1 Fixed Functions in Graphic Pipeline

1.1 Rasterization

Addition Fomulars:

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

2D rotate matrix:

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

2D matrix to rotate a vector 90 degress counter-clockwise:

$$\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

To judge vector v1 is pointing to the right side of vector v0:

$$\left(\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}\overrightarrow{v_0}\right)\cdot\overrightarrow{v_1} > 0$$

In OpenGL, default visiable triangles are counter-clockwise, thus left side of three edges form the triangle aera.

1.2 Coordinate System Transform

Describe (u,v,w) space axis in (x,y,z) space:

$$\begin{aligned} \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix} &= \begin{bmatrix} \vec{x} & \vec{y} & \vec{z} \end{bmatrix} \times \begin{bmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{bmatrix} \\ &= \begin{bmatrix} \vec{x} & \vec{y} & \vec{z} \end{bmatrix} \times \mathbf{P} \end{aligned}$$

P converts a vector from (u,v,w) space to (x,y,z) space; Inv(**P**) converts a vector from (x,y,z) space to (u,v,w) space:

$$\begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix} = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix} \times \begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix}$$

$$
\begin{aligned}
&= \begin{bmatrix} \overrightarrow{x} & \overrightarrow{y} & \overrightarrow{z} \end{bmatrix} \times \mathbf{P} \times \begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix} \\
&= \begin{bmatrix} \overrightarrow{x} & \overrightarrow{y} & \overrightarrow{z} \end{bmatrix} \times \left(\mathbf{P} \times \begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix} \right)
\end{aligned}
$$

For points transformation:

$$
\begin{bmatrix} & \mathbf{P} & & \begin{matrix} u_{root} \\ v_{root} \\ w_{root} \end{matrix} \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} & \boldsymbol{P^{-1}} & & -\boldsymbol{P^{-1}} \times \begin{bmatrix} u_{root} \\ v_{root} \\ w_{root} \end{bmatrix} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

1.3 Tangent Space

Triangle ABC in (x,y,z) space and (t,b,n) space:

$$
\begin{bmatrix} x_a & x_b & x_c \\ y_a & y_b & y_c \\ z_a & z_b & z_c \end{bmatrix} \begin{bmatrix} u_a & u_b & u_c \\ v_a & v_b & v_c \\ 0 & 0 & 0 \end{bmatrix}
$$

For 2 edges of the triangle:

$$
\begin{aligned}
\left[\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}\right] \times \begin{bmatrix} E1_x & E2_x \\ E1_y & E2_y \\ E1_z & E2_z \end{bmatrix} &= \left[\overrightarrow{t}, \overrightarrow{b}, \overrightarrow{n}\right] \times \begin{bmatrix} E1_u & E2_u \\ E1_v & E2_v \\ 0 & 0 \end{bmatrix} \\
&= \left[\overrightarrow{t}, \overrightarrow{b}\right] \times \begin{bmatrix} E1_u & E2_u \\ E1_v & E2_v \end{bmatrix}
\end{aligned}
$$

Then:

$$
\begin{aligned}
\left[\overrightarrow{t}, \overrightarrow{b}\right] &= \left[\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}\right] \times \begin{bmatrix} E1_x & E2_x \\ E1_y & E2_y \\ E1_z & E2_z \end{bmatrix} \times \begin{bmatrix} E1_u & E2_u \\ E1_v & E2_v \end{bmatrix}^{-1} \\
&= \left[\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}\right] \times \left(\begin{bmatrix} E1_x & E2_x \\ E1_y & E2_y \\ E1_z & E2_z \end{bmatrix} \times \begin{bmatrix} E1_u & E2_u \\ E1_v & E2_v \end{bmatrix}^{-1} \right)
\end{aligned}
$$

$$= \left[\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}\right] \times \left(\begin{bmatrix} E1_x & E2_x \\ E1_y & E2_y \\ E1_z & E2_z \end{bmatrix} \times \frac{1}{E1_u E2_v - E1_v E2_u} \begin{bmatrix} E2_v & -E2_u \\ -E1_v & E1_u \end{bmatrix}^{-1} \right)$$

$$= \left[\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}\right] \times \left(\frac{1}{E1_u E2_v - E1_v E2_u} \begin{bmatrix} E1_x & E2_x \\ E1_y & E2_y \\ E1_z & E2_z \end{bmatrix} \times \begin{bmatrix} E2_v & -E2_u \\ -E1_v & E1_u \end{bmatrix}^{-1} \right)$$

2 Linear Algebra

2.1 Cross Product

Cross product definition differs in right-hand coordinate system and left-hand coordinate system ensuring that:

$$\begin{aligned} \overrightarrow{x} \times \overrightarrow{y} &= \overrightarrow{z} \\ \overrightarrow{y} \times \overrightarrow{z} &= \overrightarrow{x} \\ \overrightarrow{z} \times \overrightarrow{x} &= \overrightarrow{y} \end{aligned}$$

From this, it can be inferred that:

$$\begin{aligned} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} &= \left(a_1 \overrightarrow{x} + a_2 \overrightarrow{y} + a_3 \overrightarrow{z}\right) \times \left(a_1 \overrightarrow{x} + a_2 \overrightarrow{y} + a_3 \overrightarrow{z}\right) \\ &= \begin{vmatrix} \overrightarrow{x} & \overrightarrow{y} & \overrightarrow{z} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \end{aligned}$$