Olympiad Inequalities

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§1 Fundemental Inequalities

Theorem 1.0.1

This is a list of trivial inequalities. At this point you should be pretty fimilar with these

1. If $X \in \mathbb{R}$, then $X^2 \geq 0$.
2. If $X \geq Y$ and $Y \geq Z$, then $X \geq Z$.
3. If $X \geq Y$ and $Z \geq W$, then $X + Z \geq Y + W$.
4. If $X \geq Y$ and $Y \leq W$, we cannot conclude $X \geq Z$.
5. If $X \geq Y$ and $Z \leq W$, we cannot conclude $X + Z \geq Y + W$.

Things get a bit complicated if negative numbers or inverses are involved.

Theorem 1.0.2

A warning when multiplying both sides by a negative number of taking the inverse.

1. If $X \geq Y$, then $-X \leq -Y$.
2. If $X \geq Y \geq 0$, then $\frac{1}{X} \leq \frac{1}{Y}$.

Problem 1.0.3. Compare $\frac{1}{X}$ and $\frac{1}{Y}$ if $0 \geq X \geq Y$.

In fact if we are unsure whether or not one inequality is true, we can substitute in values to verify.

§2 AM-GM

AM-GM inequlity (arithmetic mean geometric mean inequality) is the most basic, the most important, and one of the more frequently used inequality in math olympiads. Although most problems cannot be solved exclusively using AM-GM, some steps will use AM-GM.

§2.1 The Statement

Theorem 2.1.1 (AM-GM)
For nonnegative $a_1, a_2, \cdots, a_n \geq 0$, we have

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

The proof of this inequality uses induction, which is a rare method in inequality. The only problem I have seen that used similar method was 2016 ISL A1 (as an exercise later in the chapter). You can skip the proof if you would like.

Proof. First notice that the base case $n = 1$ is trivial. We will prove the inequality for $n = 2$ first, then by induction prove the inequality for n is a power of 2. Lastly, use induction to induct down (If inequality is true for $n = m$, then the inequality is true for $n = m - 1$). After these three steps, we can conclude that the inequality is true for all positive integer n.

Claim — The base case $n = 2$ is true.

We know that $(a_1 - a_2)^2 \geq 0$, we can rearrange the inequality into $(a_1 + a_2)^2 \geq 4a_1 a_2 \Longrightarrow \frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2}$.

Claim — The inequality is true for all $n = 2^k$, where $k \in \mathbb{Z}^+$.

By induction, we can assume the inequality is true for $n = 2^{k-1}$. We know that

$$\frac{a_1 + a_2 + \cdots a_{2^{k-1}}}{2^{k-1}} \geq \sqrt[2^{k-1}]{a_1 a_2 \cdots a_{2^{k-1}}}$$
$$\frac{a_{2^{k-1}+1} + a_{2^{k-1}+2} + \cdots a_{2^k}}{2^{k-1}} \geq \sqrt[2^{k-1}]{\left(a_{2^{k-1}+1}\right)\left(a_{2^{k-1}+2}\right) \cdots \left(a_{2^k}\right)}$$

Adding the two inequality together, we have

$$\frac{a_1 + a_2 + \cdots a_{2^k}}{2^{k-1}} \geq \sqrt[2^{k-1}]{a_1 a_2 \cdots a_{2^{k-1}}} + \sqrt[2^{k-1}]{\left(a_{2^{k-1}+1}\right)\left(a_{2^{k-1}+2}\right) \cdots \left(a_{2^k}\right)}$$

Apply $n = 2$ AM-GM on the right hand side, we find that

$$\sqrt[2^{k-1}]{a_1 a_2 \cdots a_{2^{k-1}}} + \sqrt[2^{k-1}]{\left(a_{2^{k-1}+1}\right)\left(a_{2^{k-1}+2}\right) \cdots \left(a_{2^k}\right)} \geq 2\sqrt{\sqrt[2^{k-1}]{a_1 a_2 \cdots a_{2^k}}} = 2\sqrt[2^k]{a_1 a_2 \cdots a_{2^k}}$$

Therefore, combining the two inequalities above, we find that

$$\frac{a_1 + a_2 + \cdots a_{2^k}}{2^k} \geq \sqrt[2^k]{a_1 a_2 \cdots a_{2^k}}$$

This completes the induction step, meaning that AM-GM is true for all $n = 2^k$, where $k \in \mathbb{Z}^+$.

Claim — If the inequality is true for $n = m$, then the inequality is true for $n = m - 1$, where $m \in \mathbb{Z}^+$.

Because the inequality for $n = m$ is true, then we have

$$\frac{a_1 + a_2 + \cdots + a_m}{m} \geq \sqrt[m]{a_1 a_2 \cdots a_m}$$

Let $a_m = \frac{1}{m-1}(a_1 + a_2 + \cdots + a_{m-1})$, and substitute it into the inequality, we find that the left hand side is

$$\frac{a_1 + a_2 + \cdots + a_{m-1} + \frac{1}{m-1}(a_1 + a_2 + \cdots + a_{m-1})}{m} = \frac{a_1 + a_2 + \cdots + a_{m-1}}{m-1} = a_m$$

Then, the inequality becomes

$$\begin{aligned}
a_m &\geq \sqrt[m]{a_1a_2\cdots a_m}\\
a_m^m &\geq a_1a_2\cdots a_m\\
a_m^{m-1} &\geq a_1a_2\cdots a_{m-1}\\
a_m &\geq \sqrt[m-1]{a_1a_2\cdots a_{m-1}}\\
\frac{a_1+a_2+\cdots+a_{m-1}}{m-1} &\geq \sqrt[m-1]{a_1a_2\cdots a_{m-1}}
\end{aligned}$$

This completes the induction step which means that if the inequality is true for $n = m$, then the inequality is true for $n = m-1$. Combining the three claims, we see that all $n \in \mathbb{Z}^+$ is true. $\square$

Now we have proven AM-GM, we need to take a look at the equality cases.

Theorem 2.1.2

AM-GM is a equality if and only if $a_1 = a_2 = \cdots = a_n$.

We see that if two values are not equal in the inequality (say $a_i \neq a_j$), then we know that $\frac{a_i+a_j}{2} > \sqrt{a_ia_j}$ TODO

Now let's see how we can use AM-GM. First, lets look at inequalities without constant terms.

§2.2 AM-GM Without Obvious Constant

Example 2.2.1

Prove that $a^2 + b^2 + c^2 \geq ab + ac + bc$ for positive reals a, b, c.

The first thing we might try is apply AM-GM on $a^2 + b^2 + c^2$ directly. However, we find that

$$a^2 + b^2 + c^2 \geq 3\sqrt[3]{a^2b^2c^2}.$$

We need term ab on the right hand side. How can we get this term? If the right hand side of AM-GM is one term, then we see that $ab \geq ab$. Although this is true, this is not helpful for us. If the right hand side of AM-GM is two terms, then their product is a^2b^2. Ah ha! We can use $a^2 + b^2$ on the left hand side, and this is what we have on the left hand side of the desired inequality. We see that

$$a^2 + b^2 \geq 2ab$$

Now, we can use the same inequality for other pair of terms, we have $a^2 + c^2 \geq 2ac$ and $b^2 + c^2 \geq 2bc$. If we add the three inequalities together, we find that

$$a^2 + b^2 + a^2 + c^2 + b^2 + c^2 \geq 2ab + 2ac + 2bc$$

If we divide both sides by 2, then we arrive at the desired inequality. Now, we can write this up nicely:

Solution to Example 2.2.1

Using AM-GM, we find the following inequalities are true:

$$\begin{cases} \frac{a^2+b^2}{2} \geq ab \\ \frac{a^2+c^2}{2} \geq ac \\ \frac{b^2+c^2}{2} \geq bc \end{cases}$$

Adding the three inequalities together we find that

$$a^2 + b^2 + c^2 \geq ab + ac + bc$$

as desired. $\square$

Example 2.2.2
Prove that $a^4+b^4+c^4 \geq a^2bc+ab^2c+abc^2$ for positive reals a, b, c.

We hope to get a^2bc on the right hand side. On the left hand side we must have twice as much a^4 term than b^4 and c^4 term. Now, we can try the simplest inequality we can have to obtain a^2bc:

$$a^4+a^4+b^4+c^4 \geq 4a^2bc$$

Apply this for other pairs of terms, we find that $a^4+b^4+b^4+c^4 \geq 4ab^2c$, and $a^4+b^4+c^4+c^4 \geq 4abc^2$. Summing the three inequalities, we find that

$$4\left(a^4+b^4+c^4\right) \geq 4\left(a^2bc+ab^2c+abc^2\right)$$

Solution to Example 2.2.2
Using AM-GM, we find the following inequalities are true:

$$\begin{cases} \frac{a^4+a^4+b^4+c^4}{4} \geq a^2bc \\ \frac{a^4+b^4+b^4+c^4}{4} \geq ab^2c \\ \frac{a^4+b^4+c^4+c^4}{4} \geq abc^2 \end{cases}$$

Adding the three inequalities together we find that

$$a^4+b^4+c^4 \geq a^2bc+ab^2c+abc^2$$

as desired. $\square$

Notice that when we are using AM-GM, the degree of both sides are preserved, and the number of terms of both sides are preserved. Whenever we arrive at an inequality with the same number of terms and the terms have the same degree, it is very likely that AM-GM can prove it. One of the counter example is the Schur's inequality $(r=1)$

$$a^3+b^3+c^3+3abc \geq a^2b+ab^2+a^2c+ac^2+b^2c+bc^2$$

These cases will be discussed in section 8, the Chinese Dumbass Notation.

§2.3 AM-GM to eliminate fractions

We can also use AM-GM to eliminate fractions.

Example 2.3.1
Prove that

$$\frac{a_1^2}{a_2}+\frac{a_2^2}{a_3}+\cdots+\frac{a_n^2}{a_1} \geq a_1+a_2+\cdots+a_n$$

The right hand side does not have any fraction. We can use $\frac{a_1^2}{a_2}+a_2$ to eliminate the fraction. We find that $\frac{a_1^2}{a_2}+a_2 \geq 2a_1$. Summing all such inequalities gives the desired result.

Solution to Example 2.3.1
Using AM-GM, we find that

$$\frac{a_1^2}{a_2}+a_2 \geq 2a_1$$

We can take the cyclic sum for all inequality in such form, we have

$$\sum_{\text{cyc}} \frac{a_1^2}{a_2} + \sum_{\text{cyc}} a_2 \geq \sum_{\text{cyc}} 2a_1$$

Therefore,

$$\sum_{\text{cyc}} \frac{a_1^2}{a_2} \geq \sum_{\text{cyc}} a_1$$

Example 2.3.2 (Canada MO 2002/3)

Prove that for all positive real numbers a, b, and c,

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a + b + c$$

and determine when equality occurs.

TODO

Example 2.3.3 (IMO 1995/2)

Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3\left(b+c\right)} + \frac{1}{b^3\left(c+a\right)} + \frac{1}{c^3\left(a+b\right)} \geq \frac{3}{2}.$$

We cannot apply AM-GM directly. We must use the condition $abc = 1$ somehow. If we multiply LHS by abc, we see that we need to show that

$$\sum_{\text{cyc}} \frac{bc}{a^2(b+c)} \geq \frac{3}{2}$$

To eliminate this fraction, we need to add a third power, but then the third power is too big to be eliminated. Now, let's see what if we multiply by another abc:

$$\sum_{\text{cyc}} \frac{b^2c^2}{a(b+c)} \geq \frac{3}{2}$$

To eliminate the fraction we can add $a(b+c)$, and the resulting inequality is also a second power, meaning this may work. Equality occur when $a = b = c = 3$, each $\frac{b^2c^2}{a(b+c)} = \frac{1}{2}$ in the equality case, so the coeffcient for $a(b+c)$ is $\frac{1}{4}$. We find that

$$\frac{b^2c^2}{a(b+c)} + \frac{1}{4}a(b+c) \geq bc$$

Summing the three inequalities we find that

$$\sum_{\text{cyc}} \frac{b^2c^2}{a(b+c)} \geq \frac{1}{2} \sum_{\text{cyc}} bc$$

This is not yet what we want. We can just show that $\frac{1}{2}(ab + ac + bc) \geq \frac{3}{2}$. Ahh! We can just apply AM-GM directly to find that

$$\frac{1}{2}(ab + ac + bc) \underset{5}{\geq} \frac{3}{2}\sqrt[3]{a^2b^2c^2} = \frac{3}{2}$$

§2.4 AM-GM motivated by equality cases

Remember when using AM-GM, the equality must be preserved. Here are some examples that require us to find the exact values.

Example 2.4.1

Prove that $x^3 + 54 \geq 27x$ for positive real number x.

First we take a look at what might be the equality case. We know that the left hand side is greater than 54, so $x > 2$. Let's try $x = 3$. Ah ha! $3^3 + 54 = 81$ and $27 \cdot 3 = 81$. This is the equality case.

Because the equality must be preserved, x^3 in equality case is 27. We must also find some other terms that is equal to 27. Therefore, we see that

$$x^3 + 27 + 27 \geq 3\sqrt[3]{27 \cdot 27 \cdot x^3} = 27x$$

Example 2.4.2

Prove that $x^5 + 43x^2 + 48 \geq 5x^4 + 7x^3 + 8x$ for positive real number x.

This is much harder than example 2.9. As before, we first consider the equality cases. We try the positive integers numbers, and we realize that $x = 4$ is the equality case. Looking at the left hand side, We hope to get x^4. This can be obtained by $x^5 + x^5 + x^2$. Because the equality must be preserved, we need a coeffcient of 64 on x^2. Therefore, we see that

$$x^5 + x^5 + 64x^2 \geq 12x^4 \Longrightarrow \frac{5}{6}x^5 + \frac{80}{3}x^2 \geq 5x^4$$

Next we hope to get x^3. This can be obtained by $x^5 + x^2 + x^2$. Because the equality must be preserved, we need a coeffcient of 64 on x^2. Therefore, we see that

$$x^5 + 64x^2 + 64x^2 \geq 48x^3 \Longrightarrow \frac{7}{48}x^5 + \frac{56}{3}x^2 \geq 7x^4$$

Now, we can subtract the two inequality that we found from the desired inequality, we see that it remains to show that

$$\frac{1}{48}x^5 + 48 \geq \frac{7}{3}x^2 + 8x$$

We can get x^2 by $x^5 + x + 1$. Because the equality must be preserved, we need a coeffcient of 1024 on 1 and a coeffcient of 256 on x. Therefore,

$$x^5 + 256x + 1024 \geq 192x^2 \Longrightarrow \frac{7}{576}x^5 + \frac{28}{9}x + \frac{112}{9} \geq \frac{7}{3}x^2$$

It remains to show that

$$\frac{5}{576}x^5 + \frac{320}{9} \geq \frac{100}{9}x$$

This is trivial since

$$\frac{5}{576}x^5 + \frac{80}{9} + \frac{80}{9} + \frac{80}{9} + \frac{80}{9} \geq 5\sqrt[5]{\frac{5}{576}\left(\frac{80}{9}\right)^2 x^5} = \frac{100}{9}x$$

Example 2.4.3 (2010 ISL A2)

§2.5 Miscellaneous use of AM-GM

TODO

Example 2.5.1 (IMO 2012/2)

§2.6 Problems for this section

Exercise 2.6.1. Prove that $a^3+b^3+c^3 \geq a^2b+b^2c+c^2a$ for positive reals a, b, c.

Exercise 2.6.2. Prove that $(a+b+c)^3 \geq a^3+b^3+c^3+24abc$ for positive reals a, b, c.

Exercise 2.6.3. Prove that $a^5+b^5+c^5 \geq a^3bc+ab^3c+abc^3 \geq a^2b^2c+a^2bc^2+ab^2c^2$ for positive reals a, b, c.

Exercise 2.6.4. Let $a_1, a_2, \cdots a_n$ be positive reals. Prove that

$$\frac{a_1^3}{a_2}+\frac{a_2^3}{a_3}+\cdots+\frac{a_n^3}{a_1} \geq a_1^2+a_2^2+\cdots+a_n^2$$

Exercise 2.6.5. Let a, b, c, d be positive reals with $(a+c)(b+d)=1$. Prove that

$$\frac{a^3}{b+c+d}+\frac{b^3}{a+c+d}+\frac{c^3}{a+b+d}+\frac{d^3}{a+b+c} \geq \frac{1}{3}$$

Exercise 2.6.6 (USAJMO 2012/3)**.** Let a, b, c be positive real numbers. Prove that

$$\frac{a^3+3b^3}{5a+b}+\frac{b^3+3c^3}{5b+c}+\frac{c^3+3a^3}{5c+a} \geq \frac{2}{3}(a^2+b^2+c^2).$$

Exercise 2.6.7 (USA TST 2010/2)**.** Let a, b, c be positive reals such that $abc=1$. Show that

$$\frac{1}{a^5(b+2c)^2}+\frac{1}{b^5(c+2a)^2}+\frac{1}{c^5(a+2b)^2} \geq \frac{1}{3}.$$

§3 RMS-AM-GM-HM

Theorem 3.0.1 (RMS-AM-GM-HM Inequality)

If $x_1, x_2, \cdots x_n$ are positive reals, then we know

$$\sqrt{\frac{x_1^2+x_2^2+\cdots+x_n^2}{n}} \geq \frac{x_1+x_2+\cdots+x_n}{n} \geq \sqrt[n]{x_1x_2\cdots x_n} \geq \frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}},$$

§4 Cauchy and Hölder

Theorem 4.0.1 (Cauchy-Schwarz Inequality)

For any real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$,

$$\left(\sum_{i=1}^n a_ib_i\right)^2 \leq \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right)$$

Theorem 4.0.2 (Engel's form of Cauchy-Schwarz Inequality)

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \cdots + b_n}.$$

Theorem 4.0.3 (Hölder's inequality)

It should look something like this

$$\sum_j \prod_i a_{ij}^{\lambda_i} \leq \prod_i \Big(\sum_j a_{ij} \Big)^{\lambda_i}.$$

§5 Weighted Power Mean

Theorem 5.0.1 (Weighted power mean)

For n positive real numbers a_i and n positive real weights w_i with sum $\sum_{i=1}^n w_i = 1$, the power mean with exponent t, where $t \in \mathbb{R}$, is defined by

$$M(t) = \begin{cases} \prod_{i=1}^n a_i^{w_i} & \text{if } t = 0 \\ \left(\sum_{i=1}^n w_i a_i^t \right)^{\frac{1}{t}} & \text{otherwise} \end{cases}.$$

§6 Other inequalities

§6.1 Rearrangement Inequality

Theorem 6.1.1

Correct order $\geq$ random order $\geq$ incorrect order

§6.2 Schur's Inequality

Theorem 6.2.1

$$a^r(a-b)(a-c) + b^r(b-a)(b-c) + c^r(c-a)(c-b) \geq 0$$

In particular $r = 1$ is especially useful

Theorem 6.2.2

$$a^3 + b^3 + c^3 + 3abc \geq a^2b + a^2c + b^2a + b^2c + c^2a + c^2b$$

§7 Inequality inside a function

This section rquires calculus!

§7.1 Jensen

Theorem 7.1.1 (Jensen's inequality)
Let F be a convex function of one real variable. Let $x_1, \ldots, x_n \in \mathbb{R}$ and let $a_1, \ldots, a_n \geq 0$ satisfy $a_1 + \cdots + a_n = 1$. Then $F(a_1x_1 + \cdots + a_nx_n) \leq a_1F(x_1) + \cdots + a_nF(x_n)$

Basically, closer is smaller on a convex function. Vice versa.

§7.2 Karamata

If sequence A majorizes sequence B, then $f(a_1) + f(a_2) + \geq f(b_1) + f(b_2) +$

§7.3 Tangent line trick

Compute the tangent line at equality and prove it. Generally stronger than Jensen.

§7.4 $n-1$ Equal value principle

One inflection point means $n-1$ values are equal.

§8 Chinese Dumbass Notation and SOS

This is not a joke, it is really called the "Chinese dumbass notation"
Similar to barycentric coordinates
TO LEARN SOS.

§9 Review Problems

Example 9.0.1 (2016 ISL A1)
Let a, b, c be positive real numbers such that $\min(ab, bc, ca) \geq 1$. Prove that

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leq \left(\frac{a+b+c}{3}\right)^2 + 1.$$

We can break down this problem into two parts.

1. Prove that if $ab \geq 1$, then
$$\sqrt{(a^2+1)(b^2+1)} \leq \left(\frac{a+b}{2}\right)^2 + 1$$
2. Use the same idea as AM-GM to show the desired inequality.

Proving 1 is beyond the reach of our current knowledge.

§10 Hints

§11 Solutions

Appendex below

§12 Calculus

Derivative and stuff

§13 Notations

Cyclic Sum

$$\sum_{\text{cyc}}$$

means to cycle through the elements exactly n times (taking elements in the format, such that the index difference remains the same). For example, if there are three elements in total, then

$$\sum_{\text{cyc}} a^2b = a^2b + b^2c + c^2a$$

Symmetric Sum

$$\sum_{\text{sym}}$$

means to cycle through the elements exactly $n!$ times (taking **all** elements in the format, regardless if index difference remains the same). For example, if there are three elements in total, then

$$\sum_{\text{sym}} a^2b = a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2$$

I think section 1 one page
section 2 10 pages
section 3 4 pages
section 4 6 pages
section 5 3 pages
section 6 3 pages
section 7 10 pages
section 8 6 pages
section 9 3 pages
section 10 2 pages
section 11 15 pages
about 63 pages