The Canadian Mathematical Olympiad

Hawsie Yan

March 4, 2022

§1 CMO 2011

Example (CMO 2011/1)
Consider 70-digit numbers with the property that each of the digits $1, 2, 3, \cdots, 7$ appear 10 times in the decimal expansion of n (and $8, 9, 0$ do not appear). Show that no number of this form can divide another number of this form.

Proof. Take $\pmod 9$, we see that the all numbers in the form is $10 \cdot \frac{(1+7)\cdot 7}{2} \equiv 280 \equiv 1 \pmod 9$. The largest number is $7\cdots 76\cdots 6\cdots 1$ is less than seven times the smallest $1\cdots 12\cdots 2\cdots 7$.

If one number is a multiple of another, say $ax = y$, where $2 \le a \le 6$ and x, y are both in the form described in the question, then we must have $y \equiv ax \equiv a \pmod 9$. However, we must have $y \equiv 1 \pmod 9$. We have reached a contradiction. Therefore, it is not possible to have one number dividing another. □

Example (CMO 2011/2)
Let $ABCD$ be a cyclic quadrilateral with opposite sides not parallel. Let X and Y be the intersections of AB, CD and AD, BC respectively. Let the angle bisector of $\angle AXD$ intersect AD, BC at E, F respectively, and let the angle bisectors of $\angle AYB$ intersect AB, CD at G, H respectively. Prove that $EFGH$ is a parallelogram.

Example (CMO 2011/3)
Amy has divided a square into finitely many white and red rectangles, each with sides parallel to the sides of the square. Within each white rectangle, she writes down its width divided by its height. Within each red rectangle, she writes down its height divided by its width. Finally, she calculates x, the sum of these numbers. If the total area of white equals the total area of red, determine the minimum of x.

Proof. Let a_i and b_i be the □

Example (CMO 2011/4)
Show that there exists a positive integer N such that for all integers $a > N$, there exists a contiguous substring of the decimal expansion of a, which is divisible by 2011.
Note. A contiguous substring of an integer a is an integer with a decimal expansion equivalent to a sequence of consecutive digits taken from the decimal expansion of a.

Example (CMO 2011/5)

Let d be a positive integer. Show that for every integer S, there exists an integer $n > 0$ and a sequence of n integers $\epsilon_1, \epsilon_2, ..., \epsilon_n$, where $\epsilon_i = \pm 1$ (not necessarily dependent on each other) for all integers $1 \leq i \leq n$, such that $S = \sum_{i=1}^{n} \epsilon_i (1 + id)^2$.

§2 CMO 2012

Example (CMO 2012/1)
Let x, y and z be positive real numbers. Show that $x^2 + xy^2 + xyz^2 \geq 4xyz - 4$.

Proof.

$$x^2 + 2\frac{1}{2}xy^2 + 4\frac{1}{4}xyz^2 + 4 \geq 16\sqrt[8]{\frac{4}{2^{10}}x^8y^8z^8} = 4xyz$$

establishing the inequality. In particular, equality holds when $(x, y, z) = (2, 2, 2)$ □

Remark. First move the -4 to the LHS. The power on the right is uniform, so we need a certain product to be uniform as well.

If we begin with $a\frac{1}{a}x^2 + b\frac{1}{b}xy^2 + c\frac{1}{c}xyz^2$, we can find that $2a + b + c = 2b + c = 2c \Longrightarrow 4a = 2b = c$. The easiest is obviously $a = 1, b = 2, c = 4$. After multiplying everything, we see that it is $Cx^8y^8z^8$, meaning we need eight terms. Ah ha! the four is the last term!

In fact, if we pick $a = 2, b = 4, c = 8$, we can use $2 + 2$ to fill in the two remaining terms.

Example (CMO 2012/2)
For any positive integers n and k, let $L(n, k)$ be the least common multiple of the k consecutive integers $n, n+1, \ldots, n+k-1$. Show that for any integer b, there exist integers n and k such that $L(n, k) > bL(n+1, k)$.

Proof. We can take $n = p > 4b, k = p - 2$ as well as $p \geq 7$. It is easy to see that

$$\begin{cases} L(n, k) = p \times \text{lcm}(p+1, p+2, \cdots, 2p-3) \\ L(n+1, k) = \text{lcm}(p+1, p+2, \cdots, 2p-2) \end{cases}$$

We know that $4|2p - 2$, let $m = \frac{p-1}{2}$. We know that $m|2p - 2 - m$, then

$$\text{lcm}(p+1, p+2, \cdots, 2p-2) \leq 4 \times \text{lcm}(p+1, p+2, \cdots, 2p-3)$$

Therefore, we see that

$$\begin{aligned} L(n, k) &= p \times \text{lcm}(p+1, p+2, \cdots, 2p-3) \\ &> 4b \times \text{lcm}(p+1, p+2, \cdots, 2p-3) \\ &\geq b \times \text{lcm}(p+1, p+2, \cdots, 2p-2) \\ &= bL(n+1, k) \end{aligned}$$

□

Remark. This is just one out of millions of constructions of n and k. There are so much more constructions.

Example (CMO 2012/3)
Let $ABCD$ be a convex quadrilateral and let P be the point of intersection of AC and BD. Suppose that $AC + AD = BC + BD$. Prove that the internal angle bisectors of $\angle ACB$, $\angle ADB$ and $\angle APB$ meet at a common point.

Example (APMO 2018/1)

Let H be the orthocenter of the triangle ABC. Let M and N be the midpoints of the sides AB and AC, respectively. Assume that H lies inside the quadrilateral $BMNC$ and that the circumcircles of triangles BMH and CNH are tangent to each other. The line through H parallel to BC intersects the circumcircles of the triangles BMH and CNH in the points K and L, respectively. Let F be the intersection point of MK and NL and let J be the incenter of triangle MHN. Prove that $FJ = FA$.

Proof. In fact, F is the circumcenter of $AMFN$.

Claim — F is the circumcenter of AMN.

Proof.

$$\begin{aligned}\angle MFN &= 180^\circ - \angle MKH - \angle NLH \\ &= 180^\circ - \angle MBH - \angle NCH \\ &= 180^\circ - (90^\circ - \angle A) - (90^\circ - \angle A) \\ &= 2\angle A\end{aligned}$$

This tells us that F lies on the circle of MOC, where O is the circumcenter of triangle AMN. Moreover, we see that

$$\begin{aligned}\angle AMF &= \angle KMB \\ &= \angle KHB \\ &= \angle CBH \\ &= 90^\circ - \angle C\end{aligned}$$

We know that $\angle AMO = 90^\circ - \angle C$. Because line from M that make $90^\circ - \angle C$ only intersect the circumcircle of MOC, then $O = F$. □

Claim — A, M, N, J are concyclic.

Proof. Let ℓ be the common tangent line of the two circumcircles. We have

$$\begin{aligned}\angle MHN &= \angle(MH, \ell) + \angle(\ell, HN) \\ &= \angle MKH + \angle HLN \\ &= 180^\circ - 2\angle A\end{aligned}$$

Therefore, $\angle MJN = 90^\circ + \frac{1}{2}\angle MHN = 180^\circ - \angle A$. The claim follows. □

Thus, joining the two above claims, we see that F is the circumcenter of (A, M, J, N), implying that $AF = FJ$. □

Remark. Points F and J are rather random. First, we can easily find the angle on F: $\angle MFN = 2\angle A$, then it is easy to guess that F is the circumcenter of $\triangle AMN$.

Now, we know that $AF = FJ$ must be true, so $AF = MF = NF = FJ$. Thus, it suffcies to show that A, M, N, F are concyclic. PoP is impossible, so we must angle chase. At this point only the tangent condiction has not been used. Using it implies the result.

Example (APMO 2015/2)

Let $S = \{2, 3, 4, \ldots\}$ denote the set of integers that are greater than or equal to 2. Does there exist a function $f : S \to S$ such that

$$f(a)f(b) = f(a^2b^2) \text{ for all } a, b \in S \text{ with } a \neq b?$$

$$\frac{f(2)f(54)}{f(3)} = f(36) = f(2)f(3) \Longrightarrow f(3)^2 = f(54)$$

$$\frac{f(108)f(4)}{f(3)} = f(144) = f(4)f(3) \Longrightarrow f(108) = f(3)^2$$

We have $f(54)f(2n) = f(108)f(n) \Longrightarrow f(2n) = f(n)$, when $n \neq 108, n \neq 27$. Therefore, we may find that $f(2) = f(4) = f(8) = f(16) = f(32) = f(64)$.

On the other hand we have $f(2)f(4) = f(64)$. However, because $f(2) = f(4) = f(64) \geq 2$, there is no solution.

Thus, no such f exist.

§3 CMO 2019

Example (CMO 2019/3)

You have a $2m$ by $2n$ grid of squares coloured in the same way as a standard checkerboard. Find the total number of ways to place mn counters on white squares so that each square contains at most one counter and no two counters are in diagonally adjacent white squares.

Proof. Trivial by European Girls Math Olympiad 2015/2. Anwser is $\binom{m+n}{m}$. □

Example (CMO 2019/4)

Prove that for $n > 1$ and real numbers $a_0, a_1, \ldots, a_n, k$ with $a_1 = a_{n-1} = 0$,

$$|a_0| - |a_n| \leq \sum_{i=0}^{n-2} |a_i - ka_{i+1} - a_{i+2}|.$$

Proof. We can let two constants, p and q satisfy $p - q = k$ and $pq = 1$. In particular, we can find that $p = \frac{1}{2}\left(\sqrt{k^2+4}+k\right), q = \frac{1}{2}\left(\sqrt{k^2+4}-k\right)$. Now, we see that

$$\sum_{i=0}^{n-2} |a_i - ka_{i+1} - a_{i+2}| = \sum_{i=0}^{n-2} |a_i - pa_{i+1} + qa_{i+1} - pqa_{i+2}|$$

We can let $d_i = a_i - pa_{i+1}$. Now, the result is easily seen:

$$\begin{aligned}
\sum_{i=0}^{n-2} |a_i - ka_{i+1} - a_{i+2}| &= \sum_{i=0}^{n-2} |a_i - pa_{i+1} + qa_{i+1} - pqa_{i+2}| \\
&= \sum_{i=0}^{n-2} |d_i + qd_{i+1}| \\
&\geq \sum_{i=0}^{n-2} |d_i| - |q| \cdot |d_{i+1}| \\
&= |d_0| - |d_{n-1}| + (1 - |q|) \sum_{i=0}^{n-2} |d_i|
\end{aligned}$$

At this point, we can easily see that $|q| = \frac{1}{2}\left(\sqrt{k^2+4}-k\right) \leq 1$. Also, $|d_0| = |a_0 - pa_1| = |a_0|$, and $|d_{n-1}| = |a_{n-1} - pn| = |pa_n| = |p| \cdot |a_n| \geq |a_n|$. Therefore, the desired result follows. □

Example (CMO 2019/5)

A 2-player game is played on $n \geq 3$ points, where no 3 points are collinear. Each move consists of selecting 2 of the points and drawing a new line segment connecting them. The first player to draw a line segment that creates an odd cycle loses. (An odd cycle must have all its vertices among the n points from the start, so the vertices of the cycle cannot be the intersections of the lines drawn.) Find all n such that the player to move first wins.

Proof. It is easy to see that a player loses on the next move when the existing graph is a complete bipartite graph on n vertices.

Claim — The second player wins when n is odd.

Proof. One of the group must be even, the other must be odd. Therefore, the there are an even number of edges, meaning that after an even number of moves, the graph is going to be bipartite. Consequently, the first player loses. □

Claim — The second player wins when n is a multiple of four.

Proof. On the first turn, the first player must draw an edge, so that the two vertices must be in different groups. Call a vertex "fixed" if it must be in different groups as some other vertices.

In particular, if player one fixes two vertices, then player two can counter it by drawing another two vertices. If player one fixes one vertex, then player two also fixed one vertex, as long as the two newly fixed vertex are in the opposite group. If player one fixes zero vertices, the player two can fix either two or zero vertices.

We see that after player 2 moves, the number of fixed vertex is always even. Moreover, the number of vertex in each group is the same. Therefore, player two wins because the number of edge in the complete bipartite graph is even. □

Claim — The first player wins when $n \equiv 2 \pmod 4$.

Proof. Similar to the strategy of player two in $4|n$ case, if player two fix one vertex, then player one would fix one vertex too. Player one can keep the number of vertex in the two groups the same. Thus, there are an odd number of edges in the complete bipartite graph. Consequently, player two loses. □

Therefore, player one wins only if $n \equiv 2 \pmod 4$, player two wins otherwise. □

§4 CMO 2020

Example (CMO 2020/3)

There are finite many coins in David's purse. The values of these coins are pair wisely distinct positive integers. Is that possible to make such a purse, such that David has exactly 2020 different ways to select the coins in his purse and the sum of these selected coins is 2020?

Proof. There is. Consider coins $(2, 4, 6, \cdots, 22, 1888, 1890, \cdots, 2000, 2014, 2016, 2018, 2020)$.

Notice that $2002, 2004, 2006, 2008, 2010, 2012$ are missing from 1888 to 2020. Assume that they are there, we can see that we can choose however many elements from 2 to 22 as we want, there is always a number that sums them to 2020. Therefore, with the six elements, we have $2^{11} = 2048$ ways of forming 2020. Now, we will remove the six elements

- 2002: (18), (2,16), (4,14), (6,12), (8,10), (2,4,12), (2,6,10), (4,6,8). We need to minus 8 from 2048.
- 2004: (16), (2,14), (4,12), (6,10), (2,4,10), (2,6,8). We need to minus 6 from 2048.
- 2006: (14), (2,12), (4,10), (6,8), (2,4,8). We need to minus 5 from 2048.
- 2008: (12), (2,10), (4,8), (2,4,6). We need to minus 4 from 2048.
- 2010: (10), (2,8), (4,6). We need to minus 3 from 2048.
- 2012: (8), (2,6). We need to minus 2 from 2048.

In total, we see that there are $2048 - 8 - 6 - 5 - 4 - 3 - 2 = 2020$ ways to form value 2020. □

Example (CMO 2020/4)

$S = \{1, 4, 8, 9, 16, ...\}$is the set of perfect integer power. $\left(S = \{n^k | n, k \in Z, k \geq 2\}\right)$. We arrange the elements in S into an increasing sequence $\{a_i\}$. Show that there are infinite many n, such that $9999 | a_{n+1} - a_n$

Proof. Assume otherwise. We can let $a_x = (9999y + 4999)^2$. a_{x+1} could be $(9999y + 5000)^2$. If it is, then we see that $9999 | a_{x+1} - a_x = (2 \cdot 9999y + 9999)$. Because we are assuming that there is a finite number of such a_x, then there must be a biggest y that satisfy there are no perfect power between $(9999y + 4999)^2$ and $(9999y + 5000)^2$. Every integer $z > y$ has a perfect power lying bwteen $(9999z + 4999)^2$ and $(9999z + 5000)^2$.

Ok, how to prove this? Lawl. □

Example (CMO 2020/5)

Simple graph G has 19998 vertices. For any subgraph $\overline{G}$ of G with 9999 vertices, $\overline{G}$ has at least 9999 edges. Find the minimum number of edges in G.

Proof. We claim that the minimum is 49995.

Proof of the bond Seperate the vertices into two groups of 9999. We color one group red and the other group blue. WLOG, we can assume that the number of red vertices that a red vertex is connected to is less than or equal the number of red vertices that a blue vertex is connected to. In particular, if the relation is not satisfied between a red vertex and a blue vertex, then they are switched. Moreover, the inequality is strict if the red vertex is connected to the blue vertex. This is because if the inequality is not strict, we can also switch them.

Assume that there is some blue vertex only connected to two red vertices. Then this means that all red vertex must have at most two edges connected to other red vertices. Because there needs to be 9999 edges formed among the red vertices, then every red vertex must have exactly two edges connected to other red vertices. However, because the blue vertex has two edge with the red vertex. The inequality is not longer strict. Contradiction!

Therefore, there must be at least three edges from the blue vertices to the red vertices. Thus, there are at least 9999 edges within red vertices, 9999 edges with the blue vertices, and $3 \cdot 9999$ edges from blue vertices to red vertices, for a total of $5 \cdot 9999 = 49995$ vertices.

Construction of the bond We can construct 3333 K_6 graphs. In total there are $3333\binom{6}{2} = 49995$ edges in G. If we choose p_i vertices from each sub K_6 graph, we will have in total

$$\sum_{i=1}^{3333} \binom{p_i}{2}$$

of edges. We know that the sum of all p_i is 9999. By Jensen, We see that

$$\sum_{i=1}^{3333} \binom{p_i}{2} \geq 3333\binom{3}{2} = 9999$$

Therefore, this construction will give at least 9999 edges regardless how we choose the vertices. □

§5 CMO 2021

Example (CMO 2021/3)

At a dinner party there are N hosts and N guests, seated around a circular table, where $N \geq 4$. A pair of two guests will chat with one another if either there is at most one person seated between them or if there are exactly two people between them, at least one of whom is a host. Prove that no matter how the $2N$ people are seated at the dinner party, at least N pairs of guests will chat with one another.

Proof. Trivial problem. □

Example (CMO 2021/4)

A function f from the positive integers to the positive integers is called *Canadian* if it satisfies

$$\gcd\left(f(f(x)), f(x+y)\right) = \gcd(x, y)$$

for all pairs of positive integers x and y. Find all positive integers m such that $f(m) = m$ for all Canadian functions f.

Proof. □

Example (CMO 2021/5)

Nina and Tadashi play the following game. Initially, a triple (a, b, c) of nonnegative integers with $a + b + c = 2021$ is written on a blackboard. Nina and Tadashi then take moves in turn, with Nina first. A player making a move chooses a positive integer k and one of the three entries on the board; then the player increases the chosen entry by k and decreases the other two entries by k. A player loses if, on their turn, some entry on the board becomes negative.

Find the number of initial triples (a, b, c) for which Tadashi has a winning strategy.