7. Riemann integral

- Riemann sum and some useful facts
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Riemann sum

Definition 7.1 A **partition** $\underline{x} = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ is a finite set such that

$$a = x_0 < x_1 < \cdots < x_n = b.$$

The **norm** of $\underline{x}$, denoted $\|\underline{x}\|$, is a number defined as

$$\|\underline{x}\| = \max\{x_1 - x_0,\ x_2 - x_1,\ \ldots,\ x_n - x_{n-1}\}.$$

Definition 7.2 let $\underline{x}$ be a partition of $[a, b]$. A **tag** of $\underline{x}$ is a finite set $\underline{\xi} = \{\xi_1, \ldots, \xi_n\}$ such that

$$a = x_0 \leq \xi_1 \leq x_1 \leq \xi_2 \leq x_2 \leq \cdots \leq x_{n-1} \leq \xi_n \leq x_n = b.$$

The pair $(\underline{x}, \underline{\xi})$ is referred to as a **tagged partition**.

example: $(\underline{x}, \underline{\xi}) = (\{1, 3/2, 2, 3\},\ \{5/4, 7/4, 5/2\})$ is a tagged partition with norm

$$\|\underline{x}\| = \max\{3/2 - 1,\ 2 - 3/2,\ 3 - 2\} = 1$$

Definition 7.3 The **Riemann sum** of f corresponding to $(\underline{x}, \underline{\xi})$ is the number

$$S_f(\underline{x}, \underline{\xi}) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}).$$

Remark 7.4 For $f \in \mathcal{C}([a,b])$ that is positive, the Riemann sum $S_f(\underline{x}, \underline{\xi})$ is an approximate area under the graph of f. As $\|\underline{x}\| \to 0$, we should expect these approximate areas to converge to some number, which we **interpret** as the area under the graph of f on the interval $[a,b]$.

Some useful facts

Definition 7.5 We define the set $\mathcal{C}([a,b]) = \{f\colon [a,b] \to \mathbf{R} \mid f \text{ is continuous}\}$.

Definition 7.6 Let $f \in \mathcal{C}([a,b])$ and $\tau > 0$, we define the **modulus of continuity** of the function f as

$$w_f(\tau) = \sup\{|f(x) - f(y)| \mid |x - y| \leq \tau\}.$$

Theorem 7.7 For all $f \in \mathcal{C}([a,b])$, we have $\lim_{\tau \to 0} w_f(\tau) = 0$, *i.e.*, for all $\epsilon > 0$, there exists some $\delta > 0$ such that for all $\tau < \delta$, we have $w_f(\tau) < \epsilon$.

proof: let $\epsilon > 0$

- $f \in \mathcal{C}([a,b]) \implies f$ is uniformly continuous on $[a,b] \implies \exists \delta > 0$ such that for all $x, y \in [a,b]$ and $|x-y| < \delta$, we have $|f(x) - f(y)| < \epsilon/2$
- let $\tau < \delta$, then for all $x, y \in [a,b]$ and $|x - y| \leq \tau$, we have $|x - y| < \delta \implies$ $|f(x) - f(y)| < \epsilon/2$ for all $x, y \in [a,b]$ and $|x - y| \leq \tau \implies \epsilon/2$ is an upper bound of the set $\{|f(x) - f(y)| \mid |x - y| \leq \tau\} \implies w_f(\tau) \leq \epsilon/2 < \epsilon$

Theorem 7.8 Let $f \in \mathcal{C}([a,b])$, then $w_f(\tau)$ has the following properties:

- For all $x, y \in [a,b]$, we have $w_f(|x-y|) \geq |f(x) - f(y)|$.
- *Monotonicity.* If $\tau_1 \leq \tau_2$, then $w_f(\tau_1) \leq w_f(\tau_2)$.

Definition 7.9 Let $(\underline{x}, \underline{\xi})$ and $(\underline{x}', \underline{\xi}')$ be tagged partitions of $[a,b]$. We say $\underline{x}'$ is a **refinement** of $\underline{x}$ if $\underline{x} \subseteq \underline{x}'$.

Theorem 7.10 Let $(\underline{x}, \underline{\xi})$ and $(\underline{x}', \underline{\xi}')$ be tagged partitions of $[a,b]$ such that $\underline{x}'$ is a refinement of $\underline{x}$. If $f \in \mathcal{C}([a,b])$, then

$$|S_f(\underline{x}, \underline{\xi}) - S_f(\underline{x}', \underline{\xi}')| \leq w_f(\|\underline{x}\|)(b-a).$$

proof: let $\underline{x} = \{x_0, \ldots, x_n\}$, $\underline{\xi} = \{\xi_1, \ldots, \xi_n\}$, $\underline{x}' = \{x'_0, \ldots, x'_n\}$, $\underline{\xi}' = \{\xi'_1, \ldots, \xi'_n\}$

- for $i = 1, \ldots, n$, let $\underline{y}^{(i)} = \{x'_q, x'_{q+1}, \ldots, x'_k\}$, $\underline{\zeta}^{(i)} = \{\xi'_{q+1}, \xi'_{q+2}, \ldots, \xi'_k\}$ s.t.

$$x_{i-1} = x'_q < x'_{q+1} < \cdots < x'_k = x_i$$

- then for all $i = 1, \ldots, n$, we have

$$
\begin{aligned}
&|f(\xi_i)(x_i - x_{i-1}) - S_f(\underline{y}^{(i)}, \underline{\zeta}^{((i)})|\\
&\quad = \left| f(\xi_i) \sum_{\ell=q+1}^{k} (x'_\ell - x'_{\ell-1}) - \sum_{\ell=q+1}^{k} f(\xi'_\ell)(x'_\ell - x'_{\ell-1}) \right| \\
&\quad = \left| \sum_{\ell=q+1}^{k} (f(\xi_i) - f(\xi'_\ell))(x'_\ell - x'_{\ell-1}) \right| \leq \sum_{\ell=q+1}^{k} |f(\xi_i) - f(\xi'_\ell)|(x'_\ell - x'_{\ell-1}) \\
&\quad \leq \sum_{\ell=q+1}^{k} w_f(x_i - x_{i-1})(x'_\ell - x'_{\ell-1}) \leq \sum_{\ell=q+1}^{k} w_f(\|\underline{x}\|)(x'_\ell - x'_{\ell-1}) \\
&\quad = w_f(\|\underline{x}\|)(x_i - x_{i-1})
\end{aligned} \tag{7.1}
$$

  - the first inequality is by lemma 4.18
  - the second inequality is from $\xi_i, \xi'_\ell \in [x_{i-1}, x_i]$
  - the third inequality is by the second statement of theorem 7.8, and $\|\underline{x}\| \geq x_i - x_{i-1}$

- put together, we have

$$\begin{aligned}|S_f(\underline{x},\underline{\xi}) - S_f(\underline{x}',\underline{\xi}')| &= \left|\sum_{i=1}^n (f(\xi_i)(x_i - x_{i-1}) - S_f(\underline{y}^{(i)},\underline{\zeta}^{(i)}))\right| \\ &\le \sum_{i=1}^n |f(\xi_i)(x_i - x_{i-1}) - S_f(\underline{y}^{(i)},\underline{\zeta}^{(i)})| \le \sum_{i=1}^n w_f(\|\underline{x}\|)(x_i - x_{i-1}) \\ &= w_f(\|x\|)(b-a),\end{aligned}$$

where the last inequality is by plugging in (7.1)

Theorem 7.11 Let $(\underline{x},\underline{\xi})$ and $(\underline{x}',\underline{\xi}')$ be any two tagged partitions of $[a,b]$ and $f \in \mathcal{C}([a,b])$, then

$$|S_f(\underline{x},\underline{\xi}) - S_f(\underline{x}',\underline{\xi}')| \le (w_f(\|\underline{x}\|) + w_f(\|\underline{x}'\|))(b-a).$$

---

proof: let $\underline{x}'' = \underline{x} \cup \underline{x}'$ and $\underline{\xi}''$ be a tag of $\underline{x}''$, then by theorem 7.10, we have

$$\begin{aligned}|S_f(\underline{x},\underline{\xi}) - S_f(\underline{x}',\underline{\xi}')| &\le |S_f(\underline{x},\underline{\xi}) - S_f(\underline{x}'',\underline{\xi}'')| + |S_f(\underline{x}'',\underline{\xi}'') - S_f(\underline{x}',\underline{\xi}')| \\ &\le (w_f(\|\underline{x}\|) + w_f(\|\underline{x}'\|))(b-a)\end{aligned}$$

Riemann integral of continuous functions

Theorem 7.12 Let $f \in \mathcal{C}([a,b])$, then there exists a unique number denoted $\int_a^b f(x)\, dx$ with the following property: For all sequences of tagged partitions $\left((\underline{x}^{(r)}, \underline{\xi}^{(r)})\right)_{r=1}^{\infty}$ such that $\lim_{r\to\infty} \|\underline{x}^{(r)}\| = 0$, we have

$$\lim_{r\to\infty} S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) = \int_a^b f(x)\, dx.$$

---

proof: uniqueness follows immediately from uniqueness of limits of sequences of real numbers, we only need to show the existence

- let $\left((\underline{y}^{(r)}, \underline{\zeta}^{(r)})\right)_{r=1}^{\infty}$ be a sequence of tagged partitions with $\lim_{r\to\infty} \|\underline{y}^{(r)}\| = 0$, we first show that $\left(S_f(\underline{y}^{(r)}, \underline{\zeta}^{(r)})\right)_{r=1}^{\infty}$ is a Cauchy sequence; let $\epsilon > 0$
  - by theorem 7.7, $\exists \delta > 0$ such that for all $\tau < \delta$, $w_f(\tau) < \frac{\epsilon}{2(b-a)}$
  - $\|\underline{y}^{(r)}\| \to 0 \implies \exists M \in \mathbf{N}$ s.t. $\forall r, s \geq M$, $\|\underline{y}^{(r)}\| < \delta$, $\|\underline{y}^{(s)}\| < \delta \implies \forall r, s \geq M$, we have $w_f(\|\underline{y}^{(r)}\|) < \frac{\epsilon}{2(b-a)}$, $w_f(\|\underline{y}^{(s)}\|) < \frac{\epsilon}{2(b-a)}$

- hence, for all $r, s \geq M$, by theorem 7.11, we have

$$
\begin{aligned}
&|S_f(\underline{y}^{(r)}, \underline{\zeta}^{(r)}) - S_f(\underline{y}^{(s)}, \underline{\zeta}^{(s)})| \\
&\qquad \leq (w_f(\|\underline{y}^{(r)}\|) + w_f(\|\underline{y}^{(s)}\|))(b-a) < \left(\frac{\epsilon}{2(b-a)} + \frac{\epsilon}{2(b-a)}\right)(b-a) = \epsilon
\end{aligned}
$$

let $L = \lim_{r\to\infty} S_f(\underline{y}^{(r)}, \underline{\zeta}^{(r)})$ (which exists by theorem 3.45)

- let $\left((\underline{x}^{(r)}, \underline{\xi}^{(r)})\right)_{r=1}^{\infty}$ be any sequence of partitions with $\lim_{r\to\infty} \|\underline{x}^{(r)}\| = 0$, we now show that $\lim_{r\to\infty} S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) = L$
  - since $\|\underline{x}^{(r)}\| \to 0$, $\|\underline{y}^{(r)}\| \to 0$, by theorem 7.7, we have

$$
\lim_{r\to\infty} (w_f(\|\underline{x}^{(r)}\|) + w_f(\|\underline{y}^{(r)}\|))(b-a) = 0
$$

  - $S_f(\underline{y}^{(r)}, \underline{\zeta}^{(r)}) \to L \implies |S_f(\underline{y}^{(r)}, \underline{\zeta}^{(r)}) - L| \to 0$
  - by theorem 7.11, we have

$$
\begin{aligned}
0 \leq |S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) - L| &\leq |S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) - S_f(\underline{y}^{(r)}, \underline{\zeta}^{(r)})| + |S_f(\underline{y}^{(r)}, \underline{\zeta}^{(r)}) - L| \\
&\leq (w_f(\|\underline{x}^{(r)}\|) + w_f(\|\underline{y}^{(r)}\|))(b-a) + |S_f(\underline{y}^{(r)}, \underline{\zeta}^{(r)}) - L|
\end{aligned}
$$

$\implies \lim_{r\to\infty} |S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) - L| = 0$ (theorem 3.21)

Remark 7.13 Let $f \in \mathcal{C}([a,b])$. We sometimes write

$$\int_a^b f(x)\; dx = \int_a^b f.$$

By convention, we also define

$$\int_a^a f = 0 \quad \text{and} \quad \int_b^a f = -\int_a^b f.$$

Properties of Riemann integral

Theorem 7.14 *Linearity.* Let $f, g \in \mathcal{C}([a,b])$ and $\alpha \in \mathbf{R}$, then

$$\int_a^b (\alpha f + g) = \alpha \int_a^b f + \int_a^b g.$$

proof: let $\left((\underline{x}^{(r)}, \underline{\xi}^{(r)})\right)_{r=1}^{\infty}$ be a sequence of tagged partitions such that $\|\underline{x}^{(r)}\| \to 0$, then we have

$$\begin{aligned}
\int_a^b (\alpha f + g) &= \lim_{r\to\infty} S_{\alpha f + g}(\underline{x}^{(r)}, \underline{\xi}^{(r)}) \\
&= \lim_{r\to\infty} (\alpha S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) + S_g(\underline{x}^{(r)}, \underline{\xi}^{(r)})) \\
&= \alpha \lim_{r\to\infty} S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) + \lim_{r\to\infty} S_g(\underline{x}^{(r)}, \underline{\xi}^{(r)}) \\
&= \alpha \int_a^b f + \int_a^b g
\end{aligned}$$

Theorem 7.15 *Additivity.* Let $f \in \mathcal{C}([a,b])$ and $a < c < b$, then we have

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

proof:

- let $\left((\underline{y}^{(r)}, \underline{\zeta}^{(r)})\right)_{r=1}^{\infty}$ be a sequence of tagged partitions of $[a,c]$ with $\|\underline{y}^{(r)}\| \to 0$
- let $\left((\underline{z}^{(r)}, \underline{\eta}^{(r)})\right)_{r=1}^{\infty}$ be a sequence of tagged partitions of $[c,b]$ with $\|\underline{z}^{(r)}\| \to 0$
- then $\left((\underline{x}^{(r)}, \underline{\xi}^{(r)})\right)_{r=1}^{\infty}$ with $\underline{x}^{(r)} = \underline{y}^{(r)} \cup \underline{z}^{(r)}$ and $\underline{\xi}^{(r)} = \underline{\zeta}^{(r)} \cup \underline{\eta}^{(r)}$ is a sequence of tagged partitions of $[a,b]$
- $\|\underline{y}^{(r)}\| \to 0$ and $\|\underline{z}^{(r)}\| \to 0 \implies \|\underline{x}^{(r)}\| \le \|\underline{y}^{(r)}\| + \|\underline{z}^{(r)}\| \to 0$
- hence, we have

$$\begin{aligned}\int_a^b f &= \lim_{r\to\infty} S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) = \lim_{r\to\infty} \left(S_f(\underline{y}^{(r)}, \underline{\zeta}^{(r)}) + S_f(\underline{z}^{(r)}, \underline{\eta}^{(r)})\right) \\ &= \lim_{r\to\infty} S_f(\underline{y}^{(r)}, \underline{\zeta}^{(r)}) + \lim_{r\to\infty} S_f(\underline{z}^{(r)}, \underline{\eta}^{(r)}) = \int_a^c f + \int_c^b f\end{aligned}$$

Theorem 7.16 Let $f, g \in \mathcal{C}([a,b])$ and $f(x) \leq g(x)$ for all $x \in [a,b]$, then we have

$$\int_a^b f \leq \int_a^b g.$$

proof: let $\left((\underline{x}^{(r)}, \underline{\xi}^{(r)})\right)_{r=1}^{\infty}$ be a sequence of tagged partitions with $\|\underline{x}^{(r)}\| \to 0$, then

$$S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) = \sum_{i=1}^{n^{(r)}} f(\xi_i^{(r)})(x_i^{(r)} - x_{i-1}^{(r)}) \leq \sum_{i=1}^{n^{(r)}} g(\xi_i^{(r)})(x_i^{(r)} - x_{i-1}^{(r)}) = S_g(\underline{x}^{(r)}, \underline{\xi}^{(r)})$$

$$\implies \lim_{r\to\infty} S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) \leq \lim_{r\to\infty} S_g(\underline{x}^{(r)}, \underline{\xi}^{(r)}) \implies \int_a^b f \leq \int_a^b g$$

Corollary 7.17 Let $f \in \mathcal{C}([a,b])$, then $\left|\int_a^b f\right| \leq \int_a^b |f|$.

proof: $\pm f(x) \leq |f(x)| \implies \int_a^b \pm f = \pm \int_a^b f \leq \int_a^b |f|$ (theorem 7.16)

Theorem 7.18 Let $f \in \mathcal{C}([a,b])$, and

$$m_f = \inf\{f(x) \mid x \in [a,b]\}, \qquad M_f = \sup\{f(x) \mid x \in [a,b]\}.$$

Then, we have

$$m_f(b-a) \leq \int_a^b f \leq M_f(b-a).$$

proof: let $\left((\underline{x}^{(r)}, \underline{\xi}^{(r)})\right)_{r=1}^{\infty}$ be a sequence of tagged partitions with $\|\underline{x}^{(r)}\| \to 0$, then

$$S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) = \sum_{i=1}^{n^{(r)}} f(\xi_i^{(r)})(x_i^{(r)} - x_{i-1}^{(r)}) \geq \sum_{i=1}^{n^{(r)}} m_f(x_i^{(r)} - x_{i-1}^{(r)}) = m_f(b-a)$$

$$S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) = \sum_{i=1}^{n^{(r)}} f(\xi_i^{(r)})(x_i^{(r)} - x_{i-1}^{(r)}) \leq \sum_{i=1}^{n^{(r)}} M_f(x_i^{(r)} - x_{i-1}^{(r)}) = M_f(b-a)$$

$\implies m_f(b-a) \leq \lim_{r\to\infty} S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) \leq M_f(b-a)$

Fundamental theorem of calculus

Theorem 7.19 *Fundamental theorem of calculus.* Let $f \in \mathcal{C}([a,b])$.

- If $F\colon [a,b] \to \mathbf{R}$ is differentiable and $F' = f$, then

$$\int_a^b f = F(b) - F(a).$$

- The function $G(x) = \int_a^x f$ is differentiable on $[a,b]$ with

$$G(a) = 0, \qquad G'(x) = f(x).$$

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proof:

- let $\left(\underline{x}^{(r)}\right)_{r=1}^{\infty}$ be a sequence of partitions with $\|\underline{x}^{(r)}\| \to 0$, by theorem 6.15, there exist tags $\underline{\xi}^{(r)}$ with $\xi_i^{(r)} \in [x_{i-1}^{(r)}, x_i^{(r)}]$, $i = 1, \ldots, n^{(r)}$, such that

$$F(x_i^{(r)}) - F(x_{i-1}^{(r)}) = F'(\xi_i^{(r)})(x_i^{(r)} - x_{i-1}^{(r)}) = f(\xi_i^{(r)})(x_i^{(r)} - x_{i-1}^{(r)})$$

hence, for the sequence of tagged partitions $\left((\underline{x}^{(r)}, \underline{\xi}^{(r)})\right)_{r=1}^{\infty}$ we have

$$S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) = \sum_{i=1}^{n^{(r)}} f(\xi_i^{(r)})(x_i^{(r)} - x_{i-1}^{(r)}) = \sum_{i=1}^{n^{(r)}} F(x_i^{(r)}) - F(x_{i-1}^{(r)}) = F(b) - F(a)$$

$\implies \int_a^b f = \lim_{r\to\infty} S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) = F(b) - F(a)$

- we only need to show that G is differentiable and $G' = f$, *i.e.*, let $c \in [a, b]$, we need to prove that $\lim_{x\to c} \frac{G(x)-G(c)}{x-c} = \lim_{x\to c} \frac{\int_a^x f - \int_a^c f}{x-c} = f(c)$; let $\epsilon > 0$
  - f continuous on $[a, b] \implies \exists\delta > 0$ such that for all $t \in [a, b]$ and $|t - c| < \delta$, we have $|f(t) - f(c)| < \epsilon/2$
  - suppose $x \in (c, c + \delta)$, then for all $t \in [c, x]$, we have $|f(t) - f(c)| < \epsilon/2$, hence,

$$\begin{aligned}
\left|\frac{\int_a^x f - \int_a^c f}{x-c} - f(c)\right| &= \left|\frac{\int_c^x f(t)\ dt}{x-c} - f(c)\right| \\
&= \left|\frac{1}{x-c}\left(\int_c^x f(t)\ dt - \int_c^x f(c)\ dt\right)\right| = \frac{1}{x-c}\left|\int_c^x (f(t) - f(c))\ dt\right| \\
&\leq \frac{1}{x-c}\int_c^x |f(t) - f(c)|\ dt \leq \frac{1}{x-c}\int_c^x \frac{\epsilon}{2}\ dt = \frac{1}{x-c}\cdot\frac{\epsilon}{2}(x-c) = \frac{\epsilon}{2} < \epsilon
\end{aligned}$$

(the first inequality is by corollary 7.17)

- suppose $x \in (c-\delta, c)$, using similar argument, we have $\left|\frac{\int_a^x f - \int_a^c f}{x-c} - f(c)\right| < \epsilon$
- put together, we conclude that for all $x \in [a,b]$ and $0 < |x-c| < \delta$, we have

$$\left|\frac{\int_a^x f - \int_a^c f}{x-c} - f(c)\right| < \epsilon$$

$$\Longrightarrow \quad \lim_{x\to c} \frac{G(x)-G(c)}{x-c} = \lim_{x\to c} \frac{\int_a^x f - \int_a^c f}{x-c} = f(c)$$

Integration by parts

Theorem 7.20 *Integration by parts.* Suppose $f, g \in \mathcal{C}([a,b])$, $f', g' \in \mathcal{C}([a,b])$, then

$$\int_a^b f'g = (f(b)g(b) - f(a)g(a)) - \int_a^b fg'.$$

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proof: let $F \in \mathcal{C}([a,b])$ with $F(x) = f(x)g(x)$, by theorem 6.8, we have

$$F'(x) = f'(x)g(x) + f(x)g'(x),$$

and hence,

$$\begin{aligned}\int_a^b f'(x)g(x)\ dx + \int_a^b f(x)g'(x)\ dx &= \int_a^b (f'(x)g(x) + f(x)g'(x))\ dx \\ &= \int_a^b F'(x)\ dx = F(b) - F(a) = f(b)g(b) - f(a)g(a)\end{aligned}$$

$\implies \int_a^b f'g = (f(b)g(b) - f(a)g(a)) - \int_a^b fg'$

Change of variables

Theorem 7.21 *Change of variables.* Let $f \in \mathcal{C}([c,d])$ and $\varphi\colon [a,b] \to [c,d]$ be continuously differentiable with $\varphi(a) = c$ and $\varphi(b) = d$. Then, we have

$$\int_c^d f(u)\ du = \int_a^b f(\varphi(x))\varphi'(x)\ dx.$$

proof:

- let $F\colon [a,b] \to \mathbf{R}$ be a function with $F' = f$, then we have

$$\int_c^d f(u)\ du = F(d) - F(c)$$

- by theorem 6.9, we have

$$(F \circ \varphi)'(x) = F'(\varphi(x))\varphi'(x) = f(\varphi(x))\varphi'(x),$$

  and hence,

$$\int_a^b f(\varphi(x))\varphi'(x)\ dx = F(\varphi(b)) - F(\varphi(a)) = F(d) - F(c) = \int_c^d f(u)\ du$$