Highlights

An Even Tighter Bound for the Shapley-Folkman-Starr Theorem

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- Tighter error bound compared to previous results.
- Application in the course allocation problem.

An Even Tighter Bound for the Shapley-Folkman-Starr Theorem

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Abstract

Based on a previous refined Shapley-Folkman lemma, we derive a tighter error bound for the Shapley-Folkman-Starr theorem and apply the result to the course allocation problem.

Keywords: Shapley-Folkman lemma, Error-estimation

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1. Introduction

The Shapley-Folkman lemma (Theorem, 1.1) characterizes the points within the convex hull of a Minkowski sum. As stated by Starr (1969), the Shapley-Folkman lemma asserts that an approximate equilibrium exists in economies with nonconvex preferences.

Theorem 1.1. *(Shapley-Folkman lemma) Let S_i be subsets of $\mathbb{R}^m$, $1 \leq i \leq n$, and $S = \sum_{i=1}^n S_i$ be the Minkowski sum of all S_i. For any $z \in convS$, there exists $z_i \in convS_i$ such that $z = \sum_{i=1}^n z_i$ with $z_i \in S_i$ for all but at most $\min\{m, n\}$ values of i.*

Zhou (1993) presented a novel proof of the Shapley-Folkman lemma, but the algebraic idea behind his proof was not explicitly expressed. This idea can be extended to derive a modification of the Shapley-Folkman lemma. To introduce this modification, it is necessary to establish the concept of the kth convex hull.

Definition 1. *Denote the kth convex hull of a set S, $conv_k S$, as the set of convex combinations of at most k elements,*

$$conv_k S = \left\{ \sum_{i=1}^k a_i v_i : v_i \in S, 0 \leq a_i \leq 1, 1 \leq i \leq k, \sum_{i=1}^k a_i = 1 \right\}.$$

The kth convex hull is a subset of the full convex hull and expands as k increases. Figure 1 illustrates the concept of the kth convex hull with a simple example. In the case that the three points set $S = \{A, B, C\}$, $\mathrm{conv}_1 S$ is just the set S, $\mathrm{conv}_2 S$ is the segments AB, AC, BC and $\mathrm{conv}_3 S$ is the whole triangle ABC. In general, $\mathrm{conv}_1 S = S$ and the $(m+1)$th convex hull is just the whole convex hull by Carathéodory's theorem (Rockafellar (1970)) in $\mathbb{R}^m$. Based on this concept, we state a modified theorem,

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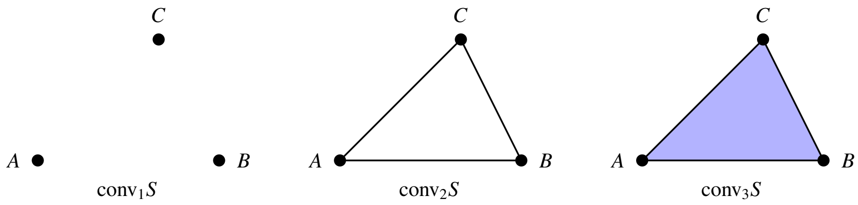

Figure 1: the kth convex hull of set $S = \{A, B, C\}$.

Theorem 1.2. *(Zhou (1993)[1]) Let S_i be subsets of $\mathbb{R}^m$, $1 \leq i \leq n$, and $S = \sum_{i=1}^n S_i$ be the Minkowski sum of all S_i. For any $z \in convS$, there exists $k_i \in \mathbb{N}$ and $\sum_{i=1}^n k_i \leq n + m$ such that $z = \sum_{i=1}^n z_i$ and $z_i \in conv_{k_i} S_i$.*

The theorem indicates a constraint on the k_i's. By examining the convexity via the concept of k-extreme points, Bi and Tang (2020) further generalized the theorem. A point in a convex set S is called k-extreme if it does not lie in the interior of any $(k+1)$-dimensional convex subset of S. The concept is formally defined as,

Definition 2. *A point z in a convex set S is called a k-extreme point of S if there does not exist $k+1$ linear independent vectors $d_1, d_2, \cdots, d_{k+1}$ such that $z \pm d_i \in S$ for $i = 1, 2, \cdots, k+1$.*

Generally, a point in a convex set is an extreme point if and only if it is 0-extreme, and on the boundary if and only if $(m-1)$-extreme. Any point in the convex hull in $\mathbb{R}^m$ is m-extreme. In the concrete example $\text{conv}S = \text{conv}\{A, B, C\}$ in the third sub-figure in figure 1: the three vertexes (A, B, C) are 0-extreme, and the points on segments AB, AC, BC including endpoints are 1-extreme, and any point in the triangle is 2-extreme. Based on the concepts of the k-extreme point and kth convex hull, Bi and Tang (2020) proposed a refined Shapley-Folkman lemma:

Theorem 1.3. *Let S_i be subsets of $\mathbb{R}^m$, $1 \leq i \leq n$, and $S = \sum_{i=1}^n S_i$ be the Minkowski sum of all S_i. If z is a k-extreme point of $convS$, there exist integers $1 \leq k_i \leq k+1$ with $\sum_{i=1}^n k_i \leq n + k$ and points $z_i \in conv_{k_i} S_i$ such that $z = \sum_{i=1}^n z_i$.*

Let k be m and recall any point in the convex hull is m-extreme; then Theorem 1.2 becomes a special case of the Theorem 1.3.

The Shapley-Folkman lemma finds applications in various fields. Tardella (1990) utilized the Shapley-Folkman lemma to establish the Lyapunov convexity theorem in measure theory. In probability theory, the law of large numbers can be directly derived from the Shapley-Folkman lemma (Artstein and Vitale (1975)). Bi and Tang (2020) employed his refinement Theorem 1.3 for the estimation of the duality gap on the nonconvex optimization problem with separable objective functions, which offered improvement compared to the original idea of applying the Shapley-Folkman lemma from Aubin and Ekeland (1976). For each theorem on existence, there is a corresponding corollary of the bound for the Hausdorff distance between the Minkowski sum of sets and its convex hull. The first notable result is derived in Starr (1969), applying the Shapely-Folkman lemma and claiming the existence of approximate equilibrium in a nonconvex

[1]The theorem is summarized from but not directly stated in Zhou's proof of the Shapley-Folkman lemma. For the purpose of completeness, the corresponding proof is attached in Appendix A.

economy. To state the result, we define the measurement of the size of a set $S \subset \mathbb{R}^m$, i.e. the (outer) radius $\text{rad}(S)$ and diameter $D(S)$:

$$\text{rad}(S) = \inf_{y\in\mathbb{R}^m} \sup_{x\in S} |x-y|,$$

$$D(S) = \sup_{x,y\in S} |x-y|,$$

where $|\cdot|$ denotes the l^2 norm in Euclidean space throughout this paper. The Hausdorff distance d_H with respect to usual Euclidean space for $X, Y \subset \mathbb{R}^m$ is defined as

$$d_H(X,Y) = \max\{\sup_{x\in X}\inf_{y\in Y}|x-y|, \sup_{y\in Y}\inf_{x\in X}|x-y|\}.$$

Corollary 1.4. *(Shapley-Folkman-Starr theorem) Let S_i be subsets of $\mathbb{R}^m$, $1 \le i \le n$, and $S = \sum_{i=1}^n S_i$ be the Minkowski sum of all S_i, then*

$$d_H^2(S, convS) \le \min\{m,n\}R^2 := \min\{m,n\} \max_{1\le i\le n} rad^2(S_i).$$

If we assume more that all S_i are compact, the Hausdorff distance is achievable, i.e., for any $z \in \text{conv}S$, there is $x \in S$ such that

$$|x-z|^2 \le \min\{m,n\}R^2.$$

Theorem 1.2 introduces the constraint on the components of the Minkowski sum, i.e., $x_i \in \text{conv}_{k_i}S_i$ and $\sum_{i=1}^n k_i \le n+m$. Building upon this constraint, Budish and Reny (2020) proposed an improvement in the error bound estimation utilizing Theorem 1.2.

Corollary 1.5. *(Budish and Reny (2020)) Let S_i be subsets of $\mathbb{R}^m$, $1 \le i \le n$, and $S = \sum_{i=1}^n S_i$ be the Minkowski sum of all S_i with additional assumption that $n > m$, then*

$$d_H^2(S, convS) \le \frac{mD^2}{4} := \frac{m}{4} \max_{1\le i\le n} D^2(S_i),$$

if all S_i are compact sets, for any $z \in convS$, there is $x \in S$ such that

$$|x-z|^2 \le \frac{mD^2}{4}.$$

Combining Jung's theorem (lemma 1), it is an improved bound for the Shapley-Folkman-Starr Corollary 1.4 with up to around 30% quantitative improvement[2].

In our study, we employ Bi and Tang's refined Shapley-Folkman lemma (Theorem 1.3) to discern the convexity of points in a convex hull of the Minkowski sum to attain a tighter error bound, considering the constraint outlined in Theorem 1.3, i.e., $x_i \in \text{conv}_{k_i}S_i$ and $\sum_{i=1}^n k_i \le n+k$. We will introduce our main result and state the consequences in Section 2. Besides, we will apply our result to the course allocation problem in Section 3, which is a typical combinatorial assignment problem discussed in Budish (2011) and Budish and Reny (2020).

[2]See section 3.1 in Budish and Reny (2020).

2. A Tighter Error Bound

By applying Theorem 1.3, we derive our main result as a new error bound. The corresponding proofs of our main theorem and corollaries are attached in Appendix A.

Theorem 2.1. *Let S_i, $1 \le i \le n$, be compact subsets of $\mathbb{R}^m$, $S = \sum_{i=1}^n S_i$ be the Minkowski sum of all S_i, and $D = \max_{1 \le i \le n} D(S_i)$. If z is a k-extreme point of $convS$, there exist $x \in S$ such that,*

$$|z - x|^2 \le \frac{1}{2} \min \left\{ \frac{nkD^2}{n+k}, \sum_{i=1}^n D^2(S_i) - \frac{\left(\sum_{i=1}^n D(S_i)\right)^2}{n+k} \right\},$$

The first term has degree $O(\min\{n,k\}D^2) = O(\min\{n,k\}R^2)$ by Jung's inequality (lemma 1), which is the same as the degree of the bound for the Shapley-Folkman-Starr theorem (Corollary 1.4) except that we categorize the convexity by the concept of k-extreme points, instead of the dimension m in the error upper bound. The latter bound is attained by jointly considering all $D(S_i)$ and constraints on k_i and will be a (qualitative) improvement compared to the first term in some particular cases. Specifically, the first term is solely concerned with the maximum value among all $D(S_i)$ (also referred to as the l_∞ norm), while the latter term emphasizes the summation of both $D(S_i)$ and $D^2(S_i)$ (corresponding to the l_1 and l_2 norms, respectively).

Results / Conditions	Starr's original result Corollary 1.4	Budish and Reny (2020) Corollary 1.5	Our results Theorem 2.1 and Corollary 2.2
In general	$\min\{n,m\}R^2$	N.A.	$\frac{1}{2}\min\left\{nmD^2/(n+m), \sum_{i=1}^n D(S_i)^2 - \left(\sum_{i=1}^n D(S_i)\right)^2/(n+k)\right\}$
$n \ge m$	mR^2	$mD^2/4$	$\frac{1}{2}\min\left\{mD^2/2, \sum_{i=1}^n D(S_i)^2 - \left(\sum_{i=1}^n D(S_i)\right)^2/(n+k)\right\}$
$n \ge m$ & k-extreme	mR^2	$mD^2/4$	$\frac{1}{2}\min\left\{kD^2/2, \sum_{i=1}^n D(S_i)^2 - \left(\sum_{i=1}^n D(S_i)\right)^2/(n+k)\right\}$
m is $O(n)$	$O(nD^2)$	N.A.	$\min\left\{O(kD^2), O\left(\sum_{i=1}^n D(S_i)^2\right)\right\}$

Table 1: Comparison of different results

Corollary 2.2. *If $n \ge k$, the estimated bound $\frac{nkD^2}{2(n+k)}$ in Theorem 2.1 can be modified to $\frac{kD^2}{4}$.*

Since k is always less than m, the dimension of given Euclidean space, and combining $D/2 \le R$, the Shapley-Folkman-Starr Corollary 1.4 is a direct result of our theorem. Furthermore, Corollary 2.2 asserts that the new bound is at least as effective as Corollary 1.5 in all cases, categorizing the convexity.

Corollary 2.3. *In general, we are unable to determine whether the first term is greater than or less than the latter term. However, the latter term is not larger than the first term in degree in the particular case that k is $O(n)$.*

It is difficult to compare the two terms without information on the sizes of components. Therefore, we naturally consider the sizes to be independent and identically distributed random variables.

Corollary 2.4. *Assume that $D(S_i)$'s are independent and identically distributed random variables, and we calculate expectation. The first term is $\frac{nk}{2(n+k)}\mathbb{E}(D^2)$ and the latter term is*

$$\frac{nk}{2(n+k)}\mathbb{E}(D^2(S_i)) + \frac{n(n-1)Var(D(S_i))}{2(n+k)}.$$

In case k is $O(n)$ or $n\mathrm{Var}(D)$ is $O(k\mathbb{E}(D^2(S_i)))$, the latter term has degree improvement $\mathbb{E}(D^2(S_i))/\mathbb{E}(D^2) = \mathbb{E}(D^2(S_i))/\mathbb{E}(\max_{1 \le i \le n} D^2(S_i))$, compared to the first term.

3. Example

We will introduce an application to the course allocation problem. We assume that there are m courses and n students in total, each of whom may enroll in no more than T courses subject to credit constraints. Consequently, feasible bundles are represented as vectors in the form of $\{0, 1\}^m$, where at most T entries are 1 and the remaining entries are 0.

Budish (2011) stated the existence of an allocation to students that is approximately competitive but may lack feasibility, with a bound on excess demands. Expanding upon the analysis, Budish and Reny (2020) provided an enhancement to this excess demand bound via his improved bound for the Shapley-Folkman-Starr theorem (Corollary 1.5). We contribute further by incorporating the concept of k-extreme points into the course allocation model and employ our findings to approximate a refined bound. The primary objective of this section is to demonstrate our bounds, comparing with the results delineated by Budish and Reny (2020) (Section 4).

To elucidate the concept of k-extreme points within our course allocation model, we introduce the concept of dominant courses, characterized by the following four specific properties: First, each dominant course has a single quota, such as individual project-based courses. Second, each student is restricted to enrolling in at most one dominant course, which reflects the realistic constraint that a college student is not allowed to work on more than one project simultaneously. Third, all students have a preference for bundles that include a dominant course over those that do not[3]. Fourth, all dominant courses are uniformly preferred by all students.

Under these premises, we posit that a competitive equilibrium exists within an $(m - l)$-dimensional convex subset of the total course allocation space (see Appendix B). As a result of these considerations, we offer a tighter bound on the excess demand with these extra assumptions. To illustrate, let's consider an example with $m = 100$ courses, of which $l = 30$ are dominant courses, and $n = 500$ students. Recognizing that not all students are in a position to take exactly T courses—owing to commitments such as part-time work or internships—we introduce a distribution for the number of courses students enroll in. Specifically, we assume a ratio of $1 : 2 : 4 : 2 : 1$ for the number of courses students will take, ranging from 1 to 5. This means that 50 students take 1 course, 100 students take 2 courses, and so on.

In assessing the feasibility of course bundles for a student i, who intends to enroll in t_i courses, we define the diameter, D_i, as the greatest distance between any two feasible bundles, calculated to be $\sqrt{2t_i}$. Based on Corollary 1.4, Budish and Reny (2020) (Section 4) presented an upper bound for the total excess demand as $\sqrt{2Tm}/2$. For the scenario previously outlined, this bound translates to $\sqrt{1000}/2 = 15.8$, which yields a maximized average excess demand per course of 1.5 students[4]. Our results, Theorem 2.1 and Corollary 2.2, posit a more refined upper bound for total excess demands as,

$$\min\left\{\sqrt{2T(m-l)}/2, \sqrt{\sum_{i=1}^{n} D_i^2 - (\sum_{i=1}^{n} D_i)^2/(n+m-l)}/\sqrt{2}\right\}$$
$$= \min\{\sqrt{700}/2, \sqrt{(3000 - 1201^2/570)/2}\} = \min\{13.2, 15.3\} = 13.2,$$

[3] One concrete scenario in which the assumption is valid, is personal research project courses are uniformly preferred by the students within a particular interest group.

[4] This scenario is realized when 50 courses have an excess demand of one student while the remaining 50 courses each have an excess demand of two students.

in the assumed case. Consequently, the maximized average excess demand per course is reduced to 1.25 students[5]. When the number of students n is reduced to 150, the upper bound presented by Budish and Reny (2020) and our first bound remain unchanged. However, our latter bound adjusts to $\sqrt{(900 - 360^2/220)/2} = 12.5$, which further lowers the maximized average excess per course to 1.18 students[6].

In conclusion, our main theorem introduces a tighter bound in the context of course allocation problem when the concept of "dominant courses" is considered. Our first bound demonstrates an enhancement compared to the findings of Budish and Reny (2020), with the consideration of "dominant courses," improving both the maximum demand for any single course (from 15.8 to 13.2) and the maximized average excess per course (from 1.51 to 1.25), irrespective of the student population size. Furthermore, when the number of students and the distribution of the number of courses they are enrolled in are factored in, our latter bound potentially provides a superior refinement in both the maximum demand for any single course (from 13.2 to 12.5) and the maximized average excess per course (from 1.25 to 1.18), particularly when the number of students is small or when the distribution is positively skewed[7].

4. Conclusion

We apply the refined Shapley-Folkman lemma (Theorem 1.3) to attain an even tighter bound for the Shapley-Folkman-Starr theorem, as presented in our main result Theorem 2.1. This approach distinguishes the convexity in the convex hull of the set $S = \sum_{i=1}^n S_i$ by using the concept of k-extreme points and categorizing the n components into k_ith convex hulls. Instead of the traditional way in which each $D(S_i)$ is directly estimated to be $\max_{1\leq i\leq n} D(S_i)$, we consider all $D(S_i)$ jointly and use the constraint on k_i to reach an even tighter bound. Our new bound presents degree improvement in certain cases, especially for the case when k is $O(n)$.

We raise a natural open question here. Under the same assumption that $S = \sum_{i=1}^n S_i \subset \mathbb{R}^m$, and for any point z in the convex hull conv$S$, we can see that there exists a vector $\mathbf{k} = (k_1, \cdots, k_n)^T \in \mathbb{N}^n, \|\mathbf{k}\|_1 \leq n+k$ such that $z \in \sum_{i=1}^n \mathrm{conv}_{k_i} S$, and we denote $A(\mathbf{k}) = \sum_{i=1}^n \mathrm{conv}_{k_i} S \subset$ conv$S$. This implies that any element in the convex hull will belong to at least one of the sets $A(\mathbf{k})$. Naturally, we are interested in exploring the relationships (e.g., Hausdorff distance, size of overlapping, etc.) among all these sets $A(\mathbf{k})$, S, and conv$S$. While the Shapely-Folkman-Starr theorem focuses on measurement about $d_H(S, \mathrm{conv}S)$, it is fair to mention that Adiprasito (2020) (Theorem 1.1) stated a relevant result of measurement about $d_H(\mathrm{conv}S, \mathrm{conv}_k S)$: Let S be a subset of finite points in $\mathbb{R}^n$ and $k \in \mathbb{N}$, then,

$$d_H(\mathrm{conv}S, \mathrm{conv}_k S) \leq \frac{D(S)}{\sqrt{2k}}.$$

[5]This outcome occurs when 75 courses have an excess demand of one student, and the remaining 25 courses each have an excess demand of two students.

[6]This is achieved when excess demand consists of one student for 82 courses and two students for the remaining 18 courses.

[7]While the results presented by Budish and Reny (2020) and our first bound consider only the maximum of the diameters, our latter bound takes into account the diameters of all feasible bundles for each student.

Appendix A. Proofs in section 2

Lemma 1. *For any set $S \subset \mathbb{R}^m$,*

$$\sqrt{\frac{2(m+1)}{m}} rad(S) \le D(S) \le \frac{rad(S)}{2}.$$

Proof. The first inequality is Jung's Theorem Jung (1901), and the last inequality is trivial in Euclidean space.

Lemma 2. *Consider the set $S = \{s_0, s_1, ..., s_{k-1}\} \subset \mathbb{R}^m$, and denote the cardinality of S as $|S|$, then $rad^2(S) \le \frac{k-1}{2k} D^2(S)$.*

Proof. For any convex combination z of S, there exists $0 \le \lambda_i \le 1$, $\sum_{i=0}^{k-1} \lambda_i = 1$, such that

$$\begin{aligned} z &= \lambda_0 s_0 + \lambda_1 s_1 + \cdots \lambda_{k-1} s_{k-1} \\ &= (1 - (\lambda_1 + \lambda_2 + \cdots + \lambda_{k-1})\, s_0 + \lambda_1 s_1 + \cdots \lambda_{k-1} s_{k-1} \\ &= s_0 + \lambda_1(s_1 - s_0) + \lambda_2(s_2 - s_0) \cdots \lambda_{k-1}(s_{k-1} - s_0) \\ &\in \{s_0\} \oplus \mathrm{span}\{s_1 - s_0, s_2 - s_0, \cdots s_{k-1} - s_0\}. \end{aligned}$$

This implies conv$S$ is contained in a vector space that is isomorphic to $\mathbb{R}^{k-1}$. By Lemma 1, $\mathrm{rad}^2(S) \le \frac{k-1}{2k} D^2(S)$.

Lemma 3. *(Cauchy-Schwarz Inequality) Given $a_1, a_2, \cdots, a_n \in \mathbb{R}^+$, $b_1, b_2, \cdots, b_n \in \mathbb{R}^+$,*

$$\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 \ge \left(\sum_{i=1}^n a_i b_i \right)^2.$$

Proof. It is a well-known result, and the proof has been omitted.

Proof of Theorem 1.2. Recall that $\mathrm{conv}S = \mathrm{conv} \sum_{i=1}^n S_i = \sum_{i=1}^n \mathrm{conv} S_i$. For any $z \in \mathrm{conv}S$, there exists $z_i \in \mathrm{conv}S_i, m_i \in \mathbb{N}, x_{ij} \in S_i, 1 \le i \le n, 1 \le j \le m_i$, such that

$$\begin{aligned} z &= z_1 + z_2 + \cdots z_n \in \mathbb{R}^m, \\ z_i &= \sum_{j=1}^{m_i} a_{ij} x_{ij}, \quad 1 \le i \le n, \sum_{j=1}^{m_i} a_{ij} = 1, a_{ij} \ge 0, 1 \le i \le n. \end{aligned}$$

Let $e_j, 1 \le j \le n$, denote the standard basis of $\mathbb{R}^n$,

$$y := z \oplus (e_1 + e_2 + \cdots + e_n)^T \in \mathbb{R}^{m+n},$$

$$y = \sum_{i=1}^n z_i \oplus e_i = \sum_{i=1}^n \left(\sum_{j=1}^{m_i} a_{ij} x_{ij} \right) \oplus e_i = \sum_{i=1}^n \sum_{j=1}^{m_i} a_{ij} \left(x_{ij} \oplus e_i \right).$$

Notice that $y \in \mathbb{R}^{m+n}$, by Carathéodory theorem (Rockafellar (1970)) for conic combination, there exist non-negative b_{ij} with at most $m+n$ of b_{ij}'s strictly greater than 0 such that,

$$y = \sum_{i=1}^n \sum_{j=1}^{m_i} b_{ij} (x_{ij} \oplus e_i).$$

By the construction of last n entries of y, we have $\sum_{j=1}^{m_i} b_{ij} = 1$ for $1 \le i \le n$. Let $k_i = \|(b_{i1}, b_{i2}, \cdots, b_{im_i})^T\|_0$ be the value of the count measure in each i, and reorder these positive term,

$$z = \sum_{i=1}^{n} x_i, \quad x_i = \sum_{j=1}^{k_i} b_{ij} x_{ij} \in \mathrm{conv}_{k_i} S_i, \quad \sum_{i=1}^{n} k_i \le m + n.$$

Proof of Theorem 2.1. For any k-extreme point z, by Theorem 1.3, there exists $\{z_i\}_{i=1}^n$ such that,

$$z = \sum_{i=1}^{n} z_i, z_i \in \mathrm{conv}_{k_i} S_i, k_i \in \mathbb{N}, \sum_{i=1}^{n} k_i \le n + k.$$

Denote $T_i \subset S_i$ such that $z_i \in \mathrm{conv} T_i$, $|T_i| = k_i$, i.e. T_i consists of the k_i elements such that z_i is in their convex combinations. Let $T = \sum_{i=1}^n T_i$ and we have $z \in \mathrm{conv} T$. We apply the Shapley-Folkman-Starr Corollary 1.4 to T and $\mathrm{conv} T$, for any $k_i \in \mathbb{N}$, $\sum_{i=1}^n k_i \le n + k$, $1 \le i \le n$,

$$\inf_{w \in T} |z - w|^2 \le \sum_{i=1}^{n} \mathrm{rad}^2(T_i) \le \sum_{i=1}^{n} \frac{k_i - 1}{2k_i} D^2(T_i),$$

the second inequality is from the lemma 2. $T_i \subset S_i$ implies $T \subset S$, hence,

$$\inf_{y \in S} |z - y|^2 \le \inf_{w \in T} |z - w|^2 \le \frac{k_i - 1}{2k_i} D^2(T_i) \le \frac{k_i - 1}{2k_i} D^2(S_i).$$

Then, we will prove the two terms of our result separately. Let $D := \max_{1 \le i \le n} D(S_i)$, for any $k_i \in \mathbb{N}$, $\sum_{i=1}^n k_i \le n + k$, $1 \le i \le n$, we firstly claim $\sum_{i=1}^n \frac{k_i - 1}{k_i} D^2(S_i) \le \frac{nk}{n+k} D^2$,

$$\sum_{i=1}^{n} \frac{k_i - 1}{k_i} D^2(S_i) \le \sum_{i=1}^{n} \frac{k_i - 1}{k_i} D^2 = nD^2 - D^2 \sum_{i=1}^{n} \frac{1}{k_i} \le D^2 \left(n - \frac{(\sum_{i=1}^n 1)^2}{\sum_{i=1}^n k_i} \right) \le \frac{nk}{n+k} D^2.$$

The second last inequality is from the Cauchy-Schwarz inequality.

Then we claim $\sum_{i=1}^n \frac{k_i - 1}{k_i} D^2(S_i) \le \sum_{i=1}^n D^2(S_i) - \frac{\left(\sum_{i=1}^n D^2(S_i)\right)^2}{n+k}$,

$$\begin{aligned}\sum_{i=1}^{n} \frac{k_i - 1}{k_i} D^2(S_i) = \sum_{i=1}^{n} D^2(S_i) - \sum_{i=1}^{k} \frac{D^2(S_i)}{k_i} &\le \sum_{i=1}^{n} D^2(S_i) - \frac{\left(\sum_{i=1}^n D(S_i)\right)^2}{\sum_{i=1}^n k_i} \\ &\le \sum_{i=1}^{n} D^2(S_i) - \frac{\left(\sum_{i=1}^n D(S_i)\right)^2}{n + k}.\end{aligned}$$

The last inequality is from the Cauchy-Schwarz inequality. Since all S_i are compact, the existence of x is guaranteed.

Proof of Corollary 2.2. it is enough to show that the following optimal program with optimal value $k/2$, given $n > k$,

$$\begin{aligned} &\max \sum_{i=1}^{n} \frac{k_i - 1}{k_i} \\ &s.t.\, k_i \in \mathbb{N}, \sum_{i=1}^{n} k_i \le n + k. \end{aligned}$$

Noticed that $f(x) = 1 - 1/x$ is concave and increasing on $[1, \infty)$ and there are at least k of k_i strictly greater than 1. By the concavity and constraints, the optimal solution is in the case that j of $k_i's$ are equal a and $n - j$ of $k_i's$ equal $a + 1$ for some positive integer a and the sum of k_i is tightly equal $n + k$.

$$ja + (n - j)(a + 1) = k \iff an - j = k.$$

Notice that $n > k$ and $j < n$, which only allows $a = 1$, hence $j = n - k$. This is the necessary condition to reach maximized value, and $a = 1, j = n-k$ is the only candidate, which claims that the optimal value is $(n - k)(1 - 1)/1 + k(2 - 1)/2 = k/2$.

Proof of Corollary 2.3.

$$\begin{aligned}
&\frac{nk}{n+k} D^2 \leq \sum_{i=1}^{n} D(S_i)^2 - \frac{\left(\sum_{i=1}^{n} D(S_i)\right)^2}{n+k} \\
\iff & nD^2 - \sum_{i=1}^{n} D^2(S_i) \leq \frac{n^2 D^2 - (\sum_{i=1}^{n} D(S_i))^2}{n+k} \\
\iff & \sum_{i=1}^{n} (D - D(S_i))(D + D(S_i)) \leq \frac{\sum_{i=1}^{n} (D + D(S_i)) \sum_{i=1}^{n} (D - D(S_i))}{n+k}.
\end{aligned}$$

It is only valid for large k. The left-hand side of the last inequality is greater than the right-hand side when $k = 0$, as Chebyshev's inequality.

In case $k \in O(n)$, the first term is in $O(nD^2)$, and the latter term is less than $\sum_{i=1}^{n} D(S_i)^2$ is not greater than nD^2 in degree.

Proof of Corollary 2.4. It is enough to show that,

$$\begin{aligned}
&\mathbb{E}\left(\sum_{i=1}^{n} D^2(S_i) - \frac{\left(\sum_{i=1}^{n} D(S_i)\right)^2}{n+k} \right) = n\mathbb{E}(D^2(S_i)) - \frac{\mathbb{E}\left[(\sum_{i=1}^{n} D(S_i)^2\right]}{n+k} \\
&= n\mathbb{E}(D^2(S_i)) - \frac{n\mathbb{E}(D^2(S_i)) + n(n-1)\mathbb{E}^2(D(S_i))}{n+k} \\
&= \frac{nk}{n+k} \mathbb{E}(D^2(S_i)) + \frac{n^2\mathbb{E}(D^2(S_i)) - n\mathbb{E}(D^2(S_i)) - n^2\mathbb{E}^2(D(S_i)) + n\mathbb{E}^2(D(S_i))}{n+k} \\
&= \frac{nk}{n+k} \mathbb{E}(D^2(S_i)) + \frac{n(n-1)\mathrm{Var}(D(S_i))}{n+k}.
\end{aligned}$$

Appendix B. Proof in Section 3

The proof here principally follows in Budish (2011) (Appendix A, Proof of Theorem 1) and modifies his proof with the additional constraints of l dominant goods.

Firstly, for any given budget vector $\mathbf{b}'$, without loss of generality, let $b'_1 \geq b'_2 \geq \cdots \geq b'_n$ and the taxes $(\tau'_{ix})_{i \in S, x \in 2^C}$ satisfy additional property that $i < j$ implies $\min_x b'_i + \tau'_{ix} > \max_x b'_j + \tau'_{jx}$. The assumption can be achieved due to the finite numbers of agents and schedules.

Secondly, without loss of generality, we assume that the first l entries of the space of allocation correspond to the l dominant goods. We restrict the value of the first l entries of the price space to the same constant value $(\min_x\{b'_l + \tau'_{lx}\} + \max_x\{b'_{l+1} + \tau'_{(l+1)x}\})/2$, which ensures only the first l agents affordable to the dominant goods, and the rest agents unaffordable. The first l entries of the constrained price space have no excess demand (first l entries of $\mathbf{z}$ always equal 0), and we can find a fixed point $\mathbf{p}^*$ in the same logic as in Budish (2011), appendix A, Proof of Theorem 1, step 2 and 3; and keep the same way in step 4 to 6.

Finally, we want to argue that there exists a $(m-l)$-extreme market-clearing excess demand vector in the convex hull of $A(6)$ in step 7, and we can employ our main theorem to find a tighter bound in step 8, comparing to $\sqrt{\sigma M}/2$. It can be done by setting the δ in step 7 small enough (less than $(\min_x\{b'_l+\tau'_{lx}\}-\max_x\{b'_{l+1}+\tau'_{(l+1)x}\})/4$) such that any point in $B_\delta(\mathbf{p}^*)$ will not change the demands of the first l dominant goods, and the non-dominant goods will have a market-clearing excess demand vector in a $(m-l)$-dimension convex hull. (The changing demands $\mathbf{z}(\mathbf{P}^\phi)$ only affect the non-dominant goods.)

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