Set theory, relations, and functions

1. Set theory

1.1. What is a set?

- A *set* is an abstract collection of distinct objects. Those objects are the *members/elements* of this set.

  In set theory, the concept 'set' is *extensional* (not *intensional*), namely, we don't bother about the ways in which the members of a set are chosen.

  Exercise: Which of the following expressions correctly describe the features of set members?

  a. Distinct to each other
  b. Ordered
  c. Relevant to each other
  d. The overall quantity is finite

- Some special sets
  - *Singleton set*: A set with exactly one member. E.g. $\{a\}, \{\{a, b\}\}$
  - *Empty/null set*: the set that has no member. Written as $\varnothing$ or $\{\ \}$.

- Convention of set-theoretic notations

  (1) a. (Ordinary) sets: $A, B, C, ...$
  b. Members of a set: $a, b, c, ..., x, y, z$
  c. Sets of sets: $\mathscr{A}, \mathscr{B}, \mathscr{C}$
  d. Cardinality of A: $|A|$ or $\#(A)$
  e. membership relation: $\in$

  Exercise: What is the cardinality of the set $\{a, a, b, b, c\}$?
  Is $\{\varnothing\}$ a singleton set or an empty set? What about $\{\varnothing, \varnothing\}$?

- Ways to define a set
  - *list/roster notation*: listing all its members

    (2) $E = \{4, 6, 8, ...\}$

    Note: The order of the elements doesn't matter. Elements can be listed multiple times.
  - *predicate/abstraction notation*: stating a condition that is satisfied by all and only the elements of the set to be defined

    Schema: $\{x \mid P(x)\}$, where x is a variable, and P is a predicate.

    (3) a. $\{x \mid x$ is an even number greater than $3\}$
    Read as: "the set of all x such that x is an even number greater than 3"
    b. $\{x \mid x$ is evenly divisible by 2 and is greater than or equal to $4\}$
  - *recursive rules*: using rules which generates/defines the members of this set.

(4) a. $4 \in E$
(the basis of recursion)
b. If $x \in E$, then $x + 2 \in E$
(to generate new elements from the elements defined before)
c. Nothing else belongs to E.
(to restrict the defined set to the elements generated by rules a and b)

1.2. Set-theoretic relations and operations

- **Relations of sets**

(5) a. **Identity/Equality**
Two sets are identical/equal iff they have exactly the same members.
$A = B$ iff for every x: $x \in A$ iff $x \in B$.
b. **Subsets**
i. A is a *subset* of B iff every element of A is an element of B.
$A \subseteq B$ iff for every x: if $x \in A$, then $x \in B$.
ii. A a *proper subset* of B iff A is a subset of B but not identical to B.
$A \subset B$ iff $A \subseteq B$ and $A \neq B$.
c. **Supersets**
i. A is a *superset* of B iff every element of B is an element of A (or say, B is a subset of A).
$A \supseteq B$ iff $B \subseteq A$
ii. A a *proper superset* of B iff B is a proper subset of A.
$A \subset B$ iff $B \subset A$
d. **Power sets**
i. Given a set A, the set of all the subsets of A is called the *power set* of A, written as $\mathscr{P}(A)$.
ii. $\varnothing$ is a subset of every set.

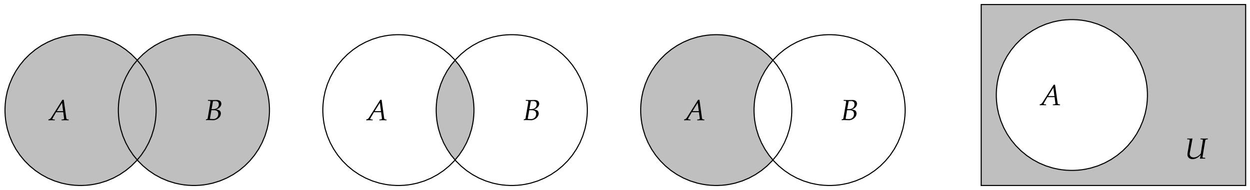
<u>Exercise:</u> Let S be an arbitrary set, consider:

a. is S a member of $\{S\}$?
b. is $\{S\}$ a member of $\{S\}$?
c. is $\{S\}$ a subset of $\{S\}$?
d. what is the set whose only member is $\{S\}$?

- **Operations on sets**

(6) a. **Union**
$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$ (or equivalently: $\bigcup\{A, B\}$)
b. **Intersection**
$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$ (or equivalently: $\bigcap\{A, B\}$)
c. **Difference**
$A - B = \{x \mid x \in A \text{ and } x \notin B\}$ also called: the *(relative) complement* of B in A
d. **(Absolute) Complement**
$A' = U - A$
$= \{x \mid x \in U \text{ and } x \notin A\}$ Other notations: $\complement A$, $A^{\complement}$, or $\overline{A}$

We can use Venn diagrams to illustrate these operations.



Union $A \cup B$ Intersection $A \cap B$ Difference $A - B$ Complement A'

<u>Exercise:</u> Simplify the following set-theoretic notations

(7) a. $\{x \mid x = a\}$
b. $\{x \mid x \neq x\}$
c. $(A - B) - B'$

1.3. Applications of set theory in natural language semantics

- The following categories denote sets of individuals:
  - common noun (*flower*, *cat*) $[\![\text{cat}]\!] = \{x \mid x \text{ is a cat}\}$
  - predictive adjective (*happy*, *gray*) $[\![\text{gray}]\!] = \{x \mid x \text{ is gray}\}$
  - VP and intransitive verb (*wears a hat*, *run*, *meow*) $[\![\text{wears a hat}]\!] = \{x \mid x \text{ wears a hat}\}$
- Quantificational determiners (e.g., *every*, *some*) denote relations of sets

(8) a. Every cat meows. $\{x \mid x \text{ is a cat}\} \subseteq \{x \mid x \text{ meows}\}$
b. Some cat meows. $\{x \mid x \text{ is a cat}\} \cap \{x \mid x \text{ meows}\} \neq \emptyset$

<u>Exercise:</u> Use set-theoretic notations to represent the meaning of the following sentences.

(9) a. Every gray cat meows.
b. No cat barks.
c. Two cats are meowing.

2. Relations

2.1. Ordered pairs and Cartesian products

- The elements of a set are not ordered. To describe functions and relations we will need the notion of an **ordered pair**, written as $\langle a, b\rangle$ (or (a, b)), where a is the first element of the pair and b is the second. Compare,

(10) If $a \neq b$, then $\{a, b\} = \{b, a\}$, but $\langle a, b\rangle \neq \langle b, a\rangle$
$\{a, a\} = \{a, a, a\}$, but $\langle a, a\rangle \neq \langle a, a, a\rangle$

- The **Cartesian product** of two sets A and B (written as $A \times B$) is the set of ordered pairs which take an element of A as the first member and an element of B as the second member.

(11) $A \times B = \{\langle x, y\rangle \mid x \in A \text{ and } y \in B\}$
E.g. Let $A = \{1, 2\}$, $B = \{a, b\}$, then $A \times B = \{\langle 1, a\rangle, \langle 1, b\rangle, \langle 2, a\rangle, \langle 2, b\rangle\}$,
$B \times A = \{\langle a, 1\rangle, \langle a, 2\rangle, \langle b, 1\rangle, \langle b, 2\rangle\}$

2.2. Relations, domain, and range

- A *relation* is a set of ordered pairs. For example:
  - Relations in math: =, $>$, $\neq$, ...
  - Relations in natural languages: the instructor of, the capital city of, ...

(12) a. $[\![$the capital city of$]\!] = \{\langle \text{USA}, \text{Washington}\rangle, \langle \text{China}, \text{Beijing}\rangle, \langle \text{France}, \text{Paris}\rangle, ...\}$
$= \{\langle x, y\rangle \mid y \text{ is the capital city of } x\}$

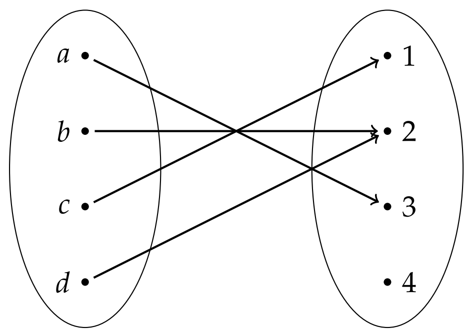
b. $[\![$invited$]\!] = \{\langle \text{Andy}, \text{Billy}\rangle, \langle \text{Cindy}, \text{Danny}\rangle, \langle \text{Emily}, \text{Flori}\rangle, ...\}$
$= \{\langle x, y\rangle \mid x \text{ invited } y\}$

- R is *a relation from A to B* iff R is a subset of the Cartesian product $A \times B$, written as $R \subseteq A \times B$.
  R is *a relation in A* iff R is a subset of the Cartesian product $A \times A$, written as $R \subseteq A \times A$.

(13) a. $[\![$the capital city of$]\!] \subseteq \{x \mid x \text{ is a country }\} \times \{y \mid y \text{ is a city}\}$

b. $[\![$the mother of$]\!] \subseteq \{x \mid x \text{ is a human }\} \times \{y \mid y \text{ is a human}\}$

- We can use a *mapping diagram* to illustrate a relation:



- A and B are the **domain** and **range** of R respectively, iff $A \times B$ is the **smallest** Cartesian product of which R is a subset. For example:

(14) Let $R = \{\langle 1, a\rangle, \langle 1, b\rangle, \langle 2, b\rangle, \langle 3, b\rangle\}$, then: $\text{Dom}(R) = \{1, 2, 3\}$, $\text{Range}(R) = \{a, b\}$

Discussion: Why is that the following definitions are problematic? Provide some counterexamples.

(15) a. A and B are the domain and range of R iff $R = A \times B$.

b. A and B are the domain and range of R iff $R \subseteq A \times B$.

2.3. Properties of relations

- **Reflexivity, symmetry, and transitivity**

  Given a set A and a relation R in A, ...

  - R is *reflexive* iff for every x in A, $\langle x, x\rangle \in R$; otherwise R is *nonreflexive*.
    If there is no pair of the form $\langle x, x\rangle$ in R, then R is *irreflexive*.
  - R is *symmetric* iff for every xy in A, if $\langle x, y\rangle \in R$, then $\langle y, x\rangle \in R$; otherwise R is *nonsymmetric*.
    If there is no pair $\langle x, y\rangle$ in R such that the pair $\langle y, x\rangle$ is in R, then R is *asymmetric*.
  - R is *transitive* iff for every xyz in A, if $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$, then $\langle x, z\rangle \in R$; otherwise R is *nontransitive*.
    If there are no pairs $\langle x, y\rangle$ and $\langle y, z\rangle$ in R such that the pair $\langle x, z\rangle$ is in R, then R is *intransitive*.

  Exercise: Identify the properties of the following relations w.r.t. reflexivity, symmetry, and transitivity. (Make the strongest possible statement. For example, call a relation 'irreflexive' instead of 'nonreflexive', if satisfied.)

  (16) a. $=$ reflective, symmetric, transitve
  b. $\leq$
  c. $\neq$
  d. $\subseteq$

  (17) a. is a sister of
  b. is a child of

- It is helpful in assimilating the notions of reflexivity, symmetry and transitivity to represent them in *relational diagrams*. If x is related to y (namely, $\langle x, y\rangle \in R$), an arrow connects the corresponding points.

$$R = \{\langle a, a\rangle, \langle a, b\rangle, \langle b, b\rangle, \langle b, c\rangle, \langle c, c\rangle\}$$

$$\begin{tikzcd} a \arrow[loop above] \arrow[r] & b \arrow[loop above] \arrow[dl] \\ c \arrow[loop below] & \end{tikzcd}$$

3. Functions

3.1. What is a function?

- A relation is a function iff each element in the domain is paired with **just one** element in the range.

  (18) $f(x) = x^2$ for $x \in \mathbb{N}$

  - a function f is *from A (in)to B* (written as 'f: $A \rightarrow B$') iff $\text{Dom}(f) = A$ and $\text{Range}(f) \subseteq B$.
  - a function f is *from A onto B* iff $\text{Dom}(f) = A$ and $\text{Range}(f) = B$.

- We may specify functions with lists, tables, or words. (Don't confuse 'tables' with 'mapping diagrams'.)

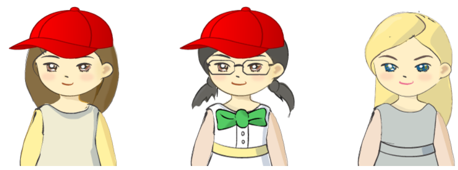
  (19) a. $F = \{\langle a, b\rangle, \langle c, b\rangle, \langle d, e\rangle\}$
  b. $F = \begin{bmatrix} a \rightarrow b \\ c \rightarrow b \\ d \rightarrow e \end{bmatrix}$

c. F is a function f with domain $\{a, b, c\}$ such that $f(a) = f(c) = b$ and $f(d) = e$.

Exercise: Define negation *it is not the case that* and conjunctive *and* as functions.

3.2. Characteristic function

- Recall that the following categories can be interpreted as sets of individuals: common nouns, predicative adjectives, intransitive verbs, VPs, ...



Annie Betty Cindy

(20) a. $[\![\text{wears a hat}]\!] = \{\text{Andy}, \text{Betty}\}$

b. $[\![\text{wears a hat}]\!] = \{x \mid x \text{ wears a hat}\}$

Alternatively, the semantics of these expressions can also be modeled as functions from the set of individual entities to the set of truth values.

(21) a. $[\![\text{wears a hat}]\!] = \{\langle\text{Andy}, 1\rangle, \langle\text{Betty}, 1\rangle, \langle\text{Cindy}, 0\rangle\}$

b. $[\![\text{wears a hat}]\!] = \begin{bmatrix} \text{Andy} \to 1 \\ \text{Betty} \to 1 \\ \text{Cindy} \to 0 \end{bmatrix}$

c. $[\![\text{wears a hat}]\!]$ = the function f from individual entities to truth values such that $f(x) = 1$ if x wears a hat, and $f(x) = 0$ otherwise.

Characteristic function: a function whose range is the set of possible truth values $\{1, 0\}$.

Exercise: provide the characteristic function for wears a bowtie.

- **Relations as functions into sets**: since a relation can be defined as a set of pairs, and any set has a characteristic function, relations can be equivalently conceptualized as functions from pairs to truth values.

(22) a. $[\![\text{is next to}]\!] = \{\langle a, b\rangle, \langle b, a\rangle, \langle b, c\rangle, \langle c, b\rangle\}$

b. $[\![\text{is next to}]\!] = \begin{bmatrix} \langle a, b\rangle \to 1 \\ \langle b, a\rangle \to 1 \\ \langle b, c\rangle \to 1 \\ \langle c, b\rangle \to 1 \\ \langle a, c\rangle \to 0 \\ \langle c, a\rangle \to 0 \end{bmatrix}$

c. $[\![\text{is next to}]\!]$ = the function f from pairs of individual entities to truth values such that $f(x, y) = 1$ if x is next to y, and $f(x, y) = 0$ otherwise.