Why the Candidates move further away? They're not crazy!

October 4, 2024

Abstract

1 Introduction

2 Deterministic Model

We denote the policy space is $\mathbb{R}^m$ and the set of voters $N = \{v_1, v_2, \cdots, v_n\}$, and their utility function $u_i : \mathbb{R}^m \to \mathbb{R}$ which is smooth(or C^2, second differentiable) and concave, where $v_i \in \mathbb{R}^m$ is the ideal position (maximum point of f_i), for the i-th voter, $1 \leq i \leq n$. For the two candidates P_1 and P_2, P_1 chooses a position first, and P_2 chooses a position later. After both candidates complete their actions, the voters will vote for the candidate close to them separately. In general, for each position x such that P_1 chooses, we could find a region $S(x)$ as a subset of policy space $\mathbb{R}^m$ such that P_2 can choose any position in this region to beat P_1.

For given information of voters (i.e. all f_i), we denote $\alpha : \mathbb{R}^m \to \mathbb{R}^+, \alpha(x) = \mu(S(x))$, where $\mu(\cdot)$ is the Lebesgue measure.

Theorem 2.1. *α is a convex function. (expected)*

We denote the collections of dominant voters as $\mathcal{M} = \{M \subset N : \#M \geq \frac{1}{2}\#N\}$. For a collection of voters M and a given position x P_1 chooses, we denote the region for P_2 could win all voters in M as V_M, denote the region for P_2 could win the ith voter as $B_i(x)$.

$$B_i(x) = \{y \in \mathbb{R}^m f_i(y) \geq f_i(x)\}, \quad V_M(x) = \bigcap_{i \in M} B_i(x)$$

$$\begin{aligned}
\alpha(x) &= \mu\left(\bigcup_{M \subset \mathcal{M}} \bigcap_{i \in M} B_i(x)\right) \\
&= \mu\left(\bigcup_{M \subset \mathcal{M}} V_M(x)\right) \\
&= \sum_{M \subset \mathcal{M}} \mu\left(V_M(x)\right) - \sum_{M_1, M_2 \subset \mathcal{M}} \mu\left(V_{M_1}(x) \cap V_{M_2}(x)\right) + \cdots \\
&= \sum_{M \subset \mathcal{M}} \mu\left(V_M(x)\right) - \sum_{M_1, M_2 \subset \mathcal{M}} \mu\left(V_{M_1 \cup M_2}(x)\right) + \cdots
\end{aligned}$$

To be proofed

Lemma 1. *The area of the intersection of circles is convex w.s.t. the first candidate's position x in R^2.*

Proof. To prove the lemma, we need several steps. Step 1: the area of is convex if the movement of x does not change the shape of the region, i.e., it's a region bounded by k arcs, and k does not change w.s.t. to the movement of x. Step 2: changes of k happen only if x is on the line of two voter's positions (hence, each domain keeping k constant is a convex set). Step 3: to complete the lemma (working).

Proof of Step 1. The area of the "cap" shape region from a circle with radius r and the "cap" with chord length L is,

$$f(r, L) = \sin^{-1} \frac{L}{2r} - \frac{L}{2}\sqrt{r^2 - L^2/4},$$

which is convex and easily checked in $\mathbb{R}^2$. Such a vertex in the intersection region is symmetric to the first candidate's position x w.s.t. the line of two voters' ideal. As a result, it could be represented by a linear transform $Ax + b$, the length of the chord x_1x_2 is

$$||A_1x + b + 1 - (A_2x + b_2)||_2,$$

which is convex. (to be done). □

2.1 Weaker Version

If all voters stand at an extreme point of the set of voters' ideal position, the conclusion is true in $\mathbb{R}^2$. (could be extended to $\mathbb{R}^m$)

3 Probablistic Model

3.1 Model given by [Sch07]

The utility function of voter i with ideal position v_i and a candidate position z_j is

$$u_{ij}(v_i, z_j) = u^*_{ij}(v_i, z_j) + \varepsilon_{ij},$$

where

$$u^*_{ij}(v_i, z_j) = \lambda_i - \beta||v_i - z||^2.$$

Here, u^*_{ij} is the observable utility for voter i associated with party/candidate j. λ_j is the valence of agent/candidate/party j, and β is a positive constant. The terms $\{\varepsilon_{ij}\}$ are the stochastic errors. We assume that all ε_{ij}'s are iid drawed from distribution Ψ, which is the Type I extreme value distribution and takes the closed form

$$\Psi(h) = \exp(-(\exp -h)).$$

The probability for a voter i to choose party j is that

$$p_{ij} = \Pr[u_{ij}(v_i, z_j) > u_{il}(x_i, z_l)], l \neq j.$$

The expected vote share for agent j is $V_j(\mathbf{z}) = \sum_{i=1}^n p_{ij}$, where the $\mathbf{z}$ is the vector of all candidates' positions.

In the model given above, the expected voter share for candidate j is,

$$V_j(\mathbf{z}) = \sum_{i=1}^n \frac{\exp(u^*_{ij}(v_i, z_j))}{\sum_j \exp(u^*_{ik}(v_i, z_k))}.$$

Remark 3.1. *(The norm is convex (by definition)) and (the function x^2 is non-decreasing (in positive half-axis) and convex) jointly imply that the square of a norm is convex, which implies that u^*_{ij} is concave. The function e^x is convex and non-decreasing We denote $G : \mathbb{R}^+ \to \mathbb{R}, x \mapsto \frac{x}{x+k}$ for some positive k. It's easy to verify that G is strictly concave and increasing. We recall that u_i is strictly concave. Hence $G \circ f_i$ is strictly concave, which implies $\mathbb{E}_z(x)$ is concave.*

$$P_{ij} = \frac{e^{f_i(x_j)}}{e^{f_i(x_1)} + e^{f_i(x_2)}}, \quad j = 1, 2$$

For a position $z \in \mathbb{R}^m$ that P_1 chooses, P_2 can choose a position to maximize his expectation:

$$\mathbb{E}_z : \mathbb{R}^m \to \mathbb{R}^+, \quad \mathbb{E}_z(x) = \sum_{i=1}^n \frac{e^{f_i(x)}}{e^{f_i(x)} + e^{f_i(z)}}$$

However, the objective function maynot be concave.

3.2 Modificated Model

All utility functions f_i are strictly concave, positive, and second-differentiable.

If the two candidate choose position x_1 and x_2, denote the probability for voter i to voter candidate P_j is P_{ij}, $1 \leq i \leq n, j = 1, 2$,

$$P_{ij} = \frac{f_i(x_j)}{f_i(x_1) + f_i(x_2)}, \quad j = 1, 2$$

For a position $z \in \mathbb{R}^m$ that P_1 chooses, P_2 can choose a position to maximize his expectation:

$$\mathbb{E}_z : \mathbb{R}^m \to \mathbb{R}^+, \quad \mathbb{E}_z(x) = \sum_{i=1}^{n} \frac{f_i(x)}{f_i(x) + f_i(z)}$$

Theorem 3.1. *Given a fixed position z, the function $\mathbb{E}_z(x)$ is (strictly) concave.*

Proof. We denote $G : \mathbb{R}^+ \to \mathbb{R}, x \mapsto \frac{x}{x+k}$ for some positive k. It's easy to verify that G is strictly concave and increasing. We recall that f_i is strictly concave. Hence $G \circ f_i$ is strictly concave, which implies $\mathbb{E}_z(x)$ is concave. □

Theorem 3.2. *(Pure Strategy Nash Equilibrium) There exist a unique $x^* \in \mathbb{R}^m$ such that (x^*, x^*) is the pure strategy Nash equilibrium, i.e.*

$$\arg\max_{x \in \mathbb{R}^m} \mathbb{E}_{x^*}(x) = \arg\max_{x \in \mathbb{R}^m} \sum_{i=1}^{n} \frac{f_i(x)}{f_i(x) + f_i(x^*)} = x^*$$

which implies $\mathbb{E}_{x^*}(x^*) = n/2$

Proof. Step 1: Let $V \subset \mathbb{R}^m$ be the convex hull of the ideal positions of voters. Denote $h : V \to \mathbb{R}^m$, $h(z) = \arg\max_{x \in \mathbb{R}^m} \mathbb{E}_z(x)$. h is well-defined due to the strictly convexity of $\mathbb{E}_z(x)$. Claim $h(z) = \arg\max_{x \in V} \mathbb{E}_z(x)$ such that $h : V \to V$.
Step 2: Prove h is continous w.r.t. z. Done by implicit function theorem with the condition $\mathbb{E}_z(x)$ is second differentiable w.r.t. x.
Step 3: h is a continuous function from a convex compact set to itself; by Brouwer's fixed-point theorem, there exists a fixed point x^* such that $h(x^*) = x^*$. Uniqueness is trivial.
Brouwer fixed-point theorem: Every continuous function from a nonempty convex compact subset K of a Euclidean space to K itself has a fixed point. □

Remark 3.2. *The theorem could be proved by the contraction mapping theorem.*

Remark 3.3. *Example:*

$$\mathbb{E}_x(y) = y^2 - x^2 + \frac{n}{2}, \, -1 < x, y < 1$$

where $x, y \in \mathbb{R}^1$ represents the two candidates' position.$(0,0)$ is a PNE here. $\mathbb{E}_x(y) + \mathbb{E}_y(x) = n, E_x(x) = \frac{n}{2}$.

Proposition 3.2.1. *the gradient of the objective function $\nabla \mathbb{E}_z$ is lipschiz continous.*

Remark 3.4. *The gradient descent algorithm solves the maximization point of $\mathbb{E}_z$.*

4 Deterministic Dynamic Game

4.1 Discrete-Time Game + Probablistic Model

Cor 6.2 states a sufficient and necessary condition for the existence of an optimal strategy. The strategies $\gamma_k^{j*}, k \in K, j = 1, 2$, where K is the total time phase, provide a saddle-point solution if and only if, there exists function $V(k, \cdot) : \mathbb{R}^{2m} \to \mathbb{R}, k \in K$ s.t.

$$V(k, x) = \min_{u_k^1} \max_{u_k^2} \{V(k+1, x + u_k^1 + u_k^2)\}, \, V(K, x) = q(x), \, V(K+1, x) = 0$$

where $x_{k+1} = f_k(x_k, u_k^1, u_k^2) = x_k + u_k^1 + u_k^2$. The unique saddle point value of the game is $V(1, x_1)$, where x_1 is the initial state.

4.2 Continuous-Time Game + Probablistic Model

We fit the probabilistic model in the dynamic game. The set of n voters is $N = \{1, 2, \cdots, n\}$ and the policy space is $\mathbb{R}^m$. Predisribed fixed time T to end the voting game. The state space is $\mathbb{R}^{2m}$, which describes the two candidates' position positions (first m-entries for P_1 and the latter for P_2). The strategy space,

$$\Gamma^1 = \{\gamma : [0, T] \times \mathbb{R}^{2m} \to \mathbb{R}^m \oplus \mathbf{0}\}\big|\gamma \text{ is continuously differentiable, Image}(\gamma) \subset B_{\mathbf{0}}(1)\}$$

$$\Gamma^2 = \{\gamma : [0, T] \times \mathbb{R}^{2m} \to \mathbf{0} \oplus \mathbb{R}^m\big|\gamma \text{ is continuously differentiable, Image}(\gamma) \subset B_{\mathbf{0}}(1)\}$$

For any pure strategy the candidates choose, it's continuously differentiable, and the speed of action is less than 1. For any given pure strategy γ^1, γ^2, the evolution function is defined as,

$$\frac{dx}{dt} := f(t, x, u_1, u_2) = u_1 + u_2 = \gamma^1(t, x) + \gamma^2(t, x)$$

the f is the general notation used in dynamic games. We denote the objective function,

$$L = \int_0^T g(t, x, u^1, u^2)\, dt + q(T, x(T)$$

at here, $g \equiv 0$ and T is fixed. To make it simple, we just write $L = q(x(T))$. We consider the probabilistic voting model and the objective function L is P_2's expectation. For example, let f_i be the utility functions for each voter and, $p_1, p_2 \in \mathbb{R}^m$ are the final position of the two candidates,

$$L = q(x(T)) = q(p_1, p_2) = \sum_{i=1}^{n} \frac{f_i(p_2)}{f_i(p_1) + f_i(p_2)}$$

then P_1 is the minimizer, and P_2 is the maximizer, since it's a zero-sum game.

Theorem 4.1. *Cor6.6 (Equation 6.75, for feedback pattern) in Tamer Basar: The sufficient condition for γ^{1*}, γ^{2*} to be the optimal (the saddle strategy) is, there exists continuously differentiable function $V : [0, T] \times \mathbb{R}^{2m} \to \mathbb{R}$ satisfies the Bellman's equation:*

$$\begin{aligned} -\frac{\partial V}{\partial t} &= \min_{u^1} \max_{u^2} \langle \frac{\partial V}{\partial x}, u^1 + u^2 \rangle \\ &= \max_{u^2} \min_{u^1} \langle \frac{\partial V}{\partial x}, u^1 + u^2 \rangle \\ &= \langle \frac{\partial V}{\partial x}, \gamma^{1*}(t, x) + \gamma^{2*}(t, x) \rangle \end{aligned}$$

where the $\langle \cdot \rangle$ denotes the general inner product in the Euclidean space.

Remark 4.1. *The interchangeability is named Issac's condition. Denote the first m-entries variables is x_1, the latter m-entries is x_2. The Issac's condition is automatically satisfied,*

$$\min_{u^1} \max_{u^2} \langle \frac{\partial V}{\partial x}, u^1 + u^2 \rangle = \min_{u^1} \max_{u^2} \left[\langle \frac{\partial V}{\partial x_1}, u^{1'} \rangle + \langle \frac{\partial V}{\partial x_2}, u^{2'} \rangle \right] = \min_{u^1} \langle \frac{\partial V}{\partial x_1}, u^{1'} \rangle + \max_{u^2} \langle \frac{\partial V}{\partial x_2}, u^{2'} \rangle$$

where $u^{1'}$ is the vector in $\mathbb{R}^m$ consisted of first m-entries of u^1. A similar definition is applied on $u^{2'}$.

Remark 4.2. *We consider the constraint that $||u^1||_2, ||u^2||_2 \leq 1$,, we can simplifize*

$$\begin{aligned} \min_{u^1} \langle \frac{\partial V}{\partial x_1}, u^{1'} \rangle \geq -||u^1||_2 \cdot ||\frac{\partial V}{\partial x_1}||_2 \geq -||\frac{\partial V}{\partial x_1}||_2 \\ \max_{u^2} \langle \frac{\partial V}{\partial x_2}, u^{2'} \rangle \leq ||\frac{\partial V}{\partial x_2}||_2 \end{aligned}$$

which are both from Cauchy-Schwarz inequality and take equality when $u^{1'}$ lies on the oppsite direction of $\frac{\partial V}{\partial x_1}$ and $u^{2'}$ lis on the same direction $\frac{\partial V}{\partial x_2}$. The PDE is simplified as,

$$
\begin{aligned}
-||\frac{\partial V}{\partial x_1}||_2 + ||\frac{\partial V}{\partial x_2}||_2 &= -\frac{\partial V}{\partial t} \\
initial\,condition: V(T,x) &= q(x), \forall x \\
boundary\,condition: V(x_1,x_2,t) &= 1\,for\,large\,|x_1|, \\
V(x_1,x_2,t) &= 0\,for\,large\,|x_2|,
\end{aligned}
$$

the boundary condition is due to x_1 being the minimizer, and x_2 being the maximizer. The ideal position should be in the convex hull of voters' ideal positions.

Remark 4.3.

$$V(x,T) = q(x) = q(x_1,x_2) = \sum_{i=1}^{n} \frac{g_i(x_2)}{g_i(x_1)+g_i(x_2)}, q_{x_1} = \sum_{i=1}^{n} \frac{-g_i(x_2)}{(g_i(x_1)+g_i(x_2))^2} \times g'(x_1)$$

$$q_{x_2} = \sum_{i=1}^{n} \frac{g_i'(x_2)g_i(x_1)}{(g_i(x_1)+g_i(x_2))^2}$$

in high dimensions, it's $||q_{x_1}||_2 = \sum_{i=1}^{n} g_i(x_2)||\nabla g_i(x_1)||_2$. In the case that the initial state (x_0^1, x_0^2) satisfies $||q_{x_1}||_2 = ||q_{x_2}||_2$, then V is time independent, i.e., $\frac{\partial V}{\partial t} \equiv 0$. This means the value of the game is given by $q(x_0^1, x_0^2)$ and it holds for any election period T.

Remark 4.4.

$$V(x_1,x_2,t)$$

the optimal value start from time t and position x_1, x_2, given fixed duration T.

4.3 The Step-function model

Suppose each voter gets fixed utility only if the candidate close enough to her position. Formally, we define the distribution of voters by a density function $p(v)$ for the probability density of voters with ideal position v in the policy space. We define the threshold $r(v)$ for voters with ideal position v. The shifted expeted value (to shift the constant-sum game into a zero-sum game) from voters with ideal positions v for candidate 2 is:

$$
w(v) = \begin{cases} p(v)/2, & \text{if}\,||P_2 - v||_2 \le r(v)\,\text{and}\,|P_1 - v||_2 > r(v), \\ -p(v)/2, & \text{if}\,||P_2 - v||_2 > r(v)\,\text{and}\,|P_1 - v||_2 \le r(v), \\ 0, else \end{cases}
$$

Remark 4.5. *No open-loop or closed-loop pattern here. All state evolsion are deterministic, instead of including a stochastic error. Theoretically, all candidates know the outcome at the beginning with given utility model, voters' distribution, initial position and voting duration.*

Theorem 4.2. *One candidate's initial position make no difference to the other's strategy.*

Proof. Recall that it's a zero-sum dynamic game and P_1 is minimizer with final positon x_1, P_2 is maximizer with final position x_2, the objective function is

$$
\begin{aligned}
\int w(v)p(v)dv = &\frac{1}{2}\mathbf{P}(\{v : ||x_2 - v||_2 \le r(v), ||x_1 - v||_2 > r(v)\}) - \\
&\frac{1}{2}\mathbf{P}(\{v : ||x_1 - v||_2 \le r(v), ||x_2 - v||_2 > r(v)\}) \\
= &\frac{1}{2}\left(\mathbf{P}(\{v : ||x_2 - v||_2 \le r(v)\}) - \mathbf{P}(v : ||x_1 - v||_2 \le r(v), ||x_2 - v||_2 \le r(v))\right) \\
&- \frac{1}{2}\left(\mathbf{P}(\{v : ||x_1 - v||_2 \le r(v)\}) - \mathbf{P}(v : ||x_1 - v||_2 \le r(v), ||x_2 - v||_2 \le r(v))\right) \\
= &\frac{1}{2}\mathbf{P}(\{v : ||x_2 - v||_2 \le r(v)\}) - \frac{1}{2}\mathbf{P}(\{v : ||x_1 - v||_2 \le r(v)\}).
\end{aligned}
$$

Hence, the value of the game is

$$\min_{P_1}\max_{P_2}\frac{1}{2}\mathbf{P}(\{v:||x_2-v||_2\le r(v)\})-\frac{1}{2}\mathbf{P}(\{v:||x_1-v||_2\le r(v)\})$$
$$=\max_{P_2}\frac{1}{2}\mathbf{P}(\{v:||x_2-v||_2\le r(v)\})-\max_{P_1}\frac{1}{2}\mathbf{P}(\{v:||x_1-v||_2\le r(v)\}).$$

The two candidates will try their best to an admissible position with maximized influence separately. As a result, the candidate P_1 and P_2 with limited moving speed ε and prescribed voting duration T and initial position x_{10}, x_{20} will move to the position

$$\arg\max_{y\in B_{x_10}(T\varepsilon)}\frac{1}{2}\mathbf{P}(\{v:||y-v||_2\le r(v)\}),$$
$$\arg\max_{y\in B_{x_20}(T\varepsilon)}\frac{1}{2}\mathbf{P}(\{v:||y-v||_2\le r(v)\}).$$

The initial and final positions determine the candidates' behaviors. □

Remark 4.6. *Given the centers and radius, the problem is finding the point/region to be covered by the most circles. Questions: in discrete cases, the problem is NP/NP-hard. Approximation/randomized algorithm.*

Remark 4.7. *In this case, why does the candidate not initially stand at the optimal point instead of doing dynamic programming?*

Remark 4.8. *What if the candidate gains benefit from each step? (instead of just considering the final value, in the sense that the current supporting rate impacts the candidates' sponsor/funding).*

Remark 4.9. *What if the threshold is dynamic? For example, the threshold with initial value w is $\max\{0, w+B_{t\varepsilon}\}$ for $t\in[0,T]$, where T is the voting duration and ε is a small number, and B refers to the standard Brownian motion.*

Remark 4.10. *In case all voters share the same threshold of 1, the two final positions change into*

$$\arg\max_{y\in B_{x_10}(T\varepsilon)}\frac{1}{2}\int_{B_y(1)}p(v)\,dv,\quad \arg\max_{y\in B_{x_20}(T\varepsilon)}\frac{1}{2}\int_{B_y(1)}p(v)\,dv.$$

Remark 4.11. *The step-function model makes sense to the more general model $u\sim\exp(-d^2)$, where d is the distance from the voters' ideal to the candidate's.*

5 Implementation

We consider the discrete-time version and based on the result in 4.1, for K time-step, $k=0,1,\cdots K$, $x_1, x_2\in\mathbb{R}^2$ represents the candidates' position separately. $V:\{0,1,\cdots,K\}\times\mathbb{R}^4\to\mathbb{R}^+$ means the saddle value from input time-step k and the input candidates' positions x_1, x_2 at this stage.

$$V(k,x_1,x_2)=\min_{u_k^1}\max_{u_k^2}\{V(k+1,x_1+u_k^1,x_2+u_k^2)\},\ V(K,x_1,x_2)=q(x_1,x_2),$$

$$V(0,x_1,x_2)=\underbrace{\min_{u_1^1}\max_{u_2^1}\min_{u_1^2}\max_{u_2^2}\cdots\min_{u_1^K}\max_{u_2^K}}_{K\ \min\max}q\left(x_1+\sum_{i=1}^K u_i^1, x_2+\sum_{i=1}^K u_i^2\right)$$

We assume the speed of each time-step is small ϵ and $u_k^j, 1\le k\le K, j=1,2$. For numerical approximation, finite random choices for each u_k^j and brute-force search for the optimal strategy.

5.1 Example

The policy space is $[-1/2, 1/2] \times [-1/2, 1/2] \subset \mathbb{R}^2$,

$$\begin{aligned}
q_1(x_1, x_2) &= \cos^2(||x_1||_2\pi) + \sin^2(||x_2||_2\pi) \\
q_2(x_1, x_2) &= \frac{\exp(x_{21}^2 - x_{22}^2 - 1)}{\exp(x_{21}^2 - x_{22}^2 - 1) + \exp(x_{11}^2 - x_{12}^2 - 1)} \\
q_3(x_1, x_2) &= \frac{\exp(\frac{1}{100||x_2||_2})}{\exp(\frac{1}{100||x_1||_2}) + \exp(\frac{1}{100||x_2||_2})} \\
q &= \frac{1}{4} \cdot (q_1 + q_2 + q_3) \in [0, 1]
\end{aligned}$$

where $x_1 = (x_{11}, x_{12})^T, x_2 = (x_{21}, x_{22})^T$ and q represents the payoff function of candidate P_2 (corrosponding to x_2). P_1 (corrosponding to x_1) is the minimizer and P_2 is the maximizer. We also have $q(x_1, x_2) + q(x_2, x_1) = 1$. In the case $K = 4$, $\epsilon = 0.02$ (moving distance in each step), the following 2 figures show the different behaviour of the two candidates.

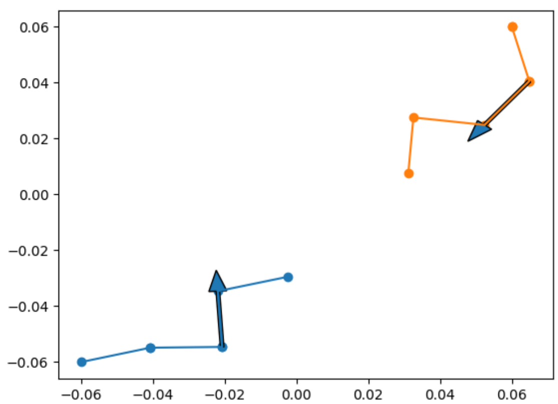

Figure 1: Initial position: $x_1 = (-0.06, -0.06)$ and $x_2 = (0.06, 0.06)$. Two candidates simultaneously move to the center (closer and closer)

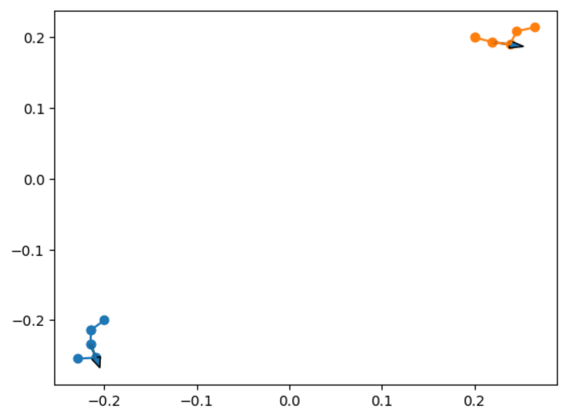

Figure 2: Initial position: $x_1 = (-0.2, -0.2)$ and $x_2 = (0.2, 0.2)$. Two candidates simultaneously move further and further

References

[Sch07] Norman Schofield. The mean voter theorem: Necessary and sufficient conditions for convergent equilibrium. *The Review of Economic Studies*, 74(3):965–980, 2007.