Hartshorne Notes/Solutions

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Contents

Chapter I

Varieties

I.1 Affine varieties

I.1.1(c). Let $F := ax^2 + bxy + cy^2 + dx + ey + f$ define the (irreducible) quadric curve C in $\mathbb{A}^2$. We can tell that $C \simeq \mathbb{A}^1$ or $C \simeq \mathbb{A}^1 \setminus \{0\}$ by inspecting a, b, c as follows (note that we heavily use $\Bbbk = \overline{\Bbbk}$). Noting that $(by)^2 - 4acy^2 = (b^2 - 4ac)y^2$, let $D = b^2 - 4ac$. If $D \neq 0$, then $C \simeq \mathbb{A}^1 \setminus \{0\}$ since there is a linear change of coordinates so that F becomes $XY = 1$. If $D = 0$, then either $a = b = c = 0$ or at least two among three are 0, or all nonzero, so that change of coordinates then gives $X = F$ or $X^2 + DX + F = Y$.

I.1.10(c,e). (c): Consider $\operatorname{Spec} \Bbbk[x]_{(x)}$ and its generic points as the open subset. (e): Nagata has an example of Noetherian ring that is infinite dimensional.

I.1.11. First, some easy general observations:

Lemma 1: If A is a domain, then $\ker : B \overset{\varphi}{\to} A$ *for any ring B is prime; i.e.* $\overline{\operatorname{im}(\varphi^\#)}$ *is irreducible.*

Lemma 2: If $\varphi : B \to A$ is finite, then $B/\ker\varphi \hookrightarrow A$ is finite, so $\dim A = \dim \operatorname{im}(\varphi^\#) \leq \dim B$.

The two lemmas immediately imply that the variety $C \subset \mathbb{A}^3$ defined by $x = t^3, y = t^4, z = t^5$ is a variety of dimension 1, and thus $\mathfrak{p} := I(C)$ is prime of height 2 ($\because$ I.1.8A). Lastly, to see that it requires three generators, consider solving for the kernel K of $[t^3\ t^4\ t^5]$ in $k(t)$, and note that there are three $k[t]$-linearly independent elements in $K \cap k[t]$.

I.1.12. $x^2 + y^2(y-1)^2 = 0$

I.2 Projective varieties

I.2.6. Let $Y_i := Y \cap U_i \neq \emptyset$, so that $A(Y_i)[x_i^{\pm}] \simeq S(Y)_{x_i}$ (as graded rings even). Thus, since Y, Y_i are irreducible, $\dim S(Y) = \dim A(Y_i)[x_i^{\pm}] = \operatorname{tr}\deg_{\Bbbk} S(Y)_{x_i} = \operatorname{tr}\deg_{\Bbbk} A(Y_i) + 1 = \dim Y_i + 1$. Now, noting that $\dim Y = \sup_i \dim Y_i$, we have that $\dim S(Y) = \dim Y + 1$, and in fact, we have a stronger statement that $\dim S(Y) = \dim Y_i + 1$ for any $Y_i \neq \emptyset$.

Remark: The statement is true even for Y a projective algebraic set (not just a projective variety). However, the stronger statement is no longer true (as $\operatorname{tr}\deg$ depends on having integral domains).

I.2.7(b). Let $Y_i := Y \cap U_i$ be one where dim is maximized so that $\dim Y = \dim Y_i$. Then $\dim Y_i = \dim(\overline{Y_i} \subset U_i) = \dim(\overline{Y} \cap U_i) = \dim \overline{Y}$ (last equality uses strong form of Ex.I.2.6).

I.2.8. One direction is easy; for the other direction: *Geometric soln:* If $\dim(Y \subset \mathbb{P}^n) = n - 1$, then (WLOG) $\dim Y_0 = n - 1$ so that $Y_0 \simeq \operatorname{Spec} \Bbbk[x_1, \ldots, x_n]/f_0$ for f_0 irreducible. Now, $Y = V(f_0^h)$ can be checked on all affine patches. *Algebraic soln:* Let's use that S is a UFD. Let $\mathfrak{p}$ be the prime such that $\operatorname{Proj} S/\mathfrak{p} = Y$, and so there is no (homogeneous) prime between $(0) \subsetneq \mathfrak{p}$. Take any nonzero homogeneous element $f \in \mathfrak{p}$, and factor it (note that each factor is homogeneous too), so that at least one irreducible factor g is in $\mathfrak{p}$. Now, $(0) \subsetneq (g) \subset \mathfrak{p}$ with (g) prime so that $(g) = \mathfrak{p}$.

I.2.12. (a): Pullback of homogeneous prime is homogeneous prime. (b): Let's re-index the y's by $\mathscr{I} = \{I \in (\binom{[n+1]}{d})\}$. Let $(y_\bullet) \in Z(\mathfrak{a})$, so that $y_I \neq 0$ for some $I \in \mathscr{I}$. WLOG assume $0 \in I$. Then we claim that $\dfrac{x_i}{x_0} = \dfrac{y_{I \setminus 0 \cup i}}{y_I}$ works (and this solution is unique up to scaling). To see this, note that for any $J \in \mathscr{I}$, we have $\dfrac{\prod_{\ell \in J} x_\ell}{x_0^d} = \dfrac{\prod_{\ell \in J} y_{I \setminus 0 \cup \ell}}{y_I^d} = \dfrac{y_J}{y_{0^d}}$ (where last equality follows from $(y_\bullet) \in Z(\mathfrak{a})$, and letting $J = I$ we have $y_{0^d} \neq 0$). (c): With part (b), we have that $D(y_{i^d})$'s ($i = 0, \dots, n$) cover Z. On $D(y_{0^d})$, part (b) solution above shows that $\frac{k[y/y_{0^d}\text{'s}]}{\mathfrak{a}} \simeq k[x_1/x_0, \dots, x_n/x_0]$. (d): The map is $[s,t] \mapsto [s^3, s^2t, st^2, t^3]$ so that on an affine patch U_s it is $[t, t^2, t^3]$, which is the twisted cubic.

I.3 Morphisms

I.3.1(c,e). (c): Let the conic be $V(f) \subset \mathbb{P}^2$. Then f_z (dehomogenize at U_z) is either $xy-1$ or x^2-y. In either case, we conclude $f = xy - z^2$ (up to coordinate change). Hence, $V(f) = \nu_2(\mathbb{P}^1)$ (the degree 2 Veronese embedding of $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$). (e): Then $A(Y) \simeq \mathscr{O}(Y) \simeq \Bbbk$, and hence $\dim Y = 0$.

I.3.3. (a): A morphism $\varphi : X \to Y$ induces $\varphi^* : \mathscr{O}(Y) \to \mathscr{O}(X)$. Define a map $\varphi_p^* : \mathscr{O}_{Y,\varphi(p)} \to \mathscr{O}_{X,p}$ by $[(U,f)] \mapsto [(\varphi^{-1}(U), f \circ \varphi)]$; that this is well-defined and is a local homomorphism of rings are easy to check. (b): If φ is an isomorphism, its inverse morphism ψ gives natural inverse to φ_p^*. For the converse, use the fact that isomorphism is local on the target, by taking an affine cover of Y and noting that $\mathscr{O}_p \simeq A_\mathfrak{p}$ for affines. (c): If $\overline{\varphi(X)} = Y$, then $\overline{\varphi(\varphi^{-1}(U))} = \overline{U}$ for any $U \subset Y$, so that φ^* is injective.

I.3.5. Let $H = V(f) \subset \mathbb{P}^n$ with $\deg f = d$. Then under $\nu_d : \mathbb{P}^n \hookrightarrow \mathbb{P}^N$ ($N = \binom{n+d}{n} - 1$), the image of H is now a hyperplane. Thus, since ν_d is isomorphism onto its image, we have that $\nu_d(\mathbb{P}^n) \setminus \nu_d(H)$ is affine, and thus $\mathbb{P}^n \setminus H$ is affine.

I.3.7(b). If $Y \subset \mathbb{P}^n \setminus H$, then Y is affine and thus is a point (contradiction to $\dim Y \geq 1$).

I.3.14(a). Fix a representative $(p_0, \dots, p_{n+1}) \in \Bbbk^{n+1}$ for the point P. WLOG let $\mathbb{P}^n = V(x_0) \subset \mathbb{P}^{n+1}$. Then for $Q = [q_0, \dots, q_{n+1}] \in \mathbb{P}^{n+1} \setminus P$, we have $\varphi(Q) = [p_0q_1 - p_1q_0, \ \dots \ p_0q_{n+1} - p_{n+1}q_0]$, which is morphism by the Lemma below.

Lemma. Let $\varphi : Y \to \mathbb{P}^n$ be a map and $\Bbbk[x_0, \dots, x_n]$ the coordinate ring of $\mathbb{P}^n$. Then φ is a morphism iff $\frac{x_i}{x_j} \circ \varphi : \varphi^{-1}(U_j) \to \Bbbk$ is a morphism for every $0 \leq i, j \leq n$. In particular, a map $\varphi : Y \to \mathbb{P}^n$ given by $n+1$ homogeneous polynomials in y_i's (homogeneous coordinates for Y) is a morphism. *Proof:* That a map is a morphism is a local property, so take distinguished affine covers of $\mathbb{P}^n$ and use Lemma I.3.6. $\square$

I.3.15(a,d). (a): Let $X \times Y = Z_1 \cup Z_2$. Define $X_i := \{x \in X \mid x \times Y \subset Z_i\}$. $X = X_1 \cup X_2$ since $x \times Y$ is irreducible (hence for each $x \in X$ we have $x \times Y \subset Z_1$ or $x \times Y \subset Z_2$). Moreover, X_1 and X_2 are closed since ideal membership test is an algebraic condition on the coefficients. But then $X = X_1$ (WLOG) so that $X \times Y \subset Z_1$. (d): Take transcendental bases $(x_1, \dots, x_d)$ and $(y_1, \dots, y_e)$ for $K(X)$ and $K(Y)$. As $K(X \times Y) = K(X)K(Y)$, we have that $(\underline{x}, \underline{y})$ is transcendental basis for $K(X \times Y)$. (Use Vakil's equivalence relation definitions for transcendental basis).

I.3.16. (a/b): Writing the coordinates of $\mathbb{P}^N$ as $\begin{bmatrix} x_0y_0 & \cdots & x_0y_n \\ \vdots & \ddots & \vdots \\ x_ny_0 & \cdots & x_ny_n \end{bmatrix}$, it is clear what the equations in $I(X)$ and $I(Y)$ become in these coordinates. (c): work via distinguished affine patches by noting that $\mathbb{P}^n \times \mathbb{P}^m \supset U_i \times U_j \simeq U_{p_{ij}} \cap \sigma \subset \mathbb{P}^N$ where σ is the image of the Segre embedding.

I.3.17. (a)/(b): UFD is integrally closed, and localization preserves integral closure. (c): $(y/x)^2 - x = 0$. (d): Suppose $A_\mathfrak{m}$ is normal for all $\mathfrak{m}$. If $f \in K(A)$ integral over A, then f is integral over $A_\mathfrak{m}$ for all $\mathfrak{m}$, and hence $f \in A_\mathfrak{m}$ for all $\mathfrak{m}$, and thus $f \in A$. (e): If $A' \supset A$ is integral closure of A, and $A \hookrightarrow B$ where B is integrally closed in $K(B)$, then $A' \subset B$.

I.3.18(b). That the twisted quartic is normal is easy (on affine patches, it is actually isomorphic to $\mathbb{A}^1$). However, it is not projective normal since $t^2u^2 = \frac{z^2}{w} \notin S(Y)$ and $(\frac{z^2}{w})^2 - xw = 0$.

I.3.20 (a): WLOG we can assume Y is affine. Let $A = A(Y)$, and $\mathfrak{m}$ correspond to P. Then $A_\mathfrak{m}$ is Noetherian normal domain. Now, for any prime $\mathfrak{p} \subset \mathfrak{m}$ of height 1, that $\dim Y \geq 2$ guarantees that $\mathfrak{p} \neq \mathfrak{m}$ and moreover there exists $\mathfrak{m} \neq \mathfrak{m}' \supset \mathfrak{p}$. Thus,

$$A_\mathfrak{m} = \bigcap_{\mathfrak{p} \subset \mathfrak{m},\ \mathrm{ht}\,\mathfrak{p}=1} A_\mathfrak{p}$$

by algebraic Hartog's lemma. Now, if $f \in \mathscr{O}(Y \setminus P)$, then $f \in \bigcap_{\mathfrak{m}' \neq \mathfrak{m}} A_{\mathfrak{m}'}$. Moreover, $f \in A_\mathfrak{m}$ by our observation above, and hence $f \in \bigcap_{\mathfrak{m}'} A_{\mathfrak{m}'} = A = \mathscr{O}(Y)$.

I.4 Rational maps

I.4.4(c) $Y_x \simeq V(y^2 = z + 1)$ and hence Y consists of points $[1, t, t^2 - 1]$ and $[0, 0, 1], [0, 1, 0]$. Thus, the image of the projection from $[0, 0, 1]$ to $V(z)$ is $\{[1, t, 0]\} \cup [0, 1, 0] \simeq \mathbb{P}^1$. The inverse map $[s, t] \mapsto [s^2, st, t^2 - s^2]$ is clearly a morphism. So, $Y \setminus P$ is in fact isomorphic to $\mathbb{P}^1$.

I.4.5. Birational is easy (they both contain $\mathbb{A}^2$ as an open set). For non-isomorphic, note that any two curves in $\mathbb{P}^2$ intersect, whereas $\mathbb{P}^1 \times \mathbb{P}^1$ has a family of lines whose members don't intersect.

I.4.6. (a,b): For $a_1a_2a_3 \neq 0$, it is easy to see that φ is an involution.

(c): The map φ is defined on $\mathbb{P}^2 \setminus S$ where $S = \{[1:0:0], [0:1:0], [0:0:1]\}$. To see that the map φ does not extend to a larger open subset U of $\mathbb{P}^2$, note that $\mathrm{Cl}\,\mathbb{P}^2 = \mathrm{Cl}\,U = \mathrm{Cl}\,\mathbb{P}^2 \setminus S$ as points are of codimension 2 in $\mathbb{P}^2$. So, it φ extends, then the linear system on U is the same as that of the original, i.e. $k\{xy, yz, zx\}$, which has three base-points (hence a contradiction).

I.4.7. WLOG, we can assume that X, Y are affine varieties. Now, taking Frac of an isomorphism $\theta : \mathscr{O}_{Y,q} \xrightarrow{\sim} \mathscr{O}_{X,p}$ gives $\widetilde{\theta} : K(Y) \xrightarrow{\sim} K(X)$, which gives a rational map $X \to Y$ by functions $\widetilde{\theta}(y_i) = \theta(y_i) \in \mathscr{O}_{X,p}$. So the rational map is defined on an open set containing p. To see that $p \mapsto q$, use the following lemma:

Lemma. Suppose $\varphi : X \to Y$ is a morphism, and p, q are points of X, Y, and there is a map of local rings $\mathscr{O}_{Y,q} \to \mathscr{O}_{X,p}$ that extends $\varphi^* : \mathscr{O}(Y) \to \mathscr{O}(X)$. Then $\varphi(p) = q$. *Proof:* Suppose $\varphi(p) = q' \neq q$. If $f \in \mathscr{O}(Y)$ vanishes on q then $\varphi^*(f)$ must vanish on p. Now, restricting to $U \subset Y$ affine if necessary (so that q, q' correspond to two different maximal ideals) we can find a function $f \in \mathscr{O}_{Y,q}$ such that $f(q) \neq 0$ but $f(q') = 0$ (f is a unit). Then $\varphi^*(f)(p) = f(q') = 0$ so that $\varphi^*(f)$ is not a unit in $\mathscr{O}_{X,p}$, which is a contradiction.

I.4.10. Let $(x, y, t : u)$ be the coordinates for $\mathbb{A}^2 \times \mathbb{P}^1$, and $X := V(xu - ty)$ be the blow-up. The curve $C' = C \setminus (0, 0)$ is equal to $C \setminus V(x)$ or $C \setminus V(y)$. In the ring $\Bbbk[x, y]/(x^3 - y^2)$, note that inverting x makes y invertible and vice versa, so the in either case the coordinate ring $\Bbbk[C']$ is $\Bbbk[x^{\pm}, y^{\pm}]/(x^3 - y^2)$. We now compute the fiber

$$\begin{array}{ccc} \varphi^{-1}(C') & \longrightarrow & X \subset \mathbb{A}^2 \times \mathbb{P}^1 \\ \big\downarrow & & \big\downarrow \\ C' & \longrightarrow & \mathbb{A}^2 \end{array}$$

in two charts: (i) where $t \neq 0$ i.e. $X \cap (\mathbb{A}^2 \times U_t)$ and (ii) $u \neq 0$ i.e. $X \cap (\mathbb{A}^2 \times U_u)$.

(i) On U_t, we have $\Bbbk[x^{\pm}, y^{\pm}, \frac{u}{t}]/(x^3 - y^2, x\frac{u}{t} - y) = \Bbbk[x^{\pm}, y^{\pm}, \frac{u}{t}]/(x - (\frac{u}{t})^2, y - (\frac{u}{t})^3)$ so that the closure of $\varphi^{-1}(C')$ inside $\mathbb{A}^2 \times U_t$ is the twisted cubic $V(x - (u/t)^2, y - (u/t)^3)$.

(ii) On the other patch U_u, we get that the closure in $\mathbb{A}^2 \cap U_u$ is $V(x(\frac{t}{u})^2 - 1, y(\frac{t}{u}) - 1)$, so that the closure does not contain a fiber over $(0,0)$. In fact, all the points of $\widetilde{C}$ belong to U_t, and the map $\widetilde{C} \to C$ is given by $\Bbbk[x,y]/(x^3 - y^2) \hookrightarrow \Bbbk[x, y, \frac{u}{t}]/(x - (\frac{u}{t})^2, y - (\frac{u}{t})^3) \simeq \Bbbk[\frac{u}{t}]$. In Ex.I.3.2. we showed that this map is homeomorphism but not an isomorphism.

I.5 Nonsingular varieties

I.5.1. They all have singularity at the origin (0,0) only. By inspecting the tangent directions at (0,0) (cf. Ex.I.5.3), we can see that (a) tacnode, (b) node, (c) cusp, (d) triple point.

I.5.3. (a): If $\mu_p(Y) = 1$ then $[df]_p$ has rank 1, so Y is smooth at p. If $\mu_p(Y) > 1$, then $[df]_p$ is a null matrix, so Y is singular at p. (b): node 2, triple point 3, cusp 2, tacnode 2.

Lemma (few things on length). The length of a module M over a Noetherian local ring $(A, \mathfrak{m}, \Bbbk)$ can be computed in many ways. One can find a composition series for M. Or equivalently, it is $\dim_{\Bbbk}(\operatorname{gr}_{\mathfrak{m}} M) = \dim_{\Bbbk} M/\mathfrak{m}M + \dim_{\Bbbk} \mathfrak{m}M/\mathfrak{m}^2M + \cdots$ (if M is Artinian, then $\mathfrak{m}^N M = 0$ for $N >> 0$). When $M = A$ with A a finite type $\Bbbk$-algebra, we are thus counting the number of (linearly independent) monomials.

Let $f = x^m$ and let n be the least such that the coefficient of y^n is nonzezo in g. Then $\mathscr{O}_0/(f,g) = mn$. *Proof:* The monomials are exactly $\{x^\alpha y^\beta : 0 \leq \alpha \leq m-1,\ 0 \leq \beta \leq n-1\}$.

I.5.4. (a): Since $f \neq g \subset \mathfrak{p}$ are irreducible, (f,g) is a regular sequence on $\mathscr{O}_p$, so that $\dim \mathscr{O}_p/(f,g) = 0$ by Krull's height theorem. Thus, $\mathscr{O}_p/(f,g)$ is Artinian and hence has finite length. Inequality I dont' really know....

(b): WLOG let $P = (0,0)$, and let $L : ax + by = 0$. If $a = 0$, then $(L \cdot Y)_p = \mu_p(Y)$ iff f_r (the nonzero homogeneous part of f of least degree) has nonzero x^r coefficient. If $a \neq 0$ (so WLOG $a = 1$), then do change of coordinates $x' = x + by, y' = y$. Then the equality holds iff f'_r has nonzero y'^r coefficient (the cases when the coefficient is zero is a polynomial equation in b, so there are only finite many such).

(c): Let $Y = V(f)$ and $L : V(z)$ where $z \not| f$. Thus, $g = f(x,y,0)$ is a polynomial in x, y of degree d. Now, note that for $p = (p_x, p_y, 0) \in Y \cap L \cap U_x$, we have $\mathscr{O}_p/(f,z) \simeq (\Bbbk[\frac{y}{x}]/f_0^x)_{\mathfrak{p}}$ where f_0^x is f dehomogenized and then z set to 0. Note that f_0^x is g^x, the dehomogenization of g. Since p is a point on $Y \cap L$, it corresponds to a root of f_0^x so that $\ell(\mathscr{O}_p/(f,z)) =$ power of $(\frac{y}{x} - \frac{p_y}{p_x})$ in the linear factorization of f_0^x, i.e. the power of $xp_y - yp_x$ in the linear factorization of g. From this it is clear that, $(L \cdot Y) = d$.

I.5.5. $x^d + y^d + z^d$ works as long as $d \neq p$. For $d = p$, consider $f = xy^{p-1} + yz^{p-1} + zx^{p-1}$. Then $[df] = [y^{p-1} + (p-1)zx^{p-2},\ z^{p-1} + (p-1)xy^{p-2},\ x^{p-1} + (p-1)yz^{p-2}]$. Setting $[df] = 0$, and noting that char $= p$, we have that the singular points are given by $f = 0$ and $xy^{p-1} = yz^{p-1} = zx^{p-1}$. The only solution is $x = y = z = 0$ if $p \neq 3$. Actually, the better thing to do is this: $f = x^p + y^p + z^p + xyz$.

I.5.6(b,d). (b): WLOG $P = (0,0)$, so let $f = xy + g$ define the nodal curve where $\deg g \geq 3$. Consider the pullback of $V(f)$ in the blow-up X in two patches $t \neq 0$ and $u \neq 0$. In $t \neq 0$, we have $f = 0$ and $xu = y$, so that plugging in we have $x^2u + g(x, xu) = 0$. But $x^2 | g(x, xu)$ since $\deg g > 2$ so that we have $x^2(u + g(x, xu)/x^2) = 0$. The $x^2 = 0$ gives the exceptional divisor E, and $u + g(x, xu)/x^2 = 0$ gives the pullback of our curve. Here, if $x = 0$ then $u = 0$ (since $\deg g > 2$), so on this affine patch the blow-up curve meets E at $u = 0$ (i.e. once). Moreover, the point $(x, u) = (0, 0)$ is clearly smooth since $\partial_u(u + g/x^2) = 1$. The same holds on the patch $u \neq 0$.

(d): Blow-up once and we get $x^2 - u^3$, the cuspidal cubic.

Lemma. Dehomogenization commutes with derivation. Precisely, if $f(x_0, \ldots, x_n)$ is a homogeneous polynomial in $\Bbbk[\underline{x}]$, then $\frac{\partial f}{\partial x_i}(x_0, \ldots, x_j = 1, \ldots, x_n) = \frac{\partial}{\partial x_i} f(x_0, \ldots, x_j = 1, \ldots, x_n)$ for $i \neq j$. *Proof:* Both operations are $\Bbbk$-linear, so suffices to check on monomials, for which the it is obvious.

I.5.7. (a): Let $X = V(f)$. That $X \setminus \{0\}$ is nonsingular follows easily from the lemma above, and moreover, that X is in fact singular at $(0,0,0)$ follows from $\deg f > 1$.

(b): $\widetilde{X}$ is possibly singular at points on $\widetilde{X} \cap E$ (where E is the exceptional divisor). Now, let $(x, y, z, t : u : v)$ be coordinates on $\mathbb{A}^3 \times \mathbb{P}^2$ and consider $\widetilde{X} \cap U_t$. The equations for $\mathrm{Bl}_0(\mathbb{A}^3)$ is $xu = ty, xv = tz, yv = uz$, so that on this patch the equations we are considering is $f(x, xu, xv)$. Now, since f is homogeneous of degree > 1, we have that $f(x, xu, xv) = x^d f(1, u, v)$. Now $x^d = 0$ gives the exceptional divisor E, and $f(1, u, v)$ gives the strict transform. $\widetilde{X} \cap U_t$ is nonsingular on points $(x, u, v) = (0, u, v)$ satisfying $f(1, u, v) = 0$, since X is singular only at $(0, 0, 0)$. Likewise for patches U_u and U_v.

(c): The three affine patches are $f(1, u, v), f(t, 1, v), f(t, u, 1)$.

I.5.8. That the rank of $[J(\underline{f})]$ is independent of the homogeneous coordinates follows from that f_i's are homogeneous (hence so are $\frac{\partial f_i}{\partial x_j}$'s). By the lemma above, passing to an affine patch (WLOG) U_{x_0} is the same as making the first column of $[J(\underline{f})]$ into a zero column. This does not change the rank however by the Euler's formula.

I.5.9. Suppose $f = gh$ is reducible (so g, h are not units). Then $Z(g) \cap Z(h) \neq \emptyset$ by Ex.I.3.7, so consider p in the intersection. Then $\mathscr{O}_p/f$ is not an integral domain. However, $V(f)$ is a nonsingular curve, and hence $\mathscr{O}_p/f$ must be a domain (see lemma below).

Lemma. A (Noetherian) regular local ring $(A, \mathfrak{m}, \Bbbk)$ is an integral domain. *Proof:* Induct on the dimension of A. If $\dim A = 0$, then $A \simeq \Bbbk$ so A is a domain. Now, if $\dim A = n$, take a regular sequence $(x_1, \ldots, x_n)$ (exists by regular-ness). Now, consider $A' = A/x_1$. By Krull's height theorem, $\dim A' = n - 1$, and moreover $\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2 \simeq (\mathfrak{m}/\mathfrak{m}^2)/(\Bbbk \cdot x_1)$, so that A' is in fact regular local ring as well. Thus, A' is a domain. Now, suppose $a, b \in A$ nonzero such that $ab = 0$. By Krull's intersection theorem, we can write $a = a'x_1^m$ and $b = b'x_1^n$ where $a', b' \notin (x_1)$. Since x_1 is a nonzero divisor, we thus have $a'b' = 0$. However, this implies that $\overline{a}', \overline{b}' \neq 0$ but $\overline{a}'\overline{b}' = 0$.

I.5.10(b,c). (b): A morphism $\varphi : X \to Y$ mapping $p \mapsto q$ induces a local ring map $\varphi_p^* : \mathscr{O}_{Y,q} \to \mathscr{O}_{X,p}$. Let $\mathfrak{n}, \mathfrak{m}$ be maximal ideals corresponding to p, q. Then we have a map of $\mathscr{O}_{Y,q}$-modules $\mathfrak{n} \to \mathfrak{m}$, from which we get a map $\mathfrak{n}/\mathfrak{n}^2 \to \mathfrak{m}/\mathfrak{m}^2$ of $\mathscr{O}_{Y,q}$-modules and hence $\mathscr{O}_{Y,q}/\mathfrak{n} \simeq \Bbbk$-vector spaces. Taking the dual, we get $(\mathfrak{m}/\mathfrak{m}^2)^\vee \to (\mathfrak{n}/\mathfrak{n}^2)^\vee$. More concretely, taking an affine open charts, suppose X, Y are affine so that the ring map is $\Bbbk[y_1, \ldots, y_m]/I(Y) \to \Bbbk[x_1, \ldots, x_n]/I(X)$ given by $y_i \mapsto f_i(\underline{x})$. Actually, while this can be done, it will have to wait a bit until we hit differentials...
(c): The map before taking duals is $\psi : \ (x)/(x^2) \to (\overline{y})/(\overline{y}^2)$ which comes from restriction of $\Bbbk[x] \hookrightarrow \Bbbk[x, y]/(x - y^2)$. This is a zero map, and so is the dual thus.

I.5.11. The two equations are $x^2 - xz = yw$ and $yz = (x+z)w$. Eliminating w, we get $y^2z - x^3 + xz^2$, the equation for the elliptic curve $E \subset \mathbb{P}^2$. For any points $(x : y : z) \in E$ satisfying such that $y \neq 0$

or $x+z \neq 0$, there exists a unique point (x,y,z,w) projecting down to (x,y,z) via φ. However, the fiber of φ over $[1,0,-1]$ (i.e. $y=0$ and $x+z=0$) is empty (the first equation then reads $2=0$). Thus, $\varphi : Y \setminus P \to E \setminus [1,0,-1]$ is a bijection. Moreover, we know check that the map is an isomorphism on two local patches $y \neq 0$ and $x+z \neq 0$. Let's do the (harder) $x+z \neq 0$ case: on U_{x+z} the equation $y^2z - x^3 + xz^2 = 0$ becomes $y^2(1-x) - x(2x-1) = 0$, whereas the fiber $\varphi^{-1}(E \cap U_{x+z})$ is affine variety in $U_{x+z} \simeq \mathbb{A}^4$ given by equations $x(2x-1) = yw$ and $y(1-x) = w$ (so plugging in the second equation to first) we have that $\mathbb{P}^2 \supset E \cap U_{x+z} \simeq Y \cap U_{x+z} \subset \mathbb{P}^3$ isomorphism.

I.5.12. (a): Follows immediately from Sylvester's law of inertia (whose proof is essentially a smart Gram-Schmidt process). (b): Use that if f factors, it must be two linear (homogeneous) polynomials. (c): The singular locus is given by $V(x_0,\ldots,x_r)$ (cf. Ex.I.5.8). (d): Quite obvious, once in the right coordinates (think "reverse of projection map").

I.5.13. The statement is local, so WLOG assume that Y is an affine variety with $A := A(Y)$. Let A' be integral closure of A where A is a finite-type $\Bbbk$-algebra domain. Theorem I.3.9A implies that for some $f_1,\ldots,f_m \in A'$, the map $A[f_1,\ldots,f_m] \hookrightarrow A'$ is in fact surjective (hence isomorphism). Now, since $f_i = a_i/b_i$ for some $a_i, b_i \neq 0 \in A$, consider $A_{b_1\cdots b_m}$. Clearly, $A' \subset A_{b_1\cdots b_m}$, and in fact, $A'_{b_1\cdots b_m} \simeq A_{b_1\cdots b_m}$. Thus, $A_{b_1\cdots b_m}$ is integrally closed. This shows that the nonnormal points are proper.

To show that normal points are in fact open, suppose $\mathfrak{m} \subset A$ corresponds to a normal point. Since $(A')_{\mathfrak{m}}$ is the integral closure of $A_{\mathfrak{m}}$, we have that the map $A \hookrightarrow A'$ induces an isomorphism $A_{\mathfrak{m}} \xrightarrow{\sim} (A')_{\mathfrak{m}}$. By Ex.I.4.7, we have that there is an open set of Y containing $\mathfrak{m}$ that is isomorphic to an open subset of $\widetilde{Y}$, the normalization of Y.

I.6 Nonsingular curves

Skip for now.

I.7 Intersections in Projective Space

I.7.1. (a): Let $Y \subset \mathbb{P}^N$ be the image of the Veronese embedding of $\mathbb{P}^n$. Then $S(Y) \simeq \Bbbk[x_1,\ldots,x_n]_{d\bullet}$, so that $P_Y(z) = P_{\mathbb{P}^n}(dz) = \binom{dz+n}{n} = \frac{d^n}{n!}z^n + \cdots$. (b): If $Y \subset \mathbb{P}^N$ is the image of the Segre embedding of $\mathbb{P}^r \times \mathbb{P}^s$, then $S(Y) \simeq \bigoplus_{i\geq 0}(\Bbbk[\underline{x}]_i \otimes \Bbbk[\underline{y}]_i)$ as graded rings. Hence, $P_Y(z) = P_{\mathbb{P}^r}(z)P_{\mathbb{P}^s}(z) = \binom{z+r}{r}\binom{z+s}{s} = \frac{1}{r!s!}z^{r+s} + \cdots$.

Lemma. Let $S(Y)$ and $S(Z)$ be coordinate rings for projective varieties $Y \subset \mathbb{P}^n$ and $Z \subset \mathbb{P}^m$. Then the coordinate ring for $Y \times Z$ as a subvariety of the Segre embedding of $\mathbb{P}^n \times \mathbb{P}^m$ is isomorphic to $\bigoplus_{i\geq 0}(S(Y)_i \otimes S(Z)_i)$ as graded rings ($\Bbbk$-algebras). In particular, $P_Y(z)\cdot P_Z(z) = P_{Y\times Z}(z)$. *Proof:* The ring $S(Y \times Z)$ is the image of the map $\Bbbk[z_{ij}\text{'s}]_\bullet \to S(Y)_\bullet \otimes_\Bbbk S(Z)_\bullet$ where $z_{ij} \mapsto \overline{x_i} \otimes \overline{y_j}$.

I.7.2(c,d,e). (c): From $0 \to S(-d) \xrightarrow{f} S \to S(H) \to 0$, we have $P_H(z) = \binom{z+n}{n} - \binom{z-d+n}{n}$ so that $p_a(H) = (-1)^{n-1}(1 - \binom{-d+n}{n} - 1) = (-1)^n\frac{(n-d)(n-d-1)\cdots(n-d-n+1)}{n!} = (-1)^{2n}\frac{(d-1)\cdots(d-n)}{n!} = \binom{d-1}{n}$. (d): Let f, g polynomials defining hypersurfaces H_1, H_2 of degree a, b. Then, from $0 \to S/f(-b) \xrightarrow{g} S/f \to S/(f,g) \to 0$, we have $P_{H_1\cap H_2}(z) = P_{H_1}(z) - P_{H_2}(z-b)$ so that $p_a(H_1 \cap H_2) = \binom{a+b-3}{3} - \binom{a-3}{3} - \binom{b-3}{3} = \frac{1}{6}(3ab(a+b) - 12ab + 6) = \frac{1}{2}ab(a+b-4) + 1$. (e): Follows immediately from the lemma above.

I.7.3. Let Y be given by a polynomial f, and suppose $p = (p_0 : p_1 : p_2) \in Y$. Let L be the line $a_0x_0 + a_1x_1 + a_2x_2 = 0$ going through the point p (i.e. $\vec{a}\cdot\vec{p} = 0$). WLOG, let $p_0 \neq 0$. Then, since

$a_0 = -\frac{a_1p_1+a_2p_2}{p_0}$, all the lines through the point p are given exactly by $a_1(\frac{x_1}{x_0} - \frac{p_1}{p_0}) + a_2(\frac{x_2}{x_0} - \frac{p_2}{p_0}) = 0$ (where a_1, a_2 not both zero), and so we can work in the affine chart U_0. Now, apply the following lemma:

Lemma. If $f(x, y)$ is a polynomial and $P = (p, q) \in V(f) \subset \mathbb{A}^2$, and $L:\ a(x-p)+b(y-q) = 0$ is a line through (p, q), then the intersection multiplicity $V(f) \cap L$ at P is the multiplicity of the root $X = 0$ of the polynomial $f(X + p, -\frac{a}{b}X + q)$ (if $b \neq 0$). In particular, if $V(f)$ is nonsingular at P, then $I(V(f), L; P) > 0$ iff $(a : b) = (f_x(P) : f_y(P))$.

Proof: The first statement follows from the fact that $\Bbbk[x]_{(x)}$ is a DVR. For the second statement, WLOG let $(p, q) = (0, 0)$. Then $f = f_x(P)x + f_y(P)y +$ (higher order). When using $ax + by = 0$ to do the coordinate change, the linear term vanishes iff $(a : b) = (f_x(P) : f_y(P))$. Otherwise, the linear term survives and the multiplicity of zero remains 1.

I.7.4. By Bezout's theorem and Ex.I.7.3 above, a line L intersects $Y \subset \mathbb{P}^2$ at exactly d points if it is not tangent to any point of Y and does not go through singular points of Y. Since $\dim Y = 1$, we have $\dim \operatorname{Sing} Y = 0$, so that $\operatorname{Sing} Y$ is finitely many points $\{\vec{p}_1, \dots, \vec{p}_m\}$. Thus, the set of lines that go through singular points or is tangent to a point of Y is given by the union of the morphism of Ex.I.7.3 (which is a proper closed subset) and $\bigcup_i V(\vec{p}_i \cdot \vec{x})$.

I.7.5(b). Note that if $p \in Y \subset \mathbb{P}^2$ is a point of multiplicity m, then a generic line through p meets Y at p with multiplicity m (i.e. there is an dense open subset U of $\mathbb{P}^1 \simeq \{\text{lines through } p\}$ such that if $L \in U$ then $i(Y, L; p) = m$. Now, WLOG let $P = (0 : 0 : 1) \in Y$ be the point with multiplicity $d-1$. Then we can consider two maps: (1) $\varphi : Y \setminus P \to \mathbb{P}^1$ which is the projection map and (2) $\psi : \mathbb{P}^1 \supset U \to Y \setminus P$ which maps a point L of U to the unique point that L meets Y other than P (uniqueness guaranteed by Bezout's theorem and the above observation). In sum we have:

$$Y \setminus P \xrightarrow{\varphi} \mathbb{P}^1 \overset{\psi}{\dashrightarrow} Y \setminus P$$

$$(a : b : c) \mapsto (a : b : 0) \mapsto (a : b : c)$$

It is clear that these are rational (dominant) maps, and thus establishes a birational equivalence of $Y \setminus P$ and $\mathbb{P}^1$.

I.7.6. Proposition I.7.6(a,b) implies that Y is in fact irreducible, so suppose Y is a variety. Also, $r = 0, n$ case is trivial, so we assume $1 \leq r \leq n-1$.

First, suppose $\dim Y = 1$. Take $P \neq Q \in Y$ and let $L = \overline{PQ}$. If there exists $R \in Y$ such that $R \notin L$, we can find a hyperplane H such that $H \supset \overline{PQ}$ and $H \not\ni R$. Then since $\deg Y = 1$, Bezout's implies that $Y \cap H$ must be a degree 1 variety with dimension 0 (dimension by Krull). But $Y \cap H$ contains two points P, Q and hence is a contradiction. Hence, $\overline{PQ} = Y$.

Now, let $\dim Y = r$. By induction hypothesis, for any $H \not\supset Y$, we have that $Y \cap H$ is a linear variety of dimension $r-1$. By dimension argument, we can find two hyperplanes H_1, H_2 such that $Z_i = Y \cap H_i$ $(i = 1, 2)$ are two different linear $(r-1)$-planes. Moreover, by Krull's height theorem, $Z_1 \cap Z_2 = Y \cap H_1 \cap H_2$ must have dimension $r-2$. Hence, $Z_1 \cap Z_2$ as a $\Bbbk$-*vector space* has dimension $r-1$ so that $Z_1 + Z_2$ as a $\Bbbk$-vector space has dimension $r+1$. Thus, $\overline{Z_1Z_2} := Z_1 + Z_2$ as a projective linear variety has dimension r. Now, if there exists $P \in Y$ such that $P \notin \overline{Z_1Z_2}$, then since $\overline{Z_1Z_2}$ has dimension $r < n$ we can find a hyperplane H such that $H \supset \overline{Z_1Z_2}$ but $H \not\ni P$. Once again $Y \cap H$ must be a variety of dimension $r-1$, but it contains Z_1 and Z_2, which is a contradiction. Hence, $Y = \overline{Z_1Z_2}$, as desired.

Alternatively, one can show that for since $Y \cap H$ is linear (or Y) for any hyperplane H, $Y \cap$ linear variety is always linear. Hence, for any two points $p, q \in Y$, we have $Y \cap \overline{pq}$ being a linear variety (hence is $\overline{pq}$). Thus, since any line through two points on Y is in Y, we have that Y must be linear.

I.7.7. Again, none of the answers online are satisfactory to me... Maybe I'm being stupid...

I.7.8. Follows immediately from Ex.I.7.7. and Ex.I.7.6. It's just quite cool that a degree 2 variety in $\mathbb{P}^n$ is thus in fact a complete intersection.

Chapter II

Schemes

II.1 Sheaves

II.1.1 A bit of category nonsense...

that Hartshorne doesn't really do.

An **additive category** is a category $\mathscr{C}$ such that (i) the Hom-sets are Abelian groups such that composition is a $\mathbb{Z}$-bilinear map, (ii) the zero object exists, and (iii) finite direct product / coproduct exist.

A few easy facts regarding additive categories:

- finite direct product is isomorphic to coproduct.
- ker/coker of mono/epi-morphism is the 0 map ($0 \to A \xrightarrow{\varphi} B$ if φ monomorphism)
- ker/coker of 0 map ($A \to 0$ / $0 \to A$) is itself, and conversely if ker / coker of a map is itself then the map is a 0 map.
- ker/coker is a mono/epi-morphism

What may *not* true is that: $(*)$ if $\ker(\varphi : A \to B) = 0$ or $\operatorname{coker}\varphi = 0$, then φ is a monomorphism or epimorphism (respectively) Find an example of this?

An **Abelian category** is an additive category $\mathscr{C}$ such that (i) kernels and cokernels exist, and (ii) **coimage** (cokernel of kernel) is isomorphic to **image** (kernel of cokernel) for any morphism. Equivalently, one can say (ii) by saying (a) every monomorphism is kernel of its cokernel, and (b) every epimorphism is cokernel of its kernel.

The relevant diagram is:

$$\begin{tikzcd}
\ker\theta \arrow[r, "i"] & A \arrow[r, "\theta"] \arrow[d] & B \arrow[r, "\pi"] & \operatorname{coker}\theta \\
& \operatorname{coim}\theta \arrow[r, "\sim"] & \operatorname{im}\theta \arrow[u] &
\end{tikzcd}$$

Now the notion of exact sequences make sense, and the statement $(*)$ above is true. For example, if $A \xrightarrow{\varphi} B \xrightarrow{0} 0$ is exact, then $B = \ker 0 = \operatorname{im}\varphi = \ker(B \to \operatorname{coker}\varphi)$, so that $\operatorname{coker}\varphi = 0$ (so far we didn't use Abelian-ness). To conclude that φ is epimorphism, we note that the diagram above becomes:

$$\begin{tikzcd} \ker\theta \arrow[r, "i"] & A \arrow[r, "\varphi"] \arrow[d] & B \arrow[r, "\pi"] & 0 \\ & \operatorname{coim}\varphi \arrow[r, "\sim"] & B \arrow[u, "\mathrm{Id}"'] & \end{tikzcd}$$

so that $A \to \operatorname{coim}\varphi \simeq B$ is a cokernel map and hence an epimorphism.

Let's show that sheaves of Abelian groups and more generally $\mathscr{O}_X$-modules form Abelian categories. Let's not be careful about which one exactly and say the category Sh_X of sheaves on a topological space X. That Sh_X is an additive category is easy to check. Moreover, that kernels and cokernels exist is also easy.

N.B. Hartshorne defines $\Im(\varphi : \mathscr{F} \to \mathscr{G})$ by sheafifying the presheaf $\Im\varphi(U) := \Im\varphi(U)(\mathscr{F}(U))$. To see that this matches the category theoretical definition, we note:

Lemma. If $\pi : \mathscr{F} \to \mathscr{G}$ is a map of sheaf to a presheaf, and $\pi^+ : \mathscr{F} \to \mathscr{G}^+$ is the map π composed with sheafification of $\mathscr{G}$, then $\ker\pi^+ \simeq (\ker\pi)^+$.

The proof is just universal property nonsense. The lesson is that to say $\ker(\pi : \mathscr{F} \to \mathscr{G})$ is a sheaf, we need *both* $\mathscr{F}$ and $\mathscr{G}$ to be sheaves.

With these set, to show that coimage is isomorphic to image note that if $\mathscr{F} \simeq \mathscr{G}$ as presheaves then $\mathscr{F}^+ \simeq \mathscr{G}^+$ naturally as sheaves. So, we can check on the *presheaf* coimage and image the isomorphism, reducing our case to statements about coimage and images in AbGrp, R-Mod, etc.

And... there we have it! Sheaves of Abelian groups or modules is an Abelian category, so that exact sequences make sense, and kernels, cokernels, images, and coimages as defined in Hartshorne match categorical theoretical viewpoint. In fact, the terms "injective" and "surjective" is now better replaced with "monomorphism" and "epimorphism."

II.1.2 Back to exercises...

The above discussion essentially takes care of the following exercises: II.1.6, II.1.7, II.1.9.

II.1.2. (a): Let's just do everything concretely. From $\ker\varphi \overset{i}{\to} \mathscr{F} \overset{\varphi}{\to} \mathscr{G}$, we have a natural map $(\ker\varphi)_p \overset{i_p}{\to} \mathscr{F}_p \overset{\varphi_p}{\to} \mathscr{G}_p$. The map i_p takes $[(f, U)] \in (\ker\varphi)_p$ and treats it as an element of $\mathscr{F}_p$. Injectivity is clear, and if $[(f, U)] \in \ker\varphi_p$ then $\varphi(f)|_V = 0$ for some $V \subset U$ so that $[(f, U)] = [(f|_V, V)] \in (\ker\varphi)_p$ since $f|_V \in \ker\varphi(V)$. For $\operatorname{im}\varphi$, since im = ker of coker, use the fact that cokernel commutes with colimits and apply the results we just proved.

(b): The sheafification construction makes it clear that for a sheaf $\mathscr{F}$, we have $\mathscr{F}_p = 0\ \forall p \in X$ iff $\mathscr{F} = 0$. Our category theory definition makes it clear that ker $= 0$ or coker $= 0$ implies injective or surjective. Hence, via part (a), that φ is injective iff φ_p is injective $\forall p$ follows. Note that $\operatorname{coker}\varphi = 0$ iff $\operatorname{im}\varphi = \mathscr{G}$, and if $(\operatorname{im}\varphi)_p = \operatorname{im}\varphi_p = \mathscr{G}_p$ for all p, then $(\operatorname{coker}\varphi)_p = 0$ for all p. (Here we secretely use that $\operatorname{coker}\varphi_p = (\operatorname{coker}\varphi)_p$.)

(c): Follows easily from (a).

II.1.3. (a): If the condition holds, then φ_p is surjective for all p. Conversely, if φ is surjective, then φ_p is surjective for all p, so for each s_p we can find $[(t_p, V_p)] \in \mathscr{F}_p$ that maps to s_p such that (shrinking V_p if necessary) we have $\varphi(t_p) = s|_{V_p}$ for each p.

(b): No idea yet.

II.1.6. (a): Since $\mathscr{F}/\mathscr{F}'$ is the cokernel of $\mathscr{F}' \to \mathscr{F}$, and $\mathscr{F}' \to \mathscr{F}$ is a monomorphism (which is a kernel of a cokernel). Piece together category stuff. (b): Same thing; piece together category stuff.

II.1.13.(Espace Étalé). The topology on $\text{Spé}(\mathscr{F})$ can be described more concretely as: $V \subset \text{Spé}(\mathscr{F})$ is open iff for every $\overline{s} : U \to \text{Spé}(\mathscr{F})$ the set $\overline{s}^{-1}(V)$ is open in U. This means that at each point $p \in X$, the set of germs of continuous *at* p is exactly $\mathscr{F}_p$ This is mysteriously hard... Ask?

Now, an element $t \in \mathscr{F}^+(U)$ is by definition a map $t : U \to \text{Spé}(\mathscr{F})$ such that $\pi \circ t = \text{Id}_U$ and locally at each point $p \in V \subset U$, the map $t|_V$ is equal to $\overline{t^p}$ for $t^p \in \mathscr{F}(V)$, and hence t is continuous. Conversely, if $t : U \to \text{Spé}(\mathscr{F})$ is a continuous section, then it is locally a section of $\mathscr{F}$ by the observation above.

II.1.16.(Flasque sheaves) (a): First, note that if U is connected, then $\underline{A}(U) = A$. If X is irreducible, then any open subset is connected.

(b): Let $\varphi : \mathscr{F} \to \mathscr{F}''$. We need a lemma:

Lemma. Let $t_1 \in \mathscr{F}(U_1)$ and $t_2 \in \mathscr{F}(U_2)$ such that $\varphi(t_1)|_{U_1 \cap U_2} = \varphi(t_2)_{U_1 \cap U_2}$. Then there exists $t \in \mathscr{F}(U_1 \cup U_2)$ such that $\varphi(t|_{U_i}) = \varphi(t_i)$, $i = 1, 2$. *Proof:* Since $t_1|_{U_{12}} - t_2|_{U_{12}} \in \ker \varphi(U_{12})$, and $\ker \varphi$ is flasque, the element lifts to $t' \in \ker \varphi(U_1)$. Now, considering $t_1 - t'$ and t_2 (which agree on intersection now), by sheaf condition we obtain t satisfying the lemma.

Now, let $s \in \mathscr{F}''(U)$. Let Σ be a poset of pairs $(U_i, t_i \in \mathscr{F}(U_i))$ where $\varphi(t_i) = s|_{U_i}$ ordered by compatible inclusion. Given any chain, sheaf condition ensures that the union of the chain is the upper bound. Hence, Σ has a maximal element. Now, by Ex.II.1.3 there exists $t_i \in \mathscr{F}(U_i)$ with $\bigcup_i U_i = U$ and $t_i \mapsto s|_{U_i}$, which implies that the maximal element of Σ must be (U, t) for some $t \in \mathscr{F}(U)$.

(c): Easy using (b). (d): Immediate from definition of pushforward.

II.1.19(c). $0 \to j_!(\mathscr{F}|_U) \to \mathscr{F} \to i_*(\mathscr{F}|_Z) \to 0$: check on stalks using parts (a) and (b).

II.1.20. (a): That $\mathscr{H}$ is a presheaf is clear. Now, if $\{s_i \in \mathscr{H}(V_i)\}$ for $\bigcup_i V_i = V$ are sections agreeing on intersections, then it uniquely patches to $s \in \mathscr{H}(V)$, and the support is still contained in $V \cap Z$ since $s_p = (s_i)_p$ for $p \in V_i$.

(b): It is clear that $\mathscr{H} = \ker \varphi$ where $\varphi : \mathscr{F} \to j_*(\mathscr{F}|_U)$.

II.2 Schemes

Lemma. Taking nilradical commutes with localization. Precisely, denote by $\mathfrak{N}(A)$ the nilradical of A, then $\mathfrak{N}(S^{-1}A) = S^{-1}\mathfrak{N}(A)$. *Proof:* $\supset$ is easy. For $\subset$, if $(a/s)^n = 0$, then $ta^n = 0$ $\exists t \in S$ so that $(ta)^n = 0$ and hence $ta \in \mathfrak{N}(A)$. Thus, $\frac{a}{s} = \frac{ta}{ts} \in S^{-1}\mathfrak{N}(A)$.

II.2.3.(Reduced schemes) (a): The above lemma implies that an affine scheme $U = \operatorname{Spec} A$ is reduced iff $A_\mathfrak{p}$ is for all $\mathfrak{p} \in U$. Moreover, reduced-ness is clearly an affine local property. *Alternate Soln:* Using the definition that $\mathscr{O}_X(U) := \{\text{compatible } s : U \to \coprod \mathscr{O}_{X,p}\}$, if all $\mathscr{O}_{X,p}$ are reduced then $\mathscr{O}_X(U)$ is for any U, and if U is reduced then so are $\mathscr{O}_{X,p}$.

(b): Some fanciness: Let $\mathfrak{N}_X$ be the sheaf (an $\mathscr{O}_X$-module) associated to presheaf defined by $\mathfrak{N}_X(U) := \mathfrak{N}(\mathscr{O}_X(U))$. The lemma above implies that $\mathfrak{N}_X$ is a quasi-coherent sheaf and that

$$0 \to \mathfrak{N}_X \to \mathscr{O}_X \to \mathscr{O}_X/\mathfrak{N}_X \to 0$$

is an exact sequence of quasi-coherent sheaves, with $\mathscr{O}_{Xred} = \mathscr{O}_X/\mathfrak{N}_X$. This is all to say that if $U \subset X$ is an affine subscheme then $\mathscr{O}_{Xred}(U) = \mathscr{O}_X(U)/\mathfrak{N}(\mathscr{O}_X(U))$, and thus letting $A := \mathscr{O}_X(U)$ we have $(U, \mathscr{O}_{Xred}|_U) \simeq (\operatorname{Spec} A/\mathfrak{N}(A), \mathscr{O}_{\operatorname{Spec} A/\mathfrak{N}(A)})$ (again by Lemma above). That there is a morphism $X_{red} \to X$ is clear: the topological map is just homeomorphism (since quotienting by the nilradical doesn't change the topology in the affine patches) and the map of sheaves is the one above $\mathscr{O}_X \to \mathscr{O}_X/\mathfrak{N}_X$.

(c): We are looking at:

$$\begin{tikzcd} Y \arrow[d] & X \arrow[l, "f"'] \arrow[dl, dashed, "\exists!"] \\ Y_{red} & \end{tikzcd}$$

The topological map is clear. For the sheaf maps, we have

$$\begin{tikzcd} \mathfrak{N}_Y \arrow[r] & \mathscr{O}_Y \arrow[r] \arrow[d] & f_*\mathscr{O}_X \\ & \mathscr{O}_{Y red} \arrow[ur, dashed] & \end{tikzcd}$$

where the top two maps compose to 0 map since X is reduced. Hence, by universal property of cokernel, there exists a map $\mathscr{O}_{Yred} \to f_*\mathscr{O}_X$.

Lemma. Let X, Y be locally ringed spaces, and suppose there is a cover $\{U_i\}$ of X with map of locally ringed spaces $(f_i, f_i^{\#}) : U_i \to Y$ such that $f_i|_{U_{ij}} = f_j|_{U_{ji}}$. Then the maps f_i glue to a morphism $f : X \to Y$. *Proof:* The topological maps indeed glue to a topological map $f : X \to Y$. For the map of sheaves, $f^{\#} : \mathscr{O}_Y \to f_*\mathscr{O}_X$ is given via the following diagram: For any $U \subset Y$ we have

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{O}_Y(U) \arrow[r] \arrow[d, "\exists!"'] & \prod_i \mathscr{O}_Y(U \cap U_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "f_i^{\#}"'] & \prod_{i,j} \mathscr{O}_Y(U \cap U_i \cap U_j) \arrow[d] \\ 0 \arrow[r] & \mathscr{O}_X(f^{-1}(U)) \arrow[r] & \prod_i \mathscr{O}_X(f^{-1}(U \cap U_i)) \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j} \mathscr{O}_X(f^{-1}(U \cap U_i \cap U_j)) \end{tikzcd}$$

Since $f^{\#}|_{U_i} = f_i^{\#}$, it is clear that $f_p^{\#}$ is a local homomorphism for all $p \in X$.

Lemma. Let X, Y be locally ringed spaces, and $\varphi, \psi : X \to Y$ morphisms. Suppose there is a cover $\{U_i\}$ of X such that $\varphi|_{U_i} = \psi|_{U_i}$ for all i. Then in fact $\varphi = \psi$. *Proof:* Use the ! part of the existence of the downward map above.

Lemma. Let $\operatorname{Spec} A, \operatorname{Spec} B \subset X$ be two affine opens in scheme X with nonempty intersection. Then for any $p \in \operatorname{Spec} A \cap \operatorname{Spec} B$, there exists an open subset U of the intersection containing p such that $\operatorname{Spec} A_f \simeq U \simeq \operatorname{Spec} B_g$ for some $f \in A$, $g \in B$. *Proof:* Take a distinguished affine open of A in the intersection, and then a smaller one inside that is distinguished in B. Now use Ex.II.2.16(a).

II.2.4. We construct the inverse map of α as follows. Cover X by affines $U_i = \operatorname{Spec} B_i$. Given a ring map $A \to \mathscr{O}_X(X)$, we have maps $\varphi_i : A \to \mathscr{O}_X(U_i)$, which gives us morphisms $f_i : U_i \to \operatorname{Spec} A$. Moreover, for any $p \in U_i \cap U_j$ take an affine $U_{ij} = \operatorname{Spec} B_{ij}$ around p distinguished in both U_i and U_j (possible by lemma above). Then the commutativity of the diagram

$$\begin{array}{ccc} A & \longrightarrow & \mathscr{O}_X(U_i) \\ \downarrow & & \downarrow \\ \mathscr{O}_X(U_j) & \longrightarrow & \mathscr{O}_X(U_{ij}) \end{array}$$

imply (via lemma above) that f_i's agree on the intersections. Hence, they glue to a unique morphism $f : X \to \operatorname{Spec} A$.

II.2.7. We prove a more general version. Note that for any $x \in X$ there is a natural morphism $\operatorname{Spec} \mathscr{O}_{X,x} \to X$. We claim that for any local ring $(A, \mathfrak{m})$, giving a morphism $\operatorname{Spec} A \to X$ is equivalent to giving a point $x \in X$ and a local ring map $\mathscr{O}_{X,x} \to A$. That is, there is a canonical map $\operatorname{Spec} \mathscr{O}_{X,x} \to X$ such that:

$$\begin{tikzcd}
 & \operatorname{Spec} \mathscr{O}_{X,x} \arrow[d] \\
\operatorname{Spec}(A, \mathfrak{m}) \arrow[ur, "\exists!"] \arrow[r] & X
\end{tikzcd}$$

Proof: If $x \in X$ is the image of $\mathfrak{m}$ under the map $\varphi : \operatorname{Spec} A \to X$, then for any open set $x \in U \subset X$ and $\mathfrak{p} \in \operatorname{Spec} A$, we have $\varphi(\mathfrak{p}) \in U$ since continuity of φ implies that $\varphi(\overline{\{\mathfrak{p}\}}) \subset \overline{\varphi(\mathfrak{p})}$ so that $x \in \overline{\varphi(\mathfrak{p})}$. So, take U to be any affine $\operatorname{Spec} B \subset X$. Then the map φ factors through $\operatorname{Spec} B$, so that we get a map $B \to A$ where $\mathfrak{m}$ pulls back to the point corresponding to x. Hence, by universal property of localization, we get a natural map (of local rings) $\mathscr{O}_{X,x} \to A$.

In the case where A is a field, the kernel of the map is $\mathfrak{m}_x$ so that the map extends to $\kappa(x) \to A$.

II.2.8. Let $\varphi : \operatorname{Spec} \Bbbk[\epsilon]/\epsilon^2 \to X$. The claim in the previous exercise Ex.II.2.7. shows that this is the same as giving a local homomorphism $\mathscr{O}_{X,x} \to \Bbbk[\epsilon]/\epsilon^2$ for some $x \in X$. To see that x is rational over $\Bbbk$, note that since X is a $\Bbbk$-scheme $\mathscr{O}_{X,x}$, so that the extension of the local map above $\kappa(x) \hookrightarrow \Bbbk$ must be an isomorphism. Lastly, note that local map $\mathscr{O}_{X,x} \to \Bbbk[\epsilon]/\epsilon^2$ induces a map of $\kappa(x) = \Bbbk$-vector spaces $\mathfrak{m}_x/\mathfrak{m}_x^2 \to (\epsilon)/(\epsilon^2)$, whose dual map gives an image of $\epsilon^\vee$ in T_x.

II.2.11. Later after NT or FT

II.2.14. (a): If every element of S_+ is nilpotent, then for any $\mathfrak{p}$ homogeneous prime and any homogeneous $f \in S_+$ we have $f^n = 0 \in \mathfrak{p}$ so that $\mathfrak{p} \supset S_+$. If there is an element $f \in S_+$ that is not nilpotent, then $S_{(f)}$ is not a zero ring, such that $D_+(f) \subset \operatorname{Proj} S_\bullet$ is nonempty.

(b): $U = \operatorname{Proj} T \setminus V(\varphi(S_+))$ by definition is open. Moreover, it has an affine open cover by $\{D_+(f) \subset U \mid f \in \varphi(S_+) \text{ homogenous}\}$. Since $S \to T$ naturally extends to $S_{(f)} \to T_{(\varphi(f))}$, we can define morphisms $\psi_f : D_+(f) \to \operatorname{Proj} S$ for each $f \in \varphi(S_+)$, and these maps agree on the intersections since $S_{(f)} \to (S_{(f)})_{\frac{g^{\deg f}}{f^{\deg g}}} \simeq S_{(fg)}$ commutes with φ.

(c): If $\mathfrak{p} \supset S_{>d}$ for some d, then $\mathfrak{p} = S_+$ in fact. Hence, for any $d > 0$, we have an open cover $\{D_+(f) \mid f \in S_{>d}\text{homogeneous}\}$ of $\operatorname{Proj} S$. This first shows that if $S \to T$ is a map of graded rings that is isomorphism after high enough degrees, then $U = \operatorname{Proj} T$, and moreover, the maps $\varphi_f : D_+(f) \to \operatorname{Proj} S$ is an isomorphism since $S_{(f)} \to T_{(\varphi(f))}$ is (since $\deg f \geq d_0$). Now use Ex.II.2.17(a) below.

(d): Patch isomorphisms $t(V \cap U_{x_i}) \simeq \operatorname{Spec} S_{(\overline{x_i})}$ together.

II.2.15. (a): WLOG V is affine so that $t(V) = \operatorname{Spec} A$ for $A = \Bbbk[x_1, \ldots, x_n]/I$, and note that localization commutes with quotients. Now, if the residue field is $\Bbbk$, then we have $\Bbbk \to (A/\mathfrak{p}) \hookrightarrow (A/\mathfrak{p})_{\mathfrak{p}} \simeq \Bbbk$ so that $\Bbbk = A/\mathfrak{p}$ already. Now, if P is a closed point then Strong Nullstellensatz implies that $\kappa(P) = A/\mathfrak{p}$ is a finite (and hence algebraic) extension of $\Bbbk$, and hence $\kappa(P) = \Bbbk$ since $\Bbbk = \overline{\Bbbk}$.

(b): We have a natural map of local rings $\mathscr{O}_{Y,f(P)} \to \mathscr{O}_{X,P}$. Now, it is easy to check that if $B \to A$ is a local $\Bbbk$-algebra homomorphism and $A/\mathfrak{m} \simeq \Bbbk$ then $B/\mathfrak{n} \simeq \Bbbk$.

(c): Injectivity is easy. Show the surjectivity for V, W affine using (a,b). Patch.

II.2.16(c,d). Just to get rid of trivialities, we assume that f is not nilpotent throughout.

(c): Write $U_i' := U_i \cap X_f$, and let $A_i = \mathscr{O}_X(U_i)$. Then since X since index i is finite, there exists n and $a_i \in A_i$'s such that $f^n b|_{U_i'} = a_i$. Now, note that for any i, j we have $(a_i - a_j)|_{U_i \cap U_j}$ vanishes on $U_i \cap U_j \cap X_f$. Now, since $U_i \cap U_j$ are all quasi-compact, and there are finitely pairwise intersection, there exists m such that replacing a_i by $f^m a_i$ we have $a_i = a_j$ on intersections. Thus, $f^{n+m} b$ is an restriction of some $a \in A$ by gluing together a_i's.

(d): Note that there is always a natural map $A_f \to \Gamma(X_f, \mathscr{O}_X)$ since f restricted to X_f is invertible (glue inverse of f, which is just $1/f$, from each affine pieces). Now, this map is injective if X is quasi-compact ($\because$ (b)), and is surjective if X further satisfies condition in (c) ($\because$ (c)).

II.2.17. (a): That f is a homeomorphism is clear, and that the sheaf map is isomorphism is clear since stalk maps are isomorphisms.

(b): There is a natural morphism $\pi : X \to \operatorname{Spec} \mathscr{O}_X(X)$. If $f_1, \ldots, f_n \in \mathscr{O}_X(X)$ such that $\operatorname{Spec} \mathscr{O}_X(X)_{f_i}$'s cover $\operatorname{Spec} \mathscr{O}_X(X)$, then $X_{f_i} = \pi^{-1}(D(f_i))$'s also cover X. And moreover, since X_{f_i}'s are affine, we have $X_{f_i} \cap X_{f_j} = \operatorname{Spec} \mathscr{O}_X(X_{f_i})_{\overline{f_j}}$ so that the intersections are also quasicompact. Hence, by Ex.II.2.16(d) we have $X_{f_i} \simeq \operatorname{Spec} \mathscr{O}_X(X)_{f_i}$, so that $\pi|_{\pi^{-1}(D(f_i))}$'s are all isomorphisms. Applying (a) above, we have that $X \simeq \operatorname{Spec} \mathscr{O}_X(X)$.

In fact, the object $\operatorname{Spec} \mathscr{O}_X(X)$ has the following universal property: For any morphism $X \to \operatorname{Spec} A$, there exists a unique map $\operatorname{Spec} \mathscr{O}_X(X) \to \operatorname{Spec} A$ such that

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & \operatorname{Spec} \mathscr{O}_X(X) \arrow[d, "\exists!"] \\ & \operatorname{Spec} A \end{tikzcd}$$

With this few, using Vakil's ACL, it is easy to show the lemma below from (b) above.

Lemma. The property of a morphism being affine is LOCT.

II.3 First properties of schemes

II.3.7. Take an affine $\operatorname{Spec} B \subset Y$ and $\operatorname{Spec} A \subset f^{-1}(\operatorname{Spec} B)$ so that we have $f|_{\operatorname{Spec} A} : \operatorname{Spec} A \to \operatorname{Spec} B$ given by $\varphi : B \to A$. Since X, Y are integral, A, B are domains. Moreover, since f is dominant, the generic point must go to the generic point, and hence φ is an injection. Let $K(A) := \operatorname{Frac}(A) = K(X)$ (likewise for B). Moreover, since f is of finite type, $A = B[a_1, \ldots, a_m]$ for some $a_i \in A$. Thus, the fiber over the generic point is $B[a_1, \ldots, a_n] \otimes_B K(B) \simeq K(B)[a_1, \ldots, a_n]$. For this to be a finite set, $\dim K(B)[a_1, \ldots, a_n] = 0$, and hence $\operatorname{tr deg} K(A)/K(B) = 0$. Thus, by Noether normalization, we have that $K(A)$ is a finite extension of $K(B)$. Localizing B by all the denominators that show in the monic polynomials (in $K(B)$) that the a_i's satisfy, we have that $B_f \to B_f[a_1, \ldots, a_n] = A_{\varphi(f)}$ is a finite ring map.

Now, since Y is irreducible, WLOG let $Y = \operatorname{Spec} B$ be affine, and moreover, shrinking $\operatorname{Spec} B$ if necessary, we can assume that $f^{-1}(\operatorname{Spec} B)$ can be covered by finitely many affines $U_i := \operatorname{Spec} A^i$ such that $U_i \to Y$ is a finite morphism. We finish with the following lemma:

Lemma. Suppose $f : X \to \operatorname{Spec} B$ is an morphism of integral schemes such that X is covered by finitely many affines $U_i := \operatorname{Spec} A^i$ such that $U_i \to \operatorname{Spec} B$ is an integral morphism. Then there exists $b \in B$ such that $f : X_b \to \operatorname{Spec} B_b$ is an affine morphism.

Proof: Since X is irreducible, $W := \bigcap_i U_i$ is nonempty. We can find an affine open $V \subset W$ that is distinguished in each U_i. In other words, $V = \operatorname{Spec} A^i_{a_i}$ $\forall i$ for some choices of $a_i \in A^i$. Since a_i is integral over B, let b_i be the nonzero constant term of the monic polynomial in $B[t]$ that a_i is a zero of. Note that $a_i \in \mathfrak{p} \subset A^i \implies b_i \in \mathfrak{p}$ so that $\operatorname{Spec} A^i_{b_i} \subset \operatorname{Spec} A^i_{a_i}$. Thus, the inverse image of $\operatorname{Spec} B_b$ where $b = \prod_i b_i$ is $\operatorname{Spec} A^i_b$ (for any i) so that $X_b \to \operatorname{Spec} B_b$ is a morphism of affine schemes (and is still finite type / integral if f was).

II.3.10. (a): Categorical nonsense reduces to the case where we have $\operatorname{Spec} A \to \operatorname{Spec} B$ and $\mathfrak{p} \subset B$. Then $\operatorname{Spec} A \times_{\operatorname{Spec} B} \operatorname{Spec} \kappa(\mathfrak{p}) = \operatorname{Spec}(A \otimes_B \kappa(\mathfrak{p}))$ which is made by localization and a quotient, which are both topological inclusions.

(b): For the last part, we compute that $\Bbbk[s,t]/(s-t^2)$ localized by multiplicative set $S = \Bbbk[s] \setminus \{0\}$ gives us $\Bbbk(s)[t]/(s-t^2)$, which is a degree 2 extension of $\Bbbk(s)$.

II.3.11. We actually do (b) first then (a). We need a short lemma: If $f : Y \to X$ is a closed immersion, and $U \subset X$, then the fiber product $f^{-1}(U) = U \cap Y \to U$ is also a closed immersion. *Proof:* That the map is homeomorphism onto a closed subset is clear, and map of stalks are surjective since it doesn't change from $f^\#$.

(b): Note that Y is quasicompact. We cover it by affines of the form $D(f_i) \cap Y$ for $f_i \in A$. For any point $p \in Y$, take an affine $\widetilde{U} \ni p$. Topologically and scheme theoretically (by Ex.II.2.2), $\widetilde{U} = U \cap Y$ for some $U \subset X$ open. Now, take $D(f) \subset U$, and note that $D(f) \cap Y = U \cap Y_f = (U \cap Y)_f$ (by Ex.II.2.16). In other words, we have:

$$\begin{tikzcd}
(U \cap Y)_f \arrow[r] \arrow[d] & U \cap Y \arrow[r] \arrow[d] & Y \arrow[d] \\
D(f) \arrow[r] & U \arrow[r] & X
\end{tikzcd}$$

where all squares are fiber diagrams and right arrows are open embeddings, so that all the down arrows are closed embeddings (by the short lemma above). Covering Y by these affines, taking finite subcover, and then extending to cover of X, we have that $f : Y \to X$ is an affine morphism. Lastly, Ex.II.2.18(d) gives $f^\# : \mathscr{O}_X(X) \to \mathscr{O}_Y(Y)$ to be a surjection, so that $\mathscr{O}_Y(Y) = A/I$ for some I.

(c): We have $f : Y \to X$ and $f' : Y' \to X$. Note that topologically, f and f' are the same. For any $U = \operatorname{Spec} A \subset X$, we have that $f^{-1}(U) = \operatorname{Spec} A/\sqrt{I}$ and $(f')^{-1}(U) = \operatorname{Spec} A/I$. Since there is a natural map such that

$$\begin{tikzcd}
A/\sqrt{I} & A \arrow[l] \arrow[dl] \\
A/I \arrow[u, "\exists!"] &
\end{tikzcd}$$

which defines a map $f^{-1}(U) \to (f')^{-1}(U)$. These maps agree for any affines $U, V \subset X$, so they patch up to give a morphism $Y \to Y'$.

(d): Topologically, $Y = \overline{f(Z)}$ clearly. If Z is reduced or quasicompact, we show that we can make Y affine locally by constructing a quasicoherent sheaf of ideals of $\mathscr{O}_X$. For $U \subset X$ affine, define $\mathscr{I}(U) := \ker(\mathscr{O}_X(U) \to \mathscr{O}_Z(f^{-1}(U)))$. For $g \in \mathscr{O}_X(U)$, that $\mathscr{I}(U_g) \simeq \mathscr{I}(U)_g$ follows from the diagram below:

$$\begin{tikzcd}
\mathscr{I}(U) \arrow[r] \arrow[d] & \mathscr{O}_X(U) \arrow[r] \arrow[d] & \mathscr{O}_Z(f^{-1}(U)) \arrow[r] \arrow[d] & \prod_i A_i \arrow[d] \\
\mathscr{I}(U_g) \arrow[r] & \mathscr{O}_X(U_g) \arrow[r] & \mathscr{O}_Z(f^{-1}(U_g)) \arrow[r] & \prod_i (A_i)_g
\end{tikzcd}$$

The universal property follows easily from the construction.

In general, given a map $f : Z \to X$, we have a map of sheaves $\mathscr{O}_X \to f_*\mathscr{O}_Z$, and let $\mathscr{I}$ be the kernel. In the case when Z is qcqs, $f_*\mathscr{O}_Z$ is in fact quasicoherent, so that $\mathscr{I}$ is already quasicoherent. In general, define $\mathscr{I}_Y := \sum_{\mathscr{I}' \subset \mathscr{I} \text{ QCoh}} \mathscr{I}'$, which is QCoh, which defines closed subscheme Y with the desired universal property.

Lemma. Let X be a finite type $\Bbbk$-scheme. If $\mathbb{l} \supset \Bbbk$ is an algebraic extension of $\Bbbk$, then a $\mathbb{l}$-valued point $\operatorname{Spec} \mathbb{l} \to X$ (over $\Bbbk$) is a closed point. In particular, the closed points of X are $x \in X$ such that $\kappa(x)$ is an algebraic extension of $\Bbbk$. *Proof:* For any map $\varphi : \Bbbk[\underline{x}]/I \to \mathbb{l}$, note that $A = \Bbbk[\underline{x}]/\ker\varphi$ is a $\Bbbk$-algebra domain of finite type that injects into $\mathbb{l}$ (an algebraic extension of $\Bbbk$) and hence is a field. Thus, the point that corresponds to a $\mathbb{l}$-valued point is a maximal ideal in every affine that contains it, so it is closed.

II.3.14. By the lemma, we may assume that $\Bbbk$ is algebraically closed. But in this case, for any affine piece $U \subset X$ given by $\Bbbk[\underline{x}]/I$, there are $\Bbbk$-valued points of U by Nullstellensatz.

II.3.16. We need show that every proper closed subset $Y$of X has property $\mathcal{P}$. Suppose $\mathcal{P}$ does not hold for some $Y_0 \subsetneq X$ closed. Then there exists a closed proper subset $Y_1 \subsetneq Y$ such that $\mathcal{P}$ does not hold, and so forth, so that we get a chain of closed subsets $Y_0 \supsetneq Y_1 \supsetneq \cdots$. But Noetherian condition implies that this chain must stabilize, which is a contradiction.

II.3.17–19. Some other time... When I need it...

II.3.20. (a): We first show that for any closed points $p, q \in X$, we have $\dim \mathscr{O}_p = \dim \mathscr{O}_q$. Note that this is true for two closed points in an affine integral finite type $\Bbbk$-scheme. Now, take affines U, V containing p, q respectively, and take a closed point in an affine open in the intersection $U \cap V$, and then note that number equality is transitive (lol). Now, if $X = Z_0 \supsetneq \cdots \supsetneq Z_d = P$ is a the maximal sequence of irreducibles giving the dimension of X, take and affine $U = \operatorname{Spec} A$ containing P. Then $\dim A = \dim X$ (use X irreducible to argue that $Z_i \cap U \supsetneq Z_{i+1} \cap U\ \forall i$), and $\dim A = \dim \mathscr{O}_p$. Combined with $\dim \mathscr{O}_p = \dim \mathscr{O}_q$ for any two closed points, we have that $\dim \mathscr{O}_p = \dim X$ for any closed point $p \in X$.

(e): If $\operatorname{Spec} A \subset U$, then from part (a) we know that $\dim X = \dim A \le \dim U \le \dim X$.

(b): Take any $\operatorname{Spec} A \subset X$. Then $\dim X = \dim A = \operatorname{tr}\deg_{\Bbbk} \operatorname{Frac}(A) = \operatorname{tr}\deg_{\Bbbk} K(X)$.

(c): Suppose ξ is the generic point for an irreducible closed subset $Z \subset X$. Take $Z = Z_0 \subsetneq \cdots \subsetneq Z_c = X$ the maximal chain that gives $c = \operatorname{codim}(Z, X)$. Take an affine open $\operatorname{Spec} A$ around Z_0, so it contains all points corresponding to Z_i's. If $\mathfrak{p} \subset A$ corresponds to Z, then $c = \dim \mathscr{O}_{X,\mathfrak{p}}$. We are done since $\operatorname{codim}(Y, X) := \inf \operatorname{codim}_{Z \subset Y}(Z, X) = \inf\{\dim \mathscr{O}_{\xi(Z),X} | \xi(Z) \in Y\}$

(d): From the previous two parts, we can assume X affine. And for any irreducible closed Y, this is true, hence true for general Y (dim is sup and codim is inf).

(f): No idea... ASK

II.3.21. For concreteness, let $R = \Bbbk[x]_{(x)}$ (in fact, $R \simeq \Bbbk[x]_{(x)}$ for $\Bbbk$ its residue field under our condition). Then $R[t] \simeq \Bbbk[x]_{(x)} \otimes_k \Bbbk[t]$. Now, localizing by x gives $\Bbbk(x)[t]$ and quotienting by x gives $\Bbbk[t]$. That is, $\operatorname{Spec} R[x]$ consists of a closed subscheme $\Bbbk[t]$ and an open subscheme $\Bbbk(x)[t]$.

II.3.22. Hold for now... Return to this later.

II.4 Separated and proper morphisms

II.4.1. Affine morphisms are separated. Finite maps are of finite type. And finite maps are closed, and are preserved under base change.

Lemma. Suppose X is reduced, and Z is a closed subscheme of X containing an open dense subset U of X. Then in fact $X = Z$. *Proof:* Since U is reduced, the scheme-theoretic image of U is $\overline{U}_{red} = X$ since U is dense and X is reduced. Thus, by universal property of scheme-theoretic image, there exists a unique j such that

$$\begin{tikzcd} Z \arrow[r] & X \\ X \arrow[u, "j"] \arrow[ur, "\mathrm{Id}"'] & \end{tikzcd}$$

from which we can deduce that $Z \simeq X$.

Warning: This is NOT true if X is not reduced. For example, consider $X = \operatorname{Spec} \Bbbk[x,y]/(x^2, xy)$, $Z = \operatorname{Spec} \Bbbk[x,y]/(x)$, and $U = \Bbbk[y^{\pm}] \simeq \left(\Bbbk[x,y]/(x^2, xy)\right)_{x+y} \simeq (\Bbbk[x,y]/(x))_{x+y}$.

Lemma. Suppose $f, g : X \to Y$ are morphisms of S-schemes, and let $h : X \to Y \times_S Y$ be the map induced from f, g. Denote by $h^{-1}(\Delta)$ (suggestively) the fiber product:

$$\begin{tikzcd} h^{-1}(\Delta) \arrow[r] \arrow[d] & Y \arrow[d, "\Delta"] \\ X \arrow[r, "h"] & Y \times_S Y \end{tikzcd}$$

Then $f = g$ iff $h^{-1}(\Delta) = X$. *Proof:* Universal property nonsense.

II.4.2. (a): Since Y is separated over S, the fiber (as in lemma above) $h^{-1}(\Delta) \hookrightarrow X$ is a closed embedding. Moreover, we have chain of fiber diagrams:

$$\begin{tikzcd} U \arrow[r] \arrow[d, equal] & h^{-1}(\Delta) \arrow[r] \arrow[d] & Y \arrow[d, "\Delta"] \\ U \arrow[r] & X \arrow[r, "h"] & Y \times_S Y \end{tikzcd}$$

so that by the first lemma above $h^{-1}(\Delta) = X$, and hence by the second lemma $f = g$, as desired.

(b): X not reduced: consider $\Bbbk[x,y] \to \Bbbk[x,y]/(x^2, xy)$ by $x \mapsto x, y \mapsto y$ and $x \mapsto x^2 = 0, y \mapsto y$. Y not separated: map $\mathbb{A}^1$ to upper and lower affine line with double origin.

Lemma. (Magic square) Let $Y \to Z, X_1 \to Y, X_2 \to Y$ be morphisms. Then

$$\begin{tikzcd} X_1 \times_Y X_2 \arrow[r] \arrow[d] & X_1 \times_Z X_2 \arrow[d] \\ Y \arrow[r, "\Delta"] & Y \times_Z Y \end{tikzcd}$$

is a fiber diagram. This is super useful in this chapter! *Proof:* (Intense) category theory nonsense

II.4.3. Let $U, V \subset X$ be affine opens. Noting that $U \times_X V \simeq U \cap V$, we have by the magic square lemma a fiber diagram:

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \times_S V \arrow[d] \\ X \arrow[r, "\Delta"] & X \times_S X \end{tikzcd}$$

where the horizontals are closed embeddings and verticals are open embeddings. Since $U \times_S V$ is affine, so is $U \cap V$ which is an open subscheme of X.

II.4.4. Note that we have $f : Z \hookrightarrow X \to Y$ and $g : Y \to S$ where $g \circ f$ is proper and g is separated. Hence, f is proper. Thus, $f(Z)$ is closed in Y. So, denote by $f(Z)$ the scheme-theoretic image of Z in Y. I have no idea why $f(Z)$ need be proper over S...

By the way, why do we need quasicompact for the proof of Lemma II.4.5.???

II.4.5.(a,b): Note that $\operatorname{Spec} K(X) \to X$ is the generic point, so that the closure of image is in fact X. Thus, if R is a valuation ring of K, then Lemma II.4.4. specializes to say: to give a map $\operatorname{Spec} R \to X$ such that

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \\ \operatorname{Spec} R \arrow[ur] & \end{tikzcd}$$

commutes is equivalently to giving a point $x \in X$ such that R dominates $\mathscr{O}_{X,x}$. Moreover, if we require R to be a valuation ring of K/k, then giving a map $\operatorname{Spec} R \to X$ so that

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur] & \operatorname{Spec} k \end{tikzcd}$$

commutes is equivalent to giving a point $x \in X$ such that $\mathscr{O}_{X,x}$ dominates R. That is, if X is separated/proper, then the center x for any valuation ring R of K/k is unique/uniquely exists.

(c) This is HOID.

(d) Let $a \in \mathscr{O}_X(X)$ such that $a \notin k$. Since $k = \overline{k}$, note that a is transcendental over k so that treating $a \in \mathscr{O}_X(X) \subset K(X)$, we have $k[a^{-1}]_{(a^{-1})} \simeq k[x]_{(x)}$, a local ring. Hence, there exists valuation ring R of K/k that dominates $k[a^{-1}]_{(a^{-1})}$. By properness, there exists $x \in X$ such that R dominates $\mathscr{O}_{X,x}$. However, $a \in \mathscr{O}_{X,x}$, so that $\mathscr{O}_{X,x} \hookrightarrow R$ gives $a, a^{-1} \in R$ so that $a^{-1} \notin \mathfrak{m}_R$, which is a contradiction.

II.4.6. We in fact show this for affine integral schemes. If $f : \operatorname{Spec} A \to \operatorname{Spec} B$ is induced by $\varphi : B \to A$, then since f is proper φ is of finite type. Thus, to show finite, we need show that it is integral. Replacing B by $\varphi(B)$, we assume that φ is injective. Suppose $R \supset B$ is a valuation ring of $K = K(A)$. Then $R \supset A$ by properness. Since the integral closure of B in K, denoted B^{cl}, is the intersection of all valuation rings of K containing B, we thus have $B^{cl} \supset A$. Thus, every element of A is integral over B, as desired.

II.4.7. Let $p \in X$ and $q = \sigma(p)$. By the additional condition there is an affine $U \subset X$ containing p, q. Note that $V := \sigma(U)$ is also affine containing p, q. Thus, since X is separated, $U \cap V$ is an affine containing p, q such that $\sigma(U \cap V) = U \cap V$. Hence, we first consider the case when X is affine $\operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n]/I$.

No fucking clue. How is this done?? For example, just take $\mathbb{C}[x, y] \to \mathbb{C}[x, y]$ via $x \mapsto ix, y \mapsto iy$.

II.4.8. (d): Universal property nonsense crazy diagram.

(e): The magic diagram (lemma above) gives us a fiber diagram:

$$\begin{array}{ccc} X & \xrightarrow{\Gamma_f} & X \times_Z Y \\ {\scriptstyle f}\downarrow & & \downarrow \\ Y & \xrightarrow{\Delta} & Y \times_Z Y \end{array}$$

Combined with the diagram

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

we have that if $X \to Z$ and Δ have property $\mathscr{P}$, then so does $X \to Y$.

(f): Note that the natural map out of X_{red} in Ex.II.3.11(c) is in fact a closed embedding. Now, consider the diagram:

$$\begin{tikzcd} X_{red} \arrow[r] \arrow[dr] & X \times_Y Y_{red} \arrow[r] \arrow[d] & Y_{red} \arrow[d] \\ & X \arrow[r] & Y \end{tikzcd}$$

II.4.9. We have closed embeddings $i : X \to \mathbb{P}^n_Y$ and $j : Y \to \mathbb{P}^m_Z$. Now, note that $\mathbb{P}^n_Y \simeq \mathbb{P}^n_Z \times_Z Y$. So, we have an closed embedding $X \xrightarrow{i} \mathbb{P}^n_Z \times_Z Y \xrightarrow{\mathrm{Id} \times j} \mathbb{P}^n_Z \times_Z \mathbb{P}^m_Z \hookrightarrow \mathbb{P}^{nm-m-n}_Z$.

II.4.10(c). This is like,, impossible...

II.4.11. Later when studying CA

II.5 Sheaves of modules

Lemma. For $\mathscr{G}$ an $\mathscr{O}_X$-modules, we have a natural $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_X^{\oplus I}, \mathscr{G}) \simeq \mathrm{Hom}(\mathscr{O}_X^{\oplus I}(X), \mathscr{G}(X))$. *Proof:* $\to$ map is clear. For $\leftarrow$, for any U, the image of the standard basis e_i of $\mathscr{O}_X(U)^{\oplus I}$ is determined by $\widetilde{e}_i$ of $\mathscr{O}_X(X)^{\oplus I} \to \mathscr{G}(X) \to \mathscr{G}(U)$ since $\widetilde{e}_i$ restricts to e_i.

II.5.1(a,d). (a): There is a natural map $\mathscr{E} \to \mathscr{E}^{\vee\vee}$ as follows: For any $U \subset X$, given $s \in \mathscr{E}(U)$, define $ev_s : \mathscr{E}^\vee|_U \to \mathscr{O}_U$ by $ev_s(W)(\varphi) = \varphi(W)(s|_W)$ for $W \subset U$ and $(\varphi : \mathscr{E}|_W \to \mathscr{O}_W) \in \mathscr{E}^\vee|_U(W)$. To show that this map is an isomorphism when $\mathscr{E}$ is locally free, we can assume $\mathscr{E}$ is in fact free since isomorphism is local condition. But then, by the lemma above, it suffices to show that $M \to M^{\vee\vee}$ is an isomorphism for M a free $\mathscr{O}_X(X)$-module of finite rank, but this is true.

(d): We will work at the presheaf-tensor level. First, there is a natural map $f_*\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{E} \to f_*(\mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{E})$ as follows: For each $V \subset Y$, the RHS is $\mathscr{F}(f^{-1}V) \underset{f_*f^{-1}\mathscr{O}_Y(V)}{\otimes} f_*f^{-1}\mathscr{E}(V)$, but since (f^{-1}, f_*) are adjoints there are natural maps $\mathscr{O}_Y(V) \to f_*f^{-1}\mathscr{O}_Y(V)$ and $\mathscr{E}(V) \to f_*f^{-1}\mathscr{E}(V)$, which gives us a map

$$\mathscr{F}(f^{-1}V) \underset{\mathscr{O}_Y(V)}{\otimes} \mathscr{E}(V) \to \mathscr{F}(f^{-1}V) \underset{f_*f^{-1}\mathscr{O}_Y(V)}{\otimes} f_*f^{-1}\mathscr{E}(V)$$

Now, for the isomorphism, since we can work locally, assume that $\mathscr{E} \simeq \mathscr{O}_Y^{\oplus n}$. The above map is just $\mathscr{F}(f^{-1}V)^{\oplus n} \to \mathscr{F}(f^{-1}V)^{\oplus n}$ where the left is a $\mathscr{O}_Y(V)$-module and the right as $f_*f^{-1}\mathscr{O}_Y(V)$-module restricted to $\mathscr{O}_Y(V)$-module. This is clearly an isomorphism.

II.5.6. (a): For $\mathfrak{p} \in X$, we have $m_\mathfrak{p} \neq 0 \iff fm \neq 0 \; \forall f \in A \setminus \mathfrak{p} \iff \mathfrak{p} \supset \mathrm{Ann}(m)$.

(b): If $\mathfrak{p} \in \mathrm{Supp}\, M$, then $M_\mathfrak{p} \neq 0 \implies fM \neq 0 \; \forall f \in A/\mathfrak{p} \iff \mathfrak{p} \supset \mathrm{Ann}(M)$ (note that $\Longleftarrow$ may not hold if M is not finitely generated!). If M if finitely generated, then $M = (m_1, \ldots, m_r)$ and $\mathrm{Ann}(M) = \bigcap_i \mathrm{Ann}(m_i)$ so that if $\mathfrak{p} \supset \mathrm{Ann}(M)$, then $\mathfrak{p} \supset \mathrm{Ann}(m_i)$ for some i so that for any $f \in A \setminus \mathfrak{p}$ we have $fm_i \neq 0$ hence $M_\mathfrak{p} \neq 0$. (We actually did not need Noetherian).

(c): This follows from part (b).

(d): First, $\mathscr{H}_Z^0(\mathscr{F})$ is quasicoherent, as it is the kernel of $\mathscr{F} \to j_*(\mathscr{F}|_U)$ (as X is Noetherian, so is U, and so $\mathscr{F}|_U$ quasicoherent implies that $j_*(\mathscr{F}|_U)$ is also). To finish, we show that over an affine $X = \operatorname{Spec} A$ now, $\Gamma_I(M) = \Gamma_Z(X, \mathscr{F})$ as modules (where $M = \Gamma(X, \mathscr{F})$). Well, for $m \in M$, $\operatorname{Supp}(m) = V(\operatorname{ann} m) \subset V(I)$ implies that $I \subset \sqrt{\operatorname{ann} m}$ so that some power of I lies in $\operatorname{ann} m$ (as I is finitely generated)—i.e. $m \in (0_M : I^\infty)$. Conversely, if $m \in (0_M : I^\infty)$, then $I^n \subset \operatorname{ann} m$ for some n so that $V(I^n) = V(I) \supset V(\operatorname{ann} m) = \operatorname{supp} m$.

(e): Follows from the same argument in (d).

II.5.7. (a): Can assume X affine so that $\mathscr{F} = \widetilde{M}$. Suppose $M_\mathfrak{p}$ is free, i.e. $A_\mathfrak{p}^{\oplus n} \xrightarrow{\sim} M_\mathfrak{p}$. This map can actually made so that it lifts to $A^{\oplus n} \to M$. Since M is coherent, we have a exact sequence:

$$A^{\oplus m} \xrightarrow{\alpha} A^{\oplus n} \xrightarrow{\beta} M \to 0$$

where $\operatorname{im} \alpha_\mathfrak{p} = 0$. But since $\operatorname{im} \alpha$ is finitely generated, there exists f such that $(\operatorname{im} \alpha)_f = 0$. Hence, localizing the sequence above by f, we obtain $A_f^{\oplus n} \xrightarrow{\sim} M_f$.

(b): Follows from part (a).

(c): Generally, there is a map $\mathscr{F} \otimes \mathscr{F}^\vee \to \mathscr{O}_X$. When $\mathscr{F}$ is invertible, so is $\mathscr{F}^\vee$ and this map is an isomorphism. For the converse, suppose $\mathscr{G}$ exists. Then for an affine patch we have $M_\mathfrak{p} \otimes N_\mathfrak{p} \simeq A_\mathfrak{p}$ at each point $\mathfrak{p}$. Now, by applying $\kappa(\mathfrak{p})$ to both sides and using Nakayama, we note that both $M_\mathfrak{p}, N_\mathfrak{p}$ are of the form $A_\mathfrak{p}/I$, so that the only way $\simeq$ holds is if $M_\mathfrak{p} \simeq N_\mathfrak{p} \simeq A_\mathfrak{p}$. Thus, we conclude that $\mathscr{F}, \mathscr{G}$ are both locally free of rank 1 by part (a,b). Does the converse not hold if we don't have coherence?

II.5.8. *Notation:* Let's denote $M \otimes_A \kappa(\mathfrak{p})$ by $M|_\mathfrak{p}$.

(a): We show that $\{x : \varphi(x) \geq \ell\}$ is closed on every trivializing affines. (so reduce to X affine and M finitely generated A-module). That is, we have an exact sequence

$$A^{\oplus m} \xrightarrow{\alpha} A^{\oplus n} \xrightarrow{\beta} M \to 0$$

With rank-nullity, one concludes that $\dim M|_\mathfrak{p}$ equals n minus the rank of the matrix $[\alpha|_\mathfrak{p}]$. Now, $\operatorname{rk}[\alpha|_\mathfrak{p}] \leq n - \ell$ iff all $(n-\ell+1)$-minors vanish. That is, if $\mathfrak{p} \in V(\{(n-\ell+1)\text{-minors of } [\alpha]\})$.

(b): The rank is constant if X is connected.

(c): Suppose $\varphi = n$ constant. We again work affine locally, which gives us $A^{\oplus m} \xrightarrow{\alpha} A^{\oplus n} \xrightarrow{\beta} M \to 0$ again, where now $[\alpha|_\mathfrak{p}] = 0$ for all $\mathfrak{p}$. That is, all the entries of $[\alpha]$ are in the nilradical of A, which is zero since A is reduced.

II.5.9. (a): For each $f \in S_1$ and $d \in \mathbb{Z}_{\geq 0}$, we have a natural map $M_d \to \Gamma(X_f, \widetilde{M}(d)) = M(d)_{(f)}$ which glues (as $\widetilde{M}(d)$ is a sheaf) to a map $\alpha_d : M_d \to \Gamma(X, \widetilde{M}(d))$.

(b): Do it w M or J?

II.5.10. (a):

II.5.11. We work affine locally: there is a natural isomorphism $(S \times_A T)_{(f \otimes g)} \xrightarrow{\sim} S_{(f)} \otimes_A T_{(g)}$. Same kind of argument goes for $\mathscr{O}(1)$ part (except we need check transition maps too now).

II.5.14.

II.5.16. Later when I need it.

II.6 Divisors

On products and Weil class group. Let X satisfy $(*)$. We first discuss $X \times \mathbb{A}^n$ and $X \times_k \mathbb{A}^n_k$ (for X a k-variety):

Observation 1: Consider $\varphi^{\#} : A \hookrightarrow A[t]$ (where A is integral domain) and the associated map $\varphi : \operatorname{Spec} A[t] \to \operatorname{Spec} A$. If $\mathfrak{p} \in \operatorname{Spec} A$, then $\mathfrak{p}[t] \in \operatorname{Spec} A[t]$, and in fact, $A[t]/\mathfrak{p}[t] \simeq (A/\mathfrak{q})[t]$. In other words, $\varphi^{-1}(V(\mathfrak{p})) = V(\mathfrak{p}[t])$. Moreover, $\mathfrak{p}[t]$ is of codimension 1, since $A[t]_{\mathfrak{p}[t]} \simeq A_{\mathfrak{p}}[t]_{\mathfrak{p}_{\mathfrak{p}}[t]}$ and if $\sum a_i t^i / \sum b_i t_i \in A[t]_{\mathfrak{p}[t]}$ then the valuation is $\max_i\{m_i : u\pi^{m_i} = a_i\}$. Thus, the map $\varphi^* : \operatorname{WDiv}(\operatorname{Spec} A) \to \operatorname{WDiv}(\operatorname{Spec} A[t])$ is a well-defined homomorphism that also sends principal divisors to principal ones. Moreover, note that this is an injection at the level of class groups.

Observation 2: For $\mathfrak{q} \in \operatorname{Spec} A[t]$, $\operatorname{codim}_{A[t]} \mathfrak{q} \geq \operatorname{codim}_A \varphi(\mathfrak{q})$ (codimension can at most decrease under φ), since any chain $(0) \subsetneq \mathfrak{p} \subsetneq \cdots \subsetneq \varphi(\mathfrak{q})$ gives a chain $(0) \subsetneq \mathfrak{p}[t] \subsetneq \cdots \subsetneq \varphi(\mathfrak{q})[t] \subset \mathfrak{q}$. Hence, if $\mathfrak{q}$ is of codimension 1, then $\varphi(\mathfrak{q})$ is either codimension 1 prime $\mathfrak{p}$ or (0) in A. When $\varphi(\mathfrak{q}) = [(0)]$, then $\mathfrak{q} \cap A = 0$ so that $A[t]_{\mathfrak{q}} \simeq K[t]_{\mathfrak{q} \otimes K}$ where $K = \operatorname{Frac}(A)$, and since $K[t]$ is a UFD, we have $\mathfrak{q} \otimes_A K = (f)$ for an irreducible $f \in K[t]$.

Observation 3: First, we note that the two observations above works the same for any (finite) number of variables. Now, note that the fiber of the generic point of the map $X \times \mathbb{A}^n \to X$ (or $X \times \mathbb{P}^n \to X$) is $\mathbb{A}^n_{K(X)}$ (or $\mathbb{P}^n_K$). In other words, we have

$$\begin{array}{ccc} \mathbb{A}^n_K(\text{or } \mathbb{P}^n_K) & \longrightarrow & X \times \mathbb{A}^n(\text{or } \mathbb{P}^n) \\ \downarrow & & \downarrow \\ \operatorname{Spec} K(X) & \longrightarrow & X \end{array}$$

Thus, we have a well-defined map $\operatorname{WDiv} \mathbb{A}^n_K \to \operatorname{WDiv} X \times \mathbb{A}^n$ via the top map (and locally it is $f \in K[\underline{x}]$ treated as a polynomial in $A_f[\underline{x}]$ for some $f \in A$). This map again sends principal divisors to principal divisors. In the case of $X \times \mathbb{P}^n$, we thus get a map $\operatorname{Cl} \mathbb{P}^n_K \to \operatorname{Cl}(X \times \mathbb{P}^n)$.

From the three observations we get the following summary

Lemma. A prime divisor $X \times \mathbb{A}^1$ either is $Y \times \mathbb{A}^1$ for $Y \subset X$ of codimension 1, or corresponds to P where $P \in \mathbb{A}^1_K$ is a closed point. By the same reasoning (with induction), a prime divisor of $X \times \mathbb{A}^n$ either is $Y \times \mathbb{A}^n$, or corresponds to $V(f) \subset \mathbb{A}^n_K$.

Lemma. $X \times \mathbb{P}^n$ as two types of prime divisors: $Y \times \mathbb{P}^n$ for Y prime divisor in X, and ones corresponding to hypersurfaces of $\mathbb{P}^n_K$.

Theorem. It thus follows that $\operatorname{Cl} X \times \mathbb{A}^n = \operatorname{Cl} X$ and $\operatorname{Cl} X \times \mathbb{P}^n = \operatorname{Cl} X \oplus \mathbb{Z}$.

II.6.1. $X \times \mathbb{P}^n$ is indeed Noetherian, separated, and integral (see TIL:7/20/2016), and since $X \times \mathbb{A}^n$ is regular in codimension 1, so is $X \times \mathbb{P}^n$ (regularity is a local condition).

Now, let $\pi_1 : X \times \mathbb{P}^n \to X$ and $\pi_2 : X \times \mathbb{P}^n \to \mathbb{P}^n$ be two projections. Let $H = V(x_0) \subset \mathbb{P}^n$ be a prime divisor, and note that $X \times \mathbb{P}^n \setminus X \times H \simeq X \times \mathbb{A}^n$. This gives us a exact sequence:

$$\operatorname{Cl} \mathbb{P}^n \simeq \mathbb{Z} \to \operatorname{Cl} X \times \mathbb{P}^n \to \operatorname{Cl} X \times \mathbb{A}^n \to 0$$

where the first map sends $[H]$ to $[X \times H]$, which is the map π_2^*. this map is injective since if $[X \times H] = 0$, then for $f \in K(X \times \mathbb{P}^n)$ such that $\operatorname{div} f = X \times H$, f restricted to $\operatorname{Spec} A \times U_i$ for any $i \neq 0$ and for any $\operatorname{Spec} A \subset X$ is just $x_0/x_i \in A[\frac{x_j}{x_i}s]$, but this is impossible. Moreover, note that composing $\operatorname{Cl} X \times \mathbb{A}^n \xrightarrow{\sim} \operatorname{Cl} X$ and π_1^*, we get a map σ that is a section of the second map in the sequence above. Hence the above sequence is short exact and splits.

II.6.2. (a) There's basically nothing to prove...

(b): Let $f \in K(\mathbb{P}^n_{\Bbbk})$, which can be written as p/q where $p, q \in \Bbbk[\underline{x}]_d$ for some d. Since components of (f) does not contain X, both p and q does not vanish when restricted to $\Bbbk[\underline{x}]/I(X)$, so that f is still a rational function $\overline{f}$ of X. Thus by definition in (a) we have $(f).X = (\overline{f})$. Now, since X is not contained in every hyperplane of $\mathbb{P}^n$, and every divisor in $\mathbb{P}^n$ is equivalent to a hyperplane, we get a map $\varphi : \mathrm{Cl}\,\mathbb{P}^n \to \mathrm{Cl}\,X$ that is well-defined.

(c): We need show that $\nu_{Y_i}(\overline{f}_i) = \mu_{\mathfrak{p}_i}(S/I(X) + (f))$ where $\mathfrak{p}_i$ is the homogeneous prime in $S := \Bbbk[\underline{x}]_\bullet$ corresponding to Y_i. Combining the two facts in TIl 7/22/2016, we have that if $\mathfrak{p}_i'$ is the prime corresponding to $Y_i \cap U_i$ in the coordinate ring $A(X \cap U_i)$, then $\nu_{Y_i}(\overline{f}_i) = \nu_{\mathfrak{p}_i'}(\overline{f}_i) = \mu_{\mathfrak{p}_i'}(A(X \cap U_i)/I(X \cap U_i) + (\overline{f})) = \mu_{\mathfrak{p}_i}(S/IX) + (f))$, as desired.

Now, for $\deg(D.X) = \deg D \cdot \deg X$, it suffices to show for $D = V$ a hypersurface. But $\deg(V.X) = \sum_i i(X, V; Y_i) \deg Y_i = (\deg V)(\deg X)$ from the general Bezout's theorem.

(d): If D is a principal divisor on X, then it is of the form $\overline{p}/\overline{q}$ where $p, q \in \Bbbk[\underline{x}]_d$ $(\exists d)$ such that $p, q \notin I(X)$. Thus, we have $\deg D = 0$ from the equation in part (c). Hence, we get a well-defined map $\mathrm{Cl}\,X \to \mathbb{Z}$. Composing with the map φ in part (b), and using part (c), we get a commutative diagram:

$$\begin{tikzcd} \mathrm{Cl}\,\mathbb{P}^n_{\Bbbk} \arrow[r, "\varphi"] \arrow[d, "\deg", "\simeq"'] & \mathrm{Cl}\,X \arrow[d, "\deg"] \\ \mathbb{Z} \arrow[r, "\times \deg X"] & \mathbb{Z} \end{tikzcd}$$

And in particular, φ must be injective!

II.6.3 (a): Let $S(V) := \Bbbk[\underline{x}]_\bullet / I(V)_\bullet$ the graded coordinate ring of V. Then $A(X) = \Bbbk[\underline{x}]/I(V)$ and $S(\overline{X}) = \Bbbk[\underline{x}, y]_\bullet / I(V)_\bullet$. Thus, on $V_i = V \cap U_{x_i}$, we have that $\pi^{-1}(V_i) = \operatorname{Spec} \Bbbk[\underline{x}/x_i, y/x_i]/I(V_i) \simeq V_i \times \mathbb{A}^1$. This in fact gives $\overline{X} \setminus P$ a structure of $\mathbb{A}^1$-bundle over V. Now, use the fact in TIL 8/26/2016 about $\mathbb{A}^n$-bundles over X in regards to class groups.

(b): $[V] \in \mathrm{Cl}\,\overline{X}$ is given by $y = 0$ (continuing with $S(\overline{X}) = \Bbbk[x, y]_\bullet / I(V)_\bullet$). If ℓ is a linear form in $\Bbbk[x]_\bullet$ that doesn't vanish on V (i.e. $\notin I(V)_\bullet$), then $y/\ell \in K(\overline{X})$ such that $[y = 0] \sim [\ell = 0]$, and the latter is equal to the class of $V.H$ pullbacked by π^*. Now, consider the exact sequence $\mathrm{Cl}\,V(y) \to \mathrm{Cl}(\overline{X} \setminus P) \to \mathrm{Cl}(X \setminus P) \to 0$. Under the isomorphism in (a), the first map becomes $\mathbb{Z} \hookrightarrow \mathrm{Cl}\,V$, $1 \mapsto H.V$ and the second map becomes π^* followed by restriction to $X \setminus P \subset X$. Hence, we obtain a short exact sequence

$$0 \to \mathbb{Z} \to \mathrm{Cl}\,V \to \mathrm{Cl}\,X \to 0$$

(c) Follows immediately from part (b) and II.6.2 as $S(Y)$ without grading is the same as $A(C(Y)) = A(X)$.

(d) The map $\operatorname{Spec} \mathscr{O}_P \to X$ is given by $\left(\Bbbk[\underline{x}, y]/I(V)\right)_{\mathfrak{m}} \leftarrow \Bbbk[\underline{x}, y]/I(V)$ (where $\mathfrak{m} = (\underline{x})$). Indeed, this gives us a map $\mathrm{Cl}\,X \twoheadrightarrow \mathrm{Cl}(\operatorname{Spec} \mathscr{O}_P)$ and a section map $\mathrm{Cl}(\operatorname{Spec} \mathscr{O}_P) \hookrightarrow \mathrm{Cl}\,X$. Now, if $\mathfrak{p} \subset A(X)$ of height 1 is not contained in $\mathfrak{m}$, then we claim that it is actually principal (see TIL 8/19/2016). I use Fulton's intersection theory here... Is there a way to avoid this???

II.6.4. As f is square-free, $(z^2 - f)$ is irreducible in $k(\underline{x})[z]$, so that $K := k(\underline{x})[z]/(z^2 - f)$ is an Galois extension of $k(\underline{x})$ of degree 2. It is also the quotient field of $A = k[\underline{x}, z]/(z^2 - f)$. We show that A is the integral closure of $k[\underline{x}]$ in K (and hence is integrally closed in K). Consider $\alpha = g + hz \in K$ where $g, h \in k(\underline{x})$ that is integral over $k[\underline{x}]$. Note that α satisfies the monic polynomial $q(t) := t^2 - 2gt + (g^2 - h^2 f) \in k(\underline{x})[t]$, so the minimal monic polynomial $p(t)$ of α (over $k[\underline{x}]$) must divide $q(t)$. If $p(t)$ is of degree 1, then $\alpha \in k[\underline{x}]$ already so we are done. If $p(t)$ is of degree 2, then $p(t) | q(t)$ implies that $p(t) = q(t)$. This implies that $2g \in A$ hence $g \in k[\underline{x}]$, and moreover, $g^2 - h^2 f \in A$ so that $g^2 - h^2 f \in k[\underline{x}]$ so that $h^2 f \in k[\underline{x}]$, implying $h \in k[\underline{x}]$ also since f is square-free.

Lemma. A short lemma if you wish to avoid the degree 1 and 2 argument: Let A be integrally closed in its quotient ring K, and $L \subset K$ be an algebraic extension. If $\alpha \in L$ is integral over A, then the minimal monic polynomial $p(t) \in A[t]$ of α over A is the same as the minimal polynomial $q(t) \in K[t]$ of α over K (after scaling it to monic). *Proof:* Clearly, $q(t)|p(t)$ in $K[t]$ by minimality of $q(t)$. But if $p(t)$ factors into two monic polynomials f, g, then both f, g has coefficients that are integral over A, and hence in A as A is integrally closed.

II.6.5. (b) Let $R := \operatorname{Spec} k[x_0, \dots, x_r]/(-x_0x_1 + x_2^2 + \cdots + x_r^2)$ with $r \neq 3$ and $X := \operatorname{Spec} R$. Let $H := V(x_1)$. Since x_1 is a nonzerdivisor in R, we have $\dim R/x_1 = \dim R - 1$, and the minimal primes of R/x_1 have height 1 over R, hence the irreducible components of $X \cap H$ are to prime divisors of X, and for $r \neq 3$ we actually have $X \cap H$ has one irreducible component. This only preserves set-theoretic information, and lets denote the divisor obtained in this way as $[X \cap_s H]$. Moreover, note that $X \setminus H \simeq \operatorname{Spec} k[x_1^{\pm}, x_2, \dots, x_n]$ so that $\operatorname{Cl} X \setminus H = 0$. Hence, we have a surjective $\mathbb{Z} \twoheadrightarrow \operatorname{Cl} X$ sending $1 \mapsto [X \cap_s H]$. We analyze the relationship of $[X \cap_s H]$ and $[X \cap H]$, the divisor associated to the scheme-theoretic intersection, for various values of $r \geq 2$. (Note that $[X \cap H] = 0$ in the class group since it is principal—it is given by function x_0).

When $r = 2$, we get $\operatorname{Spec} k[x, z]/(z^2)$, whose unique minimal prime (z) has z^2 valued to 2 so that we get $[X \cap V(x_1)] = 2[V(y, z)]$. However, when $r \geq 4$, we get instead $\operatorname{Spec} k[x_0, x_2, \dots, x_r]/(x_2^2 + \cdots + x_r^2)$, and $(x_2^2 + \cdots + x_r^2)$ is prime so that we just get $[X \cap V(x_1)] = [V(x_2^2 + \cdots + x_r^2)]$. That is, if $r = 2$, we have $2[X \cap_s H] = [X \cap H] = 0$. However, if $r \geq 4$, then $[X \cap_s H] = [X \cap H]$ so that the map $\mathbb{Z} \twoheadrightarrow \operatorname{Cl} X$ is just a zero map!

Warning: The usual excision exact sequence only works when the subtracting set is irreducible! Witness the case $r = 3$ here. A fix is as follows: if Z is a proper closed subset of X, with irreducible components $Z_1, \dots, Z_r$, then $\mathbb{Z}^{\oplus r} \to \operatorname{Cl} X \to \operatorname{Cl} X \setminus Z \to 0$ where the first map is $e_i \mapsto [Z_i]$ (where $[Z_i] = 0$ if $\operatorname{codim}_X Z_i > 1$).

II.6.8. (a): Pullback of locally free is locally free, and pullback respects tensors. Lastly, pullback of structure sheaf is the structure sheaf.

(b): Let $f : X \to Y$ be a finite map of nonsingular curves. Denote by $f_D^* : \operatorname{Cl} Y \to \operatorname{Cl} X$ the map just to distinguish it from $f^* : \operatorname{Pic} Y \to \operatorname{Pic} X$. We need show that

$$\begin{array}{ccc} \operatorname{Cl} Y & \longrightarrow & \operatorname{CL} X \\ \big\downarrow & & \big\downarrow \\ \operatorname{Pic} Y & \longrightarrow & \operatorname{pic} X \end{array}$$

is commutative. As both horizontal maps are group homomorphisms, it suffices to check for $[Q] \in \operatorname{Cl} Y$ where $Q \in Y$ is a point. As Q is locally principal, take a affine cover of Y where $[Q]$ is principal in each. On affines V not containing Q, the diagram is clearly commutative. On $U \ni Q$, let's say $U = \operatorname{Spec} B$ and $f^{-1}(U) = \operatorname{Spec} A$, and $h \in B$ exactly carves out Q (i.e. $B/h \simeq \kappa(Q)$). We have $f^\# : B \to A$, and denoting $h' := f^\#(h)$. Note that $\operatorname{Spec} A/h'$ gives the divisor $f_D^*(Q)$ on X. Now, we also see that the pullback is $A \otimes_B B \cdot h^{-1} = A \cdot (h')^{-1}$, as desired.

(c): Pretty much the same proof as part (b).

II.6.10. Note the following lemma:

Lemma. If $0 \to \mathscr{F}_0 \to \cdots \to \mathscr{F}_r \to 0$ is a long exact sequence of coherent sheaves on X, then their alternating sum (as an element in $G_0(X)$) is zero.

(b): If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an SES, then $0 \to \mathscr{F}'_\xi \to \mathscr{F}_\xi \to \mathscr{F}''_\xi \to 0$ is an SES of K-modules so that $\operatorname{rk}\mathscr{F}' + \operatorname{rk}\mathscr{F}'' = \operatorname{rk}\mathscr{F}$. Hence, the map $G_0(X) \to \mathbb{Z}$ is well-defined. For surjectivity, note that $\mathscr{O}_X^{\oplus n} = n\ \forall n \geq 0$.

(a): If R is a PID, and $X = \operatorname{Spec} R$, then the rank map $G_0(X) \to \mathbb{Z}$ is in fact an isomorphism. Suppose M is torsion R-module (i.e. $\operatorname{rk}\widetilde{M} = 0$), then there exists a resolution $0 \to R^m \to R^n \to M \to 0$ of M (as R is a PID) and $\operatorname{rk} M = 0$ forces $m = n$, so that as an element of $G_0(X)$, we have $\widetilde{M} = \mathscr{O}_X^n - \mathscr{O}_X^n = 0$.

(c): Later. Just memorize for now.

II.6.12. As X is projective, we embed X in some $\mathbb{P}^n_k$ and consider $\mathscr{F}$ to come from a finitely generated graded S-module. Setting $\deg\mathscr{F} = \chi(\mathscr{F}) - (\operatorname{rk}\mathscr{F}) \cdot \chi(\mathscr{O}_X)$ works. (1) follows by R-R, and (2) by use of Hilbert polynomial, and (3) by additiveness of Euler characteristic and rank.

II.7 Projective morphisms

II.7.1. Let $(A, \mathfrak{m}, k)$ be a local ring, and consider a surjective map of A-modules $\varphi : A \twoheadrightarrow A$. Since tensoring is right exact, we have $\varphi_k : A \otimes k \xrightarrow{\sim} A \otimes k$ so that $\varphi(1) \notin \mathfrak{m}$. Hence, the map φ is multiplication by a unit in A, hence an isomorphism.

II.7.2. Since $\{s_i\}$ and $\{t_j\}$ span the same space, there exists $j_0, \ldots, j_n \in \{0, \ldots, m\}$ such that $\operatorname{span}(t_{j_1}, \ldots, t_{j_n}) = V$ (otherwise, $\dim V > n+1$ and thus a contradiction). WLOG let $j_k = k$, and let $L = V(y_0, \ldots, y_n) \subset \mathbb{P}^m$. Since $\{t_0, \ldots, t_n\}$ generate $V \subset \Gamma(X, \mathscr{L})$, we have that $\psi^{-1}(\mathbb{P}^m \setminus L) = X$ and we can the composition with the projection $\pi : \mathbb{P}^m \setminus L \to \mathbb{P}^n$ to get $\widetilde{\psi} : X \xrightarrow{\psi} \mathbb{P}^m \setminus L \xrightarrow{\pi} \mathbb{P}^n$. Now, each t_j can be expressed as a linear combination of s_i's, and moreover, by construction there exists (exists! this is a bit subtle) an invertible $(n+1) \times (n+1)$ matrix M such that $(s_0\ \cdots\ s_n)M = (t_0\ \cdots\ t_n)$.

Lemma. Let X be a k-scheme. Then a map $\varphi : X \to \mathbb{P}^n_k$, given by $\{s_0, \ldots, s_n\} \subset \Gamma(X, \mathscr{L})$ for a line bundle $\mathscr{L}$ on X, is essentially determined by the associated linear series $V := \operatorname{span}(s_0, \ldots, s_n) \subset \Gamma(X, \mathscr{L})$. More precisely,

(a) There exists a projection map $\pi : \mathbb{P}^n_k \setminus L \to \mathbb{P}V$ from a linear subspace L of (projective) dimension $n - \dim V - 1$ such that $\pi \circ \varphi = \psi$ where $\psi : X \to \mathbb{P}V$.

(b) There is a map $\alpha : \mathbb{P}V \to \mathbb{P}^n_k$ such that $\alpha \circ \psi = \varphi$.

As a corollary, if $\{s_0, \ldots, s_m\}$ and $\{t_0, \ldots, t_n\} \subset \Gamma(X, \mathscr{L})$ generate two base-point-free linear systems V, W, with $V \subset W$ and $m \leq n$, then there exists a linear projection $\pi : \mathbb{P}^n_k - L \to \mathbb{P}^m_k$ which when composed with an automorphism ι of $\mathbb{P}^m_k$ we get $\varphi_s = \iota \circ \pi \circ \varphi_t$.

II.7.3. (a): Note that the intersection of n hyperplanes in $\mathbb{P}^n$ is non-empty. Now, let d be such that $\varphi^*(\mathscr{O}_{\mathbb{P}^m}(1)) \simeq \mathscr{O}_{\mathbb{P}^n}(d)$, and give $\mathbb{P}^m$ coordinates $x_0, \ldots, x_m$. Consider the linear system $V := \operatorname{span}(\varphi^* x_0, \ldots, \varphi^* x_m) \subset \Gamma(\mathbb{P}^n, \varphi^*(\mathscr{O}(1)))$. If $\dim V \leq n$, then V cannot be base-point-free unless $d = 0$, and hence $\varphi(\mathbb{P}^n) = pt$. Thus, to have $\varphi(\mathbb{P}^n) \neq pt$, we need $\dim V \geq n+1$, and so $n \leq m$ necessarily. FSOC, if $\dim \varphi(\mathbb{P}^n) \leq n-1$ in this case, then we can find a sequence $(H_1, \ldots, H_n)$ of hyperplanes in $\mathbb{P}^m$ such that $\varphi(\mathbb{P}^n) \cap H_1 \cap \cdots \cap H_n = \emptyset$. This would mean that $\varphi^* H_i$'s form a base-point-free linear system of dimension at most n, which is a contradiction.

(b): Use the generalization (Lemma) above.

II.7.4. (a): If such X admits a ample invertible sheaf $\mathscr{L}$, then there exists an immersion $i : X \hookrightarrow \mathbb{P}^n_A$ by $\mathscr{L}^n$.

(b): $\operatorname{Pic} X = \mathbb{Z} \oplus \mathbb{Z}/(1,1)$. And the only case when $\mathscr{O}(a,b)$ is base-point-free is $(0,0)$, i.e. $\mathscr{O}_X$.

III.7.5. Note that that if $\mathscr{F}, \mathscr{G}$ are globally generated coherent sheaves on X, then $\mathscr{F} \otimes \mathscr{G}$ is also globally generated. For the proofs below, let $\mathscr{F}$ be a coherent sheaf on X.

(a): $(\mathscr{F} \otimes \mathscr{L}^n) \otimes \mathscr{M}^n$ is globally generated for some n.

(b): $\mathscr{M} \otimes \mathscr{L}^{n-1}$ is globally generated for some n, so by part (a) above $\mathscr{M} \otimes \mathscr{L}^n$ is ample.

(c): Take n large enough so that both $\mathscr{F} \otimes \mathscr{L}^n$ and $\mathscr{M}^n$ is globally generated.

(d): Note that while $\mathscr{M}$ is a priori not finitely globally generated, we can assume so since X is Noetherian and $\mathscr{M}$ is coherent. Now, let $\mathscr{O}_X^{n+1} \twoheadrightarrow \mathscr{L}$ and $\mathscr{O}_X^{m+1} \twoheadrightarrow \mathscr{M}$ give maps $\iota : X \to \mathbb{P}^n_A$ and $\varphi : X \to \mathbb{P}^n_A$ where ι is an immersion factoring through $X \xrightarrow{\iota} \overline{X} \hookrightarrow \mathbb{P}^n_A$ (the first map is open embedding and the second map is closed embedding). As $\mathbb{P}^n_A$ is separated, the graph map $\Gamma_\varphi : X \to X \times \mathbb{P}^m_A$ is a closed embedding. Hence, $X \to \mathbb{P}^n_A \times \mathbb{P}^m_A \simeq \mathbb{P}^{mn+m+n}_A$ is a locally closed embedding, and X Noetherian means it is the same as an immersion. Now, pulling back $\mathscr{O}(1)$ on $\mathbb{P}^{mn+m+n}$ exactly gives us $\mathscr{L} \otimes \mathscr{M}$, and thus $\mathscr{L} \otimes \mathscr{M}$ is very ample.

(e): Follows from (d) as $\mathscr{L}^m$ is globally generated for large enough m.

II.7.6. (a): If $i : X \hookrightarrow \mathbb{P}^n_k$ is the embedding, then we have $\mathscr{L}(D) = i^*\mathscr{O}_{\mathbb{P}^n}(1)$, so that $\dim \Gamma(X, \mathscr{L}^n) = \dim \Gamma(X, i^*\mathscr{O}_{\mathbb{P}^n}(n)) = \Gamma_n(\mathscr{O}_X)$, the n-th degree of $\Gamma_\bullet(\mathscr{O}_X)$. Now, $\mathscr{O}_X \simeq \widetilde{S/I}$ where $S = \Bbbk[\underline{x}]_\bullet$ and $I = \Gamma_\bullet(\mathscr{I}_{X/\mathbb{P}^n_k})$. Now, we have $(S/I)_n \simeq \Gamma_n(\mathscr{O}_X)$ for all $n >> 0$ by Ex.II.5.9.

(b): We have a map $\deg : \operatorname{Cl} X \to \mathbb{Z}$, and if D is torsion, then $\deg D = 0$ indeed. Thus, if $\mathscr{L}(nD)$ has a global section s, then $\operatorname{div} s = 0$, so that $\mathscr{L}(\operatorname{div} s) \simeq \mathscr{O}_X$, i.e. $nD = 0$ or $r|n$. Otherwise, $\mathscr{L}(nD)$ has no global section.

II.7.7(c). Denote by $\vec{p}(2) := (p_0^2, p_1^2, p_2^2, p_0p_1, p_1p_2, p_2p_0)$ for a point $[p_0 : p_1 : p_2] \in \mathbb{P}^2_k$. Then, the linear system V is $\ker \vec{p}(2)$, so the linear system has rank 4 (dimension 5). Now, WLOG let $P = [0 : 0 : 1]$. Then, the map $\varphi : \mathbb{P}^2_k \setminus P \to \mathbb{P}^4$ is given by sections (x^2, y^2, xy, xz, yz) of $\Gamma(X, \mathscr{O}(2))$. On U_y and U_x, this map is clearly a closed embedding.

We blow-up the point P to get $\widetilde{X}$; locally on U_z, we get $\widetilde{X} \cap U_z = V(xv - yu)$. More precisely, we have the graded ideal $I = (xv - yu)$ in $(k[x,y])[u,v]_\bullet$ with u, v have weight 1, whose Proj is $\mathbb{A}^2_k \times_k \mathbb{P}^1_k$. U_y, U_x parts doesn't change at all, so we have $\widetilde{X} = V(xv - yu) \subset \mathbb{P}^2 \times \mathbb{P}^1$ where $(xv - yu)$ is an ideal of $k[x,y,z]_\bullet \otimes_k k[u,v]_\bullet$. Now, the map φ extends to $\widetilde{\varphi} : \widetilde{X} \to \mathbb{P}^4$ where $[0 : 0 : 1] \times [u : v] \mapsto [0 : 0 : 0 : u : v]$ (on U_z, the map is $[x : y : 1] \mapsto [x^2 : y^2 : xy : x : y]$, and which if WLOG $x \neq 0$ if $[x : y^2/x : y : 1 : y/x]$, so that since $v/u = y/x$, we obtained the map as above as (x,y) limits to $(0,0)$). To check that $\widetilde{\varphi}$ is in fact a closed embedding on U_z (it is already on U_x and U_y), we note that on U_z, we have $\widetilde{\varphi}(x,y,z,u,v) = [ux, vy, vx = uy, u, v]$, and one checks easily that on affine patches U_u and U_v, the map is indeed a closed embedding.

Lastly, to show $\deg = 3$, note that a hyperplane $V(x_2) \subset \mathbb{P}^4$ pulls back to three lines: $x/y = 0$, $y/x = 0$, and the exceptional fiber.

II.8 Differentials

II.8.1. (a): Using a Proposition from Eisenbud's book, we only need a map of k-algebras $B/\mathfrak{m} \to B/\mathfrak{m}^2$ that splits $B/\mathfrak{m}^2 \twoheadrightarrow B/\mathfrak{m}$. Well, as $B/\mathfrak{m}^2$ is complete local ring, we have $\kappa(B) \simeq K \hookrightarrow B/\mathfrak{m}^2 \to \kappa(B)$, a splitting as desired by Cohen structure theorem.

Alternate Soln: Let's do this more by hand. Note that given the choice axiom, for map of vector spaces $V \to W$ to be injective it is sufficient to show that the induced dual map $W^* \to V^*$ is surjective. We will thus show that $\operatorname{Hom}_{\kappa(B)}(\Omega_{B/k} \otimes \kappa(B), \kappa(B)) \xrightarrow{\delta^*} \operatorname{Hom}_{\kappa(B)}(\mathfrak{m}/\mathfrak{m}^2, \kappa(B))$ is surjective. Note that $\mathrm{Hom}_{\kappa(B)}(\Omega_{B/k} \otimes \kappa(B), \kappa(B)) \simeq \mathrm{Der}_k(B, \kappa(B))$. The map δ^* sends $D \in \mathrm{Der}_k(B, \kappa(B))$ to $D : \mathfrak{m}/\mathfrak{m}^2 \ni b \otimes \bar{b}' \mapsto db \otimes_B \bar{b}' \mapsto D(b)\bar{b}'$ or equivalently $\bar{b} \mapsto D(b)$ where b is any lift of $\bar{b} \in \mathfrak{m}/\mathfrak{m}^2$. Given $h \in \mathrm{Hom}_{\kappa(B)}(\mathfrak{m}/\mathfrak{m}^2, \kappa(B))$, we now wish to construct $\widetilde{h}$ such that $\delta^*(\widetilde{h}) = h$. Well, Leibniz rule implies that giving a derivation $B \to \kappa(B)$ is the same as giving a derivation $B/\mathfrak{m}^2 \to \kappa(B)$ (as $\mathfrak{m}^2$ is in the kernel as map of Abelian groups). As $B/\mathfrak{m}^2$ is complete local ring, there is a field $K \hookrightarrow B/\mathfrak{m}^2$ such that $K \xrightarrow{\sim} \kappa(B)$. We can thus build $\widetilde{h} \in \mathrm{Der}_K(B/\mathfrak{m}^2, \kappa(B)) \subset \mathrm{Der}_k(B/\mathfrak{m}^2, \kappa(B)) = \mathrm{Der}_k(b, \kappa(B))$ as in TIL 9/27/2016.

(b): Note that as B is a localization of finite type k-algebra, let's say $B = A_{\mathfrak{p}}$, we have $\dim B + \operatorname{tr}\deg_k \kappa(B) = \dim A_{\mathfrak{p}} + \operatorname{tr}\deg_k \kappa(\mathfrak{p}) = \dim A_{\mathfrak{p}} + \dim A/\mathfrak{p} = \dim A = \operatorname{tr}\deg_k K$ where K is the quotient field.

Now, If k is perfect, then $\kappa(B)$ is separably generated, so that by part (a) we have

$$0 \to \mathfrak{m}/\mathfrak{m}^2 \to \Omega_{B/k} \otimes \kappa(B) \to \Omega_{\kappa(B)/k} \to 0$$

If $\Omega_{B/k}$ is free of rank $\dim B + \operatorname{tr}\deg_k \kappa(B)$, then $\mathfrak{m}/\mathfrak{m}^2$ has $\kappa(B)$-dimension $\dim B$ so that B is regular. If B is regular, then the proof is exactly the same as Proposition II.8.8 (noting that as k is perfect, $\operatorname{tr}\deg_k K = \operatorname{rk}\Omega_{K/k}$).

(c): Follows immediately from the previous parts.

(d): We don't really need algebraic closure, just that k is perfect. For $(\Omega_{X/k})_x$ to have $\mathscr{O}_{X,x}$-rank $\dim X$, since $\operatorname{rk}_{K(X)}(\Omega_{X/k} \otimes K(X)) = \dim X$ always, from part (a) we see that we only need $\Omega_{X/k}|_x := \Omega_{X/k} \otimes \kappa(x)$ to have $\kappa(x)$-rank $= \dim X$. Let's now work affine locally, and say $X = \operatorname{Spec} A(X)$ where $A(X) := k[x_1, \ldots, x_n]/I$ so that we have

$$I/I^2 \xrightarrow{J} k[\underline{x}]\{dx_1, \ldots, dx_n\} \otimes k[\underline{x}]/I \to \Omega_{X/k} \to 0$$

where J is the (co)Jacobian matrix mod I. Tensoring by $\otimes_{A(X)}\kappa(x)$, we get:

$$I \otimes k[\underline{x}]/I \otimes \kappa(x) \xrightarrow{J(x)} k[\underline{x}]^{\oplus n} \otimes \kappa(x) \to \Omega_{X/k}|_x \to 0$$

So that we exactly need the rank of $J(x)$ to be $n - \dim X$. But its rank is always $\leq n - \dim X$, since otherwise, we would have $\operatorname{rk}\Omega_{X/k}|_x < \dim X$. Hence, that rank $\leq n - \dim X - 1$ is a closed condition implies that rank $= n - \dim X$ is an open condition.

Generalization: If $X = \bigcup_i Z_i$ is not integral, for $x \in Z_i$ not in any other irreducible components, replace $\dim X$ with $\dim Z_i$ and the same argument pulls through. If $X = \operatorname{Spec} k[x_1, \ldots, x_n]/I$, then $n - \dim Z_i = \operatorname{codim}_{\mathbb{A}^n} Z_i = \operatorname{ht} I_{\mathfrak{p}}$ where $V(\mathfrak{p}) = Z_i$.

II.8.2. We can reduce to V being finite dimensional. Let's say $\dim_k V = s$ and $\operatorname{rk}\mathscr{E} = r$. For $x \in X$ closed point, consider the map $\varphi_x : V \to \mathscr{E}|_x := \mathscr{E}_x \otimes \kappa(x)$. φ_x is surjective, and as a k-vector space $\ker\varphi_x$ thus has dimension $s - r[\kappa(x) : k]$. Thus $W \subset X \times_k |V|$ defined by $W := \{(x, s) : s_x \in \mathfrak{m}_x\mathscr{E}_x\}$ is an algebraic subset whose dimension is $\leq \sup\{\dim X + s - r[\kappa(x) : k]\} \leq n + s - r \leq s - 1$. Thus, the second projection $X \times |V| \to |V|$ show that the image of W is proper in $|V|$. Thus, there exists $s \notin p_2(W)$, i.e. $s \notin \mathfrak{m}_x\mathscr{E}_x$ $\forall x \in X$ closed. As closed points are dense in finite type k-schemes, this also means that s doesn't vanish on any $x \in X$ (not necessarily closed). Defining a map $\mathscr{O}_X \to \mathscr{E}$ by $1 \mapsto s$, over any $x \in X$, we have $s\mathscr{O}_{X,x} = \mathscr{O}_{X,x}$ so that

$$0 \to \mathscr{O}_{X,x} \to \mathscr{E}_x \to \mathscr{E}'_x \to 0$$

is a SES of free modules. Thus, we get the desired SES of locally free sheaves.

Alternate Soln See TIL 10/6/2016. The main task is to interpret $V \to \mathscr{E}|_x$ as $\mathscr{O}_X^s \twoheadrightarrow \mathscr{E}$

II.8.3. (a): A bit of CA review: if $A \to B, A \to A'$ and $B' = B \otimes_A A'$, then $\Omega_{B'/A'} \simeq \Omega_{B/A} \otimes_A A'$ via $d(b \otimes a') \leftrightarrow db \otimes a'$. Now, in our case, this implies that $\Omega_{X\times_S Y/X} \simeq p_2^* \Omega_{Y/S}$ and $\Omega_{X\times_S Y/Y} \simeq p_1^*\Omega_{X/S}$. We thus get a commutative diagram

$$\begin{tikzcd} & p_2^*\Omega_{Y/S} \arrow[d] \arrow[dr, "\simeq"] & & \\ p_1^*\Omega_{X/S} \arrow[r] \arrow[dr, "\simeq"'] & \Omega_{X\times_S Y/S} \arrow[r] \arrow[d] & \Omega_{X\times_S Y/X} \arrow[r] & 0 \\ & \Omega_{X\times_S Y/Y} & & \end{tikzcd}$$

so that the horizontal line is in fact a split short exact sequence. Thus, $\Omega_{X\times_S Y/S} \simeq p_1^*\Omega_{X/S} \oplus p_2^*\Omega_{Y/S}$.

(b): Let X^r, Y^s be smooth k-varieties, so that $\Omega_{X/k}, \Omega_{Y/k}$ are locally free of rank r, s. Then $\omega_{X\times Y} = \bigwedge^{r+s}\Omega_{X\times Y/k} \simeq p_1^*\omega_X \otimes p_2^*\omega_Y$ (check affine locally for pedantic details).

(c): Recall that $P_{X\times Y} = P_X P_Y$ (with $X \times Y$ embedded via Segre embedding), and hence $p_a(X\times Y) = p_a(X)p_b(Y) + (-1)^s p_a(X) + (-1)^r p_a(Y)$. If $X = Y =$ nonsingular plane cubic curve, then $p_a(X) = 1 - P_X(0) = 1$, and $p_a(X \times X) = -1$. However, $p_g(X) = 1$ and $p_g(X\times X) = 1$, as $\omega_X \simeq \mathscr{O}_X(3-2-1) = \mathscr{O}_X$.

II.8.4. (a): If Y is a complete intersection with $I = (f_1, \ldots, f_r)$ for $f_i \in S$, then indeed $Y = V(f_1) \cap \cdots \cap V(f_r)$ and each $V(f_i)$ is a hypersurface. For the converse, we need a lemma:

Lemma: Let H be a hypersurface (locally principal closed subscheme of codimension 1) in $\mathbb{P}^n$. Then $H = V(f)$ for some $f \in S$ homogeneous (not necessarily irreducible). *Proof:* First, note that if A is a UFD, and $I \subset A$ a (f)-primary ideal for $f \in A$ irreducible, then $I = (f^m)$ for some m; if m is the least such that $f^m \in I$, then any $g \in I$ factors to $f^k g_1 \cdots g_\ell$, and as no power of g_i's are in I ($\because f$ does not divide them) $f^k \in I$ so that $k \geq m$. Second, if $X \subset \mathbb{A}^n$ a closed subscheme such that $I(X)_{f_j}$ is principal for some open cover $\{D(f_j)\}_j$ of $\mathbb{A}^n$, then $\operatorname{Ass}(I(X)_{f_j}) = \emptyset$ or all associated primes of $I(X)_{f_i}$'s are codimension 1 by the unmixedness theorem, so that all associated primes of $I(X)$ is codimension 1. Thus, as $\Bbbk[x_1, \ldots, x_n]$ is a UFD, this implies that the primary decomposition of $I(X)$ is $\bigcap_i (g_i^{n_i}) = (\prod_i g_i^{n_i})$ for irreducible g_i's so that $I(X)$ is in fact principal. Thus, if H is a hypersurface in $\mathbb{P}^n_k$, then $H \cap U_k = V(g^{(k)})$ so that $I(H) = g$ for some g homogeneous polynomial in S.

Thus, if $Y = H_1 \cap \cdots \cap H_r$ as schemes, then $Y = V(f_1) \cap \cdots \cap V(f_r) = V(f_1, \ldots, f_r)$ for some homogeneous polynomials $f_1, \ldots, f_r$, so that $I(Y) = (f_1, \ldots, f_r)$.

(b): Let $X = \operatorname{Spec} S/I$ be the affine cone of Y. Its codimension in $\mathbb{A}^{n+1}$ is still $\operatorname{codim}_X Y$. It is indeed a (locally) complete intersection, so that $A(X)$ is Cohen-Macaulay. To check that X is regular in codimension 1, note that the cone point P is of codimension ≥ 2 since $\dim Y \geq 1$, and as $X \setminus \{P\}$ is a $\mathbb{A}^1$-bundle over Y, we have that $A(X)_\mathfrak{p} \simeq \mathscr{O}_{Y,y}[t]$ for some $y \in Y$ for any codimension 1 points $\mathfrak{p}$ in X.

(c): Follows immediately from II.5.14(d). The surjection implies that $\mathscr{O}_Y(Y)$ is of k-rank at most 1, so that Y is connected.

(d): Suppose $X \subset \mathbb{P}^n_k$ is codimension $r \leq n-2$ nonsingular subvariety. Consider the d-uple embedding of $\nu_d : \mathbb{P}^n_k \hookrightarrow \mathbb{P}_k^{\binom{n}{d}-1}$. By Bertini's theorem, there exists a hyperplane section H such that $H \not\supset \nu_d(X)$ and $H \cap \nu_d(X)$ is nonsingular variety, so that $X \cap \nu_d^{-1}H$ is a nonsingular variety of codimension $r-1$. Use this to induct and make the desired complete intersection.

(e): Use II.8.20 repeatedly.

(f): We have $\omega_X = \mathscr{O}_X(d-n-1)$. If $d < n+1$, then $\Gamma(X, \omega_X) = 0$ (as a global section would give an effective divisor of negative degree). If $d \geq n+1$, apply $- \otimes \mathscr{O}_{\mathbb{P}}(d-n-1)$ to the SES $0 \to \mathscr{O}_{\mathfrak{p}}(-d) \to \mathscr{O}_{\mathbb{P}} \to \mathscr{O}_X \to 0$ (which stays SES by part (c)) to get that $h^0(\omega_X) = h^0(\mathscr{O}_{\mathbb{P}}(d-n-1)) - h^0(\mathscr{O}_{\mathbb{P}}(-n-1)) = \binom{d-1}{n}$ (plug in $z = d-n-1$ into $P_{\mathbb{P}}(z) = \binom{z+n}{n}$). Hence, in this case $p_g(X) = p_a(X)$.

(g): Moreover, if we have nonsingular complete intersection X of type (d, e) in $\mathbb{P}^n$, (once again, let's just consider $d+e-n-1 \geq 0$ case) then the following free resolution:

$$0 \to S(-d-e) \to S(-d) \oplus S(-e) \to S \to S/I \to 0$$

and its twist by $d+e-n-1$, combined with the fact that X is also projectively normal (part (b), and hence (c) applies), we have that $p_g(X) = \binom{d-1}{n} + \binom{e-1}{n} - \binom{d+e-1}{n}$. When $n = 3$, we get the desired formula.

II.8.5. Later...

Chapter III

Cohomology

III.1 Derived Functors

Know your homological algebra.

III.2 Cohomology of Sheaves

III.2.1. (a): Let's denote the set of two k-points P, Q as Y (Y is a closed subscheme of $\mathbb{A}^1_k$). Note that $U = \mathbb{A}^1 \setminus Y$ is actually irreducible (and hence connected) in Zariski topology. Now, from $0 \to \mathbb{Z}_U \to \mathbb{Z} \to \mathbb{Z}_Y \to 0$, we get $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\varphi} H^1(\mathbb{Z}_U) \to H^1(\mathbb{Z}) \to \cdots$, and the image of the map φ is not zero since $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ cannot be surjective.

III.2.3. (a): Note that if $\varphi : \mathscr{F} \to \mathscr{G}$ is a map of sheaves, then $\Gamma_{U\cap Y}(U, \mathscr{F}) \to \Gamma_{U\cap Y}(U, \mathscr{G})$ is well-defined, as if $s \in \Gamma_{U\cap Y}(U, \mathscr{F})$, then $\varphi(U)(s)_P = \varphi_P(s_P) = \varphi_P(0) = 0$. We have shown that $\mathscr{H}^0_Y(-) : \mathsf{Sh}(X) \to \mathsf{Sh}(X)$ is a functor (where $\mathsf{Sh}(X)$ is sheaves whose objects are at least abelian groups). In particular, $\Gamma_Y(X, -) : \mathsf{Sh}(X) \to \mathsf{AbGrp}$ is a functor—let's call it F.

To see that this is left exact, let $0 \to \mathscr{F}' \xrightarrow{\alpha} \mathscr{F} \xrightarrow{\beta} \mathscr{F}'' \to 0$ be a SES of sheaves on X. That $\mathsf{F}\alpha$ is injective and $\mathsf{F}\beta \circ \mathsf{F}\alpha = 0$ is clear. Now, if $s \in \Gamma_Y(X, \mathscr{F})$ maps to 0 in $\Gamma_Y(X, \mathscr{F}'')$, then by exactness $s_p \in \mathscr{F}'_p \; \forall p \in X$, so that $s \in \mathscr{F}'(U)$ in particular, and its support is still in Y.

(b): We need show that $\Gamma_Y(X, \mathscr{F}) \to \Gamma_Y(X, \mathscr{F}'')$ is surjective. Given $s'' \in \Gamma_Y(X, \mathscr{F}'')$, considering as an element of $\mathscr{F}''(X)$, there exists $s \in \mathscr{F}(X)$ such that $s \mapsto s''$ (by II.1.16). Now, as $s|_U \in \mathscr{F}(U)$ maps to $s''|_U = 0 \in \mathscr{F}''(U)$, we have $s|_U \in \mathscr{F}'(U)$. Extend $s|_U$ to $t \in \mathscr{F}'(X)$ (possible since $\mathscr{F}'$ is flasque), then $s - t$ has support in Y since $s|_U - t|_U = 0$, and $s - t$ still maps to s'' since t mapst to 0.

(c): Exactly the same as the proof of Proposition III.2.5.

(d): From Ex.II.1.20(b), we have that $0 \to \mathscr{H}^0_Y(\mathscr{F}) \to \mathscr{F} \to j_*(\mathscr{F}|_U)$ is exact, and moreover, if $\mathscr{F}$ is flasque, the last map on any open V is $\mathscr{F}(V) \overset{\text{res}}{\twoheadrightarrow} \mathscr{F}(V \cap U)$. Now, combine this with that $\Gamma(X, -)$ is left exact to obtain the SES desired.

(e): Take an injective solution $\mathscr{I}^\bullet$ of $\mathscr{F}$, to obtain a flasque resolution $\mathscr{I}|^\bullet_U$ of of $\mathscr{F}|_U$. Then from the previous part, we have chain complex of short exact sequences:

$$0 \to \Gamma_Y(X, \mathscr{I}^\bullet) \to \Gamma(X, \mathscr{I}^\bullet) \to \Gamma(U, \mathscr{I}|^\bullet_U) \to 0$$

so that applying the homology functor and using snake lemma, we get the desired LES.

(f): Let $U = X \setminus V$. Note the exact sequence $0 \to \mathscr{F}_U \to \mathscr{F} \to \mathscr{F}_V \to 0$. We claim that $\mathscr{F}_U := j_!(\mathscr{F}|_U)$ satisfies $H^i_Y(X, \mathscr{F}_U) = 0$ for all i. Well, following the proof of III.2.2, one can construct injective resolution $\mathscr{I}^\bullet$ of $\mathscr{F}_U$ such that $(\mathscr{I}^i)_p = 0$ for all $p \in V \supset Y$ Is this actually true?. Hence, no $\mathscr{I}^i$ has a global section supported in Y, and thus $H^i_Y(X, \mathscr{F}_U) = 0$ for all i as desired.

III.3 Cohomology of Noetherian Affine Scheme

III.3.1. Let $j : X_{red} \hookrightarrow X$ be the closed embedding. If X is affine, then for any quasi-coherent sheaf $\mathscr{G}$ on X_{red}, we have $H^i(X_{red}, \mathscr{G}) = H^i(X, j_*\mathscr{G})$ which is zero for all $i > 0$ as X is affine and Noetherian-ness implies that $j_*\mathscr{G}$ is quasi-coherent. Now, suppose X_{red} is affine, and let $\mathscr{F} \in QCoh_X$ and $\mathscr{N}$ the sheaf of nilpotents. Note that we have a filtration $\mathscr{F} \supset \mathscr{N} \cdot \mathscr{F} \supset \cdots$ (which is actually a finite filtration as X is Noetherian). The successive quotients are $\mathscr{N}^i \cdot \mathscr{F} / \mathscr{N}^{i+1} \cdot \mathscr{F} \simeq \mathscr{N}^i \cdot \mathscr{F} \otimes \mathscr{O}_X/\mathscr{N} = j_*j^*(\mathscr{N}^i \cdot \mathscr{F})$ so that their nonzero cohomologies all vanish since X_{red} is affine. Once again, use LES to climb up the chain and conclude that nonzero cohomologies of $\mathscr{F}$ hence all vanish.

III.3.2. If X is affine, then so are its closed subsets. For the converse, we show a stronger statement:

Lemma: Let X be reduced Noetherian, Y, Z closed integral subschemes of X that are both affine. Then $Y \cup Z$ is also affine.

Proof: First, replace X by $Y \cup Z$ (by reduced structure on $Y \cup Z$, which makes no harm as X is reduced). If one of Y, Z is contained in another, then we are done, so suppose not, so that $\mathscr{I}_Y \mathscr{I}_Z \subset \mathscr{I}_Y \cap \mathscr{I}_Z = 0$. Note that for any $\mathscr{F} \in QCoh_X$, the higher cohomologies of $\mathscr{F} \otimes \mathscr{O}_X/\mathscr{I}_Y$ vanish. Now, let $\mathscr{F} \in QCoh_X$. From $0 \to IM/IJM \to M/IJM \to M/IM \to 0$ we have a SES $0 \to \mathscr{I}_Y\mathscr{F} \otimes \mathscr{O}_X/\mathscr{I}_Z \to \mathscr{F} \to \mathscr{F} \otimes \mathscr{O}_X/\mathscr{I}_Y \to 0$ (as $\mathscr{F} \cdot \mathscr{O}_X/\mathscr{I}_Y\mathscr{I}_Z = \mathscr{F}\mathscr{O}_X/0 \simeq \mathscr{F}$). By LES, we have that higher cohomologies of $\mathscr{F}$ all vanish, and hence X is affine.

III.3.3. Let $X = \operatorname{Spec} A$. We have shown that $\sim$: A-$\mathsf{Mod} \xrightarrow{\sim} QCoh_X$ is an equivalence of categories. And moreover, for $Y = V(\mathfrak{a})$, we have $m \in \Gamma_Y(\widetilde{M}) \iff V(\operatorname{Ann} m) \subset Y \iff \mathfrak{a} \subset \sqrt{\operatorname{Ann} m} \iff m \in (0 :_M \mathfrak{a}^\infty) = \Gamma_{\mathfrak{a}}(M)$, so that as functors we have $\Gamma_Y \circ \sim = \Gamma_{\mathfrak{a}}$. This makes all (a,b,c) parts clear.

III.3.4. (a): If $\operatorname{depth}_{\mathfrak{a}} M \geq 1$, then there is $f \in \mathfrak{a}$ a nonzerodivisor of M, which implies that $(0 :_M \mathfrak{a}^\infty) = 0$ since no nonzero element of m can kill a power of f. If M is further finitely generated, then $\operatorname{depth}_{\mathfrak{a}} M = 0$ implies that $\mathfrak{a}$ is contained in some associated prime (of which there are finitely many as M is f.g.). This associated prime is $\operatorname{Ann} m$ for some $0 \neq m \in M$, so that $m \in \Gamma_{\mathfrak{a}}(M)$, and so $\Gamma_{\mathfrak{a}}(M) \neq 0$.

(b): We induct on n. $n = 1$ is part (a) (well, $n = 0$ is empty-statement, so assume $n \geq 1$ and begin at $n = 1$). Consider the SES $0 \to M \xrightarrow{f} M \to M/f \to 0$ where $f \in \mathfrak{a}$ is a nonzerodivisor of M. Then $\operatorname{depth}_{\mathfrak{a}}(M/f) = \operatorname{depth}_{\mathfrak{a}} M - 1$, so that $\operatorname{depth}_{\mathfrak{a}} M \geq n \iff \operatorname{depth}_{\mathfrak{a}}(M/f) \geq n-1$. Moreover, we have the LES from the SES: $\cdots \to H^i_{\mathfrak{a}}(M) \xrightarrow{f} H^i_{\mathfrak{a}}(M) \to H^i_{\mathfrak{a}}(M/f) \to H^{i+1}_{\mathfrak{a}}(M) \to \cdots$. Hence, we have $H^i_{\mathfrak{a}}(M) = 0 \ \forall i < n \iff H^{i-1}_{\mathfrak{a}}(M/f) = 0 \ \forall i < n$: The $\Longrightarrow$ direction is easy; for $\Longleftarrow$ direction, note that by Exer.III.3.3.(c), for $i < n-1$, $H^i_{\mathfrak{a}}(M)$ is zero when localized at $\mathfrak{p} \not\supset \mathfrak{a}$, and for $\mathfrak{p} \supset \mathfrak{a}$, Nakayama implies that $H^i_{\mathfrak{a}}(M)$ is zero, and for $i = n-1$, note that $\times f$ cannot be an injective map as the support of every element of $H^i_{\mathfrak{a}}(M)$ is contained in $V(\mathfrak{a}) \subset V(f)$ (so some power of f annihilates that element). By induction hypothesis, we are done.

III.3.5. Let's generalize to P just a point in X, and $U - P$ means $U - \overline{\{P\}}$. For any $U \ni p$, we have a LES by Exer.III.2.3.(e): $0 \to H^0_{\overline{P}}(U, \mathscr{O}_U) \to H^0(U, \mathscr{O}_U) \to H^0(U - P, \mathscr{O}_U) \to H^1_{\overline{P}}(U, \mathscr{O}_U) \to \cdots$. Now, $\operatorname{depth} \mathscr{O}_P \geq 2 \iff H^0_{\overline{P}} = H^1_{\overline{P}} = 0$ from Exer.III.3.4.(b).

III.4 Cech Cohomology

III.3.4.1. If $f : X \to Y$ is an affine morphism, then choose a affine cover $\mathfrak{V} = \{V_i\}$ of Y. Then $\mathfrak{U} := \{U_i := f^{-1}(V_i)\}$ is an affine cover X such that $C^\bullet(\mathfrak{V}, f_*\mathscr{F}) = C^\bullet(\mathfrak{U}, \mathscr{F})$, and hence by III.4.5 we have $H^i(X, \mathscr{F}) \simeq H^i(Y, f_*\mathscr{F})\ \forall i$.

III.3.4.2. (a): Let $V := \operatorname{Spec} B \subset Y$ and $U := \operatorname{Spec} A = f^{-1}(\operatorname{Spec} B) \subset X$. Then $K(A)$ is a finite extension of $K(B)$ ($\because$algebraic+f.g.), and thus if $[K(A) : K(B)] = r$, then shrinking V if necessary (thus shrinking U), we have that $B^r \simeq A$ as B-modules. Letting $\mathscr{M} := \mathscr{O}_U$, we see that $\mathscr{M}$ is coherent since X is separated (and U is affine). Moreover, define a map $\alpha : \mathscr{O}_Y^r \to f_*(\mathscr{O}_U)$ by selecting r sections in $\Gamma(Y, f_*(\mathscr{O}_U)) = \mathscr{O}_U(U) = A$ that span A as a B-module as given by $B^r \simeq A$.

(b): Just follow the hint, and for isomorphism restrict to an affine (since quasi-coherent) and use the fact that localization commutes with Hom.

(c): Suffices to show that Y_{red} is affine, so replace Y by Y_{red}, and hence X by X_{red}. Now, we need only show that each irreducible components of Y is affine, so assume Y is integral. But then f is a closed surjective map, we can also assume that X is integral.

We will now show that Y is affine by Noetherian induction. Well, if $Z \subset Y$ is a closed subset of Y, then $f : f^{-1}(Z) \to Z$ is again a finite surjective morphism where $f^{-1}(Z)$ is affine since it is closed subscheme of X. Hence, we reduce to showing that Y is affine given that every closed subset of Y is affine.

Now, let $\mathscr{F}$ be any coherent sheaf on Y, and let $\beta : f_*\mathscr{G} \to \mathscr{F}^r$ be as in part (b). Via Cech cohomology and III.4.5, we note that $H^i(Y, \mathscr{F}^r) \simeq (H^i(Y, \mathscr{F}))^r$. Thus, as $H^i(Y, f_*\mathscr{G}) = H^i(X, \mathscr{G}) = 0\ \forall i > 0$ (X is affine), it remains to show that $H^i(Y, \mathscr{F}^r) \simeq H^i(Y, f_*\mathscr{G})\ \forall i > 0$.

Consider the exact sequence $0 \to \ker \beta \to f_*\mathscr{G} \to \mathscr{F}^r \to \operatorname{coker} \beta \to 0$. As $f_*\mathscr{G}$ and $\mathscr{F}^r$ are both coherent (since f is finite), $\ker \beta$ and $\operatorname{coker} \beta$ are both coherent, and hence their supports, let's say Y_1, Y_2, are both closed. Thus, we have $\ker \beta = (j_1)_*(j_1)^*(\ker \beta)$ and similarly for $\operatorname{coker} \beta$. Hence, as Y_1 and Y_2 are affine (by Noetherian hypothesis), the higher cohomologies of $\ker \beta$ and $\operatorname{coker} \beta$ vanish. Now, splitting the sequence above into $0 \to \ker \beta \to f_*\mathscr{G} \to \operatorname{im} \beta \to 0$ and $0 \to \operatorname{im} \beta \to \mathscr{F}^r \to \operatorname{coker} \beta \to 0$, and applying LES, we obtain $H^i(Y, f_*\mathscr{G}) \simeq H^i(Y, \operatorname{im} \beta) \simeq H^i(Y, \mathscr{F}^r)$ for $i > 0$, as desired.

III.4.3. We have $0 \to k[x^\pm, y] \times k[x, y^\pm] \to k[x^\pm, y^\pm] \to 0$ where the map is $(f, g) \mapsto f - g$. The kernel is thus $k[x, y]$, and the image (as k-vector space) is all monomials $x^\alpha y^\beta$ where at least one of α, β is non-negative. Hence, $H^0(X, \mathscr{O}_X) = k[x, y]$ and $H^1(X, \mathscr{O}_X) = k\{x_i y_i : i, j < 0\}$.

III.4.4. (a): For each p, we define $\widetilde{\lambda}^p : C^p(\mathfrak{A}, \mathscr{F}) \to C^p(\mathfrak{B}, \mathscr{F})$ by sending $\alpha \in \mathscr{F}(U_{i_0,\dots,i_p}) \mapsto \prod_J \alpha|_{U_J}$ where the J's are $(j_0, \dots, j_p)$ such that $\lambda(j_k) = i_k$. As λ is an index map (preserving the well-ordering), one checks easily that $\widetilde{\lambda}^p$'s form a map of chain complexes $C^\bullet(\mathfrak{A}, \mathscr{F}) \to C^\bullet(\mathfrak{B}, \mathscr{F})$. In fact, in the same way, we have a map $\widetilde{\lambda}^\bullet : \mathscr{C}^\bullet(\mathfrak{A}, \mathscr{F}) \to \mathscr{C}^\bullet(\mathfrak{B}, \mathscr{F})$.

(b): If $\mathscr{F} \to \mathscr{I}^\bullet$ is an injective resolution of $\mathscr{F}$, then we have maps $\varphi_\mathfrak{A} : \mathscr{C}^\bullet(\mathfrak{A}, \mathscr{F}) \to \mathscr{I}^\bullet$ and $\varphi_\mathfrak{B} : \mathscr{C}^\bullet(\mathfrak{B}, \mathscr{F}) \to \mathscr{I}^\bullet$ induced from the identity map on $\mathscr{F}$. Moreover, $\varphi_\mathfrak{B}$ is homotopic to $\varphi_\mathfrak{A} \circ \widetilde{\lambda}^\bullet$, and hence the maps $\check{H}^i(\mathfrak{A}, \mathscr{F}) \to \check{H}^i(\mathfrak{B}, \mathscr{F})$ are compatibles with maps $\check{H}^i(\mathfrak{A}, \mathscr{F}) \to H^i(X, \mathscr{F})$.

(c): Following the hint and using LES, we need show that $\varinjlim_\mathfrak{A} D^\bullet(\mathfrak{A}) \xrightarrow{\sim} \varinjlim_\mathfrak{A} C^\bullet(\mathfrak{A}, \mathscr{F})$. Well, the injectivity always holds, and for surjectivity, use Exer.II.1.3(a).

III.4.5. Define a map $\operatorname{Pic} X \to H^1(X, \mathscr{O}_X^*)$ as follows: Given $\mathscr{L}$, take any trivializing cover $\mathfrak{A} = \{U_i\}$. As $\mathscr{L}$ is a line bundle, on $U_i \cap U_j$, we have transition maps $\varphi_{ji} : \mathscr{O}_{U_i \cap U_j} \xrightarrow{\sim} \mathscr{O}_{U_i \cap U_j}$, which is a multiplication by an element of $\mathscr{O}_X(U_i \cap U_j)^*$, so that we have an element $\mathscr{L}(\mathfrak{A})$ in $C^1(\mathfrak{A}, \mathscr{O}_X^*)$. That the transition maps satisfy the cocycle condition implies that $\mathscr{L}(\mathfrak{A}) \in \ker d_1 : C^1 \to C^2$. Now, if $\mathscr{L}(\mathfrak{A}) \in \operatorname{im} d_0$, then $\mathscr{L} \simeq \mathscr{O}_X$ (can construct an explicit isomorphism by

$\mathscr{O}_X(U_i) \overset{\times f_i}{\to} \mathscr{O}_X(U_i)$ where $f_i \in \mathscr{O}_X^*(U_i)$). We thus have an element of $\check{H}^1(\mathfrak{A}, \mathscr{O}_X^*)$. Lastly, the isomorphism $\varinjlim_{mfA} \check{H}^1(\mathfrak{A}, \mathscr{O}_X^*) \xrightarrow{\sim} H^1(X, \mathscr{O}_X^*)$ implies that we can consider $\mathscr{L}(\mathfrak{A})$ as an element of $H^1(X, \mathscr{O}_X^*)$ and that the map $\operatorname{Pic} X \to H^1(X, \mathscr{O}_X^*)$ is indepndent of the choice of the cover $\mathfrak{A}$. To see that this is a group homomorphism, given $\mathscr{L}, \mathscr{M}$, taking common refinement of each trivializing cover.

III.4.7. Cech is actually cumbersome. Just use LES of $0 \to \mathscr{O}_{\mathbb{P}^2}(-d) \to \mathscr{O}_{\mathbb{P}^2} \to \mathscr{O}_X \to 0$.

III.4.8. (a): By Exer.II.5.15, a quasi-coherent sheaf $\mathscr{F}$ on X is a direct limit of coherent ones, but direct limit commutes with cohomology.

(b): Let X be open subscheme of a projective scheme $Y \subset \mathbb{P}^n_k$. Suppose $H^i(X, \mathscr{E}) = 0 \ \forall i > n$ for any locally free coherent sheaf $\mathscr{E}$. Now, let $\mathscr{F}$ be a coherent sheaf on X; by Exer.II.5.15 there exists a coherent sheaf $\mathscr{F}'$ on Y such that $\mathscr{F}'|_X = \mathscr{F}$. By Serre's theorem II.5.18 we have $\mathscr{O}_Y^r \twoheadrightarrow \mathscr{F}' \otimes \mathscr{O}_Y(m)$ for some r, m. Tensoring by $\mathscr{O}_Y(-m)$ (which is exact as $\mathscr{O}_Y(-m)$ is locally free), we obtain a SES $0 \to \mathscr{K}' \to \mathscr{E}' \to \mathscr{F}' \to 0$, where $\mathscr{E}'$ is locally free of rank r, and $\mathscr{K}'$ is coherent, as $\mathscr{E}'$ is coherent (and Y Noetherian). Restricting to X (an open subscheme), we obtain a SES $0 \to \mathscr{K} \to \mathscr{E} \to \mathscr{F} \to 0$, from which we obtain a LES that implies that $H^i(X, \mathscr{F}) \simeq H^{i+1}(X, \mathscr{K}) \ \forall i > n$. Grothendieck vanishing implies that $H^{d+1}(X, \mathscr{K}) = 0$, and hence $H^d(X, \mathscr{F}) = 0$ where $d = \dim X$. But this implies that any coherent sheaf of X has vanishing dth cohomology, so applying this to $\mathscr{K}$ we have that $H^{d-1}(X, \mathscr{F}) = 0$. Continuing this on, we conclude that $H^i(X, \mathscr{F}) = 0 \ \forall i > n$, as desired.

(c): If X is covered by $r+1$, then is no cohomology beyond r in the Cech complex.

(d): For $k = \overline{k}$ case, project from a generic point. For general case, see homogeneous Noether normalization in Eisenbud Comm. Alg.

(e): $Y \subset \mathbb{P}^n_k$ is a set-theoretic intersection of codim r if $Y = V(f_1, \ldots, f_r)$ (that is, $I(Y) = \sqrt{(f_1, \ldots, f_r)}$ in the coordinate ring). Thus, $X \setminus Y = \bigcup_{i=1}^r (X \setminus V(f_i))$, but $X \setminus V(f_i) = D^+(f_i)$ is affine (since Veronese embedding is an closed embedding for any ring).

III.4.9. If Y is a complete intersection (of codimension 2) in $X = \mathbb{A}^4_k$, then $\operatorname{cd}(X \setminus Y) \leq 1$. However, LES of Exer.III.2.3 implies that $H^2(X \setminus Y, \mathscr{O}_{X \setminus Y}) \simeq H^3_Y(X, \mathscr{O}_X)$. Setting $P = Y_1 \cap Y_2$, and suppressing some notations, we have by Mayer-Vietoris the following sequence:

$$\to H^3_P \to H^3_{Y_1} \oplus H^3_{Y_2} \to H^3_Y \to H^4_P \to H^4_{Y_1} \oplus H^4_{Y_2} \to \cdots$$

Now, as depth of P is 4, $H^3_P = 0$. Moreover, Using Exer.III.2.3 on $X \setminus Y_1$ and $X \setminus Y_2$ (note that both admit cover by 2 affines so that $\operatorname{cd}(X \setminus Y_i) \leq 1$), we have $H^\ell(X \setminus Y_i) \simeq H^{i+1}_{Y_i} \ \forall \ell > 0$. This implies that $H^\ell_{Y_i} = 0 \ \forall \ell \geq 3$. Thus, we now have that $H^3_Y \simeq H^4_P$. But $H^4_P \neq 0$ as depth of P is exactly 4. This implies that Y cannot be a complete intersection.

III.4.11. Include $\mathscr{F}$ in an injective sheaf $\mathscr{I}$, so that we have $0 \to \mathscr{F} \to \mathscr{I} \to \mathscr{G} \to 0$. By taking the derived LES of $\Gamma(V, \cdot)$ from this SES for every V, and noting that $\mathscr{I}$ is injective and $\mathscr{F}$ has all vanishing higher cohomologies, we have that $H^i(V, \mathscr{G}|_V) = 0 \ \forall i > 0$ as well. Now, the rest of the proof is exactly the same as III.4.5.

III.5 The Cohomology of Projective Space

III.5.1. Alternating sums of dimensions of LES is zero.

Lemma. Let A be a Noetherian ring of finite dimension, and $f \in A$ such that $f \notin \bigcup_{\mathfrak{p} \in \operatorname{Min}(A)} \mathfrak{p}$. Then, $\dim A/f = \dim A - 1$. *Proof:* Topologically, $\operatorname{Spec} A$ is the union of irreducible components

$V_1 \cup \cdots \cup V_r$, and $\operatorname{Spec} A/f$ is hence union of $V_i \cap V(f)$. But for each i, the dimension of $V_i \cap V(f)$ is $\dim(A/\mathfrak{p}_i)/f = \dim(A/\mathfrak{p}_i) - 1$ by Krull's PIT as f is a nonzerodivisor in $A/\mathfrak{p}_i$.

III.5.2. (a): Embed $\iota : X \hookrightarrow \mathbb{P}^n_k$ by $\mathscr{O}_X(1)$, and thus reduce to $X = \mathbb{P}^n_k$ by Exer.III.4.1 as $\iota_* \mathscr{F}$ is coherent ($\because$ ι is a closed embedding). Moreover, by II.5.15, we have $M^\sim := \Gamma_*(\mathscr{F})^\sim \simeq \mathscr{F}$, and by Serre's lemma II.5.17, we have that M is a finitely generated graded S-module. Now, we induct on the dimension of $\operatorname{Supp} \mathscr{F}$ (where dim$= -1$ means the support is empty). Note that it is a closed subscheme carved out by the sheaf of ideals $\mathscr{I} := \sqrt{\operatorname{Ann} M}^{\sim}$; for each U_i, we have $\mathscr{I}(U_i) = \sqrt{\operatorname{Ann}_{S_{(x_i)}} M_{(x_0)}} = \sqrt{\operatorname{Ann} M}_{(x_0)}$. Now, base case is the the support being empty, in which case we have a zero sheaf, so that $\chi(\widetilde{M}(n)) = 0\ \forall n \in \mathbb{Z}$. Now, suppose $\operatorname{Supp} \widetilde{M} \neq \emptyset$. Then as an S-module, the irrelevant ideal $(x_0, \ldots, x_n)$ is not a minimal homogeneous prime of M, so that there is a linear form, WLOG say x_0, such that $x_0 \notin \bigcup_{\text{homog.Min}(M)} \mathfrak{p}$ by prime avoidance. This gives a map of sheaves $0 \to \mathscr{K} \to \widetilde{M}(-1) \to \widetilde{M} \to \mathscr{G} \to 0$. Moreover, on each distinguished affines U_i, we have $0 \to K \to M_i \xrightarrow{\frac{x_0}{x_i}} M_i \to M_i/(\frac{x_0}{x_i})M_i \to 0$ (for $i \neq 0$; the $i = 0$ case is trivial). As $\frac{x_0}{x_i}$ is a not in any minimal prime of $M_i := M_{(x_i)}$, we have that $\operatorname{Supp} M_i/(\frac{x_0}{x_i})M_i \subset V(\operatorname{Ann} M + (\frac{x_0}{x_i}))$ has dimension 1 less than $\operatorname{Supp} M_i$ by the Lemma above. The same holds for $\operatorname{Supp} K$. Thus, by induction hypothesis, $\chi(\mathscr{G}(n))$ and $\chi(\mathscr{K}(n))$ are polynomials in n, and $\chi(\mathscr{F}(n))+\chi(\mathscr{G}(n+1)) = \chi(\mathscr{F}(n+1))+\chi(\mathscr{K}(n+1))$ from Exer.III.5.1, so that we have that first-difference $\chi(\mathscr{F}(n+1)) - \chi(\mathscr{F}(n))$ is a polynomial, and hence $\chi(\mathscr{F}(n))$ is a polynomial.

(b): By III.5.2, we have $\chi(\mathscr{F}(n)) = \dim_k H_0(X, \mathscr{F}(n))$ for $n >> 0$. But by definition of $M := \Gamma_*(\mathscr{F})$, we have $H^0(X, \mathscr{F}(n)) = M_n$.

III.5.3.(a): $p_a(X) = (-1)^r(\chi(\mathscr{O}_X) - 1) = -(-1)^r + \sum_{j=0}^r (-1)^{r+j} \dim_k H^j(X, \mathscr{O}_X)$ which is $\sum_{j=1}^r (-1)^{r+j} \dim_k H^j(X, \mathscr{O}_X) = \sum_{i=0}^{r-1} H^{r-i}(X, \mathscr{O}_X)$

(b): If $X = \operatorname{Proj} S/I$, then by Exer.III.5.2(b) we have $P_X(n) = \chi(\mathscr{O}_X(n))$.

(c): If $f : X \to Y$ is a birational map, then it actually extends to all of X (I.6.8) and is a surjective finite map (II.6.8). Thus, by III.4.1, we conclude that the $p_a(X) = p_a(Y)$.

(Super useful) Lemma. Let X be a quasicompact separated scheme over A Noetherian, $\mathscr{F}$ a coherent sheaf on X, and $A \to B$ a flat map. Then $H^i(X, \mathscr{F}) \otimes_A B \simeq H^i(X_B, \mathscr{F} \otimes B)$ as B-modules. *Proof:* Flatness implies that $\otimes_B$ is an exact functor. Thus, applying this to the Cech complex, and using FHHF theorem, we have our desired isomorphisms.

III.5.5. (a): We use descending induction on q. If $q = r$, there is nothing to prove. Suppose $q \geq 1$, and say $Y = \operatorname{Proj} S/(f_1, \ldots, f_{r-q})$. Then S is Cohen-Macaulay, so $\dim C(Y) = q+1 = r+1-(r-q)$ implies that $(f_1, \ldots, f_{r-q})$ is a regular sequence. Hence, $Y' = \operatorname{Proj} S/(f_1, \ldots, f_{r-q-1})$ is also a complete intersection of dimension $q + 1$. Now, we have $0 \to \mathscr{O}_{Y'}(-\deg f_{r-q}) \to \mathscr{O}_{Y'} \to \mathscr{O}_Y \to 0$. Now, the statement follows from part (c) below.

(b): $k = H^0(X, \mathscr{O}_X) \twoheadrightarrow H^0(Y, \mathscr{O}_Y)$, and hence Y must be connected. Note that as $X = \mathbb{P}^r_k$, we still have $\mathscr{O}_X(X) = k$ even if k is not necessarily algebraically closed. In fact, as $\mathscr{O}_Y(Y)$ is a nonzero (nonzero follows from base-changing to $\overline{k}$ and using the lemma above) k-algebra, we have that $\mathscr{O}_Y(Y) \simeq k$ as well.

(c): We again use descending induction on q. If $q = r$, then our statement is exactly III.5.1. Suppose true for Y' of dimension $q' > 1$, and let Y be as in (a) (of dimension $q = q' - 1$). Then again, we have the SES $0 \to \mathscr{O}_{Y'}(-d) \to \mathscr{O}_{Y'} \to \mathscr{O}_Y \to 0$ where $-d = \deg f_{r-q}$, tensoring by $\mathscr{O}_{Y'}(n)$ and taking the LES, we have $H^i(Y, \mathscr{O}_Y(n)) \simeq H^{i+1}(Y', \mathscr{O}_{Y'}(n - d)) = 0\ \forall 0 < i < q' - 1$.

(d): Combine (b) and (c), and use same proof for Exer.III.5.3(a).

Lemma (Kunnuth formula) If X, Y are quasicompact separted k-schemes, and $\mathscr{F}, \mathscr{G}$ are quasicoherent sheaves on X, Y respectively, then $H^\ell(X \times_k Y, \mathscr{F} \boxtimes \mathscr{G}) \simeq \bigoplus_{i+j=\ell} H^i(X, \mathscr{F}) \otimes_k H^j(Y, \mathscr{G})$.

III.5.6. (a): The Kunnuth formula above just kills this (and intuition too).

(b1): Recall that if Y is a effective Cartier divisor (i.e. locally principal subscheme), then $\mathscr{L}(-Y) \simeq \mathscr{I}_Y$. Now, suppose Y is such that $\mathscr{L}(Y) \simeq \mathscr{O}_Q(a, b)$ with $a, b > 0$. Then we have $0 \to \mathscr{O}_Q(-a, -b) \to \mathscr{O}_Q \to \mathscr{O}_Y \to 0$ so that by applying LES and part (a) we have $\mathscr{O}_Q(Q) \twoheadrightarrow \mathscr{O}_Y(Y)$. We have $\mathscr{O}_Q(Q) = k$ since $\Gamma_*(\mathscr{O}_Q|_{\mathbb{P}^1_k \times_k U_0}) \simeq k[v/u][x, y]_\bullet$ (only x, y has weight 1) and $\Gamma_*(\mathscr{O}_Q|_{\mathbb{P}^1_k \times_k U_1}) \simeq k[u/v][x, y]_\bullet$. Hence, as $\mathscr{O}_Y(Y)$ is again nonzero, we have Y is connected.

(b2): $\mathscr{O}_Q(a, b)$ with $a, b > 0$ is very ample and gives an embedding $i : Q \hookrightarrow \mathbb{P}^a \times \mathbb{P}^b \hookrightarrow \mathbb{P}^{ab+a+b}$. Let $N = ab + a + b$. Take a hyperplane H in $\mathbb{P}^N$ such that $i(Q) \cap H$ is regular at each point (exists by Bertini's theorem). Taking i^{-1}, we get a curve that is connected (from (b1)) and regular at each point (and hence irreducible as well).

(b3): If $|a - b| \leq 1$ as well, then by part (a), we have that $H^0(Q, \mathscr{O}_Q(n)) \twoheadrightarrow H^0(Y, \mathscr{O}_Y(n))$ $\forall n$ so that Y is projectively normal (Y is already normal as it is nonsingular). Now, if $|a - b| > 1$, then by twisting appropriately we are in the third case of (a) so that the surjection no longer holds.

(c): Again, we have $0 \to \mathscr{O}_Q(-a, -b) \to \mathscr{O}_Q \to \mathscr{O}_Y \to 0$. And from $0 \to \mathscr{O}_{\mathbb{P}^3}(-2) \to \mathscr{O}_{\mathbb{P}^2} \to \mathscr{O}_Q \to 0$, we have that $\chi(\mathscr{O}_Q) = \binom{3}{3} - \binom{1}{3} = 1$. Hence, we have $1 - \chi(\mathscr{O}_Y) = \chi(\mathscr{O}_Q(-a, -b))$. By Kunnuth formula, we have $\chi(\mathscr{F} \boxtimes \mathscr{G}) = \chi(\mathscr{F})\chi(\mathscr{I})$, and moreover, $\chi(\mathscr{O}_{\mathbb{P}^r}(n)) = \binom{n+r}{r}$ so that we end up with $(-a + 1)(-b + 1) = ab - a - b + 1$, which is the answer.

Note on Q. The Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ gives $Q = \Bbbk[x, y, z, w]_\bullet/(xw - yz)$. Keep in mind the map $\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \times \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} \mapsto \begin{bmatrix} x = a_0 b_0 & z = a_0 b_1 \\ y = a_1 b_0 & w = a_1 b_1 \end{bmatrix}$. Thus, the family of lines $\mathbb{P}^1 \times \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$ are given by $V_Q(b_1 x - b_0 z, b_1 y - b_0 w)$ as $[b_0 : b_1]$ varies, and likewise we have $\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \times \mathbb{P}^1$ corresponding to $V_Q(a_1 x - a_0 y, a_1 z - a_0 w)$. Thus, $(x, y), (z, w)$ are in the same family of lines, and $(x, z), (y, w)$ are in another family of lines.

Now, let $[s^n : s^{n-1}t : st^{n-1} : t^n]$ define the rational nth curve C on Q. We claim that its type is $(1, n-1)$. Well, let $Y = V_{\mathbb{P}^3}(w^{n-2}y - z^{n-1})$, and consider $Y \cap Q$. As a divisor, $[Y \cap Q] = (n-1, n-1)$ since $Y \sim (n-1)H$ for a hyperplane H in $\mathbb{P}^3$. But also, we claim that $[Y \cap Q] = [C] + (n-2)[V(z, w)]$. Well, $Y \cap Q \cap U_w = V_{U_w}(y - z^{n-1}, x - yz) = V_{U_w}(y - z^{n-1}, x - z^n)$, on this open set, we have a codimension one prime corresponding to the curve C (of order 1). Now, on U_y, we have $(w^{n-2} - z^{n-1}, xw - z) = (w^{n-2}(1 - wx^{n-1}), xw - z)$, and hence we have $[z = w = 0]$ of order 2, and another component that is an open part of C (already counted). Hence, we have $[Y \cap Q] = [C] + (n-2)[z = w = 0]$ as desired, and thus $(n-1, n-1) = [C] + (n-2, 0)$ so that $[C] = (1, n-1)$.

Alternate computation: Another way to solve this is to compute the intersection of C along a general line in each family. The intersection of C with $[z = w = 0]$ (i.e. $\mathbb{P}^1 \times [1 : 0]$) has degree $n - 2$; consider $\Bbbk[s^n, s^{n-1}t, st^{n-1}, t^n]/(st^{n-1}, t^n)$ and count monomials. Likewise, intersection of C with $[y = w = 0]$ has degree 1 (by same sort of counting).

III.5.7. Later for review

III.6 Ext Groups and Sheaves

III.7 The Serre Duality Theorem

III.8 Higher Direct Images of Sheaves

III.9 Flat Morphisms

Note. A few things to note about flat maps:

- An A-module M is **faithfully flat** over A if it is flat and $M \otimes_A N = 0 \implies N = 0$ for any A-module N. Note that M is faithfully flat iff $M \otimes_A N = 0 \implies N = 0$ for any N cyclic module iff $M/\mathfrak{m}M \neq 0$ for any $\mathfrak{m} \subset A$ maximal. If $\varphi : A \to B$ is faithfully flat map of rings, then $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective. In fact, φ is **strongly injective** in the sense that for $I \subset A$ we have $I^{ec} = I$.

- A flat local homomorphism of rings is faithfully flat. This implies the Going-Down theorem (a different proof is given below also).

- If $A \subset B$ is a faithfully flat injection of domains whose quotient fields are the same, then $A = B$; well, if $b \in B$, then $b = \frac{a_1}{a_2}$, and so $a_2 b = a_1 \in A$, and by faithfully flatness we have $a_2 B \cap A = (a_2)A$ so that $a_2 b = a_2 a'$ for some $a' \in A$, and thus $a_2 b = a_2 a' = a_1 \Rightarrow a' = \frac{a_1}{a_2} = b$. As a result, surjective birational map of varieties is flat iff it is an isomorphism. This shows that the normalization map is flat iff it is actually an isomorphism.

- Flat map is a nice relative notion, as follows: If $f : A \to B$ is an algebra, and M is a B-module, then M is flat over A iff $M_\mathfrak{q}$ is flat over $A_{f^{-1}(\mathfrak{q})}$ for all $\mathfrak{q} \in \operatorname{Spec} B$. *Proof:* Let $I \subset A$ be an ideal. Consider $I \otimes_A M \to M$. Note that this map is injective as a map of A-modules iff it is injective as a map of B-modules (set-wise, they are the same map!). Now, for any maximal ideal $\mathfrak{m} \subset B$ and $\mathfrak{p} := f^{-1}(\mathfrak{m}) \subset A$, note that $(I \otimes_A M) \otimes_B B_\mathfrak{m} \simeq (I \otimes_A M) \otimes_B B_\mathfrak{m} \otimes_A A_\mathfrak{p} \simeq I_\mathfrak{p} \otimes_A M_\mathfrak{m} \simeq I_\mathfrak{p} \otimes_{A_\mathfrak{p}} M_\mathfrak{m} \hookrightarrow M_\mathfrak{m}$. The *key observation* here is as follows: if M is a A-module, for which $S \subset A$ (a multiplicative subset) acts on M by isomorphisms, then M is canonically an $S^{-1}A$-module; put in another way, if M is a $S^{-1}A$-module, then ${}_AM \otimes_A S^{-1}A \simeq M$ (restriction of scalars, then extending).

- *Fun exercise:* When is $R \to R/I$ flat (for R Noetherian)? (Answer: almost never nontrivially). If $R \to R/I$ is flat, then $\operatorname{Tor}_1^R(R/J, R/I) = I \cap J/IJ = 0$ for any J, but setting $J = I$ we have $I/I^2 = 0$. Hence, $I \otimes A/I = 0 \implies \operatorname{Supp}(I) \cap \operatorname{Supp}(A/I) = \emptyset$, but $\operatorname{Spec} A = \operatorname{Supp}(I) \cup \operatorname{Supp}(A/I)$ from $0 \to I \to A \to A/I \to 0$.

III.9.1. We first prove the following Going-down theorem for flat maps:

Lemma (Going-down). Let $\varphi : A \to B$ be a flat map of (Noetherian) rings. Then Going-Down property holds for the map $\operatorname{Spec} B \to \operatorname{Spec} A$; given $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$ and $\mathfrak{p} \supset \mathfrak{p}'$, there exists $\mathfrak{q}' \subset B$ such that $\varphi^{-1}(\mathfrak{q}') = \mathfrak{p}'$. *Proof:* Base change to $A/\mathfrak{p}'$ (i.e. $A/\mathfrak{p}' \to B/\mathfrak{p}'B$) so that we can assume A domain and $\mathfrak{p}' = 0$. Let $\mathfrak{q}'$ be a minimal prime of B contained in $\mathfrak{q}$. Now, $\varphi^{-1}(\mathfrak{q}') = 0$ since $0 \neq a \in A$ is a nonzero divisor and remains nonzerodivisor in B as $A \to B$ is flat. But a minimal prime consists of zero divisors.

The lemma above implies that $f(U)$ is stable under generization. Combined with the fact the $f(U)$ is also constructible, we have that $f(U)$ is open.

III.9.2. We have the twisted cubic X parametrized as $[s^3, s^2t, st^2, t^3]$, and $P = [0:0:1:0]$. We do the local computation where $t \neq 0$, so we have X_a given by

$$x = u^3, y = u^2, z = au$$

(where $u = s/t$). Eliminating u, we have the ideal $\langle x^2 - y^3, a^3x - z^3, a^2y - z^2, ay^2 - xz, ax - yz\rangle$, and setting $a = 0$, we get $\langle x^2 - y^3, z^2, xz, yz\rangle$. The underlying set is indeed the cuspidal cubic $V(x^2 - y^3)$, and when $x \neq 0$ we have $z = 0$ from $xz = 0$ so that it is reduced everywhere except at the cusp $(0,0,0)$ on which we have $z^2 = 0$.

III.9.3. (a): Immediate from (III.9.7).

(b): The fiber over each point on the plane Y is two points except at the point over which the two planes of X meet. That's where things must go wrong. Let the map $X \to Y$ be given by $S := k[s,t] \to R := k[x,y,z,w]/(x,y) \cap (z,w)$ where $s \mapsto x + z,\ t \mapsto y + w$. Then $s \otimes y - t \otimes x$ is a nonzero element of $(s,t) \otimes_S R$ that maps to 0 in R.

(c): The picture is that the fibers over each point on the plane Y is a fuzz point of fuzz-size 2 everywhere except over the origin (fuzz-size 3). Thus, again, $x \otimes z - y \otimes w$ becomes zero but is nonzero in $(x,y) \otimes_{k[x,y]} A(X)$. So, f is not flat. However, $X_{red} = \operatorname{Spec} k[x,y,z,w]/(z,w) \simeq \operatorname{Spec} k[x,y] = Y$ indeed, and moreover, as (z,w) is annihilator of z or w, and (z,w) is a minimal prime of X, if $\mathfrak{p} = \operatorname{Ann}(r)$ for some $r \in (x,y)$, then $\mathfrak{p} \subset (z,w)$ in fact. So X has no embedded points!

III.9.4.

Chapter IV

Curves

N.B. Few things to note about divisors and line bundles on a locally factorial integral k-schemes.

- For let $U \subset X$ be affine open. Then for any $f \in \mathscr{O}_X(U) \subset K(X)$, we have that $\deg(\operatorname{div} f|_U) = 0 \iff f$ is a unit in $\mathscr{O}_X(U)$. (This follows from algebraic Hartog's lemma). Furthermore, for any $f \in f(X)$, we have $\operatorname{div} f = 0 \iff f$ is constant (i.e. $f \in k$). *Proof:* Since f is a unit in every affine piece, in particular $f \in \mathscr{O}_X(U_i)$ for all i for any cover $\{U_i\}$ of X, we have $f \in \mathscr{O}_X(X) = k$.

- Recall that we had $\mathscr{L}(D)|_{U_i} = \mathscr{O}_X \cdot f_i^{-1}$ where $\{U_i\}$ is any cover of X with $D|_{U_i}$ principal on U_i, i.e. $D|_{U_i} = \operatorname{div} f_i$ for $f_i \in \mathscr{O}_X(U_i)$. As X is integral in our case, we can also define $\mathscr{L}(D)$ as: $\Gamma(U, \mathscr{L}(D)) = \{f \in K(X)^\times : \operatorname{div} f|_U + D|_U \geq 0\} \cup \{0\}$. On a cover $\{U_i\}$ where D is locally factorial, it is easily seen that these two are the same, hence isomorphic. Thus, we'll use these two notions interchangeably when no confusion should arise; a big caution on this however: For $f \in \Gamma(X, \mathscr{L}(D)) \subset K(X)$, the notation $\operatorname{div} f$ can be ambiguous—f can be considered either as a rational function or as a section of a line bundle. To distinguish these two situations, we write $\operatorname{div} f$ if considering f as a rational function, and $(s)_0$ if considering s as a global section of a line bundle.

N.B. Also, a few things about nonsingular (integral) curves over $k = \overline{k}$:

- Let X be a nonsingular integral curve and suppose we have a morphism $\varphi : X \setminus P \to Y$ where Y is a proper k-scheme. Then φ extends uniquely to $\widetilde{\varphi} : X \to Y$.

- Furthermore, if X, Y are nonsingular complete curves, then $K(Y) \hookrightarrow K(X)$ induces a dominant rational map $\varphi : X \to Y$, which in fact uniquely extends to a surjective finite map of degree $[K(X) : K(Y)]$. (This is just combining (I.4.4) and (II.6.8)).

- As a special case of above, a nonconstant rational function $f \in K(X)$ induces $k(f) \subset K(X)$, which induces $\widetilde{f} : X \to \mathbb{P}^1_k$ where $\widetilde{f}(0) = \operatorname{div} f^+$ and $\widetilde{f}(\infty) = \operatorname{div} f^-$. As a result a nonsingular curve X is rational iff there exists $P \neq Q \in X$ such that $P \sim Q$.

IV.1 Riemann-Roch Theorem

IV.1.1. Let $n \geq g+1$. Then $l(nP) = n-g+1+l(K-nP) \geq 2$, and thus $\mathscr{L}(nP)$ has a nonconstant global section, say f, which we can consider as an element of $K(X)$. As $\operatorname{div} f + nP \geq 0$, and $\operatorname{div} f \neq 0$ (since f is nonconstant), we have that f has a pole of order at least 1 and most n on P. Moreover, f restricts to a regular function on $X \setminus P$ as $\Gamma(X \setminus P, \mathscr{L}(nP)) = \mathscr{O}_X(X \setminus P)$.

IV.1.2. We induct on r. $r = 1$ is the previous exercise. Now, suppose we have a rational function $f \in K(X)$ that is regular on $X \setminus \{P_1, \dots, P_{r-1}\}$ and poles at $P_1, \dots, P_{r-1}$. Let n_r be the order of f at P_r. Let $n \geq g+1$, so that $\Gamma(X, \mathscr{L}(nP)) \geq 2$, and let $1, f \in \Gamma(X, \mathscr{L}(nP))$ as rational functions. Then by choosing $c \in k$ carefully (as k is infinite), we can make $g := f + c \cdot 1$ that does not vanish on any of $P_1, \dots, P_{r-1}$. As g is nonconstant, we also have that g has a pole of order at least one at P_r. Now, take fg^{n_r+1} as the function that is regular on $X \setminus \{P_1, \dots, P_r\}$ with poles exactly on $P_1, \dots, P_r$.

IV.1.3. By (I.6.10), X is quasiprojective. Let's say $X \hookrightarrow \overline{X}$. As $D := \overline{X} \setminus X$ is a finite collection of points, there exists a rational function $f \in K(\overline{X})$ such that f regular except with poles on D. This f gives a map $k[t] \to \mathscr{O}_X(X)$ so that we have $f : X \to \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$. And this map extends uniquely to $\widetilde{f} : \overline{X} \to \mathbb{P}^1_k$ where points of D are exactly ones that map to the point at infinity.

IV.1.4. Just follow hint.

IV.1.5. By R-R, $\dim |D| \leq \deg D$ is equivalent to $l(K - D) \leq g$, which always holds as $l(K) = g$ and D is effective. Now, note the following lemma:

Lemma. X is rational iff there exists a point P such that $l(P) \geq 2$. *Proof:* If X is rational, then $l(P) = 2$ for any $P \in X$ in fact. Conversely, if $l(P) \geq 2$, then there is a nonconstant rational function $f \in \Gamma(X, \mathscr{L}(P)) \subset K(X)$ such that f has a pole of order 1 at P. This induces a degree 1 map $X \to \mathbb{P}^1$, and hence X is rational.

Thus, if $l(K - D) = g$, then either $D = 0$, or $l(K - P) = g - 2 + l(P) \geq g$ for some P so that $l(P) \geq 2$, which implies X is rational.

IV.I.6. Consider the divisor $D := (g+1)P$ for any $P \in X$. Then the nonconstant rational function in $\Gamma(X, \mathscr{L}(D))$ gives a map $X \to \mathbb{P}^1$ of degree equal to the order of the pole of f at P, which is $\leq g + 1$.

IV.I.7. (a): We have the following lemma:

Lemma. If $l(D) \geq 2$ and $\deg D = 2$ for a divisor D on a curve X of genus $g \geq 1$, then $\mathscr{L}(D)$ is base-point-free. *Proof:* Let s, t be two linearly independent global sections of $\mathscr{L}(D)$. Then $(s)_0 = P + Q$ and $(t)_0 = P' + Q'$, with $\{P, Q\}$ distinct from $\{P', Q'\}$ ($\because$ if $P = P'$, say, then $Q - Q' \sim 0$, so that $g \geq 1$ forces $Q = Q'$, implying $s = ct$ for some $c \in k^\times$). Thus, we have $V(s) \cap V(t) = 0$ so that $\mathscr{L}(D)$ is base-point-free, and s, t defines a degree 2 map $X \to \mathbb{P}^1$.

With the lemma, the rest of (a) is just R-R.

(b): Let $i : X \hookrightarrow Q = \mathbb{P}^1 \times \mathbb{P}^1$ be of type $(g+1, 2)$. Consider the map $\varphi : \pi_2 \circ i$. Any point of $\mathbb{P}^1$ pulls back to a line in Q that intersects X with degree 2. Hence, $\varphi : X \to \mathbb{P}^1$ is a degree 2 map.

IV.1.8. (a): Note that for a point P on a 1-dimensional integral k-scheme X, we have that $P \in X_{reg}$ iff $\mathscr{O}_{X,P}$ is integrally closed. Now, as X_{reg} an open subset of X, if $f : \widetilde{X} \to X$ is the normalization of X (which is a finite map), then $\mathscr{O}_X \to f_* \mathscr{O}_{\widetilde{X}}$ is an isomorphism at stalks $P \in X$ for all but finitely many points. Hence, we have an SES

$$0 \to \mathscr{O}_X \to f_* \mathscr{O}_{\widetilde{X}} \to \sum_{P \in X} \widetilde{\mathscr{O}}_P / \mathscr{O}_P \to 0$$

Now, since Euler characteristic is additive and f is affine, we have $\chi(\mathscr{O}_X) + \sum_{P \in X} \delta_P = \chi(\mathscr{O}_{\widetilde{X}})$ so that $1 - \chi(\mathscr{O}_X) = p_a(X)$ gives the desired equation.

(b): $p_a(X)$ implies by above equation that that $\mathscr{O}_P$ is integrally closed for all P, and hence X is already nonsingular of genus 0, so that $X \simeq \mathbb{P}^1$.

(c): Proof later, just examples now: For the cusp, note that $\dim_k k[t]/k[t^2, t^3] = 1$, and for the node, note that $\dim_k k[t]/k[t^2-1, t^3-t] = 1$.

I.1.9. (a): As $\operatorname{supp} D \in X_{reg}$, the divisor D is locally principal, and hence defines a Cartier divisor D on X so that we can consider the line bundle $\mathscr{L}(D)$. Note that $\chi(\mathscr{L}(D)) = \deg D + 1 - p_a$ holds when $D = 0$. Now, we show that the equation holds for D iff it holds for $D + P$ for any $P \in X_{reg}$. Consider the SES $0 \to \mathscr{L}(-P) \to \mathscr{O}_X \to \kappa(P) \to 0$ and tensor by $\mathscr{L}(D+P)$ to get $0 \to \mathscr{L}(D) \to \mathscr{L}(D+P) \to \kappa(P) \to 0$, and now use the additivity of Euler characteristic.

(b): As X is integral, Cartier class group is the same as Picard group. Embed $\iota : X \hookrightarrow \mathbb{P}^n$ via some line bundle $\mathscr{O}(1)$ (which corresponds to a very ample Cartier divisor). Given a line bundle $\mathscr{L}$, note that $\mathscr{L}(n)$ globally generated for some n by (II.5.17). Moreover, by Exer.II.7.5(d) we have that $\mathscr{L}(n+1)$ is then also very ample, so that we have $\mathscr{L} \otimes \mathscr{O}(n+1)$ is very ample (and $\mathscr{O}(n+1)$ is very ample).

(c): We claim that if $\mathscr{L}$ is a very ample line bundle, then it is $\simeq \mathscr{L}(D)$ where $\operatorname{supp} D \in X_{reg}$. Well, take a hyperplane $L \in \mathscr{O}(1)$ (that does not meet any of singular points of X) and pull it back to a section s of $\mathscr{L}$. Then $(s)_0$ contains none of the singular points.

(d): Yes indeed.

IV.2 Hurwitz's Theorem

IV.2.2. (a): A complete linear system of $|K|$ gives a degree 2 map $f : X \to \mathbb{P}^1$. Moreover, by Hurwitz formula, we have that $\deg R = 2 \cdot 2 - 2 - 2(0-2) = 6$. Note that if f is ramified at P, then $1 < e_P \leq 2$ so that $e_P = 2$. Hence, $R = P_1 + \cdots + P_6$ for 6 distinct points $\{P_i\}$, each with ramification index 2 (a tame ramification as $\operatorname{char} k \neq 2$).

(b): Let $h = (x - \alpha_1) \cdots (x - \alpha_6) \in k[x]$, and by change of coordinates on x if necessary, assume $\alpha_i \neq 0$. As α_i's are distinct, h is square-free, so that $K = k(x)[z]/(z^2 - h)$ is a degree 2 finite Galois extension of $k(x)$. As K is separable degree 1 over k, we can consider the projective nonsingular curve $Y := C_K$ whose function field is K. $K \supset k(x)$ then gives a map $f : Y \to \mathbb{P}^1$ of degree 2. On one chart of this map, we have $f^{-1}(U_0) \to \operatorname{Spec} k[x]$ via $k[x] \hookrightarrow k[x,z]/(z^2-h)$, as $k[x,z]/(z^2-h)$ is the integral closure of $k[x]$ in K. On the level of points, this map is $(\lambda, \pm\sqrt{h(\lambda)}) \mapsto \lambda$ for $\lambda \in k$. So on this chart, the map f is ramified exactly over $\alpha_1, \ldots, \alpha_6$. Now, note the diagram (where $z' := x^{-3} z / \sqrt{\alpha_1 \cdots \alpha_6}$)

$$\begin{array}{ccccc}
k[x^{-1}, z']/(z'^2 - \prod_i (x^{-1} - \alpha_i^{-1})) & \longrightarrow & k[x^{\pm}, z]/(z^2 - h) & \longleftarrow & k[x,z]/(z^2-h) \\
\uparrow & & \uparrow & & \uparrow \\
k[x^{-1}] & \longrightarrow & k[x^{\pm}] & \longleftarrow & k[x]
\end{array}$$

The leftmost is justified as

$$\frac{k(x)[z]}{(z^2 - h)} = \frac{k(x^{-1})[z']}{((x^6 \alpha_1 \cdots \alpha_6)(z'^2 - \widetilde{h}))} = \frac{k(x^{-1})[z']}{(z'^2 - \widetilde{h})}$$

where $\widetilde{h} = \prod_i (x^{-1} - \alpha_i^{-1})$. So on the other affine patch, the map f is exactly ramified over the points α_i as well. Thus, $f : Y \to \mathbb{P}^1$ is a degree 2 map with $\deg R = 6$, so that $g(Y) = 2$ by the Hurwitz formula. Now, the map f is given by a divisor D on Y. Then $\deg D = 2$ and $l(D) \geq 2$ implies (via R-R) that $l(K - D) \geq 1$. But $\deg(K - D) = 0$ and hence $l(K - D) \neq 0$ iff $K \sim D$.

Lemma. For k a field (not necessarily $k = \overline{k}$), we have $\operatorname{Aut} \mathbb{P}^n_k \simeq PGL_{n+1}(k)$. *Proof:* Let $f : \mathbb{P}^n_k \xrightarrow{\sim} \mathbb{P}^n_k$ be an isomorphism. Then $f^* : \operatorname{Pic} \mathbb{P}^n = \mathbb{Z} \to \operatorname{Pic} \mathbb{P}^n = \mathbb{Z}$ is an isomorphism so that

$f^*\mathscr{O}(1) \simeq \mathscr{O}(1)$. Let $s_i := f^* x_i \in \mathscr{O}(1)$ for $i = 0, \dots, n+1$. Writing $s_i = \sum_j a_{ij} x_j$, we note that $V(s_0, \dots, s_{n+1}) = \emptyset$ iff $A = (a_{ij})$ is an invertible matrix, thus we get an element of $PGL_{n+1}(k)$. It is clear that this gives an isomorphism of groups $\operatorname{Aut} \mathbb{P}^n \simeq PGL_{n+1}(k)$. Note that the map given by A where $A(e_1, \dots e_{n+1}) = (s_0, \dots, s_{n+1})$ is actually just $A[\vec{p}]$.

(c): Let $P_1 = \begin{bmatrix} a \\ b \end{bmatrix}$, $P_2 = \begin{bmatrix} c \\ d \end{bmatrix}$, $P_3 = \begin{bmatrix} e \\ f \end{bmatrix}$. As P_1, P_2 are distinct, by scaling a, b and c, d appropriately, we can assume that $a + c = e$ and $b + d = f$. The fractional linear transformation (which really is just matrix multiplication) $z \mapsto \frac{az+b}{cz+d}$ then is an automorphism of $\mathbb{P}^1_k$ that sends $0, \infty, 1$ to P_1, P_2, P_3.

(d): Yes, indeed.

(e): Just combine all the parts.

IV.2.3. Before delving into the problems, let's state (or review) some lemmas regarding nonsingular plane curves of degree d. Let $X = \operatorname{Proj} k[x, y, z]_\bullet / (f)_\bullet$.

Lemma. Note that the map $X \to (\mathbb{P}^2)^*$ is given by $p \mapsto [Df(p)]$ where $[Df] = [f_x : f_y : f_z]$ (f_x denotes $\partial f / \partial x$). That is, it is given by graded ring map $T : S_X \leftarrow k[x^*, y^*, z^*]$ where $x^* \mapsto f_x$ (likewise for y, z), as $T(x^*, y^*, z^*)$ carves out empty scheme in X as X is nonsingular.

Lemma. Let's generalize Euler's formula. Let $S = k[x_1, \dots, x_n]$, $f \in S$ homogeneous of degree d, and $T = T^\bullet(S^n) = S[t_1, \dots, t_n]$ the tensor algebra where t_i's don't commute, x_i's commute, and x_i's commute with t_j's. Consider two k-linear maps $D : T \to T$ and $\vec{x}^* : T \to T$, where D is of degree 1 and $\vec{x}^*$ is of degree -1, defined as: $D := (t_1 \cdot \partial x_1 + \dots + t_n \cdot \partial x_n) \cdot -$ and $\vec{x}^* := (x_1 \cdot t_1^* + \dots + x_n \cdot t_n^*) \cdot -$ where t_i^* is the "dual operator" to t_i in the sense that $t_i^*(t_{j_1} \cdots t_{j_k}) = t_{j_2} \cdots t_{j_k}$ if $i = j_1$ and 0 otherwise. Then, we have (as long as $\ell < d$)

$$\vec{x}^* \circ D = (x_1 \partial x_1 + \dots + x_n \partial x_n) \cdot - \text{ and hence } \vec{x}^* D^{\ell+1} f = (d - \ell)(D^\ell f)$$

In particular, $\ell = 0$ is Euler's formula, and $\ell = 1$ gives $[x\ y\ z] \begin{bmatrix} \\ \quad H \quad \\ \\ \end{bmatrix} = (d-1)[Df]$.

Proof. The subtlety is to realize that $(t_i^*(t_j(-))) = (\partial t_i^* t_j)(-)$; this is why ${}^*t_i$ is defined somewhat weirdly. After that, the proof is pretty much the proof of Euler's formula. $\square$

Lemma (Inflection point). Let $X = V(f) \subset \mathbb{P}^2_k$ (not necessarily smooth), and $L = V(ax + by + cz)$ a line (denote by $\vec{n} = [a : b : c]$). Let m_P the intersection multiplicity of X and L at a point $P \in \mathbb{P}^2_k$ (0 if not in the intersection). Our goal is to give a criterion for $m_P \geq m$ in terms of $D^m f(P)$ and $\vec{n}^m(P)$.

$m = 1$ is easy: check that $f(P) = \vec{n} \cdot P = 0$.

$m = 2$ is still easy: check that $Df(P) = \lambda \vec{n}$ for some $\lambda \in k$ (can be zero!)

$m = 3$ is where things get tricky... So, let's do this more systematically...

WLOG suppose $c \neq 0$ (we then homogenize by c later so this isn't a problem). Let $g(s, t) := f(s, t, -\frac{a}{c}s - \frac{b}{c}t) = (p_2 s - p_1 t)^{m_P}(as - bt)' things$. Then we see that $m_P \geq m$ iff $\frac{\partial g}{\partial s^i \partial t^j}(p_1, p_2) = 0\ \forall i + j \leq m$.

We immediately recover $m = 1, m = 2$ case. For $m = 3$ case, (after some rather long computation), we obtain the following:

Define $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \boxtimes \begin{bmatrix} e & f \\ g & h \end{bmatrix} := de + ah - cf - bg$. Let $I, J \subset \{1, 2, 3\}$ of size 2, and given a 3×3 matrix M, denote by $M_{I,J}$ the square submatrix given by I, J. Let $[\vec{n}^2]$ be the outer product of $\vec{n}$ with itself, and $[D^2 f]$ be the Hessian matrix of f. Then $m_P \geq 3$ if $m \geq 2$ already and $([\vec{n}^2]_{I,J} \boxtimes [D^2 f]_{I,J})(P) = 0 \; \forall I, J$.

Okay! That was quite some detour...

(a): WLOG, let $L : z = 0$, and identify $L \simeq \operatorname{Proj} k[x, y]_\bullet = \mathbb{P}^1$. Then the map $\varphi : X \to \mathbb{P}^1$ is given $p \mapsto [Df(p)] \times [0 : 0 : 1] = [f_y(p) : -f_x(p)]$ (the cross product, which is always nonzero for $p \in X$ since L is not tangent to X). In other words, the map is given by $\varphi^\# : k[x, y] \to k[x, y, z]/f$ via $x \mapsto f_y$, $y \mapsto -f_x$ (as $\varphi^\#(x, y)$ carves out $V(f_x, f_y, f)$, and L not being tangent to X implies that this is empty).

Now, let $p \in X$, so that $\varphi(p) = [f_y(p) : -f_x(p)] = V(f_x(p)x + f_y(p)y) \subset \mathbb{P}^1_k$. The function pulls back to $\varphi^\#(f_x(p)x + f_y(p)y) = f_x(p)f_y - f_y(p)f_x$, so let $h := f_y(p)f_x - f_x(p)f_y$. The map φ is ramified at p iff $V(h)$ meets $V(f)$ at p with multiplicity > 1 (note that $p \in V(h)$ indeed). This is equivalent to stating that $[Dh(p)] = \lambda[Df(p)] \; \exists \lambda \in k$. Well, we have

$$[Dh(p)] = \begin{bmatrix} & & \\ & Hf(p) & \\ & & \end{bmatrix} \begin{bmatrix} f_y(p) \\ -f_x(p) \\ 0 \end{bmatrix}$$

So, if $p \in L$ to begin with, $[f_y(p) : -f_x(p) : 0] = p$ so that the generalized Euler's formula (see Lemma above) does the job.

Now for $p = [a : b : 1] \notin L$ case: The condition is equivalent to saying that $[Dh(p)] \times [Df(p)] = 0$, which when computed out says

$$f_y^2 f_{xx} - 2f_x f_y f_{xy} + f_x^2 f_{yy}, \; -f_x f_y f_{xz} + f_x^2 f_{yz} - f_y f_z f_{xx} - f_x f_z f_{xy}, \; f_z f_y f_{xy} - f_x f_z f_{yy} - f_y^2 f_{xz} + f_x f_y f_{yz}$$

(evaluation at P is ommitted for convenience). This corresponds to $(I, J) = ([12], [12]), ([12], [13]), ([12], [23])$ in the inflection point lemma above where $\vec{n} = Df(p)$. We could have at the very beginning arranged so that L is $z = 0$ while $p = [0 : 0 : 1]$ (i.e. do the coordinate change $X = x - az, Y = y - bz, Z = z$), so we can assume $a = b = 0$. Then $f_z(p) = f_{zz}(p) = 0$ (both repeated Euler's formula), so that the equation for $(I, J) = ([23], [23])$ is also automatically satisfied, and so p is an inflection point.

(b): Finish other parts later...

IV.3 Embeddings in Projective Space

Note. Let D be a divisor of degree d and rank $r := \dim |D| = h^0(X, \mathscr{L}(D)) - 1$ on a (nonsingular) curve X. Some facts to know in a chart:

$d\backslash r$	-1	0	1	2	3
< 0	Not effective	can't; is $\Longleftarrow$	$\Longleftarrow$	$\Longleftarrow$	$\Longleftarrow$
0	e.g. $P - Q$ on $g > 0$	$D \sim 0$	$\Longleftarrow$	$\Longleftarrow$	$\Longleftarrow$
1	...	$D \sim P$ $\exists P$	Is very ample, so $\iota : X \xrightarrow{\sim} \mathbb{P}^1$ via $\mathscr{L}(D) \simeq \iota^*\mathscr{O}(1)$	$\Longleftarrow$	$\Longleftarrow$
2	...	...	Either: (1) b-p-f hence hyperelliptic or (2) rational	$\mathscr{L}(D)$ makes X the 2-uple embedding of $\mathbb{P}^1$	$\Longleftarrow$
3	...	...	Either: (1) b-p-f hence trigonal, or (2) hyperelliptic or rational	Either: (1) very ample hence hyperelliptic (in fact elliptic) or (2) only b-p-f so rational (maps to a singular plane cubic) or (3) not b-p-f hence rational	$\mathscr{L}(D)$ makes X the 3-uple embedding of $\mathbb{P}^1$

IV.3.1. If $\deg D \geq 5 = 2 \cdot 2 + 1$, then it is very ample. Now, if D is very ample on a $g = 2$ curve X, then $\dim |D| \geq 3$ as there are no smooth planes curves of genus 2. Moreover, we have $\deg D \geq 3$; if $\deg D = 2$, note either one of the following: (1) if $E \sim D$ effective, then $\dim |D - E| = 0$ implying that $\dim |D| \leq 2$, which is a contradiction, (2) $\deg D = 2$ implies by Bezout's Theorem that $X \subset H$ for some linear space H of dimension 2 (just take any three points on X) and hence X is a conic in $\mathbb{P}^2$ and hence rational. Now using R-R, we have that $\deg D = \dim |D| - \dim |K - D| + 2 - 1 \geq 3 - \dim |K - D| + 1$, but $\dim |K - D| = -1$ since $\deg(K - D) \leq 2 - 3 = -1$. Hence, $\deg D \geq 5$, as desired.

IV.3.2. (a): Adjunction formula $K_X = (X + K_{\mathbb{P}^2})|_X = (4H - 3H)|_X = H|_X = H \cdot X$ where H is a hyperplane section in $\mathbb{P}^2$ (which is a line).

(b): By R-R, we have $\dim |D| = \dim |K - D|$. Now, we showed in part (a) that K_X is the very ample line bundle giving the embedding $X \hookrightarrow \mathbb{P}^2$ (i.e. $\omega_X \simeq \mathscr{O}_X(1)$), and hence $\dim |K - D| = 2 - 2 = 0$ by (IV.3.1).

(c): For X to be hyperelliptic, there must exist an effective divisor D of degree 2 and rank 1 (this looks only necessary, but is actually sufficient if $g \geq 1$). But in part (b), we showed that every degree 2 effective divisor has rank 0.

IV.3.3. Suppose $X = \bigcap_i H_i$ where $\deg H_i = d_i$. Since $\omega_X = \mathscr{O}_X(\sum_i d_i - n - 1)$ and $2g - 2 > 0$ if $g > 2$, we have that $\sum_i d_i - n - 1 := m$ for some $m > 0$. If $\mathbb{P}^n \hookrightarrow \mathbb{P}^N$ is m-tuple embedding, then ω_X is $\mathscr{O}_{\mathbb{P}^N}(1)$ pulled back to X, so that it is very ample. Now, if $g = 2$, we know that K is not very ample, and hence geus 2 curves is never a complete intersection.

IV.3.4. (a): Toric geometry FTW.

(b): If $d < n$, then take any $d + 1$ distinct points on X, which determines a linear subspace of dimension at least d. By Bezout's Theorem, X must be contained in this linear subspace, and hence X is degenerate. Thus, if X is non-degenerate, then $d \geq n$. Now, if $d = n$, so that $\dim |D| = \deg D = n$, then by (Ex.IV.1.5), we conclude that $g(X) = 0$, and $\mathscr{O}_X(1) \simeq \mathscr{O}_{\mathbb{P}^1}(d)$ under the isomorphism $X \simeq \mathbb{P}^1$. Thus, X is a rational normal curve.

(c): Yes.

(d): A curve of degree 3 in $\mathbb{P}^n$ is already contained in some $\mathbb{P}^3 \subset \mathbb{P}^n$. So it is either plane elliptic curve, or twisted cubic in $\mathbb{P}^3$.

IV.3.5 (a): Suppose $X' := \varphi(X)$ is regular. Then the birational map $X' \dashrightarrow X$ extends to a full morphism $X' \to X$, and thus the projection is an isomorphism $X \simeq X'$. Thus, we have the following:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, hook] & \mathbb{P}^3 \setminus O \arrow[d] \\ & \mathbb{P}^2 \end{tikzcd}$$

If L is a line of $\mathbb{P}^2$, which pulls back to a hyperplane section H of $\mathbb{P}^3$, we see that $|X.H| = |X.L|$. Well, this is convoluted way of just saying that $\mathscr{O}_X(1) \simeq \mathscr{O}_{X'}(1)$. But $h^0(\mathscr{O}_X(1)) \geq 4$, while $h^0(\mathscr{O}_{X'}(1)) \leq h^0(\mathscr{O}_{\mathbb{P}^2}(1)) = 3$ since $X' \subset \mathbb{P}^2$ is a complete intersection (hence by (Ex.III.5.5.) we have $H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1)) \twoheadrightarrow H^0(X', \mathscr{O}_{X'}(1)))$. This is a contradiction. Hence, we conclude that a non-planar curve in $\mathbb{P}^3$ cannot be embedded in $\mathbb{P}^2$.

(b): As X is the normalization of X', we have that $p_a(X') = p_a(X) + \sum_P \delta_P$. Note that $\deg X = \deg X' = d$ as projection is a degree 1 map, and hence $g = \binom{d-1}{2} - \sum_P \delta_P < \binom{d-1}{2}$ as X' is singular.

(c): Since $X_0^{red} \simeq \varphi(X)$, from part (a) we have $p_a(X_0^{red}) = p_a(\varphi(X)) > p_a(X)$. However, by (III.9.10) we have $p_a(X_0) = p_a(X_1) = p_a(X)$ so that $X_0^{red} \not\simeq X_0$. Hence X_0 is not reduced.

IV.3.6. (a): Again, by selecting 5 distinct points of X, by Bezout's Theorem we can assume that $X \subset \mathbb{P}^4$. If $X \subset \mathbb{P}^2$ then it is a plane curve of degree 4 and genus 3. If $X \subset \mathbb{P}^4$, by (Ex.IV.3.4) above we have X is the rational normal quartic. For the $X \subset \mathbb{P}^3$ case, we generalize:

Lemma: A degree $d+1$ curve X in $\mathbb{P}^d$ is either rational or elliptic ($d > 1$ indeed). *Proof:* Let D be the effective divisor giving the embedding $X \subset \mathbb{P}^d$, so that $\dim |D| \geq d$ and $\deg D = d+1$. Then R-R implies that $g \leq h^0(K-D) + 1$. But if D is a special divisor, then $\dim |D| \leq \frac{1}{2} \deg D$ by Clifford's theorem, which is only possible if $d \leq 1$. Hence, $h^0(K-D) = 0$, and thus $g \leq 1$.

Hence, either $X \subset \mathbb{P}^3$ as a elliptic curve (in which case, it is a complete intersection of two quadrics as shown below), or $X \subset \mathbb{P}^3$ is rational.

(b): Consider the LES associated to $0 \to \mathscr{I}_X(2) \to \mathscr{O}_{\mathbb{P}^3}(2) \to \mathscr{O}_X(2) \to 0$. As $h^0(\mathscr{O}_{\mathbb{P}^3}(2)) = \binom{5}{2} = 10$ and $h^1(\mathscr{O}_{\mathbb{P}^3}(2)) = 0$, and moreover $h^0(\mathscr{O}_X(2)) = 2 \cdot 4$ (since the deg is $8 > 2 \cdot 1 - 2 = 0$), we conclude that $h^0(\mathscr{I}_X(2)) \geq 2$. In other words, if I_X is the homogeneous ideal of X, then there exist $f_1, f_2 \in I_X$ (not the same). If any of the two factors into linear factors (since they are quadrics), X would be degenerate, so that f_1 and f_2 are irreducible and don't share any factors. Now, by Bezout's theorem, $C := V(f_1, f_2)$ is a degree 4 curve containing X, which is also of degree 4, and hence by (I.7.6) we conclude that C is irreducible and hence $C = X$.

IV.3.7. Any plane curve X of degree 4 with a single node is not a projection of a smooth curve $\widetilde{X}$ in $\mathbb{P}^3$. This is because $3 = p_a(X) = p_a(\widetilde{X}) + 1$ so that $p_a(\widetilde{X}) = 2$, but curves of degree 4 in $\mathbb{P}^3$ are either $g = 0, 1$, or 3. Now, $V(xyz^2 + x^4 + y^4) \subset \mathbb{P}^2$ is a plane curve with a single node at $[0:0:1]$.

IV.3.8.

IV.3.9. For general H, $H \cap X$ is d distinct points by Bertini's theorem (or, observe that the family of tangent lines to X in $\mathbb{G}(1,3)$ is of dimension 1, so that planes containing tangent lines is at most of dimension 2). Now, we show that the family of multisecant lines of X in $\mathbb{G}(1,3)$ is of dimension 1 as well. stuck here...

IV.3.10. We say that $P_1, \ldots, P_m \subset \mathbb{P}^n$ are in general position if there is no linear space L^{m-2} of dimension $m-2$ such that $P_1, \ldots, P_m \in L^{m-2}$. Now, let $V_m \subset \prod_{i=1}^m \mathbb{P}^n$ be subset of points *not* in general position. V_m is a closed subset since it is the vanishing locus of $m \times m$ minors of a $n \times m$ matrix. In particular, when $m = n$, we have that V_n is irreducible closed subset. If $X \subset \mathbb{P}^n$, then $(\prod_{i=1}^n X) \cap V_n$ is proper closed subset of $\prod_{i=1}^n X$, as X is irreducible and thus the intersection is proper, or every set of n points in X is non-general position in which case $X \subset \mathbb{P}^{n-1}$.

IV.3.11.

IV.4 Elliptic Curves

IV.4.1. Essentially a repeat of the proof of (IV.4.6.)...

IV.4.2. Let $\deg D = d \geq 3$ and fix an embedding $i : X \hookrightarrow \mathbb{P}^{d-1}$ given by $|D|$. Since X is nonsingular, it is normal, and hence to show projective normality is equivalent to showing $H^0(\mathscr{O}_{\mathbb{P}^{d-1}}(n)) \twoheadrightarrow H^0(\mathscr{O}_X(n))\ \forall n \geq 0$. Now, from the SES $0 \to \mathscr{I}_X \to \mathscr{O}_{\mathbb{P}^{d-1}} \to \mathscr{O}_X \to 0$, we need show that $H^1(\mathscr{I}_X(n)) = 0\ \forall n \geq 0$. This holds for $n = 0, 1$ since $I_0, I_1 = 0$ since $i(X)$ is nonempty and nondegenerate (where $I = \Gamma_*(\mathscr{I}_X)$). Now,

IV.5 The Canonical Embedding

IV.5.1. Curves of genus 0 and 1 admits a degree 2 map to $\mathbb{P}^1$ and can be embedded as a complete intersection. So really, by hyperelliptic we mean $g \geq 2$ automatically. For $g \geq 2$ complete intersections, the canonical divisor is very ample (Ex.IV.3.3), and hence by (IV.5.2) it cannot be hyperelliptic.

IV.5.3. The same reasoning as (Ex.IV.2.2) shows that the hyperelliptic genus g curves form an irreducible family of dimension $2g-1$. So hyperelliptic genus 4 curves are a family of dimension 7.

Now, for nonhyperelliptic genus 4, we have a canonical embedding $X \hookrightarrow \mathbb{P}^3$ that is a complete intersection of an (irreducible) quadric and a (irreducible) cubic by (IV.5.2.2). For the quadric, we have $\binom{3+2}{2} - 1 = 9$ dimensional space, and the subsequence cubic that does that vanish on the quadric chosen is of dimension $\binom{3+3}{3} - 4 - 1 = 15$. Lastly, $PGL(3)$ has dimension 15, so we have $9 + 15 - 15 = 9$.

Now, for the nonhyperelliptic genus 4 to have a unique g_3^1, the quadric is a rank 2 quadric, which forms a family of dimension 8—it's really a (irreducible) hypersurface in $(\mathbb{P}^9)^*$ given by the determinant of the symmetric matrix.

IV.5.4. We know that genus 4 curve X canonically embedded is a complete intersection of a quadric Q and a cubic C. Now, this irreducible quadric is either the nonsingular quadric $\mathbb{P}^1 \times \mathbb{P}^1$ or the singular cone. In the first case, there are two g_3^1's coming from the two family of lines (note that g_3^1 is equivalent to three collinear points with the canonical embedding of X by R-R); these g_3^1's are actually different since one collapses the three points along one family of lines while the other keeps them distinct (the maps $X \to \mathbb{P}^1$ are from the projections $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$). Furthermore, the sum of these two g_3^1's is the canonical divisor. In the singular conic case, there is a unique g_3^1 since the lines in the family are all linear equivalent, and by the following lemma, and two times this g_3^1 also gives the canonical divisor in this case (as the pullback of hyperplane is the double line on the singular cone).

Lemma. Let $X \subset Y$ closed embedding of schemes satisfying $(*)$+locally-factorial of (II.6). Suppose $D, D' \in \operatorname{Div} X$ such that $D = X \cap E,\ D' = X \cap E'$ for some codimension-1 integral subschemes

E, E' of Y. Then $E \sim E'$ on Y implies that $D \sim D'$ on X. *Proof:* By assumption E, E' does not contain X. Let $f \in K(Y)$ be the rational function such that $\operatorname{div} f = E - E'$. Then affine locally $f = a/b$ where b does not vanish on X. Hence, it makes sense to map f to a rational function $\overline{f} \in K(X)$ by $\overline{a}/\overline{b}$ (if $A \to A/\mathfrak{p}$ induces $X \hookrightarrow Y$, then we have $A_{\mathfrak{p}} \to \kappa(\mathfrak{p})$). Then $\overline{f}$ now induces $D \sim D'$ since intersecting E (for e.g.) is the same is looking at where $\overline{a}$ vanishes.

Now for the actual problem:

(a): If X has two g^1_3's, then it is necessarily on the nonsingular quadric when canonically embedded, so project away from a point P on the curve to get $X \setminus P \to \mathbb{P}^2$ that is injective except at over two points in the image where P is a part of three collinear points. Thus, the closure of the image X' is birational to X, and has two nodes, and thus by (Ex.IV.1.8) we have $p_a(X') = p(X) + 2 = 6$ and hence $\deg X' = 5$.

(b): If X is on the quadric singular cone however, and we take a point P on X, then there is a plane through P that meet two other points Q, R with multiplicity 2 each (meets P with multiplicity 2 as well) where P, Q, R are collinear. Hence the projection has a tacnode and is of degree 5 (note that $\delta_P = 2$ for tacnode as twice blow-up resolves the singularity). To only have nodes, we need project from a point not on the singular cone. But then the number of secants through the point is 6 By some intersection theory stuff....

IV.6 Classification of Curves in $\mathbb{P}^3_k$

Note. Suppose $X \subset \mathbb{P}^n$ a curve and and $P \in \mathbb{P}^n$ such that projection from P induces $X \to X'$ a birational map. Then the degree of X' is one less than X if $P \in X$ and same as X if $P \notin X$.

Lemma. Let Y be a complete intersection in $\mathbb{P}^n_k$, and $X \subset Y \subset \mathbb{P}^n_k$ projective k-variety that is a closed subscheme of Y. Suppose $\dim X = \dim Y = 1$ and $\deg X = \deg Y$. Then $X = Y$. *Proof:* Y is ACM, and hence equidimensional, and moreover $C(Y)$ the affine cone is reduced since it is 2-dimensional CM ring (hence normal). Now, $\dim X = \dim Y$ implies that X is an irreducible component of $Y_{red} = Y$, and thus $\deg Y = \deg X$ implies that Y has no other components, thus $Y = X$.

IV.6.1. Let $X \subset \mathbb{P}^3$ be a rational curve of degree 4. Then taking LES of $0 \to \mathscr{I}_X(2) \to \mathscr{O}_{\mathbb{P}^3}(2) \to \mathscr{O}_X(2) \to 0$ and counting dimensions gives us $h^0(\mathscr{I}_X(2)) - h^1(\mathscr{I}_X(2)) = 1$, and hence $h^0(\mathscr{I}_X(2)) \geq 1$. Now, X cannot be contained in a plane since it is rational, and hence, if $h^0(\mathscr{I}_x(2)) \geq 2$, so that X lies on two different irreducible quadrics F_1, F_2 (irreducible since X is not in a plane), then $Y := V(F_1, F_2)$ is a 1-dimensional k-scheme of degree 4 containing X and hence $X = Y$. But this would mean that X is a genus 1 curve. Hence, X lies on a unique quadric. If it lies on the singular quadric, then a projection from a point on the quadric but not on the curve gives a birational map to X' with a tacnode. But then, $p_a(\widetilde{X}') + 2 = 0 + 2 = p_a(X')$, which is not possible.

IV.6.2. The same argument as (Ex.IV.6.1) gives $h^0(\mathscr{I}_X(3)) \geq 4$, and hence X lies on a cubic surface. $[s^5, s^4t, st^4, t^5]$ lies on a quadric surface, but $[s^5, s^4t + s^3t^2, s^2t^3 + st^4, t^5]$ does not.

IV.6.3. Once again, one can compute that $h^0(\mathscr{I}_X(2)) \geq 1$, so X lies on a quadric surface (a unique one since if X is in two quadric surfaces then its degree is at most 4). Now, given an abstract genus 2 curve X, note that any divisor D of degree 5 is very ample since $2g + 1 = 5$ here, and has rank $5 - 2 = 3$, so $|D|$ gives an embedding $X \hookrightarrow \mathbb{P}^3$.

Now, a (not supposed to be correct) proof that X cannot be embedded on a singular quadric: If X is on a singular quadric Q, then $X \cap H$ for a plane H tangent to Q is $X \cap 2L$ where $2L$ is the double line. Thus, $\deg X \cap 2L$ must be even but $\deg X \cap H = 5$ by Bezout. (This faulty proof shows that doing intersection theory on singular surfaces gets really tricky!)

However, here is an example of genus 2 curve of degree 5 in $\mathbb{P}^3$ lying on a singular quadric. Let $S(X) := S/I$ have a free resolution:

$$0 \longrightarrow S(-4)^2 \xrightarrow{\begin{bmatrix} x & y \\ y & z \\ w^2 & z^2 \end{bmatrix}} S(-3)^2 \oplus S(-2) \xrightarrow{I_2(M)} S \longrightarrow S/I \longrightarrow 0$$

For nonsingular quadric, any curve of type $(2,3)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ will do.

IV.6.4. Suppose X a genus 11 degree 9 curve in $\mathbb{P}^3$. X cannot lie on a quadric surface since none of the cases in (IV.6.4.1) work out. Now, let's consider the sequence $0 \to \mathscr{I}_X(3) \to \mathscr{O}_{\mathbb{P}^3}(3) \to \mathscr{O}_X(3) \to 0$. As $\mathscr{O}_X(3)$ is nonspecial, then we have $h^0(\mathscr{I}_X(3)) \geq 20 - (27 - 11 + 1) = 3$. Thus, X lies in two independent cubics that are necessarily irreducible (since X does not lie in a plane or a quadric). But then X is the intersection of these cubic surfaces, in which case the genus will be 10. Thus, X of degree 9 and genus 11 does not exist in $\mathbb{P}^3$.

Chapter V

Surfaces

V.1 Geometry on a Surface

V.1.1. Just plug into Riemann-Roch.

V.1.2. Note that $P_X(t) = \chi(\mathscr{O}_X(tH)) = \frac{1}{2}tH(tH - K) + p_a + 1 = \frac{1}{2}H^2t^2 - (H.K)t + p_a + 1$, and that $2\pi - 2 = H.(H + K) = H^2 + H.K$.

V.1.3. (a): (Note that we can't just use adjunction formula since D may be singular). Take χ of the SES $0 \to \mathscr{O}_X(-D) \to \mathscr{O}_X \to \mathscr{O}_D \to 0$ to get $1 - \chi(\mathscr{O}_D) = 1 + \chi(\mathscr{O}_X(-D)) - \chi(\mathscr{O}_X) = \frac{1}{2}D.(D + K)$.

(b): Follows from (a).

(c): Follows from (a).

V.1.4. (a): As $X \subset \mathbb{P}^3$ is a hypersurface of degree d, we have $K_X = dH - 4H$ where H is the hyperplane class (restricted to X). By adjunction formula, $-2 = C.(C + K) = C.(C + K) = C.(C + (d - 4)H) = C^2 + d - 4$ (as $C.H = 1$).

(b): Set $X = V(f = x^d + xz^{d-1} + y^d + yw^{d-1})$. Note that $\mathrm{Sing}(f) = V(dx^{d-1} + z^{d-1},\ dy^{d-1} + w^{d-1}, xz^{d-2}, yw^{d-2}) = \emptyset$.

V.1.5. (a): $(d - 4)^2H^2 = d(d - 4)^2$.

(b):