Notes on Basic Formal Tools for Semantics

Sets

Set: 'a collection of distinct objects, considered as an object in its own right' (Wiki)

Defining sets

- Extensionally: $\{\mathsf{a}, \mathsf{b}, \mathsf{c}\}$
- Intensionally: $\{x \mid x \text{ is a girl}\}$

Empty set: $\emptyset$ or $\{\}$

Set membership: $\mathsf{a} \in \{\mathsf{a}, \mathsf{b}, \mathsf{c}\}$

Relations between sets

- Proper subset: $A \subset B$ iff $A \subseteq B$ and $B \nsubseteq A$
- Equivalence: $A = B$ iff $A \subseteq B$ and $B \subseteq A$
- Superset: $B \supseteq A$ iff $A \subseteq B$

Power sets $\wp(A) = \{B \mid B \subseteq A\}$

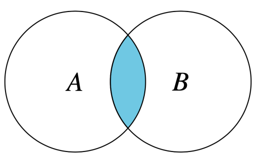
$\wp(A)$ is **partially ordered** iff for any X, Y, and Z in $\wp(A)$,

- Reflexivity: $X \subseteq X$.
- Transitivity: if $X \subseteq Y$ and $Y \subseteq Z$, then $Z \subseteq Z$.
- Antisymmetry: if $X \subseteq Y$ and $Y \subseteq X$, then $X = Y$.

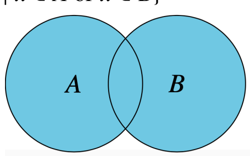
Pairs $(\mathsf{a}, \mathsf{b}) \neq (\mathsf{b}, \mathsf{a})$ $\qquad$ n**-tuples** $(\mathsf{a}, \mathsf{b}, \mathsf{c}, \ldots)$

Set operations

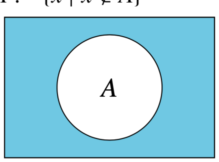
Intersection: $A \cap B := \{x \mid x \in A \text{ and } x \in B\}$



Union $A \cup B := \{x \mid x \in A \text{ or } x \in B\}$



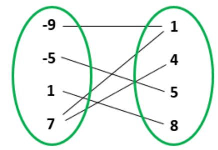
Complementation $\overline{A} := \{x \mid x \notin A\}$



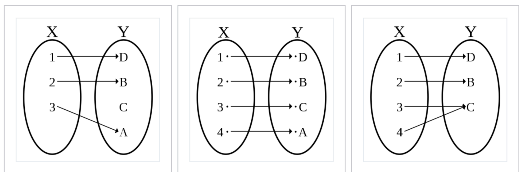
Relations and Functions

Relations as sets of pairs (Cartesian products)

$$A \times B := \{(x, y) \mid x \in A \text{ and } y \in B\}$$



Function: any relation where each input is paired with at most one output



injection $\qquad$ bijection $\qquad$ surjection

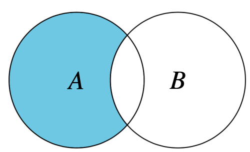
Characteristic function: a function whose range is the set of truth values: $f : A \to \{1, 0\}$

Identity function: a function mapping anything to itself: $f(x) = x$

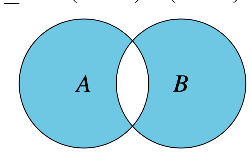
Currying: Given a function $f : (X \times Y) \to Z$, Currying constructs a new function:

$$g : X \to Y \to Z; \text{ or } g' : Y \to X \to Z$$

Difference $A - B := \{x \mid x \in A \text{ and } x \notin B\}$



Exclusive union $A \veebar B := (A \cup B) - (A \cap B)$



Lambda Calculus

Lambda notation: Consider a function:

$$f : \mathbb{N} \to \mathbb{N} \text{ for every } x \in \mathbb{N}.\ f(x) = x + 1$$

The function can be expressed as follows:

$$f = \lambda x : x \in \mathbb{N}. x + 1$$

The function which maps every x such that $x \in \mathbb{N}$ to $x + 1$

Semantics of Lambda terms

$\lambda x \in A.E$ is a function f mapping any objects u in A to $E[u/x]$.

Lambda reduction

η**-equivalence**: $f = \lambda x.f(x)$
Two functions are equivalent iff they return the same values for every argument.

β**-reduction**: $(\lambda x.E_1)(E_2) = E_1[E_2/x]$
"$E_1[E_2/x]$" is the expression just like E_1, but where every free occurrence of x has been replaced by E_2.

α**-equivalence**: $\lambda x.E = \lambda y.E[y/x]$
A specific choice of a bound variable doesnt matter.

Types of Lambda terms

Types are nothing more than sets of objects.

basic types | functional types $\qquad$ $a, b, \ldots \mid a \to b$

If x is of type a and E is of type b, then $\lambda x.E$ is a function of type $a \to b$.

Lambda abstraction

If x is of type a and E is of type b, then $\lambda x.E[x/y]$ is a function f of $a \to b$ for any variable y of type a.

Compositional rules

Functional Application

$$\frac{f :: a \to b \qquad x :: a}{f(x) :: b}$$

Predicate Modification

$$\frac{f :: a \to \mathsf{t} \qquad g :: a \to \mathsf{t}}{\lambda x.f(x) \wedge g(x) :: a \to \mathsf{t}}$$