

1 Fixed Functions in Graphic Pipeline

1.1 Rasterization

Addition Formulas:

$$\begin{aligned}\cos(x+y) &= \cos x \cos y - \sin x \sin y \\ \sin(x+y) &= \sin x \cos y + \cos x \sin y\end{aligned}$$

2D rotate matrix:

$$\begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$$

2D matrix to rotate a vector 90 degrees counter-clockwise:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

To judge vector v_1 is pointing to the right side of vector v_0 :

$$\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \vec{v}_0 \right) \cdot \vec{v}_1 > 0$$

In OpenGL, default visible triangles are counter-clockwise, thus left side of three edges form the triangle area.

1.2 Coordinate System Transform

Describe (u,v,w) space axis in (x,y,z) space:

$$\begin{aligned} \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix} &= \begin{bmatrix} \vec{x} & \vec{y} & \vec{z} \end{bmatrix} \times \begin{bmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{bmatrix} \\ &= \begin{bmatrix} \vec{x} & \vec{y} & \vec{z} \end{bmatrix} \times \mathbf{P} \end{aligned}$$

\mathbf{P} converts a vector from (u,v,w) space to (x,y,z) space; $\text{Inv}(\mathbf{P})$ converts a vector from (x,y,z) space to (u,v,w) space:

$$\begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix} = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix} \times \begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix}$$

$$\begin{aligned}
&= [\vec{x} \quad \vec{y} \quad \vec{z}] \times \mathbf{P} \times \begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix} \\
&= [\vec{x} \quad \vec{y} \quad \vec{z}] \times \left(\mathbf{P} \times \begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix} \right)
\end{aligned}$$

For points transformation:

$$\begin{bmatrix} & & & u_{root} \\ & \mathbf{P} & & v_{root} \\ & & & w_{root} \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} & & & u_{root} \\ & \mathbf{P}^{-1} & & -\mathbf{P}^{-1} \times \begin{bmatrix} u_{root} \\ v_{root} \\ w_{root} \end{bmatrix} \\ & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1.3 Tangent Space

Triangle ABC in (x,y,z) space and (t,b,n) space:

$$\begin{bmatrix} x_a & x_b & x_c \\ y_a & y_b & y_c \\ z_a & z_b & z_c \end{bmatrix} \begin{bmatrix} u_a & u_b & u_c \\ v_a & v_b & v_c \\ 0 & 0 & 0 \end{bmatrix}$$

For 2 edges of the triangle:

$$\begin{aligned}
[\vec{x}, \vec{y}, \vec{z}] \times \begin{bmatrix} E1_x & E2_x \\ E1_y & E2_y \\ E1_z & E2_z \end{bmatrix} &= [\vec{t}, \vec{b}, \vec{n}] \times \begin{bmatrix} E1_u & E2_u \\ E1_v & E2_v \\ 0 & 0 \end{bmatrix} \\
&= [\vec{t}, \vec{b}] \times \begin{bmatrix} E1_u & E2_u \\ E1_v & E2_v \end{bmatrix}
\end{aligned}$$

Then:

$$\begin{aligned}
[\vec{t}, \vec{b}] &= [\vec{x}, \vec{y}, \vec{z}] \times \begin{bmatrix} E1_x & E2_x \\ E1_y & E2_y \\ E1_z & E2_z \end{bmatrix} \times \begin{bmatrix} E1_u & E2_u \\ E1_v & E2_v \end{bmatrix}^{-1} \\
&= [\vec{x}, \vec{y}, \vec{z}] \times \left(\begin{bmatrix} E1_x & E2_x \\ E1_y & E2_y \\ E1_z & E2_z \end{bmatrix} \times \begin{bmatrix} E1_u & E2_u \\ E1_v & E2_v \end{bmatrix}^{-1} \right)
\end{aligned}$$

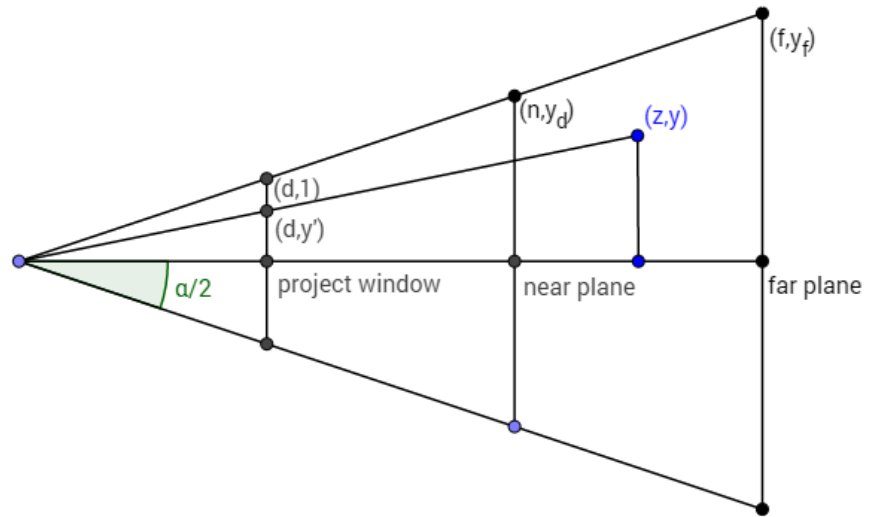
$$\begin{aligned}
&= [\vec{x}, \vec{y}, \vec{z}] \times \left(\begin{bmatrix} E1_x & E2_x \\ E1_y & E2_y \\ E1_z & E2_z \end{bmatrix} \times \frac{1}{E1_u E2_v - E1_v E2_u} \begin{bmatrix} E2_v & -E2_u \\ -E1_v & E1_u \end{bmatrix}^{-1} \right) \\
&= [\vec{x}, \vec{y}, \vec{z}] \times \left(\frac{1}{E1_u E2_v - E1_v E2_u} \begin{bmatrix} E1_x & E2_x \\ E1_y & E2_y \\ E1_z & E2_z \end{bmatrix} \times \begin{bmatrix} E2_v & -E2_u \\ -E1_v & E1_u \end{bmatrix}^{-1} \right)
\end{aligned}$$

1.4 Perspective Projection

In perspective projection, $r, n, f[\alpha, \beta]$ defines a frustum:

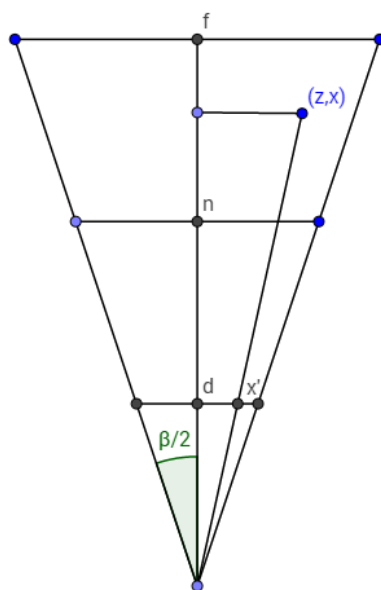
- f: z of far plane
- n: z of near plane
- r: ratio of width to height of view window
- d: z of view window
- α : view angle in the y axis direction
- β : view angle in the x axis direction

In the y axis direction:



$$y' = y \frac{d}{z} = \frac{y}{z \tan \frac{\alpha}{2}}$$

And in the x axis direction:



$$x' = x \frac{d}{z}$$

2 Linear Algebra

2.1 Cross Product

Cross product definition differs in right-hand coordinate system and left-hand coordinate system ensuring that:

$$\begin{aligned}\vec{x} \times \vec{y} &= \vec{z} \\ \vec{y} \times \vec{z} &= \vec{x} \\ \vec{z} \times \vec{x} &= \vec{y}\end{aligned}$$

From this, it can be inferred that:

$$\begin{aligned}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} &= (a_1 \vec{x} + a_2 \vec{y} + a_3 \vec{z}) \times (a_1 \vec{x} + a_2 \vec{y} + a_3 \vec{z}) \\ &= \begin{vmatrix} \vec{x} & \vec{y} & \vec{z} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}\end{aligned}$$