1 Fixed Function in Graphic Pipeline

1.1 Rasterization

Addition formulas:

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

2D rotate matrix:

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

2D matrix to rotate a vector 90 degress counter-clockwise:

$$\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

To judge vector v1 is pointing to the right side of vector v0:

$$\left(\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}\vec{v_0}\right)\cdot\vec{v_1} > 0$$

In OpenGL, default visiable triangles are counter-clockwise, thus left side of three edges form the triangle aera.

1.2 Coordinate System Transform

Describe (u,v,w) space axis in (x,y,z) space:

$$\begin{aligned} \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix} &= \begin{bmatrix} \vec{x} & \vec{y} & \vec{z} \end{bmatrix} \times \begin{bmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{bmatrix} \\ &= \begin{bmatrix} \vec{x} & \vec{y} & \vec{z} \end{bmatrix} \times \mathbf{P} \end{aligned}$$

P converts a vector from (u,v,w) space to (x,y,z) space; Inv(**P**) converts a vector from (x,y,z) space to (u,v,w) space:

$$\begin{aligned} \begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix} &= \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix} \times \begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix} \\ &= \begin{bmatrix} \vec{x} & \vec{y} & \vec{z} \end{bmatrix} \times \mathbf{P} \times \begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix} \\ &= \begin{bmatrix} \vec{x} & \vec{y} & \vec{z} \end{bmatrix} \times \left(\mathbf{P} \times \begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix}\right) \end{aligned}$$

For points transformation:

$$\begin{bmatrix} & \mathbf{P} & & u_{root} \\ & & & v_{root} \\ & & & w_{root} \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} & \boldsymbol{P}^{-1} & & -\boldsymbol{P}^{-1} \times \begin{bmatrix} u_{root} \\ v_{root} \\ w_{root} \end{bmatrix} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1.3 Tangent Space

Triangle ABC in (x,y,z) space and (t,b,n) space:

$$\begin{bmatrix} x_a & x_b & x_c \\ y_a & y_b & y_c \\ z_a & z_b & z_c \end{bmatrix} \begin{bmatrix} u_a & u_b & u_c \\ v_a & v_b & v_c \\ 0 & 0 & 0 \end{bmatrix}$$

For 2 edges of the triangle:

$$\begin{aligned} \left[\vec{x}, \vec{y}, \vec{z}\right] \times \begin{bmatrix} E1_x & E2_x \\ E1_y & E2_y \\ E1_z & E2_z \end{bmatrix} &= \left[\vec{t}, \vec{b}, \vec{n}\right] \times \begin{bmatrix} E1_u & E2_u \\ E1_v & E2_v \\ 0 & 0 \end{bmatrix} \\ &= \left[\vec{t}, \vec{b}\right] \times \begin{bmatrix} E1_u & E2_u \\ E1_v & E2_v \end{bmatrix} \end{aligned}$$

Then:

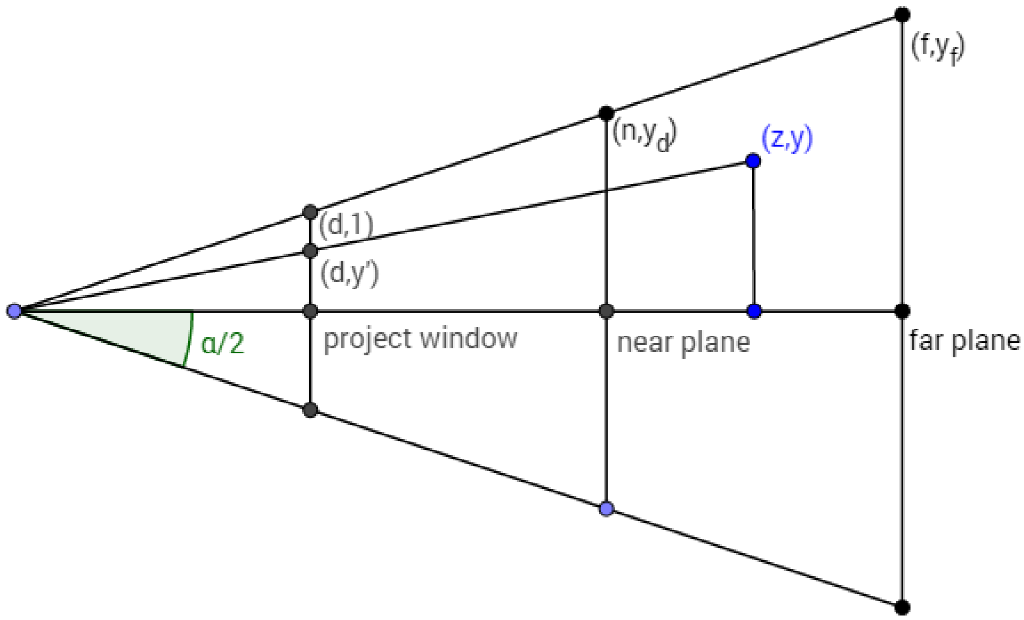
$$\begin{aligned} \left[\vec{t}, \vec{b}\right] &= \left[\vec{x}, \vec{y}, \vec{z}\right] \times \begin{bmatrix} E1_x & E2_x \\ E1_y & E2_y \\ E1_z & E2_z \end{bmatrix} \times \begin{bmatrix} E1_u & E2_u \\ E1_v & E2_v \end{bmatrix}^{-1} \\ &= \left[\vec{x}, \vec{y}, \vec{z}\right] \times \left(\begin{bmatrix} E1_x & E2_x \\ E1_y & E2_y \\ E1_z & E2_z \end{bmatrix} \times \begin{bmatrix} E1_u & E2_u \\ E1_v & E2_v \end{bmatrix}^{-1} \right) \\ &= \left[\vec{x}, \vec{y}, \vec{z}\right] \times \left(\begin{bmatrix} E1_x & E2_x \\ E1_y & E2_y \\ E1_z & E2_z \end{bmatrix} \times \frac{1}{E1_u E2_v - E1_v E2_u} \begin{bmatrix} E2_v & -E2_u \\ -E1_v & E1_u \end{bmatrix}^{-1} \right) \\ &= \left[\vec{x}, \vec{y}, \vec{z}\right] \times \left(\frac{1}{E1_u E2_v - E1_v E2_u} \begin{bmatrix} E1_x & E2_x \\ E1_y & E2_y \\ E1_z & E2_z \end{bmatrix} \times \begin{bmatrix} E2_v & -E2_u \\ -E1_v & E1_u \end{bmatrix}^{-1} \right) \end{aligned}$$

1.4 Perspective Projection

In perspective projection, r,n,f[α, β] defines a frustum:

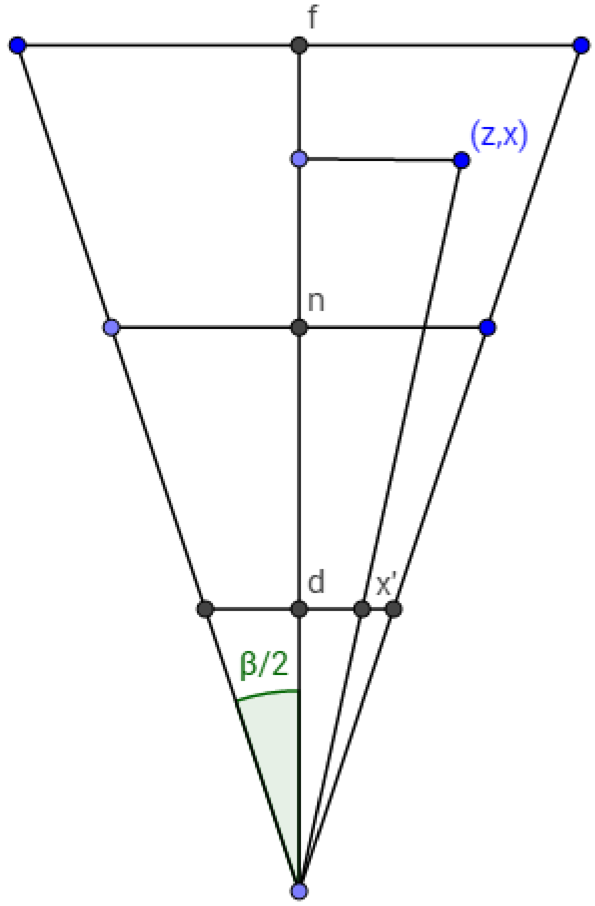
- f: z of far plane
- n: z of near plane
- r: ratio of width to height of view wnidow
- d: z of view window
- α: view angle in the y axis direction
- β: view angle in the x axis direction

In the y axis direction:



$$y' = y\frac{d}{z} = \frac{y}{z\tan\frac{\alpha}{2}}$$

And in the x axis direction:



$$x' = x\frac{d}{z} = \frac{x}{z \bullet r\tan\frac{\alpha}{2}}$$

Put together:

$$\begin{bmatrix} \frac{1}{r\tan\frac{\alpha}{2}} & & & \\ & \frac{1}{\tan\frac{\alpha}{2}} & & \\ & & A & B \\ & & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{x}{r\tan\frac{\alpha}{2}} \\ \frac{y}{\tan\frac{\alpha}{2}} \\ A \bullet z + B \\ z \end{bmatrix} \rightarrow \begin{bmatrix} \frac{x}{r \bullet z\tan\frac{\alpha}{2}} \\ \frac{y}{z\tan\frac{\alpha}{2}} \\ A + \frac{B}{z} \\ 1 \end{bmatrix}$$

To make A+B/z 0 at near plane and -1 at far plane:

- $A = \frac{-f}{f-n}$
- $B = \frac{nf}{f-n}$

2 Linear Algebra

2.1 Cross Product

Cross product definition differs in right-hand coordinate system and left-hand coordinate system ensuring that:

$$\overrightarrow{x} \times \overrightarrow{y} = \overrightarrow{z}$$
$$\overrightarrow{y} \times \overrightarrow{z} = \overrightarrow{x}$$
$$\overrightarrow{z} \times \overrightarrow{x} = \overrightarrow{y}$$

From this, it can be inferred that:

$$\begin{aligned}
\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} &= \left(a_1 \overrightarrow{x} + a_2 \overrightarrow{y} + a_3 \overrightarrow{z}\right) \times \left(a_1 \overrightarrow{x} + a_2 \overrightarrow{y} + a_3 \overrightarrow{z}\right) \\
&= \begin{vmatrix} \overrightarrow{x} & \overrightarrow{y} & \overrightarrow{z} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}
\end{aligned}$$