Ideal Position in a Voting Model
SCIE4500 Final Report

Haoyu Wu
Supervisor: Prof. Kevin Tang & Prof. Jianfeng Cai

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Abstract

In this paper, we focus on the theoretical explanation of the equilibria (i.e. the ideal position) in the voting problem. It is well known that the Median Voter theorem concludes an equilibrium in the one-dimensional, two-party case under the majority voting rule and with a single-peak preference assumption. However, the rule generally fails and the concept of 'median' also extends in the high dimension cases. Previous results have claimed some conservative sufficient and necessary conditions in high dimensions, under different assumptions like voting preference and rules, most of which require symmetric voters' positions to discuss the local and global equilibria. We explore the strongest position or area in the given policy space, which is with the least possible area to beat our chosen position, under the simplest model. There exists an increasing number of people standing in extreme positions in the United States and our paper is inspired by these phenomena.

1 Introduction & Model Setting

In recent years, there has been a tendency for more and more people to become extreme in American politics, especially during the US presidential election. This trend motivates us to explore the equilibria in the voting problem. The famous median voter theorem states the equilibria of a 1-dimension voting model always lie on the median of voters.

Theorem 1.1. *(Median Voter Theorem) Assume the policy space is* $[0, 1]$ *and the preferences of voters are represented by the distance to their ideal position. If there is a total of* 2 *candidates, then the ideal candidate position is in the median of the ideal positions of voters.*

In the case that the number of voters is even, the median position represents the interval between the 2 median voters. Under this assumption, no other position can beat the candidate with the median position. Naturally, the median position fails in more than 2-candidate case. Under 2-candidate assumption, however, the median position generally fails in high dimensions [Plo67], and even we cannot guarantee the existence of the strongest candidate position[MW76]. Before we investigate the high-dimension cases, it's necessary to formally define the equilibria.

For a convex policy space $X \subset \mathbb{R}^m$, let $N = \{v_1, v_2, \cdots, v_n\}$ denote the set of voters. For each voter v_i, we assume a complete binary $\prec_i \subset X \times X$ representing his preference and $\sim_i$ representing indifference. Then we define several types of equilibria.

Definition 1. *Types of equilibria:*
1. (Strong) Majority Condorect: $E_1 = \{x \in X : \forall y \neq x, |y \prec x| > n/2\}$.
2. (Weak) Majority Condorect: $E_2 = \{x \in X : \forall y \neq x, |y \prec x| \geq n/2\}$.

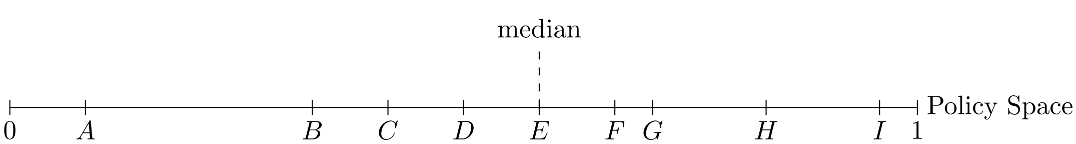

Figure 1: Median position in 1-dimension

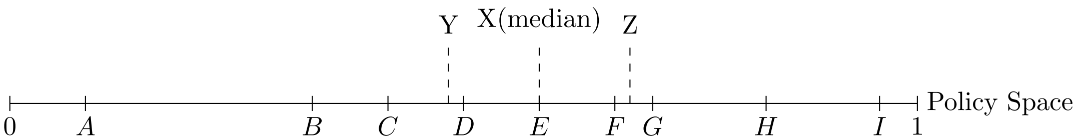

Figure 2: Median position fails in the 3-candidate case

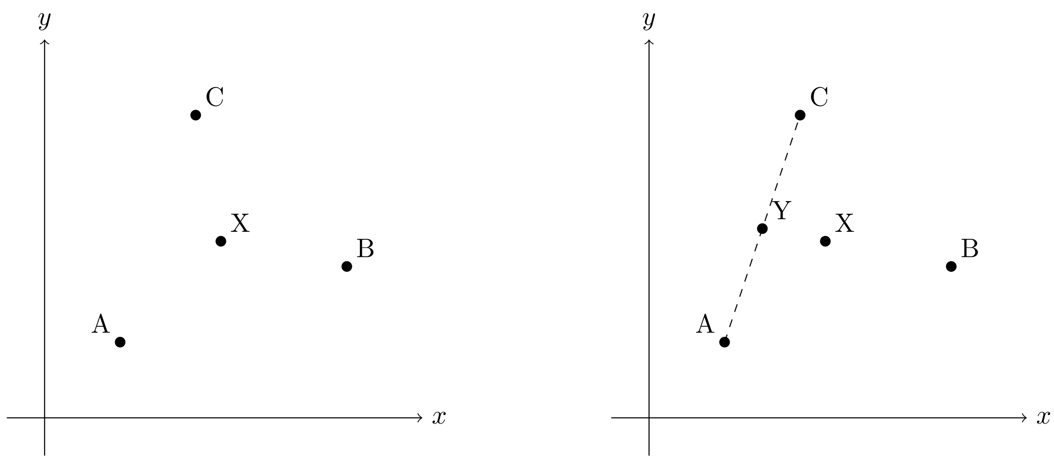

Figure 3: Voters A, B, C; For given candidate X, we could find position Y to beat X

3. (Strong) Plurality Condorect: $E_3 = \{x \in X : \forall y \neq x, |y \prec x| > |x \prec y|\}$.
4. (Weak) Plurality Condorect: $E_4 = \{x \in X : \forall y \neq x, |y \prec x| \geq |x \prec y|\}$.
5. (Strong) Majority Core: $E_5 = \{x \in X : \forall y, |x \prec y| < n/2$
6. (Weak) Majority Core: $E_6 = \{x \in X : \forall y, |x \prec y| \leq n/2$

Exactly, these types are equivalent under convex preferences.

Theorem 1.2. *Assuming for any distinguished $x, y \in X$, $y \succeq_i x$ implies $tx + (1-t)y \succ_i x$ for any $0 < t < 1$. Then $E_1 = E_3 = E_5$ and $E_2 = E_4 = E_6$.*

The assumption here is the convex preference constraint and noticed that it does not necessarily require transitivity or that the preferences are represented by utility functions. Plott [Plo67] first mentions these definitions and computes the local equilibria via gradient by assuming preferences are represented by smooth utility functions. Later, Mckelvey studied the equilibria problem under different situations: in multi-dimension [MW76], under intransitivity [McK79], in arbitrary voting rules [MS87] and his collaborator Schofield extended the results to probabilistic model [Sch07] after Mckelvey passed. We will see some necessary and sufficient conditions of the equilibria in the following section 2.

2 Sufficient and Necessary Conditions

Let N be the set of voters and $A \subset N$, we denote $x \succ_A y \iff x \succeq_i y$ for all $i \in A$ and at least one of $\succeq_i$ is strict. Denote the set of Pareto optimal points in the policy space X,

$$P_A := \{x \in X : \nexists y \in X, y \succ_A x\}$$

and the contrast set $C(i, j)$ for voters $i, j \in N$ denotes as $P_{\{i,j\}}$. It means that for a given point in the contrast set, any change helping one will hurt the other. The definition does even not require preferences to be represented by utility functions and transitivity [MW76]. In the case that the preferences of i, j are represented by smooth utility functions, the contrast set $C(i, j)$ is a path jointing i, j and each point on the path shares the same gradient direction as the two utility functions up to an opposite direction.

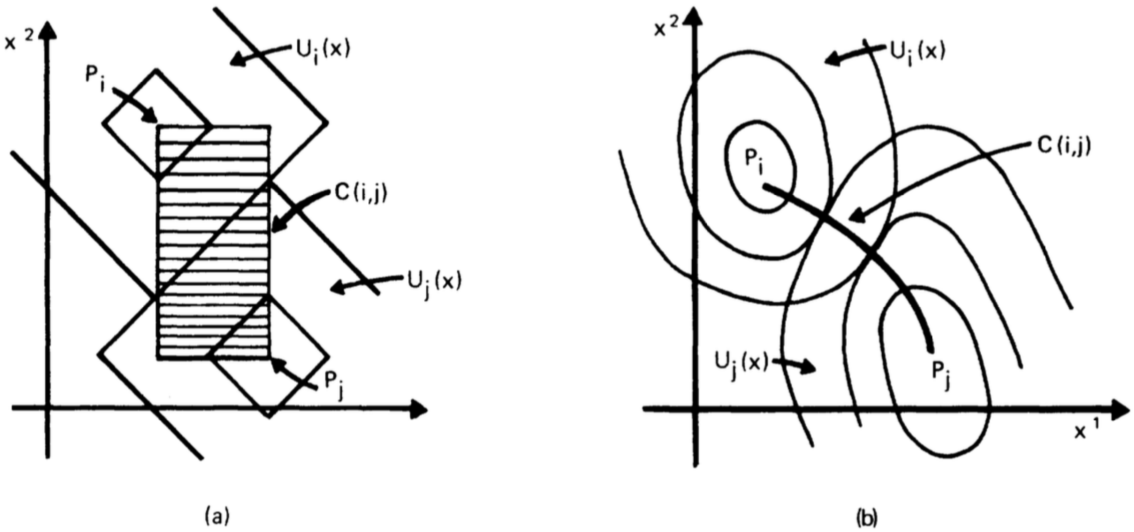

Figure 4: The contrast set $C(i, j)$ of voters P_i, P_j with utility functions U_i, U_j

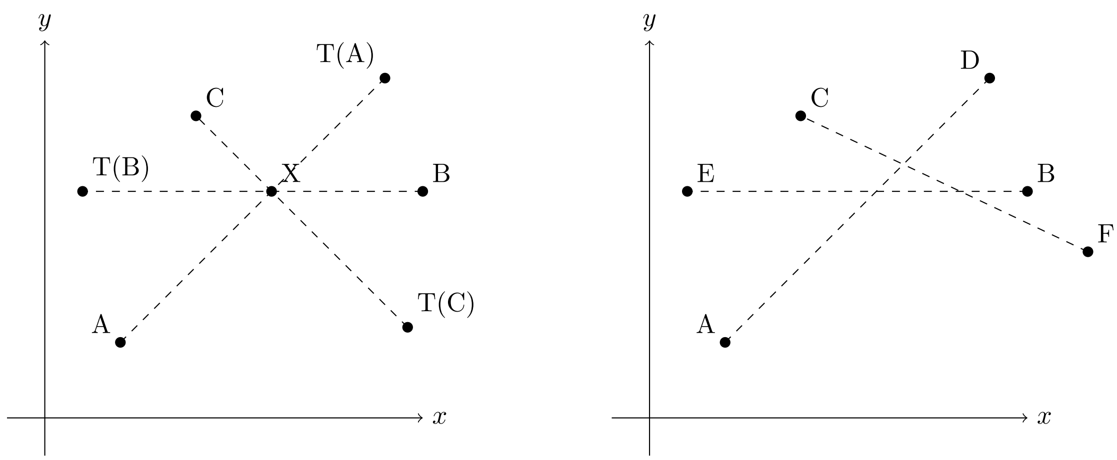

Figure 5: Left: In the model with 6 voters, there exists a pairing satisfying conditions and X is the equilibria. Right: there does not exist any good pairing.

Theorem 2.1. *It's sufficient for a policy point x to be a weak Condorcet point (i.e. $x \in E_4$) is that there exists a bijective mapping $T : N \to N$ such that,*

$$x \in \bigcap_{i \in N} C(i, T(i))$$

In addition, if there exists $i_0 \in N$ such that $T(i_0) = i_0$ and $x \in C(i_0, i_0)$, then x is a strong Condorect point (i.e. $x \in E_3$). With some additional assumptions, the sufficient conditions could be transformed into a geometrical form. For voters i, j, we say that they are weakly symmetric about a position x if there exists $t \in [0, 1]$ such that $x = tP_i + (1 - t)P_j$.

Corollary 2.1.1. *Suppose the preference of each voter is represented by a utility function in form $U_i(x) = -||x - P_i||_{(i)}$. If there exists a pairing $T : N \to N$ such that voters i and T_i are weakly symmetric about x and T is norm preserving, i.e. $|| \cdot ||_{(i)} = || \cdot ||_{T(i)}$, then x is a weak Condorect point (i.e. $x \in E_4$). If there exists i_0 such that $T(i_0) = i_0$, then x is a strong Condorect point (i.e. $x \in E_3$).*

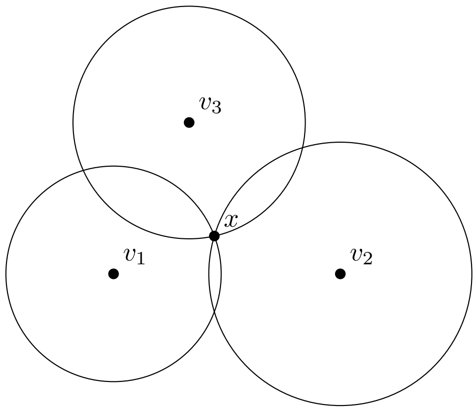

Figure 6: the Model of 3 Voters and in $\mathbb{R}^2$

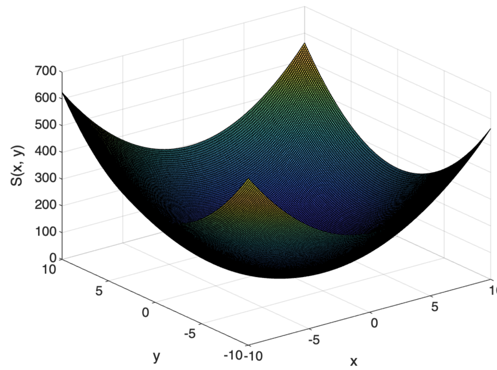

Figure 7: One of the separated components of f

3 Compute the Ideal Position

3.1 Convex Measurement Function

We see that the existence of equilibria always requires such a conservative symmetric condition and it is not always guaranteed in practice. However, for any given position, we can compute the measurement of the region that can beat our chosen position. We want to find the strongest position with minimized measurement of the region. For a given policy space $X \subset \mathbb{R}^m$ and the set of voters N, we define the function of the Lebesgue measurement of the area to beat the chosen position:

$$f : X \to \mathbb{R}^+ \cup \{0\}, \quad x \mapsto \mu\,(\text{region that are able to beat } x)$$

where μ is the Lebesgue measure. In addition, we assume the preferences of each voter are represented by a utility function U_i. We assume that $B_i(r)$, the ball centered at v_i with a distance less than r, is convex. The function f is with form,

$$M = \{A \subset 2^N : |A| \geq n/2\}$$

$$f : X \to \mathbb{R}^+ \cup \{0\}, \quad x \mapsto \mu\left(\bigcup_{A \in M} \bigcap_{v_i \in A} B_i(U_i(x))\right)$$

Conjecture 3.1. *The function f is convex.*

It has been verified in the simplest case that 3 voters in total and in policy space $\mathbb{R}^2$. For voters v_1, v_2, v_3 and the policy position x in the convex hull of voters, we consider the circles with center v_1, v_2, v_3 and radius $||x - v_1||_2, ||x - v_2||_2, ||x - v_3||_2$ respectively. The region that beats the chosen position is the overlapping part of arbitrary two circles. The function could be written in the explicit form,

$$\begin{aligned}
f(x, y) &= u^2 \cos^{-1} \frac{p^2 + u^2 - v^2}{2pu} + v^2 \cos^{-1} \frac{p^2 + v^2 - u^2}{2pv} - \frac{1}{2}\sqrt{(u + v + p)(u + v - p)(p + u - v)(p + v - u)} \\
&+ w^2 \cos^{-1} \frac{q^2 + w^2 - v^2}{2qu} + v^2 \cos^{-1} \frac{q^2 + v^2 - w^2}{2qv} - \frac{1}{2}\sqrt{(q + v + w)(q + v - w)(q + w - v)(u + v - q)} \\
&+ u^2 \cos^{-1} \frac{r^2 + u^2 - w^2}{2pu} + w^2 \cos^{-1} \frac{r^2 + w^2 - u^2}{2rw} - \frac{1}{2}\sqrt{(r + u + w)(r + u - w)(r + w - u)(u + w - r)}
\end{aligned}$$

where u, v, w are the distances between x and v_1, v_2, v_3 respectively and p, q, r are distances $||v_1 - v_2||_2, ||v_2 - v_3||_2, ||v_1 - v_3||_2$ respectively. The graphs of the function of the area of one of the 3 leaf-shape regions have been plotted in a convex shape and the total area is the sum of the areas of the 3 leaves (hence it's also convex). With the assumption that the function is convex, we have lots of methods or tools to derive the unique optimal position.

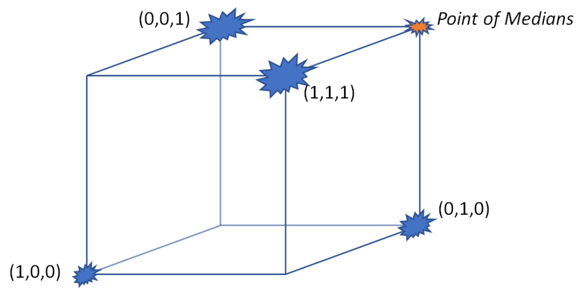

Figure 8: the Marginal Median of the set of nine vectors

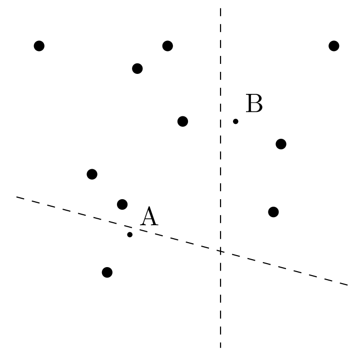

Figure 9: depth(A)=1 and depth(B)=3

3.2 Median in High dimensions

There are several median concepts in high dimensions and all of them degenerate into the common median concept in one dimension.

3.2.1 Marginal Median

The marginal median is the direct extension of the concept of median in one dimension to the high dimensions. Each entry of the marginal median vector is the median in its dimension respectively.

Definition 2. *For a given set of n vectors $\{v_i = (v_i^1, \cdots, v_i^m)\}_{i=1,\cdots n} \subset \mathbb{R}^m$, the marginal median is defined as $v_{mm} = (w_1, \cdots, w_m)$ where w_k is the median of $\{v_1^k, \cdots, v_n^k\}$.*

However, it does not provide a reasonable estimation of the data and even it may be out of the convex hull of the data. Consider the case that a data set with 9 vectors, one of them is $(1,0,0)^T$, two of them are $(0,1,0)^T$, and three of them are $(0,0,1)^T$ and the remaining three are $(1,1,1)^T$. We could separate each dimension and derive that the marginal median is $(0,1,1)^T$, out of the convex hull of all the nine vectors.

3.2.2 L_1 Median

The L_1 median is the most widely acceptable concept in high dimensions, which is the center minimized the sum of Euclidean distance, for a given set of vectors $\{v_1, v_2, \cdots, v_n\}$ in Euclidean space,

$$v_{lm} = \arg_{v\in\mathbb{R}^m} \min \sum_{i=1}^{n} ||v - v_i||_2$$

The existence and uniqueness are guaranteed by the convexity of the objective function. We will investigate the relationship between the L_1 median and the ideal position we desire.

3.2.3 Halfspace Median

The halfspace median is a median related to the depth of data. Given a data set $N \subset \mathbb{R}^m$, the halfspace median v_{hm} is the point maximizing the minimum number of data standing in the same halfspace with v_{hm}. It is formally defined as,

$$v_{hm} := \arg \max_{v\in\mathbb{R}^m} \min_{w\in\mathbb{R}^m} |\{v \in N : (w \cdot v)(w \cdot m) \geq 0\}|$$

there exists an algorithm to compute the half-space median in $\mathbb{R}^2$ in time complexity $O(n^2 \log n)$ [PJR98].

3.3 Minimized Measurement respect to Deminsions

We are interested in the behavior of the minimized measurement of the region when the voters follow some given distribution. We conjecture that the ratio of the minimized measurement to the whole policy space or the convex hull of the voters is decreasing, as the dimension increases.

References

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