AP Physics C: Mechanics

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Overview

§1 Unit 1: Kinematics

Recommandation

This unit weighs about 14-20% on the AP exam. The recommand time spend on this unit is about 11 to 22 hours.

§1.1 Motion in One Dimension

Displacement vs. Distance & Speed vs. Velocity

Distance is a scaler (only has value). Displacement is a vector (has both direction and value). Say that you walked 10m east, then 20m west. Your distance travelled in 30m, whereas your displacement is 10m[west].

Speed is a scaler, and velocity is a vector. Your average speed is given by $\frac{\text{distance}}{\text{time}}$, your average velocity is given by $\frac{\text{displcement}}{\text{time}}$. Say that you walked 10m east, then 20m west in 5s, then your average speed is $6\,\frac{\text{m}}{\text{s}}$, and your velocity is $2\frac{\text{m}}{\text{s}}$[west].

Lastly, we usually use signs to replace the direction at the end of the vector. See claim 1.1.3 for general sign convention. Distance and speed are always nonnegative, because they do not have a direction.

Kinematic Equations

Theorem 1.1.1 (Kinamatic Equations)
If the acceleration on an object is constant, then the following are true:

$$v_f = v_0 + at$$
$$x_f = x_0 + v_0 t + \frac{1}{2}at^2$$
$$v_f^2 = v_0^2 + 2a(x_f - x_0)$$

Where a represents the acceleration, v_0 represents the initial velocity, v_f represents the final velocity, x_0 represents the initial position of the object, and x_f represent the final position of the object.

In fact, we have a more general result if the acceleration is not constant.

Theorem 1.1.2 (Relations with position, velocity, and acceleration)
If the function of position of the object with respect to time is given by $x(t)$, the function of velocity with respect to time is given by $v(t)$, and the function of acceleration of the object with respect to time is given by $a(t)$, then we have

$$x'(t) = v(t) \iff \int v(t)dt = x(t)$$
$$v'(t) = a(t) \iff \int a(t)dt = v(t)$$

Claim 1.1.3 — We usually consider an object travelling right/east travelling to the positive direction, whereas left/west is negative. We usually use variable x for the east/west direction motion. Also, up/north is considered to be positive and down/south is considered to be negative. We usually use variable y for the north/south direction motion.

Theorem 1.1.4 (Average acceleration)
If the acceleration is nonconstant, the average acceleration is the change in velocity divided time.

Problem

Car one moves at a constant speed of $40\frac{\text{m}}{\text{s}}$. Car two is at a full stop. As soon as car one passes car two, car two starts to accelerate at a constant rate of $2\frac{\text{m}}{\text{s}^2}$, but only until it reaches its maximum speed of $60\frac{\text{m}}{\text{s}}$.

(a) How much time passes before car two catches us to car one?

(b) How far has car two travelled by this time?

Solution

(a) Let the initial position of car two be the origin, we are albe to find that the function of the position of car one is $40t$. Also, the function of car two is

$$\begin{cases} t^2 & \text{for } t \leq 30 \\ 900 + 60t & \text{for } t > 30 \end{cases}$$

If $40t = t^2$, we find that $t = 0\text{s}$ or $t = 40\text{s}$. $t = 0\text{s}$ is when car one passes car two which is extraneous, $t = 40\text{s}$ is out of domain which is also extraneous. Then we look at if $40t = 900 + 60t$. We find that $t = 45\text{s}$. This is an acceptable solution.

(b) We can compute the distence that car one has travelled or car two. Car one is easier to compute, so the distence that both cars travelled is $40 \cdot 45 = 1800\text{m}$.

Acceleration Due to Gravity

Whenever we drop some object the acceleration is constant (assume air resistance is negligible). This constant is $9.8\frac{\text{m}}{\text{s}^2}$. In most area on Earth, the value is $9.81\frac{\text{m}}{\text{s}^2}$. However, some regions are not. This value can be computed by $\frac{GM}{r^2}$ in Unit 3 and 7. College Board claims it to be $10\frac{\text{m}}{\text{s}^2}$ for the calculations to be easier. Because we are doing AP, we will also use $10\frac{\text{m}}{\text{s}^2}$.

Problem

Suppose you are 2m tall and you are standing on top of a 78m tall building and throw a ball upward with a speed of $25\frac{\text{m}}{\text{s}}$.

(a) What is the maximum height that the ball will reach?

(b) How much time does it take to reach the ground?

(c) What is the velocity of the ball when it reaches the ground?

(d) What is the speed of the ball when it reaches the ground?

Solution

(a) Set the bottom of the building to be the origin. The function of the position of the ball with respect to time is $y = 80 + 25t - 5t^2$. The vertex of this parabola is at $t = 2.5$s, and the height of the ball is 111.25m.

(b) When the ball reaches the ground, the y value is 0. We just need to solve the equation $80+25t-5t^2 = 0$. We may find that $t = 7.22$s or $t = -2.22$s by quadratic formula. $t = -2.22$s is extraneous, so we know that $t = 7.22$s.

(c) We can use the time that we found in (b) to find the velocity. The velocity is $v = v_0 + at = 25 - 10(7.22) = -47.2\frac{\text{m}}{\text{s}}$.

Alternatively, we can use the third kinetmatic equation. $v_f^2 = v_0^2+2a(x_f-x_0) = 25^2+2(-10)(0-80) = 2225$. Therefore, $v_f = \pm 47.2\frac{\text{m}}{\text{s}}$. We know that the object is travelling downwards when it gets to the ground, then $v_f = -47.2\frac{\text{m}}{\text{s}}$.

(d) The speed is just the absolute value of the velocity. Thus, the ball is travelling with speed $47.2\frac{\text{m}}{\text{s}}$ when it reaches the ground.

§1.2 Motion in Two Dimensions

Vectors

The vectors in section 1.1 only has one direction. What if it has more than one direction? Say I walked 50m East and 50m North. Now we can find the total displacement by Pythagorean's Theorem. The displacement is $\sqrt{50^2 + 50^2} = 70.71$m, and the angle can be computer by $\arctan\left(\frac{50}{50}\right) = 45°$. Therefore, the displacement vector is 70.71m[E45°N]. The distance in this case is 100m.

Problem

Compute the displacement vector if you walk 35m west and 75m north.

Solution

The total displacement is $\sqrt{35^2 + 75^2} = 82.76$m. The angle is given by $\arctan\left(\frac{35}{75}\right) = 25°$. Thus the vector is 82.76m[N25°W].

We can also decompose vectors into their individual components.

Problem

Decompose the vector 63m[S34°E]

Solution

The East/West direction is given by $63 \cdot \sin(34°) = 35.23$m[E], and the North/South direction is given by $63 \cdot \cos(34°) = 52.23$m[S].

Two Dimensional motion

Theorem 1.2.1

If we break down the a two dimensional motion into two one dimensional motion, then each components is indenpendent from the other. The only thing they share is time.

Problem

If you are standing on top of a 45m tall building and kicks a ball down (meaning that the ball is initially travelling parallel to the ground) with initial velocity $15\frac{\text{m}}{\text{s}}$.

(a) How long does it take the ball to reach the ground?

(b) What is the velocity when the ball reach the ground?

(c) How far away will it land from the building?

Solution

(a) Let the bottom of the building be 0. The initial y velocity is 0 and acceleration is -10. We know that the function of the y position with respect to time is $y = 45 - 5t^2$. The ball reaches height 0 at $t = 3$s or $t = -3$s. We know that $t = -3$ is extraneous, so it takes 3s for the ball to reach the ground.

(b) The final y velocity is $v_f = v_0 + at = -30\frac{\text{m}}{\text{s}}$. Because there is no acceleration on the x component, then the x velocity stays the same. Thus, the final x velocity is $15\frac{\text{m}}{\text{s}}$.

Now we combine the two vectors together. The total velocity is $\sqrt{15^2 + 30^2} = 33.54\frac{\text{m}}{\text{s}}$. The angle is given by $\arctan\left(\frac{30}{15}\right) = 63°$. Therefore, the final velocity is $33.54\frac{\text{m}}{\text{s}}$[63° below the horizon].

(c) Because the x velocity stays the same, the total x displacement is $15 \cdot 3 = 45$m. Therefore, the ball will land 45m away from the building.

Projectile motion

Theorem 1.2.2

When an object is in the projectile motion. The acceleration on the y component is the acceleration due to gravity, and the acceleration on the x component is 0.

Problem

If you are standing on top of a 60m tall building and throw a ball with velocity $23\frac{\text{m}}{\text{s}}$[76° above horizontal]. How far will it land from the building?

Solution

We will first decompose the velocity vector, then use y component to find the time, and lastly find the x displacement.
We find that $23\frac{\text{m}}{\text{s}}$[76° above horizontal] $= 22.32\frac{\text{m}}{\text{s}}$[vertical]$+5.56\frac{\text{m}}{\text{s}}$[horizontal]. The y position with respect to time is given by $y = 60 + 22.32t - 5t^2$. Then $t = 6.353$s. Therefore, the object travelled $6.353 \cdot 5.56 = 35.32$m in the x component.

Problem

If you are standing on the ground, and throws a ball umltiple times. Assume the initial velocity is the same. At what angle will the ball travel the furthest before landing?

Solution

Let the initial velocity be v and the direction be θ above horizontal, the y component of the velocity is $v\sin\theta$. Then the function of the y position with respect to time is $y = 0 - v\sin\theta \cdot t + \frac{1}{2}at^2$. We find that $t = \frac{2v\sin\theta}{a}$.
The x component of the initial velocity is $v\cos\theta$. Therefore the x displacement is $\frac{2v^2\sin\theta\cos\theta}{a}$. Because v and a are constants, then it is up to $\sin\theta\cos\theta$ for maximum.
We know that $\sin\theta\cos\theta = \frac{\sin 2\theta}{2}$. $\sin 2\theta$ achieves maximum when $2\theta = 90°$. Therefore, $\theta = 45°$ if we want to ball to travel the furthest. In particular, the ball will travel $\frac{v^2}{a}$ before hitting the ground.

Lemma 1.2.3

Objects will travel the furthest when we throw the them at $45°$ above the horizon.

§2 Unit 2: Newton's Laws of Motion

Recommandation

This unit weighs about 17-23% on the AP exam. The recommand time spend on this unit is about 12 to 24 hours.

§2.1 First and Second Law

Forces

There are many forces acting on an object. Force of gravity $F_g = mg$, normal force F_N, force of tension F_t, force of friction F_f, spring force F_s, etc. The forces named above are the ones we will be dealing with in this course. The unit of force is N which is equal to $\frac{\text{kg}\cdot\text{m}}{\text{s}^2}$.

We draw the free body diagram with every force components exerted on the center of mass of an object to analyze what the net force on the object is.

Theorem 2.1.1 (Direction of Forces)

Each force has its own directions. The following forces has an fixed direction.

- The normal force is always perpandicular to the surface.
- The force of gravity is always pointing downward.
- The force of friction is always parallel to the surface.

Theorem 2.1.2 (Magnitude of the gravitational force)

he magnitute is given by $F_g = mg$, where the m is the mass of the object and g is the constant of acceleration sue to gravity. Unless otherwise specified, we are assuming the object is on Earth (where $g = 10\frac{\text{m}}{\text{s}^2}$).

Theorem 2.1.3 (Magnitude of force of friction)

There exist the static frictional force and the kinetic frictional force. Static frictional force is for object at rest, and kinetic frictional force is for objects that are moving.

The ***maximum*** force of friction is given by the coeffcient of friction multiplied by the magnitute of the normal force. **The force of friction is never greater than the sum of the other forces in the same direction, because it will not make the object accelerate in the opposite direction that the force is applied**

The static coeffcient of friction is μ_s and the kinetic coeffcient of friction is μ_k. Thus, the inequalities are given by

$$\begin{cases} F_{f(\text{static})} \leq \mu_s F_N \\ F_{f(\text{kinetic})} \leq \mu_k F_N \end{cases}$$

Also, the static friction is greater than or equal to the kinetic friction. In other words $\mu_s \geq \mu_k$.

The other forces can be exerted in any direction. They will usually be given in a question

The force of friction theory can be confusing. Here is an example:

Problem

If a person is pulling an 100kg box to the left with 50N of force, but the box is not moving. Given that the static coeffcient of friction $\mu_s = 0.24$. Find the force of friction acting on this object.

Solution

Theorem 2.1.4 (Newton's First law of motion)

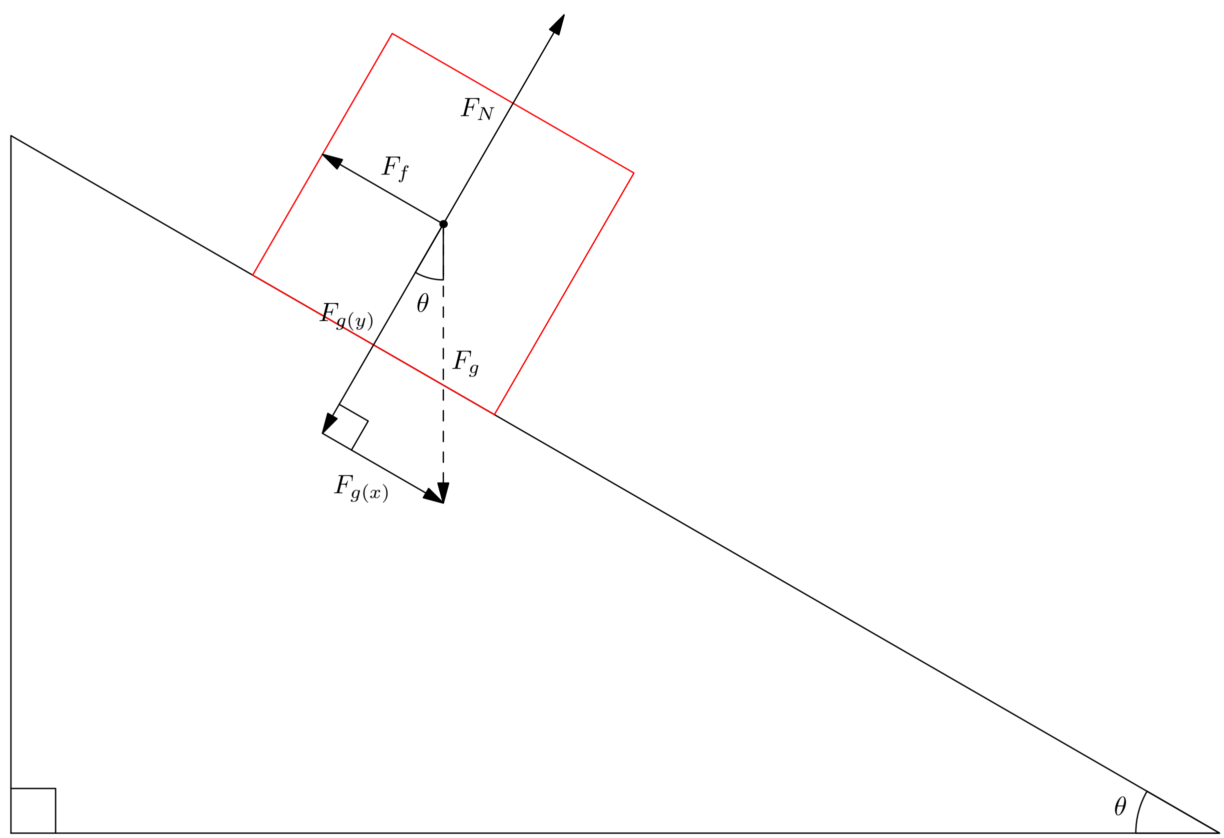
The object stays in current motion (either at rest or moving at a constant velocity), if and only if there are no net external forces on the object.

Problem

If an object with mass 5kg is resting on a inclined plane at an angle with 22°. Find the normal force, the force of gravity, and the force of friction.

Solution

The object remains at rest, so the net force on the object. We can break down the force of gravity into two components. $F_{g(x)} = F_g \sin(\theta)$ and $F_{g(y)} = F_g \cos(\theta)$. See diagram.
The total force of gravity is F_g is $10 \cdot 5 = 50$N. Then $F_{g(x)} = F_g \sin(22°) = 18.73$N and $F_{g(y)} = F_g \cos(22°) = 46.36$N. Therefore, the normal force is 46.36N, and the force of friction is 18.73N.



general second law stuff (free body diagram etc)

Terminal velocity

Some forces are dependent on the velocity that the object is moving. For example, air resistance.

For example if an object is falling, but the force the air resistance exerts is $F_r = kv$, where k is a constant, then the termial velocity (in other words maximum velocity) occur when $F_r = F_g$. Therefore, one may compute that the termial velocity is $v = \frac{mg}{k}$.

Problem

If an object is dropping from in the Earth's gravitational field, a resistive force is acting on the object with magnitute kv, where k is the a constant, and v is the velocity that the object is travelling. Write the differential equation that represents Newton's second law. Then find velocity as a function of time. (Here, g is negative, k is positive)

Solution

We know that $\sum F = ma$, we can subsutite in $\sum F = mg - kv$, and $ma = m\frac{dv}{dt}$ to find that $\frac{dv}{dt} = \frac{mg-kv}{m}$. Now we compute. The equation can be rewritten as

$$\begin{aligned} \int \frac{mdv}{mg - kv} &= \int dt \\ -\frac{m}{k}\ln(mg - kv) &= t + C \end{aligned}$$

We know that when $t = 0$ the velocity is 0. Then $C = -\frac{m}{k}\ln(mg)$. Thus, we see that

$$\begin{aligned} -\frac{m}{k}\ln(mg - kv) + \frac{m}{k}\ln(mg) &= t \\ \ln\left(\frac{mg - kv}{mg}\right) &= -\frac{kt}{m} \\ \frac{-kv + mg}{mg} &= e^{-\frac{kt}{m}} \\ v &= \frac{mg}{k}\left(1 - e^{-\frac{kt}{m}}\right) \end{aligned}$$

Remark. The intergral can also be calculated with limits:

$$\begin{aligned} \int_0^v \frac{mdv}{mg - kv} &= \int_0^t dt \\ \left(-\frac{m}{k}\ln(mg - kv)\right)\Big|_0^v &= t \\ -\frac{m}{k}\ln(mg - kv) + \frac{m}{k}\ln(mg) &= t \end{aligned}$$

The rest follows.

Remark. If we see that acceleration is associated with velocity, then we know that the resistive force in involved.

Practice

2010 FRQ #1 & 2013 FRQ #2 & 2017 FRQ #2

§2.2 Circular Motion

Problem

A conial pendulum consists of a mass attached to a string and being whirled in a horizontal circle at constant speed. Suppose the length of the string is L, the angle the string makes with the vertical is θ, and the object has mass m. What is the speed of the object going around the circle?

Solution

First, we break down the force of tension into two components the vertical and the horizontal. The vertical components is $F_t \cos(\theta)$ which has the same magnitute as the force of gravity. Thus, $mg = F_t \cos(\theta)$. The horizontal component of the tension force is $F_t \sin(\theta) = F_t \cos(\theta) \tan(\theta) = mg \tan\theta$. This is also the centripetal force of the moving object. We know that $F_c = \frac{mv^2}{r}$. Therefore, $mg \tan\theta = \frac{mv^2}{r} \implies v = \sqrt{rg \tan(\theta)}$.

Problem

A car of mass m is driving around a curve such that the car's path is a horizontal circle with radius r. The roadway is banked at an angle θ. If the road way is covered with ice so that friction is negligible, then the car must go exactly speed v in order to not slide off one side of the roadway. Find the velocity v.

Solution

This problem is not much different than the problem we are give previously. The difference is replaceing pendulum with a car and force of tension with the normal force. Thus, the answer is still $v = \sqrt{rg \tan(\theta)}$.

There also exist tangential acceleration. It is the change in the magnitute of the velocity of the object. We can think of this as apeeding up while making a turn. The total acceleration is $\sqrt{a_t^2 + a_c^2}$.

§2.3 Third Law

The five parts that ***fully*** describe a force are

1. Agent – The thing exerting the force
2. Object – The object that the force is exerted on.
3. "Push" or "Pull" – if the object is accelerating towards the agent, then it is a pull. Vice versa.
4. Type of force – could be force of tension, force of friction, normal force, etc.
5. General direction – left/right/up/down.

For example if a person is standing on the floor then:

- The Earth pulls downward on the person using gravitational force.

- The floor pushes upward on the person using the normal force.

The five components are underlined.

Misconception

Every action has an equal and opposite reaction.

This is totally wrong!

Theorem 2.3.1 (Newton's third law)

If object A exerts a force on object B, then object B exerts a force on object A. In particular, there two forces

- have the same magnitute;
- are in opposite directions;
- occur at the same time;
- are both the same type of force;
- are both "push" or "pull".

Now let's describe the person standing on the floor:

- The Earth pulls downward on the person using gravitational force.
- The person pulls upward on the Earth using gravitational force.
- The floor pushes upward on the person using the normal force.
- The person pushes downward on the floor using the normal force.

Horizontal train of blocks

If we have two blocks connected by a string, the left blocj is pulling on the right block leftward using force od tension.

§3 Unit 3: Work, Energy, and Power

Recommandation

This unit weighs about 14-17% on the AP exam. The recommand time spend on this unit is about 10 to 20 hours.

§3.1 Work-Energy Theorem

Work and energy is measured in joules, which is equivalent to N·m. Work is a scaler.

Theorem 3.1.1

The work done on an object by a force is given by

$$W = \int_a^b F(r) \cdot dr$$

where $F(r)$ is a function of force with respect to position. In particular, if the force remains the same, then the equation is $F \cdot r$.

Theorem 3.1.2 (Kinetic Energy)

The amount of kinetic in am moving object is

$$K = \frac{1}{2}mv^2$$

Proof. If the object is at rest, and an force is exerted so that the object reaches velocity v. Then we know that the total work done by the force is

$$\int F(r)dr = \int madr = m\int \frac{dv}{dt}dr = m\int \frac{dr}{dt}dv = m\int vdv = \frac{1}{2}mv^2 + C.$$

We know that when the object is not moving the kinetic energy is 0. Thus, $C = 0$. □

The change in energy is work.

Theorem 3.1.3 (Gravitational potential energy)

The gravitational potential energy when the object is close to the surface of Earth is given by $U_g = mgh$, where h is the distance of the object with the surface.

If the object is far away. The the potential energy is given by $-\frac{GMm}{r}$, where M is the mass of the planet, and m is the mass of the object. **REMEMBER, THE DENOMINATOR IS NOT SQUARED.**

Proof. The first part we set the 0 energy when the object is on the surface. Because the force of gravity is constant (because the mass of the object does not change), then the work done by gravity is $F \cdot d = mgh$.

The second part, we set infinitly far away as 0 energy. The work done by the potential energy on the object is given by

$$\int \frac{GMm}{r^2}dr = -\frac{GMm}{r} + C$$

We can easily compute that $C = 0$ □

Theorem 3.1.4
Work done by friction is negative.

Theorem 3.1.5 (Spring)
The force of the spring is the given by $F_s = kx$, where x is the diaplacement of the spring. The energy is given by $U_e = \frac{1}{2}kx^2$.

If the force of the spring is not constant say cubic, then the energy stored in the srping must be computed by the integral.

Theorem 3.1.6

$$-\frac{dU}{dx} = F$$

$$U = mgh \text{ and } U = -\frac{GMm}{r}$$

Problem

If the function force on a spring with respect to displacement is $F_s = -20000x^3$. Find the energy stored in the spring when it is streched out from 0m to 0.2m.

Solution

The work done by the force when the spring is streched d out is given by

$$\int_0^{0.2} -20000x^3 dx = -5000x^4 \Big|_0^{0.2} = 8.$$

The energy stored in the spring is 8J.

§3.2 Forces and Potential Energy

Problem

An object with mass 4kg is initially at $x = 6$m, moving right with a velocity of $4\frac{\text{m}}{\text{s}}$. The potential energy function is $U(x) = \frac{144}{x} + 4x - 48$. What is the maximum and minimum position that the object can reach?

Solution

At the start the object has kinetic energy $\frac{1}{2}mv^2 = 32$J, and potential energy $\frac{144}{6} + 4 \cdot 6 - 48 = 0$J. Therefore the maximum and minimum occur when $U = 32$J. We find that $x = 2, 18$. Therefore, the object can only be between 2 and 18.

§3.3 Conservation of Energy

The total energy stays the same in the isolated system. Rememeber that Earth is usually part of the system.

Problem

A block with mass m slides from rest of height h. There is a spring with force $F = kx^3$. Find the maximum compression of the spring. Friction is negligible.

Solution

Gravitational potential energy is turned entirely into potential energy in the spring. $U_g = mgh$, and the energy in the spring is

$$\int kx^3 dx = \frac{1}{4}kx^4$$

Therefore, $mgh = \frac{1}{4}kx^4 \implies x = \sqrt[4]{\frac{4mgh}{k}}$.

Problem

A pendulum is released from rest attached on a string with length L. Find the tension of the string when the pendulum is at the bottom of the swing.

Solution

If we draw the free body diagram we find that $F_t + F_g = F_c$ when the pendulum is at the bottom of the spring. We know F_g, so we need to find v.
Because energy is conserved, then potential energy is transferred entirely into kinetic energy. $mgL = \frac{1}{2}mv^2 \implies v = \sqrt{2gL} \implies F_c = \frac{mv^2}{r} = 2mg$.
Thus, $F_t = 3mg$.

§3.4 Power

Power is defined as the rate that work is being done by, or the rate that the energy is being transformed.

Theorem 3.4.1
If W and U is described as a function, then

$$\frac{dW}{dt} = P \text{ or } \frac{dU}{dt} = P$$

If they are expressed as constant, then

$$\frac{W}{t} = P \text{ or } \frac{U}{t} = P$$

Power in AP is always positive. Thus units of power is $\frac{\mathrm{J}}{\mathrm{s}}$ =Watt $(\mathbf{W})$. Unfortunally, this unit represents work as well.

Theorem 3.4.2

If an object is moving at an constant velocity of v while being pushed by a constant force F (there exist some sort of resistive force), the

$$P = \frac{W}{t} = \frac{F \cdot x}{t} = F \cdot v$$

Problem

A boat, powered by a motor, is moving at an constant speed. The over all resistive force is directly proportional to the square of the velocity. By what factor should the power of motor be increaed if the velocity of the board were to be tripled?

Solution

The force is multiplied by 9 when the velocity is tripled. Then the force that the motor exerts must be tripled. We know that once the boat reaches equlibrium, then the force and the velocity is constant. So $P = Fv$, meaning that the power must be multiplied by 27.

Problem

One person is working with power output $P = bt^3$, where b is a constant. How much work will this person be able to preform between $t = 0$ and $t = T$.

Solution

The amount of work done is given by

$$W = \int_0^T bt^3 dt = \frac{1}{4} bT^4$$

These problems did not take too much WORK and also not too much TIME, but some POWER.

§4 Unit 4: Systems of Particles and Linear Momentum

Recommandation

This unit weighs about 14-17% on the AP exam. The recommand time spend on this unit is about 10 to 20 hours.

§4.1 Center of Mass

The forces are acting on the center of mass. Even if the force is not exerted directly on the center of mass. The object could be rotating, but the center of mass follows a curve.

Theorem 4.1.1

If an object has a symmetric shape and uniform density, then the center of mass is the geometric mass of the object.

Theorem 4.1.2

If the density is not uniform or the shape is not symmetrical, then the center of mass is given by

$$\frac{\sum x_i m_i}{\sum m} \text{ or } \frac{\int x dm}{\int dm}$$

The density of an object at any given point is $\lambda = \frac{dm}{dx}$. In particular, if the density is uniform, the $\lambda = \frac{m}{L}$.

Problem

Find the mass and the location of the center of mass of a bar with length L and linear density given by formula $\lambda = bx^2$, where b is a constant, and x is the distance from the left to right.

Solution

The center of mass is given by

$$\frac{\int_0^L x dm}{\int_0^L dm} = \frac{\int_0^L x\lambda dx}{\int_0^L \lambda dx} = \frac{\int_0^L bx^3 dx}{\int_0^L bx^2 dx} = \frac{\frac{1}{4}bL^4}{\frac{1}{3}bL^3} = \frac{3}{4}L$$

Which is $\frac{3}{4}$ of the length from left to right.

§4.2 Impulse and Momentum

Theorem 4.2.1

Momentum is given by

$$p = mv$$

Impulse is

$$J = \int F dt \Longleftrightarrow \frac{dp}{dt} = F$$

The change in momentum is impulse.

Theorem 4.2.2

If the mass of an object is non constant, the we need to use

$$\frac{mdv}{dt} + \frac{vdm}{dt} = p$$

Problem

A block of mass m is sliding to the left with speed v_0, a force is applied by the formula $F = bt^2$, where b is a constant. How long eill it take for the object to be moveing right with speed v_0.

Solution

The initial momentum is $-v_0 m$ and the final momentum is $v_0 m$. The impulse is $2v_0 m$, then we find that

$$2v_0 m = \int_o^T bt^2 dt \Longrightarrow 2v_0 m = \frac{1}{3} bT^3 \Longrightarrow t = \sqrt[3]{\frac{6v_0 m}{b}}$$

§4.3 Conservation of Linear Momentum, Collisions

If there are no extermal force: Momentum is always conserved in an isloated system. Energy is conserved, but some energy is in the form of thermal energy. We consider this as energy not conserved.

Theorem 4.3.1

In an elastic collisions, both energy and momentum are conserved. In inelastic collisions, only momentum is conserved. Energy is not conserved (kinetic energy decreased). In an explosion, momentum is conserved, but energy is not conserved (kinetic energy increased).

Problem

A 30g ball is launched into a 0.5kg wood block upported by a light string. The ball passes through the block and leaves with velocity of $15\frac{\text{m}}{\text{s}}$. The block rises to a final velocity of 8.5cm. Find the initial velocity of the ball.

Solution

First, when the block swings up, the energy is conserved. $\frac{1}{2}mv^2 = K = U_g = mgh \implies v = \sqrt{2gh} = \sqrt{2 \cdot 0.085 \cdot 9.8} = 1.29\frac{\text{m}}{\text{s}}$.
Because momentum is conserved during the Collision between the ball the the wood block, we know that $0.03 \cdot v + 0.5 \cdot 0 = 0.03 \cdot 15 + 0.5 \cdot 1.29 \implies v = 36.5\frac{\text{m}}{\text{s}}$.

2D collision

§5 Unit 5: Rotation

Recommandation

This unit weighs about 14-20% on the AP exam. The recommand time spend on this unit is about 10 to 20 hours.

§5.1 Torque and Rotational Statics

Torque τ is given by the cross product of force F and the distance between the object and the pivot point r.

Theorem 5.1.1 (Direction of torque)

Wrist at the pivot point, palm pointing in the direction of force that the force is exerted, and the thumb is the direction of torque.

Theorem 5.1.2 (Newton's second law in the angular form)

$$\tau = I\alpha$$

Theorem 5.1.3 (Rotational Inertia)

$$I = \sum r^2 m$$

$$I = \int r^2 dm$$

Two conditions of equlibrium are $\sum F = 0$ and $\sum \tau = 0$. In fact if the object is not rotating, then the pivot point can be placed anywhere on the object.

Theorem 5.1.4 (Parallel Axis Theorem)

$$I = I_0 + Md^2$$

where I_0 is the rotaional inertia when the pivot point is at the center of mass.

§5.2 Rotational Kinematics

same as linear, except different variable. covered in centripital force.

§5.3 Rotational Dynamics and Energy

Energy of an rotating object is given by

$$E = \frac{1}{2} I\omega^2$$

Energy conserved.

§5.4 Angular Momentum and Its Conservation

Angular momentum is given by $L = pr$ for a object travelling staight, also $L = I\omega$ for an rotating object. Angular momentum conserved in collision.

Also
$$\frac{dL}{dt} = \tau$$

§6 Unit 6: Oscillations

Recommandation

This unit weighs about 6-14% on the AP exam. The recommand time spend on this unit is about 5 to 10 hours.

§6.1 Simple Harmonic Motion, Springs, and Pendulums

Harmonic motion is repeating motion where acceleration is proportional to displacement. The could be a mass on a string or a pendulum.

Definition 6.1.1. Amplitude is the distance from equilibrium to the maximum. We use A to represent amplitude.

Definition 6.1.2. Period is the time that the object completes one oscillation. We use T to represent period.

Definition 6.1.3. Angular frequence is given by $\omega = \frac{2\pi}{T}$.

General equation for a simple harmonic motion is $x(t) = A\sin(\omega t)$.

Problem

A mass on a string is undergoing simple harmonic motion such that its position as a function of time, $x(t)$, can be represented by $p\sin(qt)$ where $p = 0.47\text{m}$ and $q = 3.14\frac{\text{rad}}{\text{s}}$.

(a) Determine the amplitude.

(b) Determine the distance travelled in one complete oscillation.

(c) Determine the period of the mass's oscillation.

Solution

(a) 0.47m

(b) $0.47 \times 4 = 1.88\text{m}$

(c) $\frac{2\pi}{T} = 3.14 \implies T = 2\text{s}$.

From position time function, we get velocity time function: $v(t) = A\omega\cos(\omega t)$. Also, acceleration time function: $a(t) = -A\omega^2\sin(\omega t)$.

Theorem 6.1.4

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Proof. From the functions, we find that $x(t) = -\omega^2 a(t)$. Now we look at forces, the total force exerted by the spring is $F = -kx$ and it also equal to $F = ma$. Therefore, $-kx = ma \implies -\frac{x}{a} = \frac{m}{k} \implies \frac{1}{\omega^2} = \sqrt{\frac{m}{k}} \implies T = 2\pi\sqrt{\frac{m}{k}}$ $\square$

Springs

If we hang the object vertically, the only thing that changed is the equilibrium position. Nothing else changes. The new equilibriumis at $mg - kx = 0$.

Theorem 6.1.5 (Spring is series)

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n}$$

Theorem 6.1.6 (Spring is parallel)

$$k = k_1 + k_2 + \cdots + k_n$$

This is completly opposite from circuits.

Pendulums

Spring restore and oscillate from the force of the spring. Pendulums restore and oscillate from force of gravity.

simple pendulum

If the length of the string attached to the pendulum is L, and the angle between where the pendulum is released and the resting position is θ. We know that $F_g = Lg\sin\theta$. Also, we know that $s = L\theta$, where s denotes the arc length. Take the second derivative of both sides, we find that $a = L\frac{d^2\theta}{dt^2}$. If we look at forces, we notice that $-mg\sin\theta = ma$. Therefore, $-g\sin\theta = L\frac{d^2\theta}{dt^2}$. $(\theta < 15°)$

We can approximate $\sin\theta \approx \theta$. Because the pendulum is doing an simple harnomic motion, the we can know that $\frac{d^2\theta}{dt^2} = -\omega^2\theta$. Thus, $\omega = \sqrt{\frac{g}{L}}$. Because $T = \frac{2\pi}{\omega}$, then $T = 2\pi\sqrt{\frac{L}{g}}$.

physical pendulum

This is an object with a hole hanged up on the wall. Period is given by

$$T = 2\pi\sqrt{\frac{I}{mgD}}$$

where I is the rotational inertia about where the pendulum is swinging about, and D is the distance from the point of swinging and the center of mass of the object.

Torsional Pendulum

$$T = 2\pi\sqrt{\frac{I}{\kappa}}$$

I is the rotational inertia, and κ (Kappa) is the torsional constant. We can think of this as spring constant, but instead of providing an linear force, it provides torsional force.

Energy

Theorem 6.1.7 (Potential Energy)

$$U_s = \frac{1}{2}kx^2$$

Theorem 6.1.8 (Kinetic Energy)

$$KE = \frac{1}{2}mv^2$$

We know that $x(t) = A\sin(\omega t)$ and $v(t) = \omega A\cos(\omega t)$. Because $\omega^2 = \frac{k}{m}$, then we see that

$$U_s = \frac{1}{2}kA^2\sin^2(\omega t), \text{ and } KE = \frac{1}{2}kA^2\cos^2(\omega t)$$

The total mechanical energy is their sum, which is $\frac{kA^2}{2}$.

Problem

An object with mass m of 1.34kg oscillates at the end of a spring with period T of 0.89s. In going from far left to the far right of its motion, it travels a distance d of 0.56m. Determine the position, velocity, acceleration as a function of time. Then calculate

(a) spring constant of the spring

(b) angular frequency

(c) total energy of the system

(d) maximum speed of the mass

(e) position of the mass when it is travelling at 1.27 $\frac{\text{m}}{\text{s}}$.

Solution

First we can determine $\omega = \frac{2\pi}{T} = 7.06$. We also know that the amplitude is 0.28m. Therefore we find that

$$\begin{cases} x(t) = 0.28\sin(7.06t) \\ v(t) = 1.98\cos(7.06t) \\ a(t) = -14.0\sin(7.06t) \end{cases}$$

From $T = 2\pi\sqrt{\frac{m}{k}}$, we find that $k = 66.8\frac{\text{N}}{\text{m}}$. Angular frequency is $\omega = 7.06\frac{\text{rad}}{\text{s}}$. Total energy is $E_{tot} = \frac{kA^2}{2} =$ 2.62J. The maximum speed is $E_{tot} = \frac{1}{2}mv^2 \implies v = 1.98\frac{\text{m}}{\text{s}}$.

$$\frac{1}{2}kx^2 + \frac{1}{2}mv^2 = E_{tot}$$

From $v = 1.27$, we find that $x = 0.215$m.

§7 Unit 7: Gravitation

Recommandation

This unit weighs about 6-14% on the AP exam. The recommand time spend on this unit is about 5 to 10 hours.

§7.1 Gravitational Forces

Theorem 7.1.1

The Gravitational force between the two masses is

$$F_g = \frac{Gm_1m_2}{r^2}$$

The strength of a Gravitational field is $g = \frac{Gm}{r^2}$. This is the acceleration due to gravity. In particular, g on earth is $9.8\frac{\mathrm{m}}{\mathrm{s}^2}$. This is outside of the planet.

Theorem 7.1.2

If an object is inside an planet, then the mass within r contribute to gravity. Therefore, if the object is inside the planet, then the force is $\frac{4}{3}\pi G\rho \cdot r$.

Proof. The mass on a planet is uniform. Let ρ be the density. We know that $\rho = \frac{m}{\frac{4}{3}\pi R^3}$. The mass inside the radius r is

$$\frac{4}{3}\pi r^3 \cdot \rho = m\frac{r^3}{R^3}$$

Therefore, the acceleration due to gravity at r is

$$\frac{G\left(m\frac{r^3}{R^3}\right)}{r^2} = \frac{Gm}{R^3} \cdot r = \frac{4}{3}\pi G\rho \cdot r$$

□

Theorem 7.1.3

The gravity inside a hollow sphere is 0.

§7.2 Orbits of Planets and Satellites

Theorem 7.2.1

From the centripital force equal to the force of gravity, we may find that

$$\frac{GMm}{r^2} = \frac{mv^2}{r} \Longrightarrow GM = v^2r \Longrightarrow T^2 = \frac{4\pi^2}{GM}r^3$$

Where T is the period that the object travels a full cycly, and $T = \frac{2\pi r}{v}$.

Theorem 7.2.2 (Kepler's First law)
The orbits of planets are ellipses with the sun at one of the foci.

Theorem 7.2.3 (Kepler's Second law)
When the planet is orbiting the sun, the area covered during a period of time (the sector joining joining the Sun and position of the planet) is the same.

Theorem 7.2.4 (Kepler's Third law)
The angular momentum for an object in orbit stays constant. This is especially useful when the path is not a circle.

Theorem 7.2.5
The Gravitational potential energy of an object is $\frac{GMm}{r}$. **REMEBER NOT SQUARED!**.

Theorem 7.2.6
The escape velocity is $\sqrt{\frac{2GM}{r}}$.

Proof.

$$\begin{aligned} U_g + K &= 0 \\ -\frac{GMm}{r} + \frac{1}{2}mv^2 &= 0 \\ v^2 &= 2\frac{GM}{r} \\ v &= \sqrt{\frac{2GM}{r}} \end{aligned}$$

□

Theorem 7.2.7
The energy to put into orbit is exactly half the potential energy. In other words, the energy in an object orbiting planet at height r is $-\frac{GMm}{2r}$

§8 The Exam

§8.1 MCQ section

§8.2 FRQ section

2010 FRQ

Solution to Problem 1

We know that the object reached terminal velocity when the drag force is equal to the force of gravity. Then $F_d = F_g$, meaning that $Cv^2 = mg$. Therefore, $v_T^2 = \frac{mg}{C}$.

2011 FRQ

2012 FRQ

2013 FRQ

2014 FRQ

2015 FRQ