Multi. Stat. HW1

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1 Problem 1

If we let the distribution to be $N(0,1)$, the percentage the the data that lies outside the outside bars is about to be 0.7%.

Proof.

$$Q_3 = \Phi^{-1}(0.75) \approx 0.6745$$
$$Q_1 = -Q_3 \approx -0.6745$$
$$IQR = Q_3 - Q_1 \approx 1.349$$

$$\begin{aligned} upper\_outlier &= Q_3 + 1.5 \cdot IQR \\ &= 0.6745 + 1.5 \times 1.349 \\ &= 2.698 \\ &\approx 2.7 \end{aligned}$$

So the portion that is bigger than the upper outside bar is about

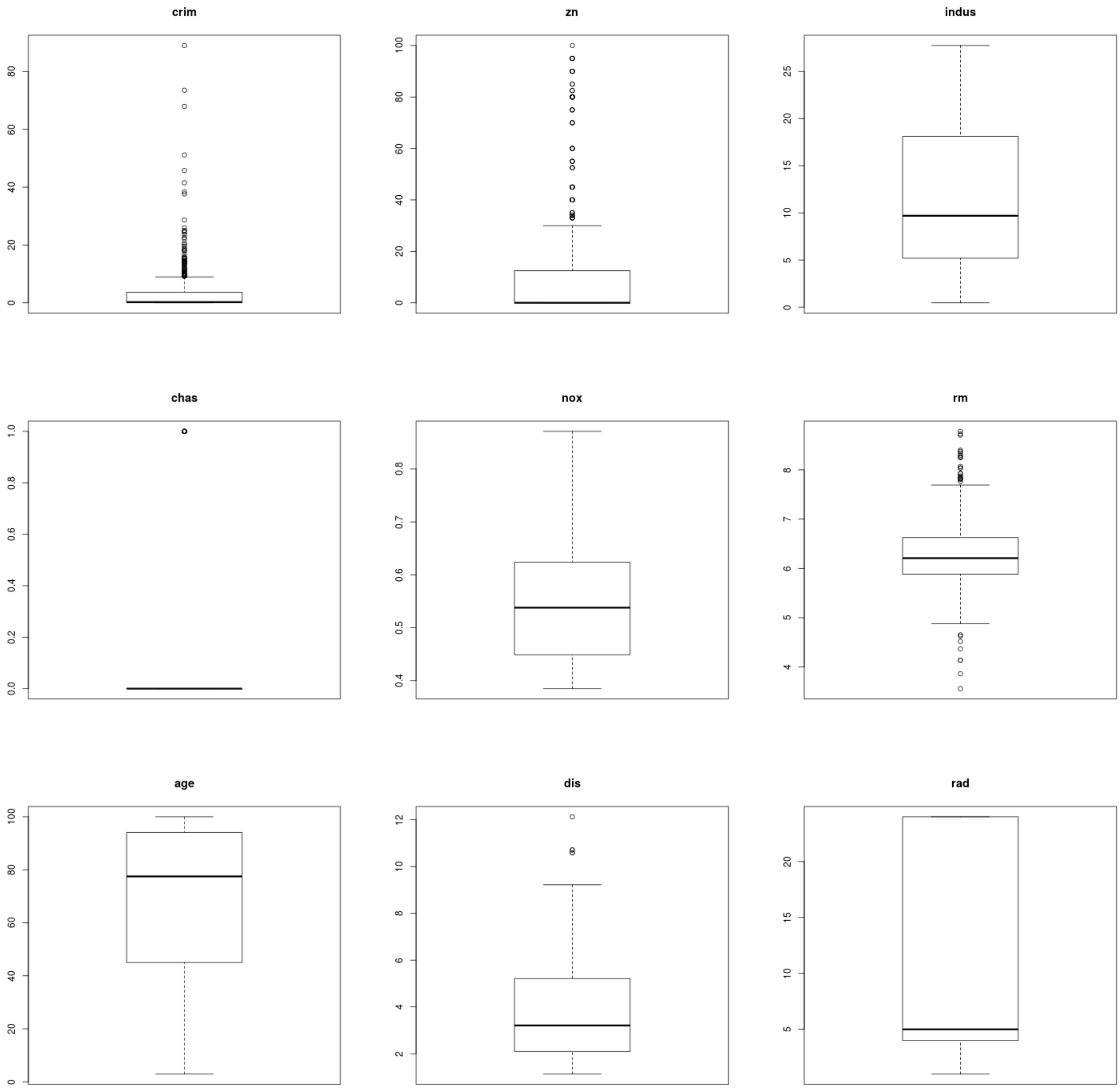
$$1 - \Phi^{-1}(2.7) \approx 0.0035$$

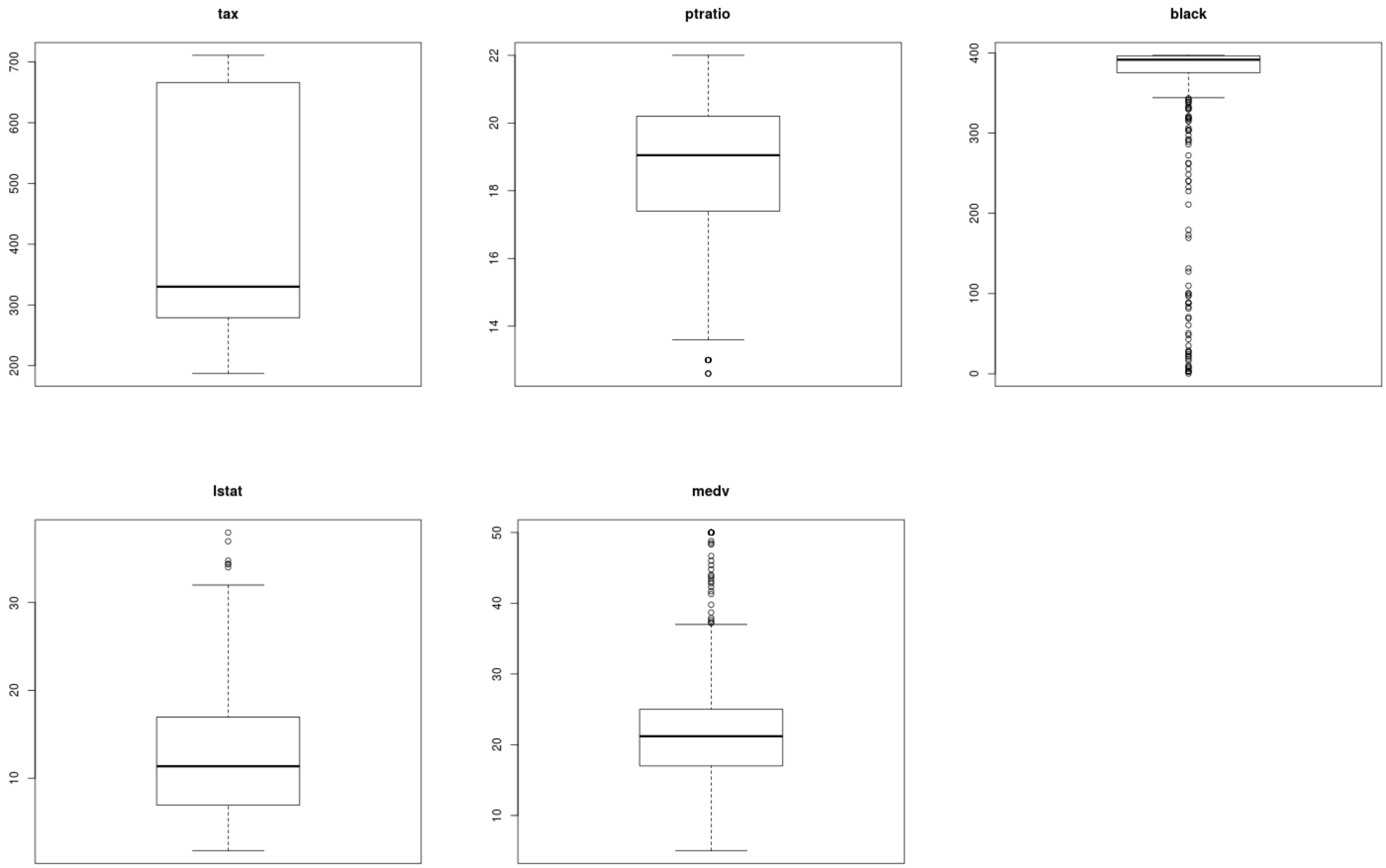
So the portion that lie outside the outside bars should be about $0.007 \approx 0.7\%$. □

If the data follow the distribution $N(0,\sigma^2)$, then the portion should also be about 0.7%.
We can get the data following the distribution $N(0,\sigma^2)$ by multiplying the standard normal distribution by σ, and the quantiles, IQR, and the value of the outside bars are also multiplied by σ after doing the transformation. The data follows $N(0,\sigma^2)$ that lies outside the outside bars also lies outside the outisde bars (of the dist. $N(0,1)$) when it is devided by σ, so the percentage of data that lies outside the outside bars in this problem is the same as the previous problem.

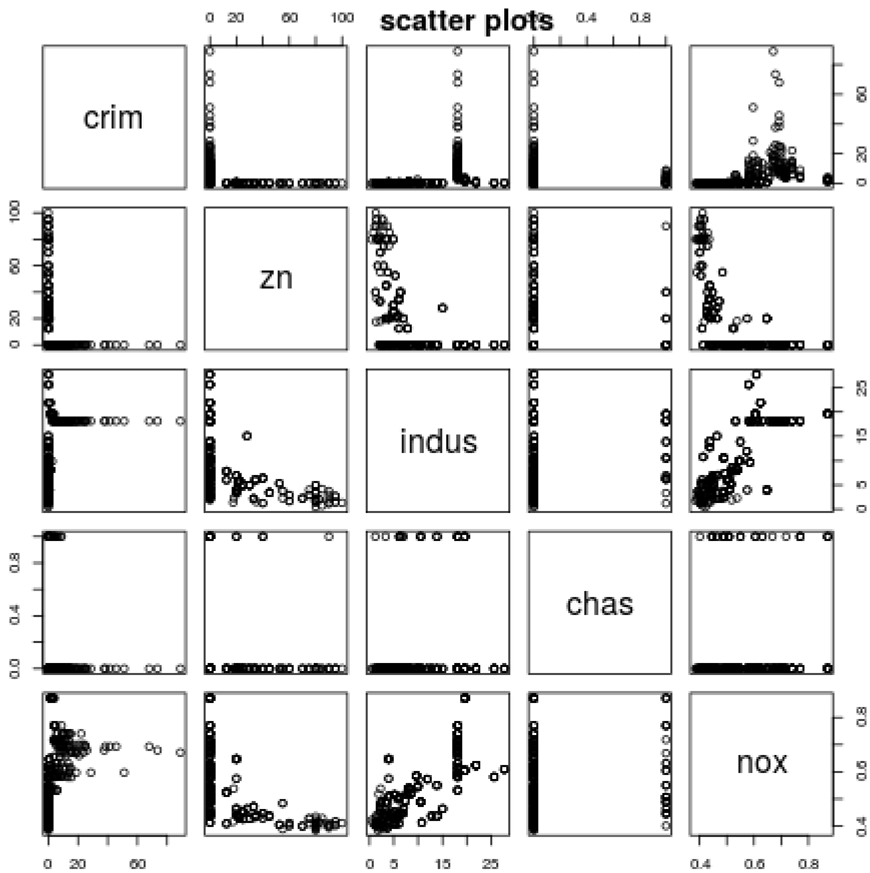
2 Problem 2

a.


crim
0
20
40
60
80
zn
0
20
40
60
80
100
indus
0
5
10
15
20
25
chas
0.0
0.2
0.4
0.6
0.8
1.0
nox
0.4
0.5
0.6
0.7
0.8
rm
4
5
6
7
8
age
0
20
40
60
80
100
dis
2
4
6
8
10
12
rad
5
10
15
20


tax
ptratio
black
lstat
medv

b.



The figure above is the matrix scatter plots for the first five variables.

c.
The correlation matrix with 4 digits is showned below.

```
> round(cor(data),4)
           crim      zn   indus    chas     nox      rm     age     dis     rad     tax ptratio   black   lstat    medv
crim     1.0000 -0.2005  0.4066 -0.0559  0.4210 -0.2192  0.3527 -0.3797  0.6255  0.5828  0.2899 -0.3851  0.4556 -0.3883
zn      -0.2005  1.0000 -0.5338 -0.0427 -0.5166  0.3120 -0.5695  0.6644 -0.3119 -0.3146 -0.3917  0.1755 -0.4130  0.3604
indus    0.4066 -0.5338  1.0000  0.0629  0.7637 -0.3917  0.6448 -0.7080  0.5951  0.7208  0.3832 -0.3570  0.6038 -0.4837
chas    -0.0559 -0.0427  0.0629  1.0000  0.0912  0.0913  0.0865 -0.0992 -0.0074 -0.0356 -0.1215  0.0488 -0.0539  0.1753
nox      0.4210 -0.5166  0.7637  0.0912  1.0000 -0.3022  0.7315 -0.7692  0.6114  0.6680  0.1889 -0.3801  0.5909 -0.4273
rm      -0.2192  0.3120 -0.3917  0.0913 -0.3022  1.0000 -0.2403  0.2052 -0.2098 -0.2920 -0.3555  0.1281 -0.6138  0.6954
age      0.3527 -0.5695  0.6448  0.0865  0.7315 -0.2403  1.0000 -0.7479  0.4560  0.5065  0.2615 -0.2735  0.6023 -0.3770
dis     -0.3797  0.6644 -0.7080 -0.0992 -0.7692  0.2052 -0.7479  1.0000 -0.4946 -0.5344 -0.2325  0.2915 -0.4970  0.2499
rad      0.6255 -0.3119  0.5951 -0.0074  0.6114 -0.2098  0.4560 -0.4946  1.0000  0.9102  0.4647 -0.4444  0.4887 -0.3816
tax      0.5828 -0.3146  0.7208 -0.0356  0.6680 -0.2920  0.5065 -0.5344  0.9102  1.0000  0.4609 -0.4418  0.5440 -0.4685
ptratio  0.2899 -0.3917  0.3832 -0.1215  0.1889 -0.3555  0.2615 -0.2325  0.4647  0.4609  1.0000 -0.1774  0.3740 -0.5078
black   -0.3851  0.1755 -0.3570  0.0488 -0.3801  0.1281 -0.2735  0.2915 -0.4444 -0.4418 -0.1774  1.0000 -0.3661  0.3335
lstat    0.4556 -0.4130  0.6038 -0.0539  0.5909 -0.6138  0.6023 -0.4970  0.4887  0.5440  0.3740 -0.3661  1.0000 -0.7377
medv    -0.3883  0.3604 -0.4837  0.1753 -0.4273  0.6954 -0.3770  0.2499 -0.3816 -0.4685 -0.5078  0.3335 -0.7377  1.0000
>
```

3 Problem 3

Proof.

$$r_{xy} = \frac{1}{n} \frac{\sum_{i=1}^{n} (x_i - \overline{x})(y_i - \overline{y})}{s_x \cdot s_y}$$

According to the conditions, the data was changed linearly, and the standard deviation is multiplied by a coefficient, we can get:

$$\begin{aligned}
s_i &= ax_i + b \\
t_i &= cy_i + d \\
\overline{s} &= a\overline{x} + b \\
\overline{t} &= c\overline{y} + d \\
s_s &= as_x \\
s_t &= cs_y
\end{aligned}$$

Then we can get:

$$\begin{aligned}
r_{st} &= \frac{\sum_{i=1}^{n} (s_i - \overline{s})\left(t_i - \overline{t}\right)}{n \cdot s_s \cdot s_t} \\
&= \frac{\sum_{i=1}^{n} (ax_i + b - a\overline{x} - b)(cy_i + d - c\overline{y} - d)}{n \cdot a \cdot s_x \cdot c \cdot s_y} \\
&= \frac{1}{n} \frac{\sum_{i=1}^{n} a \cdot (x_i - \overline{x}) \cdot c \cdot (y_i - \overline{y})}{a \cdot s_x \cdot c \cdot s_y} \\
&= \frac{1}{n} \frac{\sum_{i=1}^{n} (x_i - \overline{x})(y_i - \overline{y})}{s_x \cdot s_y} \\
&= r_{xy}
\end{aligned}$$

So the linear transformation does not change the sample correlation. □

Problem 4

2.8

The characteristic polynomial of the matrix is

$$f(\lambda) = (1 - \lambda)(-2 - \lambda) - 2^2.$$

Set $f(\lambda) = 0$ and we can get the root of $f(\lambda)$,

$$\lambda_1 = 2, \lambda_2 = -3.$$

Let

$$\mathbf{e}_1 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix}$$

We can see that

$$\mathbf{A}\mathbf{e}_1 = \lambda_1 \mathbf{e}_1, \mathbf{A}\mathbf{e}_2 = \lambda_2 \mathbf{e}_2, \|\mathbf{e}_1\| = \|\mathbf{e}_2\| = 1,$$

So $\mathbf{e}_1$ and $\mathbf{e}_2$ are 2 normalized eigenvectors of $\mathbf{A}$. And according to the sqectual theorem,

$$A = \lambda_1 \mathbf{e}_1 \mathbf{e}_1^T + \lambda_2 \mathbf{e}_2 \mathbf{e}_2^T = 2 \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \left[\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right] - 3 \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix} \left[\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right].$$

2.9

a.

$$B = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{6} \end{bmatrix}$$

Then we can find that $AB = BA = I$, so $A^{-1} = B$.

b.

$$S = [\mathbf{e}_1, \mathbf{e}_2], \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}, \Sigma^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} \end{bmatrix}.$$

Then we have $A = S\Sigma S^{-1}$. Because $S\Sigma S^{-1}S\Sigma^{-1}S^{-1} = S\Sigma\Sigma^{-1}S^{-1} = I$, we can know that $A^{-1} = S\Sigma^{-1}S^{-1}$, so the eigenvalues of A^{-1} is,

$$\mu_1 = \frac{1}{2}, \mu_2 = -\frac{1}{3}.$$

And the normalized eigenvectors remain the same as the eigenvectors of A, so

$$\mathbf{v}_1 = \mathbf{e}_1, \mathbf{v}_2 = \mathbf{e}_2.$$

c.

Write the matrix A^{-1} by spectral theorem,

$$A^{-1} = \mu_1 \mathbf{v}_1 \mathbf{v}_1^T + \mu_2 \mathbf{v}_2 \mathbf{v}_2^T = \frac{1}{2} \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \left[\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right] - \frac{1}{3} \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix} \left[\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right].$$

2.15

The quadratic form

$$\mathbf{Q} = \mathbf{x}^T \mathbf{A} \mathbf{x}, \mathbf{x}^T = [x_1, x_2], \mathbf{A} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

We can compute the eigenvalues of $\mathbf{A}$,

$$\lambda_1 = 4, \lambda_2 = 2.$$

The eigenvalues of the matrix is all positive, so it's positive definite.