta_f K^\bullet) &\iff f^i\alpha \in B^i(\eta_f K^\bullet) \\
&\iff \exists f^{i-1}\beta \in f^{i-1}K^{i-1},\, d(f^{i-1}\beta) = f^i\alpha \\
&\iff \exists \beta \in K^{i-1},\, d\beta = f\alpha \\
&\iff f\alpha \in B^i(K^\bullet) \\
&\iff [\alpha] \in H^i(K^\bullet)[f].
\end{aligned}$$

This completes the proof. $\square$

Corollary. If $K^\bullet \xrightarrow{\text{qis}} M^\bullet$ is a quasi-isomorphism between complexes of f-torsionfree A-modules, then this induces a quasi-isomorphism $\eta_f K^\bullet \xrightarrow{\text{qis}} \eta_f M^\bullet$.

By the corollary, we get a functor

$$L\eta_f : D(A) \to D(A),$$

which we call the *decalage functor*. Note that for any $K \in D(A)$ and any $i \in \mathbb{Z}$, we have

$$H^i(L\eta_f K) \simeq H^i(K)/H^i(K)[f].$$

Example. (1) For any $K \in D(A)$, we have $L\eta_f(K)[\frac{1}{f}] \simeq K[\frac{1}{f}]$ in $D(A)$.
(2) If M is an f-torsionfree A-module, then we have $L\eta_f M = \eta_f M = M$.
(3)(cf. [1, Warning 5.5]) Let $A = \mathbb{Z}$ and $f = p$. Then, for any integer $n > 0$, we have

$$L\eta_p(\mathbb{Z}/p^n\mathbb{Z}) = \mathbb{Z}/p^{n-1}\mathbb{Z}.$$

In particular, the decalage functor $L\eta_p : D(\mathbb{Z}) \to D(\mathbb{Z})$ is not exact.
(4)(Koszul complex, cf. [1, Example 5.17]) Let A be a ring and $(g_1, \dots, g_d)$ a d-tuple of elements of A. The *Koszul complex* $K^\bullet = K^\bullet(g_1, \dots, g_d) = K^\bullet(A; g_1 \dots, g_d)$ is defined in the following way. Let A^d be the free A-module of rank d with $\{e_i\}_{i=1}^d$ the canonical basis. Let $K^n \overset{\text{def}}{=} \wedge^n A^d$, which is a free A-module of rank $\binom{d}{n}$ and define the nth differential by

$$d^n : K^n \to K^{n+1} \,;\, x \mapsto \sum_{i=1}^{d} g_i e_i \wedge x.$$

Recall the following.

- If $d = 1$, then $K^\bullet = \cdots \to 0 \to A \xrightarrow{f} A \to 0 \to \cdots$.
- In general, $K^\bullet(g_1, \dots, g_d) \simeq \otimes_{i=1}^d K^\bullet(g_i)$.
- The cohomology groups $H^n(K^\bullet(A; g_1, \dots, g_d))$ are g_i-torsion for any i.

More general, for any comlex $M^\bullet$ of A-modules, we define the Koszul complex $K^\bullet(M^\bullet; g_1, \dots, g_d)$ by

$$K^\bullet(M^\bullet; g_1, \dots, g_d) \overset{\text{def}}{=} M^\bullet \otimes_A K^\bullet(A; g_1, \dots, g_d).$$

Let $f \in A$ be a regular element and suppose that $M^\bullet$ is f-torsionfree. The complex $\eta_f(K^\bullet(M^\bullet; g_1, \dots, g_d))$ can be understood in the following two cases.

(i) If $g_j \mid f$ for some j, then $\eta_f(K^\bullet(M^\bullet; g_1, \dots, g_d)) \simeq 0$.

(ii) If $f \mid g_j$ for any j, then $\eta_f(K^\bullet(M; g_1, \dots, g_d)) \simeq K^\bullet(\eta_f M^\bullet; g_1/f, \dots, g_d/f)$.

Indeed, since

$$K^\bullet(M^\bullet; g_1, \dots, g_d) \simeq M^\bullet \otimes_A K^\bullet(g_1) \otimes_A \cdots \otimes_A K^\bullet(g_d).$$

We have only to consider the case $d = 1$. Then $M^\bullet \otimes_A K(g)$ is given by

$$\begin{array}{rcl} \cdots \to M^n \oplus M^{n-1} & \to & M^{n+1} \oplus M^n \to \cdots \\ (x, y) & \to & (dx, dy + (-1)^n gx). \end{array}$$

Then it is not difficult to show that the multiplication-by-g map on $M^\bullet \otimes_A K^\bullet(g)$ is homotopic to the zero map, which implies (i). Let us show (ii). Notice that

$$\begin{aligned} \eta_f(K^\bullet(M^\bullet; g))^n &= \left\{(x, y) \in f^n M^n \oplus f^n M^{n-1} \,\middle|\, (dx, dy + (-1)^n gx) \in f^{n+1} M^{n+1} \oplus f^{n+1} M^n\right\} \\ &= (\eta_f M^\bullet)^n \oplus f(\eta_f M^\bullet)^{n-1}, \end{aligned}$$

where the last equality holds thanks to the assumption that f divides g. Moreover, for any n, the diagram

$$\begin{tikzcd} M^n \oplus M^{n-1} \arrow[r, "(x,y)\mapsto(dx,dy+(-1)^n(g/f)x)"] \arrow[d, "(\mathrm{id},f)"'] & M^{n+1} \oplus M^n \arrow[d, "(\mathrm{id},f)"] \\ M^n \oplus M^{n-1} \arrow[r, "(x,y)\mapsto(dx,dy+(-1)^n gx)"'] & M^{n+1} \oplus M^n \end{tikzcd}$$

is commutative. Hence, the map $(\mathrm{id}, f) : K^\bullet(\eta_f M^\bullet; g/f) = \eta_f M^\bullet \otimes_A K^\bullet(g/f) \to \eta_f K^\bullet(M^\bullet; g)$ gives an isomorphism of complexes. This implies (ii).

2 Basic properties

Let A, f be as above.

Lemma. (cf. [1, Lemmas 5.12, 5.13 and 5.19]) We have the following.

(i) Let $\alpha : A \to B$ be a ring map with $\alpha(f) \in B$ regular. Then, for any $M \in D(B)$, we have a natural isomorphism
$$\alpha_*(L\eta_{\alpha(f)}M) \simeq L\eta_f(\alpha_* M).$$

(ii) Let $g \in A$ be another regular element. Then, for any $M \in D(A)$, we have a natural isomorphism
$$L\eta_f(L\eta_g M) \simeq L\eta_{fg}(M).$$

(iii) Let $I \subset A$ be a finitely generated ideal. If $K \in D(A)$ is I-adically complete, then so is $L\eta_f K$.

Proof of (iii). Before the proof, recall the next things (cf. [1, 1.4]).

- In this seminar, the completeness always means the one in the derived sense.
- An object $K \in D(A)$ is I-adically complete if and only if all the cohomology groups $H^i(K)$ are I-adically complete.
- The category of I-adically complete A-modules form an abelian subcategory of all A-modules.
- Any A-module M is I-adically complete in the classical sense, i.e. $M \xrightarrow{\simeq} \varprojlim_n M/I^n M$ if and only if M is I-adically complete and $\cap_n I^n M = 0$.

Let us begin the proof. We have to show that all the cohomology groups $H^i(L\eta_f K)$ are I-adically complete. Fix i. By assumption, $H^i(K)$ is I-adically complete, hence so is $H^i(L\eta_f K) \simeq H^i(K)/H^i(K)[f]$. This completes the proof. □

Lemma. (cf. [1, Lemma 5.20]) For any integer $d \geq 0$ and any $K \in D^{[0,d]}(A)$ with $H^0(K)$ f-torsionfree, there exist natural maps $\alpha : L\eta_f K \to K$ and $\beta : K \to L\eta_f K$ such that $\alpha \circ \beta = f^d$ and $\beta \circ \alpha = f^d$.

Proof. Let $M^\bullet$ be an f-torsionfree representative of K. Let $\tau_{[0,d]}M^\bullet$ be the canonical truncation of $M^\bullet$. Namely, it is a complex of $M^\bullet$ such that

$$(\tau_{[0,d]}M^\bullet)^i = \begin{cases} 0 & \text{if } i > d, \\ \mathrm{Ker}(d^d) & \text{if } i = d, \\ M^i & \text{if } 0 < i < d, \\ \mathrm{Coker}(d^{-1}) & \text{if } i = 0, \\ 0 & \text{if } i < 0. \end{cases}$$

Since $H^0(K)$ and $M^\bullet$ are f-torsionfree, so is $\mathrm{Coker}(d^{-1})$. Moreover, $M^\bullet$ and $\tau_{[0,d]}M^\bullet$ are quasi-isomorphic. Therefore, we may assume that $M^i = 0$ for $i \notin [0,d]$. Then, $\eta_f M^\bullet$ is a subcomplex of $M^\bullet$ and the multiplication-by-f^d map $f^d : M^\bullet \to M^\bullet$ factors though the subcomplex $\eta_f M^\bullet$,

$$\begin{tikzcd} \eta_f M^\bullet \arrow[r, hook] \arrow[d, "f^d"'] & M^\bullet \arrow[d, "f^d"] \arrow[dl, dashed] \\ \eta_f M^\bullet \arrow[r, hook] & M^\bullet. \end{tikzcd}$$

This completes the proof. $\square$

3 The Bockstein construction

Let A and f be as above.

Definition. (The Bockstein construction, cf. [1, Construction 5.6, Remark 5.7]) For any $K \in D(A)$, we have the following associated *Bockstein complex*

$$(H^\bullet(K/f), \beta_f) \overset{\text{def}}{=} (\cdots \to H^i(K \otimes_A^L A/fA) \xrightarrow{\beta_f^i} H^{i+1}(K \otimes_A^L A/fA) \to \cdots),$$

where the differential β_f^i is the boundary map associated with the distinguished triangle obtained by tensoring the canonical triangle

$$A/fA \to A/f^2A \to A/fA.$$

Lemma. With the above notation, we have $\beta_f^{i+1} \circ \beta_f^i = 0$.

Proof. Indeed, consider the commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & A \arrow[r] \arrow[d] & A/fA \arrow[d, equal] \\ A/fA \arrow[r, "f"] & A/f^2A \arrow[r] & A/fA, \end{tikzcd}$$

of exact triangles, we get a commutative diagram

$$\begin{tikzcd} H^i(K) \arrow[r, "f"] & H^i(K) \arrow[r, "a_i"] & H^i(K \otimes_A^L A/f) \arrow[r, "b_i"] \arrow[d, equal] & H^{i+1}(K) \arrow[r, "f"] \arrow[d, "a_{i+1}"] & H^{i+1}(K) \ (\text{exact}) \\ & & H^i(K/f) \arrow[r, "\beta_f^i"'] & H^{i+1}(K/f), & \end{tikzcd}$$

which implies that

$$\beta_f^{i+1} \circ \beta_f^i = (a_{i+2} \circ b_{i+1}) \circ (a_{i+1} \circ b_i) = a_{i+2} \circ (b_{i+1} \circ a_{i+1}) \circ b_i = 0.$$

This completes the proof. □

Example. (cf. [1, Example 5.8]) For $K \in D(A)$ such that $fH^i(K) = 0$ for any i, the associated Bockstein complex $(H^\bullet(K/f), \beta_f)$ is acyclic. With the same notation as in the proof of the previous lemma, we have

$$\mathrm{Ker}(\beta_f^i) = \mathrm{Ker}(a_{i+1} \circ b_i) = \mathrm{Ker}(b_i) = \mathrm{Im}(a_i) = \mathrm{Im}(a_i \circ b_{i-1}) = \mathrm{Im}(\beta_f^{i-1}).$$

This completes the proof.

Lemma. (cf. [1, Lemma 5.9]) For any $K \in D(A)$, there exists a canonical isomorphism

$$L\eta_f(K)/f \simeq (H^\bullet(K/f), \beta_f).$$

Sketch. Choose a representative $K^\bullet$ of K with f-torsionfree term. For any $f^i\alpha \in (\eta_f K^\bullet)^i \subset f^i K^i$, we have $d\alpha \in fK^{i+1}$, hence $\alpha \in Z^i(K/f)$, which gives a class $[\alpha] \in H^i(K/f)$. This map $(\eta_f K^\bullet)^i \ni f^i\alpha \mapsto [\alpha] \in H^i(K/f)$ defines a morphism $\eta_f K \to (H^\bullet(K/f), \beta_f)$ in $D(A)$. This morphism induces an isomorphism $L\eta_f(K)/f \xrightarrow{\simeq} (H^\bullet(K/f), \beta_f)$. □

Lemma. (cf. [1, Lemma 5.14]) Let $K \to L \to M$ be a distinguished triangle in $D(A)$. Assume that the condition

(*) The boundary maps $H^i(M/f) \to H^{i+1}(K/f)$ are the zero maps.

is satisfied. Then the sequence

$$L\eta_f K \to L\eta_f L \to L\eta_f M$$

is a distinguished triangle in $D(A)$.

Proof. Since it is a distinguished triangle after inverting f, it is enough to show the claim holds after reduction modulo f. Therefore, by the previous lemma, it suffices to show the sequence of the Bockstein complexes

$$(H^\bullet(K/f), \beta_f) \to (H^\bullet(L/f), \beta_f) \to (H^\bullet(M/f), \beta_f)$$

is a distinguished triangle. However, by virtue of the condition (*), more strongly, it is an exact sequence of complexes of A-modules. □

Example. (cf. [1, Example 5.15]) Let us see that the decalage functor $L\eta$ maps an almost isomorphism into an isomorphism in the following situation. Let $A = \mathscr{O}_C$ and $f = p$. Let $K \in D(\mathscr{O}_C)$ be a perfect complex. Let $f : K \to L$ be any morphism into an object $L \in D(\mathscr{O}_C)$ such that the mapping cone $M \stackrel{\text{def}}{=} C(f)$ is almost zero, i.e. the cohomology groups $H^i(M)$ are almost zero. Then, one can see that the induced map $L\eta_p K \to L\eta_p L$ is a quasi-isomorphism. Since

$$H^i(L\eta_p M) = pH^i(M) = 0$$

for any i, it suffices to show that the induced sequence

$$L\eta_p K \to L\eta_p L \to L\eta_p M$$

is a distinguished triangle. By the previous lemma, we are reduced to checking the condition (*) for the given triangle $K \to L \to M$, namely showing that the boundary maps

$$H^i(M/p) \to H^{i+1}(K/p)$$

are the zero maps. Fix i. We will show that $H^i(K/p)$ does not contain almost zero elements, which implies that the boundary map is zero because $H^{i-1}(M/p)$ is almost zero.

Since K is perfect, by [3, Lemma 066W], K/p is also perfect in $D(\mathscr{O}_C/p)$. Since $\mathscr{O}_C/p$ is coherent (cf. [2, Example 2.4(i)]), this implies that each cohomology group $H^i(K/p)$ is a finitely presented $\mathscr{O}_C/p$-module, whence a finitely presented torsion $\mathscr{O}_C$-module. The condition that $H^i(K/p)$ is finitely presented implies that there exists an isomorphism $H^i(K/p) \simeq M \otimes_R \mathscr{O}_C$ of $\mathscr{O}_C$-modules for some discrete valuation subring R of $\mathscr{O}_C$ and for some finite torsion R-module M. Then we can find that there exists an isomorphism $H^i(K/f) \simeq \oplus_{j=1}^n \mathscr{O}_C/g_j$ of $\mathscr{O}_C$-module for some nonzero elements $g_1, \ldots, g_n \in \mathscr{O}_C$. Therefore, we are reduced to showing that for any nonzero element $g \in \mathscr{O}_C$, the module $\mathscr{O}_C/g$ has no almost zero elements. Indeed, let $f \in \mathscr{O}_C$ be an element such that $f \notin g\mathscr{O}_C$. Then, we have $g/f \in \mathfrak{m}$. However, since $|C^*|$ is p-divisible, there exists an element $h \in K^*$ such that $|h|^p = |g/f|$. Then we have $h \in \mathfrak{m}$ and $hf \notin g\mathscr{O}_C$. This implies that the image of f in $\mathscr{O}_C/g$ is not an almost zero element. This implies that $\mathscr{O}_C/g$ does not contain nonzero almost zero elements.

Lemma. (cf. [1, Lemma 5.16]) Let $g \in A$ be another regular element. Then for any $K \in D(A)$, a natural map

$$\alpha : L\eta_f(K)/g \to L\eta_f(K/g)$$

is an isomorphism if $H^i(K/f)$ has no g-torsion for any i.

Proof. The assertion is equivalent to saying that the following sequence induced from the natural triangle $K \xrightarrow{g} K \to K/g$ is a distinguished triangle

$$L\eta_f K \xrightarrow{g} L\eta_f K \to L\eta_f(K/g).$$

Again, it is enough to check the condition (*). However, since $H^{i+1}(K/f)$ has no g-torsion, the boundary maps must be the zero maps. □

Lemma. (cf. [1, Lemma 5.18]) Let $K \in D^{\leq 1}(A)$ and $M \in D^{\geq 0}(A)$ with $H^0(M)$ being f-torsionfree. Fix a map $\alpha : K \to M$. Then we have the following.

(1) The canonical map $L\eta_f M \to M$ induces an isomorphism $H^1(L\eta_f M) \simeq fH^1(M)$.

(2) The map α has at most one factorization of the form $K \xrightarrow{\alpha'} L\eta_f M \xrightarrow{\text{cano.}} M$.

(3) The following conditions are equivalent.

   (a) The factorization of α as in (2) exists.

   (b) The induced map $H^1(\alpha) : H^1(K) \to H^1(M)$ factors through $fH^1(M)$.

Proof. Without loss of generality, we may assume that $M \in D^{[0,1]}(A)$ with $H^0(M)$ being f-torsionfree, hence M can be represented by a f-torsionfree complex $M^\bullet$ of the form

$$\cdots \to 0 \to M^0 \xrightarrow{d} M^1 \to 0 \to \cdots .$$

Thus, $L\eta_f M$ can be represented by the complex

$$\cdots \to 0 \to d^{-1}(fM^1) \xrightarrow{d} fM^1 \to 0 \to \cdots .$$

Consider the distinguished triangle

$$L\eta_f M \xrightarrow{a} M \to C(a).$$

Here, the mapping cone $C(a)$ has the form

$$\cdots \to d^{-1}(fM^1) \xrightarrow{x \mapsto (-dx, x)} fM^1 \oplus M^0 \xrightarrow{(x,y) \mapsto x+dy} M^1 \to 0 \to \cdots$$

and it is quasi-isomorphic to the complex $M^1/(fM^1 + dM^0)[-1] \simeq H^1(M)/fH^1(M)[-1]$, hence we get a distinguished triangle

$$H^1(M)/fH^1(M)[-2] \to L\eta_f M \xrightarrow{a} M.$$

The long exact sequence associated with this triangle gives a short exact sequence

$$0 \to H^1(L\eta_f M) \to H^1(M) \to H^1(M)/fH^1(M) \to 0,$$

whence the claim (1). Moreover, by applying $\mathrm{Hom}(K, \quad)$, we get an exact sequence

$$\mathrm{Hom}(K, H^1(M)/fH^1(M)[-2]) \to \mathrm{Hom}(K, L\eta_f M) \to \mathrm{Hom}(K, M) \to \mathrm{Hom}(K, H^1(M)/fH^1(M)[-1]).$$

Since $K \in D^{\leq 1}(A)$, we have $\mathrm{Hom}(K, H^1(M)/fH^1(M)[-2]) = 0$, the uniqueness of the factorization of α is true, whence the claim (2). Furthermore, the exactness implies that α factors through $a : L\eta_f M \to M$ if and only if the composition $K \xrightarrow{\alpha} M \to H^1(M)/fH^1(M)[-1]$ is trivial, or equivalently the induced map $H^1(\alpha) : H^1(K) \to H^1(M)$ factors through $fH^1(M)$. Therefore, the equivalence between (a) and (b) in (3) is true. $\square$

References

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