Model-theoretical Semantics: Connectives

Connectives

Negation: Tom didn't leave.
Conjunction: Tom danced and Andy sang.
Disjunction: Tom danced or Andy sang.
Conditional: If Andy sang, then Tom danced.

Truth table

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

p	$\neg p$
1	0
0	1

p	q	$p \leftrightarrow q$
1	1	1
1	0	0
0	1	0
0	0	1

Meaning

$$[\![\text{and}]\!] = \begin{bmatrix} 1 \mapsto \begin{bmatrix} 1 \mapsto 1 \\ 0 \mapsto 0 \end{bmatrix} \\ 0 \mapsto \begin{bmatrix} 1 \mapsto 0 \\ 0 \mapsto 0 \end{bmatrix} \end{bmatrix} \qquad [\![\text{not}]\!] = \begin{bmatrix} 1 \mapsto 0 \\ 0 \mapsto 1 \end{bmatrix}$$

$$[\![\text{or}]\!] = \begin{bmatrix} 1 \mapsto \begin{bmatrix} 1 \mapsto 1 \\ 0 \mapsto 1 \end{bmatrix} \\ 0 \mapsto \begin{bmatrix} 1 \mapsto 1 \\ 0 \mapsto 0 \end{bmatrix} \end{bmatrix}$$

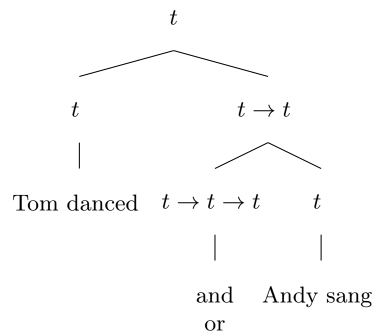
Type

$[\![\text{not}]\!] :: t \to t$

$[\![\text{and}]\!] :: t \to t \to t$

$[\![\text{or}]\!] :: t \to t \to t$

Composition



Logical equivalences

1. Double negation elimination

- $\neg\neg p = p$

2. Distributive laws

- $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$
- $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

3. Associative laws

- $(p \wedge q) \wedge r = p \wedge (q \wedge r)$
- $(p \vee q) \vee r = p \vee (q \vee r)$

4. DeMorgan's laws

- $\neg(p \wedge q) = \neg p \vee \neg q$
- $\neg(p \vee q) = \neg p \wedge \neg q$

5. Eliminability of the material conditional

- $p \to q = \neg p \vee q$

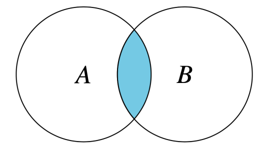
A sample proof

$(p \wedge q) \to r = p \to (q \to r)$

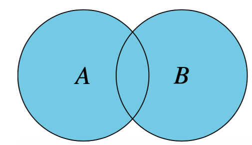
Proof:
$$\begin{aligned}(p \wedge q) \to r &= \neg(p \wedge q) \vee r && (5)\\ &= (\neg p \vee \neg q) \vee r && (4)\\ &= \neg p \vee (\neg q \vee r) && (3)\\ &= p \to (\neg q \vee r) && (5)\\ &= p \to (q \to r) && (5) \quad \square\end{aligned}$$

Operations on sets

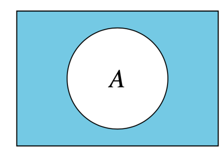
Intersection $A \cap B := \{x \mid x \in A \text{ and } x \in B\}$



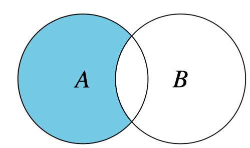
Union $A \cup B := \{x \mid x \in A \text{ or } x \in B\}$



Complementation $\overline{A} := \{x \mid x \notin A\}$



Difference $A - B := \{x \mid x \in A \text{ and } x \notin B\}$



Exclusive union $A \underline{\cup} B := (A \cup B) - (A \cap B)$