The Clairaut-Legendre method for slow differential rotation

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Introduction

Consider a star in permanent rotation around a fixed axis, neglecting friction and electromagnetic forces. The star may rotate rigidly, with the angular velocity $\Omega(\mathbf{r}) = \Omega$ constant. Alternatively, the star could rotate *differentially*, with $\Omega(\mathbf{r})$ dependent on position.

If we assume a simple form for the differential rotation (given below), it is possible (at least in principle) to determine equilibria for *slow* rotation by extending the methods used for slow *rigid* rotation. In what follows, we also assume that the star is axisymmetric and has an equatorial plane of symmetry perpendicular to the axis of rotation.

Differential rotation

A simple form of differential rotation is given by [1] as

$$\Omega(r, \theta) = \frac{A^2 \Omega_c}{A^2 + r^2 \sin^2 \theta}, \tag{1}$$

where A is the *rotation parameter* and Ω_c (plugging in $\theta = 0$) is the angular velocity on the rotation axis.* Equivalently, we may write

$$\Omega(\varpi) = \frac{A^2 \Omega_c}{A^2 + \varpi^2} = \frac{\Omega_c}{1 + (\varpi/A)^2}$$

with $\varpi = r \sin \theta$ the cylindrical radial coordinate. In the limit $A \to \infty$, we have $\Omega \to \Omega_c$. Thus, $A \to \infty$ lets us recover rigid rotation.

*This corresponds to a linear relation $j(\Omega)$, where j is the specific angular momentum of the fluid. See [1]. We will henceforth refer to this rotation law as "j-linear".

Since Ω does not depend on z, the effective gravity (see [2]) is derived from a potential

$$
\begin{aligned}
\Phi(\varpi, z) &= V(\varpi, z) - \int^{\varpi} \Omega^2(\varpi')\varpi' \, d\varpi' \\
&= V(\varpi, z) - A^4\Omega_c^2 \int^{\varpi} \frac{\varpi' \, d\varpi'}{(A^2 + \varpi'^2)^2} \\
&= V(\varpi, z) + \frac{A^4\Omega_c^2}{2(A^2 + \varpi^2)} + C \qquad (2) \\
&= V(\varpi, z) + \frac{1}{2}A^2\Omega_c\Omega + C \qquad (3)
\end{aligned}
$$

for some arbitrary constant C. Let $A > \varpi$ (so we can use a geometric series) and consider the centrifugal potential term:

$$
\begin{aligned}
\frac{1}{2}A^2\Omega_c\Omega &= \frac{\Omega_c^2 A^2}{2(1 + (\varpi/A)^2)} \\
&= \frac{1}{2}\Omega_c^2 A^2(1 - (\varpi/A)^2 + \mathcal{O}(A^{-4})) \\
&= \frac{1}{2}\Omega_c^2(A^2 - \varpi^2) + \mathcal{O}(A^{-2}).
\end{aligned}
$$

Thus, for $A > \varpi$, we obtain

$$
\begin{aligned}
\Phi(\varpi, z) &= V(\varpi, z) + \frac{1}{2}A^2\Omega_c\Omega + C \\
&= V(\varpi, z) + C + \frac{1}{2}\Omega_c^2 A^2 - \frac{1}{2}\Omega_c^2\varpi^2 + \mathcal{O}(A^{-2}) \\
&= V'(\varpi, z) - \frac{1}{2}\Omega_c^2\varpi^2 + \mathcal{O}(A^{-2})
\end{aligned}
$$

where $V' = V + C + \frac{1}{2}\Omega_c^2 A^2$ is physically equivalent to V since it differs by a constant. As expected, in the limit $A \to \infty$ we recover the potential for rigid rotation, $\Phi = V - \frac{1}{2}\Omega^2\varpi^2$.

Potential terms

We now apply the Clairaut-Legendre method [2] to find equilibria for slow differential rotation of the form (1). Since $\partial\Omega/\partial z = 0$, level surfaces of Φ coincide with isobaric and isopycnic surfaces. We can label the level surfaces of Φ by their *mean* radius a, and denote the equipotential at the surface of the star by a_s, so $\rho(a_s) = 0$.

By axial symmetry, the distance of points on an equipotential a from the centre of mass should depend only on θ, i.e., we have a relation $r(a, \theta)$. Fixing a and expanding $r_a(\theta) = r(a, \theta)$ in Legendre polynomials, we obtain

$$r(a, \theta) = a \left[1 - \sum_{n=1}^{\infty} \epsilon_{2n}(a) P_{2n}(\cos\theta) \right]. \tag{4}$$

We only keep even components due to equatorial symmetry. The function $\epsilon_{2n}(a)$ describes† the strength of the P_{2n} deformation for the equipotential surface a. In the slow rotation regime, we assume the first-order deformation will dominate; hence, we find $\epsilon(a) = \epsilon_2(a)$ and ignore higher order terms. This is equivalent to assuming *spheroidal* level surfaces (as discussed in [2]).

To determine ϵ, we will expand (3) in Legendre polynomials and use a standard perturbative argument. At a point $\mathbf{r}'$ on an equipotential with mean radius a', we can split the gravitational potential into an external contribution V_e from mass elements with $a > a'$, and an internal contribution V_i with $a < a'$. Then, integrating over mass,

$$V_e(\mathbf{r}') = -G \int_{M(a')}^{M(a_s)} \frac{dm}{|\mathbf{r} - \mathbf{r}'|}, \tag{5}$$

$$V_i(\mathbf{r}') = -G \int_{0}^{M(a')} \frac{dm}{|\mathbf{r} - \mathbf{r}'|}. \tag{6}$$

Here, M depends on a' via

$$M(a') = 4\pi \int_0^{a'} \rho(a) a^2 da.$$

We can rewrite the differential mass element as

$$dm = \rho(a)\, dV = \rho(a) r^2 (\partial r / \partial a) \sin\theta d\theta\, d\phi\, da. \tag{7}$$

We expand the separation vector in Legendre polynomials in the usual fashion. Let $\mathbf{r}' = (r', \phi', \theta')$ (fixed) and $\mathbf{r} = (r, \phi, \theta)$ (varying) in spherical coordinates, and note that

$$\cos\gamma = \frac{\mathbf{r} \cdot \mathbf{r}'}{rr'} = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi - \phi'), \tag{8}$$

†The negative signs are conventional, but anticipate polar flattening, i.e., $r < a$ when $\theta \approx 0$ and hence $P_{2n}(\cos\theta) \approx 1$.

where γ is the angle between $\mathbf{r}$ and $\mathbf{r}'$. Then

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{n=0}^{\infty} \frac{r^n}{r'^{n+1}} P_n(\cos\gamma) \qquad (r < r') \tag{9}$$

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{n=0}^{\infty} \frac{r'^n}{r^{n+1}} P_n(\cos\gamma) \qquad (r' < r). \tag{10}$$

We need two more formulas. The first is the addition formula for spherical harmonics, which we write using the associated Legendre functions ${P_n}^m$:

$$\begin{aligned} P_n(\cos\gamma) &= P_n(\cos\theta)P_n(\cos\theta') \\ &\quad + 2\sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} {P_n}^m(\cos\theta){P_n}^m(\cos\theta')\cos(m(\phi-\phi')). \end{aligned} \tag{11}$$

The second follows from (11) and the orthogonality relations for Legendre polynomials:

$$\begin{aligned} \int_0^{2\pi}\int_0^{\pi} P_n(\cos\gamma)P_l(\cos\theta)\sin\theta\, d\theta\, d\phi &= \int d\Omega_{\mathrm{SA}}\, P_n(\cos\gamma)P_l(\cos\theta) \\ &= \frac{4\pi}{2n+1} P_n(\cos\theta')\delta_{ln}, \end{aligned} \tag{12}$$

with θ, θ', and γ as for (8) and $d\Omega_{\mathrm{SA}}$ denoting the solid angle element. The addition theorem and orthogonality relations can be found in any standard text on mathematical physics, e.g., [3].

Assuming that products of ϵ_{2n} terms disappear, we can use formulas (7)–(12) to rewrite (5) and (6):

$$V_e(\mathbf{r}') = -\frac{4\pi G}{3}\int_{a'}^{a_s} da\, \rho(a) \frac{\partial}{\partial a}\left[\frac{3a^2}{2} - \sum_{n=1}^{\infty} \frac{3}{4n+1} \frac{r'^{2n}}{a^{2n-2}} \epsilon_{2n}(a) P_{2n}(\cos\theta')\right], \tag{13}$$

$$V_i(\mathbf{r}') = -\frac{4\pi G}{3}\int_{0}^{a'} da\, \rho(a) \frac{\partial}{\partial a}\left[\frac{a^3}{r'} - \sum_{n=1}^{\infty} \frac{3}{4n+1} \frac{a^{2n+3}}{r'^{2n+1}} \epsilon_{2n}(a) P_{2n}(\cos\theta')\right]. \tag{14}$$

See the appended derivation. Thus, $V_e(\mathbf{r}')$ and $V_i(\mathbf{r}')$ depend only on (r', θ').

The Clairaut-Legendre equation

We know that $\Phi = \Phi(a)$. Using (3) and dropping primes for clarity, it follows that

$$\begin{aligned}\Phi(a) = V(r,\theta) + \frac{1}{2}A^2\Omega_c\Omega &= V(r,\theta) + \frac{A^4\Omega_c^2}{2(A^2 + r^2\sin^2\theta)} \\ &\approx V(r,\theta) + \frac{A^4\Omega_c^2}{2[A^2 + \frac{2}{3}a^2(1 - P_2(\cos\theta))]} \end{aligned} \tag{15}$$

is a function of a only. Note that we have used the first-order approximation $r \approx a$, and the fact that $\sin^2\theta = \frac{2}{3}(1 - P_2(\cos\theta))$. We expand the $\cos\theta$ dependence in Legendre polynomials. Dropping odd terms (the dependence on $\cos\theta$ is even), we have

$$\frac{A^4\Omega_c^2}{2[A^2 + \frac{2}{3}a^2(1 - P_2(\cos\theta))]} = c_0(a) + \sum_{n=1}^{\infty} c_n(a)P_{2n}(\cos\theta). \tag{16}$$

Given our approximation, independence of θ translates into the requirement that the coefficient of P_2 in (15) vanishes. We already have $V = V_i + V_e$ in a suitable form. We must also find $c(a) = c_1(a)$ in (16). We use the orthogonality of Legendre polynomials

$$\int_{-1}^{1} dx\, P_n(x)P_m(x) = \frac{2}{2n+1}\delta_{nm},$$

with $x = \cos\theta$. Then, restoring primes and setting $\alpha = 2(a'/A)^2$,

$$\begin{aligned} c(a') &= \frac{5A^4\Omega_c^2}{4}\int_{-1}^{1} dx\, \frac{P_2(x)}{(A^2 + \frac{2}{3}a'^2) - \frac{2}{3}a'^2 P_2(x)} \\ &= \frac{5A^2\Omega_c^2}{4}\int_{-1}^{1} dx\, \frac{P_2(x)}{(1 + \frac{1}{3}\alpha) - \frac{1}{3}\alpha P_2(x)} \\ &= \frac{15A^4\Omega_c^2}{8a'^2}\left[\frac{2(1+\frac{1}{3}\alpha)\tanh^{-1}\left(x\sqrt{\alpha(2+\alpha)^{-1}}\right)}{\sqrt{\alpha(2+\alpha)}} - x\right]_{-1}^{1} \\ &= \frac{15A^4\Omega_c^2}{4a'^2}\left[\frac{2(1+\frac{1}{3}\alpha)\tanh^{-1}\left(\sqrt{\alpha(2+\alpha)^{-1}}\right)}{\sqrt{\alpha(2+\alpha)}} - 1\right]. \end{aligned} \tag{17}$$

Using (13) and the approximation $r' \approx a'$ (which gives first-order accuracy), the coefficient c_e of P_2 in V_e is

$$\begin{aligned} c_e &= \frac{4\pi G}{3} \int_{a'}^{a_s} da\, \rho(a) \frac{\partial}{\partial a} \frac{3}{5} r'^2 \epsilon(a) \\ &= \frac{4\pi G}{5} r'^2 \int_{a'}^{a_s} da\, \rho(a) \epsilon_a(a) \\ &\approx \frac{4\pi G}{5} a'^2 \int_{a'}^{a_s} da\, \rho(a) \epsilon_a(a). \end{aligned} \tag{18}$$

Using (14), and expanding $(r')^{-n}$ to first-order as

$$\frac{1}{(r')^n} = \frac{1}{(a')^n (1 - \epsilon(a) P_2)^n} \approx \frac{1}{(a')^n} (1 + n\epsilon(a') P_2),$$

the coefficient c_i of P_2 in V_i is

$$\begin{aligned} c_i &\approx -\frac{4\pi G}{3} \int_0^{a'} da\, \rho(a) \frac{\partial}{\partial a} \left(\frac{a^3}{a'} \epsilon(a') - \frac{3}{5} \frac{a^5}{a'^3} \epsilon(a) \right) \\ &= -\frac{4\pi G \epsilon(a')}{a'} \int_0^{a'} da\, \rho(a) a^2 - \frac{4\pi G}{5a'^3} \int_0^{a'} da\, \rho(a) \frac{\partial}{\partial a} [a^5 \epsilon(a)]. \end{aligned} \tag{19}$$

Since the sum of (17), (18) and (19) must vanish, we have

$$\frac{c}{4\pi G} = -\frac{c_e + c_i}{4\pi G}$$

or

$$\frac{1}{4\pi G} c(a') = \frac{1\epsilon(a')}{a'} \int_0^{a'} da\, \rho(a) a^2 - \frac{1}{5a'^3} \int_0^{a'} da\, \rho(a) \frac{d}{da} [a^5 \epsilon(a)] - \frac{a'^2}{5} \int_{a'}^{a_s} da\, \rho(a) \epsilon'(a) \tag{20}$$

where $\epsilon' = d\epsilon/da$.

In what follows, we swap primed and unprimed variables for clarity. We multiply by a^3 and differentiate with respect to a. Considering the right-hand side first, we get

$$(a^2 \epsilon'(a) + 2a\epsilon(a)) \int_0^a da'\, \rho(a') a'^2 - a^4 \int_a^{a_s} da'\, \rho(a') \epsilon'(a'). \tag{21}$$

In terms of $c(a)$, the left-hand side gives

$$\frac{\partial}{\partial a}\left(\frac{a^3 c(a)}{4\pi G}\right) = \frac{a^2}{4\pi G}(3c(a) + ac'(a)) \tag{22}$$

where $c' = dc/da$.

Define the *mean density* of matter within mean radius a by

$$\rho_m(a) = \frac{3}{a^3}\int_0^a da'\, \rho(a')a'^2. \tag{23}$$

Then, if we divide (21) and (22) by a^4, differentiate, and multiply by $3a/\rho_m(a)$, we obtain for the left-hand side

$$a^2\epsilon''(a) + 6\frac{\rho(a)}{\rho_m(a)}\,(a\epsilon'(a) + \epsilon(a)) - 6\epsilon(a). \tag{24}$$

On the right-hand side, we get

$$\frac{3}{4\pi G\rho_m(a)a^2}(a^2c''(a) + ac'(a) - 4c(a)). \tag{25}$$

Finally, we consider the change of variable

$$\eta(a) = \frac{d\log\epsilon}{d\log a} = \frac{a}{\epsilon}\epsilon'(a).$$

We interpret η as the *order of growth of* ϵ, which can be seen by making the substitution $\epsilon = a^n$. Substituting in (24) and combining with (25), we obtain

$$a\eta'(a) + 6\frac{\rho(a)}{\rho_m(a)}(\eta + 1) + \eta(\eta - 1) - 6 = \frac{3(a^2c''(a) + ac'(a) - 4c(a))}{4\pi G\rho_m(a)a^2}. \tag{26}$$

The appendix below shows that this equation is regular at $a = 0$ and reduces to the rigid expression when $a \to 0$ or $A \to \infty$. As a result, the boundary condition is identical to the rigid case:

$$\eta(0) = 0. \tag{27}$$

Finally, we need to additional boundary condition to link η and ϵ. This is usually supplied by evaluating (20) at the surface $a = a_s$.

A1. Derivation of (14)

Legendre polynomials P_n with argument suppressed are functions of $\cos\theta$ (if they appear in the $\Omega_{\rm SA}$ integral) and $\cos\theta'$ otherwise. Similarly, ϵ_{2k} is always a function of a. Finally, recall that we assume the ϵ_{2k} are small enough to neglect their products. Entries on the right refer to formulae used:

$$
\begin{aligned}
V_i(\mathbf{r}') &= -G\int_0^{M(a')}\frac{dm}{|\mathbf{r}-\mathbf{r}'|} \\
&= -G\int_0^{M(a')} dm\sum_{n=0}^{\infty}\frac{r^n}{r'^{n+1}}P_n(\cos\gamma) && (9)\\
&= -G\int_0^{a'} da\,\rho(a)\sum_{n=0}^{\infty}\int d\Omega_{\rm SA}\left(\frac{\partial r}{\partial a}\right)\frac{r^{n+2}}{r'^{n+1}}P_n(\cos\gamma) && (7)\\
&= -G\int_0^{a'} da\,\rho(a)\frac{\partial}{\partial a}\left[\sum_{n=0}^{\infty}\frac{1}{n+3}\int d\Omega_{\rm SA}\frac{r^{n+3}}{r'^{n+1}}P_n(\cos\gamma)\right] \\
&= -G\int_0^{a'} da\,\rho(a)\frac{\partial}{\partial a}\left[\sum_{n=0}^{\infty}\frac{1}{n+3}\frac{a^{n+3}}{r'^{n+1}}\int d\Omega_{\rm SA}\left(1-\sum_{k=1}^{\infty}\epsilon_{2k}P_{2k}\right)^{n+3}P_n(\cos\gamma)\right] && (4)\\
&\approx -G\int_0^{a'} da\,\rho(a)\frac{\partial}{\partial a}\left[\sum_{n=0}^{\infty}\frac{1}{n+3}\frac{a^{n+3}}{r'^{n+1}}\int d\Omega_{\rm SA}\left(1-(n+3)\sum_{k=1}^{\infty}\epsilon_{2k}P_{2k}\right)P_n(\cos\gamma)\right] \\
&= -G\int_0^{a'} da\,\rho(a)\frac{\partial}{\partial a}\left[\sum_{n=0}^{\infty}\frac{1}{n+3}\frac{a^{n+3}}{r'^{n+1}}\left(4\pi\delta_{0n}-(n+3)\frac{4\pi}{2n+1}\sum_{k=1}^{\infty}\epsilon_{2k}P_n\delta_{2k,n}\right)\right] && (12)\\
&= -G\int_0^{a'} da\,\rho(a)\frac{\partial}{\partial a}\left[\frac{4\pi}{3}\frac{a^3}{r'}-\sum_{k=1}^{\infty}\frac{4\pi}{4k+1}\frac{a^{2k+3}}{r'^{2k+1}}\epsilon_{2k}P_{2k}\right].
\end{aligned}
$$

This is equivalent to (14). We leave the similar derivation of (13) to the reader.

A2. Results about the function $c(a)$

We list some results about the function in (17),

$$c(a) = \frac{15A^4\Omega_c^2}{4a^2}\left[\frac{2(1+\frac{1}{3}\alpha)\tanh^{-1}\left(\sqrt{\alpha(2+\alpha)^{-1}}\right)}{\sqrt{\alpha(2+\alpha)}} - 1\right], \quad \alpha = 2(a/A)^2.$$

We first check that the right-hand side of (26) is regular as $a \to 0$. Using the fact that $\alpha \to 0$ and the Maclaurin series

$$\tanh^{-1}(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots$$

we can expand to second order:

$$\begin{aligned}
c(a) &= \frac{15A^4\Omega_c^2}{4a^2}\left[\frac{2(1+\frac{1}{3}\alpha)}{\sqrt{\alpha(2+\alpha)}}\tanh^{-1}\sqrt{\frac{\alpha}{2+\alpha}} - 1\right] \\
&\approx \frac{15A^4\Omega_c^2}{4a^2}\left[\frac{2(1+\frac{1}{3}\alpha)}{\sqrt{\alpha(2+\alpha)}}\left(\sqrt{\frac{\alpha}{2+\alpha}} + \frac{1}{3}\left(\frac{\alpha}{2+\alpha}\right)^{3/2} + \frac{1}{5}\left(\frac{\alpha}{2+\alpha}\right)^{5/2}\right) - 1\right] \\
&= \frac{15A^4\Omega_c^2}{4a^2}\left[2(1+\tfrac{1}{3}\alpha)\left(\frac{1}{2+\alpha} + \frac{\alpha}{3(2+\alpha)^2} + \frac{\alpha^2}{5(2+\alpha)^3}\right) - 1\right] \\
&\approx \frac{15A^4\Omega_c^2}{4a^2}\left[(1+\tfrac{1}{3}\alpha)\left(1 - \tfrac{1}{3}\alpha + \tfrac{2}{15}\alpha^2\right) - 1\right] \\
&\approx \frac{15A^4\Omega_c^2}{4a^2}\frac{\alpha^2}{45} = \frac{a^2\Omega_c^2}{3}.
\end{aligned} \tag{28}$$

Thus, we see that as $\alpha \to 0$, the RHS of (26)

$$\frac{3(a^2c''(a) + ac'(a) - 4c(a))}{4\pi G\rho_m(a)a^2} \to 0. \tag{29}$$

It follows that if $a \to 0$, $\alpha \to 0$ and the solution is regular at $a = 0$. It also follows that in the rigid limit $A \to \infty$, $\alpha \to 0$ and expression (29) vanishes for *all* a. This is expected since in the case of rigid rotation, c is identically 0 and the RHS of (26) is trivial.

However, we want to know the leading-order behaviour of $c(a)$ for small a. If we push the series expansion one step further (calculation omitted), we pick up an additional term

$$c(a) \approx \frac{a^2\Omega_c^2}{3} - \frac{8a^4\Omega_c^2}{21A^2}. \tag{30}$$

The RHS of (26) now gives

$$\frac{3(a^2c''(a)+ac'(a)-4c(a))}{4\pi G\rho_m(a)a^2} \to -\frac{3}{4\pi G\rho_c a^2}\frac{8\Omega_c^2}{21A^2}(12a^4+4a^4-4a^4)$$
$$= -\frac{24\Omega_c^2 a^2}{7\pi G\rho_c A^2} = \vartheta(a) \tag{31}$$

where $\rho_c = \rho(0) = \rho_m(0)$ is the central density. This expression vanishes when $A \to \infty$, but for finite A gives the leading order behaviour due to differential rotation.

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