Introduction to Multi-Structured Signal Recovering

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IRE3500 Presentation, Dec 2022

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Problem Introduction

In signal recovery, given the original data x_0, we want to find a non-negative convex function f, always to be a norm, through some gaussian measurement:

Definition

Consider the convex program

$$\hat{x} = \text{argmin}_{x \in C} f(x) \quad s.t.\, Ax = Ax_0$$

when $C \subset \mathbb{R}^n$ and A is an $m \times n$ random matrix. m is the number of gaussian measurements. We want to find good f such that $\hat{x} = x_0$ at high probability when m is greater than some required number(complexity).

Well-Known Result for one structure signal

List of Well-Known Result			
Model	Convex Method	Complexity	Degree of Freedom
s-Sparse vector in $\mathbb{R}^n$	l_1 norm	$O(s\log(n/s))$	s
$n\times n$ rank-r matrix	Nuclear norm	$O(nr)$	$r(2n-r)$
$n\times n$ permutation matrix	Norm induced by Birkhoff polytope	$O(n\log n)$	$n!$
$n\times n$ orthogonal matrix	Spectral norm	$O(n^2)$	$n(n-1)/2$

- There is a norm corresponding to each structure, as the convex method.
- If a signal has multi structures, it's natural to think of a combination of these norms.
- Next, we handle the recovery problem of the multi-structured signal by the combination of their corresponding norms
- However, we will conclude that applying multi norms to recover the signal does not work than the dominant norm in complexity

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Simultaneously Structured Models

Consider a signal x_0 with several low-dimensional structured $S_1, S_2, \cdots, S_d$, which correspond to norm $||\cdot||_{(1)}, ||\cdot||_{(2)}, \cdots, ||\cdot||_{(d)}$. Such an x_0 is defined as a simultaneously structured model.

Definition

Denote the convex recovery program:

$$\arg\min_{x \in C} f(x) = h\left(||x||_{(1)}, ||x||_{(2)}, \cdots, ||x||_{(d)}\right) \quad s.t.\, Ax = Ax_0 \tag{1}$$

where $h : \mathbb{R}_+ \to \mathbb{R}_+$ is convex and non-decreasing in each measurement.

Deterministic Failure [OJF+15]

Deterministic Failure

Suppose $C = \mathbb{R}^n$ and,

$$\inf_{g \in \partial f(x_0)} |\bar{g}^T \bar{x}_0| > \frac{||A\bar{x}_0||_2}{\sigma_{min}(A^T)}$$

Then x_0 is NOT a minimizer of (1).

- where $\partial f(x_0)$ is the subgradient of f at x_0, define as,

$$\partial f(x) = \{g \in C : f(z) \geq f(x) + \langle g, z - x \rangle\}$$

Also could be written as,

$$\partial f(x) = \{g \in C : \langle g, x \rangle = f(x), \sup_{f(u) \leq 1} \langle g, u \rangle \leq 1\}$$

- the subgradient cone is the polar cone of the decent cone.

Proof of Deterministic Failure

- Mostly the proof is from KKT condition of x_0 to be the minimizer.
- Suppose not, from KKT condition of x_0 to be a minizer of (1), v, $z, g \in \partial f(x_0)$,s.t.

$$g - v - A^T z = 0, \quad x_0^T v = 0$$

- Then, by some usual argument, we will have the inequality:

$$\frac{|\bar{x_0}^T g|}{||g||_2} \leq \frac{||A\bar{x}_0||_2||z||_2}{||g||_2} \leq \frac{||A\bar{x}_0||_2||z||_2}{||P(A^T z)||_2} \leq \frac{||A\bar{x}_0||_2}{\sigma_{min}(A^T)}$$

  which violates the original condition, where P is the project to span$\{x_0\}$.
- Noticed deterministic failure says that if $\sigma_{min}(A^T)$ satisfies some condition, then the recovery program would fail for any function f in the class.

Gaussian Lower Bound

- it's natural to ask us to combine some results in random matrix theory.

Definition of Lipschitz constant of a norm $||\cdot||$

$$L = \sup_{x \neq y \in \mathbb{R}^{\ltimes}} \frac{||x|| - ||y||}{||x - y||_2}$$

Lemma

Let L_i be the Lipschitz constant of the i'th norm and $\kappa_i = \frac{||\bar{x}_0||_{(i)}}{L_i}$ for $1 \leq i \leq d$, $\kappa_{min} = \min \kappa_i$, then

$$\inf_{g \in \partial f(x_0)} |\bar{g}^T \bar{x}_0| \geq \kappa_{min}$$

for any f in the class $h\left(||x||_{(1)}, ||x||_{(2)}, \cdots, ||x||_{(d)}\right)$.

Gaussian Lower Bound

Gaussian Lower Bound[OJF+15]

Assume A is a random matrix, i.e. entries of A are independent and $\mathcal{N}(0,1)$. Then x_0 would not be minimizer of any f in (1) with probability at least $1 - 10\exp(-\frac{1}{16}\min\{m_{low}, (1-\bar{\mathrm{D}}(C))^2)\}$, where

$$m_{low} := \frac{(1-\bar{\mathrm{D}}(C))n\kappa_{min}^2}{100} \sim O(n\kappa_{min}^2)$$

Noticed $\bar{\mathrm{D}}(C) = 0$ when $C = \mathbb{R}^n$. The result conclude $m_{low} = O(n\kappa_{min}^2)$.

- Actually, we see that it does not help to contain the domain in a cone, which is only different from a constant factor.

Proof of Gaussian Lower Bound

It mostly applies deterministic failure, and some result in random matrix. We focus on the case $C = \mathbb{R}^n$. By theorem from the random matrix,

$$\begin{aligned} &P(\sigma_{max} P(A^T)) \leq 1.5\sqrt{m} + 1.5\sqrt{1} \leq 2.5\sqrt{m}) \\ &= P(\sigma_{min}(A^T) \geq \sqrt{n} - 1.5\sqrt{m}) \\ &= 1 - 2e^{-m/8} \end{aligned}$$

such that

$$\kappa_{min} \leq \frac{||A\bar{x}_0||_2}{\sigma_{min} A^T} = \frac{\sigma_{max}(P(A^T))}{\sigma_{min} A^T} \leq \frac{2.5\sqrt{m}}{\sqrt{n} - 1.5\sqrt{m}}$$

Recall $\inf_{g \in \partial f(x_0)} |\bar{g}^T \bar{x}_0| \geq \kappa_{min}^2$, hence

$$(\sqrt{n} - 1.5\sqrt{m})\kappa_{min} \leq 2.5\sqrt{m} \Longrightarrow c\sqrt{n}\kappa_{min} \leq \sqrt{m} \Longrightarrow m \geq c' n \kappa_{min}$$

Performance

Gaussian Lower Bound Method		
Model	the function f	$n\kappa^2$
s-Sparse vector in $\mathbb{R}^n$	l_1 norm	s
r-rank matrix in $\mathbb{R}^{n\times n}$	Nuclear norm	nr
k, k-sparse, r-rank matrix	$h(\|\|\cdot\|\|_*, \|\|\cdot\|\|_1)$	$\min\{k^2, nr\}$

- In the meantime, it shows that $O(m_{low}) \geq O(dominant\ norm)$, meaning if the gaussian measurement is lower than the recover complexity of the dominant norm, we would have a high probability that all functions in (1) fail to recover the original signal.
- Applying multi norms to recover the signal would not work than the dominant norm in complexity.

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What is Low-rank Tensor

- We now focus the case on tensor, only with low-rank structure. First, we need to denote what is the rank of a tensor.
- Usually, there are 2 definition, called the CP rank and Tucker rank.

Definition of CP rank

For a k-way tensor $\mathcal{X} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_k}$:

$$\text{rank}_{cp}(\mathcal{X}) = \min \left\{ r : \mathcal{X} = \sum_{i=1}^{r} a_1^{(i)} \circ a_2^{(i)} \circ \cdots \circ a_k^{(i)} \right\}$$

- It's more like the traditional definition of rank. But the nuclear norm of a tensor in CP rank is intractable since it's NP-hard in general, out of the polynomial time.[HL09]

Tucker Rank

- We focus on a numerical computation on another structure, recovering tensor with a small tucker rank. Consider the Tucker decomposition:

$$\mathcal{X} = \mathcal{C} \times_1 \mathcal{U}_1 \times_2 \mathcal{U}_2 \times_3 \cdots \times_k \mathcal{U}_k$$

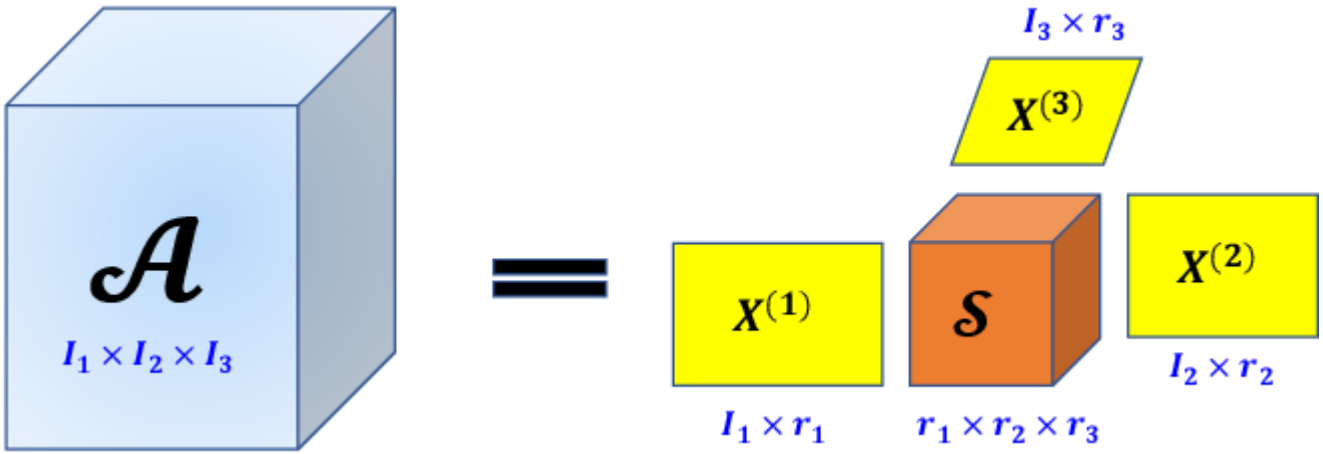
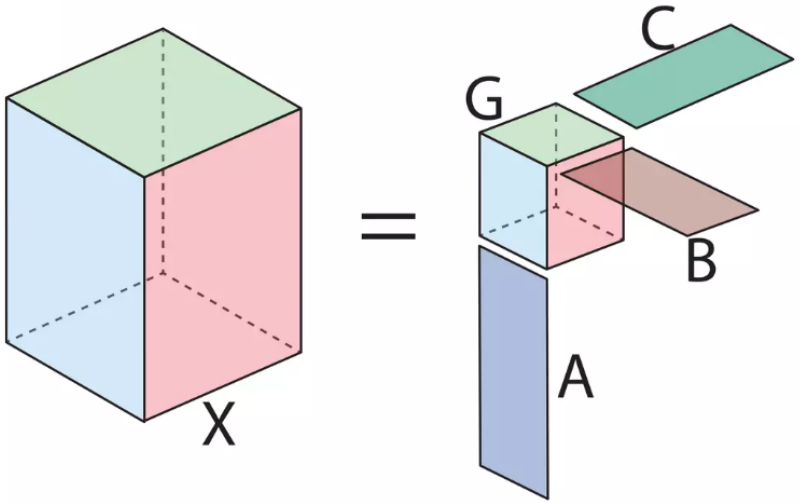


Figure: Tucker Rank-1

Tucker Rank

Tensor rank: Tucker-rank



Tucker-decomposition (Tucker, 1966)

$\mathcal{X}_{ijk} = \sum_{l=1}^{r_1} \sum_{m=1}^{r_2} \sum_{n=1}^{r_3} g_{lmn} a_{il} b_{jm} c_{kn} =: [[G; A, B, C]]$.

- G is called *core tensor*.
- Tucker-rank $= (r_1, r_2, r_3)$

Tucker Rank

- We focus on a numerical computation on another structure, recovering tensor with a small tucker rank. Consider the Tucker decomposition:

$$\mathcal{X} = \mathcal{C} \times_1 \mathcal{U}_1 \times_2 \mathcal{U}_2 \times_3 \cdots \times_k \mathcal{U}_k$$

Unfold the tensor in different directions; we can obtain

$$\mathcal{X}_{(j)} = (U_j)\mathcal{C}_{[j]}(\mathcal{U}_k \otimes \cdots \mathcal{U}_{j+1} \otimes \mathcal{U}_{j-1} \otimes \cdots \otimes \mathcal{U}_1)^* \in \mathbb{R}^{n_j \times \prod_{i \neq j} n_i}$$

Definition of Tucker Rank [Tuc66]

$$\text{rank}_{tc}(\mathcal{X}) := \left(\text{rank}(\mathcal{X}_{(1)}), \text{rank}(\mathcal{X}_{(2)}), \cdots, \text{rank}(\mathcal{X}_{(k)})\right)$$

- we say tensor $\mathcal{X}$ is low-rank, i.e. tucker rank r means $\text{rank}(\mathcal{X}_{(j)}) \leq r$ for $1 \leq j \leq k$.

Recovery of Low-rank Tensor

Definition of Tucker Rank

$$\text{rank}_{tc}(\mathcal{X}) := \left(\text{rank}(\mathcal{X}_{(1)}, \text{rank}(\mathcal{X}_{(2)}, \cdots, \text{rank}(\mathcal{X}_{(k)}\right)$$

- For a low-rank tensor, actually, we have k structured on it. If we just use one of them by convex relaxation and use nuclear norm to recover it, we can recover the low-rank tensor at complexity $O(n^{k-1}r)$. It's the same as recovering a rank r, $n \times n^{k-1}$ matrix.
- But only one structure is used in this method; we want to make some improvements. One way is to find a structure that includes all the k structures, i.e., a set that includes the intersection of the k corresponding set and make convex relaxation on the new set. We want to use nuclear norm to recover the signal, which depends on how large the matrix is. The method is to make the matrix more square.

Square Norm

For the tensor $\mathcal{X}$ with Tucker decomposition:

$$\mathcal{X} = \mathcal{C} \times_1 \mathcal{U}_1 \times_2 \mathcal{U}_2 \times_3 \cdots \times_k \mathcal{U}_k \in \mathbb{R}^{n_1 \times n_2 \times, \cdots, \times n_k}$$

Denote

$$\mathcal{X}_{[j]} = (U_j \otimes \cdots \otimes U_1)\mathcal{C}_{[1,2,\cdots,j]}(\mathcal{U}_k \otimes \cdots \mathcal{U}_{j+1})^* \in \mathbb{R}^{\prod_{i=1}^{j} n_i \times \prod_{i=j+1}^{k} n_i}$$

$\text{rank}(A \otimes B) = \text{rank}(A)\ \text{rank}(B)$, $\text{rank}(\mathcal{X}_{[j]}) \leq \min\{\prod_{i=1}^{j} r_i, \prod_{i=j+1}^{k} r_i\}$. And we write $\mathcal{X}_\square = \mathcal{X}_{[[\frac{k}{2}]]}$ and $||\mathcal{X}_\square|| := ||\mathcal{X}_{[[\frac{k}{2}]]}||_*$. Since $\mathcal{X}_\square$ is a low-rank matrix, we recover it by nuclear norm,

Square Norm [MHWG13]

$$\text{minimize}\, ||\mathcal{X}||_\square \quad s.t.\, A\mathcal{X} = A\mathcal{X}_0$$

with $m \geq Cr^{\lfloor \frac{k}{2} \rfloor} n^{\lceil \frac{k}{2} \rceil}$, we can recover $\mathcal{X}$ with a high probability.

Optimality?

- We discuss the optimality of the square norm method. Assuming that k is an even number, it divides the k into two parts and makes convex relaxation for each part, i.e. the $\frac{k}{2}$ tucker rank structure becomes the rank of their tensor product is smaller than $r^{k/2}$. And noticed the two-part have the same convex relaxation.
- In the process, there are two steps that may loss information (make the set of convex relaxation too large). The first is dividing the structures into 2 parts, and make convex relaxation separately, and the second is the union of the two sets of convex relaxation. The square norm method will not enlarge in the second step, meaning somehow optimality.

Prospective-I

- Consider the special case $k = 3$, then the square method does not work and the traditional convex nuclear norm method requires complexity $O(rn^2)$, i.e. recover a rank r, $n \times n^2$ matrix. But the non-convex method could do it better, in $O(rn^{1.5})$. We want to explore whether the convex method could do it either.
- On the one hand, is the best function; it shows direct use of nuclear norm in tucker decomposition structures will recover the problem in high probability, but $O(rn^2)$. We want to make more improvements to the structures. Simply combining the 3 structures would not help since 3 is an odd number concerning the square norm method.

Prospective-II

- One possible way is to use 2 of the 3 structures, i.e. the intersection of two structures, and make convex relaxation on the intersection set.
- One possible way is try to reshape the $\mathbb{R}^{n_1 \times n_2 \times n_3}$ tensor in a $n^{1.5} \times n^{1.5}$, even rank r^2 matrix. Then applying the nuclear norm, we can recover the matrix in complexity $O(r^2 n^{1.5})$. It's related to some fundamental properties of the tensor theory.
- Another way is to jump out of the nuclear norm to find out another way to recover the low-rank signal.

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Definition of Atomic Norm [CRPW12]

It was first mention by V. Chandrasekaran in 2012, many people use atomic norm to compute many things after that.
Consider a collection of atoms, named as an atom set $\mathcal{A}$, is a compact subset of $\mathbb{R}^p$. We can assume that the element in $\mathcal{A}$ is all extreme points of conv($\mathcal{A}$). Let $||x||_{\mathcal{A}}$ denote the gauge of $\mathcal{A}$:

$$||x||_{\mathcal{A}} = \inf\{t > 0 : x \in t\, conv(\mathcal{A})\}$$

where, WLOG, the centroid of conv(A) is at origin. We assume more than the value of the gauge is $+\infty$ when x does not lie in the affine hull of conv(A). Furthermore, the gauge can be written as

$$||x||_{\mathcal{A}} = \inf\{\sum_{a\in\mathcal{A}} c_a : x = \sum_{a\in\mathcal{A}} c_a a, c_a \geq 0, \forall a \in \mathcal{A}\}$$

we have $||\cdot||_{\mathcal{A}}$ is a norm if A is centrally symmetry, called as atomic norm induced by $\mathcal{A}$.

Example of Performance of Atomic Norm

- Given a structured signal, we can find a suitable atom set $\mathcal{A}$, usually consist of the simplest models, then use the corresponding atomic norm to recover it. In many cases, the atomic norm is the same as the traditional norm, the convex method we use to recover the signal.

Sparse Vector

Consider the atomic set $\mathcal{A} \subset \mathbb{R}^p$ consist of all the unit vectors $\{\pm e_i\}$, it's same as l^1 norm in $\mathbb{R}^p$

Low-Rank Matrix

Consider the set $\mathcal{A}$ of rank-one matrices with unit length under Euclidean norm. Then $\text{conv}(\mathcal{A})$ is the nuclear norm ball, consisting of matrices with the sum of singular values, i.e. the length under the nuclear norm, less than or equal to one.

Important Property

Given original data x_0, $\hat{x} = x_0$ is the unique optimal solution of

$$\arg\min ||x||_{\mathcal{A}} \quad s.t.\ \Phi x = \Phi x_0$$

if and only if $\text{null}(\Phi) \cap C(||\cdot||_{\mathcal{A}}, x_0) = \{0\}$, where $C(||\cdot||_{\mathcal{A}}, x_0)$ is the decent cone of $||\cdot||_{\mathcal{A}}$ at x_0, $C(||\cdot||_{\mathcal{A}}, x_0) = cone\{v : ||x_0 + v||_{\mathcal{A}} \leq ||x_0||_{\mathcal{A}}\}$.

Proof.

We can rewrite the above recovery program in the form:

$$\min_z ||x_0 + z||_{\mathcal{A}} \quad s.t.\ z \in \text{null}(\Phi)$$

Suppose $\text{null}(\Phi) \cap C(||\cdot||_{\mathcal{A}}, x_0) = \{0\}$. Notice $||x_0 + z||_{\mathcal{A}} \leq ||x_0||_{\mathcal{A}}$ implies $z \in C(||\cdot||_{\mathcal{A}}, x_0)$, we have $||x_0 + z||_{\mathcal{A}} \geq ||x_0||_{\mathcal{A}}, \forall z \in \text{null}(\Phi)\backslash\{0\}$.
Conversely, x_0 is the unique optimal solution means $||x_0 + z||_{\mathcal{A}} \geq ||x_0||_{\mathcal{A}}$, $\forall z \in \text{null}(\Phi)\backslash\{0\}$, implies the other direction.

Why Atomic Norm?-I

- Smaller decent cone at x_0, easier to satisfy the empty intersection condition.
- Points in $conv(\mathcal{A})$ with smaller decent cones corresponding to simpler models, the larger corresponds to the more complicated. Extreme points of $conv(\mathcal{A})$ is the simplest models.
- The low-dimensional faces of $conv(\mathcal{A})$ consist of those elements of a linear combination of a few basic atoms of $\mathcal{A}$. It means points in low-dimensional faces have smaller decent cones than those points in high-dimensional faces.

Why Atomic Norm?-II

- The atomic norm is somehow the best possible convex method. Consider atom set $\mathcal{A}$ consist of all simplest models, any reasonable convex recovery method should have same value on all these points. These means no any other atom is preferred.
- For any $a \in \mathcal{A}$, all $a' - a, a' \in \mathcal{A}$ is the decent directions. A best method should have smallest decent cone at same point, and this is just the atomic norm.

Prospective-I

- For a multi-structured signal, there's an atomic set corresponding to each norm (consist of the simplest models). In some cases, the intersection of these atomic sets will be the atomic set we want for the multi-structured signal.
- for example, the case of $2k$-sparse vector in $\mathbb{R}^{2p}$, consider strongly, consider the signal has 2 structures, k-sparse in first p entries and k-sparse in latter p entries. The corresponding atomic set for the first structure is:

$$(0, 0, \cdots, 0, \pm 1_{\text{(the k-th term)}}, 0, \cdots, 0_{\text{(the p-th term)}}, \mathbb{R}^p), 1 \leq k \leq p$$

and similar to the latter one. Or easily written as $e_k \oplus \mathbb{R}^p$ and $\mathbb{R}^p \oplus e_k$, where e_k is the standard bases of $\mathbb{R}^p$. Their intersection is just $\{\pm e_i \oplus \pm e_j\}_{i,j=1}^{2p}$, and the atomic norm is $\max\{||x_1||_{l^1}, ||x_2||_{l^1}\}$, $x = x_1 \oplus x_2$, it has a good performance in recovering a signal simultaneously with the 2 sparse structure. And it's a little similar to what we want for the $2k$-sparse structure in $\mathbb{R}^{2p}$.

Prospective-II

- But we noticed that the 2 separated structures are slightly stronger than the original $2k$-sparse structure, hence the norm is less than the l^1 norm, $conv(\mathcal{A})$ contains the unit ball w.s.t. l^1 norm.
- And more generally, this method will fail for some cases, like the case sparse and low-rank. This method just easily intersects the two atomic sets but does not distinguish which structure is dominant. We want to find out how and when this method would work. For example, how is the strictly definition of "dominant".

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