Anti-Intelletual Waterloo Euclid

Doing such anti-intelletual contest will reduce your life by 10 years

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April 5, 2022

§1 Solution to 7(b)

Because $\frac{AC}{AD} = \frac{3}{4}$, we may let $AC = 3x$ and $AD = 4x$.

First, using cosine law, we find that

$$\begin{aligned} AD^2 &= AC^2 + CD^2 - 2\cos(\angle ACD) \cdot AC \cdot CD \\ 16x^2 &= 9x^2 + 1 + \frac{6}{5}(3x) \\ 35x^2 - 18x - 5 &= 0 \\ (7x-5)(5x+1) &= 0 \\ x &= \frac{5}{7}, -\frac{1}{5} \end{aligned}$$

Because length can only be positive, then $x = \frac{5}{7}$ only. It follows that $AC = \frac{15}{7}$ and $AB = \frac{20}{7}$.

Now, use Stewart's theorem, let $AB = y$, we find that

$$\begin{aligned} y^2 + 2\left(\frac{20}{7}\right)^2 &= 3\left(\frac{15}{7}\right)^2 + 1 \cdot 2 \cdot 3 \\ y^2 &= \frac{675 + 6 \cdot 49 - 800}{49} \\ y^2 &= \frac{169}{49} \\ y &= \frac{13}{7} \end{aligned}$$

Therefore, the length of AB is $\frac{13}{7}$.

§2 Solution to 8(a)

We first find the points of intersection:

$$\begin{aligned} ax^2 + 2 &= -x + 4a \\ ax^2 + x + 2 - 4a &= 0 \\ (x+2)(ax + (1-2a)) &= 0 \\ x &= -2, \frac{2a-1}{a} \end{aligned}$$

Now, we compute the area of the triangle. We will use the y-axis as the base of the triangle, then the height is $2 + \frac{2a-1}{a}$. Then, the length of the base is $4a - 2$. Thus, the area of the triangle is $\frac{(2a-1)(4a-1)}{a}$. In particular, we know that

$$\begin{aligned} \frac{(2a-1)(4a-1)}{a} &= \frac{72}{5} \\ \frac{8a^2 - 6a + 1}{a} &= \frac{72}{5} \\ 40a^2 - 30a + 5 &= 72a \\ 40a^2 - 102a + 5 &= 0 \\ (20a-1)(2a-5) &= 0 \\ a &= \frac{5}{2}, \frac{1}{20} \end{aligned}$$

Because $a > \frac{1}{2}$, we know that $a = \frac{5}{2}$.

§3 Solution to 8(b)

Because the angle form an arithmetic sequence, one of the angle must be 60°. We let the other two angles be $60^\circ - \theta$ and $60^\circ + \theta$. By sine law, we see that

$$\frac{x}{\sin(60^\circ)} = \frac{y}{\sin(60^\circ - \theta)} = \frac{z}{\sin(60^\circ + \theta)}$$

where x, y, z indicated the side length of the triangle.

If x, y, z form an geometric sequence, we need to have $\sin(60^\circ - \theta), \sin(60^\circ), \sin(60^\circ + \theta)$ to form an geometric sequence. Thus,

$$\begin{aligned}
\sin^2(60^\circ) &= \sin(60^\circ - \theta)\sin(60^\circ + \theta) \\
\frac{3}{4} &= \Big(\sin(60^\circ)\cos\theta + \sin\theta\cos(60^\circ)\Big)\Big(\sin(60^\circ)\cos\theta - \sin\theta\cos(60^\circ)\Big) \\
\frac{3}{4} &= \frac{3}{4}\cos^2\theta - \frac{1}{4}\sin^2\theta \\
3 &= 3\cos^2\theta - (1 - \cos^2\theta) \\
1 &= \cos^2\theta
\end{aligned}$$

This is impossible because $\cos\theta = \pm 1$ iff $\theta = 180^\circ k$, where $k \in \mathbb{Z}$. However, we must have $0^\circ < \theta < 60^\circ$.

§4 Solution to 9 (d)

First, we compute the sum of all elements.

Claim — The sum of (m, n)-sawtooth is $nm^2 - n + 1$.

Proof. Each $1, 2, 3, 4, \cdots, m-1, m, m-1, \cdots 1$ has sum

$$\frac{(m+1)(m)}{2} + \frac{m(m-1)}{2} = m^2$$

There are in total of n such reptitions, therefore, the sum is nm^2. However, we have counter the middle $n-1$ 1's twice. We need to subtract them. Thus the sum is $nm^2 - (n-1)$, and the result follows. $\square$

In total there are $n(2m-1) - (n-1) = 2mn - 2n + 1$ numbers. Then the average is

$$\begin{aligned}\frac{nm^2 - n + 1}{2mn - 2n + 1} &= \frac{nm^2 - mn + \frac{1}{2}m + mn - \frac{1}{2}m - n + 1}{2mn - 2n + 1} \\ &= \frac{m}{2} + \frac{mn - \frac{1}{2}m - n + 1}{2mn - 2n + 1} \quad (1) \\ &= \frac{m+1}{2} + \frac{-\frac{1}{2}m + \frac{1}{2}}{2mn - 2n + 1} \quad (2)\end{aligned}$$

In (1), we see that $mn - \frac{1}{2}m - n + 1 > 0$ (because $(m-1)\left(n - \frac{1}{2}\right) + \frac{1}{2} > 0$), the denominator is greater than 0. Thus, we see that the average is greater than $\frac{m}{2}$.

In (2), we see that $-\frac{1}{2}m + \frac{1}{2} < 0$, the denominator is greater than 0. Thus, we see that the average is less than $\frac{m+1}{2}$.

It follows that the average lies strictly between $\frac{m}{2}$ and $\frac{m+1}{2}$, meaning that the average cannot be an integer.

§5 Solution to 10(b)

After the first two toppings are places, there could be an $0^\circ \le \theta \le 180^\circ$. Now we draw in line ℓ_3. If this line passes through the overlapping region, then the probability of containing all three toppings is 1. If ℓ_3 does not passes through the region, the probability is $\frac{1}{2}$.

Therefore, the probability of there exist a all three toping is

$$\begin{aligned}
&\frac{1}{(180^\circ)^2}\int_0^{180^\circ} \theta + \frac{1}{2}(180^\circ - \theta)d\theta \\
=&\frac{1}{(180^\circ)^2}\left(90^\circ\theta + \frac{1}{4}\theta^2\right)\Bigg|_0^{180^\circ} \\
=&\frac{1}{(180^\circ)}\left(90^\circ + \frac{1}{4}180^\circ\right) \\
=&\frac{3}{4}
\end{aligned}$$