Ideal Position in a Voting Model

SCIE4500 Final Report

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Median Voter Theorem

Assume the policy space is $[0,1]$ and the preferences of voters are represented by the distance to their ideal position. If there is a total of 2 candidates, then the ideal candidate position is in the median of the ideal positions of voters.

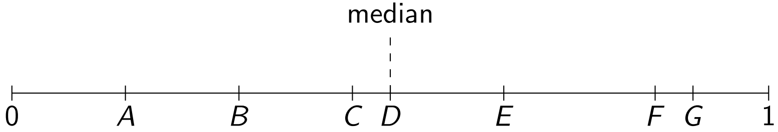

Figure: Median position in 1-dimension

MVT Fails in 3-Candidate Model

Remark

In the case that the number of voters is even, the median position represents the interval between the 2 median voters. Naturally, the median position fails in more than 2-candidate case. Under 2-candidate assumption, however, the median position generally fails in high dimensions [Plo67].

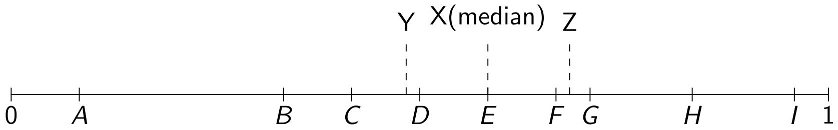

Figure: Suppose $A, B, C, D, E, F, G, H, I$ are voters and X, Y, Z are candidates. Y wins A, B, C, D, Z wins F, G, I and the median X only wins E

MVT fails in High Dimensions

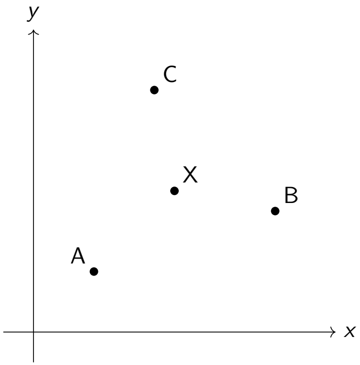

Figure: The A, B, C are voters and for any chosen position X, we could find the other position Y that beats the chosen X.

Model Setting

Definition of Preference

For a convex policy space $X \subset \mathbb{R}^m$, let $N = \{v_1, v_2, \cdots, v_n\}$ denote the set of voters. For each voter v_i, we assume a complete binary $\prec_i \subset X \times X$ representing his preference and $\sim_i$ representing indifference.

Type of Equilibria

1. (Strong) Majority Condorect: $E_1 = \{x \in X : \forall y \neq x, |y \prec x| > n/2\}$.
2. (Weak) Majority Condorect: $E_2 = \{x \in X : \forall y \neq x, |y \prec x| \geq n/2\}$.
3. (Strong) Plurality Condorect: $E_3 = \{x : \forall y \neq x, |y \prec x| > |x \prec y|\}$.
4. (Weak) Plurality Condorect: $E_4 = \{x \in X : \forall y \neq x, |y \prec x| \geq |x \prec y|\}$.
5. (Strong) Majority Core: $E_5 = \{x \in X : \forall y, |x \prec y| < n/2$
6. (Weak) Majority Core: $E_6 = \{x \in X : \forall y, |x \prec y| \leq n/2$

- Convex Preference implies different types of equilibria are equivalent.

Theorem

Assuming for any distinguished $x, y \in X$, $y \succeq_i x$ implies $tx + (1-t)y \succ_i x$ for any $0 < t < 1$. Then $E_1 = E_3 = E_5$ and $E_2 = E_4 = E_6$.

- convex preference constraint and noticed that it does not necessarily require transitivity or that the preferences are represented by utility functions.
- Plott [Plo67] first mentions these definitions and computes the local equilibria via gradient by assuming preferences are represented by smooth utility functions.

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Definition

Let N be the set of voters and $A \subset N$, we denote $x \succ_A y \iff x \succeq_i y$ for all $i \in A$ and at least one of $\succeq_i$ is strict. Denote the set of Pareto optimal points in the policy space X,

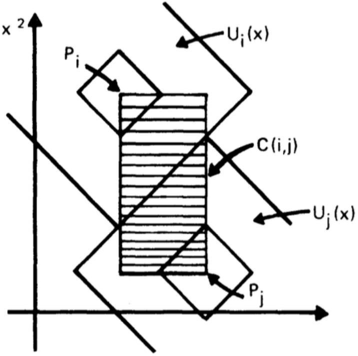
$$P_A := \{x \in X : \nexists y \in X, y \succ_A x\}$$

and the contrast set $C(i,j)$ for voters $i, j \in N$ denotes as $P_{\{i,j\}}$.

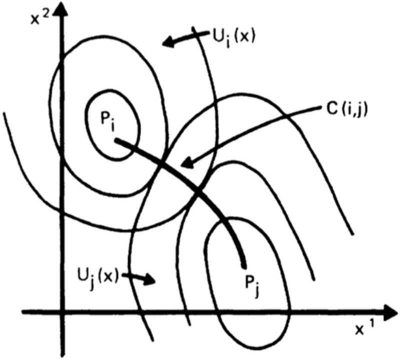
- It means that for a given point in the contrast set, any change helping one will hurt the other.
- The definition does even not require preferences to be represented by utility functions and transitivity [MW76].

Examples

- In the case that the preferences of i, j are represented by smooth utility functions, the contrast set $C(i, j)$ is a path jointing i, j and each point on the path shares the same gradient direction as the two utility functions up to an opposite direction.



(a)



(b)

Sufficient Conditions

Theorem

It's sufficient for a policy point x to be a weak Condorcet point (i.e. $x \in E_4$) is that there exists a bijective mapping $T : N \to N$ such that,

$$x \in \bigcap_{i \in N} C(i, T(i))$$

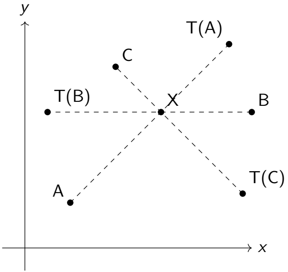
In addition, if there exists $i_0 \in N$ such that $T(i_0) = i_0$ and $x \in C(i_0, i_0)$, then x is a strong Condorect point (i.e. $x \in E_3$).

Remark

With some additional assumptions, the sufficient conditions could be transformed into a geometrical form. For voters i, j, we say that they are weakly symmetric about a position x if there exists $t \in [0, 1]$ such that $x = tP_i + (1-t)P_j$.

Corollary

Suppose the preference of each voter is represented by a utility function in form $U_i(x) = -||x - P_i||_{(i)}$. If there exists a pairing $T : N \to N$ such that voters i and T_i are weakly symmetric about x and T is norm preserving, i.e. $|| \cdot ||_{(i)} = || \cdot ||_{T(i)}$, then x is a weak Condorect point (i.e. $x \in E_4$). If there exists i_0 such that $T(i_0) = i_0$, then x is a strong Condorect point (i.e. $x \in E_3$).



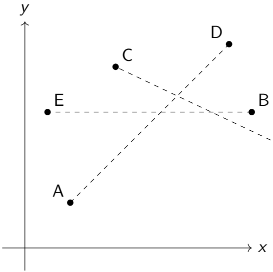


Figure: Left: In the model with 6 voters, there exists a mapping satisfying conditions and X is the equilibria. Right: there does not exist any good mapping.

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Convex Measurement Function

- the existence of equilibria always requires such a conservative symmetric condition and it is not always guaranteed in practice.
- However, for any given position, we can compute the measurement of the region that can beat our chosen position. We want to find the strongest position with minimized measurement of the region.

Definition

For any given position, we can compute the measurement of the region that can beat our chosen position. For a given policy space $X \subset \mathbb{R}^m$ and the set of voters N,

$$f : X \to \mathbb{R}^+ \cup \{0\}, \quad x \mapsto \mu\,(\text{region that are able to beat } x)$$

where μ is the Lebesgue measure.

Alternative Form (Con't)

$$f : X \to \mathbb{R}^+ \cup \{0\}, \quad x \mapsto \mu\,(\text{region that are able to beat } x)$$

In addition, we assume the preferences of each voter are represented by a utility function U_i. We assume that $B_i(r)$, the ball centered at v_i with a distance less than r, is convex. The function f is with form,

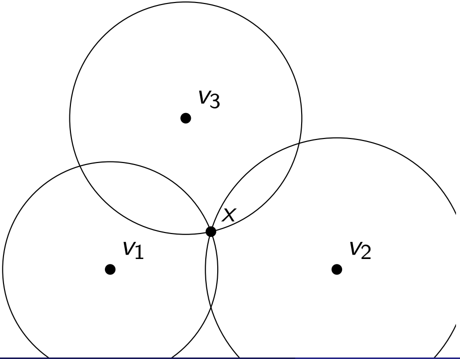
$$M = \{A \subset 2^N : |A| \geq n/2\}$$

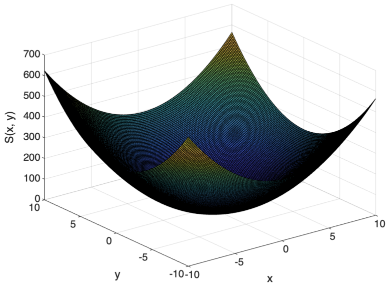
$$f : X \to \mathbb{R}^+ \cup \{0\}, \quad x \mapsto \mu\left(\bigcup_{A \in M} \bigcap_{v_i \in A} B_i(U_i(x))\right)$$

Conjecture

The measurement function f is convex (or convex in the convex hull of voters).

- It has been verified in the simplest case that 3 voters in total and in policy space $\mathbb{R}^2$. For voters v_1, v_2, v_3 and the policy position x in the convex hull of voters, we consider the circles with center v_1, v_2, v_3 and radius $||x - v_1||_2, ||x - v_2||_2, ||x - v_3||_2$ respectively.
- The graphs of the function of the area of one of the 3 leaf-shape regions have been plotted in a convex shape and the total area is the sum of the areas of the 3 leaves (hence it's also convex).





Median in High Dimensions

- There are several median concepts in high dimensions and all of them degenerate into the common median concept in one dimension.
- Here we introduce three of them, the marginal median v_{mm}, the L_1 median v_{lm} and the halfspace median v_{hm}.

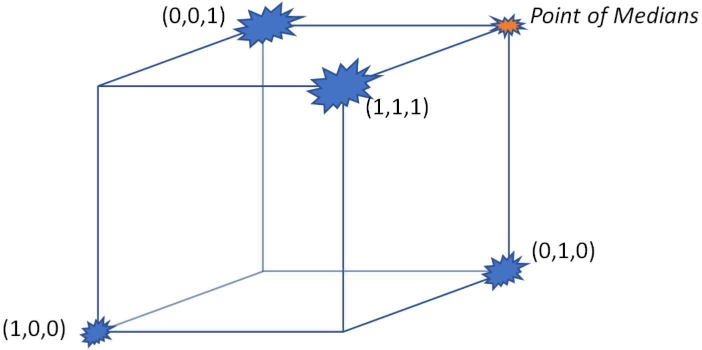
Marginal Median

The marginal median is the direct extension of the concept of median in one dimension to the high dimensions. Each entry of the marginal median vector is the median in its dimension respectively.
or a given set of n vectors $\{v_i = (v_i^1, \cdots, v_i^m)\}_{i=1,\cdots n} \subset \mathbb{R}^m$, the marginal median is defined as $v_{mm} = (w_1, \cdots, w_m)$ where w_k is the median of $\{v_1^k, \cdots, v_n^k\}$.

- However, it does not provide a reasonable estimation of the data and even it may be out of the convex hull of the data.

Poor Performance of Marginal Median

- Consider the case that a data set with 9 vectors, one of them is $(1,0,0)^T$, two of them are $(0,1,0)^T$, and three of them are $(0,0,1)^T$ and the remaining three are $(1,1,1)^T$. We could separate each dimension and derive that the marginal median is $(0,1,1)^T$, out of the convex hull of all the nine vectors.



L_1 Median

The L_1 median is the most widely acceptable concept in high dimensions, which is the center minimized the sum of Euclidean distance, for a given set of vectors $\{v_1, v_2, \cdots, v_n\}$ in Euclidean space,

$$v_{lm} = \arg_{v \in \mathbb{R}^m} \min \sum_{i=1}^{n} ||v - v_i||_2$$

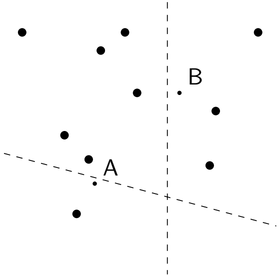
The existence and uniqueness are guaranteed by the convexity of the objective function. We will investigate the relationship between the L_1 median and the ideal position we desire.

Halfspace Median

- The halfspace median is a median related to the depth of data. Given a data set $N \subset \mathbb{R}^m$, the halfspace median v_{hm} is the point maximizing the minimum number of data standing in the same halfspace with v_{hm}. It is formally defined as,

$$v_{hm} := \arg\max_{v \in \mathbb{R}^m} \min_{w \in \mathbb{R}^m} |\{v \in N : (w \cdot v)(w \cdot m) \geq 0\}|$$

- There exists an algorithm to compute the half-space median in $\mathbb{R}^2$ in time complexity $O(n^2 \log n)$ [PJR98].



$\text{depth}(A) = 1$

$\text{depth}(B) = 3$

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