

8. Sequences of functions

- power series
- pointwise and uniform convergence
- interchange of limits
- Weierstrass M-test
- properties of power series

Power series

Definition 8.1 A **power series** about $x_0 \in \mathbf{R}$ is a series of the form

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m.$$

Definition 8.2 Let $\sum_{m=0}^{\infty} a_m (x - x_0)^m$ be a power series, if the limit

$$R = \lim_{m \rightarrow \infty} |a_m|^{1/m}$$

exists, we define the **radius of convergence** ρ as

$$\rho = \begin{cases} 1/R & R > 0 \\ \infty & R = 0. \end{cases}$$

Theorem 8.3 Let $\sum_{m=0}^{\infty} a_m(x - x_0)^m$ be a power series and suppose $R = \lim_{m \rightarrow \infty} |a_m|^{1/m}$ exists. If $R = 0$, the series converges absolutely for all $x \in \mathbf{R}$. If $R > 0$, the series converges absolutely if $|x - x_0| < \rho$ and diverges if $|x - x_0| > \rho$.

proof: consider the root test (theorem 4.27), we have

$$L = \lim_{m \rightarrow \infty} |a_m(x - x_0)^m|^{1/m} = |x - x_0| \lim_{m \rightarrow \infty} |a_m|^{1/m} = R|x - x_0|$$

- suppose $R = 0$, then we have $L = 0 < 1$ for all $x \in \mathbf{R}$
 $\implies \sum_{m=0}^{\infty} a_m(x - x_0)^m$ converges absolutely for all $x \in \mathbf{R}$
- suppose $R > 0$
 - if $|x - x_0| < \rho \implies L < R\rho = 1 \implies \sum_{m=0}^{\infty} a_m(x - x_0)^m$ converges absolutely
 - if $|x - x_0| > \rho \implies L > R\rho = 1 \implies \sum_{m=0}^{\infty} a_m(x - x_0)^m$ diverges

Remark 8.4 Let $\sum_{m=0}^{\infty} a_m(x - x_0)^m$ be a power series with radius of convergence ρ . Define $f: (x_0 - \rho, x_0 + \rho) \rightarrow \mathbf{R}$ such that

$$f(x) = \sum_{m=0}^{\infty} a_m(x - x_0)^m,$$

then, the function f is the limit of a sequence of functions $(f_n)_{n=1}^{\infty}$, given by

$$f(x) = \lim_{n \rightarrow \infty} f_n(x), \quad f_n(x) = \sum_{m=0}^n a_m(x - x_0)^m.$$

Example 8.5 Consider the geometric series $\sum_{m=0}^{\infty} x^m$ (which is a power series with $a_m = 1$, $x_0 = 0$), we have $f: (-1, 1) \rightarrow \mathbf{R}$ given by

$$f(x) = \frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = \lim_{n \rightarrow \infty} f_n(x), \quad f_n(x) = \sum_{m=0}^n x^m.$$

Example 8.6 *Exponential function.* Consider the power series with $a_m = \frac{1}{m!}$, $x_0 = 0$, we have the exponential function $f(x): \mathbf{R} \rightarrow \mathbf{R}$, given by

$$f(x) = \exp(x) = \sum_{m=0}^{\infty} \frac{x^m}{m!} = \lim_{n \rightarrow \infty} f_n(x), \quad f_n(x) = \sum_{m=0}^n \frac{x^m}{m!}.$$

Remark 8.7 Based on remark 8.4, we may ask several questions.

(1) Is the function f continuous?

(2) If (1) is true, is f differentiable, and does $f' = \lim_{n \rightarrow \infty} f'_n$?

(3) If (1) is true, does $\int_a^b f = \lim_{n \rightarrow \infty} \int_a^b f_n$?

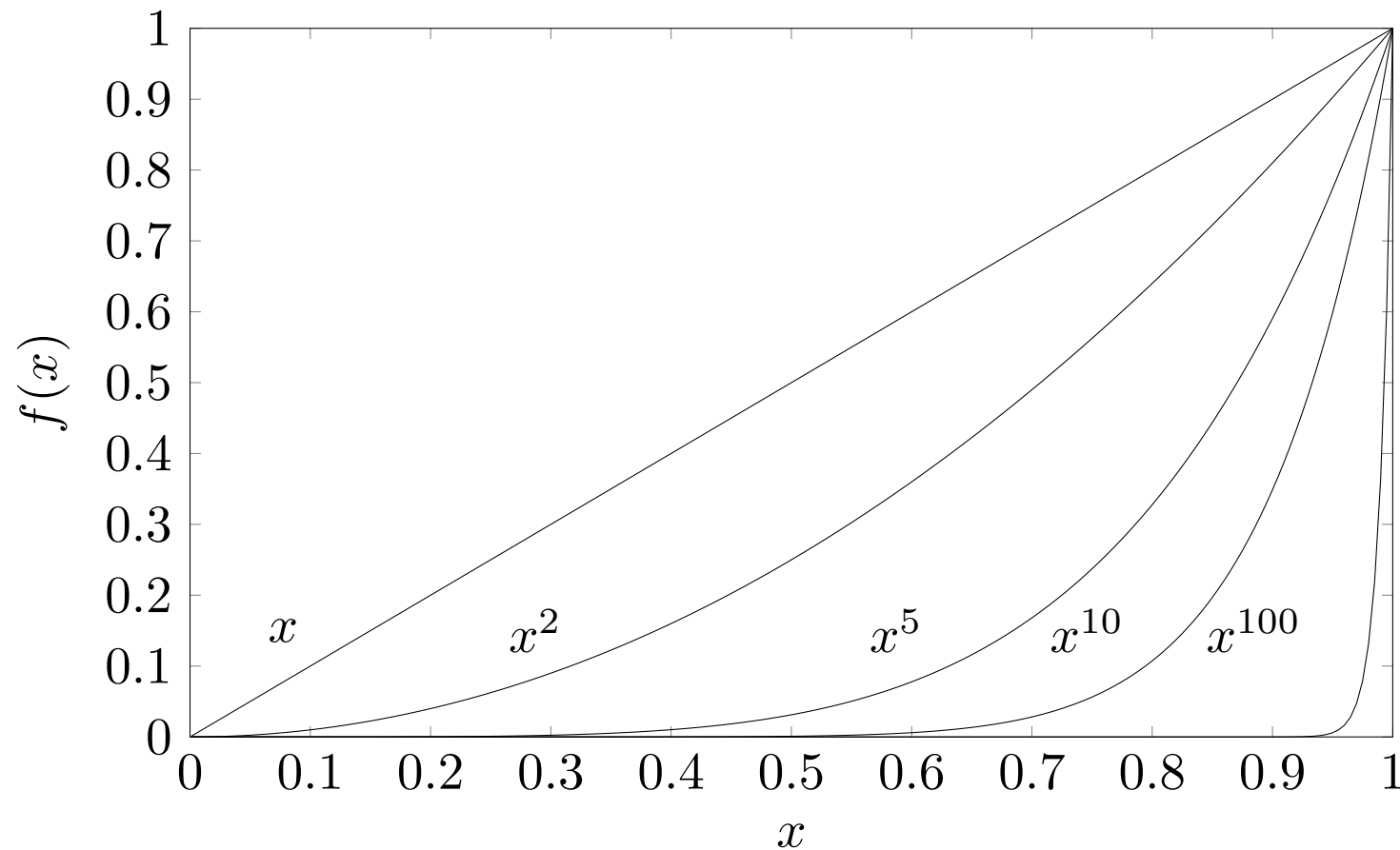
Pointwise convergence

Definition 8.8 Let $(f_n)_{n=1}^{\infty}$ with $f_n: S \rightarrow \mathbf{R}$ for all $n \in \mathbf{N}$ be a sequence of functions, and let $f: S \rightarrow \mathbf{R}$ be a function. We say that $(f_n)_{n=1}^{\infty}$ **converges pointwise** (or just **converges**) to f if for all $x \in S$, we have $\lim_{n \rightarrow \infty} f_n(x) = f(x)$.

Example 8.9 Let $f_n(x) = x^n$ be defined on $[0, 1]$, then we have the sequence of functions $(f_n)_{n=1}^{\infty}$ converges pointwise to $f(x) = \begin{cases} 0 & x \in [0, 1) \\ 1 & x = 1 \end{cases}$.

proof:

- if $x \in [0, 1)$: $\lim_{n \rightarrow \infty} x^n = 0$
- if $x = 1$: $\lim_{n \rightarrow \infty} 1^n = 1$



Remark 8.10 A sequence of continuous functions may not converge pointwise to a continuous function.

Example 8.11 Let $f_n(x): [0, 1] \rightarrow \mathbf{R}$ be defined by

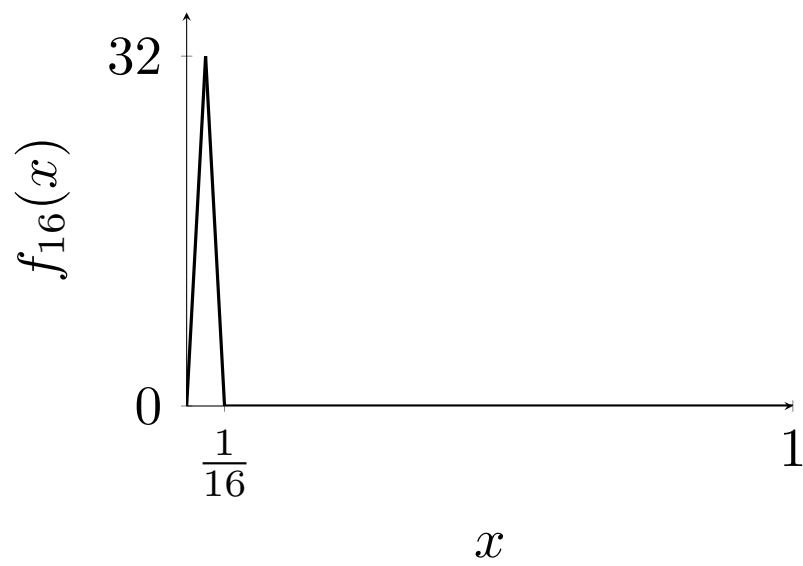
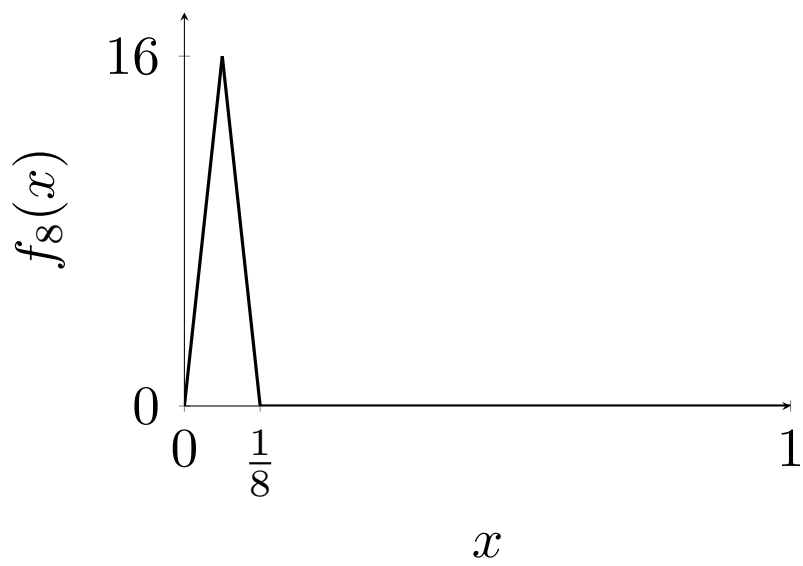
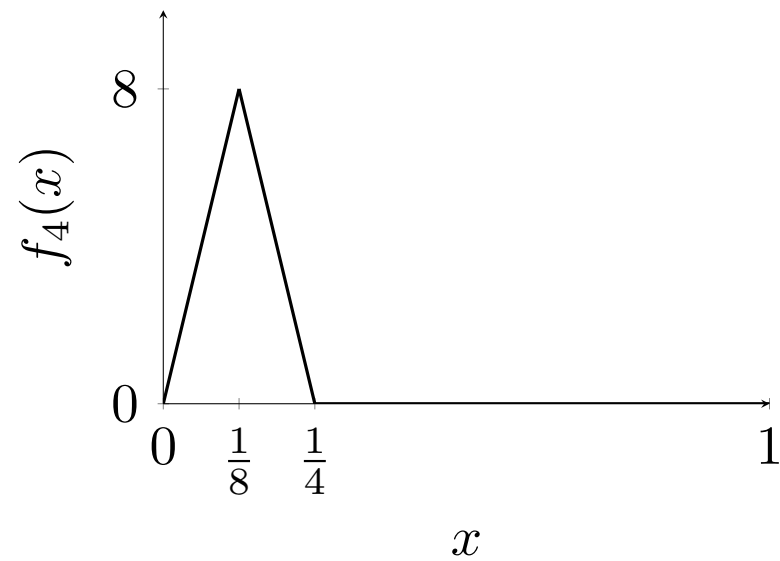
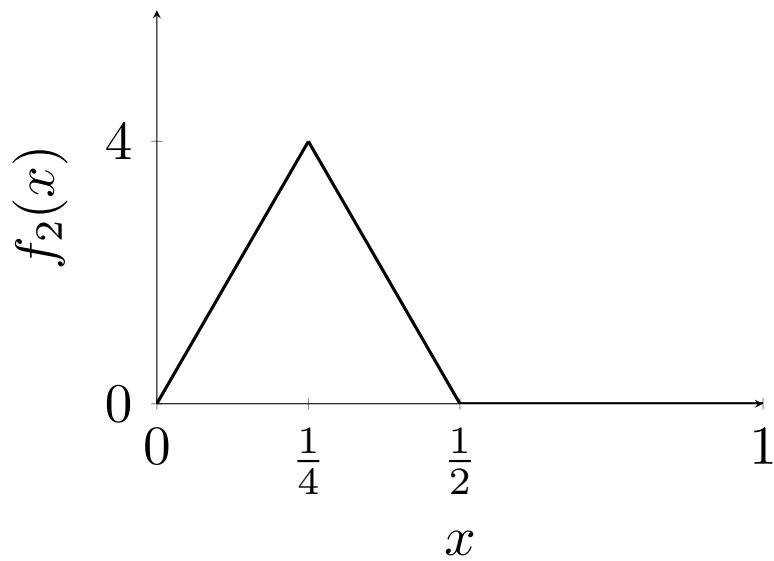
$$f_n(x) = \begin{cases} 4n^2x & x \in [0, \frac{1}{2n}] \\ 4n - 4n^2x & x \in [\frac{1}{2n}, \frac{1}{n}] \\ 0 & x \in [\frac{1}{n}, 1] \end{cases},$$

then $(f_n)_{n=1}^{\infty}$ converges pointwise to $f(x) = 0$ ($x \in [0, 1]$).

proof:

- if $x = 0$, we have $\lim_{n \rightarrow \infty} f_n(0) = 0$
- if $x \in (0, 1]$, then $\exists M \in [0, 1]$ such that $\forall n \geq M, \frac{1}{n} < x$, and hence,

$$(f_n(x))_{n=1}^{\infty} = f_1(x), \dots, f_{M-1}(x), 0, 0, 0, \dots \implies \lim_{n \rightarrow \infty} f_n(x) = 0$$



Uniform convergence

Definition 8.12 Let $(f_n)_{n=1}^{\infty}$ with $f_n: S \rightarrow \mathbf{R}$ for all $n \in \mathbf{N}$ be a sequence of functions, and let $f: S \rightarrow \mathbf{R}$ be a function. We say that $(f_n)_{n=1}^{\infty}$ **converges uniformly** to f if for all $\epsilon > 0$, there exists some $M \in \mathbf{N}$ such that for all $n \geq M$ and $x \in S$, we have $|f_n(x) - f(x)| < \epsilon$.

Theorem 8.13 Let $f: S \rightarrow \mathbf{R}$, $f_n: S \rightarrow \mathbf{R}$ for all $n \in \mathbf{N}$ be functions. If the sequence of functions $(f_n)_{n=1}^{\infty}$ converges uniformly to f , then $(f_n)_{n=1}^{\infty}$ converges pointwise to f .

proof: let $c \in S$, $\epsilon > 0$

- $(f_n)_{n=1}^{\infty}$ converges uniformly to $f \implies \exists M \in \mathbf{N}$ such that for all $n \geq M$ and $x \in S$, $|f_n(x) - f(x)| < \epsilon$
- hence, $\forall n \geq M$, $|f_n(c) - f(c)| < \epsilon \implies (f_n)_{n=1}^{\infty}$ converges pointwise to f

Remark 8.14 Let $f: S \rightarrow \mathbf{R}$, $f_n: S \rightarrow \mathbf{R}$ for all $n \in \mathbf{N}$ be functions. The sequence $(f_n)_{n=1}^{\infty}$ does not converge to f uniformly if there exists some $\epsilon > 0$ such that for all $M \in \mathbf{N}$, there exist some $n \geq M$ and some $x \in S$, so that $|f_n(x) - f(x)| \geq \epsilon$.

Theorem 8.15 Let $f_n(x) = x^n$, $n \in \mathbf{N}$, and let

$$f(x) = \begin{cases} 0 & x \in [0, 1) \\ 1 & x = 1. \end{cases}$$

- The sequence $(f_n)_{n=1}^{\infty}$ converges uniformly to f on $[0, b]$ for all $0 < b < 1$.
 - The sequence $(f_n)_{n=1}^{\infty}$ does not converge to f uniformly on $[0, 1]$.
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proof:

- let $\epsilon > 0$, $b \in (0, 1)$, then $b^n \rightarrow 0 \implies \exists M \in \mathbf{N}$ such that $\forall n \geq M$, $b^n < \epsilon \implies \forall n \geq M$ and $x \in [0, b]$, we have

$$|f_n(x) - f(x)| = x^n \leq b^n < \epsilon$$

- choose $\epsilon = 1/2$, then $\forall M \in \mathbf{N}$, choose $n = M$, $x = (1/2)^{1/M} < 1$, we have

$$|f_M(x) - f(x)| = x^M = 1/2 \geq \epsilon$$

Interchange of limits

Example 8.16 In general, limits cannot be interchanged. For example,

$$\lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} \frac{n/k}{n/k + 1} = \lim_{n \rightarrow \infty} 0 = 0, \quad \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{n/k}{n/k + 1} = \lim_{k \rightarrow \infty} 1 = 1.$$

Remark 8.17 Based on example 8.16, we may ask the following questions.

- If $f_n: S \rightarrow \mathbf{R}$ with f_n continuous for all $n \in \mathbf{N}$ and $(f_n)_{n=1}^{\infty}$ converges to f uniformly or pointwise, then is f continuous?
- If $f_n: [a, b] \rightarrow \mathbf{R}$ with f_n differentiable for all $n \in \mathbf{N}$, and $(f_n)_{n=1}^{\infty}$ converges to f , $(f'_n)_{n=1}^{\infty}$ converges to g uniformly or pointwise, then is f differentiable and does $f' = g$?
- If $f_n: [a, b] \rightarrow \mathbf{R}$, $n \in \mathbf{N}$, $f: [a, b] \rightarrow \mathbf{R}$, with f_n and f continuous, and $(f_n)_{n=1}^{\infty}$ converges to f uniformly or pointwise, then does $\int_a^b f = \lim_{n \rightarrow \infty} \int_a^b f_n$?

Remark 8.18 If convergence is only pointwise, the answer is *no* for all questions in remark 8.17.

- Let $f_n(x) = x^n$ on $[0, 1]$, $n \in \mathbf{N}$. Example 8.9 shows that $(f_n)_{n=1}^{\infty}$ converges pointwise to a noncontinuous function.

- Let $f_n(x) = \frac{x^{n+1}}{n+1}$ on $[0, 1]$, then $(f_n)_{n=1}^{\infty}$ converges to $f(x) = 0$ pointwise on $[0, 1]$ and $(f'_n)_{n=1}^{\infty}$ converges pointwise to g given by
$$g(x) = \begin{cases} 0 & x \in [0, 1) \\ 1 & x = 1 \end{cases}, \text{ but } f'(1) = 0 \neq g(1) = 1.$$

- Let $f_n: [0, 1] \rightarrow \mathbf{R}$ be given by
$$f_n(x) = \begin{cases} 4n^2x & x \in [0, \frac{1}{2n}] \\ 4n - 4n^2x & x \in [\frac{1}{2n}, \frac{1}{n}] \\ 0 & x \in [\frac{1}{n}, 1] \end{cases},$$

then $(f_n)_{n=1}^{\infty}$ converges to $f(x) = 0$ pointwise on $[0, 1]$ (example 8.11), but

$$\int_0^1 f = 0 \neq \lim_{n \rightarrow \infty} \int_0^1 f_n = \lim_{n \rightarrow \infty} \left(\frac{1}{2} \cdot \frac{1}{n} \cdot 2n \right) = 1.$$

Theorem 8.19 If $f_n: S \rightarrow \mathbf{R}$ is continuous for all $n \in \mathbf{N}$, $f: S \rightarrow \mathbf{R}$, and $(f_n)_{n=1}^{\infty}$ converges to f uniformly, then f is continuous.

proof: let $c \in S$, $\epsilon > 0$

- f_n continuous on S , $c \in S \implies \exists \delta > 0$ such that for all $x \in S$ and $|x - c| < \delta$, we have $|f_n(x) - f_n(c)| < \epsilon/3$
- $f_n \rightarrow f$ uniformly $\implies \exists M \in \mathbf{N}$ such that for all $n \geq M$ and $x \in S$, we have $|f(x) - f_n(x)| < \epsilon/3$
- hence, for all $x \in S$ and $|x - c| < \delta$, we have

$$\begin{aligned} |f(x) - f(c)| &= |f(x) - f_M(x) + f_M(x) - f_M(c) + f_M(c) - f(c)| \\ &\leq |f(x) - f_M(x)| + |f_M(x) - f_M(c)| + |f_M(c) - f(c)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

Theorem 8.20 If $f_n: [a, b] \rightarrow \mathbf{R}$ is continuous for all $n \in \mathbf{N}$, $f: [a, b] \rightarrow \mathbf{R}$, and $(f_n)_{n=1}^{\infty}$ converges to f uniformly, then $\int_a^b f = \lim_{n \rightarrow \infty} \int_a^b f_n$.

proof: let $\epsilon > 0$

- by theorem 8.19, we know that f is continuous on $[a, b]$
- $(f_n)_{n=1}^{\infty}$ converges uniformly to $f \implies \exists M \in \mathbf{N}$ such that for all $n \geq M$ and $x \in [a, b]$, we have $|f_n(x) - f(x)| < \frac{\epsilon}{b-a}$
- hence, for all $n \geq M$, we have

$$\left| \int_a^b f_n - \int_a^b f \right| = \left| \int_a^b (f_n - f) \right| \leq \int_a^b |f_n - f| < \int_a^b \frac{\epsilon}{b-a} = \epsilon,$$

where the first inequality is by corollary 7.17

Remark 8.21 Notationally, theorem 8.20 says that

$$\int_a^b f = \int_a^b \lim_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} \int_a^b f_n.$$

Theorem 8.22 If $f_n: [a, b] \rightarrow \mathbf{R}$ is continuously differentiable, $f: [a, b] \rightarrow \mathbf{R}$, $g: [a, b] \rightarrow \mathbf{R}$, and

- $(f_n)_{n=1}^{\infty}$ converges to f pointwise,
- $(f'_n)_{n=1}^{\infty}$ converges to g uniformly,

then f is continuously differentiable and $f' = g$.

proof: let $x \in [a, b]$

- by theorem 8.19, we know that g is continuous on $[a, b]$
- by theorem 7.19, we have

$$\int_a^x f'_n = f_n(x) - f(a) \implies \lim_{n \rightarrow \infty} \int_a^x f'_n = \lim_{n \rightarrow \infty} f_n(x) - \lim_{n \rightarrow \infty} f_n(a)$$

- $f_n \rightarrow f$ pointwise $\implies \lim_{n \rightarrow \infty} f_n(x) - \lim_{n \rightarrow \infty} f_n(a) = f(x) - f(a)$
- $f'_n \rightarrow g$ uniformly $\implies \lim_{n \rightarrow \infty} \int_a^x f'_n = \int_a^x g$ (theorem 8.20)
- put together, we have

$$\int_a^x g = f(x) - f(a) \implies \left(\int_a^x g \right)' = g(x) = f'(x)$$

Weierstrass M-test

Theorem 8.23 *Weierstrass M-test.* Let $f_k: S \rightarrow \mathbf{R}$ for all $k \in \mathbf{N}$. Suppose there exists $M_k > 0$, $k \in \mathbf{N}$, such that

(a) $|f_k(x)| \leq M_k$ for all $x \in S$,

(b) $\sum_{k=1}^{\infty} M_k$ converges.

Then, we have the following conclusion.

(1) The series $\sum_{k=1}^{\infty} f_k(x)$ converges absolutely for all $x \in S$.

(2) Let $f(x) = \sum_{k=1}^{\infty} f_k(x)$ for all $x \in S$, then the sequence $(\sum_{k=1}^n f_k)_{n=1}^{\infty}$ converges to f uniformly on S .

proof:

(1) $|f_k(x)| \leq M_k$, $\sum_{k=1}^{\infty} M_k$ converges $\implies \sum_{k=1}^{\infty} |f_k(x)|$ converges
(theorem 4.21) $\implies \sum_{k=1}^{\infty} f_k(x)$ converges absolutely

(2) let $\epsilon > 0$; $\sum_{k=1}^{\infty} M_k$ converges $\implies \exists M \in \mathbf{N}$ s.t. $\forall n \geq M$, we have

$$\sum_{k=n+1}^{\infty} M_k = \left| \sum_{k=1}^{\infty} M_k - \sum_{k=1}^n M_k \right| < \epsilon$$

then, for all $n \geq M$ and $x \in S$, we have

$$\begin{aligned} \left| \sum_{k=1}^{\infty} f_k(x) - \sum_{k=1}^n f_k(x) \right| &= \left| \sum_{k=n+1}^{\infty} f_k(x) \right| \\ &\leq \sum_{k=n+1}^{\infty} |f_k(x)| \\ &\leq \sum_{k=n+1}^{\infty} M_k \\ &< \epsilon \end{aligned}$$

Properties of power series

Theorem 8.24 Let $\sum_{k=0}^{\infty} a_k(x - x_0)^k$ be a power series with radius of convergence $\rho \in (0, \infty]$, then for all $r \in (0, \rho)$, the series $\sum_{k=0}^{\infty} a_k(x - x_0)^k$ converges uniformly on $[x_0 - r, x_0 + r]$.

proof:

- note that we have $|x - x_0| \leq r$ for all $x \in [x_0 - r, x_0 + r]$
- let $f_k = a_k(x - x_0)^k$, choose $M_k = |a_k|r^k$, $k \in \mathbf{N}$, then for all $x \in [x_0 - r, x_0 + r]$, we have

$$|f_k(x)| = |a_k(x - x_0)^k| = |a_k||x - x_0|^k \leq |a_k|r^k = M_k$$

- consider the root test (theorem 4.27) for $\sum_{k=0}^{\infty} M_k$, we have

$$L = \lim_{k \rightarrow \infty} M_k^{1/k} = \lim_{k \rightarrow \infty} \left(|a_k| r^k \right)^{1/k} = \lim_{k \rightarrow \infty} |a_k|^{1/k} r = \begin{cases} r/\rho & \rho < \infty \\ 0 & \rho = \infty, \end{cases}$$

since $r \in (0, \rho)$, we have $L < 1 \implies \sum_{k=0}^{\infty} M_k$ converges absolutely

- put together, by theorem 8.23, we have

$$\left(\sum_{k=0}^n f_k \right)_{n=1}^{\infty} = \left(\sum_{k=0}^n a_k (x - x_0)^k \right)_{n=1}^{\infty}$$

converges uniformly on $[x_0 - r, x_0 + r]$

Theorem 8.25 Let $\sum_{k=0}^{\infty} a_k(x - x_0)^k$ be a power series with radius of convergence $\rho \in (0, \infty]$, then we have the following conclusion.

- For all $c \in (x_0 - \rho, x_0 + \rho)$, the function given by the series $\sum_{k=0}^{\infty} a_k(x - x_0)^k$ is differentiable at c , and

$$\left. \frac{d}{dx} \left(\sum_{k=0}^{\infty} a_k(x - x_0)^k \right) \right|_{x=c} = \sum_{k=0}^{\infty} \left. \frac{d}{dx} (a_k(x - x_0)^k) \right|_{x=c}.$$

- For all a, b such that $x_0 - \rho < a < b < x_0 + \rho$,

$$\int_a^b \sum_{k=0}^{\infty} a_k(x - x_0)^k dx = \sum_{k=0}^{\infty} \int_a^b a_k(x - x_0)^k dx.$$