UROP1000 2021 Summer Report

Supervisor: Frederick Fong Tzh Ho

WU, Haoyu

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1 abstract

This report focus on some basic knowledge about flow geometry. It's a statement that there is a evolution, i.e. the heat equation, for a closed convex plain curve shrinking to a point with spect to time $t \in [0, T)$, where t is finite. The curve is also becoming more and more circular in the evolution. Firstly, this report will state some basic lemma in the calculation of some value of the plane curve. Then mainly, this report gives a way to prove that the curve is becoming more and more circular during the evolution. Using some geometric techniques, we show that the ratio of the maximun curvature and minimun curvature of the curve goes to 1.

2 The plane curve

Consider a plane curve, we take $M = S^1$ with parameter u (modulo 2π) and write the curve:

$$(x,y) = \gamma(u), u \in M \subset \mathbb{R}$$

And the arclength reparameteration:

$$s = s(u), |\gamma'(s)| = 1$$

$$v := \sqrt{(\frac{\partial x}{\partial u})^2 + (\frac{\partial y}{\partial u})^2} = \left|\frac{\partial \gamma}{\partial u}\right| \Longrightarrow \frac{\partial}{\partial s} = \frac{1}{v}\frac{\partial}{\partial u}$$

And let T and N be the unit tangent vector and the inward pointing vector, consider the Frenet equations:

$$\frac{\partial T}{\partial u} = vkN, \quad \frac{\partial N}{\partial u} = -vkT$$

2.1

2.1.1 Lemma $\frac{\partial v}{\partial t} = -k^2 v$

Proof.

$$2v\frac{\partial v}{\partial t} = \frac{\partial}{\partial t}v^2 = \frac{\partial}{\partial t}(\frac{\partial \gamma}{\partial N} \cdot \frac{\partial \gamma}{\partial N})$$
$$= 2vT \cdot \frac{\partial}{\partial u}\frac{\partial \gamma}{\partial t}$$
$$= 2vT \cdot \frac{\partial}{\partial u}\left(\frac{\partial k}{\partial u}N - vk^2 T\right)$$
$$= -2v^2 k$$

$$\implies \frac{\partial v}{\partial t} = -k^2 v$$

□

2.1.2 Lemma $\frac{\partial L}{\partial t} = -\int_0^{2\pi} k^2 \, ds$

Proof.

$$\frac{\partial L}{\partial t} = \frac{\partial}{\partial t}\int_0^{2\pi} v \, du = \int_0^{2\pi} \frac{\partial v}{\partial t} = -\int_0^{2\pi} k^2 \, ds$$

□

2.1.3 Lemma $\frac{\partial}{\partial t}\frac{\partial}{\partial s} = \frac{\partial}{\partial s}\frac{\partial}{\partial t} + k^2 \frac{\partial}{\partial s}$

Proof.

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s} = \frac{\partial}{\partial t}\frac{1}{v}\frac{\partial}{\partial u} = k^2 \frac{1}{v}\frac{\partial}{\partial u} + \frac{1}{v}\frac{\partial}{\partial u}\frac{\partial}{\partial t} = k^2 \frac{\partial}{\partial s} + \frac{\partial}{\partial s}\frac{\partial}{\partial t}$$

□

2.1.4 Lemma $\frac{\partial T}{\partial t} = \frac{\partial k}{\partial s} N$ *and* $\frac{\partial N}{\partial t} = -\frac{\partial k}{\partial s} T$

Proof.

$$\begin{aligned}\frac{\partial T}{\partial t} = \frac{\partial}{\partial t}\frac{\partial}{\partial s}\gamma = k^2 T + \frac{\partial}{\partial s} + \frac{\partial}{\partial t}\gamma \\ = k^2 T + \frac{\partial}{\partial s} kN \\ = k^2 T + \frac{\partial k}{\partial s} N - k^2 T = \frac{\partial k}{\partial s} N\end{aligned}$$

$$\begin{aligned}0 = \frac{\partial}{\partial t} T \cdot N = \frac{\partial T}{\partial t} \cdot N + \frac{\partial N}{\partial t} \cdot T \\ = \frac{\partial k}{\partial t} N \cdot N + \frac{\partial N}{\partial t} \cdot T\end{aligned}$$

$$\frac{\partial N}{\partial t} \mathbin{//} T \Longrightarrow \frac{\partial N}{\partial t} = -\frac{\partial k}{\partial s} T$$

□

Let θ be the angle between the tangent vector and the x axis. Then

2.1.5 Lemma $\frac{\partial \theta}{\partial t} = \frac{\partial k}{\partial s}$ *and* $\frac{\partial \theta(s)}{\partial s} = k(s)$

Proof. Since $T = (\cos\theta, \sin\theta)$ and apply lemma 2.1.4:

$$\frac{\partial T}{\partial \theta}\frac{\partial \theta}{\partial t} = \frac{\partial T}{\partial t} = -\frac{\partial k}{\partial s} N = \frac{\partial k}{\partial s}(-\sin\theta, \cos\theta)$$

$$\frac{\partial T}{\partial \theta} = (-\sin\theta, \cos\theta) \Longrightarrow \frac{\partial \theta}{\partial t} = \frac{\partial k}{\partial s}$$

Consider the Frenet equation proves the second relation.

□

3

3.1 Lemma

Denote:

$$k_w^* := \sup\{b : k(\theta) > b \text{ on some interval of length } w\ \}$$

then:

$$k_w^*(t) r_{in}(t) < \frac{1}{1 - K(w)(\frac{r_{out}}{r_{in}} - 1)}$$

where r_{in} and r_{out} are respectively the radiu of the largest insribed circle and the smallest circumscribebed of the curve defined by $k(\cdot,t)$. K is a positive decreasing function of w with $K(0) = \infty$ and $K(\pi) = 0$.

Proof. For any constant $M < k_w^*(t)$, the set $\{\theta : k(\theta,t) > M\}$ contains an interval of length greater than or equal M. By changing the parametrization, we assume that the length contains the interval $(-\frac{w}{2}, \frac{w}{2})$.

Than we construct a circular arc with curvature M(radiu $1/M$), arc angle w, passes and tangent to the curve at $\theta = 0$, point P. The two tangent lines of the circular arc cross at point X.

By the convexity, γ is contained between the tangent lines of the points $k(\frac{w}{2},t)$, $k(-\frac{w}{2},t)$. So that γ is contained in the two tangent lines of the end points of the circular arc.

Consider the straight line PX, we take a point A on it such that the distance between A and the tangent lines is just the r_{in}. The orginal center O_1 of the largest insribed circle must out of the circle with center P and radiu AP. Otherwise the insribed circle will be out of γ.

Then let the straight line PO_1 intersects γ with the other point Q. Denote the distance of $PX = d$, $AP = a$, $\frac{1}{M} = b$.

We obtains the Figure 1:

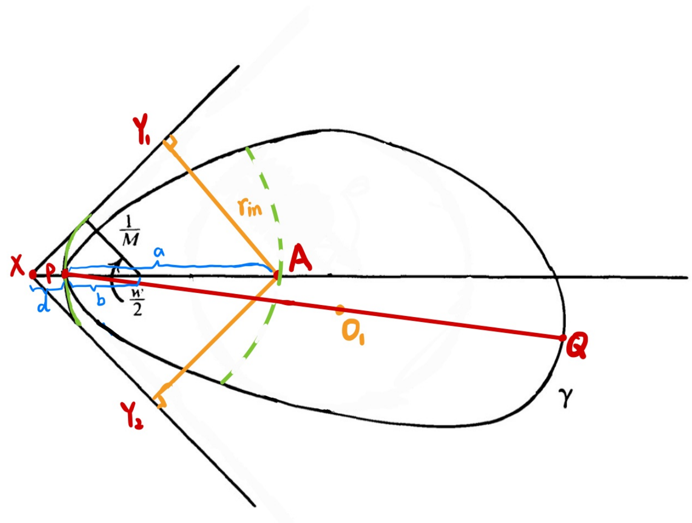

Figure 1:

Noticed:

$$\cos\left(\frac{w}{2}\right) = \frac{1/M}{1/M + d} = \frac{r_{in}}{a+d} \tag{3.1.1}$$

$$2r_{out} \geq PQ = PO_1 + O_1Q \geq a + r_{in} \tag{3.1.2}$$

From (3.1.2):

$$\frac{r_{out}}{r_{in}} \geq \frac{1}{2} + \frac{a}{2r_{in}} \Longrightarrow \frac{r_{out}}{r_{in}} - 1 \geq -\frac{1}{2} + \frac{a}{2r_{in}} \tag{3.3}$$

From (3.1.1):

$$d = \frac{1/M}{\cos(w/2)} - 1/M = \frac{1}{M}\left(\frac{1}{\cos(w/2)} - 1\right)$$

$$a = \frac{r_{in}}{\cos(w/2)} - d = \frac{r_{in}}{\cos(w/2)} - \frac{1}{M}\left(\frac{1}{\cos(w/2)} - 1\right) \tag{3.1.4}$$

Substituting (3.1.4) in (3.1.3):

$$\begin{aligned}
\frac{r_{out}}{r_{in}} - 1 &\geq -\frac{1}{2} + \frac{\frac{r_{in}}{\cos(w/2)} - \frac{1}{M}\left(\frac{1}{\cos(w/2)} - 1\right)}{2r_{in}} \\
\frac{r_{out}}{r_{in}} - 1 &\geq -\frac{1}{2} + \frac{1}{2\cos(w/2)} - \frac{1}{Mr_{in}}\left(\frac{1}{2\cos(w/2)} - \frac{1}{2}\right) \\
\frac{1}{Mr_{in}} &\geq 1 + \frac{\frac{r_{out}}{r_{in}} - 1}{\frac{1}{2\cos(w/2)} - \frac{1}{2}} \\
Mr_{in} &\leq \frac{1}{1 - K(w)(r_{out}/r_{in} - 1)}
\end{aligned}$$

where

$$K(w) = \left(\frac{1}{2\cos(w/2)} - \frac{1}{2}\right)^{-1} = \frac{2\cos(w/2)}{1 - \cos(w/2)}$$

Since M can be chose arbitrarily close to $k_W^*(t)$, this proves the lemma.

□

3.2 Theorem

$k(\theta,t)r_{in}(t)$ converges uniformly to 1.

Proof. Mutatis mutandis [1]4.3(M.Gage 1984), we can easily prove that the family $k(\theta,t)r_{in}(t)$ is equicontinuous and also uniformly bounded by lemma above. By the Azerla-Ascoli theorem:

$$\exists\{t_i\}_{i=1}^{\infty}, t_i \to T, \exists f(\theta) \leq 1, k(\theta,t_i) \rightrightarrows f(\theta)$$

And we also have $(k(\theta,t_i)r_{in}(t_i))^{-1}$ converges pointwise to $f(\theta)^{-1}$ in $\mathbb{R} \cup \{\infty\}$.

3.2.1 Lemma (based on idea of Fatou's Lemma)

$$\int \frac{d\theta}{f(\theta)} \leq \liminf_{i\to\infty} \int \frac{d\theta}{k(\theta,t_i)r_{in}(t_i)}$$

Proof. Denote:

$$g_n(\theta) := \inf_{i\geq n}(k(\theta,t_i)r_{in}(t_i))^{-1}$$

$$h(\theta) := \lim_{n\to\infty} g_n(\theta) = \lim_{n\to\infty}\inf_{i\geq n}(k(\theta,t_i)r_{in}(t_i))^{-1} = \liminf_{i\to\infty}(k(\theta,t_i)r_{in}(t_i))^{-1}$$

Since $\int \inf_{i\geq n}(k(\theta,t_i)r_{in}(t_i))^{-1} \leq \int (k(\theta,t_i)r_{in}(t_i))^{-1}$ for all $i \geq n$:

$$\Longrightarrow \int \inf_{i\geq n}(k(\theta,t_i)r_{in}(t_i))^{-1} \leq \inf_{i\geq n} \int (k(\theta,t_i)r_{in}(t_i))^{-1}$$

Since g_n is increasing and pointwise converges to h, by MCT:

$$\int h = \lim_{n\to\infty} \int g_n(\theta)$$

$$\int \frac{d\theta}{f(\theta)} = \int \liminf_{i\to\infty} (k(\theta,t_i)r_{in}(t_i))^{-1} = \int h = \lim_{n\to\infty} \int g_n(\theta)$$
$$\leq \lim_{n\to\infty} \inf_{i\geq n} \int (k(\theta,t_i)r_{in}(t_i))^{-1} = \liminf_{i\to\infty} \int (k(\theta,t_i)r_{in}(t_i))^{-1}$$

□

By lemma 2.1.5, and noticed that:

$$\liminf_{i\to\infty} \int \frac{d\theta}{(k(\theta,t_i)r_{in}(t_i))} = \liminf_{i\to\infty} \int \frac{ds}{r_{in}(t_i)} = \liminf_{i\to\infty} \frac{L(t_i)}{r_{in}(t_i)} = 2\pi$$

$$\implies \int \frac{d\theta}{f(\theta)} \leq 2\pi$$

However, since $0 < f(\theta) \leq 1$:

$$\int \frac{d\theta}{f(\theta)}\, d\theta \geq \int 1\, d\theta \geq 2\pi$$

$$\implies f \equiv 1$$

Since every convergent subsequence converges uniformly to 1, $k(\theta,t)r_{in}(t)$ converges uniformly to 1. That means that $k(\theta,t)$ converges to constant r_{in} for fixed t for all θ, i.e. the curve is becoming more and more circular.

□

Reference

[1] Gage M.E., Hamilton R.: The heat equation shrinking convex plane curves. J. Differential Geom. 23, 69–96 (1986)