

Network Science Project Report

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Abstract

1 Introduction

1.1 Introduction to the Problem

In this project, we study a problem that is related to both online decision problem, or more specifically, the multi-armed bandit, and the information synchronization problem.

1.2 Related Works

1.3 Basic Settings and Problem Formulation

1.4 Hardness of the Problem

1.4.1 Exploration vs. Exploitation

1.4.2 Exponential Number of Arms

2 Methods to Solve the Problem

2.1 General Methods

To tackle the problem of the unknown distribution of the edges, we apply the framework in multi-arm bandit. However, in the network information synchronization setting, the number of possible result will be exponentially large with respect to the number of nodes. Fortunately, [1] gave a framework of combinatorial multi-armed bandit to deal with the problem of exponentially number of ‘arms’. We apply the framework, and the algorithm in that framework, and combine it with the network information synchronization problem. The algorithm is shown in **Algorithm 1**

2.2 Minimize the Total Cost

In this section, we consider the setting when we want to minimize the cost of information synchronization in the network. Now suppose that the random variable X_i on edge i represents the cost of select edge i . In this setting, we just consider the total cost but not consider the other measurements. This setting is reasonable in many real life cases when the information synchronization does not have constraints on time but the cost of communication is large, especially when the information synchronization does not happen so often and the cost of synchronization is high. We have the following simple theorem under this setting, and we assume that the random variables that represent the cost on each edge are positive.

Theorem 2.1. *Under the senerio of minimizing the total cost while synchronizing the information, the edges that are chosen by the optimal strategy will form a spanning tree.*

Algorithm 1 Algorithm to solve the online information synchronization problem

Input: The graph structure(without weight) $G(V, E)$, and the algorithm \mathcal{A} we want to run online.

Output: The action of each round

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1: procedure ALG( $G(V, E), \mathcal{A}$ )
2:    $\hat{\mu}_i \leftarrow$  the empirical expectation of edge  $i$ .
3:    $T_i \leftarrow$  the number of time the edge  $i$  is used.
4:   Every time we use an edge  $i$ , we will update the empirical expectation  $\hat{\mu}_i$  and the counter  $T_i$ .
5:   for  $t = 1, 2, \dots, |E|$  do
6:     Play an instance from all possible instances which contains the edge  $i$ .
7:   end for
8:   for  $t = |E| + 1, \dots$  do
9:      $\bar{\mu}_i \leftarrow \hat{\mu}_i - \sqrt{\frac{3 \ln t}{2T_i}}$ .
10:    Play an instance which minimize the result of algorithm  $\mathcal{A}$  when each edge has weight  $\bar{\mu}_i$ .
11:   end for
12: end procedure

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Proof. This theorem is really simple. First because we want to synchronize the information for all nodes, the chosen edges will let all of the nodes to be connected. Then because all of the edges have positive cost, then if there is a cycle it must be non-optimal. So the optimal strategy will choose a spanning tree of the graph. \square

Now it is obvious that the best strategy tries to find the minimum expectation cost spanning tree, which is the spanning tree that minimize the total cost in the long term. We have the following algorithm, see **Algorithm 2**, which follows the framework **Algorithm 1**.

Algorithm 2 Algorithm to solve the problem under the min cost setting

Input: The graph structure(without weight) $G(V, E)$.

Output: The action of each round, where the edge selected in each round forms a spanning tree of all the nodes and then the algorithm tries to minimize the long term total cost of the spanning tree.

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1: procedure ALG( $G(V, E)$ )
2:    $\hat{\mu}_i \leftarrow$  the empirical expectation of edge  $i$ .
3:    $T_i \leftarrow$  the number of time the edge  $i$  is used.
4:   Every time we use an edge  $i$ , we will update the empirical expectation  $\hat{\mu}_i$  and the counter  $T_i$ .
5:   for  $t = 1, 2, \dots, |E|$  do
6:     Find an arbitrary spanning tree that contains edge  $t$ .
7:   end for
8:   for  $t = |E| + 1, \dots$  do
9:      $\bar{\mu}_i \leftarrow \hat{\mu}_i - \sqrt{\frac{3 \ln t}{2T_i}}$ .
10:    Find the minimum spanning tree with  $\bar{\mu}_i$  as the weight on edge  $i$ .
11:   end for
12: end procedure

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Now the only problem is how to find the minimum spanning tree of a graph. The problem is simple and we can use Kruskal's algorithm. Then given a spanning tree of the graph, we can transfer the information through the edges of the tree and we can synchronize the information.

2.3 Minimize the Maximum Distance

In this subsection, we consider the case when we want to minimize the time when we synchronize all the information. Here we assume that the weight of each edge $e = (u, v)$ represent the time that information goes from u to v or v to u .

2.3.1 A very simple case

Now we start from a simple case of our problem, we add an assumption about our problem.

Assumption 1: The cost of each edge is a constant, and we know the constant.

Under this setting, the fastest way to complete the information synchronization is that $\forall u, v \in V$, u send the information to v through the shortest path between u, v and v send the information to u along the shortest path from v to u . However, finding the union of the shortest path of all pair of points usually has $O(n^2)$ edges, and it is very wasting if every edge just transfer little information.

Here we use another method to transfer the information, and we need the following definition.

Definition 2.1. (Center) Given a graph $G(V, E)$, we call a vertex $v \in V$ the center of graph G , such that

$$v = \arg \min_{u \in V} \{ \max_{u' \in V} \text{dist}(u, u') \},$$

where $\text{dist}(u, u')$ denotes the shortest distance from u to u' .

Now given $G(V, E)$, denote $v \in V$ as the center of graph G , then denote T as the shortest path tree rooted at v . The shortest path tree only has $|V| - 1$ edges so it is really saving when we do not have so much cost. Moreover, we have the following theorem which shows that given the center and the shortest path tree rooted at the center, the strategy is not so bad.

Theorem 2.2. *If $v \in V$ is the center of graph $G(V, E)$ and T is the shortest path tree of G rooted at v , then consider the following strategy, all of the node transfer its own information to v through tree T , and after collecting all the information, the node v transfer all of the information to each nodes. Then this strategy is a 2 approximation with respect to the time, i.e. it cost at most twice the time of the optimal strategy.*

Proof. The proof is really simple. Denote $d = \min_{u \in V} \{ \max_{u' \in V} \text{dist}(u, u') \}$, which is the maximum distance from the center v to any other nodes. Then denote $d' = \max_{u, u'} \text{dist}(u, u')$, there dist denote the shortest path length between u, u' . Then we know that d is the time required by our strategy, and d' is the time required by the optimal strategy. Then we have

$$d' = \text{dist}(u, u') \leq \text{dist}(u, v) + \text{dist}(v, u') \leq d + d = 2d.$$

□

Now given the theorem, we know that if we can find the center of the graph and then figure out the shortest path tree rooted at the center, then the solution is a good approximation of the optimal solution, and the number of edges is minimized.

Now the only problem is how to find the center and the shortest path tree. First we call the ‘All Pair Shortest Path’ algorithm to find the shortest distance between every 2 nodes, and we denote the distance matrix D , such that $d_{ij} = \text{dist}(v_i, v_j)$. Then we find $c = \arg \min_i \max_j d_{ij}$, then c is the center of the graph. At last we call the shortest path algorithm to find the shortest path tree rooted at c .

We know that ‘All Pair Shortest Path’ problem can be solved in $O(n^3)$ by the Floyd algorithm, and the shortest path tree can be found by Dijkstra algorithm, and the total running time is $O(n^3)$.

2.3.2 A not so simple case

Now we consider a not so simple case, which we have the assumption that the each random variable is a constant but we do not know the constant. Then our solution is to call the algorithm framework (**Algorithm 1**), and turned it into the online version, see **Algorithm 3**.

Algorithm 3 Algorithm to solve the problem under the min time setting

Input: The graph structure(without weight) $G(V, E)$.

Output: The action of each round, where the edge selected in each round forms a spanning tree of all the nodes and then the algorithm tries to find the minimize the information transfer time in the long term.

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1: procedure ALG( $G(V, E)$ )
2:    $\hat{\mu}_i \leftarrow$  the empirical expectation of edge  $i$ .
3:    $T_i \leftarrow$  the number of time the edge  $i$  is used.
4:   Every time we use an edge  $i$ , we will update the empirical expectation  $\hat{\mu}_i$  and the counter  $T_i$ .
5:   for  $t = 1, 2, \dots, |E|$  do
6:     Find an arbitrary spanning tree that contains edge  $t$ .
7:   end for
8:   for  $t = |E| + 1, \dots$  do
9:      $\bar{\mu}_i \leftarrow \hat{\mu}_i - \sqrt{\frac{3 \ln t}{2T_i}}$ .
10:    Find the center of the graph  $G$  and the corresponding shortest path tree rooted at the center with  $\bar{\mu}_i$  as the weight on edge  $i$ .
11:   end for
12: end procedure

```

2.3.3 Another not so simple case

Now we consider the case that the distribution of each random variable on the edge is known. Similar as the previous discussion, we also want to find a ‘center’ of the graph and a spanning tree rooted at the center, such that the spanning tree minimize the largest distance from the center we choose. Formally, we want to find a spanning tree T rooted at v such that

$$\mathbb{E}[\max_{u, T} \text{dist}(v, u)],$$

is minimized, where the distance function dist denote the shortest path from v to u on the tree T .

Note that computing the exact spanning tree that minimize the objective function may be difficult, and we try an approximate solution. We compute the mean weight of each edge and we find the center and the corresponding shortest path tree rooted at the center with respect to the mean of each weight. Our solution is approximate because of the following fact of the objective functions,

$$\mathbb{E}[\max_{u, T} \text{dist}(v, u)] \geq \max_{u, T} \mathbb{E}[\text{dist}(v, u)],$$

since the max function is convex. Then the algorithm is the same as the most simple case, given the mean of each edge.

2.3.4 The general case

Then in the general case, we do not know the distribution of the random variables. We also use the approximate solution as shown in the previous case. Then combined with the online settings, we can thus reuse the algorithm when the random variables are constants, see **Algorithm 3**.

3 Simulation

3.1 Minimize the Total Cost

We test the result of our method on a complete graph with 20 nodes. We first test the situation when the weight random variable of each edge is a uniform random variable. We plot the total cost of our action, the best action, and the difference between them. The best action is the action that is optimal in the long term, i.e. $\mathbb{E}[\sum_{e \in T} x_e]$ is minimized. We also plot the average difference, as shown in **Figure 1**. We also simulate

the case when the random variable of weight on each edge is $m_i \cdot x_i$ where m_i, x_i are standard Weibull distribution with shape parameter 0.8, and the results are shown in **Figure 2**.

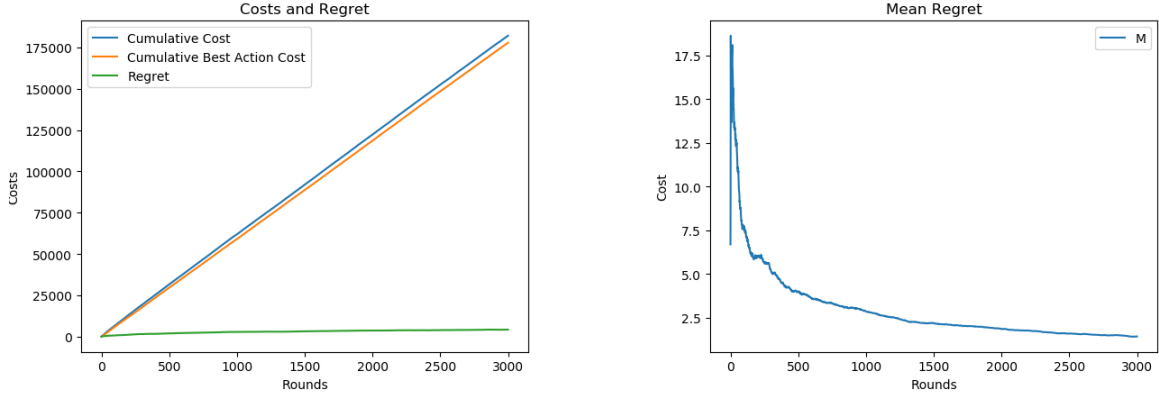


Figure 1: The simulation result on a complete graph with 20 nodes. The weight of each edge is a uniform random variable with support $[m_i - 3, m_i + 3]$, where m_i is chosen uniformly from $[3, 5]$. The left figure shows the cost of the best action, our action, and the difference between them (regret). The right figure shows the average regret through time.

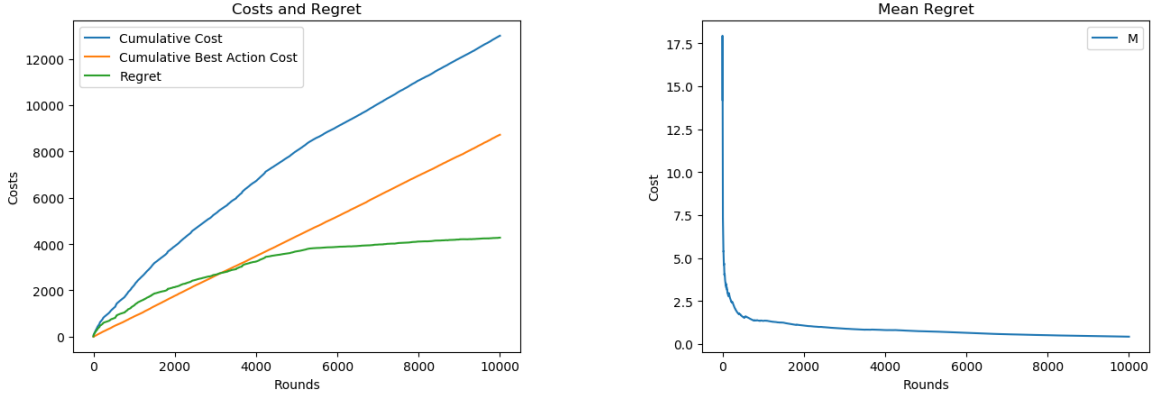


Figure 2: The simulation result on a complete graph with 20 nodes. The weight of each edge is random variable $m_i \cdot x_i$, where m_i, x_i are standard Weibull distribution with parameter 0.8. The left figure and the right figure shows the same things as in the previous figure.

3.2 Minimize the Maximum Distance

3.2.1 The case when the time is constant but unknown

We test the result of our method on a complete graph with 20 nodes. We test the situation when the weight random variable of each edge is a uniform random variable, and the results are shown in **Figure 3**. We also simulate the case when the random variable of weight on each edge is $m_i \cdot x_i$ where m_i, x_i are standard Weibull distribution with shape parameter 2 and 0.7, and the results are shown in **Figure 4, Figure 5**.

3.2.2 The general case

We test our algorithm on the general case. Because our algorithm just find the best solution in the long term approximately, so it is not reasonable to test the difference between our action and the approximate solution.

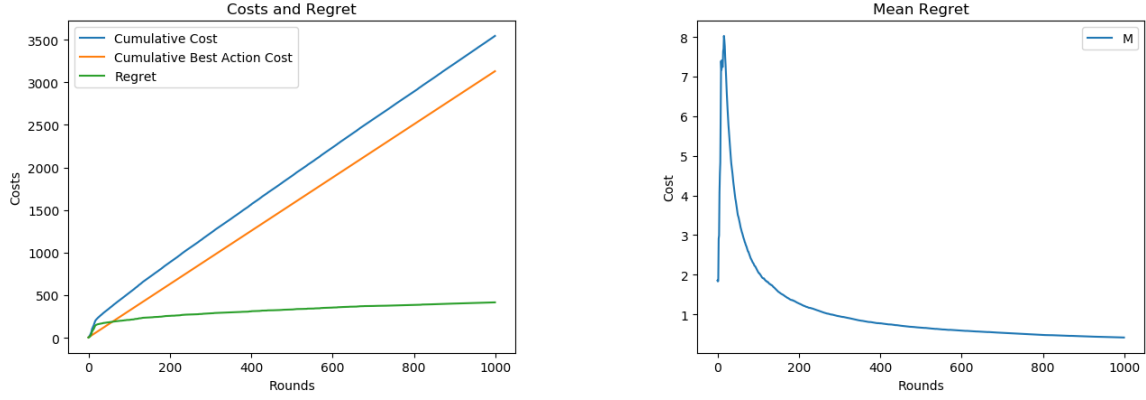


Figure 3: The simulation result on a complete graph with 20 nodes. The weight of each edge is a constant m_i , which is chosen uniformly from $[1, 5]$. The left figure and the right figure shows the same things as in the previous figure.

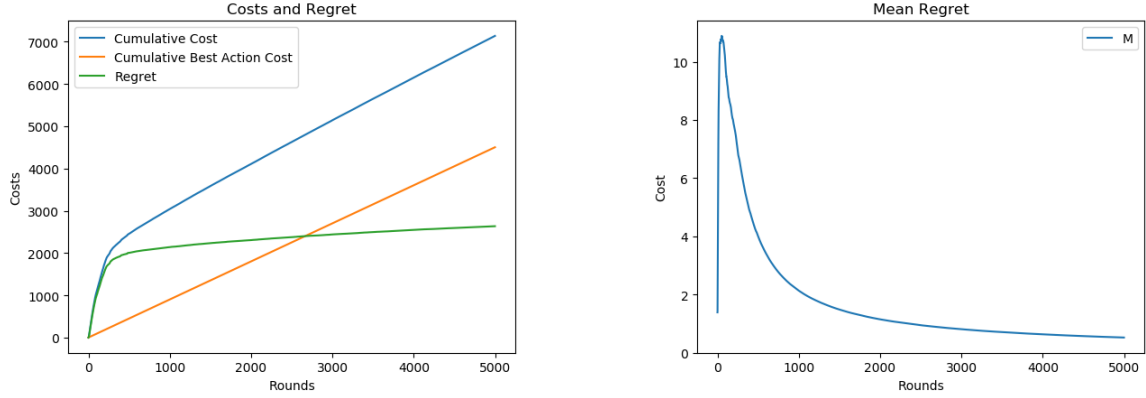


Figure 4: The simulation result on a complete graph with 20 nodes. The weight of each edge is random variable $m_i \cdot x_i$, where m_i, x_i are standard Weibull distribution with parameter 2. The left figure and the right figure shows the same things as in the previous figure.

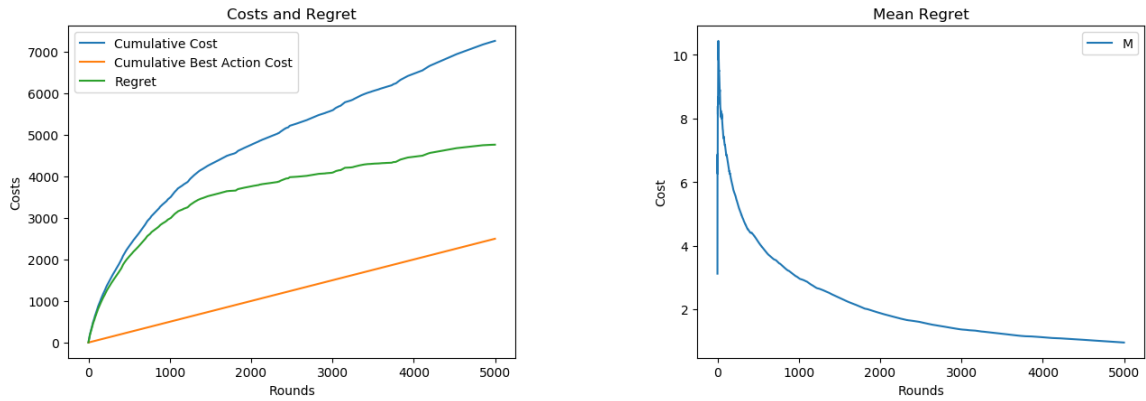


Figure 5: The simulation result on a complete graph with 20 nodes. The weight of each edge is random variable $m_i \cdot x_i$, where m_i, x_i are standard Weibull distribution with parameter 0.7. The left figure and the right figure shows the same things as in the previous figure.

Here we test on the ‘cheated’ best action, which knows the exact value of the random variables in each round and find the best solution on each round. The test results are shown in **Figure 6**.

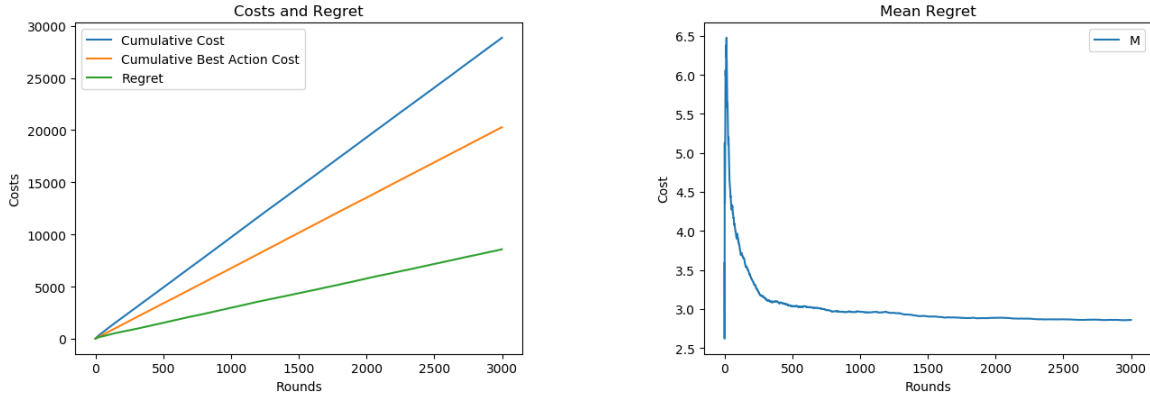


Figure 6: The simulation result on a complete graph with 20 nodes. The weight of each edge is a uniform random variable with support $[m_i - 3, m_i + 3]$, where m_i is chosen uniformly from $[4, 8]$. The left figure shows the cost of the best action which ‘cheats’ by find the best action each round, our action, and the difference between them. The right figure shows the average differenece through time.

3.3 Analysis and Further Discussion

First from the simulation results, we find that the algorithm will converge almost to the best action except for the general case. We think that this is due to the fact that the previous cases only cares about the mean of each edge, which we try to approximate in our algorithm. But in the general case when the solution is not only based on the mean of each random variable but also the distribution of each random variable, our method may not converge to the best action.

But the simulation results shows that our algorithm may also approximate the ‘cheating’ solution to some constant, which may depend on the graph topology and the distribution of the random variable. Then combine our previous argument that when the information transfer through the shortest path tree rooted at the center, the time is at most 2 times of the optimal strategy. So we can say that our strategy can solve this general case approximately.

Moreover, we can find that the speed of convergence are differnet. When the distrubution has larger variance or the random variable is long tail, the speed of convergence is slow.

4 Conclusion

In this report, we study the online information synchronization problem, with known weight of the communication graph. We apply the framework in [1] and let it fit our problem settings. We study the problem from 2 different aspect, one from the minimum cost, and the other from the minimum time spent during the synchronization process. In the minimum cost version, we show that our method can tackle this problem successfully by simulation. However under the minimum time spent setting, our algorithm can successfully solve the problem when the random variables are constant and can only get a approximate solution of the general case.

References

- [1] W. Chen, Y. Wang, and Y. Yuan, “Combinatorial multi-armed bandit: general framework, results and applications,” 2013.