# Ideal Position in Voting Model

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**Abstract**

# 1 Introduction

# 2 Deterministic Model

We denote the policy space is $\mathbb{R}^m$ and the set of voters $N = \{v_1, v_2, \cdots, v_n\}$, and their utility function $u_i : \mathbb{R}^m \to \mathbb{R}$ which is smooth(or $C^2$, second differentiable) and concave, where $v_i \in \mathbb{R}^m$ is the ideal position (maximum point of $f_i$), for the $i$-th voter, $1 \leq i \leq n$. For the two candidates $P_1$ and $P_2$, $P_1$ chooses a position first, and $P_2$ chooses a position later. After both candidates complete their actions, the voters will vote for the candidate close to them separately. In general, for each position $x$ such that $P_1$ chooses, we could find a region $S(x)$ as a subset of policy space $\mathbb{R}^m$ such that $P_2$ can choose any position in this region to beat $P_1$.

For given information of voters (i.e. all $f_i$), we denote $\alpha : \mathbb{R}^m \to \mathbb{R}^+, \alpha(x) = \mu(S(x))$, where $\mu(\cdot)$ is the Lebesgue measure.

**Theorem 2.1.** *$\alpha$ is a convex function. (expected)*

We denote the collections of dominant voters as $\mathcal{M} = \{M \subset N : \#M \geq \frac{1}{2}\#N\}$. For a collection of voters $M$ and a given position $x$ $P_1$ chooses, we denote the region for $P_2$ could win all voters in $M$ as $V_M$, denote the region for $P_2$ could win the $i$th voter as $B_i(x)$.

$$B_i(x) = \{y \in \mathbb{R}^m f_i(y) \geq f_i(x)\}, \quad V_M(x) = \bigcap_{i \in M} B_i(x)$$

$$\begin{aligned}
\alpha(x) &= \mu\left(\bigcup_{M \subset \mathcal{M}} \bigcap_{i \in M} B_i(x)\right) \\
&= \mu\left(\bigcup_{M \subset \mathcal{M}} V_M(x)\right) \\
&= \sum_{M \subset \mathcal{M}} \mu\left(V_M(x)\right) - \sum_{M_1, M_2 \subset \mathcal{M}} \mu\left(V_{M_1}(x) \cap V_{M_2}(x)\right) + \cdots \\
&= \sum_{M \subset \mathcal{M}} \mu\left(V_M(x)\right) - \sum_{M_1, M_2 \subset \mathcal{M}} \mu\left(V_{M_1 \cup M_2}(x)\right) + \cdots
\end{aligned}$$

To be proofed

## 2.1 Weaker Version

If all voters stand at an extreme point of the set of voters' ideal position, the conclusion is true in $\mathbb{R}^2$. (could be extended to $\mathbb{R}^m$)

## 3 Probablistic Model

### 3.1 Model given by [Sch07]

The utility function of voter $i$ with ideal position $v_i$ and a candidate position $z_j$ is

$$u_{ij}(v_i, z_j) = u^*_{ij}(v_i, z_j) + \varepsilon_{ij},$$

where

$$u^*_{ij}(v_i, z_j) = \lambda_i - \beta||v_i - z||^2.$$

Here, $u^*_{ij}$ is the observable utility for voter $i$ associated with party/candidate $j$. $\lambda_j$ is the valence of agent/candidate/party $j$, and $\beta$ is a positive constant. The terms $\{\varepsilon_{ij}\}$ are the stochastic errors. We assume that all $\varepsilon_{ij}$'s are iid drawed from distribution $\Psi$, which is the Type $I$ extreme value distribution and takes the closed form

$$\Psi(h) = \exp(-(\exp -h)).$$

The probability for a voter $i$ to choose party $j$ is that

$$p_{ij} = \Pr[u_{ij}(v_i, z_j) > u_{il}(x_i, z_l)], l \neq j.$$

The expected vote share for agent $j$ is $V_j(\mathbf{z}) = \sum_{i=1}^n p_{ij}$, where the $\mathbf{z}$ is the vector of all candidates' positions.

In the model given above, the expected voter share for candidate $j$ is,

$$V_j(\mathbf{z}) = \sum_{i=1}^n \frac{\exp(u^*_{ij}(v_i, z_j))}{\sum_j \exp(u^*_{ik}(v_i, z_k))}.$$

**Remark 3.1.** *(The norm is convex (by definition)) and (the function $x^2$ is non-decreasing (in positive half-axis) and convex) jointly imply that the square of a norm is convex, which implies that $u^*_{ij}$ is concave. The function $e^x$ is convex and non-decreasing We denote $G : \mathbb{R}^+ \to \mathbb{R}, x \mapsto \frac{x}{x+k}$ for some positive $k$. It's easy to verify that $G$ is strictly concave and increasing. We recall that $u_i$ is strictly concave. Hence $G \circ f_i$ is strictly concave, which implies $\mathbb{E}_z(x)$ is concave.*

$$P_{ij} = \frac{e^{f_i(x_j)}}{e^{f_i(x_1)} + e^{f_i(x_2)}}, \quad j = 1, 2$$

For a position $z \in \mathbb{R}^m$ that $P_1$ chooses, $P_2$ can choose a position to maximize his expectation:

$$\mathbb{E}_z : \mathbb{R}^m \to \mathbb{R}^+, \quad \mathbb{E}_z(x) = \sum_{i=1}^n \frac{e^{f_i(x)}}{e^{f_i(x)} + e^{f_i(z)}}$$

However, the objective function maynot be concave.

### 3.2 Modificated Model

All utility functions $f_i$ are strictly concave, positive, and second-differentiable.

If the two candidate choose position $x_1$ and $x_2$, denote the probability for voter $i$ to voter candidate $P_j$ is $P_{ij}$, $1 \leq i \leq n, j = 1, 2$,

$$P_{ij} = \frac{f_i(x_j)}{f_i(x_1) + f_i(x_2)}, \quad j = 1, 2$$

For a position $z \in \mathbb{R}^m$ that $P_1$ chooses, $P_2$ can choose a position to maximize his expectation:

$$\mathbb{E}_z : \mathbb{R}^m \to \mathbb{R}^+, \quad \mathbb{E}_z(x) = \sum_{i=1}^n \frac{f_i(x)}{f_i(x) + f_i(z)}$$

**Theorem 3.1.** *Given a fixed position $z$, the function $\mathbb{E}_z(x)$ is (strictly) concave.*

*Proof.* We denote $G : \mathbb{R}^+ \to \mathbb{R}, x \mapsto \frac{x}{x+k}$ for some positive $k$. It's easy to verify that $G$ is strictly concave and increasing. We recall that $f_i$ is strictly concave. Hence $G \circ f_i$ is strictly concave, which implies $\mathbb{E}_z(x)$ is concave. $\square$

**Theorem 3.2.** *(Pure Strategy Nash Equilibrium) There exist a unique $x^* \in \mathbb{R}^m$ such that $(x^*, x^*)$ is the pure strategy Nash equilibrium, i.e.*

$$\arg\max_{x \in \mathbb{R}^m} \mathbb{E}_{x^*}(x) = \arg\max_{x \in \mathbb{R}^m} \sum_{i=1}^{n} \frac{f_i(x)}{f_i(x) + f_i(x^*)} = x^*$$

*which implies $\mathbb{E}_{x^*}(x^*) = n/2$*

*Proof.* Step 1: Let $V \subset \mathbb{R}^m$ be the convex hull of the ideal positions of voters. Denote $h : V \to \mathbb{R}^m$, $h(z) = \arg\max_{x \in \mathbb{R}^m} \mathbb{E}_z(x)$. $h$ is well-defined due to the strictly convexity of $\mathbb{E}_z(x)$. Claim $h(z) = \arg\max_{x \in V} \mathbb{E}_z(x)$ such that $h : V \to V$.
Step 2: Prove $h$ is continous w.r.t. $z$. Done by implicit function theorem with the condition $\mathbb{E}_z(x)$ is second differentiable w.r.t. $x$.
Step 3: $h$ is a continuous function from a convex compact set to itself; by Brouwer fixed-point theorem, there exists a fixed point $x^*$ such that $h(x^*) = x^*$. Uniqueness is trivial.
Brouwer fixed-point theorem: Every continuous function from a nonempty convex compact subset K of a Euclidean space to K itself has a fixed point. $\square$

**Remark 3.2.** *Example:*

$$\mathbb{E}_x(y) = y^2 - x^2 + \frac{n}{2}, \ -1 < x, y < 1$$

*where $x, y \in \mathbb{R}^1$ represents the two candidates' position.$(0,0)$ is a PNE here. $\mathbb{E}_x(y) + \mathbb{E}_y(x) = n, E_x(x) = \frac{n}{2}$.*

**Proposition 3.2.1.** *the gradient of the objective function $\nabla \mathbb{E}_z$ is lipschiz continous.*

**Remark 3.3.** *The gradient descent algorithm solves the maximization point of $\mathbb{E}_z$.*

# 4 Deterministic Dynamic Game

## 4.1 Discrete-Time Game + Probablistic Model

Cor 6.2 states a sufficient and necessary condition for existence of optimal strategy. The strategies $\gamma_k^{j*}, k \in K, j = 1, 2$, where $K$ is the total time phase, provide a saddle-point solution if and only if, there exist function $V(k, \cdot) : \mathbb{R}^{2m} \to \mathbb{R}, k \in K$ s.t.

$$V(k, x) = \min_{u_k^1} \max_{u_k^2} \{V(k+1, x + u_k^1 + u_k^2)\}, \ V(K, x) = q(x), \ V(K+1, x) = 0$$

where $x_{k+1} = f_k(x_k, u_k^1, u_k^2) = x_k + u_k^1 + u_k^2$. The unique saddle point value of the game is $V(1, x_1)$, where $x_1$ is the initial state.

## 4.2 Continuous-Time Game + Probablistic Model

We fit the probabilistic model in the dynamic game. The set of $n$ voters is $N = \{1, 2, \cdots, n\}$ and the policy space is $\mathbb{R}^m$. Predisribed fixed time $T$ to end the voting game. The state space is $\mathbb{R}^{2m}$, which describes the two candidates' position positions (first $m$-entries for $P_1$ and the latter for $P_2$). The strategy space,

$$\Gamma^1 = \{\gamma : [0, T] \times \mathbb{R}^{2m} \to \mathbb{R}^m \oplus \mathbf{0}\} \big| \gamma \text{ is continuously differentiable, Image}(\gamma) \subset B_{\mathbf{0}}(1)\}$$

$$\Gamma^2 = \{\gamma : [0, T] \times \mathbb{R}^{2m} \to \mathbf{0} \oplus \mathbb{R}^m \big| \gamma \text{ is continuously differentiable, Image}(\gamma) \subset B_{\mathbf{0}}(1)\}$$

for any pure strategy the candidates choose, it's continuously differentiable, and the speed of action is less than 1. For any given pure strategy $\gamma^1, \gamma^2$, the evolution function is defined as,

$$\frac{dx}{dt} := f(t, x, u_1, u_2) = u_1 + u_2 = \gamma^1(t, x) + \gamma^2(t, x)$$

the $f$ is the general notation used in dynamic games. We denote the objective function,

$$L = \int_0^T g(t, x, u^1, u^2)\, dt + q(T, x(T)$$

at here, $g \equiv 0$ and $T$ is fixed. To make it simple, we just write $L = q(x(T))$. We consider the probabilistic voting model and the objective function $L$ is $P_2$'s expectation. For example, let $f_i$ be the utility functions for each voter and, $p_1, p_2 \in \mathbb{R}^m$ are the final position of the two candidates,

$$L = q(x(T)) = q(p_1, p_2) = \sum_{i=1}^{n} \frac{f_i(p_2)}{f_i(p_1) + f_i(p_2)}$$

then $P_1$ is the minimizer, and $P_2$ is the maximizer, since it's a zero-sum game.

**Theorem 4.1.** *Cor6.6 (Equation 6.75, for feedback pattern) in Tamer Basar: The sufficient condition for $\gamma^{1*}, \gamma^{2*}$ to be the optimal (the saddle strategy) is, there exists continuously differentiable function $V : [0, T] \times \mathbb{R}^{2m} \to \mathbb{R}$ satisfies the Bellman's equation:*

$$\begin{aligned}
-\frac{\partial V}{\partial t} &= \min_{u^1} \max_{u^2} \langle \frac{\partial V}{\partial x}, u^1 + u^2 \rangle \\
&= \max_{u^2} \min_{u^1} \langle \frac{\partial V}{\partial x}, u^1 + u^2 \rangle \\
&= \langle \frac{\partial V}{\partial x}, \gamma^{1*}(t, x) + \gamma^{2*}(t, x) \rangle
\end{aligned}$$

*where the $\langle \cdot \rangle$ denotes the general inner product in the Euclidean space.*

**Remark 4.1.** *The interchangeability is named Issac's condition. Denote the first $m$-entries variables is $x_1$, the latter $m$-entries is $x_2$. The Issac's condition is automatically satisfied,*

$$\min_{u^1} \max_{u^2} \langle \frac{\partial V}{\partial x}, u^1 + u^2 \rangle = \min_{u^1} \max_{u^2} \left[ \langle \frac{\partial V}{\partial x_1}, u^{1'} \rangle + \langle \frac{\partial V}{\partial x_2}, u^{2'} \rangle \right] = \min_{u^1} \langle \frac{\partial V}{\partial x_1}, u^{1'} \rangle + \max_{u^2} \langle \frac{\partial V}{\partial x_2}, u^{2'} \rangle$$

*where $u^{1'}$ is the vector in $\mathbb{R}^m$ consisted of first $m$-entries of $u^1$. A similar definition is applied on $u^{2'}$.*

**Remark 4.2.** *We consider the constraint that $||u^1||_2, ||u^2||_2 \leq 1,$, we can simplifize*

$$\begin{aligned}
\min_{u^1} \langle \frac{\partial V}{\partial x_1}, u^{1'} \rangle &\geq -||u^1||_2 \cdot ||\frac{\partial V}{\partial x_1}||_2 \geq -||\frac{\partial V}{\partial x_1}||_2 \\
\max_{u^2} \langle \frac{\partial V}{\partial x_2}, u^{2'} \rangle &\leq ||\frac{\partial V}{\partial x_2}||_2
\end{aligned}$$

*which are both from Cauchy-Schwarz inequality and take equality when $u^{1'}$ lies on the oppsite direction of $\frac{\partial V}{\partial x_1}$ and $u^{2'}$ lis on the same direction $\frac{\partial V}{\partial x_2}$. The PDE is simplified as,*

$$\begin{aligned}
-||\frac{\partial V}{\partial x_1}||_2 + ||\frac{\partial V}{\partial x_2}||_2 &= -\frac{\partial V}{\partial t} \\
initial\, condition : V(T, x) &= q(x), \forall x \\
boundary\, condition : V(x_1, x_2, t) &= 1\, for\, large\, |x_1|, \\
V(x_1, x_2, t) &= 0\, for\, large\, |x_2|,
\end{aligned}$$

*the boundary condition is due to $x_1$ is the minimizer and $x_2$ is the maximizer. The ideal position should be in the convex hull of voters' ideal positions.*

**Remark 4.3.**

$$V(x,T)=q(x)=q(x_1,x_2)=\sum_{i=1}^{n}\frac{g_i(x_2)}{g_i(x_1)+g_i(x_2)}, q_{x_1}=\sum_{i=1}^{n}\frac{-g_i(x_2)}{(g_i(x_1)+g_i(x_2))^2}\times g'(x_1)$$

$$q_{x_2}=\sum_{i=1}^{n}\frac{g_i'(x_2)g_i(x_1)}{(g_i(x_1)+g_i(x_2))^2}$$

*in high dimensions, it's* $||q_{x_1}||_2=\sum_{i=1}^{n}g_i(x_2)||\nabla g_i(x_1)||_2$. *In the case that the initial state* $(x_0^1,x_0^2)$ *satisfies* $||q_{x_1}||_2=||q_{x_2}||_2$, *then* $V$ *is time independent, i.e.,* $\frac{\partial V}{\partial t}\equiv 0$. *This means the value of the game is given by* $q(x_0^1,x_0^2)$ *and it holds for any election period* $T$.

**Remark 4.4.**

$$V(x_1,x_2,t)$$

*the optimal value start from time* $t$ *and position* $x_1,x_2$, *given fixed duration* $T$.

# 5 Implementation

We consider the discrete-time version and based on the result in 4.1, for $K$ time-step, $k=0,1,\cdots K$, $x_1,x_2\in\mathbb{R}^2$ represents the candidates' position separately. $V:\{0,1,\cdots,K\}\times\mathbb{R}^4\to\mathbb{R}^+$ means the saddle value from input time-step $k$ and the input candidates' positions $x_1,x_2$ at this stage.

$$V(k,x_1,x_2)=\min_{u_k^1}\max_{u_k^2}\{V(k+1,x_1+u_k^1,x_2+u_k^2)\},\ V(K,x_1,x_2)=q(x_1,x_2),$$

$$V(0,x_1,x_2)=\underbrace{\min_{u_1^1}\max_{u_2^1}\min_{u_1^2}\max_{u_2^2}\cdots\min_{u_1^K}\max_{u_2^K}}_{K\ \text{min max}}q\left(x_1+\sum_{i=1}^{K}u_i^1,x_2+\sum_{i=1}^{K}u_i^2\right)$$

We assume the speed of each time-step is small $\epsilon$ and $u_k^j, 1\le k\le K, j=1,2$. For numerical approximation, finite random choices for each $u_k^j$ and brute-force search for the optimal strategy.

## 5.1 Example

The policy space is $[-1/2,1/2]\times[-1/2,1/2]\subset\mathbb{R}^2$,

$$q_1(x_1,x_2)=\cos^2(||x_1||_2\pi)+\sin^2(||x_2||_2\pi)$$

$$q_2(x_1,x_2)=\frac{\exp(x_{21}^2-x_{22}^2-1)}{\exp(x_{21}^2-x_{22}^2-1)+\exp(x_{11}^2-x_{12}^2-1)}$$

$$q_3(x_1,x_2)=\frac{\exp(\frac{1}{100||x_2||_2})}{\exp(\frac{1}{100||x_1||_2})+\exp(\frac{1}{100||x_2||_2})}$$

$$q=\frac{1}{4}\cdot(q_1+q_2+q_3)\in[0,1]$$

where $x_1=(x_{11},x_{12})^T, x_2=(x_{21},x_{22})^T$ and $q$ represents the payoff function of candidate $P_2$ (corrosponding to $x_2$). $P_1$ (corrosponding to $x_1$) is the minimizer and $P_2$ is the maximizer. We also have $q(x_1,x_2)+q(x_2,x_1)=1$. In the case $K=4$, $\epsilon=0.02$ (moving distance in each step), the following 2 figures show the different behaviour of the two candidates.

# References


[Sch07] Norman Schofield. The mean voter theorem: Necessary and sufficient conditions for convergent equilibrium. *The Review of Economic Studies*, 74(3):965–980, 2007.

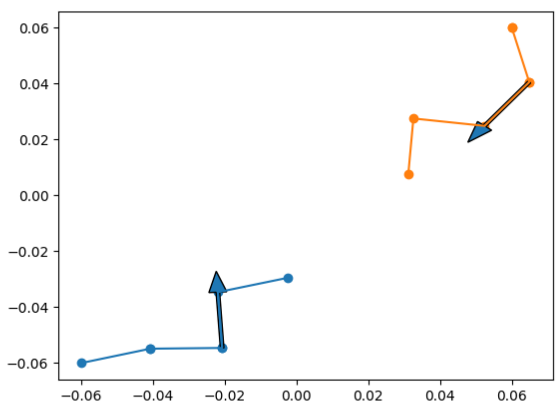


Figure 1: Initial position: $x_1 = (-0.06, -0.06)$ and $x_2 = (0.06, 0.06)$. Two candidates simultaneously move to the center (closer and closer)

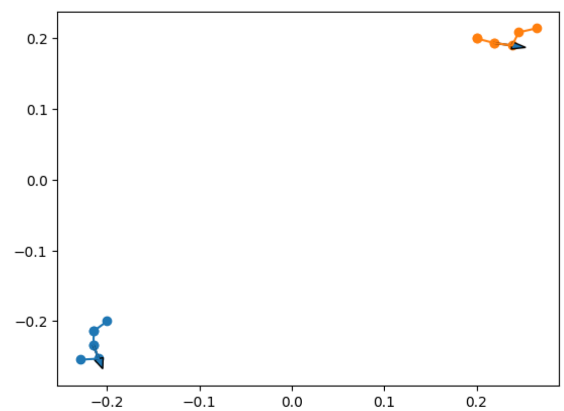


Figure 2: Initial position: $x_1 = (-0.2, -0.2)$ and $x_2 = (0.2, 0.2)$. Two candidates simultaneously move further and further