Lecture Notes for Real Analysis

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Contents

1. Basic set theory

- sets
- mathematical induction
- functions
- cardinality

Sets

Definition 1.1 A **set** is a collection of objects called elements or members. A set with no objects is called the **empty set** and is denoted by $\emptyset$ (or sometimes by $\{\}$).

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notation:

- $a \in S$ means that 'a is an element in S'
- $a \notin S$ means that 'a is not an element in S'
- $\forall$ means 'for all'
- $\exists$ means 'there exists'
- $\exists!$ means 'there exists a unique'
- $\implies$ means 'implies'
- $\iff$ means 'if and only if'

Definition 1.2

- A set A is a **subset** of a set B if $x \in A$ implies $x \in B$, denoted as $A \subseteq B$.
- Two sets A and B are **equal** if $A \subseteq B$ and $B \subseteq A$, denoted as $A = B$.
- A set A is a **proper subset** of B if $A \subseteq B$ and $A \neq B$, denoted as $A \subsetneq B$.

set building notation: we write

$$\{x \in A \mid P(x)\} \quad \text{or} \quad \{x \mid P(x)\}$$

to mean 'all $x \in A$ that satisfies property $P(x)$'

examples:

- $\mathbf{N} = \{1, 2, 3, 4, \ldots\}$: the set of natural numbers
- $\mathbf{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$: the set of integers
- $\mathbf{Q} = \{m/n \mid m, n \in \mathbf{Z},\ n \neq 0\}$: the set of rational numbers
- $\mathbf{R}$: the set of real numbers

it follows that $\mathbf{N} \subseteq \mathbf{Z} \subseteq \mathbf{Q} \subseteq \mathbf{R}$

Definition 1.3 Given sets A and B:

- The **union** of A and B is the set $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.
- The **intersection** of A and B is the set $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.
- The **set difference** of A and B is the set $A \setminus B = \{x \in A \mid x \notin B\}$.
- The complement of A is the set $A^c = \{x \mid x \notin A\}$.
- A and B are **disjoint** if $A \cap B = \emptyset$.

Theorem 1.4 *De Morgan's Laws.* If A, B, C are sets, then

- $(B \cup C)^c = B^c \cap C^c$;
- $(B \cap C)^c = B^c \cup C^c$;
- $A \setminus (B \cup C) = A \setminus B \cap A \setminus C$;
- $A \setminus (B \cap C) = A \setminus B \cup A \setminus C$.

we prove the first statement:

- let B, C be sets, we need to show that

$$(B \cup C)^c \subseteq B^c \cap C^c \quad \text{and} \quad B^c \cap C^c \subseteq (B \cup C)^c$$

- $x \in (B \cup C)^c \implies x \notin B \cup C \implies x \notin B$ and $x \notin C$
  $\implies x \in B^c$ and $x \in C^c \implies x \in B^c \cap C^c \implies (B \cup C)^c \subseteq B^c \cap C^c$

- $x \in B^c \cap C^c \implies x \in B^c$ and $x \in C^c \implies x \notin B$ and $x \notin C$
  $\implies x \notin B \cup C \implies x \in (B \cup C)^c \implies B^c \cap C^c \subseteq (B \cup C)^c$

Mathematical induction

Axiom 1.5 *Well ordering property.* If the set $S \subseteq \mathbf{N}$ is nonempty, then there exists some $x \in S$ such that $x \leq y$ for all $y \in S$, *i.e.*, the set S always has a **least element**.

Theorem 1.6 *Induction.* Let $P(n)$ be a statement depending on $n \in \mathbf{N}$. Assume that we have:

1. *Base case.* The statement $P(1)$ is true.
2. *Inductive step.* If $P(m)$ is true then $P(m+1)$ is true.

Then, $P(n)$ is true for all $n \in \mathbf{N}$.

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proof:

- suppose $S \neq \emptyset$, then S has a least element $m \in S$
- since $P(1)$ is true, we have $m \neq 1$, *i.e.*, $m > 1$
- since m is a least element, we have $m - 1 \notin S \implies P(m-1)$ is true
- this implies that $P(m)$ is true $\implies m \notin S$, which is a contradiction
- hence, $S = \emptyset$, *i.e.*, $P(n)$ is true for all $n \in \mathbf{N}$

Example 1.7 For all $c \in \mathbf{R}$, $c \neq 1$, and for all $n \in \mathbf{N}$,

$$1 + c + c^2 + \cdots + c^n = \frac{1 - c^{n+1}}{1 - c}.$$

proof:

- the base case ($n = 1$): the left hand side of the equation is $1 + c$; the right hand side is $\frac{1-c^2}{1-c} = \frac{(1+c)(1-c)}{1-c} = 1 + c$, which equals to the left hand side
- the inductive step: assume that the equation is true for $k \in \mathbf{N}$, $i.e.$,

$$1 + c + c^2 + \cdots + c^k = \frac{1 - c^{k+1}}{1 - c},$$

we have

$$\begin{aligned} 1 + c + c^2 + \cdots + c^k + c^{k+1} &= \frac{1 - c^{k+1}}{1 - c} + c^{k+1} \\ &= \frac{1 - c^{k+1} + c^{k+1} - c^{(k+1)+1}}{1 - c} = \frac{1 - c^{(k+1)+1}}{1 - c} \end{aligned}$$

Example 1.8 *Bernoulli's inequality.* For all $c \geq -1$, $(1 + c)^n \geq 1 + nc$ for all $n \in \mathbf{N}$.

proof:

- for the base case ($n = 1$), we have $(1 + c)^1 \geq 1 + 1 \cdot c$
- the inductive step: suppose $m \in \mathbf{N}$, $m > 1$ and $(1 + c)^m \geq 1 + mc$, then

$$(1 + c)^{m+1} \geq (1 + mc)(1 + c) = 1 + (m + 1)c + mc^2 \geq 1 + (m + 1)c$$

Functions

Definition 1.9 If A and B are sets, a **function** $f\colon A \to B$ is a mapping that assigns each $x \in A$ to a unique element in B denoted $f(x)$.

Definition 1.10 Consider a function $f\colon A \to B$. Define the **image** (or direct image) of a subset $C \subseteq A$ as

$$f(C) = \{f(x) \in B \mid x \in C\}.$$

Define the **inverse image** of a subset $D \subseteq B$ as

$$f^{-1}(D) = \{x \in A \mid f(x) \in D\}.$$

examples:

- $f\colon \{1,2,3,4\} \to \{a,b\}$ where $f(1) = f(2) = a$, $f(3) = f(4) = b$, we have $f(\{1,2\}) = \{a\}$, $f^{-1}(\{b\}) = \{3,4\}$
- $f\colon \mathbf{R} \to \mathbf{R}$ where $f(x) = \sin(\pi x)$, we have $f([0,1/2]) = [0,1]$, $f^{-1}(\{0\}) = \mathbf{Z}$

Definition 1.11 Let $f\colon A \to B$ be a function.

- The function f is **injective** or **one-to-one** if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.
- The function f is **surjective** or **onto** if $f(A) = B$.
- The function f is **bijective** if f is both surjective and injective. In this case, the function $f^{-1}\colon B \to A$ is the **inverse function** of f, which assigns each $y \in B$ to the unique $x \in A$ such that $f(x) = y$.

- if the function f is a bijection, then $f(f^{-1}(x)) = x$
- example: for the bijection $f\colon \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^3$, we have $f^{-1}(x) = \sqrt[3]{x}$

Definition 1.12 Consider $f\colon A \to B$ and $g\colon B \to C$. The **composition** of the functions f and g is the function $g \circ f\colon A \to C$ defined as

$$(g \circ f)(x) = g(f(x)).$$

- example: if $f(x) = x^3$ and $g(y) = \sin(y)$, then $(g \circ f)(x) = \sin(x^3)$

Cardinality

Definition 1.13 We state that the two sets A and B have the same **cardinality** if there exists a bijection $f\colon A \to B$.

notation:

- $|A|$ denotes the cardinality of the set A
- $|A| = |B|$ if the sets A and B have the same cardinality
- $|A| = n$ if $|A| = |\{1, \ldots, n\}|$
- $|A| \leq |B|$ if there exists an injection $f\colon A \to B$
- $|A| < |B|$ if $|A| \leq |B|$ and $|A| \neq |B|$

Theorem 1.14

- If $|A| = |B|$, then $|B| = |A|$.
- If $|A| = |B|$, and $|B| = |C|$, then $|A| = |C|$.

proof:

- show that the inverse function $f^{-1}\colon B \to A$ of $f\colon A \to B$ is a bijection
- show that the composition $g \circ f\colon A \to C$ of functions $f\colon A \to B$ and $g\colon B \to C$ is a bijection

Theorem 1.15 *Cantor-Schröder-Bernstein.* If $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$.

Definition 1.16 The set A is **countably finite** if $|A| = |\mathbf{N}|$. Specifically, the set A is **finite** if $|A| = n \in \mathbf{N}$. The set A is **countable** if A is finite or countably infinite. Otherwise, we say A is **uncountable**.

Example 1.17 The set of even natural numbers and the set of odd natural numbers have the same cardinality as $\mathbf{N}$, $i.e.$, $|\{2n \mid n \in \mathbf{N}\}| = |\{2n-1 \mid n \in \mathbf{N}\}| = |\mathbf{N}|$.

proof: consider the bijection $f\colon \mathbf{N} \to \{2n \mid n \in \mathbf{N}\}$ given by $f(n) = 2n$ and $g\colon \mathbf{N} \to \{2n-1 \mid n \in \mathbf{N}\}$ given by $g(n) = 2n-1$

Example 1.18 The set of all integers has the same cardinality as $\mathbf{N}$, $i.e.$, $|\mathbf{Z}| = |\mathbf{N}|$.

proof: consider the bijection $f\colon \mathbf{Z} \to \mathbf{N}$ given by

$$f(n) = \begin{cases} 2n & n \geq 0 \\ -(2n+1) & n < 0 \end{cases}$$

Definition 1.19 The **powerset** of a set A, denoted by $\mathcal{P}(A)$, is the set of all subsets of A, $i.e.$, $\mathcal{P}(A) = \{B \mid B \subseteq A\}$.

- for a finite set A of cardinality n, the cardinality of $\mathcal{P}(A)$ is 2^n

examples:

- $A = \emptyset$ then $\mathcal{P}(A) = \{\emptyset\}$
- $A = \{1\}$ then $\mathcal{P}(A) = \{\emptyset, \{1\}\}$
- $A = \{1,2\}$ then $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

Theorem 1.20 *Cantor.* If A is a set, then $|A| < |\mathcal{P}(A)|$.

- therefore, $|\mathbf{N}| < |\mathcal{P}(\mathbf{N})| < |\mathcal{P}(\mathcal{P}(\mathbf{N}))| < \cdots$, $i.e.$, there are infinite number of infinite sets

proof:
we first show that $|A| \leq |\mathcal{P}(A)|$

- consider the function $f\colon A \to \mathcal{P}(A)$ given by $f(x) = \{x\}$
- the function f is a injection since

$$f(x_1) = f(x_2) \implies \{x_1\} = \{x_2\} \implies x_1 = x_2$$

we now show that $|A| \neq |\mathcal{P}(A)|$ by contradiction

- suppose $|A| = |\mathcal{P}(A)|$, then there is a surjection $g\colon A \to \mathcal{P}(A)$

- consider the set $B \subseteq A$ given by

$$B = \{x \in A \mid x \notin g(x)\} \in \mathcal{P}(A)$$

- since g is surjective and $B \in \mathcal{P}(A)$, there exists a $b \in A$ such that $g(b) = B$

- there are two cases
  1. $b \in B \implies b \notin g(b) \implies b \notin B$
  2. $b \notin B \implies b \notin g(b) \implies b \in B$

  where in either case we obtain a contradiction

- hence, g is not surjective $\implies |A| \neq |\mathcal{P}(A)|$

Corollary 1.21 For all $n \in \mathbf{N} \cup \{0\}$, $n < 2^n$.

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2. Real numbers

- ordered sets
- least upper bound property
- fields
- real numbers
- archimedian property
- using supremum and infimum
- absolute value
- triangle inequality
- uncountabality of the real numbers

Ordered sets

Definition 2.1 An **ordered set** is a set S with a relation $<$ called an 'ordering' such that:

1. *Trichotomy.* For all $x, y \in S$, either $x < y$, $x = y$, or $x > y$.
2. *Transitivity.* If $x, y, z \in S$ have $x < y$ and $y < z$, then $x < z$.

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examples:

- $\mathbf{Z}$ is an ordered set with ordering $m > n \iff m - n \in \mathbf{N}$
- $\mathbf{Q}$ is an ordered set with ordering $p > q \iff p - q = m/n$ for some $m, n \in \mathbf{N}$
- $\mathbf{Q} \times \mathbf{Q}$ is an ordered set with dictionary ordering $(q, r) > (s, t) \iff q > s$, or $q = s$ and $r > t$
- the set $\mathcal{P}(\mathbf{N})$ with ordering defined by $A \prec B$ if $A \subseteq B$ is *not* an ordered set

Least upper bound property

Definition 2.2 Let S be an ordered set and let $E \subseteq S$, then:

- If there exists some $b \in S$ such that $x \leq b$ for all $x \in E$, then E is **bounded above** and b is an **upper bound** of E.

- If there exists some $c \in S$ such that $x \geq c$ for all $x \in E$, then E is **bounded below** and c is a **lower bound** of E.

- If there exists an upper bound b_0 of E such that $b_0 \leq b$ for all upper bounds b of E, then b_0 is the **least upper bound** or the **supremum** of E, written as

$$b_0 = \sup E.$$

- If there exists a lower bound c_0 of E such that $c_0 \geq c$ for all lower bounds c of E, then c_0 is the **greatest lower bound** or the **infimum** of E, written as

$$c_0 = \inf E.$$

examples:

- $S = \mathbf{Z}$ and $E = \{-2, -1, 0, 1, 2\}$, then $\inf E = -2$ and $\sup E = 2$

- $S = \mathbf{Q}$ and $E = \{q \in \mathbf{Q} \mid 0 \leq q < 1\}$, then $\inf E = 0$ and $\sup E = 1 \notin E$, *i.e.*, the supremum or infimum need not be in E

- $S = \mathbf{Z}$ and $E = \mathbf{N}$, then $\inf E = 1$ but $\sup E$ does not exist

Definition 2.3 *Least upper bound property.* An ordered set S has the least upper bound property if every $E \subseteq S$ which is nonempty and bounded above has a supremum in S.

example: $-\mathbf{N} = \{-1, -2, -3, \ldots\}$, to show this (informally), suppose $E \subseteq -\mathbf{N}$ is bounded above, then $-E \subseteq \mathbf{N}$ is bounded below and according to the well ordering principle, $-E$ has a least element $x \in -E$, and thus $-x = \sup E$

Theorem 2.4 If $x \in \mathbf{Q}$ and

$$x = \sup\{q \in \mathbf{Q} \mid q > 0,\ q^2 < 2\},$$

then $x \geq 1$ and $x^2 = 2$.

proof: let $E = \{q \in \mathbf{Q} \mid q > 0,\ q^2 < 2\}$

- $x \geq 1$ since $1 \in E \implies \sup E \geq 1$
- we show $x^2 \geq 2$ by contradiction: suppose $x^2 < 2$, let $h = \min\{\frac{1}{2},\ \frac{2-x^2}{2(2x+1)}\}$
  - since $x \geq 1$ and $x^2 < 2$, we have $0 < h \leq 1/2 < 1$
  - $h < 1 \implies (x+h)^2 = x^2 + 2hx + h^2 < x^2 + 2hx + h$
  - since $h \leq \frac{2-x^2}{2(2x+1)}$, we have

    $$(x+h)^2 < x^2 + (2x+1)h \leq x^2 + \frac{1}{2}(2-x^2) < x^2 + 2 - x^2 = 2 \implies x + h \in E$$

  - $h > 0 \implies x + h > x$, but $x + h \in E \implies x$ is not an upper bound for E, *i.e.*, $x \neq \sup E$, which is a contradiction

- we now show $x^2 \not> 2$ by contradiction: suppose $x^2 > 2$, let $h = \frac{x^2-2}{2x}$
  - since $x^2 > 2$ and $x \geq 1$, we have $h > 0$
  - $h > 0 \implies (x-h)^2 = x^2 - 2hx + h^2 > x^2 - 2hx = x^2 - (x^2 - 2) = 2$
  - let $q \in E$, then $q^2 < 2 < (x-h)^2$, hence

    $$(x-h)^2 - q^2 = ((x-h)+q)((x-h)-q) > 0 \implies (x-h) - q > 0,$$

    i.e., $x - h > q$ for all $q \in E \implies x - h$ is an upper bound for E
  - $h > 0 \implies x > x - h \implies x \neq \sup E$, which is a contradiction
- therefore, $x^2 = 2$

Theorem 2.5 The set $E = \{q \in \mathbf{Q} \mid q > 0,\ q^2 < 2\}$ does not have a supremum in $\mathbf{Q}$.

proof (by contradiction): suppose there exists some $x \in \mathbf{Q}$ such that $x = \sup E$

- by theorem 2.4, we have $x \geq 1$ and $x^2 = 2$
- in particular, $x > 1$ since if $x = 1 \implies x^2 = 1 \neq 2$
- $x \in \mathbf{Q} \implies$ there exist $m, n \in \mathbf{N}$ $(m > n)$ such that $x = m/n$, *i.e.*, $m = nx \in \mathbf{N}$
- let $S = \{k \in \mathbf{N} \mid kx \in \mathbf{N}\} \subseteq \mathbf{N}$, then $S \neq \emptyset$ since $n \in S$
- by the well ordering property, there is a least element $k_0 \in S$
- let $k_1 = k_0(x-1) = k_0 x - k_0 \in \mathbf{Z}$, in particular, $k_1 \in \mathbf{N}$ since $x > 1 \implies k_1 > 0$
- $x^2 = 2 \implies x < 2$ as otherwise $x^2 \geq 4$, hence

$$k_1 = k_0(x-1) < k_0(2-1) = k_0 \implies k_1 \notin S$$

- $k_1 = k_0(x-1) \implies k_1 x = k_0 x^2 - k_0 x$, since $x^2 = 2$, we have

$$k_1 x = 2k_0 - k_0 x = k_0 - k_0(x-1) = k_0 - k_1 \in \mathbf{N} \implies k_1 \in S,$$

which is a contradiction

Fields

Definition 2.6 A set F is a **field** if it has two operations: addition $(+)$ and multiplication $(\cdot)$ with the following properties.

(A1) If $x, y \in F$ then $x + y \in F$.

(A2) *Commutativity.* For all $x, y \in F$, $x + y = y + x$.

(A3) *Associativity.* For all $x, y, z \in F$, $(x + y) + z = x + (y + z)$.

(A4) There exists an element $0 \in F$ such that $0 + x = x = x + 0$ for all $x \in F$.

(A5) For all $x \in F$, there exists a $y \in F$ such that $x + y = 0$, denoted by $y = -x$.

(M1) If $x, y \in F$ then $x \cdot y \in F$.

(M2) *Commutativity.* For all $x, y \in F$, $x \cdot y = y \cdot x$.

(M3) *Associativity.* For all $x, y, z \in F$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

(M4) There exists an element $1 \in F$ such that $1 \cdot x = x = x \cdot 1$ for all $x \in F$.

(M5) For all $x \in F \setminus \{0\}$, there exists an $x^{-1} \in F$ such that $x \cdot x^{-1} = 1$.

(D) *Distributativity.* For all $x, y, z \in F$, $(x + y) \cdot z = x \cdot z + y \cdot z$.

examples:

- $\mathbf{Q}$ is a field

- $\mathbf{Z}$ is not a field since it fails (M5)

- $\mathbf{Z}_2 = \{0, 1\}$ where $1 + 1 = 0$ (mod 2) is a field

- $\mathbf{Z}_3 = \{0, 1, 2\}$ with $c = a + b$ (mod 3), *i.e.*,

$$2 + 1 = 3 = 0 \quad \text{and} \quad 2 \cdot 2 = 4 = 3 + 1 = 1,$$

  is a field

Theorem 2.7 If $x \in F$ where F is a field then $0x = 0$.

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proof: $xx = (x + 0)x = xx + 0x \implies 0x = 0$

Definition 2.8 A field F is an **ordered field** if F is also an ordered set with ordering $<$ and satisfies:

1. For all $x, y, z \in F$, $x < y \implies x + z < y + z$.
2. If $x > 0$ and $y > 0$ then $xy > 0$.

If $x > 0$ we say x is **positive**, and if $x \geq 0$ we say x is **nonnegative**.

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examples:

- $\mathbf{Q}$ is an ordered field

- $\mathbf{Z}_2 = \{0, 1\}$ where $1 + 1 = 0$ is not a ordered field
  (if $0 > 1 \implies 0 + 1 > 1 + 1 \implies 1 > 0$; if $1 > 0 \implies 1 + 1 > 1 + 0 \implies 0 > 1$)

Theorem 2.9 Let F be an ordered field and $x, y, z, w \in F$, then:

- If $x > 0$ then $-x < 0$ (and vice versa).
- If $x > 0$ and $y < z$ then $xy < xz$.
- If $x < 0$ and $y < z$ then $xy > xz$.
- If $x \neq 0$ then $x^2 > 0$.
- If $0 < x < y$ then $0 < 1/y < 1/x$.
- If $0 < x < y$ then $x^2 < y^2$.
- If $x \leq y$ and $z \leq w$ then $x + z \leq y + w$.

Theorem 2.10 Let $x, y \in F$ where F is an ordered field. If $x > 0$ and $y < 0$ or $x < 0$ and $y > 0$, then $xy < 0$.

proof:

- $x > 0,\ y < 0 \implies x > 0,\ -y > 0 \implies -xy > 0 \implies xy < 0$
- $x < 0,\ y > 0 \implies -x > 0,\ y > 0 \implies -xy > 0 \implies xy < 0$

Theorem 2.11 *Greatest lower bound.* Let F be an ordered field with the least upper bound property. If $A \subseteq F$ is nonempty and bounded below, then $\inf A$ exists in F.

proof: let $B = \{-x \mid x \in A\}$

- $A \subseteq F$ bounded below $\implies \exists a \in F$, $\forall x \in A$, $a \leq x \implies \exists a \in F$, $\forall x \in A$, $-a \geq -x \implies \exists a \in F$, $\forall x \in B$, $-a \geq x \implies B \subseteq F$ has an upper bound $-a$ (this also shows that if a is a lower bound of A then $-a$ is an upper bound of B)
- F has the least upper bound property $\implies \sup B \in F$
- let $c = \sup B$, then $c \geq x$, $\forall x \in B \implies -c \leq -x$, $\forall x \in B \implies -c \leq x$, $\forall x \in A \implies -c \in F$ is an lower bound of A
- we also have $c \leq -a$ with a being a lower bound of $A \implies -c \geq a \implies -c \in F$ is the greatest lower bound of A, $i.e.$, $-c = \inf A \in F$

Real nubmers

Theorem 2.12 There exists a "unique" ordered field, labeled $\mathbf{R}$, such that $\mathbf{Q} \subseteq \mathbf{R}$ and $\mathbf{R}$ has the least upper bound property.

- one can construct $\mathbf{R}$ using Dedekind cuts or as equivalence classes of Cauchy sequences.

Theorem 2.13 There exists a unique $r \in \mathbf{R}$ such that $r \geq 1$ and $r^2 = 2$, $i.e.$, $\sqrt{2} \in \mathbf{R}$ but $\sqrt{2} \notin \mathbf{Q}$.

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proof: let $E = \{x \in \mathbf{R} \mid x > 0,\ x^2 < 2\} \subseteq \mathbf{R}$

- we have $x < 2$ for all $x \in E$ (since if $x \geq 2 \implies x^2 \geq 4$) $\implies$ E is bounded above $\implies$ $\sup E$ exists in $\mathbf{R}$
- let $r = \sup E$, using the same proof for theorem 2.4 we have $r \geq 1$ and $r^2 = 2$
- to show the uniqueness, suppose $\tilde{r} \geq 1$, $\tilde{r}^2 = 2$, then

$$r^2 - \tilde{r}^2 = 0 \implies (r + \tilde{r})(r - \tilde{r}) = 0 \implies r - \tilde{r} = 0 \implies r = \tilde{r}$$

(since $r \geq 1$, $\tilde{r} \geq 1 \implies r + \tilde{r} > 0$)

Theorem 2.14 If $x \in \mathbf{R}$ satisfies $x < \epsilon$ for all $\epsilon \in \mathbf{R}$, $\epsilon > 0$, then $x \leq 0$.

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proof by contradiction:

- suppose $x > 0$ satisfies $x \leq \epsilon$ for all $\epsilon > 0$
- $x > 0 \implies 2x > x > 0 \implies x > x/2 > 0$
- take $\epsilon = x/2$ we have $x > \epsilon > 0$, which is a contradiction

Archimedian property

Theorem 2.15 *Archimedian property.* If $x, y \in \mathbf{R}$ and $x > 0$, then there exists an $n \in \mathbf{N}$ such that $nx > y$.

proof by contradiction:

- suppose $nx \leq y$ for all $n \in \mathbf{N} \implies \forall n \in \mathbf{N}$, $n \leq y/x \implies \mathbf{N}$ is bounded above by $y/x \implies$ there exists $\sup \mathbf{N} \in \mathbf{R}$
- let $a = \sup \mathbf{N} \implies a - 1 < a$ is not an upper bound of $\mathbf{N} \implies \exists m \in \mathbf{N}$, $a - 1 < m \implies a < m + 1 \in \mathbf{N} \implies a$ is not an upper bound of $\mathbf{N}$, which is a contradiction

Theorem 2.16 *Density of* $\mathbf{Q}$. If $x, y \in \mathbf{R}$ and $x < y$ then there exists some $r \in \mathbf{Q}$ such that $x < r < y$.

proof:

- first suppose $0 \leq x < y$, by the Archimedian property, we have

$$n(y - x) > 1 \implies ny > nx + 1$$

  for some $n \in \mathbf{N}$

- let $S = \{k \in \mathbf{N} \mid k > nx\} \subseteq \mathbf{N}$, by Archimedian property, there exists some $p \in \mathbf{N}$ such that $p > nx \implies S \neq \emptyset$
- by the well ordering property, there is a least element $m \in S$ such that $m > nx$
- $m \in \mathbf{N} \implies m \geq 1$
- if $m = 1$, then $m - 1 = 0 \implies nx \geq m - 1 = 0$ since $x \geq 0$
- if $m > 1$, then $m - 1 \in \mathbf{N}$ but $m - 1 \notin S$ since $m > m - 1$ is the least element $\implies nx \geq m - 1 \implies m \leq nx + 1 < ny$
- hence, we have

$$nx < m < ny \implies x < m/n < y$$

  for some $m, n \in \mathbf{N}$, *i.e.*, there exists an $r = m/n \in \mathbf{Q}$ such that $x < r < y$

- now suppose $x < 0$, if $x < 0 < y$ then simply take $r = 0$; if $x < y \leq 0$, we have $0 \leq -y < -x$, thus there exists some $\tilde{r} \in \mathbf{Q}$ such that

$$-y < \tilde{r} < -x \implies x < -\tilde{r} < y$$

  (by the first case), *i.e.*, we have $x < r < y$ by taking $r = -\tilde{r}$

Theorem 2.17 Suppose $S \subseteq \mathbf{R}$ is nonempty and bounded above. Then, $x = \sup S$ if and only if:

1. x is an upper bound of S.
2. For all $\epsilon > 0$, there exists some $y \in S$ such that $x - \epsilon < y \leq x$.

proof:

- first suppose $x = \sup S$
  - obviously, x is an upper bound of S
  - for all $\epsilon > 0$, we have $x > x - \epsilon \implies x - \epsilon$ is not an upper bound of S, $i.e.$, there exists some $y \in S$ such that $x - \epsilon < y \leq x$
- now suppose x is an upper bound of S, and satisfies $x - \epsilon < y \leq x$ for all $\epsilon > 0$ and for some $y \in S$, we only need to show that for all z that is an upper bound of S, we have $x \leq z$
  - assume there exists an upper bound z of S smaller than x, $i.e.$, $y \leq z < x$ for all $y \in S$
  - take $\epsilon = x - z > 0$ (since $x > z$) $\implies x \geq y > x - \epsilon = x - x + z = z \implies y > z$ for some $y \in S$, $i.e.$, z is not an upper bound of S, which is a contradiction

Theorem 2.18 Let $S = \{1 - \frac{1}{n} \mid n \in \mathbf{N}\}$, then $\sup S = 1$.

proof:

- if $n \in \mathbf{N}$, then $1 - \frac{1}{n} < 1 \implies 1$ is an upper bound of S
- let $\epsilon > 0$, then by the Archimedian property, for some $n \in \mathbf{N}$, we have

$$n\epsilon > 1 \implies \epsilon > \frac{1}{n} \implies -\epsilon < -\frac{1}{n} \implies 1 - \epsilon < 1 - \frac{1}{n} \leq 1$$

  by theorem 2.17, we have $\sup S = 1$

Remark 2.19 We have similar property as theorem 2.17 for infimum. Suppose $S \subseteq \mathbf{R}$ is nonempty and bounded below, then $x = \inf S$ if and only if:

- x is a lower bound of S.
- For all $\epsilon > 0$, there exists some $y \in S$ such that $x \leq y < x + \epsilon$.

Using supremum and infimum

Definition 2.20 For $x \in \mathbf{R}$ and $A \subseteq \mathbf{R}$, define

$$x + A = \{x + a \mid a \in A\}, \qquad xA = \{xa \mid a \in A\}.$$

Theorem 2.21 Let $A \subseteq \mathbf{R}$ be nonempty, we have:

- If $x \in \mathbf{R}$ and A is bounded above, then $\sup(x + A) = x + \sup A$.
- If $x > 0$ and A is bounded above, then $\sup(xA) = x \sup A$.

proof:

- suppose $x \in \mathbf{R}$ and A is bounded above:
  - for all $a \in A$, we have $a \leq \sup A \implies x + a \leq x + \sup A$, *i.e.*, the set $x + A$ is bounded by $x + \sup A$
  - let $\epsilon > 0$, for some $b \in A$, we have

    $$\sup A - \epsilon < b \leq \sup A \implies (x + \sup A) - \epsilon < x + b \leq x + \sup A,$$

    $i.e.$, $\sup(x + A) = x + \sup A$

- suppose $x > 0$ and A is bounded above:
  - for all $a \in A$, $a \leq \sup A \implies xa \leq x \sup A$, $i.e.$, the set xA is bounded by $x \sup A$
  - let $\epsilon > 0 \implies \epsilon/x > 0$, for some $b \in A$, we have

    $$\sup A - \epsilon/x < b \leq \sup A \implies x \sup A - \epsilon < xb \leq x \sup A,$$

    $i.e.$, $\sup(xA) = x \sup A$

Remark 2.22 Similarly, we can also show that:

- If $x \in \mathbf{R}$ and A is bounded below, then $\inf(x + A) = x + \inf A$.
- If $x > 0$ and A is bounded below, then $\inf(xA) = x \inf A$.
- If $x < 0$ and A is bounded below, then $\sup(xA) = x \inf A$.
- If $x < 0$ and A is bounded above, then $\inf(xA) = x \sup A$.

Theorem 2.23 Let $A, B \subseteq \mathbf{R}$ where $x \leq y$ for all $x \in A$, $y \in B$, then $\sup A \leq \inf B$.

proof: for all $x \in A$, $y \in B$, $x \leq y \implies B$ is bounded below by $x \implies x \leq \inf B$ $\implies A$ is bounded above by $\inf B \implies \sup A \leq \inf B$

Absolute value

Definition 2.24 If $x \in \mathbf{R}$, we define the **absolute value** of x as

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases}$$

Theorem 2.25

- $|x| \geq 0$, and, $|x| = 0$ if and only if $x = 0$.
- $|-x| = |x|$ for all $x \in \mathbf{R}$.
- $|xy| = |x||y|$ for all $x, y \in \mathbf{R}$.
- $|x|^2 = x^2$ for all $x \in \mathbf{R}$.
- $|x| \leq y$ if and only if $-y \leq x \leq y$.
- $-|x| \leq x \leq |x|$ for all $x \in \mathbf{R}$.

Triangle inequality

Theorem 2.26 *Triangle inequality.* For all $x, y \in \mathbf{R}$,

$$|x + y| \leq |x| + |y|.$$

proof: let $x, y \in \mathbf{R}$

- $x + y \leq |x| + |y|$
- $-x + -y \leq |-x| + |-y| = |x| + |y| \implies -(|x| + |y|) \leq x + y$
- hence, we have

$$-(|x| + |y|) \leq x + y \leq |x| + |y| \implies |x + y| \leq |x| + |y|$$

Corollary 2.27 *Reverse triangle inequality.* For all $x, y \in \mathbf{R}$,

$$\big||x| - |y|\big| \leq |x - y|.$$

Uncountabality of the real numbers

Definition 2.28 Let $x \in (0,1]$ and let $d_{-i} \in \{0,1,\ldots,9\}$. We say that x is represented by the digits $\{d_{-i} \mid i \in \mathbf{N}\}$, *i.e.*, $x = 0.d_{-1}d_{-2}\cdots$, if

$$x = \sup\{10^{-1}d_{-1} + 10^{-2}d_{-2} + \cdots + 10^{-n}d_{-n} \mid n \in \mathbf{N}\}.$$

example: $0.2500\cdots = \sup\{\frac{2}{10},\ \frac{2}{10}+\frac{5}{100},\ \frac{2}{10}+\frac{5}{100}+\frac{0}{1000},\ \ldots\} = \sup\{\frac{1}{5},\ \frac{1}{4}\} = \frac{1}{4}$

Theorem 2.29

- For all set of digits $\{d_{-i} \mid i \in \mathbf{N}\}$, there exists a unique $x \in [0,1]$ such that $x = 0.d_{-1}d_{-2}\cdots$.
- For all $x \in (0,1]$, there exists a unique sequence of digits d_{-i} such that $x = 0.d_{-1}d_{-2}\cdots$ and

$$0.d_{-1}d_{-2}\cdots d_{-n} < x \le 0.d_{-1}d_{-2}\cdots d_{-n} + 10^{-n}, \quad \text{for all } n \in \mathbf{N}. \qquad (2.1)$$

- the second part indicates that the digital representation of $1/2$ is $0.4999\cdots$

Theorem 2.30 *Cantor.* The set $(0,1]$ is uncountable.

proof (by contradiction):

- assume $(0,1]$ is countable, then there exists a bijection $x\colon \mathbf{N} \to (0,1]$, let

$$x(n) = 0.d_{-1}^{(n)}d_{-2}^{(n)}\cdots, \quad n \in \mathbf{N},$$

  where $d_{-i}^{(n)}$ denotes the ith decimal of the real number $x(n) \in (0,1]$, and let

$$e_{-i} = \begin{cases} 1 & d_{-i}^{(i)} \ne 1 \\ 2 & d_{-i}^{(i)} = 1 \end{cases} \qquad (2.2)$$

- let $y = 0.e_{-1}e_{-2}\cdots$, since all e_{-i} are nonzero, $e_{-1}, e_{-2}, \ldots$ satisfies (2.1); according to theorem 2.29, we have $0.e_{-1}e_{-2}\cdots$ being the unique decimal representation of y
- again according to theorem 2.29 and all e_{-i} are nonzero, we have $y \in (0,1] \implies \exists m \in \mathbf{N}$, $y = x(m) = 0.d_{-1}^{(m)}d_{-2}^{(m)}\cdots = 0.e_{-1}e_{-2}\cdots$, however, we have $e_{-m} \ne d_{-m}^{(m)}$ since (2.2), *i.e.*, for all $m \in \mathbf{N}$, $x(m) \ne y$, which is a contradiction

Corollary 2.31 The set of real numbers $\mathbf{R}$ is uncountable.

3. Sequences

- sequences and limits
- monotone sequences and subsequences
- inequalities and operations involving limits
- limit superior and limit inferior
- Bolzano-Weierstrass theorem
- Cauchy sequences

Sequences and limits

Definition 3.1 A **sequence** (of real numbers) is a function $x\colon \mathbf{N} \to \mathbf{R}$. To denote a sequence we write $(x_n)_{n=1}^{\infty}$, where x_n is the nth element in the sequence.

- sequence need not start at $n = 1$, *e.g.*, the sequence $x\colon \{n \in \mathbf{Z} \mid n \geq m\} \to \mathbf{R}$ is denoted $(x_n)_{n=m}^{\infty}$

Definition 3.2 A sequence $(x_n)_{n=1}^{\infty}$ is **bounded** if there exists some $B \geq 0$ such that $|x_n| \leq B$ for all $n \in \mathbf{N}$.

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examples:

- the sequence $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ is bounded since $\frac{1}{n} \leq 1$ for all n
- the sequence $(n)_{n=1}^{\infty}$ is not bounded since for all $B \geq 0$ there exists some $n \geq B$ according to the Archimedian property

Definition 3.3 A sequence $(x_n)_{n=1}^{\infty}$ is said to **converge** to $x \in \mathbf{R}$ if for all $\epsilon > 0$, there exists an $M \in \mathbf{N}$ such that for all $n \geq M$, we have $|x_n - x| < \epsilon$.

The number x is called a **limit** of the sequence. If the limit x is unique, we write

$$x = \lim_{n \to \infty} x_n.$$

A sequence that converges is said to be **convergent**, and otherwise is **divergent**.

Remark 3.4 A sequence $(x_n)_{n=1}^{\infty}$ is divergent if for all $x \in \mathbf{R}$, there exists some $\epsilon > 0$, such that for all $M \in \mathbf{N}$, there exists an $n \geq M$, so that $|x_n - x| \geq \epsilon$.

Theorem 3.5 Let $x, y \in \mathbf{R}$. If for all $\epsilon > 0$, $|x - y| < \epsilon$, then $x = y$.

proof: assume $x \neq y \implies |x - y| > 0$; take $\epsilon = \frac{1}{2}|x - y| \implies |x - y| < \frac{1}{2}|x - y|$ $\implies |x - y| < 0$, which is a contradiction

Theorem 3.6 If $(x_n)_{n=1}^{\infty}$ converges to x and y, then $x = y$, $i.e.$, a convergent sequence has a unique limit.

proof: let $\epsilon > 0$

- $(x_n)_{n=1}^{\infty}$ converges to $x \implies \exists M_1 \in \mathbf{N}$, $\forall n \geq M_1$, $|x_n - x| < \epsilon/2$
- $(x_n)_{n=1}^{\infty}$ converges to $y \implies \exists M_2 \in \mathbf{N}$, $\forall n \geq M_2$, $|x_n - y| < \epsilon/2$
- let $M = M_1 + M_2$, then $M \geq M_1$ and $M \geq M_2$, then we have

  $$|x_M - x| < \epsilon/2 \quad \text{and} \quad |x_M - y| < \epsilon/2,$$

  hence,

  $$\begin{aligned} |x - y| &= |(x - x_M) + (x_M - y)| \\ &\leq |x - x_M| + |y - x_M| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon \end{aligned}$$

- according to theorem 3.5, we have $x = y$

Remark 3.7 Sometimes we write '$x_n \to x$ as $n \to \infty$' to mean $x = \lim_{n\to\infty} x_n$. We may also avoid the 'as $n \to \infty$' part if the limiting process is clear from the context.

Example 3.8 Given the sequence $(x_n)_{n=1}^{\infty}$ with $x_n = c \in \mathbf{R}$ for all $n \in \mathbf{N}$, we have $\lim_{n\to\infty} x_n = c$.

proof: let $\epsilon > 0$, $M = 1$, then for all $n \geq M$, we have $|x_n - c| = |c - c| = 0 < \epsilon$

Example 3.9 The sequence $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ converges to $x = 0$, $i.e.$, $\lim_{n\to\infty} \frac{1}{n} = 0$.

proof: let $\epsilon > 0$, choose an $M \in \mathbf{N}$ such that $M > 1/\epsilon$ (such an M exists according to the Archimedian property), then for all $n \geq M$, we have $\left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| \leq \frac{1}{M} < \epsilon$

Example 3.10 The sequence $\left(\frac{1}{n^2+2n+100}\right)_{n=1}^{\infty}$ converges to $x = 0$.

proof: let $\epsilon > 0$ choose $M \in \mathbf{N}$ such that $M \geq \epsilon^{-1}/2$, then for all $n \geq M$, we have

$$\left|\frac{1}{n^2+2n+100} - 0\right| = \frac{1}{n^2+2n+100} \leq \frac{1}{2n} \leq \frac{1}{2M} < \epsilon$$

Example 3.11 The sequence $(x_n)_{n=1}^{\infty}$ where $x_n = (-1)^n$ is divergent.

proof: let $x \in \mathbf{R}$, $M \in \mathbf{N}$, then

$$\begin{aligned} & |x_M - x_{M+1}| = \left|(-1)^M - (-1)^{M+1}\right| = 2 \\ \Longrightarrow \quad & 2 = |(x_M - x) + (x - x_{M+1})| \leq |x_M - x| + |x_{M+1} - x| \\ \Longrightarrow \quad & |x_M - x| \geq 1 \quad \text{or} \quad |x_{M+1} - x| \geq 1, \end{aligned}$$

$i.e.$, let $\epsilon = 1$, $n = M$, we have either $|x_n - x| \geq \epsilon$ or $|x_{n+1} - x| \geq \epsilon$

Theorem 3.12 If $(x_n)_{n=1}^{\infty}$ is convergent, then $(x_n)_{n=1}^{\infty}$ is bounded.

proof:

- suppose $(x_n)_{n=1}^{\infty}$ converges to x, let $\epsilon = 1$, then there exists some $M \in \mathbf{N}$ such that for all $n \geq M$, $|x_n - x| < 1 \implies x_n < |x| + 1$
- let $B = \max\{|x_1|, |x_2|, \ldots, |x_M|, |x| + 1\}$, since $x_n \leq |x_n|$ for all $n \in \mathbf{N}$, $n \leq M$, and $x_n < |x| + 1$ for all $n \geq M$, we have $B \geq |x_n|$ for all $n \in \mathbf{N}$

Monotone sequences

Definition 3.13

- A sequence $(x_n)_{n=1}^{\infty}$ is **monotone increasing** if $x_n \leq x_{n+1}$ for all $n \in \mathbf{N}$.
- A sequence $(x_n)_{n=1}^{\infty}$ is **monotone decreasing** if $x_n \geq x_{n+1}$ for all $n \in \mathbf{N}$.
- If $(x_n)_{n=1}^{\infty}$ is either monotone increasing or monotone decreasing, we say the sequence $(x_n)_{n=1}^{\infty}$ is **monotone** (or monotonic).

examples:

- the sequence $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ is monotone decreasing
- the sequence $\left(-\frac{1}{n}\right)_{n=1}^{\infty}$ is monotone increasing
- the sequence $((-1)^n)_{n=1}^{\infty}$ is not monotone

Theorem 3.14 A monotone sequence $(x_n)_{n=1}^{\infty}$ converges if and only if it is bounded.

- If the sequence $(x_n)_{n=1}^{\infty}$ is monotone increasing and bounded, then

$$\lim_{n\to\infty} x_n = \sup\{x_n \mid n \in \mathbf{N}\}.$$

- If the sequence $(x_n)_{n=1}^{\infty}$ is monotone decreasing and bounded, then

$$\lim_{n\to\infty} x_n = \inf\{x_n \mid n \in \mathbf{N}\}.$$

proof: we prove for monotone increasing sequences, the other case is similar

- suppose $(x_n)_{n=1}^{\infty}$ is convergent, according to theorem 3.12, it is bounded
- suppose $(x_n)_{n=1}^{\infty}$ is monotone increasing and bounded
  - $(x_n)_{n=1}^{\infty}$ is monotone increasing $\implies$ $x_n \leq x_{n+1}$ for all $n \in \mathbf{N}$
  - $(x_n)_{n=1}^{\infty}$ is bounded $\implies$ the set $\{x_n \mid n \in \mathbf{N}\}$ has supremum $x = \sup\{x_n \mid n \in \mathbf{N}\}$
  - let $\epsilon > 0$, according to theorem 2.17, there exists some $M \in \mathbf{N}$ such that $x - \epsilon < x_M \leq x$, then for all $n \geq M$, we have

$$x - \epsilon < x_M \leq x_n \leq x < x + \epsilon \implies |x_n - x| < \epsilon$$

Example

recall the following lemma from example 1.8 for the proof of the next theorem:

Lemma 3.15 *Bernoulli's inequality.* If $x \geq -1$ then $(x+1)^n \geq 1 + nx$ for all $n \in \mathbf{N}$.

Theorem 3.16 If $c \in (0,1)$ then the sequence $(c^n)_{n=1}^{\infty}$ converges and $\lim_{n\to\infty} c^n = 0$. If $c > 1$, the sequence $(c^n)_{n=1}^{\infty}$ does not converge.

proof:

- if $c > 1$, we show that the sequence $(c^n)_{n=1}^{\infty}$ is unbounded (and hence does not converge):
  - let $B \geq 0$, then there exists some $n \in \mathbf{N}$, $n > \frac{B}{c-1}$ such that

$$c^n = ((c-1)+1)^n \geq 1 + n(c-1) > n(c-1) > B$$

(the first inequality is because of lemma 3.15)

- if $c \in (0,1)$, we first show that $(c^n)_{n=1}^{\infty}$ is monotone decreasing and bounded (and hence, convergent), $i.e.$, show that $c^{n+1} \leq c^n \leq c$ for all $n \in \mathbf{N}$ by induction:
  - suppose $n = 1 \implies c^2 \leq c \leq c$, the first inequality holds since $0 < c < 1$
  - suppose $n > 1$, and $c^{n+1} \leq c^n \leq c$, then we have $c^{n+2} \leq c^{n+1} \leq c^n \leq c$

  let $\lim_{n\to\infty} c^n = L$, we now show that $L = 0$
  - let $\epsilon > 0$, then there exists some $M \in \mathbf{N}$ such that for all $n \geq M$ such that

$$|c^n - L| < \frac{1}{2}(1-c)\epsilon$$

  - hence, we have

$$\begin{aligned}
(1-c)|L| &= |L - cL| \\
&= |(L - c^{M+1}) + (c^{M+1} - cL)| \\
&\leq |L - c^{M+1}| + c|c^M - L| \\
&< |L - c^{M+1}| + |c^M - L| \\
&< \frac{1}{2}(1-c)\epsilon + \frac{1}{2}(1-c)\epsilon \\
&= (1-c)\epsilon,
\end{aligned}$$

$i.e.$, $|L| < \epsilon$ for all $\epsilon > 0$ (according to theorem 2.14) $\implies |L| \leq 0 \implies L = 0$

Subsequences

Definition 3.17 Let $(x_n)_{n=1}^{\infty}$ be a sequence and $(n_i)_{i=1}^{\infty}$ be a strictly increasing sequence of natural numbers. The sequence $(x_{n_i})_{i=1}^{\infty}$ is called a **subsequence** of $(x_n)_{n=1}^{\infty}$.

example: consider the sequence $(x_n)_{n=1}^{\infty} = (n)_{n=1}^{\infty}$, $i.e.$, $1, 2, 3, 4, \ldots$

- the following are subsequences of $(x_n)_{n=1}^{\infty}$:
  - $1, 3, 5, 7, 9, 11, \ldots$, described with $(x_{n_i})_{i=1}^{\infty} = (x_{2i-1})_{i=1}^{\infty}$
  - $2, 4, 6, 8, 10, 12, \ldots$, described with $(x_{n_i})_{i=1}^{\infty} = (x_{2i})_{i=1}^{\infty}$
  - $2, 3, 5, 7, 11, 13, \ldots$, described with $(x_{n_i})_{i=1}^{\infty}$ where n_i are primes
- the following are not subsequences of $(x_n)_{n=1}^{\infty}$:
  - $1, 1, 1, 1, 1, 1, \ldots$
  - $1, 1, 3, 3, 5, 5, \ldots$

Theorem 3.18 If $\lim_{n\to\infty} x_n = x$, then all subsequences of $(x_n)_{n=1}^{\infty}$ converge to x.

proof:

- let $(x_{n_i})_{i=1}^{\infty}$ be a subsequence of $(x_n)_{n=1}^{\infty}$
- let $\epsilon > 0$, then there exists some $M_0 \in \mathbf{N}$ such that $|x_n - x| < \epsilon$ for all $n \geq M_0$
- let $M = M_0$, then for all $i \geq M$, since $n_i \geq i \geq M = M_0$, we have

$$|x_{n_i} - x| < \epsilon$$

Remark 3.19 Theorem 3.18 implies that the sequence $((-1)^n)_{n=1}^{\infty}$ is divergent.

Inequalities involving limits

Theorem 3.20 The sequence $(x_n)_{n=1}^{\infty}$ converges with $\lim_{n\to\infty} x_n = x$ if and only if the sequence $(|x_n - x|)_{n=1}^{\infty}$ converges with $\lim_{n\to\infty} |x_n - x| = 0$.

proof: let $\epsilon > 0$

- suppose $\lim_{n\to\infty} x_n = x$, then $\exists M_0 \in \mathbf{N}$ such that $\forall n \geq M_0$, $|x_n - x| < \epsilon$; let $M = M_0$, then $\forall n \geq M = M_0$, $|x_n - x - 0| = |x_n - x| < \epsilon$
- suppose $\lim_{n\to\infty} |x_n - x| = 0$, then $\exists M \in \mathbf{N}$, $\forall n \geq M$, $|x_n - x - 0| < \epsilon$, *i.e.*, $|x_n - x| < \epsilon$

Theorem 3.21 *Squeeze theorem.* Let $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$, and $(x_n)_{n=1}^{\infty}$ be sequences such that

$$a_n \leq x_n \leq b_n$$

for all $n \in \mathbf{N}$. Suppose that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ converge and

$$\lim_{n\to\infty} a_n = x = \lim_{n\to\infty} b_n.$$

Then $(x_n)_{n=1}^{\infty}$ converges and $\lim_{n\to\infty} x_n = x$.

proof: let $\epsilon > 0$

- $a_n \to x \implies \exists M_1 \in \mathbf{N}$ such that $\forall n \geq M_1$, $|a_n - x| < \epsilon$
- $b_n \to x \implies \exists M_2 \in \mathbf{N}$ such that $\forall n \geq M_2$, $|b_n - x| < \epsilon$
- $a_n \leq x_n \leq b_n \implies a_n - x \leq x_n - x \leq b_n - x$
- take $M = \max\{M_1, M_2\}$, then $\forall n \geq M$, we have

$$-\epsilon < a_n - x \leq x_n - x \leq b_n - x < \epsilon \implies |x_n - x| < \epsilon$$

Example 3.22 The sequence $\left(\frac{n^2}{n^2+n+1}\right)_{n=1}^{\infty}$ converges with $\lim_{n\to\infty} \frac{n^2}{n^2+n+1} = 1$.

proof:

- let $\epsilon > 0$, we have

$$0 \leq \left|\frac{n^2}{n^2+n+1} - 1\right| = \left|\frac{n+1}{n^2+n+1}\right| \leq \frac{n+1}{n^2+n} = \frac{1}{n}$$

- $0 \to 0$ and $\frac{1}{n} \to 0 \implies \left|\frac{n^2}{n^2+n+1} - 1\right| \to 0 \implies \frac{n^2}{n^2+n+1} \to 1$

Theorem 3.23 Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be sequences.

- If $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ converge and $x_n \le y_n$ for all $n \in \mathbf{N}$, then we have $\lim_{n\to\infty} x_n \le \lim_{n\to\infty} y_n$.
- If $(x_n)_{n=1}^{\infty}$ converges and $a \le x_n \le b$ for all $n \in \mathbf{N}$, then $a \le \lim_{n\to\infty} x_n \le b$.

proof: we show the first statement since the second statement can then be proved by considering sequences $(y_n)_{n=1}^{\infty}$ and $(z_n)_{n=1}^{\infty}$ where $y_n = a \le x_n \le b = z_n$

- let $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, suppose $x > y$
- $x > y \implies x - y > 0$, let $\epsilon = \frac{x-y}{2} > 0$
- $x_n \to x \implies \exists M_1 \in \mathbf{N}$ s.t. $\forall n \ge M_1$, $|x_n - x| < \frac{x-y}{2}$
- $y_n \to y \implies \exists M_2 \in \mathbf{N}$ s.t. $\forall n \ge M_2$, $|y_n - y| < \frac{x-y}{2}$
- let $M = \max\{M_1, M_2\}$, we have $x_M - x > -\frac{x-y}{2}$ and $y_M - y < \frac{x-y}{2}$, hence,

$$x_M > x - \frac{x-y}{2} = \frac{x+y}{2} = y + \frac{x-y}{2} > y_M,$$

  which contradicts to $x_n \le y_n$ for all $n \in \mathbf{N}$

Operations involving limits

Theorem 3.24 Suppose $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.

- The sequence $(x_n + y_n)_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty}(x_n + y_n) = x + y$.
- For all $c \in \mathbf{R}$, the sequence $(cx_n)_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} cx_n = cx$.
- The sequence $(x_n y_n)_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} x_n y_n = xy$.
- If $y_n \ne 0$ for all $n \in \mathbf{N}$ and $y \ne 0$, then the sequence $\left(\frac{x_n}{y_n}\right)_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{x}{y}$.

proof:

- to show $x_n \to x$, $y_n \to y \implies x_n + y_n \to x + y$, let $\epsilon > 0$
  - $x_n \to x \implies \exists M_1 \in \mathbf{N}$ such that $\forall n \ge M_1$, $|x_n - x| < \epsilon/2$
  - $y_n \to y \implies \exists M_2 \in \mathbf{N}$ such that $\forall n \ge M_2$, $|y_n - y| < \epsilon/2$
  - let $M = \max\{M_1, M_2\}$, then for all $n \ge M$, we have

$$|(x_n + y_n) - (x + y)| \le |x_n - x| + |y_n - y| < \epsilon/2 + \epsilon/2 = \epsilon$$

- to show $x_n \to x \implies cx_n \to cx$, let $\epsilon > 0$
  - $x_n \to x \implies \exists M \in \mathbf{N}$ such that $\forall n \geq M$, $|x_n - x| < \frac{1}{|c|+1}\epsilon$
  - then for all $n \geq M$, we have $|cx_n - cx| = |c||x_n - x| < \frac{|c|}{|c|+1}\epsilon < \epsilon$
- we show that $x_n \to x$, $y_n \to y \implies x_n y_n \to xy$:
  - $x_n \to x \implies |x_n - x| \to 0$
  - $y_n \to y \implies |y_n - y| \to 0$, and $(y_n)_{n=1}^{\infty}$ is bounded, *i.e.*, $\exists B \geq 0$, $|y_n| \leq B$
  - hence, we have

$$\begin{aligned} 0 \leq |x_n y_n - xy| &= |x_n y_n + x y_n - x y_n - xy| \\ &= |(x_n - x) y_n + (y_n - y) x| \\ &\leq |x_n - x||y_n| + |y_n - y||x| \\ &\leq |x_n - x| B + |y_n - y||x| \end{aligned}$$

  - according to the previous statements, $|x_n - x| \to 0 \implies |x_n - x|B \to 0$, $|y_n - y| \to 0 \implies |y_n - y||x| \to 0$, then $|x_n - x|B + |y_n - y||x| \to 0$
  - hence, according to theorem 3.21, $|x_n y_n - xy| \to 0$

- to prove $x_n \to x$, $y_n \to y$ $(y_n \neq 0$ for all $n \in \mathbf{N}$, $y \neq 0) \implies \frac{x_n}{y_n} \to \frac{x}{y}$, we first show that there exists some $b > 0$ such that $|y_n| \geq b$:
  - let $\epsilon = \frac{|y|}{2}$, then $y_n \to y \implies \exists M \in \mathbf{N}$ s.t. $\forall n \geq M$, $|y_n - y| < \frac{|y|}{2}$
  - then for all $n \geq M$, we have

$$\frac{|y|}{2} > |y_n - y| \geq ||y_n| - |y|| \implies |y_n| > \frac{|y|}{2}$$

    (the second inequality is from the reverse triangle inequality)
  - take $b = \min\{|y_1|, \dots, |y_M|, |y|/2\}$, we have $|y_n| \geq b$ for all $n \in \mathbf{N}$

we then show that $\left(\frac{1}{y_n}\right)_{n=1}^{\infty}$ converges with $\lim_{n\to\infty} \frac{1}{y_n} = \frac{1}{y}$: note that

$$0 \leq \left|\frac{1}{y_n} - \frac{1}{y}\right| = \left|\frac{y_n - y}{y_n y}\right| = \frac{|y_n - y|}{|y_n||y|} \leq \frac{|y_n - y|}{b|y|},$$

and $y_n \to y \implies \frac{|y_n - y|}{b|y|} \to 0$, hence, $\left|\frac{1}{y_n} - \frac{1}{y}\right| \to 0$, *i.e.*, $\frac{1}{y_n} \to \frac{1}{y}$

put together, $x_n \to x$ and $\frac{1}{y_n} \to \frac{1}{y} \implies \frac{x_n}{y_n} \to \frac{x}{y}$

Theorem 3.25 If $(x_n)_{n=1}^{\infty}$ is a convergent sequence with $\lim_{n\to\infty} x_n = x$, and $x_n \geq 0$ for all $n \in \mathbf{N}$, then the sequence $\left(\sqrt{x_n}\right)_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x}$.

proof:

- suppose $x = 0$, let $\epsilon > 0$, then we have $x_n \to 0 \implies \exists M \in \mathbf{N}$ s.t. $\forall n \geq M$, $|x_n - 0| = |x_n| < \epsilon^2 \implies \forall n \geq M$, $|\sqrt{x_n} - \sqrt{x}| = |\sqrt{x_n}| < \sqrt{\epsilon^2} < \epsilon$
- suppose $x > 0$, we have

$$0 \leq |\sqrt{x_n} - \sqrt{x}| = \left|\frac{(\sqrt{x_n} - \sqrt{x})(\sqrt{x_n} + \sqrt{x})}{\sqrt{x_n} + \sqrt{x}}\right| = \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \leq \frac{|x_n - x|}{\sqrt{x}},$$

  hence, $x_n \to x \implies |x_n - x| \to 0 \implies \frac{|x_n - x|}{\sqrt{x}} \to 0 \implies |\sqrt{x_n} - \sqrt{x}| \to 0$

Remark 3.26 Suppose the sequence $(x_n)_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} x_n = x$. One can prove that $\lim_{n\to\infty} x_n^k = x^k$ by induction. Moreover, if $x_n \geq 0$ for all $n \in \mathbf{N}$, one can also prove that $\lim_{n\to\infty} \sqrt[k]{x_n} = \sqrt[k]{x}$.

Theorem 3.27 If $(x_n)_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} x_n = x$, then $(|x_n|)_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} |x_n| = |x|$.

proof: let $\epsilon > 0$

- $x_n \to x \implies \exists M \in \mathbf{N}$ such that $\forall n \geq M$, $|x_n - x| < \epsilon$
- by reverse triangle inequality, for all $n \geq M$, we have

$$||x_n| - |x|| \leq |x_n - x| < \epsilon$$

Some special sequences

Theorem 3.28 If $p > 0$ then $\lim_{n\to\infty} n^{-p} = 0$.

proof: let $\epsilon > 0$, choose $M \in \mathbf{N}$ such that $M > (1/\epsilon)^{1/p}$, then for all $n \geq M$, $|n^{-p} - 0| = 1/n^p \leq 1/M^p < \epsilon$

Theorem 3.29 If $p > 0$ then $\lim_{n\to\infty} p^{1/n} = 1$.

proof:

- if $p = 1$, $\lim_{n\to\infty} p^{1/n} = \lim_{n\to\infty} 1^{1/n} = 1$
- suppose $p > 1$
  - $p > 1 \implies p^{1/n} > 1^{1/n} = 1 \implies p^{1/n} - 1 > 0$
  - according to the Bernoulli's inequality (example 1.8), we have

$$\Big(1 + (p^{1/n} - 1)\Big)^n \geq 1 + n(p^{1/n} - 1) \implies \frac{p-1}{n} \geq p^{1/n} - 1 > 0$$

  - $\frac{p-1}{n} \to 0 \implies p^{1/n} - 1 \to 0 \implies p^{1/n} \to 1$
- if $0 < p < 1 \implies 1/p > 1$, hence, $\lim_{n\to\infty} p^{1/n} = \lim_{n\to\infty} \frac{1}{(1/p)^{1/n}} = 1/1 = 1$

Theorem 3.30 The sequence $\left(n^{1/n}\right)_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} n^{1/n} = 1$.

proof:

- one can simply show that $n^{1/n} \geq 1$ by induction $\implies n^{1/n} - 1 > 0$
- according to the binomial theorem, for all $x, y \in \mathbf{R}$ and $n \in \mathbf{N}$, we have $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$, where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- let $x = 1$, $y = n^{1/n} - 1$, for all $n > 1$, we have

$$\begin{aligned}
& n = (1 + n^{1/n} - 1)^n = \sum_{k=0}^{n} \binom{n}{k} (n^{1/n} - 1)^k \geq \binom{n}{2} (n^{1/n} - 1)^2 \\
\implies \quad & n \geq \frac{n!}{2!(n-2)!} (n^{1/n} - 1)^2 = \frac{1}{2} n(n-1)(n^{1/n} - 1)^2 \\
\implies \quad & \sqrt{\frac{2}{n-1}} \geq n^{1/n} - 1 > 0 \\
\implies \quad & n^{1/n} - 1 \to 0 \implies n^{1/n} \to 1
\end{aligned}$$

Limit superior and limit inferior

Definition 3.31 Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence. Define, if the limits exist,

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} (\sup\{x_k \mid k \geq n\}) \quad \text{and} \quad \liminf_{n\to\infty} x_n = \lim_{n\to\infty} (\inf\{x_k \mid k \geq n\}).$$

They are called the **limit superior** and **limit inferior**, respectively.

Theorem 3.32 Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence, and let

$$a_n = \sup\{x_k \mid k \geq n\} \quad \text{and} \quad b_n = \inf\{x_k \mid k \geq n\}.$$

Then:

- The sequence $(a_n)_{n=1}^{\infty}$ is monotone decreasing and bounded.
- The sequence $(b_n)_{n=1}^{\infty}$ is monotone increasing and bounded.
- We have $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$.

proof:

- we first prove the following lemma:

  Lemma 3.33 Let $A, B \subseteq \mathbf{R}$, $A, B \neq \emptyset$, and A, B are bounded. If $A \subseteq B$ then we have $\inf B \leq \inf A \leq \sup A \leq \sup B$.

  – $A \subseteq B \implies \sup B$ is an upper bound of $A \implies \sup A \leq \sup B$
  – similarly, $\inf B$ is an lower bound of $A \implies \inf B \leq \inf A$
  – $A, B \neq \emptyset \implies \inf A \leq \sup A \implies \inf B \leq \inf A \leq \sup A \leq \sup B$

- we now show the first two statements in the theorem
  – $(x_n)_{n=1}^{\infty}$ is bounded $\implies$ there exists some $B \geq 0$ such that $-B \leq x_n \leq B$
  – for all $n \in \mathbf{N}$, we have $\{x_k \mid k \geq n+1\} \subseteq \{x_k \mid k \geq n\} \subseteq \{x_n \mid n \in \mathbf{N}\}$, according to lemma 3.33, this implies that

  $$-B \leq b_n \leq b_{n+1} \leq a_{n+1} \leq a_n \leq B,$$

  i.e., $(a_n)_{n=1}^{\infty}$ is bounded monotone decreasing and $(b_n)_{n=1}^{\infty}$ is bounded monotone increasing ($\implies$ $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ converge)

- according to the previous inequalities, we have $b_n \leq a_n$ for all $n \in \mathbf{N} \implies$ $\lim_{n\to\infty} b_n \leq \lim_{n\to\infty} a_n$ (theorem 3.23), *i.e.*, $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$

Example 3.34 We have $\limsup_{n\to\infty}(-1)^n = 1$ and $\liminf_{n\to\infty}(-1)^n = -1$.

proof: $\forall n \in \mathbf{N}$, the set $\{(-1)^k \mid k \geq n\} = \{-1, 1\} \implies \sup\{(-1)^k \mid k \geq n\} = 1$, $\inf\{(-1)^k \mid k \geq n\} = -1 \implies \limsup_{n\to\infty}(-1)^n = 1$ and $\liminf_{n\to\infty}(-1)^n = -1$

Example 3.35 We have $\limsup_{n\to\infty}\frac{1}{n} = \liminf_{n\to\infty}\frac{1}{n} = 0$.

proof: for all $n \in \mathbf{N}$, we have $\sup\{1/k \mid k \geq n\} = 1/k$ and $\inf\{1/k \mid k \geq n\} = 0$, hence,

$$\limsup_{n\to\infty}\frac{1}{n} = \lim_{n\to\infty}\frac{1}{k} = 0 \quad \text{and} \quad \liminf_{n\to\infty}\frac{1}{n} = \lim_{n\to\infty} 0 = 0$$

Bolzano-Weierstrass theorem

Theorem 3.36 Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence. Then, there exists subsequences $(x_{n_i})_{i=1}^{\infty}$ and $(x_{m_i})_{i=1}^{\infty}$ such that

$$\lim_{i\to\infty} x_{n_i} = \limsup_{n\to\infty} x_n \quad \text{and} \quad \lim_{i\to\infty} x_{m_i} = \liminf_{n\to\infty} x_n.$$

proof: let $a_n = \sup\{x_k \mid k \geq n\}$

- $a_1 = \sup\{x_k \mid k \geq 1\} \implies \exists n_1 \geq 1$ such that $a_1 - 1 < x_{n_1} \leq a_1$
- $a_{n_1+1} = \sup\{x_k \mid k \geq n_1 + 1\} \implies \exists n_2 > n_1$ s.t. $a_{n_1+1} - \frac{1}{2} < x_{n_2} \leq a_{n_1+1}$
- $a_{n_2+1} = \sup\{x_k \mid k \geq n_2 + 1\} \implies \exists n_3 > n_1$ s.t. $a_{n_2+1} - \frac{1}{3} < x_{n_3} \leq a_{n_2+1}$
- repeatedly, we can find a sequence of integers $n_1 < n_2 < \cdots$ such that

  $$a_{n_{i-1}+1} - \frac{1}{i} < x_{n_i} \leq a_{n_{i-1}+1}$$

  (defining $n_0 = 0$)
- $\left(a_{n_{i-1}+1}\right)_{i=1}^{\infty}$ is a subsequence of $(a_n)_{n=1}^{\infty}$, and $\lim_{n\to\infty} a_n = \limsup_{n\to\infty} x_n$ $\implies \lim_{n\to\infty} a_{n_{i-1}+1} = \limsup_{n\to\infty} x_n \implies \lim_{n\to\infty} x_{n_i} = \limsup_{n\to\infty} x_n$
- similarly, we can find a subsequence of $(x_n)_{n=1}^{\infty}$ that converges to $\liminf_{n\to\infty} x_n$

Theorem 3.37 *Bolzano-Weierstrass.* Every bounded sequence consisting of real numbers has a convergent subsequence.

Theorem 3.38 Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence. Then, $(x_n)_{n=1}^{\infty}$ converges if and only if $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$.

proof:

- suppose $\lim_{n\to\infty} x_n = x$, then the subsequences that converge to $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$ must converge to x (theorem 3.18)

- suppose $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = x$, for all $n \in \mathbf{N}$, according to the squeeze theorem,

$$\inf\{x_k \mid k \geq n\} \leq x_n \leq \sup\{x_k \mid k \geq n\} \implies \lim_{n\to\infty} x_n = x$$

Cauchy sequences

Definition 3.39 A sequence $(x_n)_{n=1}^{\infty}$ is **Cauchy** if for all $\epsilon > 0$, there exists an $M \in \mathbf{N}$ such that for all $n, k \geq M$, we have $|x_n - x_k| < \epsilon$.

Remark 3.40 A sequence $(x_n)_{n=1}^{\infty}$ is not Cauchy if there exists some $\epsilon > 0$, such that for all $M \in \mathbf{N}$, there exists some $n, k \geq M$, so that $|x_n - x_k| \geq \epsilon$.

Example 3.41 The sequence $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ is Cauchy.

proof: let $\epsilon > 0$, choose $M \in \mathbf{N}$ such that $M > 2/\epsilon$, then for all $n, k \geq M$, we have

$$\left|\frac{1}{n} - \frac{1}{k}\right| \leq \frac{1}{n} + \frac{1}{k} \leq \frac{2}{M} < \epsilon$$

Example 3.42 The sequence $((-1)^n)_{n=1}^{\infty}$ is not Cauchy.

proof: let $\epsilon = 1$, $M \in \mathbf{N}$, $n = M$, $k = M + 1$, then $\left|(-1)^n - (-1)^k\right| = 2 \geq \epsilon$

Theorem 3.43 If the sequence $(x_n)_{n=1}^{\infty}$ is Cauchy, then $(x_n)_{n=1}^{\infty}$ is bounded.

proof:

- let $\epsilon = 1$, $(x_n)_{n=1}^{\infty}$ is Cauchy $\implies$ $\exists M \in \mathbf{N}$ such that $\forall n, k \geq M$, $|x_n - x_k| < 1$
- let $k = M$ $\implies$ $\forall n \geq M$, $|x_n - x_M| < 1$ $\implies$ $\forall n \geq M$, $|x_n| < |x_M| + 1$
- take $B = \max\{|x_1|, |x_2|, \ldots, |x_M|, |x_M| + 1\}$, then $|x_n| \leq B$ for all $n \in \mathbf{N}$

Theorem 3.44 If the sequence $(x_n)_{n=1}^{\infty}$ is Cauchy and a subsequence $(x_{n_i})_{i=1}^{\infty}$ converges, then $(x_n)_{n=1}^{\infty}$ converges.

proof: let $\epsilon > 0$

- $(x_n)_{n=1}^{\infty}$ is Cauchy $\implies$ $\exists M_1 \in \mathbf{N}$ such that $\forall n, k \geq M_1$, $|x_n - x_k| < \epsilon/2$
- let $\lim_{i\to\infty} x_{n_i} = x$ $\implies$ $\exists M_2 \in \mathbf{N}$ such that $\forall i \geq M_2$, $|x_{n_i} - x| < \epsilon/2$
- let $M = \max\{M_1, M_2\}$, then $\forall k \geq M$, $n_k \geq k \geq M_1$, $n_k \geq k \geq M_2$, hence,

$$|x_k - x| \leq |x_k - x_{n_k}| + |x_{n_k} - x| < \epsilon/2 + \epsilon/2 = \epsilon$$

Theorem 3.45 *Completeness of the real numbers.* A sequence of real numbers $(x_n)_{n=1}^{\infty}$ is Cauchy if and only if the sequence $(x_n)_{n=1}^{\infty}$ is convergent.

proof:

- suppose $(x_n)_{n=1}^{\infty}$ is Cauchy $\implies$ $(x_n)_{n=1}^{\infty}$ is bounded (theorem 3.43) $\implies$ there exists convergent subsequence of $(x_n)_{n=1}^{\infty}$ (theorem 3.37) $\implies$ $(x_n)_{n=1}^{\infty}$ is convergent (theorem 3.44)
- suppose $\lim_{n\to\infty} x_n = x$, let $\epsilon > 0$, then $\exists M \in \mathbf{N}$, $\forall n \geq M$, $|x_n - x| < \epsilon/2$; let $k \geq M$, then $|x_n - x_k| \leq |x_n - x| + |x - x_k| < \epsilon/2 + \epsilon/2 = \epsilon$

Remark 3.46 We say a set is **Cauchy-complete**, or just **complete**, if all Cauchy sequence of elements in the set converges to some point in the set. Theorem 3.45 indicates that $\mathbf{R}$ is complete.

Remark 3.47 The set $\mathbf{Q}$ is *not* complete. Since $\mathbf{Q}$ does not have the least upper bound property, then, $e.g.$, $\sup\{x_n \mid n \in \mathbf{N}\}$, $\sup\{x_k \mid k \geq n\}$, $etc.$, might not exist in $\mathbf{Q}$.

4. Series

- series
- Cauchy series
- linearity of series
- absolute convergence
- comparison, ratio, and root tests
- alternating series
- rearrangements

Series

Definition 4.1 Given a sequence $(x_n)_{n=1}^{\infty}$, the formal object $\sum_{n=1}^{\infty} x_n$ is called a **series**. A series **converges** if the sequence $(s_m)_{m=1}^{\infty}$ defined by

$$s_m = \sum_{n=1}^{m} x_n = x_1 + \cdots + x_m$$

converges. The numbers s_m are called **partial sums**. If the series converges, we write

$$\sum_{n=1}^{\infty} x_n = \lim_{m\to\infty} s_m.$$

In this case, we treat $\sum_{n=1}^{\infty} x_n$ as a number.

If the sequence $(s_m)_{m=1}^{\infty}$ diverges, we say the series is **divergent**. In this case, $\sum_{n=1}^{\infty} x_n$ is simply a formal object and not a number.

- series need not start at $n = 1$

Example 4.2 The series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges.

proof: the sequence of partial sums $(s_m)_{m=1}^{\infty}$ is given by:

$$\begin{aligned} s_m &= \sum_{n=1}^{m} \frac{1}{n(n+1)} \\ &= \sum_{n=1}^{m} \frac{1}{n} - \frac{1}{n+1} \\ &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{m} - \frac{1}{m+1} \\ &= 1 - \frac{1}{m+1}, \end{aligned}$$

hence, $s_m \to 1 \implies \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges and $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

Theorem 4.3 If $|r| < 1$, then $\sum_{n=0}^{\infty} r^n$ converges and $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$.

proof:

- the sequence of partial sums $(s_m)_{m=1}^{\infty}$ is given by:

$$s_m = \sum_{n=0}^{m} r^n = \frac{\left(\sum_{n=0}^{m} r^n\right)(1-r)}{1-r} = \frac{\sum_{n=0}^{m}(r^n - r^{n+1})}{1-r} = \frac{1-r^{m+1}}{1-r}$$

- $|r| < 1 \implies r^n \to 0$ (theorem 3.16) $\implies s_m \to \frac{1}{1-r}$

Remark 4.4 Series of the form $\sum_{n=0}^{\infty} \alpha r^n$ with $\alpha, r \in \mathbf{R}$ are called **geometric series**.

Theorem 4.5 Let $(x_n)_{n=1}^{\infty}$ be a sequence and let $M \in \mathbf{N}$. Then, $\sum_{n=1}^{\infty} x_n$ converges if and only if $\sum_{n=M}^{\infty} x_n$ converges.

proof:

- for all $m \geq M$, we have

$$\sum_{n=1}^{m} x_n = \sum_{n=1}^{M-1} x_n + \sum_{n=M}^{m} x_n$$

- suppose $\sum_{n=1}^{\infty} x_n$ converges, we have

$$\lim_{m\to\infty} \sum_{n=M}^{m} x_n = \lim_{m\to\infty} \left(\sum_{n=1}^{m} x_n - \sum_{n=1}^{M-1} x_n \right) = \lim_{m\to\infty} \left(\sum_{n=1}^{m} x_n \right) - \sum_{n=1}^{M-1} x_n$$

- suppose $\sum_{n=M}^{\infty} x_n$ converges, we have

$$\lim_{m\to\infty} \sum_{n=1}^{m} x_n = \lim_{m\to\infty} \left(\sum_{n=M}^{m} x_n + \sum_{n=1}^{M-1} x_n \right) = \lim_{m\to\infty} \left(\sum_{n=M}^{m} x_n \right) + \sum_{n=1}^{M-1} x_n$$

Cauchy series

Definition 4.6 The series $\sum_{n=1}^{\infty} x_n$ is **Cauchy** if the sequence of partial sums $(s_m)_{m=1}^{\infty}$ is Cauchy.

Theorem 4.7 The series $\sum_{n=1}^{\infty} x_n$ is Cauchy if and only if $\sum_{n=1}^{\infty} x_n$ is convergent.

proof: according to theorem 3.45

- suppose $\sum_{n=1}^{\infty} x_n$ is Cauchy $\implies$ $(s_m)_{m=1}^{\infty}$ is Cauchy $\implies$ $(s_m)_{m=1}^{\infty}$ is convergent $\implies$ $\sum_{n=1}^{\infty} x_n$ is convergent

- suppose $\sum_{n=1}^{\infty} x_n$ is convergent $\implies$ $(s_m)_{m=1}^{\infty}$ is convergent $\implies$ $(s_m)_{m=1}^{\infty}$ is Cauchy $\implies$ $\sum_{n=1}^{\infty} x_n$ is Cauchy

Theorem 4.8 The series $\sum_{n=1}^{\infty} x_n$ is Cauchy if and only if for all $\epsilon > 0$, there exists an $M \in \mathbf{N}$ such that for all $m \geq M$ and $k > m$, we have $\left|\sum_{n=m+1}^{k} x_n\right| < \epsilon$.

proof: let $\epsilon > 0$

- suppose $\sum_{n=1}^{\infty} x_n$ is Cauchy $\implies$ $(\sum_{n=1}^{m} x_n)_{m=1}^{\infty}$ is Cauchy $\implies$ $\exists M \in \mathbf{N}$ such that $\forall m, k \geq M$ (assume $k > m$), we have

$$\left|\sum_{n=1}^{m} x_n - \sum_{n=1}^{k} x_n\right| < \epsilon \implies \left|\sum_{n=m+1}^{k} x_n\right| < \epsilon$$

- suppose $\exists M \in \mathbf{N}$ such that for all $k > m \geq M$, $\left|\sum_{n=m+1}^{k} x_n\right| < \epsilon$, then we have

$$\left|\sum_{n=1}^{m} x_n - \sum_{n=1}^{k} x_n\right| = \left|\sum_{n=m+1}^{k} x_n\right| < \epsilon,$$

$i.e.$, $(\sum_{n=1}^{m} x_n)_{m=1}^{\infty}$ is Cauchy $\implies$ $\sum_{n=1}^{\infty} x_n$ is Cauchy

Theorem 4.9 If the series $\sum_{n=1}^{\infty} x_n$ converges then $\lim_{n\to\infty} x_n = 0$.

proof: let $\epsilon > 0$, $\sum_{n=1}^{\infty} x_n$ converges $\implies$ $\sum_{n=1}^{\infty} x_n$ is Cauchy $\implies$ $\exists M_0 \in \mathbf{N}$ such that $\forall k > m \geq M_0$, we have $\left|\sum_{n=m+1}^{k} x_n\right| < \epsilon$ (theorem 4.8); choose $M = M_0 + 1$, then $\forall m \geq M$, by taking $k = m > m - 1 \geq M_0$, we have

$$|x_m - 0| = |x_m| = \left|\sum_{n=m-1+1}^{m} x_n\right| < \epsilon \implies \lim_{n\to\infty} x_n = 0$$

Remark 4.10 The converse of theorem 4.9 does not hold.

Theorem 4.11 If $|r| \geq 1$ then the series $\sum_{n=0}^{\infty} r^n$ diverges.

proof: If $|r| \geq 1$, then $\lim_{n\to\infty} r^n \neq 0$, according to theorem 4.9, $\sum_{n=0}^{\infty} r^n$ diverges

Corollary 4.12 The series $\sum_{n=0}^{\infty} \alpha r^n$ with $\alpha, r \in \mathbf{R}$ converges if and only if $|r| < 1$.

Theorem 4.13 The **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge.

proof: we show that a subsequence of $(s_m)_{m=1}^{\infty}$ is unbounded

- consider the subsequence $(s_{2^i})_{i=1}^{\infty}$, given by

$$\begin{aligned} s_{2^i} &= \sum_{n=1}^{2^i} \frac{1}{n} = 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{i-1}+1} + \cdots + \frac{1}{2^i}\right) \\ &= 1 + \sum_{k=1}^{i} \sum_{n=2^{k-1}+1}^{2^k} \frac{1}{n} \\ &\geq 1 + \sum_{k=1}^{i} \sum_{n=2^{k-1}+1}^{2^k} \frac{1}{2^k} = 1 + \sum_{k=1}^{i} \frac{1}{2^k}(2^k - (2^{k-1}+1) + 1) \\ &= 1 + \sum_{k=1}^{i} \frac{2^{k-1}}{2^k} = 1 + \sum_{k=1}^{i} \frac{1}{2} = 1 + \frac{i}{2} \end{aligned}$$

- $(1 + i/2)_{i=1}^{\infty}$ is unbounded $\implies$ $(s_{2^i})_{i=1}^{\infty}$ is unbounded $\implies$ $(s_m)_{m=1}^{\infty}$ is unbounded $\implies$ $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge

Linearity of series

Theorem 4.14 Let $\alpha \in \mathbf{R}$ and $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ be convergent series. Then the series $\sum_{n=1}^{\infty}(\alpha x_n + y_n)$ converges and

$$\sum_{n=1}^{\infty}(\alpha x_n + y_n) = \alpha \sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} y_n.$$

proof: consider the partial sums of $\sum_{n=1}^{\infty}(\alpha x_n + y_n)$, we have

$$\begin{aligned} & \sum_{n=1}^{m}(\alpha x_n + y_n) = \alpha \sum_{n=1}^{m} x_n + \sum_{n=1}^{m} y_n \\ \implies \quad & \lim_{m\to\infty} \sum_{n=1}^{m}(\alpha x_n + y_n) = \alpha \lim_{m\to\infty} \sum_{n=1}^{m} x_n + \lim_{m\to\infty} \sum_{n=1}^{m} y_n \\ \implies \quad & \sum_{n=1}^{\infty}(\alpha x_n + y_n) = \alpha \sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} y_n \end{aligned}$$

Absolute convergence

Theorem 4.15 If $x_n \geq 0$ for all $n \in \mathbf{N}$, then the series $\sum_{n=1}^{\infty} x_n$ converges if and only if the sequence of partial sums $(s_m)_{m=1}^{\infty}$ is bounded.

proof:

- suppose $\sum_{n=1}^{\infty} x_n$ converges $\implies$ $(s_m)_{m=1}^{\infty}$ converges $\implies$ $(s_m)_{m=1}^{\infty}$ is bounded
- suppose $(s_m)_{m=1}^{\infty}$ is bounded, since $x_n \geq 0$ for all $n \in \mathbf{N}$, we have

$$s_m = \sum_{n=1}^{m} x_n \leq \sum_{n=1}^{m} x_n + x_{n+1} = s_{m+1},$$

  $i.e.$, $(s_m)_{m=1}^{\infty}$ is monotone increasing $\implies$ $(s_m)_{m=1}^{\infty}$ converges $\implies$ $\sum_{n=1}^{\infty} x_n$ converges

Definition 4.16 The series $\sum_{n=1}^{\infty} x_n$ **converges absolutely** if $\sum_{n=1}^{\infty} |x_n|$ converges.

Theorem 4.17 If the series $\sum_{n=1}^{\infty} x_n$ converges absolutely then $\sum_{n=1}^{\infty} x_n$ converges.

proof:

- we first prove the following claim by induction:

  Lemma 4.18 For all $x_1, \ldots, x_n \in \mathbf{R}$, we have $|\sum_{i=1}^{n} x_i| \leq \sum_{i=1}^{n} |x_i|$.

  - suppose $n = 2$, we have the triangle inequality $|x_1 + x_2| \leq |x_1| + |x_2|$
  - suppose $n > 2$, and $|\sum_{i=1}^{n} x_i| \leq \sum_{i=1}^{n} |x_i|$ holds, we have

$$\left|\sum_{i=1}^{n+1} x_i\right| \leq \left|\sum_{i=1}^{n} x_i\right| + |x_{n+1}| \leq \sum_{i=1}^{n} |x_i| + |x_{n+1}| = \sum_{i=1}^{n+1} |x_i|$$

- $\sum_{n=1}^{\infty} x_n$ converges absolutely $\implies$ $\sum_{n=1}^{\infty} |x_n|$ converges $\implies$ let $\epsilon > 0$, $\exists M \in \mathbf{N}$ s.t. $\forall k > m \geq M$, $|\sum_{n=m+1}^{k} |x_n|| = \sum_{n=m+1}^{k} |x_n| < \epsilon$
- hence, for all $k > m \geq M$, we have $\left|\sum_{n=m+1}^{k} x_n\right| \leq \sum_{n=m+1}^{k} |x_n| < \epsilon \implies \sum_{n=1}^{\infty} x_n$ converges

Remark 4.19 The converse of theorem 4.17 does not hold.

Comparison test

Theorem 4.20 *Comparison test.* Suppose $0 \le x_n \le y_n$ for all $n \in \mathbf{N}$.

- If $\sum_{n=1}^{\infty} y_n$ converges then $\sum_{n=1}^{\infty} x_n$ converges.
- If $\sum_{n=1}^{\infty} x_n$ diverges then $\sum_{n=1}^{\infty} y_n$ diverges.

---

proof:

- suppose $\sum_{n=1}^{\infty} y_n$ converges $\implies$ $(\sum_{n=1}^{m} y_n)_{m=1}^{\infty}$ is bounded $\implies$ $\exists B \ge 0$ s.t. $\forall m \in \mathbf{N}$, $|\sum_{n=1}^{m} y_n| = \sum_{n=1}^{m} y_n \le B$ $\implies$ $\forall m \in \mathbf{N}$, we have

$$0 \le \sum_{n=1}^{m} x_n \le \sum_{n=1}^{m} y_n \le B$$

  $\implies$ $(\sum_{n=1}^{m} x_n)_{m=1}^{\infty}$ is bounded $\implies$ $\sum_{n=1}^{\infty} x_n$ converges (theorem 4.15)
- suppose $\sum_{n=1}^{\infty} x_n$ diverges $\implies$ $(\sum_{n=1}^{m} x_n)_{m=1}^{\infty}$ is unbounded (theorem 4.15) $\implies$ $\forall B \ge 0$, $\exists m \in \mathbf{N}$ such that $|\sum_{n=1}^{m} x_n| = \sum_{n=1}^{m} x_n > B$, hence, for this m,

$$\sum_{n=1}^{m} y_n \ge \sum_{n=1}^{m} x_n > B$$

  $\implies$ $(\sum_{n=1}^{m} y_n)_{m=1}^{\infty}$ is unbounded $\implies$ $\sum_{n=1}^{\infty} y_n$ diverges

---

Theorem 4.21 For $p \in \mathbf{R}$, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

---

proof:

- suppose $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges, assume $p \le 1$, then we have $0 < \frac{1}{n} \le \frac{1}{n^p}$; the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $\implies$ $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges (theorem 4.20), which is a contradiction
- suppose $p > 1$, let $s_m = \sum_{n=1}^{m} \frac{1}{n^p}$
  - we first show that $s_m \le s_{2^m}$ for all $m \in \mathbf{N}$: by induction, we have $2^m > m$ for all $m \in \mathbf{N}$ $\implies$ $s_m = \sum_{n=1}^{m} \frac{1}{n^p} \le \sum_{n=1}^{2^m} \frac{1}{n^p} = s_{2^m}$
  - we now show that s_{2^m} is bounded by $1 + \frac{1}{1-2^{-(p-1)}}$:

$$\begin{aligned} s_{2^m} &= \sum_{n=1}^{2^m} \frac{1}{n^p} \\ &= 1 + \left(\frac{1}{2^p}\right) + \left(\frac{1}{3^p} + \frac{1}{4^p}\right) + \cdots + \left(\frac{1}{(2^{m-1}+1)^p} + \cdots + \frac{1}{(2^m)^p}\right) \\ &= 1 + \sum_{k=1}^{m} \sum_{n=2^{k-1}+1}^{2^k} \frac{1}{n^p} \le 1 + \sum_{k=1}^{m} \sum_{n=2^{k-1}+1}^{2^k} \frac{1}{(2^{k-1}+1)^p} \end{aligned}$$

$$
\begin{aligned}
&\le 1 + \sum_{k=1}^{m} \sum_{n=2^{k-1}+1}^{2^k} \frac{1}{(2^{k-1})^p} = 1 + \sum_{k=1}^{m} 2^{-p(k-1)}(2^k - (2^{k-1}+1) + 1) \\
&= 1 + \sum_{k=1}^{m} 2^{-(p-1)(k-1)} = 1 + \sum_{k=0}^{m-1} 2^{-(p-1)k} \\
&\le 1 + \sum_{k=0}^{\infty} 2^{-(p-1)k} = 1 + \sum_{k=0}^{\infty} \left(2^{-(p-1)}\right)^k \\
&= 1 + \frac{1}{1-2^{-(p-1)}},
\end{aligned}
$$

where the last equality is from the fact that $p - 1 > 0$, and using the properties of geometric series (theorem 4.3)

– put together, we have $0 < s_m \le s_{2^m} \le 1 + \frac{1}{1-2^{-(p-1)}} \implies (s_m)_{m=1}^{\infty}$ is monotone increasing and bounded $\implies (s_m)_{m=1}^{\infty}$ converges $\implies \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges

Ratio test

Theorem 4.22 *Ratio test.* Suppose $x_n \neq 0$ for all n and the limit

$$L = \lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|}$$

exists.

- If $L > 1$ then $\sum_{n=1}^{\infty} x_n$ diverges.
- If $L < 1$ then $\sum_{n=1}^{\infty} x_n$ converges absolutely.

---

proof:

- suppose $L > 1$, then $\exists M \in \mathbf{N}$ such that $\forall n \ge M$, $\frac{|x_{n+1}|}{|x_n|} \ge 1 \implies \forall n \ge M$, $|x_{n+1}| \ge |x_n| \implies \lim_{n\to\infty} x_n \neq 0 \implies \sum_{n=1}^{\infty} x_n$ diverges (theorem 4.9)
- suppose $L < 1$, let $L < \alpha < 1$
  – $\exists M \in \mathbf{N}$ such that $\forall n \ge M$, $\frac{|x_{n+1}|}{|x_n|} \le \alpha \implies \forall n \ge M$, $|x_{n+1}| \le \alpha |x_n| \implies$

$$|x_n| \le \alpha |x_{n-1}| \le \alpha^2 |x_{n-2}| \le \cdots \le \alpha^{n-M} |x_M| \implies |x_n| \le \alpha^{n-M} |x_M|, \ \forall n \ge M$$

– consider the partial sums of the series $\sum_{n=1}^{\infty}|x_n|$, assume $m > M$, we have

$$\begin{aligned}\sum_{n=1}^{m}|x_n| &= \sum_{n=1}^{M-1}|x_n| + \sum_{n=M}^{m}|x_n| \le \sum_{n=1}^{M-1}|x_n| + \sum_{n=M}^{\infty}|x_n| \\ &\le \sum_{n=1}^{M-1}|x_n| + \sum_{n=M}^{\infty}\alpha^{n-M}|x_M| = \sum_{n=1}^{M-1}|x_n| + |x_M|\sum_{n=0}^{\infty}\alpha^n \\ &= \sum_{n=1}^{M-1}|x_n| + \frac{|x_M|}{1-\alpha},\end{aligned}$$

where the last equality is from the properties of geometric series and $0 < \alpha < 1$

– hence, the sequence of partial sums $(\sum_{n=1}^{m}|x_n|)_{m=1}^{\infty}$ is monotone increasing and bounded $\implies$ $\sum_{n=1}^{\infty}|x_n|$ converges $\implies$ $\sum_{n=1}^{\infty}x_n$ converges absolutely

---

Remark 4.23 If $L = 1$ in theorem 4.22 then the test doesn't apply. For example, $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges.

Example 4.24 The series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}$ converges absolutely.

---

proof:

$$\left|\frac{(-1)^n}{n^2+1}\right| = \frac{1}{n^2+1} < \frac{1}{n^2} \implies \lim_{n\to\infty}\frac{\left|\frac{(-1)^{n+1}}{(n+1)^2+1}\right|}{\left|\frac{(-1)^n}{n^2+1}\right|} < \lim_{n\to\infty}\frac{n^2}{(n+1)^2} = 1$$

Example 4.25 The series $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ converges absolutely for all $x \in \mathbf{R}$.

---

proof:

$$\lim_{n\to\infty}\frac{\left|\frac{x^{n+1}}{(n+1)!}\right|}{\left|\frac{x^n}{n!}\right|} = \lim_{n\to\infty}\frac{|x|}{n+1} = 0 < 1$$

Root test

Theorem 4.26 *Root test.* Let $\sum_{n=1}^{\infty} x_n$ be a series and suppose that the limit

$$L = \lim_{n\to\infty} |x_n|^{1/n}$$

exists.

- If $L > 1$ then $\sum_{n=1}^{\infty} x_n$ diverges.
- If $L < 1$ then $\sum_{n=1}^{\infty} x_n$ converges absolutely.

proof:

- suppose $L > 1$, then $\exists M \in \mathbf{N}$ s.t. $\forall n \geq M$, $|x_n|^{1/n} \geq 1 \implies \forall n \geq M$, $|x_n| \geq 1$ $\implies \lim_{n\to\infty} x_n \neq 0 \implies \sum_{n=1}^{\infty} x_n$ diverges (theorem 4.9)
- suppose $L < 1$, let $L < \alpha < 1$
  - $\exists M \in \mathbf{N}$ such that $\forall n \geq M$, $|x_n|^{1/n} \leq \alpha \implies \forall n \geq M$, $|x_n| \leq \alpha^n$

  - consider the partial sums of the series $\sum_{n=1}^{\infty} |x_n|$, assume $m > M$, we have

$$\begin{aligned}
\sum_{n=1}^{m} |x_n| &= \sum_{n=1}^{M-1} |x_n| + \sum_{n=M}^{m} |x_n| \leq \sum_{n=1}^{M-1} |x_n| + \sum_{n=M}^{\infty} |x_n| \\
&\leq \sum_{n=1}^{M-1} |x_n| + \sum_{n=M}^{\infty} \alpha^n = \sum_{n=1}^{M-1} |x_n| + \sum_{n=0}^{\infty} \alpha^{M+n} \\
&= \sum_{n=1}^{M-1} |x_n| + \alpha^M \sum_{n=0}^{\infty} \alpha^n \\
&= \sum_{n=1}^{M-1} |x_n| + \frac{\alpha^M}{1-\alpha},
\end{aligned}$$

    where the last equality is from the properties of geometric series and $0 < \alpha < 1$
  - hence, the sequence of partial sums $(\sum_{n=1}^{m} |x_n|)_{m=1}^{\infty}$ is monotone increasing and bounded $\implies \sum_{n=1}^{\infty} |x_n|$ converges $\implies \sum_{n=1}^{\infty} x_n$ converges absolutely

Remark 4.27 Similarly, if $L = 1$ in theorem 4.26 then the test doesn't apply.

Alternating series

Theorem 4.28 Let $(x_n)_{n=1}^\infty$ be a monotone decreasing sequence with $\lim_{n\to\infty} x_n = 0$. Then the series $\sum_{n=1}^\infty (-1)^n x_n$ converges.

proof: consider the partial sums of $\sum_{n=1}^\infty (-1)^n x_n$, given by $s_m = \sum_{n=1}^m (-1)^n x_n$

- $(x_n)_{n=1}^\infty$ is monotone decreasing and $x_n \to 0 \implies \forall n \in \mathbf{N}$, $x_n \geq x_{n+1} \geq 0$
- we first show that the subsequence $(s_{2m})_{m=1}^\infty$ converges, notice that

$$s_{2m} = \sum_{n=1}^{2m} (-1)^n x_n = -x_1 + x_2 - x_3 + \cdots - x_{2m-1} + x_{2m} \tag{4.1}$$

  - rearranging the terms in (4.1), since $x_{n+1} \leq x_n$, $\forall n \in \mathbf{N}$, we have

$$\begin{aligned} s_{2m} &= (x_2 - x_1) + (x_4 - x_3) + \cdots + (x_{2m} - x_{2m-1}) \\ &\geq (x_2 - x_1) + (x_3 - x_2) + \cdots + (x_{2m} - x_{2m-1}) + (x_{2m+2} - x_{2m+1}) \\ &= s_{2(m+1)} \end{aligned}$$

  $\implies (s_{2m})_{m=1}^\infty$ is monotone decreasing

  - rearranging the terms in (4.1) differently, since $x_n \geq x_{n+1} \geq 0$, $\forall n \in \mathbf{N}$, we have

$$s_{2m} = -x_1 + (x_2 - x_3) + (x_4 - x_5) + \cdots + (x_{2m-2} - x_{2m-1}) + x_{2m} \geq -x_1$$

  $\implies (s_{2m})_{m=1}^\infty$ is bounded below

  - put together, we conclude that $(s_{2m})_{m=1}^\infty$ converges, let $s_{2m} \to x$
- we now show that $(s_m)_{m=1}^\infty$ also converges to x, let $\epsilon > 0$
  - $s_{2m} \to x \implies \exists M_1 \in \mathbf{N}$ such that $\forall m \geq M_1$, $|s_{2m} - x| < \epsilon/2$
  - $x_n \to 0 \implies \exists M_2 \in \mathbf{N}$ such that $\forall m \geq M_2$, $|x_m| < \epsilon/2$

  let $M = \max\{2M_1 + 1, M_2\}$, then $\forall m \geq M$, $m \geq 2M_1 + 1$ and $m \geq M_2$
  - if m is even $\implies \frac{m}{2} > M_1$, hence

$$|s_m - x| = \left|s_{2\cdot\frac{m}{2}} - x\right| < \epsilon/2 < \epsilon$$

  - if m is odd, then $m-1$ is even and $m - 1 \geq 2M_1 \implies \frac{m-1}{2} \geq M_1$, hence

$$\begin{aligned} |s_m - x| = |s_{m-1} - x + x_m| &= \left|s_{2\cdot\frac{m-1}{2}} - x + x_m\right| \\ &\leq \left|s_{2\cdot\frac{m-1}{2}} - x\right| + |x_m| < \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

  put together, we have $(s_m)_{m=1}^\infty$ converges $\implies \sum_{n=1}^\infty (-1)^n x_n$ converges

Corollary 4.29 The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges but does not converge absolutely.

proof:

- since $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ is monotone decreasing with $\lim_{n\to\infty} \frac{1}{n} = 0$, it follows immediately from theorem 4.28 that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges
- since $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$, and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, we conclude that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ does not converge absolutely

Rearrangements

Theorem 4.30 Suppose $\sum_{n=1}^{\infty} x_n$ converges absolutely and $\sum_{n=1}^{\infty} x_n = x$. Let $\sigma\colon \mathbf{N} \to \mathbf{N}$ be a bijective function. Then, the series $\sum_{n=1}^{\infty} x_{\sigma(n)}$ is absolutely convergent and $\sum_{n=1}^{\infty} x_{\sigma(n)} = x$. In other words, absolute convergence implies, if we rearrange the sequence, the new series will still converge to the same value of the original series.

proof:

- we first show $\sum_{n=1}^{\infty} |x_{\sigma(n)}|$ converges, *i.e.*, $\left(\sum_{n=1}^{m} |x_{\sigma(n)}|\right)_{m=1}^{\infty}$ is bounded
  - $\sum_{n=1}^{\infty} |x_n|$ converges $\implies$ $(\sum_{n=1}^{m} |x_n|)_{m=1}^{\infty}$ is bounded $\implies$ $\exists B \geq 0$ such that $\forall m \in \mathbf{N}$, $\sum_{n=1}^{m} |x_n| \leq B$
  - $\forall m \in \mathbf{N}$, $\{1, \ldots, m\}$ is a finite set $\implies$ $\exists k \in \mathbf{N}$ such that
$$\sigma(\{1, \ldots, m\}) \subseteq \{1, \ldots, k\},$$
    hence,
$$\sum_{n=1}^{m} |x_{\sigma(n)}| = \sum_{n \in \sigma(\{1,\ldots,m\})} |x_n| \leq \sum_{n=1}^{k} |x_n| \leq B$$
    $\implies$ $\forall m \in \mathbf{N}$, $\sum_{n=1}^{m} |x_{\sigma(n)}|$ is bounded

- we now show that $\sum_{n=1}^{\infty} x_{\sigma(n)} = x$, let $\epsilon > 0$
  - $\sum_{n=1}^{\infty} x_n = x \implies \exists M_0 \in \mathbf{N}$ such that for all $k > m \geq M_0$, we have

$$\left|\sum_{n=1}^{m} x_n - x\right| < \epsilon/2 \quad \text{and} \quad \left|\sum_{n=m+1}^{k} x_n\right| < \epsilon/2$$

  - the set $\{1, \ldots, M_0\}$ is finite $\implies \exists M \in \mathbf{N}$, $M > M_0$ such that

$$\{1, \ldots, M_0\} \subseteq \sigma(\{1, \ldots, M\}),$$

    hence, for all $m \geq M$, let $p = \max(\sigma(\{1, \ldots, m\})) > M_0$, we have

$$\sigma(\{1, \ldots, m\}) = \{1, \ldots, M_0\} \cup \{M_0 + 1, \ldots, p\}$$

  - consider the partial sums of $\sum_{n=1}^{\infty} x_{\sigma(x)}$, for all $m \geq M$, we have

$$\begin{aligned}\left|\sum_{n=1}^{m} x_{\sigma(n)} - x\right| &= \left|\sum_{n \in \sigma(\{1,\ldots,m\})} x_n - x\right| = \left|\sum_{n=1}^{M_0} x_n - x + \sum_{n=M_0+1}^{p} x_n\right| \\ &\leq \left|\sum_{n=1}^{M_0} x_n - x\right| + \left|\sum_{n=M_0+1}^{p} x_n\right| < \epsilon/2 + \epsilon/2 = \epsilon\end{aligned}$$

    $\implies \lim_{m\to\infty} \sum_{n=1}^{m} x_{\sigma(n)} = x \implies \sum_{n=1}^{\infty} x_{\sigma(n)} = x$

5. Continuous functions

- cluster points of sets
- limits of functions and sequential properties
- left and right limits
- continuous functions
- operations that preserves continuity
- extreme value theorem
- intermediate value theorem
- uniform and Lipschitz continuity

Cluster points of sets

Definition 5.1 Let $S \subseteq \mathbf{R}$. We say that the point $c \in \mathbf{R}$ is a **cluster point** of S if for all $\delta > 0$, we have $(c-\delta, c+\delta) \cap S \setminus \{c\} \neq \emptyset$, *i.e.*, for all $\delta > 0$, there exists some $x \in S$, such that $0 < |x - c| < \delta$.

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examples:

- $S = \{1/n \mid n \in \mathbf{N}\}$ has a cluster point $c = 0$
- $S = (0, 1)$ has a set of cluster points given by $[0, 1]$
- $S = \mathbf{Q}$ has a set of cluster points given by $\mathbf{R}$
- $S = \{0\}$ has no cluster points
- $S = \mathbf{Z}$ has no cluster points

Theorem 5.2 Let $S \subseteq \mathbf{R}$. Then c is a cluster point of S if and only if there exists a sequence $(x_n)_{n=1}^{\infty}$ of elements in $S \setminus \{c\}$ such that $\lim_{n\to\infty} x_n = c$.

proof:

- suppose c is a cluster point of S, then $\forall \delta > 0$, $\exists x \in S$ such that $0 < |x - c| < \delta$
  - $\forall n \in \mathbf{N}$, choose $x_n \in S$ such that $0 < |x_n - c| < \frac{1}{n}$
  - $\frac{1}{n} \to 0 \implies |x_n - c| \to 0 \implies x_n \to c$
- suppose there exists a sequence $(x_n)_{n=1}^{\infty}$ with $x_n \in S \setminus \{c\}$ for all $n \in \mathbf{N}$ such that $x_n \to c$, let $\delta > 0$
  - $x_n \to c$ with $x_n \in S \setminus \{c\} \implies \exists M \in \mathbf{N}$ such that $\forall n \geq M$, $0 < |x_n - c| < \delta$
  - choose $x = x_M$, then we have $0 < |x - c| < \delta \implies S$ has cluster point c

Limits of functions

Definition 5.3 Let $f \colon S \to \mathbf{R}$ be a function and c be a cluster point of $S \subseteq \mathbf{R}$. Suppose there exists an $L \in \mathbf{R}$, and for all $\epsilon > 0$, there exists some $\delta > 0$ such that for all $x \in S$ and $0 < |x - c| < \delta$, we have $|f(x) - L| < \epsilon$. We then say $f(x)$ **converges** to L as x goes to c, and we write

$$f(x) \to L \quad \text{as} \quad x \to c.$$

We say L is a **limit** of $f(x)$ as x goes to c, and if L is unique, we write

$$\lim_{x\to c} f(x) = L.$$

Remark 5.4 The function $f \colon S \to \mathbf{R}$ does not converge to $L \in \mathbf{R}$ as x goes to a cluster point c of S implies that there exists some $\epsilon > 0$, such that for all $\delta > 0$, there exists some $x \in S$ and $0 < |x - c| < \delta$, so that $|f(x) - L| \geq \epsilon$.

Theorem 5.5 Let $f\colon S \to \mathbf{R}$ be a function and c be a cluster point of $S \subseteq \mathbf{R}$. If $f(x) \to L_1$ and $f(x) \to L_2$ as $x \to c$, then $L_1 = L_2$.

proof: let $\epsilon > 0$

- $f(x) \to L_1$ as $x \to c \implies \exists \delta_1 > 0$ such that for all $x \in S$ and $0 < |x - c| < \delta_1$, $|f(x) - L_1| < \epsilon/2$
- $f(x) \to L_2$ as $x \to c \implies \exists \delta_2 > 0$ such that for all $x \in S$ and $0 < |x - c| < \delta_2$, $|f(x) - L_2| < \epsilon/2$
- choose $\delta = \min\{\delta_1, \delta_2\}$, then for all $x \in S$ and $0 < |x - c| < \delta$, we have

  $$|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \le |f(x) - L_1| + |f(x) - L_2| < \epsilon/2 + \epsilon/2 = \epsilon$$

  $$\implies L_1 = L_2$$

Example 5.6 Let $f(x) = ax + b$. Then, for all $c \in \mathbf{R}$, we have $\lim_{x\to c} f(x) = ac + b$.

proof: let $\epsilon > 0$, choose $\delta = \frac{\epsilon}{|a|+1}$, then for all $x \in \mathbf{R}$ and $0 < |x - c| < \delta$, we have

$$|f(x) - (ac + b)| = |(ax + b) - (ac + b)| = |a||x - c| < |a|\delta = \frac{|a|}{|a| + 1}\epsilon \le \epsilon$$

Example 5.7 Let $f\colon (0, \infty) \to \mathbf{R}$ with $f(x) = \sqrt{x}$. Then, for all $c > 0$, we have $\lim_{x\to c} f(x) = \sqrt{c}$.

proof: let $\epsilon > 0$, choose $\delta = \epsilon\sqrt{c}$, then for all $x > 0$ and $0 < |x - c| < \delta$, we have

$$|f(x) - \sqrt{c}| = |\sqrt{x} - \sqrt{c}| = \left|\frac{(\sqrt{x} - \sqrt{c})(\sqrt{x} + \sqrt{c})}{\sqrt{x} + \sqrt{c}}\right| = \left|\frac{x - c}{\sqrt{x} + \sqrt{c}}\right| \le \frac{|x - c|}{\sqrt{c}} < \frac{\delta}{\sqrt{c}} < \epsilon$$

Example 5.8 Let $f(x) = \begin{cases} 1 & x \neq 0 \\ 2 & x = 0 \end{cases}$. Then, $\lim_{x\to 0} f(x) = 1$ $(\neq f(0))$.

proof: let $\epsilon > 0$, choose $\delta = 1$, then $\forall x$ satisfies $0 < |x| < \delta$, we have $x \neq 0 \implies \forall x$ satisfies $0 < |x| < \delta$, we have $|f(x) - 1| = |1 - 1| = 0 < \epsilon$

Theorem 5.9 Let $f\colon S \to \mathbf{R}$ be a function and c be a cluster point of $S \subseteq \mathbf{R}$. Then, the following statements are equivalent:

- The function $f(x)$ converges to $L \in \mathbf{R}$ as x goes to c, *i.e.*, $\lim_{x\to c} f(x) = L$.
- For all sequences $(x_n)_{n=1}^{\infty}$ in $S \setminus \{c\}$ such that $\lim_{n\to\infty} x_n = c$, we have $\lim_{n\to\infty} f(x_n) = L$.

proof:

- suppose $\lim_{x\to c} f(x) = L$, let $\epsilon > 0$
  - $\exists\delta > 0$, such that for all $x \in S$ and $0 < |x - c| < \delta$, we have $|f(x) - L| < \epsilon$
  - $x_n \to c$, $x_n \in S \setminus \{c\} \implies \exists M \in \mathbf{N}$ such that $\forall n \geq M$, $0 < |x_n - c| < \delta \implies \forall n \geq M$, we have $|f(x_n) - L| < \epsilon$, *i.e.*, $f(x_n) \to L$
- suppose for all sequences in $S \setminus \{c\}$ s.t. $x_n \to c$, we have $f(x_n) \to L$
  - assume $\lim_{x\to c} f(x) \neq L \implies \exists\epsilon > 0$ s.t. $\forall\delta > 0$, there exists some $x \in S$ and $0 < |x - c| < \delta$, so that $|f(x) - L| \geq \epsilon$
  - choose a sequence $(x_n)_{n=1}^{\infty}$ s.t. $\forall n \in \mathbf{N}$, $x_n \in S \setminus \{c\}$, $0 < |x_n - c| < \frac{1}{n}$, and $|f(x_n) - L| \geq \epsilon$ for all $n \in \mathbf{N}$
  - however, $\frac{1}{n} \to 0 \implies x_n \to c \implies f(x_n) \to L \implies \exists M \in \mathbf{N}$ s.t. $\forall n \geq M$, $|f(x_n) - L| < \epsilon$, which is a contradiction

Theorem 5.10 For all $c \in \mathbf{R}$, we have $\lim_{x\to c} x^2 = c^2$.

proof: let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbf{R} \setminus \{c\}$ such that $x_n \to c$, then according to theorem 3.24, we have $x_n^2 \to c^2 \implies \lim_{x\to c} x^2 = c^2$ (theorem 5.9)

Theorem 5.11 The limit $\lim_{x\to 0} \sin(1/x)$ does not exist, but $\lim_{x\to 0} x\sin(1/x) = 0$.

proof:

- we first show that $\lim_{x\to 0} x\sin(1/x) = 0$: let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbf{R} \setminus \{0\}$ such that $x_n \to 0$; since $0 \leq |x_n \sin(1/x_n)| \leq |x_n|$ for all $n \in \mathbf{N}$, and $x_n \to 0$, we have $|x_n \sin(1/x_n)| \to 0 \implies \lim_{x\to 0} x\sin(1/x) = 0$
- we now show that $\lim_{x\to 0} \sin(1/x)$ does not exist:
  - choose a sequence $(x_n)_{n=1}^{\infty}$ where $x_n = \frac{2}{(2n-1)\pi}$, then we have $x_n \to 0$
  - consider the sequence $(\sin(1/x_n))_{n=1}^{\infty}$, we have

$$\sin(1/x_n) = \sin\left(\frac{(2n-1)\pi}{2}\right) = (-1)^{n+1}$$

$\implies (\sin(1/x_n))_{n=1}^{\infty}$ does not converge $\implies \lim_{x\to 0} \sin(1/x)$ does not exist

Sequential properties

Theorem 5.12 Let $f, g\colon S \to \mathbf{R}$ be functions and c be a cluster point of $S \subseteq \mathbf{R}$. Suppose $f(x) \le g(x)$ for all $x \in S$, and we have $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, then $\lim_{x\to c} f(x) \le \lim_{x\to c} g(x)$.

proof: let $(x_n)_{n=1}^\infty$ be a sequence in $S \setminus \{c\}$ such that $x_n \to c$

- $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist $\implies$ $(f(x_n))_{n=1}^\infty$ and $(g(x_n))_{n=1}^\infty$ converges
- let $f(x_n) \to L_1$, $g(x_n) \to L_2$, since $f(x) \le g(x)$ for all $x \in S$, we have $L_1 \le L_2$, $i.e.$, $\lim_{x\to c} f(x) \le \lim_{x\to c} g(x)$

similarly, we can prove the following theorems using the properties of sequences:

Theorem 5.13 Let $f\colon S \to \mathbf{R}$ be a function and c be a cluster point of $S \subseteq \mathbf{R}$. Suppose the limit $\lim_{x\to c} f(x)$ exists, and there exists $a, b \in \mathbf{R}$ such that $a \le f(x) \le b$ for all $x \in S \setminus \{c\}$, then $a \le \lim_{x\to c} f(x) \le b$.

Theorem 5.14 Let c be a cluster point of $S \subseteq \mathbf{R}$, and $f, g, h\colon S \to \mathbf{R}$ be functions such that $f(x) \le g(x) \le h(x)$ for all $x \in S\setminus\{c\}$. Suppose $\lim_{x\to c} f(x) = \lim_{x\to c} h(x)$, then $\lim_{x\to c} g(x) = \lim_{x\to c} f(x) = \lim_{x\to c} h(x)$.

Theorem 5.15 Let c be a cluster point of $S \subseteq \mathbf{R}$, and $f, g\colon S \to \mathbf{R}$ be functions such that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, we have:

- $\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$;
- $\lim_{x\to c}(f(x) \cdot g(x)) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$;
- if $\lim_{x\to c} g(x) \ne 0$ and $g(x) \ne 0$ for all $x \in S \setminus \{c\}$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)};$$

Theorem 5.16 Let c be a cluster point of $S \subseteq \mathbf{R}$ and $f\colon S \to \mathbf{R}$ be a function such that $\lim_{x\to c} f(x)$ exists, then we have $\lim_{x\to c} |f(x)| = |\lim_{x\to c} f(x)|$.

Left and right limits

Definition 5.17 Let $S \subseteq \mathbf{R}$ and $f\colon S \to \mathbf{R}$ be a function.

Suppose c is a cluster point of $S \cap (-\infty, c)$, we say $f(x)$ converges to L as $x \to c^-$, if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in S$ and $c - \delta < x < c$, we have $|f(x) - L| < \epsilon$. We call such a limit the **left limit** of f at c, denoted $\lim_{x\to c^-} f(x)$.

Suppose c is a cluster point of $S \cap (c, \infty)$, we say $f(x)$ converges to L as $x \to c^+$, if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in S$ and $c < x < c + \delta$, we have $|f(x) - L| < \epsilon$. We call such a limit the **right limit** of f at c, denoted $\lim_{x\to c^+} f(x)$.

Example 5.18 Consider the function f given by

$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0, \end{cases}$$

we have $\lim_{x\to 0^-} f(x) = 0$ and $\lim_{x\to 0^+} f(x) = 1$, even if $f(0)$ is undefined.

Continuous functions

Definition 5.19 Let $S \subseteq \mathbf{R}$ and $c \in S$. We say the function f is **continuous** at c if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in S$ and $|x - c| < \delta$, we have $|f(x) - f(c)| < \epsilon$.

We say the function f is continuous on the set U for $U \subseteq S$ if f is continuous at every point of U.

Remark 5.20 The function f is not continuous at point $c \in S$ if there exists some $\epsilon > 0$ such that for all $\delta > 0$, there exists some $x \in S$ and $|x - c| < \delta$, so that $|f(x) - f(c)| \geq \epsilon$.

Example 5.21 The function $f(x) = ax + b$ is continuous on $\mathbf{R}$.

proof: let $c \in \mathbf{R}$, $\epsilon > 0$, choose $\delta = \frac{\epsilon}{|a|+1}$, then for all $x \in \mathbf{R}$ and $|x - c| < \delta$, we have

$$|f(x) - f(c)| = |ax + b - ac - b| = |a||x - c| < |a|\delta = \frac{|a|}{|a|+1}\epsilon \leq \epsilon$$

Example 5.22 The function f given by

$$f(x) = \begin{cases} 1 & x \neq 0 \\ 2 & x = 0 \end{cases}$$

is not continuous at $c = 0$.

proof: choose $\epsilon = 1$ and let $\delta > 0$, then $x = \delta/2$ satisfies $|x| < \delta$, but

$$|f(x) - f(0)| = |1 - 0| = 1 \geq \epsilon$$

Theorem 5.23 Let $S \subseteq \mathbf{R}$ be a set, $c \in S$ be a point, and $f\colon S \to \mathbf{R}$ be a function.

- If c is not a cluster point of S, then the function f is continuous at c.
- If c is a cluster point of S, then the function f is continuous at c if and only if $\lim_{x\to c} f(x) = f(c)$.
- The function f is continuous at c if and only if for all sequences $(x_n)_{n=1}^{\infty}$ in S with $\lim_{n\to\infty} x_n = c$, we have $\lim_{n\to\infty} f(x_n) = f(c)$.

proof: to show the first statement, let $\epsilon > 0$

- $c \in S$ and c is not a cluster point of $S \implies \exists \delta > 0$ s.t. $(c - \delta, c + \delta) \cap S = \{c\}$
- then for all $x \in S$ such that $|x - c| < \delta$, we have $x = c$, and hence,

$$|f(x) - f(c)| = |f(c) - f(c)| = 0 < \epsilon$$

we now show the second statement:

- suppose f is continuous at c, let $\epsilon > 0$
  - f is continuous at $c \implies \exists \delta > 0$ such that for all $x \in S$ and $|x - c| < \delta$, we have $|f(x) - f(c)| < \epsilon$
  - then $\forall x \in S$ s.t. $0 < |x - c| < \delta$, $|f(x) - f(c)| < \epsilon \implies \lim_{x\to c} f(x) = f(c)$

- suppose $\lim_{x \to c} f(x) = f(c)$, let $\epsilon > 0$
  - $f(x) \to f(c)$ as $x \to c \implies \exists \delta > 0$ such that for all $x \in S$ and $0 < |x - c| < \delta$, we have $|f(x) - f(c)| < \epsilon$
  - then for all $x \in S$ such that $|x - c| < \delta$: if $x = c$, we have

$$|f(x) - f(c)| = |f(c) - f(c)| = 0 < \epsilon$$

    if $x \neq c$, we have $0 < |x - c| < \delta \implies |f(x) - f(c)| < \epsilon$
  - put together, we conclude that the function f is continuous at c

we now show the third statement

- suppose f is continuous at c, let $(x_n)_{n=1}^{\infty}$ be a sequence in S, $x_n \to c$, let $\epsilon > 0$
  - f is continuous at $c \implies \exists \delta > 0$ such that for all $x \in S$ and $|x - c| < \delta$, we have $|f(x) - f(c)| < \epsilon$
  - $x_n \to c \implies \exists M \in \mathbf{N}$ such that $\forall n \geq M$, $|x_n - c| < \delta \implies \forall n \geq M$, $|f(x_n) - f(c)| < \epsilon \implies (f(x_n))_{n=1}^{\infty} \to f(c)$
- suppose for all $(x_n)_{n=1}^{\infty}$ in S such that $x_n \to c$, we have $f(x_n) \to f(c)$
  - assume f is not continuous at $c \implies \exists \epsilon > 0, \forall \delta > 0, \exists x \in S$ such that $|x - c| < \delta$, but $|f(x) - f(c)| \geq \epsilon$
  - choose $x_n \in S$ such that $\forall n \in \mathbf{N}$, $0 \leq |x_n - c| < \frac{1}{n}$ but $|f(x_n) - f(x)| \geq \epsilon$
  - $\frac{1}{n} \to 0 \implies x_n \to c \implies f(x_n) \to f(c) \implies \exists M \in \mathbf{N}$ such that $\forall n \geq M$, $|f(x_n) - f(c)| < \epsilon$, which is a contradiction

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Theorem 5.24 The functions $\sin x$ and $\cos x$ are continuous functions on $\mathbf{R}$.

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proof:

- recall the following properties of $\sin x$ and $\cos x$ for all $x \in \mathbf{R}$:
  - $\sin^2(x) + \cos^2(x) = 1 \implies |\sin x| \leq 1$ and $|\cos x| \leq 1$
  - $|\sin x| \leq |x|$
  - $\sin(a + b) = \cos(a)\sin(b) + \sin(a)\cos(b)$
  - $\sin(a) - \sin(b) = 2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)$
- we first show that $\sin x$ is continuous, let $c \in \mathbf{R}$, let $\epsilon > 0$, choose $\delta = \epsilon$, then for all $x \in \mathbf{R}$ such that $|x - c| < \delta$, we have

$$|\sin x - \sin c| = \left|2\sin\left(\frac{x-c}{2}\right)\cos\left(\frac{x+c}{2}\right)\right| \leq 2\left|\sin\left(\frac{x-c}{2}\right)\right| \leq 2\frac{|x-c|}{2} = |x - c| < \epsilon$$

- we now show that $\cos x$ is continuous, let $c \in \mathbf{R}$, let $(x_n)_{n=1}^{\infty}$ be a sequence with $x_n \to c$, then we have $x_n + \frac{\pi}{2} \to c + \frac{\pi}{2}$, and hence,

$$\lim_{n\to\infty} \cos x_n = \lim_{n\to\infty} \sin\left(x_n + \frac{\pi}{2}\right) = \sin\left(c + \frac{\pi}{2}\right) = \cos c$$

Theorem 5.25 *Dirichlet function.* The Dirichlet function given by

$$f(x) = \begin{cases} 1 & x \in \mathbf{Q} \\ 0 & x \notin \mathbf{Q} \end{cases}$$

is not continuous on all of $\mathbf{R}$.

proof: let $c \in \mathbf{R}$

- if $c \in \mathbf{Q}$, then for all $n \in \mathbf{N}$, there exists $x_n \notin \mathbf{Q}$ such that $c < x_n < c + \frac{1}{n}$; $\frac{1}{n} \to 0 \implies x_n \to c$, however,

$$\lim_{n\to\infty} f(x_n) = 0 \neq f(c) = 1$$

$\implies (f(x_n))_{n=1}^{\infty}$ does not converge to $f(c)$

- if $c \notin \mathbf{Q}$, then for all $n \in \mathbf{N}$, there exists $x_n \in \mathbf{Q}$ such that $c < x_n < c + \frac{1}{n}$; $\frac{1}{n} \to 0 \implies x_n \to c$, however,

$$\lim_{n\to\infty} f(x_n) = 1 \neq f(c) = 0$$

$\implies (f(x_n))_{n=1}^{\infty}$ does not converge to $f(c)$

Operations that preserves continuity

Theorem 5.26 Let $f, g\colon S \to \mathbf{R}$ be functions on $S \subseteq \mathbf{R}$ and are continuous at $c \in S$.

- The function $f + g$ is continuous at c.
- The function $f \cdot g$ is continuous at c.
- If $g(x) \neq 0$ for all $x \in S$, then the function f/g is continuous at c.

proof: we show that the function $f + g$ is continuous at c, the other two statements can be proved similarly; let $(x_n)_{n=1}^{\infty}$ be a sequence in S with $x_n \to c$

- f is continuous at $c \implies \lim_{n\to\infty} f(x_n) = f(c)$
- g is continuous at $c \implies \lim_{n\to\infty} g(x_n) = g(c)$
- hence, $\lim_{n\to\infty}(f(x_n) + g(x_n)) = f(c) + g(c) \implies f + g$ is continuous at c

Theorem 5.27 Let $f\colon B \to \mathbf{R}$ and $g\colon A \to B$ be functions on $A, B \subseteq \mathbf{R}$. If g is continuous at $c \in A$ and f is continuous at $g(c) \in B$, then $f \circ g$ is continuous at c.

proof: let $(x_n)_{n=1}^{\infty}$ be a sequence in A and $x_n \to c \implies g(x_n) \to g(c) \implies f(g(x_n)) \to f(g(c)) \implies f \circ g$ is continuous at c

Theorem 5.28 Let f be a polynomial function of the form

$$f(x) = a_p x^p + \cdots + a_1 x + a_0.$$

Then, the function f is continuous on $\mathbf{R}$.

proof: let $c \in \mathbf{R}$, let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbf{R}$ and $x_n \to c$, then we have

$$\begin{aligned}
\lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} (a_p x_n^p + \cdots + a_1 x_n + a_0) \\
&= a_p \lim_{n\to\infty} x_n^p + \cdots + a_1 \lim_{n\to\infty} x_n + a_0 \\
&= a_p c^p + \cdots + a_1 c + a_0 = f(c)
\end{aligned}$$

Example 5.29 Theorems 5.26 and 5.27 allows us to show that some given function is continuous without a huge $\epsilon - \delta$ proof, for example:

- The function $1/x^2$ is continuous on $(0, \infty)$, since x^2 is continuous on $(0, \infty)$.
- The function $(\cos(1/x^2))^2$ is continuous on $(0, \infty)$, since $\cos x$ is continuous on $\mathbf{R}$, and x^2 is continuous on $(0, \infty)$.

Extreme value theorem

Definition 5.30 A function $f \colon S \to \mathbf{R}$ is **bounded** if there exists some $B \geq 0$ such that for all $x \in S$, we have $|f(x)| \leq B$.

Theorem 5.31 If the function $f \colon [a, b] \to \mathbf{R}$ is continuous then f is bounded.

proof:

- suppose f is unbounded, then $\forall B \geq 0$, $\exists x \in [a, b]$ such that $|f(x)| > B$
- let $(x_n)_{n=1}^{\infty}$ be a sequence in $[a, b]$ such that for all $n \in \mathbf{N}$, $|f(x_n)| > n$
- $(x_n)_{n=1}^{\infty}$ is in $[a, b]$ $\implies$ $(x_n)_{n=1}^{\infty}$ is bounded $\implies$ there exists a subsequence $(x_{n_i})_{i=1}^{\infty}$ (theorem 3.37) that converges to $c \in \mathbf{R}$
- $a \leq x_n \leq b \implies a \leq x_{n_i} \leq b \implies c \in [a, b]$
- f is continuous on $[a, b]$ $\implies$ $f(x_{n_i}) \to f(c)$ $\implies$ $(f(x_{n_i}))_{i=1}^{\infty}$ is bounded
- however, $|f(x_{n_i})| > n_i \implies (n_i)_{i=1}^{\infty}$ is bounded, which is a contradiction

Definition 5.32 Let $f\colon S \to \mathbf{R}$ be a function. We say the function f achieves an **absolute minimum** at c if $f(x) \geq f(c)$ for all $x \in S$. We say the function f achieves an **absolute maximum** at d if $f(x) \leq f(d)$ for all $x \in S$.

Theorem 5.33 *Extreme value theorem.* Let $f\colon [a,b] \to \mathbf{R}$ be a function on a closed, bounded interval $[a,b]$. If the function f is continuous on $[a,b]$, then f achieves absolute maximum and absolute minimum on $[a,b]$.

proof: we show the case for absolute maximum

- f is continuous on $[a,b] \implies f$ is bounded $\implies$ the set $E = \{f(x) \mid x \in [a,b]\}$ is bounded $\implies \sup E \in \mathbf{R}$ exists
- $\sup E$ is the supremum of $\{f(x) \mid x \in [a,b]\} \implies \forall x \in [a,b],\ f(x) \leq \sup E$, and, there exists some sequence $(f(x_n))_{n=1}^{\infty}$ with $x_n \in [a,b]$ such that $f(x_n) \to \sup E$
- $(x_n)_{n=1}^{\infty}$ is in $[a,b] \implies$ there exists a subsequence $(x_{n_i})_{i=1}^{\infty}$ such that $x_{n_i} \to d$ and $d \in [a,b] \implies f(x_{n_i}) \to f(d)$ (since f is continuous)
- $f(x_n) \to \sup E \implies f(x_{n_i}) \to \sup E \implies \sup E = f(d) \implies$ there exists a point $d \in [a,b]$ such that $f(x) \leq f(d)$ for all $x \in [a,b]$

Remark 5.34 To apply the extreme value theorem, the function f has to be continuous on a closed, bounded interval.

If the function $f\colon [a,b] \to \mathbf{R}$ is not continuous, consider the function given by

$$f(x) = \begin{cases} \frac{1}{2} & x = 0 \text{ or } x = 1 \\ x & x \in (0,1), \end{cases}$$

which neither achieves an absolute maximum nor an absolute minimum on $[0,1]$.

If the function $f\colon S \to \mathbf{R}$ is continuous but S not closed and bounded, consider the function given by

$$f(x) = \frac{1}{x} - \frac{1}{1-x}, \quad S = (0,1),$$

which neither achieves an absolute maximum nor an absolute minimum on $[0,1]$.

Intermediate value theorem

Theorem 5.35 Let $f\colon [a,b] \to \mathbf{R}$ be a continuous function. If $f(a) < 0$ and $f(b) > 0$, then there exists some $c \in (a,b)$ such that $f(c) = 0$.

proof: let $a_1 = a$, $b_1 = b$, for all $n \in \mathbf{N}$, given a_n and b_n, define a_{n+1} and b_{n+1} as:

- $a_{n+1} = a_n$, $b_{n+1} = \frac{a_n+b_n}{2}$, if $f\left(\frac{a_n+b_n}{2}\right) \geq 0$
- $a_{n+1} = \frac{a_n+b_n}{2}$, $b_{n+1} = b_n$, if $f\left(\frac{a_n+b_n}{2}\right) < 0$

then the sequences $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ has the following properties:

- $a \leq a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \leq b$ for all $n \in \mathbf{N}$ $\implies$ $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ are monotone and bounded $\implies$ $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ converge, let $a_n \to c$, $b_n \to d$
- $f(a_n) \leq 0$, $f(b_n) \geq 0$ for all $n \in \mathbf{N}$, since f is continuous, $c, d \in [a,b]$ $\implies$ $\lim_{n\to\infty} f(a_n) = f(c) \leq 0$ and $\lim_{n\to\infty} f(b_n) = f(d) \geq 0$
- $b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2} = \frac{b_{n-1} - a_{n-1}}{2^2} = \cdots = \frac{b-a}{2^n}$ $\implies$ $b_n - a_n = \frac{1}{2^{n-1}}(b-a)$ $\implies$ $\lim_{n\to\infty}(b_n - a_n) = \lim_{n\to\infty} \frac{1}{2^{n-1}}(b-a) = 0 = \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n$ $\implies$ $\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n$ $\implies$ $c = d$

put together, we have $f(c) \leq 0$, $f(d) \geq 0$, and $f(c) = f(d)$ $\implies$ $f(c) = f(d) = 0$ $\implies$ $\exists c \in (a,b)$ such that $f(c) = 0$

Theorem 5.36 *Bolzano's intermediate value theorem.* Let $f\colon [a,b] \to \mathbf{R}$ be a continuous function. Suppose $y \in \mathbf{R}$ such that $f(a) < y < f(b)$ or $f(b) < y < f(a)$, then there exists a $c \in (a,b)$ such that $f(c) = y$.

proof: we consider the case for $f(a) < y < f(b)$, the other case is similar

- let $g\colon [a,b] \to \mathbf{R}$ be a function given by $g(x) = f(x) - y$, then g is continuous on $[a,b]$ (theorem 5.26)
- $f(a) < y < f(b)$ $\implies$ $g(a) = f(a) - y < 0$, $g(b) = f(b) - y > 0$ $\implies$ $\exists c \in (a,b)$ such that $g(c) = f(c) - y = 0$ (theorem 5.35) $\implies$ $\exists c \in (a,b)$ such that $f(c) = y$

Theorem 5.37 Let $f\colon [a,b] \to \mathbf{R}$ be a continuous function. Suppose the function f achieves absolute minimum at $c \in [a,b]$, and achieves absolute maximum at $d \in [a,b]$. Then, we have $f([a,b]) = [f(c), f(d)]$, $i.e.$, every value between the absolute minimum value and the absolute maximum value is achieved.

proof:

- according to theorem 5.33, we have $f([a,b]) \subseteq [f(c), f(d)]$
- according to theorem 5.36, we have $[f(c), f(d)] \subseteq f([c,d]) \subseteq f([a,b])$
- hence, $f([a,b]) = [f(c), f(d)]$

Remark 5.38 Similarly, theorem 5.36 is false if the function f is not continuous.

Example 5.39 The polynomial given by $f(x) = x^{2021} + x^{2020} + 9.03x + 1$ has at least one real root.

proof: we have $f(0) = 1 > 0$ and $f(-1) = -8.03 < 0$, hence, by theorem 5.36, there exists some $c \in (-1, 0)$ such that $f(c) = 0$

Uniform continuity

Example 5.40 The function $f(x) = \frac{1}{x}$ is continuous on $(0, 1)$.

proof: let $c \in (0, 1)$ and $\epsilon > 0$, choose $\delta = \min\left\{\frac{c}{2}, \frac{c^2}{2}\epsilon\right\}$, then $\forall x \in (0, 1)$ such that $|x - c| < \delta$, we have

- $||x| - |c|| \leq |x - c| < \delta \leq \frac{c}{2} \implies -\frac{c}{2} < |x| - c \implies \frac{1}{|x|} < \frac{2}{c}$
- hence, $\left|\frac{1}{x} - \frac{1}{c}\right| = \frac{|x-c|}{|x|c} < \frac{\delta}{|x|c} < \frac{2\delta}{c^2} \leq \frac{2}{c^2} \cdot \frac{c^2}{2}\epsilon = \epsilon$

Remark 5.41 Example 5.40 shows that in the definition of function continuity, the number δ can depend on both the number ϵ and the point c.

Definition 5.42 Let $f\colon S \to \mathbf{R}$ be a function. We say the function f is **uniformly continuous** on S if for all $\epsilon > 0$, there exists some $\delta > 0$ such that for all $x, c \in S$ and $|x - c| < \delta$, we have $|f(x) - f(c)| < \epsilon$.

Remark 5.43 In the definition of uniform continuity, the number δ only depends on ϵ.

Example 5.44 The function $f(x) = x^2$ is uniformly continuous on $[0, 1]$.

proof: let $\epsilon > 0$, choose $\delta = \frac{\epsilon}{2}$, then for all $x, c \in [0, 1]$ and $|x - c| < \delta$, we have $|x + c| \leq 2$, and hence,

$$|f(x) - f(c)| = |x^2 - c^2| = |x + c||x - c| < |x + c|\delta \leq 2\delta = 2 \cdot \epsilon = \epsilon$$

Remark 5.45 Let $f\colon S \to \mathbf{R}$ be a function. We say the function f is not uniformly continuous on S if there exists some $\epsilon > 0$ such that for all $\delta > 0$, there exists some $x, c \in S$ and $|x - c| < \delta$ so that $|f(x) - f(c)| \geq \epsilon$.

Example 5.46 The function $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1)$.

proof: choose $\epsilon = 2$, let $\delta > 0$, choose $c = \min\left\{\delta, \frac{1}{2}\right\}$, $x = \frac{c}{2}$, then we have

- $x, c \in (0, 1)$ and $|x - c| = \frac{c}{2} \leq \frac{\delta}{2} < \delta$
- $\left|\frac{1}{x} - \frac{1}{c}\right| = \frac{|x-c|}{|x||c|} = \frac{c}{2} \cdot \frac{2}{c^2} = \frac{1}{c} \geq 2 = \epsilon$

Example 5.47 The function given by $f(x) = x^2$ is not uniformly continuous on $\mathbf{R}$.

proof: choose $\epsilon = 2$, let $\delta > 0$, choose $c = \frac{2}{\delta}$, $x = c + \frac{\delta}{2}$, then we have

- $x, c \in \mathbf{R}$ and $|x - c| = \frac{\delta}{2} < \delta$
- $|x^2 - c^2| = |x + c||x - c| = (2c + \frac{\delta}{2}) \cdot \frac{\delta}{2} = (\frac{4}{\delta} + \frac{\delta}{2}) \cdot \frac{\delta}{2} = 2 + \frac{\delta^2}{4} \geq 2 = \epsilon$

Theorem 5.48 Let $f\colon [a, b] \to \mathbf{R}$ be a function. Then, the function f is continuous on $[a, b]$ if and only if f is uniformly continuous on $[a, b]$.

proof:

- suppose f is uniformly continuous on $[a, b]$: let $c \in [a, b]$, $\epsilon > 0$, then according to uniform continuity, $\exists \delta > 0$ such that for all $x \in [a, b]$ and $|x - c| < \delta$, we have $|f(x) - f(c)| < \epsilon$
- suppose f is continuous on $[a, b]$
  - assume f is not uniformly continuous on $[a, b]$, then $\exists \epsilon > 0$ such that $\forall \delta > 0$, there exists $x, c \in [a, b]$ such that $|x - c| < \delta$ but $|f(x) - f(c)| \geq \epsilon$

- choose sequences $(x_n)_{n=1}^{\infty}$ and $(c_n)_{n=1}^{\infty}$ such that for all $n \in \mathbf{N}$, $x_n, c_n \in [a, b]$, $|x_n - c_n| < \frac{1}{n}$, but $|f(x_n) - f(c_n)| \geq \epsilon$
- since $x_n \in [a, b]$ for all $n \in \mathbf{N}$, there exists a subsequence $(x_{n_i})_{i=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that $x_{n_i} \to c$ and $c \in [a, b]$ (theorem 3.37)
- take subsequence $(c_{n_i})_{i=1}^{\infty}$ of $(c_n)_{n=1}^{\infty}$ according to the indexes n_i of $(x_{n_i})_{i=1}^{\infty}$, then $c_{n_i} \in [a, b]$ for all $n \in \mathbf{N} \implies$ there exists a subsequence $\left(c_{n_{i_j}}\right)_{j=1}^{\infty}$ such that $c_{n_{i_j}} \to d$ and $d \in [a, b]$
- take subsequence $\left(x_{n_{i_j}}\right)_{j=1}^{\infty}$ of $(x_{n_i})_{i=1}^{\infty}$ according to the indexes n_{i_j} of $\left(c_{n_{i_j}}\right)_{j=1}^{\infty}$, then $x_{n_{i_j}} \to c$ since $x_{n_i} \to c$
- $0 \leq |x_{n_{i_j}} - c_{n_{i_j}}| < \frac{1}{n_{i_j}}$ and $\frac{1}{n_{i_j}} \to 0 \implies \lim_{j\to\infty} |x_{n_{i_j}} - c_{n_{i_j}}| = 0 \implies \lim_{j\to\infty} x_{n_{i_j}} = \lim_{j\to\infty} c_{n_{i_j}} \implies c = d$
- since f is continuous on $[a, b]$ and $x_{n_{i_j}} \to c$, $c_{n_{i_j}} \to c$, we have

$$\begin{aligned} & \lim_{j\to\infty} f(x_{n_{i_j}}) = \lim_{j\to\infty} f(c_{n_{i_j}}) = f(c) \\ \implies \quad & 0 = |f(c) - f(c)| = \lim_{j\to\infty} |f(x_{n_{i_j}}) - f(c_{n_{i_j}})| \geq \epsilon, \end{aligned}$$

which is a contradiction

Lipschitz continuity

Definition 5.49 Let $f\colon S \to \mathbf{R}$ be a function. We say the function f is **Lipschitz continuous** on S if there exists some $K \geq 0$ such that for all $x, y \in S$, we have $|f(x) - f(y)| \leq K|x - y|$.

Remark 5.50 Geometrically, the function f is Lipschitz continuous if and only if all lines intersects the graph of f in at least two distinct points has slope in absolute value less than or equal to K.

Theorem 5.51 Let $f\colon S \to \mathbf{R}$ be a function. If the function f is Lipschitz continuous, then f is uniformly continuous.

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proof: let $\epsilon > 0$

- f is Lipschitz continuous $\implies \exists K \geq 0$ such that for all $x, y \in S$, we have $|f(x) - f(y)| \leq K|x - y|$
- choose $\delta = \epsilon/(K + 1)$, then for all $x, y \in S$ and $|x - y| < \delta$, we have

$$|f(x) - f(y)| \leq K|x - y| < K\delta = \frac{K}{K + 1}\epsilon < \epsilon$$

Example 5.52 The function $f(x) = \sqrt{x}$ is Lipschitz continuous on $[1, \infty)$, but is not Lipschitz continuous on $[0, \infty)$.

proof:

- consider the function $f\colon [1, \infty) \to \mathbf{R}$ given by $f(x) = \sqrt{x}$, then $\forall x, y \in [1, \infty)$:
  - $x \geq 1$, $y \geq 1 \implies \sqrt{x} + \sqrt{y} \geq 2$
  - hence,

$$|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}} \leq \frac{1}{2}|x-y|$$

  $\implies f$ is Lipschitz continuous with $K = 1/2$

- consider the function $g\colon [0, \infty) \to \mathbf{R}$ given by $g(x) = \sqrt{x}$, let $K \geq 0$, choose $x = 0$, $y = \frac{1}{K^2+1}$, then

$$\left|\frac{f(x)-f(y)}{x-y}\right| = \left|\frac{\sqrt{x}-\sqrt{y}}{x-y}\right| = \frac{\sqrt{y}}{y} = \frac{1}{\sqrt{y}} = \sqrt{K^2+1} > \sqrt{K^2} = K$$

$\implies |f(x) - f(y)| > K|x-y|$

6. Derivative

- definition and basic properties
- differentiation rules
- Rolle's theorem and mean value theorem
- Taylor's theorem

Derivative of functions

Definition 6.1 Let I be an interval, let $f\colon I \to \mathbf{R}$ be a function, and let $c \in I$. We say the function f is **differentiable** at c if the limit

$$L = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists. We call L the **derivative** of f at c, and we write $f'(c) = L$.

If f is differentiable at all $c \in I$, then we say the function f is differentiable, and we write f' or $\frac{df}{dx}$ for the function $f'(x)$, $x \in I$.

Example 6.2 Consider the function $f(x) = ax + b$, then $f'(c) = a$ for all $c \in \mathbf{R}$.

proof: let $x, c \in \mathbf{R}$, then we have

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{ax + b - (ac + b)}{x - c} = \lim_{x \to c} \frac{a(x - c)}{x - c} = \lim_{x \to c} a = a$$

Example 6.3 Consider the function $f(x) = x^2$, then $f'(c) = 2c$ for all $c \in \mathbf{R}$.

proof: let $x, c \in \mathbf{R}$, then we have

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{x^2 - c^2}{x - c} = \lim_{x \to c} \frac{(x + c)(x - c)}{x - c} = \lim_{x \to c}(x + c) = 2c$$

Theorem 6.4 Suppose the function $f \colon I \to \mathbf{R}$ is differentiable at $c \in I$, then f is continuous at c.

proof: f is differentiable at $c \in I \implies$ the limit $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists, hence,

$$\lim_{x \to c} f(x) = \lim_{x \to c} \left(\frac{f(x) - f(c)}{x - c}(x - c) + f(c) \right) = f'(c) \cdot 0 + f(c) = f(c)$$

Remark 6.5 The converse of theorem 6.4 does not hold.

Example 6.6 The function $f(x) = |x|$ is not differentiable at 0.

proof: let $(x_n)_{n=1}^{\infty}$ be a sequence with $x_n = \frac{(-1)^n}{n}$ for all $n \in \mathbf{N}$

- $0 \leq \left| \frac{(-1)^n}{n} \right| \leq \frac{1}{n}$ and $\frac{1}{n} \to 0 \implies x_n \to 0$
- consider the sequence $\left(\frac{f(x_n) - f(0)}{x_n - 0} \right)_{n=1}^{\infty}$, we have

$$\frac{f(x_n) - f(0)}{x_n - 0} = \frac{|x_n|}{x_n} = \frac{\left| \frac{(-1)^n}{n} \right|}{\frac{(-1)^n}{n}} = (-1)^n$$

- $\lim_{n \to \infty} (-1)^n$ does not exist $\implies \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$ does not exist

Remark 6.7 There exist functions that are continuous but nowhere differentiable.

Differentiation rules

Theorem 6.8 Let I be an interval, let $f\colon I \to \mathbf{R}$ and $g\colon I \to \mathbf{R}$ be differentiable functions at $c \in I$.

- *Linearity.* Let $\alpha \in \mathbf{R}$. Define $h(x) = \alpha f(x) + g(x)$, then $h'(c) = \alpha f'(c) + g'(c)$.
- *Product rule.* Define $h(x) = f(x)g(x)$, then $h'(c) = f'(c)g(c) + f(c)g'(c)$.
- *Quotient rule.* If $g(x) \neq 0$ for all $x \in I$, define $h(x) = f(x)/g(x)$, then

$$h'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}.$$

proof: f, g differentiable at $c \implies \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$, $\lim_{x\to c} \frac{g(x)-g(c)}{x-c}$ exists, and f, g continuous at $c \implies \lim_{x\to c} f(x) = f(c)$, $\lim_{x\to c} g(x) = g(c)$

- if $h(x) = \alpha f(x) + g(c)$, then we have

$$\begin{aligned}
\lim_{x\to c} \frac{h(x)-h(c)}{x-c} &= \lim_{x\to c} \frac{\alpha f(x) + g(x) - \alpha f(c) - g(c)}{x-c} \\
&= \alpha \lim_{x\to c} \frac{f(x)-f(c)}{x-c} + \lim_{x\to c} \frac{g(x)-g(c)}{x-c} = \alpha f'(c) + g'(c)
\end{aligned}$$

- if $h(x) = f(x)g(x)$, then we have

$$\begin{aligned}
\lim_{x\to c} \frac{h(x)-h(c)}{x-c} &= \lim_{x\to c} \frac{f(x)g(x) - f(c)g(c)}{x-c} \\
&= \lim_{x\to c} \frac{f(x)g(x) - f(c)g(c) + f(x)g(c) - f(x)g(c)}{x-c} \\
&= \lim_{x\to c} \frac{g(c)(f(x)-f(c)) + f(x)(g(x)-g(c))}{x-c} \\
&= g(c) \lim_{x\to c} \frac{f(x)-f(c)}{x-c} + \lim_{x\to c} f(x) \frac{g(x)-g(c)}{x-c} = f'(c)g(c) + f(c)g'(c)
\end{aligned}$$

- if $h(x) = f(x)/g(x)$, then we have

$$\begin{aligned}
\lim_{x\to c} \frac{h(x)-h(c)}{x-c} &= \lim_{x\to c} \frac{f(x)/g(x) - f(c)/g(c)}{x-c} = \lim_{x\to c} \frac{1}{g(x)g(c)} \frac{f(x)g(c) - f(c)g(x)}{x-c} \\
&= \lim_{x\to c} \frac{1}{g(x)g(c)} \frac{f(x)g(c) - f(c)g(x) + f(x)g(x) - f(x)g(x)}{x-c} \\
&= \lim_{x\to c} \frac{1}{g(x)g(c)} \frac{g(x)(f(x)-f(c)) - f(x)(g(x)-g(c))}{x-c} \\
&= \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}
\end{aligned}$$

Theorem 6.9 *Chain rule.* Let I_1, I_2 be two intervals. Let $g\colon I_1 \to \mathbf{R}$ be differentiable at $c \in I_1$ and $f\colon I_2 \to \mathbf{R}$ be differentiable at $g(c)$. Define $h\colon I_1 \to \mathbf{R}$ by $h = f \circ g$, then h is differentiable at c, and

$$h'(c) = f'(g(c))g'(c).$$

proof: let $d = g(c)$

- define the following functions:

$$u(y) = \begin{cases} \frac{f(y)-f(d)}{y-d} & y \neq d \\ f'(d) & y = d \end{cases} \quad \text{and} \quad v(x) = \begin{cases} \frac{g(x)-g(c)}{x-c} & x \neq c \\ g'(c) & x = c, \end{cases}$$

  then we have

$$\lim_{y\to d} u(y) = \lim_{y\to d} \frac{f(y)-f(d)}{y-d} = f'(d) = u(d)$$
$$\lim_{x\to c} v(x) = \lim_{x\to c} \frac{g(x)-g(c)}{x-c} = g'(c) = v(c),$$

  $i.e.$, u is continuous at d, v is continuous at c

- note that $f(y) - f(d) = u(y)(y-d)$ and $g(x) - d = v(x)(x-c)$, we have

$$h(x) - h(c) = f(g(x)) - f(d) = u(g(x))(g(x)-d) = u(g(x))v(x)(x-c)$$

- put together, we have

$$\lim_{x\to c} \frac{h(x)-h(c)}{x-c} = \lim_{x\to c} u(g(x))v(x) = u(g(c))v(c) = f'(g(c))g'(c)$$

Rolle's theorem

Definition 6.10 Let $f\colon S \to \mathbf{R}$ with $S \subseteq \mathbf{R}$.

The function f is said to have a **relative maximum** at $c \in S$ if there exists some $\delta > 0$ such that for all $x \in S$ and $|x - c| < \delta$, we have $f(x) \leq f(c)$.

The function f is said to have a **relative minimum** at $c \in S$ if there exists some $\delta > 0$ such that for all $x \in S$ and $|x - c| < \delta$, we have $f(x) \geq f(c)$.

Theorem 6.11 If the function $f\colon [a,b] \to \mathbf{R}$ has a relative maximum or minimum at $c \in (a,b)$ and f is differentiable at c, then $f'(c) = 0$.

proof: we show the case for c being a relative maximum point

- $c \in (a,b)$ is an relative maximum point $\implies$ $\exists \delta > 0$ such that for all $x \in [a,b]$ and $|x - c| < \delta$, we have $f(x) \leq f(c)$
- let $(x_n)_{n=1}^{\infty}$ be a sequence with $x_n = c - \frac{\delta}{2n}$ for all $n \in \mathbf{N}$, then we have $x_n < c$, $x_n \to c$, and $|x_n - c| < \delta$ for all $n \in \mathbf{N}$ $\implies$ $f'(c) = \lim_{n\to\infty} \frac{f(x_n)-f(c)}{x_n - c} \geq 0$
- let $(y_n)_{n=1}^{\infty}$ be a sequence with $y_n = c + \frac{\delta}{2n}$ for all $n \in \mathbf{N}$, then we have $y_n > c$, $y_n \to c$, and $|y_n - c| < \delta$ for all $n \in \mathbf{N}$ $\implies$ $f'(c) = \lim_{n\to\infty} \frac{f(y_n)-f(c)}{y_n - c} \leq 0$

Remark 6.12 In theorem 6.11, the function f does not necessarily have to be defined on a closed interval, but the point c where the relative extremum is achieved has to be on the open interval (a,b).

Remark 6.13 Absolute extremum is a special case of relative extremum.

Theorem 6.14 *Rolle.* Let the function $f\colon [a,b] \to \mathbf{R}$ be continuous and differentiable on (a,b). If $f(a) = f(b)$, then there exists some $c \in (a,b)$ such that $f'(c) = 0$.

proof: let $f(a) = f(b) = K$; f is continuous on $[a,b]$ $\implies$ there exists an absolute maximum point $c_1 \in [a,b]$ and an absolute minimum point $c_2 \in [a,b]$ (theorem 5.33)

- if $c_1 > K$, then $c_1 \in (a,b)$ $\implies$ $f'(c_1) = 0$ (theorem 6.11)
- if $c_2 < K$, then $c_2 \in (a,b)$ $\implies$ $f'(c_2) = 0$ (theorem 6.11)
- if $c_1 = c_2 = K$, then $K \leq f(x) \leq K$ for all $x \in [a,b]$ $\implies$ $f(x) = K$ for all $x \in [a,b]$ $\implies$ $f'(c) = 0$ for all $c \in (a,b)$

Mean value theorem

Theorem 6.15 *Mean value theorem.* Let the function $f\colon [a,b] \to \mathbf{R}$ be continuous and differentiable on (a,b), then there exists some $c \in (a,b)$ such that

$$f(b) - f(a) = f'(c)(b-a).$$

proof:

- define $g\colon [a,b] \to \mathbf{R}$ with $g(x) = f(x) - f(b) + \frac{f(b)-f(a)}{b-a}(b-x)$
- since $g(a) = g(b) = 0$, by theorem 6.14, there exists $c \in (a,b)$ such that

$$g'(c) = 0 = f'(c) - \frac{f(b)-f(a)}{b-a} \quad \implies \quad f(b) - f(a) = f'(c)(b-a)$$

Theorem 6.16 If the function $f\colon I \to \mathbf{R}$ is differentiable and $f'(x) = 0$ for all $x \in I$, then f is constant.

proof: let $a, b \in I$ with $a < b$, then f is continuous on $[a,b]$ and differentiable on $(a,b) \implies \exists c \in (a,b)$ such that $f(b) - f(a) = f'(c)(b-a) = 0$ (since $f'(x) = 0$ for all $x \in I$) $\implies f(b) = f(a)$

Theorem 6.17 Let $f\colon I \to \mathbf{R}$ be a differentiable function.

- The function f is increasing if and only if $f'(x) \geq 0$ for all $x \in I$.
- The function f is decreasing if and only if $f'(x) \leq 0$ for all $x \in I$.

proof: we prove the first statement

- suppose $f'(x) \geq 0$ for all $x \in I$, let $a, b \in I$ with $a < b$, then f is continuous on $[a,b]$ and differentiable on $(a,b) \implies \exists c \in (a,b)$ s.t. $f(b) - f(a) = f'(c)(b-a)$ (theorem 6.15) and $f'(c) \geq 0 \implies f(b) - f(a) \geq 0 \implies f(a) \leq f(b)$
- suppose f is increasing, let $c \in I$, then we can find a sequence $(x_n)_{n=1}^{\infty}$ with either $x_n < c$ or $x_n > c$ for all $n \in \mathbf{N}$ such that $x_n \to c$
  - if $x_n < c$ for all $n \in \mathbf{N} \implies f(x_n) \leq f(c)$ for all $n \in \mathbf{N}$, and hence

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{n \to \infty} \frac{f(x_n) - f(c)}{x_n - c} \geq 0$$

  - if $x_n > c$ for all $n \in \mathbf{N} \implies f(x_n) \geq f(c)$ for all $n \in \mathbf{N}$, and hence

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{n \to \infty} \frac{f(x_n) - f(c)}{x_n - c} \geq 0$$

  in either case, we have $f'(c) \geq 0$

Taylor's theorem

Definition 6.18 We say the function $f\colon I \to \mathbf{R}$ is n-**times differentiable** on $J \subseteq I$ if $f', f'', \ldots, f^{(n)}$ exist at every point in J, where $f^{(n)}$ denotes the nth derivative of f.

Theorem 6.19 *Taylor.* Suppose the function $f\colon [a,b] \to \mathbf{R}$ is continuous and has n continuous derivatives on $[a,b]$ such that $f^{(n+1)}$ exists on (a,b). Given $x_0, x \in [a,b]$, there exists some $c \in (x_0, x)$ such that

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

We denote

$$P_n(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k \quad \text{and} \quad R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

as the n**th order Taylor polynomial** and the n**th order remainder** of f, respectively.

proof: let $x, x_0 \in [a,b]$ and $x \neq x_0$ (if $x = x_0$ then any c satisfies the theorem)

- let $M_{x,x_0} = \frac{f(x) - P_n(x)}{(x - x_0)^{n+1}}$, then we have

$$f(x) = P_n(x) + M_{x,x_0}(x - x_0)^{n+1}$$

- note that for all $0 \leq k \leq n$, we have $f^{(k)}(x_0) = P_n^{(k)}(x_0)$
- let $g(s) = f(s) - P_n(s) - M_{x,x_0}(s - x_0)^{n+1}$, then we have

$$\begin{aligned}
g(x_0) &= f(x_0) - P_n(x_0) - M_{x,x_0}(x_0 - x_0)^{n+1} = 0 \\
g'(x_0) &= f'(x_0) - P_n'(x_0) - M_{x,x_0}(n+1)(x_0 - x_0)^{n} = 0 \\
&\vdots \\
g^{(n)}(x_0) &= f^{(n)}(x_0) - P_n^{(n)}(x_0) - M_{x,x_0}(n+1)!(x_0 - x_0) = 0
\end{aligned}$$

- by theorem 6.15:

$$\begin{aligned}
g(x_0) = g(x) = 0 \quad &\Longrightarrow \quad \exists x_1 \text{ between } x_0 \text{ and } x \text{ s.t. } g'(x_1) = 0 \\
g'(x_0) = g'(x_1) = 0 \quad &\Longrightarrow \quad \exists x_2 \text{ between } x_0 \text{ and } x_1 \text{ s.t. } g''(x_2) = 0 \\
&\vdots \\
g^{(n-1)}(x_0) = g^{(n-1)}(x_{n-1}) = 0 \quad &\Longrightarrow \quad \exists x_n \text{ between } x_0 \text{ and } x_{n-1} \text{ s.t. } g^{(n)}(x_n) = 0 \\
g^{(n)}(x_0) = g^{(n)}(x_n) = 0 \quad &\Longrightarrow \quad \exists c \text{ between } x_0 \text{ and } x_n \text{ s.t. } g^{(n+1)}(c) = 0
\end{aligned}$$

- note that

$$\frac{d^{n+1}}{ds^{n+1}} M_{x,x_0}(s - x_0)^{n+1} = M_{x,x_0}(n+1)! \quad \text{and} \quad P_n^{(n+1)}(c) = 0$$

- we have the $(n+1)$-times derivative of g at c given by

$$0 = g^{(n+1)}(c) = f^{(n+1)}(c) - M_{x,x_0}(n+1)! \implies M_{x,x_0} = \frac{f^{(n+1)}(c)}{(n+1)!}$$

- hence, we have

$$\begin{aligned} f(x) &= P_n(x) + M_{x,x_0}(x - x_0)^{n+1} \\ &= P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1} \end{aligned}$$

Theorem 6.20 *Second derivative test.* Suppose the function $f\colon (a,b) \to \mathbf{R}$ has two continuous derivatives. If $x_0 \in (a,b)$ such that $f'(x_0) = 0$ and $f''(x_0) > 0$, then f has a strict relative minimum at x_0.

---

proof:

- it is easy to show that f'' is continuous and $f''(x_0) > 0 \implies$ there exists some $\delta > 0$ such that for all $c \in (x_0 - \delta, x_0 + \delta)$, we have $f''(c) > 0$
- then for all $x \in (x_0 - \delta, x_0 + \delta)$, by theorem 6.19, there exists some c_0 between x and x_0 such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(c_0)(x - x_0)^2$$

- c_0 between x and $x_0 \implies c_0 \in (x_0 - \delta, x_0 + \delta) \implies f''(c) > 0$, and since $f'(x_0) = 0$, we have

$$f(x) - f(x_0) = \frac{1}{2} f''(c_0)(x - x_0)^2 > 0 \implies f(x) > f(x_0)$$

7. Riemann integral

- Riemann sum and some useful facts
- Riemann integral of continuous functions
- properties of Riemann integral
- fundamental theorem of calculus
- integration by parts
- change of variables

Riemann sum

Definition 7.1 A **partition** $\underline{x} = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ is a finite set such that

$$a = x_0 < x_1 < \cdots < x_n = b.$$

The **norm** of $\underline{x}$, denoted $\|\underline{x}\|$, is a number defined as

$$\|\underline{x}\| = \max\{x_1 - x_0,\ x_2 - x_1,\ \ldots,\ x_n - x_{n-1}\}.$$

Definition 7.2 let $\underline{x}$ be a partition of $[a, b]$. A **tag** of $\underline{x}$ is a finite set $\underline{\xi} = \{\xi_1, \ldots, \xi_n\}$ such that

$$a = x_0 \le \xi_1 \le x_1 \le \xi_2 \le x_2 \le \cdots \le x_{n-1} \le \xi_n \le x_n = b.$$

The pair $(\underline{x}, \underline{\xi})$ is referred to as a **tagged partition**.

example: $(\underline{x}, \underline{\xi}) = (\{1, 3/2, 2, 3\},\ \{5/4, 7/4, 5/2\})$ is a tagged partition with norm

$$\|\underline{x}\| = \max\{3/2 - 1,\ 2 - 3/2,\ 3 - 2\} = 1$$

Definition 7.3 The **Riemann sum** of f corresponding to $(\underline{x}, \underline{\xi})$ is the number

$$S_f(\underline{x}, \underline{\xi}) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}).$$

Remark 7.4 For $f \in \mathcal{C}([a, b])$ that is positive, the Riemann sum $S_f(\underline{x}, \underline{\xi})$ is an approximate area under the graph of f. As $\|\underline{x}\| \to 0$, we should expect these approximate areas to converge to some number, which we **interpret** as the area under the graph of f on the interval $[a, b]$.

Some useful facts

Definition 7.5 We define the set $\mathcal{C}([a, b]) = \{f \colon [a, b] \to \mathbf{R} \mid f \text{ is continuous}\}$.

Definition 7.6 Let $f \in \mathcal{C}([a, b])$ and $\tau > 0$, we define the **modulus of continuity** of the function f as

$$w_f(\tau) = \sup\{|f(x) - f(y)| \mid |x - y| \le \tau\}.$$

Theorem 7.7 For all $f \in \mathcal{C}([a, b])$, we have $\lim_{\tau \to 0} w_f(\tau) = 0$, $i.e.$, for all $\epsilon > 0$, there exists some $\delta > 0$ such that for all $\tau < \delta$, we have $w_f(\tau) < \epsilon$.

proof: let $\epsilon > 0$

- $f \in \mathcal{C}([a, b]) \implies f$ is uniformly continuous on $[a, b] \implies \exists \delta > 0$ such that for all $x, y \in [a, b]$ and $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon/2$
- let $\tau < \delta$, then for all $x, y \in [a, b]$ and $|x - y| \le \tau$, we have $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon/2$ for all $x, y \in [a, b]$ and $|x - y| \le \tau \implies \epsilon/2$ is an upper bound of the set $\{|f(x) - f(y)| \mid |x - y| \le \tau\} \implies w_f(\tau) \le \epsilon/2 < \epsilon$

Theorem 7.8 Let $f \in \mathcal{C}([a,b])$, then $w_f(\tau)$ has the following properties:

- For all $x, y \in [a,b]$, we have $w_f(|x-y|) \geq |f(x) - f(y)|$.
- *Monotonicity.* If $\tau_1 \leq \tau_2$, then $w_f(\tau_1) \leq w_f(\tau_2)$.

Definition 7.9 Let $(\underline{x}, \underline{\xi})$ and $(\underline{x}', \underline{\xi}')$ be tagged partitions of $[a,b]$. We say $\underline{x}'$ is a **refinement** of $\underline{x}$ if $\underline{x} \subseteq \underline{x}'$.

Theorem 7.10 Let $(\underline{x}, \underline{\xi})$ and $(\underline{x}', \underline{\xi}')$ be tagged partitions of $[a,b]$ such that $\underline{x}'$ is a refinement of $\underline{x}$. If $f \in \mathcal{C}([a,b])$, then

$$|S_f(\underline{x}, \underline{\xi}) - S_f(\underline{x}', \underline{\xi}')| \leq w_f(\|\underline{x}\|)(b-a).$$

proof: let $\underline{x} = \{x_0, \ldots, x_n\}$, $\underline{\xi} = \{\xi_1, \ldots, \xi_n\}$, $\underline{x}' = \{x_0', \ldots, x_n'\}$, $\underline{\xi}' = \{\xi_1', \ldots, \xi_n'\}$

- for $i = 1, \ldots, n$, let $\underline{y}^{(i)} = \{x_q', x_{q+1}', \ldots, x_k'\}$, $\underline{\zeta}^{(i)} = \{\xi_{q+1}', \xi_{q+2}', \ldots, \xi_k'\}$ s.t.

$$x_{i-1} = x_q' < x_{q+1}' < \cdots < x_k' = x_i$$

- then for all $i = 1, \ldots, n$, we have

$$\begin{aligned}
&|f(\xi_i)(x_i - x_{i-1}) - S_f(\underline{y}^{(i)}, \underline{\zeta}^{(i)})| \\
&\quad = \left| f(\xi_i) \sum_{\ell=q+1}^{k} (x_\ell' - x_{\ell-1}') - \sum_{\ell=q+1}^{k} f(\xi_\ell')(x_\ell' - x_{\ell-1}') \right| \\
&\quad = \left| \sum_{\ell=q+1}^{k} (f(\xi_i) - f(\xi_\ell'))(x_\ell' - x_{\ell-1}') \right| \leq \sum_{\ell=q+1}^{k} |f(\xi_i) - f(\xi_\ell')|(x_\ell' - x_{\ell-1}') \\
&\quad \leq \sum_{\ell=q+1}^{k} w_f(x_i - x_{i-1})(x_\ell' - x_{\ell-1}') \leq \sum_{\ell=q+1}^{k} w_f(\|\underline{x}\|)(x_\ell' - x_{\ell-1}') \\
&\quad = w_f(\|\underline{x}\|)(x_i - x_{i-1})
\end{aligned} \tag{7.1}$$

  - the first inequality is by lemma 4.18
  - the second inequality is from $\xi_i, \xi_\ell' \in [x_{i-1}, x_i]$
  - the third inequality is by the second statement of theorem 7.8, and $\|\underline{x}\| \geq x_i - x_{i-1}$

- put together, we have

$$
\begin{aligned}
|S_f(\underline{x},\underline{\xi}) - S_f(\underline{x}',\underline{\xi}')| &= \left|\sum_{i=1}^{n}(f(\xi_i)(x_i - x_{i-1}) - S_f(\underline{y}^{(i)},\underline{\zeta}^{(i)}))\right| \\
&\le \sum_{i=1}^{n}|f(\xi_i)(x_i - x_{i-1}) - S_f(\underline{y}^{(i)},\underline{\zeta}^{(i)})| \le \sum_{i=1}^{n} w_f(\|\underline{x}\|)(x_i - x_{i-1}) \\
&= w_f(\|x\|)(b-a),
\end{aligned}
$$

  where the last inequality is by plugging in (7.1)

Theorem 7.11 Let $(\underline{x},\underline{\xi})$ and $(\underline{x}',\underline{\xi}')$ be any two tagged partitions of $[a,b]$ and $f \in \mathcal{C}([a,b])$, then

$$|S_f(\underline{x},\underline{\xi}) - S_f(\underline{x}',\underline{\xi}')| \le (w_f(\|\underline{x}\|) + w_f(\|\underline{x}'\|))(b-a).$$

---

proof: let $\underline{x}'' = \underline{x} \cup \underline{x}'$ and $\underline{\xi}''$ be a tag of $\underline{x}''$, then by theorem 7.10, we have

$$
\begin{aligned}
|S_f(\underline{x},\underline{\xi}) - S_f(\underline{x}',\underline{\xi}')| &\le |S_f(\underline{x},\underline{\xi}) - S_f(\underline{x}'',\underline{\xi}'')| + |S_f(\underline{x}'',\underline{\xi}'') - S_f(\underline{x}',\underline{\xi}')| \\
&\le (w_f(\|\underline{x}\|) + w_f(\|\underline{x}'\|))(b-a)
\end{aligned}
$$

Riemann integral of continuous functions

Theorem 7.12 Let $f \in \mathcal{C}([a,b])$, then there exists a unique number denoted $\int_a^b f(x)\,dx$ with the following property: For all sequences of tagged partitions $\left((\underline{x}^{(r)},\underline{\xi}^{(r)})\right)_{r=1}^{\infty}$ such that $\lim_{r\to\infty}\|\underline{x}^{(r)}\| = 0$, we have

$$\lim_{r\to\infty} S_f(\underline{x}^{(r)},\underline{\xi}^{(r)}) = \int_a^b f(x)\,dx.$$

---

proof: uniqueness follows immediately from uniqueness of limits of sequences of real numbers, we only need to show the existence

- let $\left((\underline{y}^{(r)},\underline{\zeta}^{(r)})\right)_{r=1}^{\infty}$ be a sequence of tagged partitions with $\lim_{r\to\infty}\|\underline{y}^{(r)}\| = 0$, we first show that $\left(S_f(\underline{y}^{(r)},\underline{\zeta}^{(r)})\right)_{r=1}^{\infty}$ is a Cauchy sequence; let $\epsilon > 0$
  - by theorem 7.7, $\exists\delta > 0$ such that for all $\tau < \delta$, $w_f(\tau) < \frac{\epsilon}{2(b-a)}$
  - $\|\underline{y}^{(r)}\| \to 0 \implies \exists M \in \mathbf{N}$ s.t. $\forall r,s \ge M$, $\|\underline{y}^{(r)}\| < \delta$, $\|\underline{y}^{(s)}\| < \delta \implies \forall r,s \ge M$, we have $w_f(\|\underline{y}^{(r)}\|) < \frac{\epsilon}{2(b-a)}$, $w_f(\|\underline{y}^{(s)}\|) < \frac{\epsilon}{2(b-a)}$

– hence, for all $r, s \geq M$, by theorem 7.11, we have

$$\begin{aligned}
&|S_f(\underline{y}^{(r)}, \underline{\zeta}^{(r)}) - S_f(\underline{y}^{(s)}, \underline{\zeta}^{(s)})| \\
&\qquad \leq (w_f(\|\underline{y}^{(r)}\|) + w_f(\|\underline{y}^{(s)}\|))(b-a) < \left(\frac{\epsilon}{2(b-a)} + \frac{\epsilon}{2(b-a)}\right)(b-a) = \epsilon
\end{aligned}$$

let $L = \lim_{r\to\infty} S_f(\underline{y}^{(r)}, \underline{\zeta}^{(r)})$ (which exists by theorem 3.45)

- let $\left((\underline{x}^{(r)}, \underline{\xi}^{(r)})\right)_{r=1}^{\infty}$ be any sequence of partitions with $\lim_{r\to\infty} \|\underline{x}^{(r)}\| = 0$, we now show that $\lim_{r\to\infty} S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) = L$
  – since $\|\underline{x}^{(r)}\| \to 0$, $\|\underline{y}^{(r)}\| \to 0$, by theorem 7.7, we have

$$\lim_{r\to\infty} (w_f(\|\underline{x}^{(r)}\|) + w_f(\|\underline{y}^{(r)}\|))(b-a) = 0$$

  – $S_f(\underline{y}^{(r)}, \underline{\zeta}^{(r)}) \to L \implies |S_f(\underline{y}^{(r)}, \underline{\zeta}^{(r)}) - L| \to 0$
  – by theorem 7.11, we have

$$\begin{aligned}
0 &\leq |S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) - L| \leq |S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) - S_f(\underline{y}^{(r)}, \underline{\zeta}^{(r)})| + |S_f(\underline{y}^{(r)}, \underline{\zeta}^{(r)}) - L| \\
&\leq (w_f(\|\underline{x}^{(r)}\|) + w_f(\|\underline{y}^{(r)}\|))(b-a) + |S_f(\underline{y}^{(r)}, \underline{\zeta}^{(r)}) - L|
\end{aligned}$$

$\implies \lim_{r\to\infty} |S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) - L| = 0$ (theorem 3.21)

Remark 7.13 Let $f \in \mathcal{C}([a,b])$. We sometimes write

$$\int_a^b f(x)\, dx = \int_a^b f.$$

By convention, we also define

$$\int_a^a f = 0 \quad \text{and} \quad \int_b^a f = -\int_a^b f.$$

Properties of Riemann integral

Theorem 7.14 *Linearity.* Let $f, g \in \mathcal{C}([a,b])$ and $\alpha \in \mathbf{R}$, then

$$\int_a^b (\alpha f + g) = \alpha \int_a^b f + \int_a^b g.$$

proof: let $\left((\underline{x}^{(r)}, \underline{\xi}^{(r)})\right)_{r=1}^{\infty}$ be a sequence of tagged partitions such that $\|\underline{x}^{(r)}\| \to 0$, then we have

$$\begin{aligned}
\int_a^b (\alpha f + g) &= \lim_{r\to\infty} S_{\alpha f+g}(\underline{x}^{(r)}, \underline{\xi}^{(r)}) \\
&= \lim_{r\to\infty} (\alpha S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) + S_g(\underline{x}^{(r)}, \underline{\xi}^{(r)})) \\
&= \alpha \lim_{r\to\infty} S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) + \lim_{r\to\infty} S_g(\underline{x}^{(r)}, \underline{\xi}^{(r)}) \\
&= \alpha \int_a^b f + \int_a^b g
\end{aligned}$$

Theorem 7.15 *Additivity.* Let $f \in \mathcal{C}([a,b])$ and $a < c < b$, then we have

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

proof:

- let $\left((\underline{y}^{(r)}, \underline{\zeta}^{(r)})\right)_{r=1}^{\infty}$ be a sequence of tagged partitions of $[a,c]$ with $\|\underline{y}^{(r)}\| \to 0$
- let $\left((\underline{z}^{(r)}, \underline{\eta}^{(r)})\right)_{r=1}^{\infty}$ be a sequence of tagged partitions of $[c,b]$ with $\|\underline{z}^{(r)}\| \to 0$
- then $\left((\underline{x}^{(r)}, \underline{\xi}^{(r)})\right)_{r=1}^{\infty}$ with $\underline{x}^{(r)} = \underline{y}^{(r)} \cup \underline{z}^{(r)}$ and $\underline{\xi}^{(r)} = \underline{\zeta}^{(r)} \cup \underline{\eta}^{(r)}$ is a sequence of tagged partitions of $[a,b]$
- $\|\underline{y}^{(r)}\| \to 0$ and $\|\underline{z}^{(r)}\| \to 0 \implies \|\underline{x}^{(r)}\| \le \|\underline{y}^{(r)}\| + \|\underline{z}^{(r)}\| \to 0$
- hence, we have

$$\begin{aligned}
\int_a^b f = \lim_{r\to\infty} S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) &= \lim_{r\to\infty} (S_f(\underline{y}^{(r)}, \underline{\zeta}^{(r)}) + S_f(\underline{z}^{(r)}, \underline{\eta}^{(r)})) \\
= \lim_{r\to\infty} S_f(\underline{y}^{(r)}, \underline{\zeta}^{(r)}) + \lim_{r\to\infty} S_f(\underline{z}^{(r)}, \underline{\eta}^{(r)}) &= \int_a^c f + \int_c^b f
\end{aligned}$$

Theorem 7.16 Let $f, g \in \mathcal{C}([a,b])$ and $f(x) \leq g(x)$ for all $x \in [a,b]$, then we have

$$\int_a^b f \leq \int_a^b g.$$

proof: let $\left((\underline{x}^{(r)}, \underline{\xi}^{(r)})\right)_{r=1}^{\infty}$ be a sequence of tagged partitions with $\|\underline{x}^{(r)}\| \to 0$, then

$$S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) = \sum_{i=1}^{n^{(r)}} f(\xi_i^{(r)})(x_i^{(r)} - x_{i-1}^{(r)}) \leq \sum_{i=1}^{n^{(r)}} g(\xi_i^{(r)})(x_i^{(r)} - x_{i-1}^{(r)}) = S_g(\underline{x}^{(r)}, \underline{\xi}^{(r)})$$

$\implies \lim_{r\to\infty} S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) \leq \lim_{r\to\infty} S_g(\underline{x}^{(r)}, \underline{\xi}^{(r)}) \implies \int_a^b f \leq \int_a^b g$

Corollary 7.17 Let $f \in \mathcal{C}([a,b])$, then $\left|\int_a^b f\right| \leq \int_a^b |f|$.

proof: $\pm f(x) \leq |f(x)| \implies \int_a^b \pm f = \pm \int_a^b f \leq \int_a^b |f|$ (theorem 7.16)

Theorem 7.18 Let $f \in \mathcal{C}([a,b])$, and

$$m_f = \inf\{f(x) \mid x \in [a,b]\}, \qquad M_f = \sup\{f(x) \mid x \in [a,b]\}.$$

Then, we have

$$m_f(b-a) \leq \int_a^b f \leq M_f(b-a).$$

proof: let $\left((\underline{x}^{(r)}, \underline{\xi}^{(r)})\right)_{r=1}^{\infty}$ be a sequence of tagged partitions with $\|\underline{x}^{(r)}\| \to 0$, then

$$S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) = \sum_{i=1}^{n^{(r)}} f(\xi_i^{(r)})(x_i^{(r)} - x_{i-1}^{(r)}) \geq \sum_{i=1}^{n^{(r)}} m_f(x_i^{(r)} - x_{i-1}^{(r)}) = m_f(b-a)$$

$$S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) = \sum_{i=1}^{n^{(r)}} f(\xi_i^{(r)})(x_i^{(r)} - x_{i-1}^{(r)}) \leq \sum_{i=1}^{n^{(r)}} M_f(x_i^{(r)} - x_{i-1}^{(r)}) = M_f(b-a)$$

$\implies m_f(b-a) \leq \lim_{r\to\infty} S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) \leq M_f(b-a)$

Fundamental theorem of calculus

Theorem 7.19 *Fundamental theorem of calculus.* Let $f \in \mathcal{C}([a,b])$.

- If $F\colon [a,b] \to \mathbf{R}$ is differentiable and $F' = f$, then

$$\int_a^b f = F(b) - F(a).$$

- The function $G(x) = \int_a^x f$ is differentiable on $[a,b]$ with

$$G(a) = 0, \qquad G'(x) = f(x).$$

proof:

- let $\left(\underline{x}^{(r)}\right)_{r=1}^{\infty}$ be a sequence of partitions with $\|\underline{x}^{(r)}\| \to 0$, by theorem 6.15, there exist tags $\underline{\xi}^{(r)}$ with $\xi_i^{(r)} \in [x_{i-1}^{(r)}, x_i^{(r)}]$, $i = 1, \dots, n^{(r)}$, such that

$$F(x_i^{(r)}) - F(x_{i-1}^{(r)}) = F'(\xi_i^{(r)})(x_i^{(r)} - x_{i-1}^{(r)}) = f(\xi_i^{(r)})(x_i^{(r)} - x_{i-1}^{(r)})$$

hence, for the sequence of tagged partitions $\left((\underline{x}^{(r)}, \underline{\xi}^{(r)})\right)_{r=1}^{\infty}$ we have

$$S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) = \sum_{i=1}^{n^{(r)}} f(\xi_i^{(r)})(x_i^{(r)} - x_{i-1}^{(r)}) = \sum_{i=1}^{n^{(r)}} F(x_i^{(r)}) - F(x_{i-1}^{(r)}) = F(b) - F(a)$$

$\implies \int_a^b f = \lim_{r\to\infty} S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) = F(b) - F(a)$

- we only need to show that G is differentiable and $G' = f$, *i.e.*, let $c \in [a,b]$, we need to prove that $\lim_{x\to c} \frac{G(x)-G(c)}{x-c} = \lim_{x\to c} \frac{\int_a^x f - \int_a^c f}{x-c} = f(c)$; let $\epsilon > 0$
  - f continuous on $[a,b]$ $\implies$ $\exists \delta > 0$ such that for all $t \in [a,b]$ and $|t-c| < \delta$, we have $|f(t) - f(c)| < \epsilon/2$
  - suppose $x \in (c, c+\delta)$, then for all $t \in [c,x]$, we have $|f(t) - f(c)| < \epsilon/2$, hence,

$$\begin{aligned}
\left|\frac{\int_a^x f - \int_a^c f}{x-c} - f(c)\right| &= \left|\frac{\int_c^x f(t)\,dt}{x-c} - f(c)\right| \\
&= \left|\frac{1}{x-c}\left(\int_c^x f(t)\,dt - \int_c^x f(c)\,dt\right)\right| = \frac{1}{x-c}\left|\int_c^x (f(t) - f(c))\,dt\right| \\
&\le \frac{1}{x-c}\int_c^x |f(t) - f(c)|\,dt \le \frac{1}{x-c}\int_c^x \frac{\epsilon}{2}\,dt = \frac{1}{x-c}\cdot\frac{\epsilon}{2}(x-c) = \frac{\epsilon}{2} < \epsilon
\end{aligned}$$

(the first inequality is by corollary 7.17)

– suppose $x \in (c-\delta, c)$, using similar argument, we have $\left|\frac{\int_a^x f - \int_a^c f}{x-c} - f(c)\right| < \epsilon$

– put together, we conclude that for all $x \in [a,b]$ and $0 < |x-c| < \delta$, we have

$$\left|\frac{\int_a^x f - \int_a^c f}{x-c} - f(c)\right| < \epsilon$$

$$\Longrightarrow \quad \lim_{x\to c} \frac{G(x)-G(c)}{x-c} = \lim_{x\to c} \frac{\int_a^x f - \int_a^c f}{x-c} = f(c)$$

Integration by parts

Theorem 7.20 *Integration by parts.* Suppose $f, g \in \mathcal{C}([a,b])$, $f', g' \in \mathcal{C}([a,b])$, then

$$\int_a^b f'g = (f(b)g(b) - f(a)g(a)) - \int_a^b fg'.$$

---

proof: let $F \in \mathcal{C}([a,b])$ with $F(x) = f(x)g(x)$, by theorem 6.8, we have

$$F'(x) = f'(x)g(x) + f(x)g'(x),$$

and hence,

$$\int_a^b f'(x)g(x)\ dx + \int_a^b f(x)g'(x)\ dx = \int_a^b (f'(x)g(x) + f(x)g'(x))\ dx$$

$$= \int_a^b F'(x)\ dx = F(b) - F(a) = f(b)g(b) - f(a)g(a)$$

$\Longrightarrow\ \int_a^b f'g = (f(b)g(b) - f(a)g(a)) - \int_a^b fg'$

Change of variables

Theorem 7.21 *Change of variables.* Let $f \in \mathcal{C}([c,d])$ and $\varphi\colon [a,b] \to [c,d]$ be continuously differentiable with $\varphi(a) = c$ and $\varphi(b) = d$. Then, we have

$$\int_c^d f(u)\ du = \int_a^b f(\varphi(x))\varphi'(x)\ dx.$$

proof:

- let $F\colon [a,b] \to \mathbf{R}$ be a function with $F' = f$, then we have

$$\int_c^d f(u)\ du = F(d) - F(c)$$

- by theorem 6.9, we have

$$(F \circ \varphi)'(x) = F'(\varphi(x))\varphi'(x) = f(\varphi(x))\varphi'(x),$$

  and hence,

$$\int_a^b f(\varphi(x))\varphi'(x)\ dx = F(\varphi(b)) - F(\varphi(a)) = F(d) - F(c) = \int_c^d f(u)\ du$$

8. Sequences of functions

- power series
- pointwise and uniform convergence
- interchange of limits
- Weierstrass M-test
- properties of power series

Power series

Definition 8.1 A **power series** about $x_0 \in \mathbf{R}$ is a series of the form

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m.$$

Definition 8.2 Let $\sum_{m=0}^{\infty} a_m (x - x_0)^m$ be a power series, if the limit

$$R = \lim_{m \to \infty} |a_m|^{1/m}$$

exists, we define the **radius of convergence** ρ as

$$\rho = \begin{cases} 1/R & R > 0 \\ \infty & R = 0. \end{cases}$$

Theorem 8.3 Let $\sum_{m=0}^{\infty} a_m(x-x_0)^m$ be a power series and $R = \lim_{m\to\infty} |a_m|^{1/m}$ exists. If $R = 0$, the series converges absolutely for all $x \in \mathbf{R}$. If $R > 0$, the series converges absolutely if $|x-x_0| < \rho$ and diverges if $|x-x_0| > \rho$.

proof: consider the root test (theorem 4.26), we have

$$L = \lim_{m\to\infty} |a_m(x-x_0)^m|^{1/m} = |x-x_0| \lim_{m\to\infty} |a_m|^{1/m} = R|x-x_0|$$

- suppose $R = 0$, then we have $L = 0 < 1$ for all $x \in \mathbf{R} \implies \sum_{m=0}^{\infty} a_m(x-x_0)^m$ converges absolutely for all $x \in \mathbf{R}$
- suppose $R > 0$
  - if $|x-x_0| < \rho \implies L < R\rho = 1 \implies \sum_{m=0}^{\infty} a_m(x-x_0)^m$ converges absolutely
  - if $|x-x_0| > \rho \implies L > R\rho = 1 \implies \sum_{m=0}^{\infty} a_m(x-x_0)^m$ diverges

Remark 8.4 Let $\sum_{m=0}^{\infty} a_m(x-x_0)^m$ be a power series with radius of convergence ρ. Define $f\colon (x_0-\rho, x_0+\rho) \to \mathbf{R}$ such that

$$f(x) = \sum_{m=0}^{\infty} a_m(x-x_0)^m,$$

then, the function f is the limit of a sequence of functions $(f_n)_{n=1}^{\infty}$, given by

$$f(x) = \lim_{n\to\infty} f_n(x), \quad f_n(x) = \sum_{m=0}^{n} a_m(x-x_0)^m.$$

Example 8.5 Consider the geometric series $\sum_{m=0}^{\infty} x^m$ (which is a power series with $a_m = 1$, $x_0 = 0$), we have $f\colon (-1,1) \to \mathbf{R}$ given by

$$f(x) = \frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = \lim_{n\to\infty} f_n(x), \quad f_n(x) = \sum_{m=0}^{n} x^m.$$

Example 8.6 *Exponential function.* Consider the power series with $a_m = \frac{1}{m!}$, $x_0 = 0$, we have the exponential function $f(x)\colon \mathbf{R} \to \mathbf{R}$, given by

$$f(x) = \exp(x) = \sum_{m=0}^{\infty} \frac{x^m}{m!} = \lim_{n\to\infty} f_n(x), \quad f_n(x) = \sum_{m=0}^{n} \frac{x^m}{m!}.$$

Remark 8.7 Based on remark 8.4, we may ask several questions.

(1) Is the function f continuous?

(2) If (1) is true, is f differentiable, and does $f' = \lim_{n\to\infty} f_n'$?

(3) If (1) is true, does $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$?

Pointwise convergence

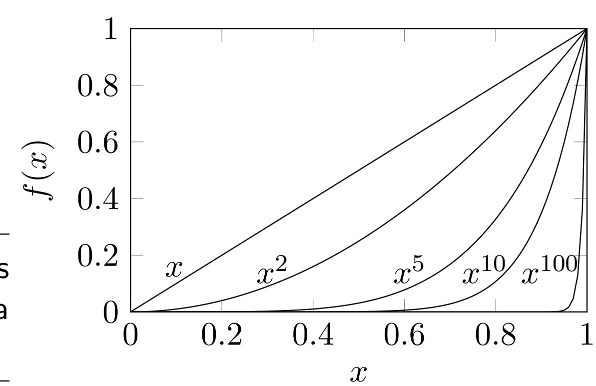
Definition 8.8 Let $(f_n)_{n=1}^{\infty}$ with $f_n\colon S \to \mathbf{R}$ for all $n \in \mathbf{N}$ be a sequence of functions, and let $f\colon S \to \mathbf{R}$ be a function. We say that $(f_n)_{n=1}^{\infty}$ **converges pointwise** (or just **converges**) to f if for all $x \in S$, we have $\lim_{n\to\infty} f_n(x) = f(x)$.

Example 8.9 Let $f_n(x) = x^n$ be defined on $[0,1]$, then we have the sequence of functions $(f_n)_{n=1}^{\infty}$ converges pointwise to $f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$.

proof:

- if $x \in [0,1)$: $\lim_{n\to\infty} x^n = 0$
- if $x = 1$: $\lim_{n\to\infty} 1^n = 1$

Remark 8.10 A sequence of continuous functions may not converge pointwise to a continuous function.



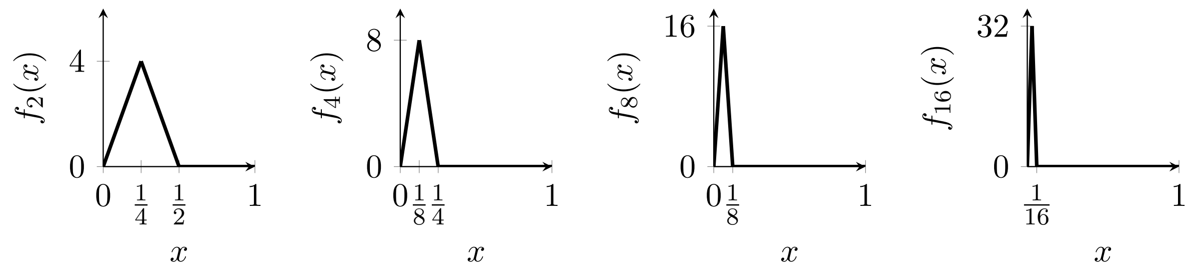
Example 8.11 Let $f_n(x)\colon [0,1] \to \mathbf{R}$ be defined by

$$f_n(x) = \begin{cases} 4n^2x & x \in \left[0, \frac{1}{2n}\right] \\ 4n - 4n^2x & x \in \left[\frac{1}{2n}, \frac{1}{n}\right] \\ 0 & x \in \left[\frac{1}{n}, 1\right], \end{cases}$$

then $(f_n)_{n=1}^{\infty}$ converges pointwise to $f(x) = 0$ $\big(x \in [0,1]\big)$.

proof: if $x = 0$, we have $\lim_{n\to\infty} f_n(0) = 0$; if $x \in (0,1]$, then $\exists M \in [0,1]$ such that $\forall n \geq M$, $\frac{1}{n} < x$, and hence,

$$(f_n(x))_{n=1}^{\infty} = f_1(x), \ldots, f_{M-1}(x), 0, 0, 0, \ldots \implies \lim_{n\to\infty} f_n(x) = 0$$



Uniform convergence

Definition 8.12 Let $(f_n)_{n=1}^{\infty}$ with $f_n\colon S \to \mathbf{R}$ for all $n \in \mathbf{N}$ be a sequence of functions, and let $f\colon S \to \mathbf{R}$ be a function. We say that $(f_n)_{n=1}^{\infty}$ **converges uniformly** to f if for all $\epsilon > 0$, there exists some $M \in \mathbf{N}$ such that for all $n \geq M$ and $x \in S$, we have $|f_n(x) - f(x)| < \epsilon$.

Theorem 8.13 Let $f\colon S \to \mathbf{R}$, $f_n\colon S \to \mathbf{R}$ for all $n \in \mathbf{N}$ be functions. If the sequence of functions $(f_n)_{n=1}^{\infty}$ converges uniformly to f, then $(f_n)_{n=1}^{\infty}$ converges pointwise to f.

proof: let $c \in S$, $\epsilon > 0$

- $(f_n)_{n=1}^{\infty}$ converges uniformly to $f \implies \exists M \in \mathbf{N}$ such that for all $n \geq M$ and $x \in S$, $|f_n(x) - f(x)| < \epsilon$
- hence, $\forall n \geq M$, $|f_n(c) - f(c)| < \epsilon \implies (f_n)_{n=1}^{\infty}$ converges pointwise to f

Remark 8.14 Let $f\colon S \to \mathbf{R}$, $f_n\colon S \to \mathbf{R}$ for all $n \in \mathbf{N}$ be functions. The sequence $(f_n)_{n=1}^{\infty}$ does not converge to f uniformly if there exists some $\epsilon > 0$ such that for all $M \in \mathbf{N}$, there exist some $n \geq M$ and some $x \in S$, so that $|f_n(x) - f(x)| \geq \epsilon$.

Theorem 8.15 Let $f_n(x) = x^n$, $n \in \mathbf{N}$, and let $f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$.

- The sequence $(f_n)_{n=1}^{\infty}$ converges uniformly to f on $[0,b]$ for all $0 < b < 1$.
- The sequence $(f_n)_{n=1}^{\infty}$ does not converges to f uniformly on $[0,1]$.

proof:

- let $\epsilon > 0$, $b \in (0,1)$, then $b^n \to 0 \implies \exists M \in \mathbf{N}$ such that $\forall n \geq M$, $b^n < \epsilon \implies \forall n \geq M$ and $x \in [0,b]$, we have

$$|f_n(x) - f(x)| = x^n \leq b^n < \epsilon$$

- choose $\epsilon = 1/2$, then $\forall M \in \mathbf{N}$, choose $n = M$, $x = (1/2)^{1/M} < 1$, we have

$$|f_M(x) - f(x)| = x^M = 1/2 \geq \epsilon$$

Interchange of limits

Example 8.16 In general, limits cannot be interchanged. For example,

$$\lim_{n\to\infty} \lim_{k\to\infty} \frac{n/k}{n/k+1} = \lim_{n\to\infty} 0 = 0, \qquad \lim_{k\to\infty} \lim_{n\to\infty} \frac{n/k}{n/k+1} = \lim_{k\to\infty} 1 = 1.$$

Remark 8.17 Based on example 8.16, we may ask the following questions.

- If $f_n \colon S \to \mathbf{R}$ with f_n continuous for all $n \in \mathbf{N}$ and $(f_n)_{n=1}^{\infty}$ converges to f uniformly or pointwise, then is f continuous?
- If $f_n \colon [a,b] \to \mathbf{R}$ with f_n differentiable for all $n \in \mathbf{N}$, and $(f_n)_{n=1}^{\infty}$ converges to f, $(f_n')_{n=1}^{\infty}$ converges to g uniformly or pointwise, then is f differentiable and does $f' = g$?
- If $f_n \colon [a,b] \to \mathbf{R}$, $n \in \mathbf{N}$, $f \colon [a,b] \to \mathbf{R}$, with f_n and f continuous, and $(f_n)_{n=1}^{\infty}$ converges to f uniformly or pointwise, then does $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$?

Remark 8.18 If convergence is only pointwise, the answer is *no* for all questions in remark 8.17.

- Let $f_n(x) = x^n$ on $[0,1]$, $n \in \mathbf{N}$. Example 8.9 shows that $(f_n)_{n=1}^{\infty}$ converges pointwise to a noncontinuous function.
- Let $f_n(x) = \frac{x^{n+1}}{n+1}$ on $[0,1]$, then $(f_n)_{n=1}^{\infty}$ converges to $f(x) = 0$ pointwise on $[0,1]$ and $(f_n')_{n=1}^{\infty}$ converges pointwise to g given by $g(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$, but $f'(1) = 0 \neq g(1) = 1$.
- Let $f_n\colon [0,1] \to \mathbf{R}$ be given by $f_n(x) = \begin{cases} 4n^2x & x \in \left[0, \frac{1}{2n}\right] \\ 4n - 4n^2x & x \in \left[\frac{1}{2n}, \frac{1}{n}\right] \\ 0 & x \in \left[\frac{1}{n}, 1\right] \end{cases}$, then $(f_n)_{n=1}^{\infty}$ converges to $f(x) = 0$ pointwise on $[0,1]$ (example 8.11), but

$$\int_0^1 f = 0 \neq \lim_{n\to\infty} \int_0^1 f_n = \lim_{n\to\infty} (\frac{1}{2} \cdot \frac{1}{n} \cdot 2n) = 1.$$

Theorem 8.19 If $f_n\colon S \to \mathbf{R}$ is continuous for all $n \in \mathbf{N}$, $f\colon S \to \mathbf{R}$, and $(f_n)_{n=1}^{\infty}$ converges to f uniformly, then f is continuous.

proof: let $c \in S$, $\epsilon > 0$

- f_n continuous on S, $c \in S \implies \exists \delta > 0$ such that for all $x \in S$ and $|x - c| < \delta$, we have $|f_n(x) - f_n(c)| < \epsilon/3$
- $f_n \to f$ uniformly $\implies \exists M \in \mathbf{N}$ such that for all $n \geq M$ and $x \in S$, we have $|f(x) - f_n(x)| < \epsilon/3$
- hence, for all $x \in S$ and $|x - c| < \delta$, we have

$$\begin{aligned} |f(x) - f(c)| &= |f(x) - f_M(x) + f_M(x) - f_M(c) + f_M(c) - f(c)| \\ &\leq |f(x) - f_M(x)| + |f_M(x) - f_M(c)| + |f_M(c) - f(c)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

Theorem 8.20 If $f_n\colon [a,b] \to \mathbf{R}$ is continuous for all $n \in \mathbf{N}$, $f\colon [a,b] \to \mathbf{R}$, and $(f_n)_{n=1}^{\infty}$ converges to f uniformly, then $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.

proof: let $\epsilon > 0$

- by theorem 8.19, we know that f is continuous on $[a,b]$
- $(f_n)_{n=1}^{\infty}$ converges uniformly to $f \implies \exists M \in \mathbf{N}$ such that for all $n \geq M$ and $x \in [a,b]$, we have $|f_n(x) - f(x)| < \frac{\epsilon}{b-a}$
- hence, for all $n \geq M$, we have

$$\left|\int_a^b f_n - \int_a^b f\right| = \left|\int_a^b (f_n - f)\right| \leq \int_a^b |f_n - f| < \int_a^b \frac{\epsilon}{b-a} = \epsilon,$$

  where the first inequality is by corollary 7.17

Remark 8.21 Notationally, theorem 8.20 says that

$$\int_a^b f = \int_a^b \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_a^b f_n.$$

Theorem 8.22 If $f_n\colon [a,b] \to \mathbf{R}$ is continuously differentiable, $f\colon [a,b] \to \mathbf{R}$, $g\colon [a,b] \to \mathbf{R}$, and

- $(f_n)_{n=1}^{\infty}$ converges to f pointwise,
- $(f_n')_{n=1}^{\infty}$ converges to g uniformly,

then f is continuously differentiable and $f' = g$.

proof: let $x \in [a,b]$

- by theorem 8.19, we know that g is continuous on $[a,b]$
- by theorem 7.19, we have

$$\int_a^x f_n' = f_n(x) - f(a) \implies \lim_{n\to\infty} \int_a^x f_n' = \lim_{n\to\infty} f_n(x) - \lim_{n\to\infty} f_n(a)$$

- $f_n \to f$ pointwise $\implies \lim_{n\to\infty} f_n(x) - \lim_{n\to\infty} f_n(a) = f(x) - f(a)$
- $f_n' \to g$ uniformly $\implies \lim_{n\to\infty} \int_a^x f_n' = \int_a^x g$ (theorem 8.20)
- put together, we have

$$\int_a^x g = f(x) - f(a) \implies \left(\int_a^x g\right)' = g(x) = f'(x)$$

Weierstrass M-test

Theorem 8.23 *Weierstrass M-test.* Let $f_k\colon S \to \mathbf{R}$ for all $k \in \mathbf{N}$. Suppose there exists $M_k > 0$, $k \in \mathbf{N}$, such that

(a) $|f_k(x)| \leq M_k$ for all $x \in S$,

(b) $\sum_{k=1}^{\infty} M_k$ converges.

Then, we have the following conclusion.

(1) The series $\sum_{k=1}^{\infty} f_k(x)$ converges absolutely for all $x \in S$.

(2) Let $f(x) = \sum_{k=1}^{\infty} f_k(x)$ for all $x \in S$, then the series $(\sum_{k=1}^{n} f_k)_{n=1}^{\infty}$ converges to f uniformly on S.

---

proof:

(1) $|f_k(x)| \leq M_k$, $\sum_{k=1}^{\infty} M_k$ converges $\implies$ $\sum_{k=1}^{\infty} |f_k(x)|$ converges (theorem 4.20) $\implies$ $\sum_{k=1}^{\infty} f_k(x)$ converges absolutely

(2) let $\epsilon > 0$; $\sum_{k=1}^{\infty} M_k$ converges $\implies$ $\exists M \in \mathbf{N}$ s.t. $\forall n \geq M$, we have

$$\sum_{k=n+1}^{\infty} M_k = \left| \sum_{k=1}^{\infty} M_k - \sum_{k=1}^{n} M_k \right| < \epsilon$$

then, for all $n \geq M$ and $x \in S$, we have

$$\left| \sum_{k=1}^{\infty} f_k(x) - \sum_{k=1}^{n} f_k(x) \right| = \left| \sum_{k=n+1}^{\infty} f_k(x) \right| \leq \sum_{k=n+1}^{\infty} |f_k(x)| \leq \sum_{k=n+1}^{\infty} M_k < \epsilon$$

Properties of power series

Theorem 8.24 Let $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ be a power series with radius of convergence $\rho \in (0,\infty]$, then for all $r \in (0,\rho)$, the series $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ converges uniformly on $[x_0-r, x_0+r]$.

proof:

- note that we have $|x-x_0| \le r$ for all $x \in [x_0-r, x_0+r]$
- let $f_k = a_k(x-x_0)^k$, choose $M_k = |a_k|r^k$, $k \in \mathbf{N}$, then $\forall x \in [x_0-r, x_0+r]$,

$$|f_k(x)| = |a_k(x-x_0)^k| = |a_k||x-x_0|^k \le |a_k|r^k = M_k$$

- consider the root test (theorem 4.26) for $\sum_{k=0}^{\infty} M_k$, we have

$$L = \lim_{k\to\infty} M_k^{1/k} = \lim_{k\to\infty} \left(|a_k|r^k\right)^{1/k} = \lim_{k\to\infty} |a_k|^{1/k} r = \begin{cases} r/\rho & \rho < \infty \\ 0 & \rho = \infty \end{cases}$$

  since $r \in (0,\rho)$, we have $L < 1 \implies \sum_{k=0}^{\infty} M_k$ converges absolutely
- put together, by theorem 8.23, we have $(\sum_{k=0}^{n} f_k)_{n=1}^{\infty} = \sum_{k=0}^{n} a_k(x-x_0)^k$ converges uniformly on $[x_0-r, x_0+r]$

Theorem 8.25 Let $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ be a power series with radius of convergence $\rho \in (0,\infty]$, then we have the following conclusion.

- For all $c \in (x_0-\rho, x_0+\rho)$, the function given by the series $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ is differentiable at c, and

$$\frac{d}{dx}\left(\sum_{k=0}^{\infty} a_k(x-x_0)^k\right)\Bigg|_{x=c} = \sum_{k=0}^{\infty} \frac{d}{dx}(a_k(x-x_0)^k)\Big|_{x=c}.$$

- For all a, b such that $x_0-\rho < a < b < x_0+\rho$,

$$\int_a^b \sum_{k=0}^{\infty} a_k(x-x_0)^k \, dx = \sum_{k=0}^{\infty} \int_a^b a_k(x-x_0)^k \, dx.$$

9. Metric spaces

- metric spaces
- Cauchy-Schwarz inequality
- open and closed sets
- closure and boundary
- sequences and convergence in metric spaces
- convergence properties of topology
- Cauchy sequences and completeness

Metric spaces

Definition 9.1 Let A and B be sets. The **Cartesian product** is the set of tuples defined as

$$A \times B = \{(x, y) \mid x \in A,\ y \in B\}.$$

examples:

- $\{a, b\} \times \{c, d\} = \{(a, c), (a, d), (b, c), (b, d)\}$
- the set $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ is the Cartesian plane
- the set $[0, 1]^2 = [0, 1] \times [0, 1]$ is a subset of the Cartesian plane bounded by a square with vertices $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$

Remark 9.2 To denote an element in the set $\mathbf{R}^n$, we write $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, or simply $x \in \mathbf{R}^n$, where the subscripts $i = 1, \ldots, n$ denote the ith entry of the tuple $(x_1, \ldots, x_n)$ that describes x.

We also simply write $0 \in \mathbf{R}^n$ to mean the point $(0, 0, \ldots 0) \in \mathbf{R}^n$.

Definition 9.3 Let X be a set, and let $d\colon X \times X \to \mathbf{R}$ be a function such that for all $x, y, z \in X$, we have

- $d(x, y) \geq 0$, (*nonnegativity*)
- $d(x, y) = 0$ if and only if $x = y$,
- $d(x, y) = d(y, x)$, and (*symmetry*)
- $d(x, z) \leq d(x, y) + d(y, z)$. (*triangle inequality*)

Then the pair (X, d) is called a **metric space**. The function d is called the **metric** or the **distance function**. Sometimes we just write X as the metric space if the metric is clear from context.

Example 9.4 The real numbers $\mathbf{R}$ is a metric space with the metric $d(x, y) = |x - y|$.

proof:

- the first three properties follows immediately from the properties of the absolute value (theorem 2.25)
- to show the triangle inequality, let $x, y, z \in \mathbf{R}$, then we have

$$d(x, z) = |x - z| = |x - y + y - z| \leq |x - y| + |y - z| = d(x, y) + d(x, z)$$

Definition 9.5 Let (X, d) be a metric space. A set $S \subseteq X$ is said to be **bounded** if there exists a point $p \in X$ and some number $B \in \mathbf{R}$ such that

$$d(p, x) \leq B \quad \text{for all } x \in S.$$

We say (X, d) is bounded if X is a bounded set.

Cauchy-Schwarz inequality

Theorem 9.6 *Cauchy-Schwarz inequality.* Suppose $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, $y = (y_1, \ldots, y_n) \in \mathbf{R}^n$, then

$$\left(\sum_{i=1}^n x_i y_i\right)^2 \leq \left(\sum_{i=1}^n x_i^2\right)\left(\sum_{i=1}^n y_i^2\right).$$

proof:

$$\begin{aligned}
0 &\leq \sum_{i=1}^n \sum_{j=1}^n (x_i y_j - x_j y_i)^2 = \sum_{i=1}^n \sum_{j=1}^n (x_i^2 y_j^2 - 2x_i y_j x_j y_i + x_j^2 y_i^2) \\
&= \left(\sum_{i=1}^n x_i^2\right)\left(\sum_{j=1}^n y_j^2\right) + \left(\sum_{i=1}^n y_i^2\right)\left(\sum_{j=1}^n x_j^2\right) - 2\left(\sum_{i=1}^n x_i y_i\right)\left(\sum_{j=1}^n x_j y_j\right) \\
\Longrightarrow \ & \left(\sum_{i=1}^n x_i y_i\right)^2 \leq \left(\sum_{i=1}^n x_i^2\right)\left(\sum_{i=1}^n y_i^2\right)
\end{aligned}$$

Theorem 9.7 The function $f\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ given by

$$f(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$$

is a metric for $\mathbf{R}^n$.

proof: we show that f satisfies the triangle inequality, by theorem 9.6, we have

$$\begin{aligned}
(f(x,z))^2 &= \sum_{i=1}^n (x_i - z_i)^2 = \sum_{i=1}^n (x_i - y_i + y_i - z_i)^2 \\
&= \sum_{i=1}^n (x_i - y_i)^2 + 2\sum_{i=1}^n (x_i - y_i)(y_i - z_i) + \sum_{i=1}^n (y_i - z_i)^2 \\
&\leq \sum_{i=1}^n (x_i - y_i)^2 + 2\sqrt{\sum_{i=1}^n (x_i - y_i)^2 \sum_{i=1}^n (y_i - z_i)^2} + \sum_{i=1}^n (y_i - z_i)^2 \\
&= \left(\sqrt{\sum_{i=1}^n (x_i - y_i)^2} + \sqrt{\sum_{i=1}^n (y_i - z_i)^2}\right)^2 = (f(x,y) + f(y,z))^2
\end{aligned}$$

n-dimensional Euclidean space

Definition 9.8 The n-**dimensional Euclidean space** is the metric space $(\mathbf{R}^n, d)$ with the metric d defined by

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}. \tag{9.1}$$

Remark 9.9 For $n = 1$, the n-dimensional Euclidean space reduces to the real numbers and the metric given by (9.1) agrees with the standard metric for the set of real numbers $d(x,y) = |x - y|$ in example 9.4.

Open and closed sets

Definition 9.10 Let (X, d) be a metric space, $x \in X$, and $\delta > 0$. Define the **open ball** and **closed ball**, of radius δ around x as

$$B(x,\delta) = \{y \in X \mid d(x,y) < \delta\} \quad \text{and} \quad C(x,\delta) = \{y \in X \mid d(x,y) \le \delta\},$$

respectively.

Example 9.11 Consider the metric space $\mathbf{R}$, for $x \in \mathbf{R}$ and $\delta > 0$, we have

$$B(x,\delta) = (x - \delta, x + \delta) \quad \text{and} \quad C(x,\delta) = [x - \delta, x + \delta].$$

Example 9.12 Consider the metric space $\mathbf{R}^2$, for $x \in \mathbf{R}^2$ and $\delta > 0$, we have

$$B(x,\delta) = \{y \in \mathbf{R}^2 \mid (x_1 - y_1)^2 + (x_2 - y_2)^2 < \delta^2\}.$$

Definition 9.13 Let (X, d) be a metric space. A subset $V \subseteq X$ is **open** if for all $x \in V$, there exists some $\delta > 0$ such that $B(x, \delta) \subseteq V$. A subset $E \subseteq X$ is **closed** if the complement $E^c = X \setminus E$ is open.

examples:

- $(0, \infty) \subseteq \mathbf{R}$ is open; $[0, \infty) \subseteq \mathbf{R}$ is closed
- $[0, 1) \subseteq \mathbf{R}$ is neither open nor closed
- the singleton $\{x\}$ with $x \in X$ is closed

Theorem 9.14 Let (X, d) be a metric space.

(1) The sets $\emptyset$ and X are open.

(2) If $V_1, \ldots, V_k$ are subsets of X, then $\bigcap_{i=1}^k V_i$ is open, $i.e.$, a *finite* intersection of open sets is open.

(3) Let $\{V_i \subseteq X \mid i \in I\}$ be a collection of open subsets of X, where I is an arbitrary index set, then $\bigcup_{i \in I} V_i$ is open, $i.e.$, a union of open sets is open.

proof:

- the sets $\emptyset$ and X are obviously open

- let $x \in \bigcap_{i=1}^k V_i$, then $x \in V_1, \ldots, V_k$
  - $V_1, \ldots, V_k$ are open $\implies \exists \delta_1, \ldots, \delta_k > 0$ s.t. $B(x, \delta_1) \subseteq V_1, \ldots, B(x, \delta_k) \subseteq V_k$
  - choose $\delta = \min\{\delta_1, \ldots, \delta_k\}$, then $B(x, \delta) \subseteq V_1, \ldots, V_k \implies B(x, \delta) \subseteq \bigcap_{i=1}^k V_i$
- let $x \in \bigcup_{i \in I} V_i$, then $\exists V_k \in \{V_i \mid i \in I\}$ such that $x \in V_k$
  - V_k is open $\implies \exists \delta > 0$ such that $B(x, \delta) \subseteq V_k \subseteq \bigcup_{i \in I} V_i$

Theorem 9.15 Let (X, d) be a metric space.

(1) The sets $\emptyset$ and X are closed.

(3) Let $\{V_i \subseteq X \mid i \in I\}$ be a collection of closed subsets of X, where I is an arbitrary index set, then $\bigcap_{i \in I} V_i$ is closed, $i.e.$, an intersection of closed sets is closed.

(2) If $V_1, \ldots, V_k$ are subsets of X, then $\bigcup_{i=1}^k V_i$ is closed, $i.e.$, a *finite* union of closed sets is closed.

Remark 9.16 Note that in theorem 9.14, the statement (2) is not true for an arbitrary intersection. For example, $\bigcap_{n=1}^\infty (-1/n, 1/n) = \{0\}$, which is not open in $\mathbf{R}$.

Similarly, in theorem 9.15, the statement (3) is not true for an arbitrary intersection. For example, $\bigcup_{n=1}^\infty [1/n, \infty) = (0, \infty)$, which is not closed in $\mathbf{R}$.

Theorem 9.17 Let (X, d) be a metric space, $x \in X$, and $\delta > 0$. Then $B(x, \delta)$ is open and $C(x, \delta)$ is closed.

proof: we show that $B(x, \delta)$ is open; let $z \in B(x, \delta)$, then $d(x, z) < \delta$

- choose $\epsilon = \delta - d(x, z)$, let $B(z, \epsilon) = \{y \in X \mid d(y, z) < \epsilon\}$ be an open ball
- let $y \in B(z, \epsilon)$, we have $d(y, z) < \epsilon$, and hence

$$d(x, y) \leq d(x, z) + d(z, y) < d(x, z) + \epsilon = d(x, z) + \delta - d(x, z) = \delta$$

$$\implies y \in B(x, \delta) \implies B(z, \epsilon) \subseteq B(x, \delta)$$

Closure and boundary

Definition 9.18 Let (X, d) be a metric space and $A \subseteq X$. The **closure** of A is the set

$$\mathbf{cl}\, A = \bigcap \{E \subseteq X \mid E \text{ is closed and } A \subseteq E\},$$

i.e., $\mathbf{cl}\, A$ is the intersection of all closed sets that contain A.

Definition 9.19 Let (X, d) be a metric space and $A \subseteq X$. The **interior** of A is the set

$$\mathbf{int}\, A = \{x \in A \mid B(x, \delta) \subseteq A \text{ for some } \delta > 0\}.$$

The **boundary** of A is the set

$$\mathbf{bd}\, A = \mathbf{cl}\, A \setminus \mathbf{int}\, A.$$

example: consider $A = (0, 1]$ and $X = \mathbf{R}$, then we have $\mathbf{cl}\, A = [0, 1]$, $\mathbf{int}\, A = (0, 1)$, and $\mathbf{bd}\, A = \{0, 1\}$

Remark 9.20 Notationally, in some textbooks, the closure, interior, and boundary of some set A are denoted as

$$\overline{A} = \mathbf{cl}\, A, \quad A^\circ = \mathbf{int}\, A, \quad \text{and} \quad \partial A = \mathbf{bd}\, A,$$

respectively.

Theorem 9.21 Let (X, d) be a metric space and $A \subseteq X$.

- The closure $\mathbf{cl}\, A$ is closed and $A \subseteq \mathbf{cl}\, A$.
- If A is closed, then $\mathbf{cl}\, A = A$.

proof: let $\mathbf{cl}\, A = \bigcap\{E \subseteq X \mid E \text{ is closed and } A \subseteq E\}$

- the first statement follows directly from the definition of closure and theorem 9.15
- if A is closed, then $A \in \{E \subseteq X \mid E \text{ is closed and } A \subseteq E\} \implies \mathbf{cl}\, A \subseteq A \implies A = \mathbf{cl}\, A$

Theorem 9.22 Let (X, d) be a metric space and $A \subseteq X$, then $x \in \mathbf{cl}\, A$ if and only if for all $\delta > 0$, we have $B(x, \delta) \cap A \neq \emptyset$.

proof: we show the following claim: $x \notin \mathbf{cl}\, A$ if and only if there exists some $\delta > 0$ such that $B(x, \delta) \cap A = \emptyset$

- suppose $x \notin \mathbf{cl}\, A$, then $x \in (\mathbf{cl}\, A)^c$
  - $\mathbf{cl}\, A$ is closed $\implies$ $(\mathbf{cl}\, A)^c$ is open $\implies$ $\exists \delta > 0$ s.t. $B(x, \delta) \subseteq (\mathbf{cl}\, A)^c \subseteq A^c \implies B(x, \delta) \cap A = \emptyset$
- suppose $\exists \delta > 0$ such that $B(x, \delta) \cap A = \emptyset$, let $x \in X$
  - $B(x, \delta)$ is open $\implies$ $(B(x, \delta))^c$ is closed
  - $B(x, \delta) \cap A = \emptyset \implies A \subseteq (B(x, \delta))^c \implies \mathbf{cl}\, A \subseteq (B(x, \delta))^c$
  - $x \in B(x, \delta) \implies x \notin (B(x, c))^c$
  - put together, we have $x \notin \mathbf{cl}\, A$

Theorem 9.23 Let (X, d) be a metric space and $A \subseteq X$, then $\mathbf{int}\, A$ is open and $\mathbf{bd}\, A$ is closed.

proof:

- let $x \in \mathbf{int}\, A$
  - $x \in \mathbf{int}\, A \implies \exists\delta > 0$ such that $B(x, \delta) \subseteq A$
  - let $z \in B(x, \delta)$; $B(x, \delta)$ open $\implies \exists\epsilon > 0$ such that $B(z, \epsilon) \subseteq B(x, \delta) \subseteq A \implies z \in \mathbf{int}\, A \implies B(x, \delta) \subseteq \mathbf{int}\, A \implies \mathbf{int}\, A$ is open
- $\mathbf{int}\, A$ open $\implies (\mathbf{int}\, A)^c$ closed $\implies \mathbf{bd}\, A = \mathbf{cl}\, A \setminus \mathbf{int}\, A = \mathbf{cl}\, A \cap (\mathbf{int}\, A)^c$ is closed (theorem 9.15)

Theorem 9.24 Let (X, d) be a metric space and $A \subseteq X$, then $x \in \mathbf{bd}\, A$ if and only if for all $\delta > 0$, we have the sets $B(x, \delta) \cap A$ and $B(x, \delta) \cap A^c$ are both nonempty.

proof:

- suppose $x \in \mathbf{bd}\, A$, let $\delta > 0$
  - $x \in \mathbf{bd}\, A \implies x \in \mathbf{cl}\, A$, and hence, by theorem 9.22, we have $B(x, \delta) \cap A \neq \emptyset$
  - assume $B(x, \delta) \cap A^c = \emptyset$, then we have $B(x, \delta) \subseteq A \implies x \in \mathbf{int}\, A$, which is a contradiction

- suppose $B(x, \delta) \cap A \neq \emptyset$ and $B(x, \delta) \cap A^c \neq \emptyset$ for all $\delta > 0$, assume $x \notin \mathbf{bd}\, A$
  - $x \notin \mathbf{bd}\, A \implies x \notin \mathbf{cl}\, A$ or $x \in \mathbf{int}\, A$
  - if $x \notin \mathbf{cl}\, A \implies \exists\delta_0 > 0$ such that $B(x, \delta_0) \cap A = \emptyset$, which is a contradiction
  - if $x \in \mathbf{int}\, A \implies \exists\delta_0 > 0$ such that $B(x, \delta_0) \subseteq A \implies B(x, \delta_0) \cap A^c = \emptyset$, which is a contradiction

Theorem 9.25 Let (X, d) be a metric space and $A \subseteq X$, then $\mathbf{bd}\, A = \mathbf{cl}\, A \cap \mathbf{cl}(A^c)$.

proof: let $x \in \mathbf{bd}\, A$, $\delta > 0$

- by theorem 9.24, we have $B(x, \delta) \cap A$ and $B(x, \delta) \cap A^c$ nonempty
- by theorem 9.22, $B(x, \delta) \cap A \neq \emptyset \implies x \in \mathbf{cl}\, A$ and $B(x, \delta) \cap A^c \neq \emptyset \implies x \in \mathbf{cl}\, A^c$
- hence, we have $\mathbf{bd}\, A = \mathbf{cl}\, A \cap \mathbf{cl}(A^c)$

Sequences in metric spaces

Definition 9.26 A **sequence** in a metric space (X, d) is a function $x\colon \mathbf{N} \to X$. To denote a sequence we write $(x_n)_{n=1}^{\infty}$, where x_n is the nth element in the sequence.

A sequence $(x_n)_{n=1}^{\infty}$ is **bounded** if there exists a point $p \in X$ and $B \in \mathbf{R}$ such that $d(p, x_n) \leq B$ for all $n \in \mathbf{N}$.

Let $(n_i)_{i=1}^{\infty}$ be a strictly increasing sequence of natural numbers, then the sequence $(x_{n_i})_{i=1}^{\infty}$ is called a **subsequence** of $(x_n)_{n=1}^{\infty}$.

Definition 9.27 A sequence $(x_n)_{n=1}^{\infty}$ in a metric space (X, d) is said to **converge** to a point $p \in X$ if for all $\epsilon > 0$, there exists some $M \in \mathbf{N}$ such that for all $n \geq M$, we have $d(x_n, p) < \epsilon$.

The point p is called a **limit** of $(x_n)_{n=1}^{\infty}$. If the limit p is unique, we write

$$\lim_{n\to\infty} x_n = p.$$

A sequence that converges is said to be **convergent**, and otherwise is **divergent**.

Theorem 9.28 A convergent sequence in a metric space has a unique limit.

proof: let $x, y \in X$ such that $x_n \to x$ and $x_n \to y$; let $\epsilon > 0$

- $x_n \to x \implies \exists M_1 \in \mathbf{N}$ such that $\forall n \geq M_1$, $d(x_n, x) < \epsilon/2$
- $x_n \to y \implies \exists M_2 \in \mathbf{N}$ such that $\forall n \geq M_2$, $d(x_n, y) < \epsilon/2$
- hence, for all $n \geq M$, we have

$$d(x, y) \leq d(x_n, x) + d(x_n, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$$\implies d(x, y) = 0 \implies x = y$$

Theorem 9.29 A convergent sequence in a metric space is bounded.

proof: suppose $x_n \to p \in X$

- let $\epsilon > 0$, $x_n \to p \implies \exists M \in \mathbf{N}$ such that $\forall n \geq M$, $d(x_n, p) < \epsilon$
- choose $B = \max\{d(x_1, p), \ldots, d(x_M, p), \epsilon\}$, then for all $n \in \mathbf{N}$, $d(x_n, p) \leq B$

Theorem 9.30 A sequence $(x_n)_{n=1}^{\infty}$ in a metric space (X,d) converges to $p \in X$ if and only if there exists a sequence $(a_n)_{n=1}^{\infty}$ of real numbers such that for all $n \in \mathbf{N}$, we have

$$d(x_n,p) \le a_n \quad \text{and} \quad \lim_{n\to\infty} a_n = 0.$$

proof:

- suppose $x_n \to p$
  - $x_n \to p \implies \forall \epsilon > 0$, $\exists M \in \mathbf{N}$ s.t. $\forall n \ge M$, $d(x_n,p) < \epsilon \implies d(x_n,p) \to 0$
  - choose $a_n = d(x_n,p)$ for all $n \in \mathbf{N}$, then we have $d(x_n,p) \le a_n$ and $a_n \to 0$
- suppose $a_n \to 0$ with $a_n \in \mathbf{R}$ and $d(x_n,p) \le a_n$, let $\epsilon > 0$
  - $0 \le d(x_n,p) \le a_n$, $a_n \to 0 \implies d(x_n,p) \to 0$ (theorem 3.21)
  - $d(x_n,p) \to 0 \implies \exists M \in \mathbf{N}$ such that $\forall n \ge M$, $d(x_n,p) < \epsilon \implies x_n \to p$

Theorem 9.31 Let $(x_n)_{n=1}^{\infty}$ be a sequence in a metric space (X,d). If $(x_n)_{n=1}^{\infty}$ converges to $p \in X$, then all subsequences of $(x_n)_{n=1}^{\infty}$ converges to p.

proof: let $\epsilon > 0$

- let $x_n \to p$, then $\exists M \in \mathbf{N}$ such that $\forall n \ge M$, $d(x_n,p) < \epsilon$
- let $(x_{n_i})_{i=1}^{\infty}$ be a subsequence of $(x_n)_{n=1}^{\infty}$, then we have $n_i \ge i$
- hence, for all $i \ge M$, we have $n_i \ge M \implies \forall i \ge M$, $d(x_{n_i},p) < \epsilon$

Convergence in Euclidean space

Theorem 9.32 Let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbf{R}^k$, where $x_n \in \mathbf{R}^k$ for all $n \in \mathbf{N}$. Then $(x_n)_{n=1}^{\infty}$ converges if and only if $(x_{n,i})_{n=1}^{\infty}$ converges for all $i = 1, \ldots, k$, *i.e.*,

$$\lim_{n\to\infty} x_n = \left(\lim_{n\to\infty} x_{n,1},\ \ldots,\ \lim_{n\to\infty} x_{n,k} \right).$$

proof:

- suppose $x_n \to p \in \mathbf{R}^k$, let $\epsilon > 0$
  - $x_n \to p \implies \exists M \in \mathbf{N}$ such that $\forall n \geq M$, $d(x_n, p) < \epsilon$
  - hence, $\forall n \geq M$, we have

$$(d(x_n, p))^2 = \sum_{i=1}^{k} (x_{n,i} - p_i)^2 < \epsilon^2 \implies (x_{n,i} - p_i)^2 < \epsilon^2, \quad i = 1, \ldots, k$$

$$\implies |x_{n,i} - p_i| < \epsilon \text{ for all } i = 1, \ldots, k \implies x_{n,i} \to p_i \text{ for all } i = 1, \ldots, k$$

- suppose $x_{n,i} \to p_i$ for all $i = 1, \ldots, k$, let $\epsilon > 0$, $p = (p_1, \ldots, p_k)$
  - $x_{n,i} \to p_i$, $i = 1, \ldots, k \implies \exists M_1, \ldots, M_k \in \mathbf{N}$ such that $\forall n \geq M_i$, we have $|x_{n,i} - p_i| < \epsilon/\sqrt{k}$, $i = 1, \ldots, k$
  - choose $M = \max\{M_1, \ldots, M_k\}$, then $\forall n \geq M$, we have

$$d(x_n, p) = \sqrt{\sum_{i=1}^{k} (x_{n,i} - p_i)^2} < \sqrt{\sum_{i=1}^{k} \left(\frac{\epsilon}{\sqrt{k}} \right)^2} = \sqrt{\sum_{i=1}^{k} \frac{\epsilon^2}{k}} = \sqrt{\epsilon^2} = \epsilon$$

$$\implies x_n \to p$$

Convergence properties of topology

Theorem 9.33 Let (X, d) be a metric space and $(x_n)_{n=1}^{\infty}$ be a sequence in X, then $(x_n)_{n=1}^{\infty}$ converges to $p \in X$ if and only if for all open sets $U \subseteq X$ with $p \in U$, there exists some $M \in \mathbf{N}$ such that for all $n \geq M$, we have $x_n \in U$.

proof:

- suppose $x_n \to p$, let $U \subseteq X$ be open and $p \in U$
  - U is an open set contains $p \implies \exists \delta > 0$ such that $B(p, \delta) \subseteq U$
  - $x_n \to p \implies \exists M \in \mathbf{N}$ s.t. $\forall n \geq M$, $d(x_n, p) < \delta \implies \forall n \geq M$, $x_n \in B(p, \delta)$ $\implies \forall n \geq M$, $x_n \in U$
- suppose for all open sets $U \subseteq X$ with $p \in U$, there exists some $M \in \mathbf{N}$ such that $x_n \in U$ for all $n \geq M$; let $\epsilon > 0$
  - choose $U = B(p, \epsilon)$, then $\exists M \in \mathbf{N}$ such that $\forall n \geq M$, $x_n \in B(p, \epsilon)$
  - hence, $\forall n \geq M$, $d(x_n, p) < \epsilon \implies x_n \to p$

Theorem 9.34 Let (X, d) be a metric space, $E \subseteq X$ be a closed set, and $(x_n)_{n=1}^{\infty}$ be a sequence in E that converges to some $p \in X$, then we have $p \in E$.

proof: assume $(x_n)_{n=1}^{\infty}$ in E converges to p but $p \notin E$

- $p \notin E \implies p \in E^c$
- E is closed $\implies E^c$ is open, then by theorem 9.33, $\exists M \in \mathbf{N}$ such that $\forall n \geq M$, $x_n \in E^c \implies \forall n \geq M$, $x_n \notin E$, which is a contradiction

Theorem 9.35 Let (X, d) be a metric space and $A \subseteq X$, then $p \in \mathbf{cl}\, A$ if and only if there exists a sequence $(x_n)_{n=1}^{\infty}$ of elements in A such that $\lim_{n\to\infty} x_n = p$.

proof:

- suppose $p \in \mathbf{cl}\, A$, then by theorem 9.22, we have $B(p, \delta) \cap A \neq \emptyset$ for all $\delta > 0$
  - choose $(x_n)_{n=1}^{\infty}$ such that $x_n \in A$ and $d(x_n, p) < \frac{1}{n}$ for all $n \in \mathbf{N}$
  - $0 \leq d(x_n, p) < \frac{1}{n}$ and $\frac{1}{n} \to 0 \implies d(x_n, p) \to 0 \implies x_n \to p$ (theorem 9.30)
- suppose $(x_n)_{n=1}^{\infty}$ in A and $x_n \to p$, let $\delta > 0$
  - $x_n \to p \implies \exists M \in \mathbf{N}$ s.t. $\forall n \geq M$, $d(x_n, p) < \delta \implies \forall n \geq M$, $x_n \in B(p, \delta)$
  - then, since $x_n \in A$, we have $B(p, \delta) \cap A \neq \emptyset \implies p \in \mathbf{cl}\, A$ (theorem 9.22)

Cauchy sequences and completeness

Definition 9.36 Let (X, d) be a metric space. A sequence $(x_n)_{n=1}^{\infty}$ in X is **Cauchy** if for all $\epsilon > 0$, there exists some $M \in \mathbf{N}$ such that for all $n, k \geq M$, we have $d(x_n, x_k) < \epsilon$.

Theorem 9.37 A convergent sequence in a metric space is Cauchy.

proof: let $x_n \to p$, $\epsilon > 0$, then $\exists M \in \mathbf{N}$ such that $\forall n, k \geq M$, $d(x_n, p) < \epsilon/2$ and $d(x_k, p) < \epsilon/2$, and hence $\forall n, k \geq M$, we have

$$d(x_n, x_k) \leq d(x_n, p) + d(x_k, p) < \epsilon/2 + \epsilon/2 = \epsilon$$

Definition 9.38 We say a metric space (X, d) is **complete** or **Cauchy-complete** if all Cauchy sequences in X converges to some point in X.

Theorem 9.39 The Euclidean space $\mathbf{R}^k$ is a complete metric space.

proof: let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence with $x_n \in \mathbf{R}^k$ for all $n \in \mathbf{N}$; let $\epsilon > 0$

- $(x_n)_{n=1}^{\infty}$ is Cauchy $\implies$ $\exists M \in \mathbf{N}$ such that $\forall m, n \geq M$, $d(x_m - x_n) < \epsilon$
- hence, for all $m, n \geq M$, we have

$$(d(x_m, x_n))^2 = \sum_{i=1}^{k} (x_{m,i} - x_{n,i})^2 < \epsilon^2 \implies |x_{m,i} - x_{n,i}| < \epsilon, \quad i = 1, \ldots, k$$

$\implies$ the sequence of real numbers $(x_{n,i})_{n=1}^{\infty}$ is Cauchy for all $i = 1, \ldots, k$

- by theorem 3.45, we conclude that $(x_{n,i})_{n=1}^{\infty}$ converges for all $i = 1, \ldots, k$
- then, by theorem 9.32, we conclude that the sequence $(x_n)_{n=1}^{\infty}$ converges