

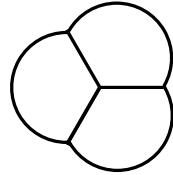
Why is a soap bubble like a railway?

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Abstract

At a certain infamous tea party, the Mad Hatter posed the riddle: why is a raven like a writing desk? We do not answer this question. Instead, we solve a related, apparently nonsensical query: why is a soap bubble like a railway? One answer is that both *minimize over combinatorial structures*. We give a self-contained introduction to this genre of minimization problem, starting with minimal networks on the Euclidean plane and ending with tetrahedrally close-packed structures for three-dimensional foams. Along the way, we touch on algorithms and complexity, graph theory, curvature, physics, space-filling polyhedra, and bees from other dimensions. The only prerequisites are high school geometry and algebra, and a spirit of adventure.



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1 Introduction

What do soap bubbles, railway lines, and beehives have in common? All three involve some choice of minimal structure. Surface tension pulls a soap bubble taut so as to minimize area; rail costs money, so railway networks minimize length; and evolution selects for bees which use as little wax as possible in building hive walls. Intriguingly, all three problems are related! Two-dimensional bubbles solve the railway problem, and a large foam is hexagonal for the same reason as a beehive. One of our goals will be to understand these connections.

A second goal, however, is to turn minimization into a mathematical theme park featuring graph theory, algorithms and complexity, the physics of computation, curvature, polyhedra, tessellations, and various unsolved conjectures. This is a maximalist take on minimization. The only prerequisites are high school algebra and geometry, so these notes should be suitable for high school and undergraduate mathematics enrichment. In consequence, we will frequently be sloppy, diagrammatic, or even “physical” in our reasoning. Some readers will no doubt object. But the price of admission is lower, and we hope the rides are no less fun.

Overview. In §2, we initiate our study of making things small with the surprisingly rich problem of minimal networks on a triangle. In §2.1, we analyze the equilateral triangle using symmetry, and argue that a hub should be placed in the center. We deform this solution in §2.2, and give some loose arguments that the hub collides with a vertex when an internal angle opens to 120° . This is generalized in §2.3 to give the 120° rule for general minimal networks. Finally, in §2.4, we give a brief history of minimal networks and related problems.

In §3, we use tools from graph theory to take the 120° rule, which is a local constraint, and turn it into a global constraint on the structure of the network. Trees and their basic properties are introduced in §3.1, and exploited in §3.2 to put a bound on the maximum number of hubs. This allows us to solve some small but nontrivial networks. In §3.3, the bound is turned into a rough argument for the computational hardness of finding minimal networks, while §3.4 provides some easily computable alternatives, namely the minimal spanning tree and Steiner insertion heuristic.

With §4, we move laterally into the realm of two-dimensional foams, constructing soap bubble computers in §4.1 to solve our minimal network problems. Our computational hardness results resurface as predictions about the physics of soap bubbles. In §4.2, we introduce Euler’s formula and apply it to bubble networks, while in §4.3, we make a simple scaling argument that most bubbles in a large foam are hexagonal. This is related to the fact that bees build hexagonal hives, and the honeycomb theorem that bees know the best way to partition the plane into cells of equal size. The planar bubble configuration problem make its appearance in §4.4, along with special case like the isoperimetric inequality and planar double and triple bubbles.

The last section, §5, requires a little more pluck. After defining mean curvature in §5.1, we state Plateau’s laws in §5.2, after motivating them by analogy with bubble networks. With §5.3, we describe the three-dimensional bubble configuration problem, and dip our toes into the soapy water of Plateau’s problem. Finally, in §5.4 we generalize Euler’s formula to study network constraints on three-dimensional foams, and conclude with a whirlwind tour of regular tessellations of space, the Kelvin problem, the Weaire-Phelan surprise, and the related tetrahedrally closed-packed structures.

Exercises. There are 40 problems of varying length and difficulty. Many of these are essential and used subsequently in the text. I hope this is not a weakness, but rather than an incentive to solve them! Difficult exercises are labelled with a mountain (\blacktriangle), or an icy mountain (\blacktriangleleft) in the case of greater abstraction or required background. Longer labels (\blacktriangle and \blacktriangleleft) inflect for length. For solutions, please contact me [by email](#).

2 Trains and triangles

Suppose we want to join up three towns A , B and C by rail. Building railways is expensive, since we not only need to design and build the rail itself, but acquire the land beneath it. In contrast, stations can be reasonably cheap: we just slap together some sidings, a platform, and a bench or two. To minimize cost we should make the total length of the rail network as short as possible.

If the railway lets us travel from one town to any other, we say that the rail network is *connected*. A *hub* is a station built solely to connect rails. A connected rail network of minimal length is called a *minimal network* or *Steiner tree*.² Two possible networks for the three towns are shown in Fig. 1. The “triangle” network is built from two sides of the triangle formed by the three towns, while the “trident” network adds a hub (*Steiner point*) in the middle.

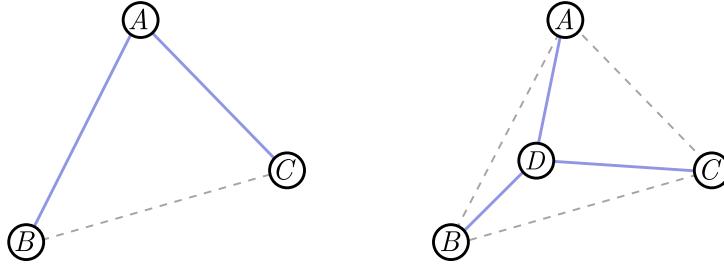


Figure 1: Rail networks (triangle and trident) connecting three towns.

Exercise 2.1. Choosing sides.

Suppose A , B and C are separated by distances AB , AC and BC . A triangular network consists of two sides of the triangle. Which ones should we choose?

Exercise 2.2. Triangle or trident?

In Fig. 1, we have two networks connecting the same towns: a triangular network, and a trident-shaped network with a hub station D in the middle. Which is shorter?

Hint. Measure lengths with a ruler. Simple but it works!

Already, there is a surprise. Although the simplest network consists of two sides of the triangle, this is not minimal, since the trident in Fig. 1 is shorter. We can go further and optimize the placement of the hub D . The case for a general triangle is tricky, but we can build most of the intuition we need by focusing on the special case of an *equilateral triangle*.

2.1 Equilateral triangles

Suppose A , B and C sit on the corners of an equilateral triangle of side length d , as in Fig. 2. The triangular network has total length $L_A = 2d$. For the trident network on the right, we place the

²We will see where the term “tree” comes from in §3.

hub D directly in the middle. Let's trade our engineering for math hats, and find the length of the trident network using trigonometry.

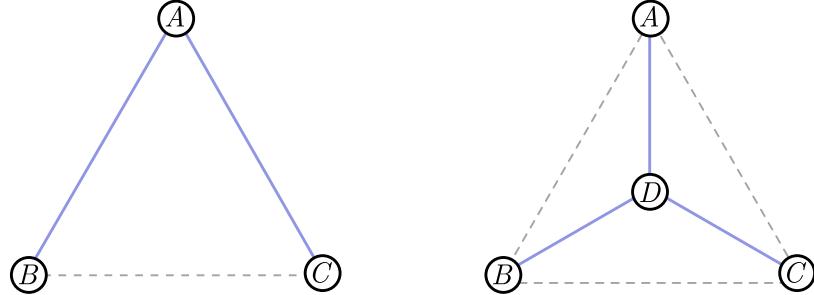


Figure 2: Rail networks on an equilateral triangle.

Exercise 2.3. Equilateral trident.

Show that the length of the trident network is

$$L_Y = \sqrt{3}d.$$

Since $\sqrt{3} \approx 1.7 < 2$, the trident is shorter than the triangle.

Although this beats the triangle network, it's possible that placing D somewhere other than the center could make the network even shorter. But as it turns out, the center is optimal, and we can argue this from *symmetry*. We draw one of the triangle's axes of symmetry³ in red in Fig. 3. We can wiggle the hub D left and right along the dark blue line in Fig. 3.

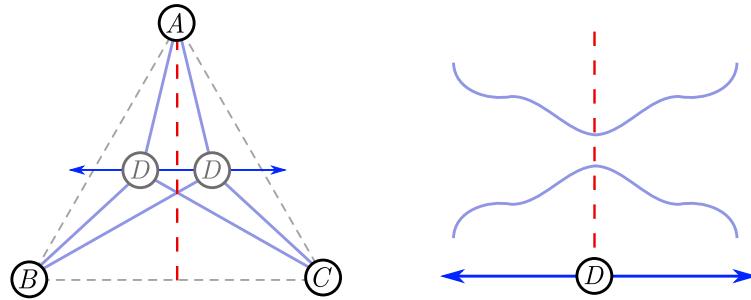


Figure 3: *Left.* Wiggling the hub. *Right.* Length is an even function of wiggle.

Because of symmetry, the total length of the network (light blue lines) is an *even function* of how far we have moved D along the dark blue line. On the right in Fig. 3, we depict two possibilities for an even function. Length could either be a *minimum* on the red line, like the curve on top, or a *maximum*, like the curve on the bottom. Of course, if we move the hub along the blue line outside

³This cuts the triangle into two mirror-image halves.

the triangle, the network becomes very long. This suggests it is a minimum⁴ on the red line, and for a minimal network it should lie on that line, as in Fig. 4 (left). But there are two other axes of symmetry, associated with B and C . All three intersect at the center of the triangle, as shown in Fig. 4 (right). Since D should lie on each of these lines, it must lie at the center!

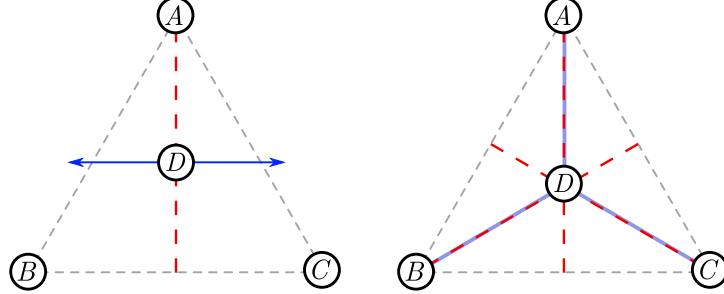


Figure 4: *Left.* Length is minimized on the red line. *Right.* Total length is minimized at the intersection of the red lines.

2.2 Deforming the triangle

We are now going to take our solution to the equilateral triangle and slowly deform it. In other words, slide the corners so that the triangle is no longer equilateral. What will happen to the optimal position of the hub D ? Since everything is sliding continuously, the optimal hub should slide continuously as well. In Fig. 5, the paths of the corners are depicted in purple, and the corresponding continuous change of hub in green.

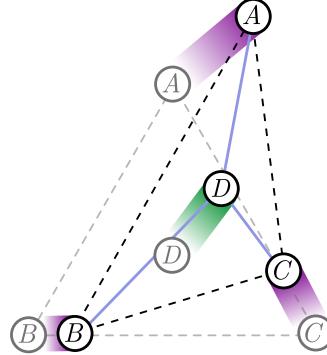


Figure 5: Optimal hub position slides as we slide the corners of the triangle.

Since the hub position changes continuously, it should stay inside the triangle for small deformations of the corners. But for triangles which are far from equilateral, the sliding hub might *collide* with a corner! In this case, the trident network collapses into a simpler triangular network, formed from two sides of the triangle. In Fig. 6, B remains fixed in position, but A and C lower symmetrically and open out the angle of the triangle, with the optimal hub D moving vertically down as they do so. At some critical angle θ_{crit} , it will coincide with B .

⁴This does not *prove* it is a minimum, since there may be other minima we have missed. See Exercise 2.10 for a rigorous proof.

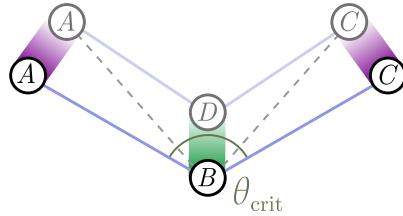


Figure 6: At some critical angle θ_{crit} , D collides with B .

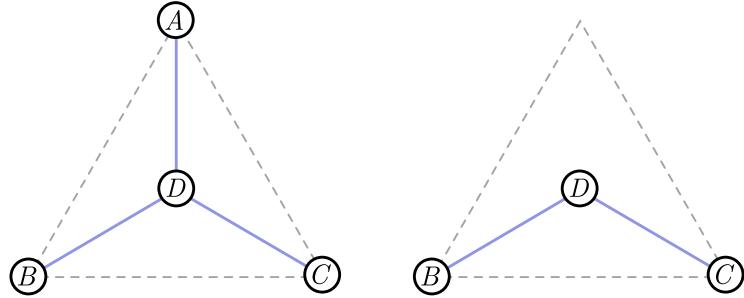


Figure 7: Removing a corner city removes a leg from the equilateral trident.

It turns out this critical angle is $\theta_{\text{crit}} = 120^\circ$. Although we won't provide a watertight proof just yet, we can give a plausibility argument. Let's return to the equilateral triangle. Instead of adding a hub in the middle, suppose that D is in fact a *fourth city* fixed in place. Clearly, the solution in Fig. 7 (left) is still optimal, since if we could add more hubs to reduce the total length, we could add more hubs to improve the network for the equilateral triangle. If we now remove a corner city, such as A , the optimal network removes the corresponding leg of the trident, as in Fig. 7 (right).

Exercise 2.4. *Cutting corners.*

Suppose that in Fig. 7 (right), we can add a new hub E which reduces the total length of the network. Explain how adding E could reduce the length of the network in Fig. 7 (left), and thereby improve our solution for the equilateral triangle.

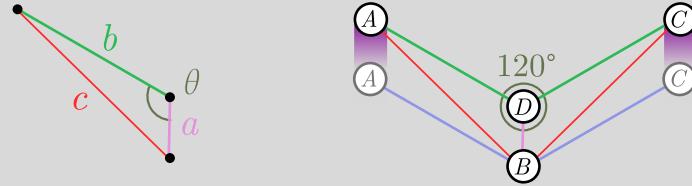
Exercise 2.4 is an example of a *proof by contradiction*, a favourite proof method among mathematicians. To show something is false, we assume it is true and use it to derive a contradiction with known facts. We then reason backwards to conclude that it cannot be true.

Exercise 2.5. *Critical isosceles.* \blacktriangle

The argument above really only establishes that $\theta_{\text{crit}} \leq 120^\circ$. In principle, the triangular network might become optimal at some angle $\theta_{\text{crit}} < 120^\circ$. In this exercise, we will show for an isosceles triangle that this is not the case. We will need the *law of cosines*,

$$c^2 = a^2 + b^2 - 2ab \cos \theta,$$

for the triangle depicted below left:



Above right, we have a triangular network (blue lines) ABC , forming an angle of 120° . We now raise the two nodes A and B symmetrically so that the angle ABC is less than 120° . You can prove that the green and purple lines are shorter than the red lines, so that an interior hub D , making an angle 120° with green and purple lines, yields a shorter network.

(a) Show using the law of cosines (or otherwise) that

$$c^2 = a^2 + b^2 + ab.$$

(b) From part (a), argue that

$$a + 2b < 2c.$$

(c) Conclude that for an isosceles triangle ABC , the critical angle is $\theta_{\text{crit}} = 120^\circ$.

2.3 The 120° rule

Let's state the general, n -city version of the problem we've been studying.

Box 2.1. *Minimal networks.*

Suppose we have n cities on the plane. The *minimal network* or *Steiner tree* is a configuration of edges connecting these cities which has minimal length. We can introduce additional hubs in order to minimize total distance.

Our work with triangles pays off with a remarkable conclusion about minimal networks connecting *any* number of cities called *the 120° rule*. Readers who are not interested in the proof may simply internalize the contents of the following and move on.

Box 2.2. The 120° rule.

In a minimal network, every hub has three edges separated by angles of 120° .

The argument is simple and ingenious. Our first step is to show that it is impossible for a hub to have edges separated by less than 120° . Suppose we have cities or *fixed nodes* A_1, A_2, \dots, A_n connected by a minimal network. Imagine there is a hub station H with incoming rail lines separated

by less than $\theta_{\text{crit}} = 120^\circ$, as on the left in Fig. 8. There may be other incoming lines, but these will play no role in our proof and can be ignored.

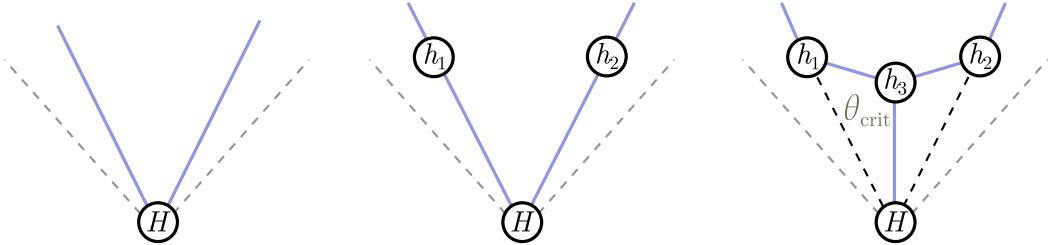


Figure 8: *Left.* A hub with incoming angle less than θ_{crit} . *Middle.* Adding two extra stations. *Right.* A shorter network.

We can build two new stations on these outgoing legs, h_1 and h_2 , without changing the length of track. For simplicity, we take to be the same distance from H , as in Fig. 8 (middle). But from our work in the previous section, we know that the minimal network connecting h_1 , h_2 and H is not the triangle we have drawn! Instead, it is a trident with another hub h_3 in the middle, Fig. 8 (right). This strictly decreases the length of the network, and our original network was not minimal.

So, we know that any hub must have spokes separated by at least 120° . How do we know that there are three, separated by exactly 120° ? This is very simple. Suppose two lines enter H , separated by *more* than 120° . Then there can only be two incoming edges, joining H to some cities A and B , since any additional lines would have to be closer than 120° to one of these lines. We have the situation depicted on the left of Fig. 9.



Figure 9: *Left.* A hub with incoming angle greater than θ_{crit} . *Right.* A shorter network.

Hopefully you can see what goes wrong: if there is a “kink” in the blue line, then we can obtain a shorter network by deleting H and directly connecting A and B . Once again, we have a contradiction! Strictly speaking, we can have hubs with only two incoming edges, separated by 180° . But such a hub is always unnecessary, since all it does is sit on a straight line. If we delete these useless hubs, we have the general result advertised above, namely that any hub in a minimal network has three equally spaced spokes.

Exercise 2.6. Outer rim. \blacktriangle

Our proof applies to hubs only, but the same type of argument lets you deduce properties of the cities A_1, A_2, \dots, A_n . Prove the following:

- (a) No incoming edges can be separated by less than 120° .
- (b) The number of incoming edges is between one and three.
- (c) A city has three edges only if it lies at the center of an equilateral triangle.

The upshot is that the fixed nodes can have from one to three edges.

Exercise 2.7. *Easy polygons.*

Consider cities on the corners of a regular n -sided polygon. Show that for $n \geq 6$, the minimal network is just the perimeter with one edge removed.

2.4 A minimal history

French mathematician PIERRE DE FERMAT (1607–1665) was the first to ask about minimal networks on the triangle, though he framed it as a geometric problem:

Box 2.3. Fermat's problem.

Given three points A, B, C in the plane, find the point D such that the sum of lengths $|DA| + |DB| + |DC|$ is minimal.

He solved the problem himself, but according to the mathematical custom of the day, sent a letter to Galileo's student EVANGELISTA TORRICELLI (1608–1647), challenging him to solve it. Torricelli used a different method, so the position of the hub is called the *Fermat-Torricelli point*. JAKOB STEINER (1796–1863) generalized Fermat's question to n points on the plane:

Box 2.4. Steiner's problem.

Given n points A_1, \dots, A_n in the plane, find the point D such that the sum of lengths $|DA_1| + \dots + |DA_n|$ is minimal.

Although minimal networks are also called Steiner trees, Steiner's problem is very different from the one we've been considering. Steiner wanted a *single* point such that the sum of lengths to that point is minimal, rather than a connected network of minimal length. Put differently, it is the minimal network when you are allowed to add at most one hub. The two problems only agree for $n = 3$.

In 1836, 200 years later, the great German mathematician CARL FRIEDRICH GAUSS (1777–1855) considered the problem of designing a minimal rail network between four German cities (Exercise 3.9). The French mathematician JOSEPH DIEZ GERGONNE (1771–1859) considered the general n city problem (connecting them via canals rather than railways) and discovered the 120° rule. The world evidently paid no attention until 1934, when Czech mathematicians VOJTEČH JARNÍK (1897–1970) and MILOŠ KÖSSLER (1884–1961) independently rediscovered Gergonne's results [21]. The Gergonne-Jarník-Kössler version was later popularized under the name *Steiner trees* by RICHARD COURANT and HERBERT ROBBINS in their classic 1941 text, *What is Mathematics?* [5].

Exercise 2.8. *Easy points.*

Use the reasoning in §2.1 to argue that, for a regular polygon of n sides, the center of the polygon solves Steiner's problem.

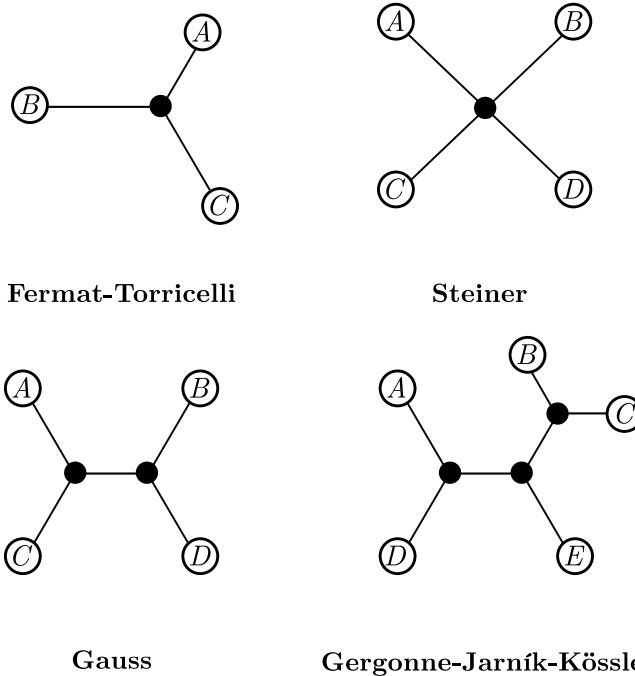
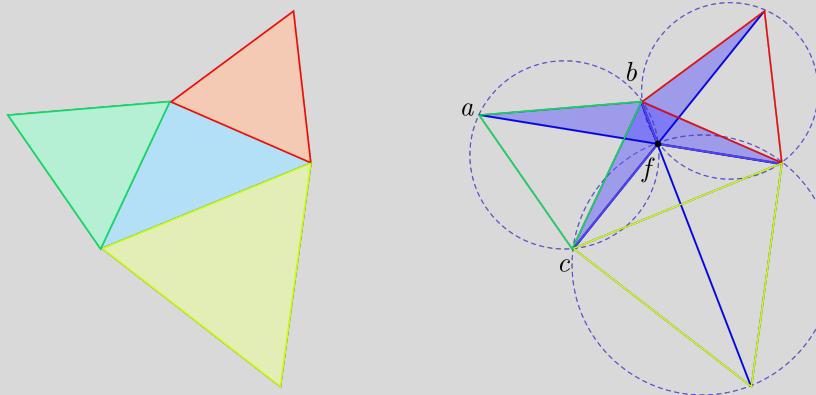


Figure 10: A visual history of minimal networks.

Exercise 2.9. Searching for Fermat-Torricelli. \blacktriangleleft

Here, we give a geometric construction of the Fermat-Torricelli point for any triangle. Proceed if you like geometry! So, we're going to find the interior point satisfying the 120° rule for the blue triangle (below left). Start by attaching equilateral triangles (green, red, yellow) on each side of the triangle, and drawing lines (dark blue) from the outer corners of the equilateral triangles to the opposite corner of our original triangle, as shown below right.



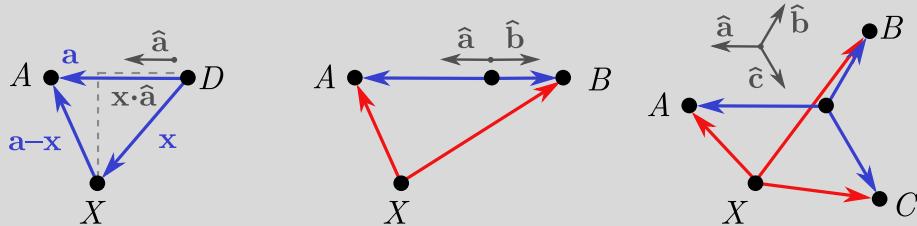
We claim these lines intersect at the point f , and moreover, are separated by angles of 120° . To prove this, draw the dotted circles circumscribing each equilateral

triangle. The exercises guide you through a demonstration that the circles intersect at 120° angles at f , using the [inscribed angle theorem](#).

- (a) Show that the shaded triangles are congruent. Argue that, in consequence, the three blue lines do intersect at a single point.
 - (b) From part (a), argue that $\angle baf = \angle bcf$.
 - (c) From (b) and the inscribed angle theorem, argue that a, b, c, f lie on a circle.
 - (d) Since the triangle is equilateral, $\angle cab = 60^\circ$. Using the inscribed angle theorem once more, show that $\angle cfb = 120^\circ$. Repeating this argument for the remaining two triangles gives our result!
- This construction works provided all angles in the blue triangle are $< 120^\circ$.
- (e) What goes wrong if an angle is $\geq 120^\circ$?

Exercise 2.10. *From straight lines to Steiner's problem.*

Here, we give a rigorous proof of the 120° rule, and immediately extend it to find the analogous rule for Steiner's problem. The proof makes use of vectors and the dot product, hence the higher difficulty rating. Recall that $|\mathbf{v}|$ is the length of the vector \mathbf{v} , and $\hat{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|$ is the unit vector pointing in the same direction.



Choose a point D on the plane, which will act as the “origin”. Consider another point, A , making a vector $\mathbf{a} = DA$, with unit vector $\hat{\mathbf{a}}$.

- (a) Prove (visually or however you like) that for any other point X , with $\mathbf{x} = DX$,

$$|\mathbf{a}| \leq |\mathbf{a} - \mathbf{x}| + \mathbf{x} \cdot \hat{\mathbf{a}},$$

where as in the image above, $\mathbf{x} \cdot \hat{\mathbf{a}}$ is the length of \mathbf{x} projected onto \mathbf{a} .

- (b) Consider two points A and B on the plane. Using the previous exercise, show that for any point X ,

$$|DA| + |DB| \leq |XA| + |XB| + \mathbf{x} \cdot (\hat{\mathbf{a}} + \hat{\mathbf{b}}).$$

- (c) Conclude that if we choose D so that $\hat{\mathbf{a}} + \hat{\mathbf{b}} = \mathbf{0}$, the sum $|DA| + |DB|$ will be minimized. Geometrically, what does correspond to? Does this make sense?

- (d) Let's now introduce *three* points A, B, C on the plane, with origin D . Generalize (c) to establish that $|DA| + |DB| + |DC|$ is minimized when

$$\hat{\mathbf{a}} + \hat{\mathbf{b}} + \hat{\mathbf{c}} = \mathbf{0}.$$

Show that this is precisely the 120° rule.

- (e) Finally, consider points A_1, \dots, A_n and corresponding vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$. Generalize (d) to conclude that if a point D exists such that

$$\hat{\mathbf{a}}_1 + \dots + \hat{\mathbf{a}}_n = \mathbf{0},$$

then it solves Steiner's problem.

- (f) Exploit (e) to solve Steiner's problem for an arbitrary quadrilateral.

3 Graphs

In this section, we will learn more about the connectivity and layout of minimal networks. This is the domain of *graph theory*, so we will provide an introduction to some of its basic notions first. A graph is a bunch of dots connected by lines, drawn on a page. The technical term for these dots is *vertices* or *nodes*, and *edges* for the lines. If an edge joins two nodes, we say they are *neighbours*. Edges must start and end at different vertices, and are allowed to overlap. Vertices can be attached to any number of edges, including zero. The rules are illustrated in Fig. 11. We let E denote the number of edges and N the number of nodes.

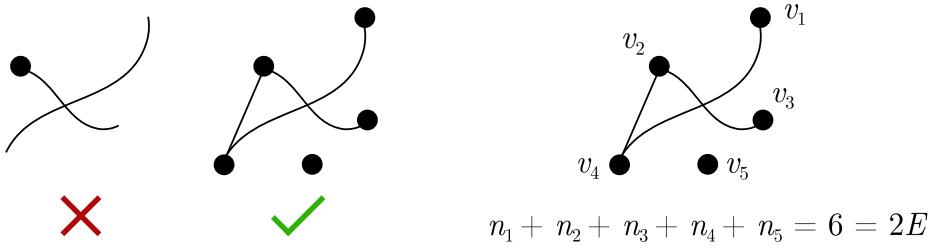


Figure 11: *Left*. “Illegal” and “legal” graphs. *Right*. The handshake lemma in action.

We will need a simple, general result called the *handshake lemma*. There are two ways to count edges. The first is simply to count the edges directly, yielding a number E . But our rules tell us that edges attach to a vertex at each end. So, instead we can go through the vertices and count the number of edges which attach to them. But this will hit each edge *twice*, once the vertex at each end. So this way of counting gives $2E$. More precisely, suppose there are N vertices v_1, v_2, \dots, v_N . If these have n_1, n_2, \dots, n_N edges attached, the handshake lemma states that

$$n_1 + n_2 + \dots + n_N = 2E. \quad (1)$$

The name, incidentally, comes from the fact that if vertices v_1, \dots, v_N are people, and they perform n_1, \dots, n_N handshakes each, adding these numbers will give *twice* the total number of handshakes.

3.1 Trees and leaves

In a rail network, we can list all the routes from A to B . If the network is connected, then by definition, there is at least one route. But if the network is minimal, there is a *unique* route. If there is more than one way to get from A to B , the network has unnecessary edges and can be “pruned” to get something shorter. We might worry that pruning redundant paths between A and B could disconnect other cities, but this is never the case. Fig. 12 shows why.

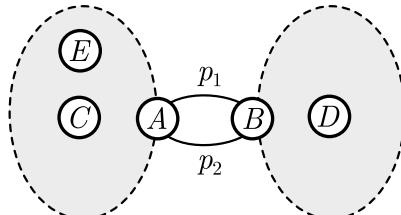


Figure 12: Pruning unnecessary paths.

Suppose A and B are connected by two paths, labelled p_1 and p_2 , and potentially consisting of more than one edge. The blob to the left is all the vertices whose paths to B go through A first, and similarly, vertices on the right connect to A through B . If two nodes are in the same blob, such as C and E , then pruning path p_2 has no effect on whether they are connected. If two nodes are in different blobs, like C and D , they can still reach each other using path p_1 . So, removing the redundant path p_2 joining A and B does not affect the connectivity of any other nodes.

Exercise 3.1. *Pruning along the path.*

Generalize the argument above to account for nodes that lie between A and B . More precisely, these are nodes which can connect to either A or B without passing through the other, and schematically lie on paths p_1 or p_2 .

Once we have completely pruned the network, there is only a single path connecting any two nodes A and B . Such a network is called a *tree* because if you squint the edges resemble branches. This explains the “tree” in Steiner tree! An example is shown in Fig. 13. Every tree has a special node called a *leaf*. As the name suggests, this is at the “end” of the tree’s branches. More formally, a leaf is a node with a single edge, like J , D , G , and I in Fig. 13. It may seem intuitive, but as an exercise in reasoning about trees, let’s *prove* they must have leaves.

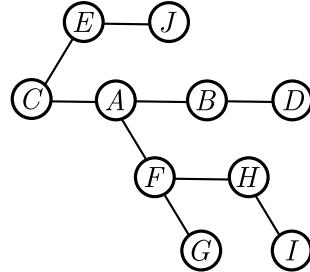
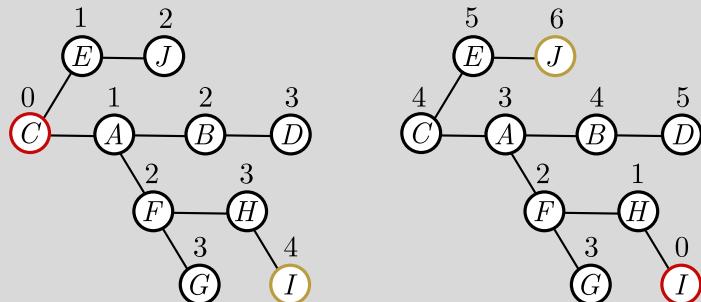


Figure 13: A tree network, with a unique path between each node.

Exercise 3.2. *Finding leaves.*

To begin our proof that each tree has a leaf, we choose a node at random (red, below left) and count the number of steps to each other node.



- (a) Explain why the number of steps from the red node to any other node is well-defined in a tree.
- (b) Consider the node or nodes furthest from the red node (orange, above left). Argue that these must be leaves. *Hint.* If they are not, what is the distance from red node to their neighbours?
- (c) In fact, we can prove something stronger. The previous question tells us how to find a leaf. Repeat the same procedure, but start with the leaf and find the furthest node. Conclude that every tree has *two* leaves.
- (d) Show, using an example, that a tree need not have more than two leaves.

3.2 Hub caps

In Fig. 13, you may have noticed the number of edges $E = 9$ is one less than the number of nodes, $N = 10$. This is not a coincidence. For any tree, it turns out that $E = N - 1$. We can prove this fact using the existence of leaves. The idea is simple: keep removing the leaf, and the single edge joining it to the rest of the tree, until you have a single node left. This requires the removal of $N - 1$ nodes, and hence $N - 1$ edges. Since there are *no* edges now, and we removed one each time, we must have started with $N - 1$ edges. Hence,

$$E = N - 1 \quad (2)$$

for trees in general. Equation (2), along with the handshake lemma (1), will allow us to place a cap on the maximum number of hubs that can occur in the network.

Suppose we are trying to connect n cities, and introduce h hubs in order to do so. The total number of nodes is then $N = n + h$. The 120° rule tells that each hub attaches to exactly three edges. Each of the n cities attaches to at least one edge to ensure it is connected to the rest of the network. Thus, (1) gives

$$2E = n_1 + n_2 + \cdots + n_N \geq n + 3h. \quad (3)$$

From (2), we know that $E = N - 1 = n + h - 1$. Combining this with (3), we find

$$2(n + h - 1) = 2n + 2h - 2 \geq n + 3h \implies n - 2 \geq h. \quad (4)$$

In other words, the number of hubs h is at most $n - 2$.

Exercise 3.3. Hubs and nubs.

While hubs always have three attached edges, Exercise 2.6 tells us that cities (fixed nodes) have between one and three edges.

- (a) Show it is always possible to arrange n cities so that $h = 0$.
- (b) At the other end of the spectrum, argue that the maximum number of hubs, $h = n - 2$, occurs when the fixed nodes are exactly the leaves of the network.

The 120° rule and hub cap together give us a simple tool for building minimal networks. For n fixed nodes, pick $h = n - 2$ hubs, with spokes emerging at angles of 120° , and connect them together to form a tree, with the fixed nodes as leaves. Although the angles are fixed, we can extend the spokes and legs, and perform overall rotations of the network. We call this extendable configuration of hubs and spokes a *tinkertoy*, after the modular children's toy it vaguely resembles (Fig. 14).⁵

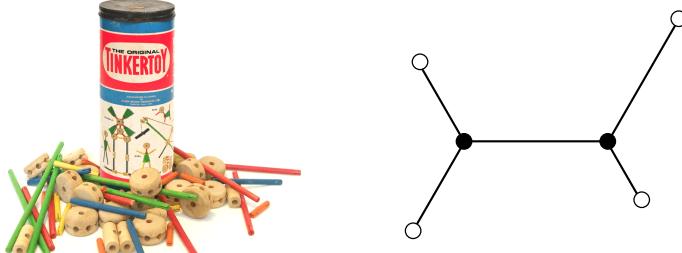
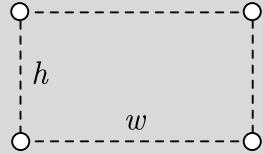


Figure 14: *Left.* A real TinkertoyTM. *Right.* A network tinkertoy.

We can play with our network tinkertoys, or program a computer to play with them, until they do what we want. The strategy of playing with tinkertoys to find the minimal network was first proposed by ZDZISLAW ALEXANDER MELZAK⁶ [22]. We give some examples in the following exercises.

Exercise 3.4. *Minimal rectangular network.*

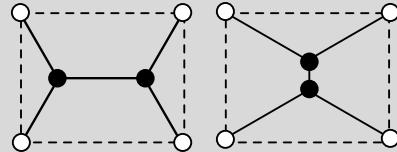
Consider four cities on a rectangle of height h and width $w \geq h$:



- (a) Draw the single tinkertoy for $n = 4$, and argue from Exercise 2.6 that this should describe the minimal network.
- (b) Fit the tinkertoy to the city, and deduce that the minimal network has length

$$L = w + \sqrt{3}h.$$

- (c) Show that the tinkertoy can be oriented in *two* ways when $h < w < \sqrt{3}h$. Explain why the horizontal orientation is always minimal.



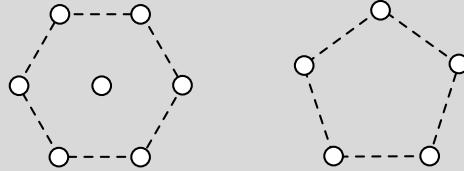
Part (c) tells us something very important. Even if a tinkertoy fits, the configuration isn't necessarily the true minimum!

⁵In the mathematics literature, a tinkertoy graph is called a *topology*, and the different ways of stretching the edges the *degeneracies* of that topology.

⁶Apparently, Melzak was a professor at UBC, but remarkably little information exists about him now.

Exercise 3.5. Harder polygons.

When $h = n - 2$, you can use a single tinkertoy, but if $h < n - 2$, you will need multiple tinkertoys.



- (a) Find the minimal network connecting a regular hexagon with a node in the center. You will need multiple tinkertoys!
- (b) Draw the single tinkertoy for $n = 5$, and schematically indicate what the minimal network on a regular pentagon looks like.

3.3 Avoiding explosions

For a small number of hubs, tinkertoys are useful. But are they useful for many hubs? Suppose that fiddling with tinkertoys is a quick operation, i.e. once a tinkertoy is selected, a human or a computer can quickly check whether the tinkertoy can be made to hit our fixed points. If there are many tinkertoys, finding the right one could take a while. In Fig. 15, we show a few tinkertoys for $h = 6$, suggesting that with more hubs, enumerating them all may turn out to be hard.

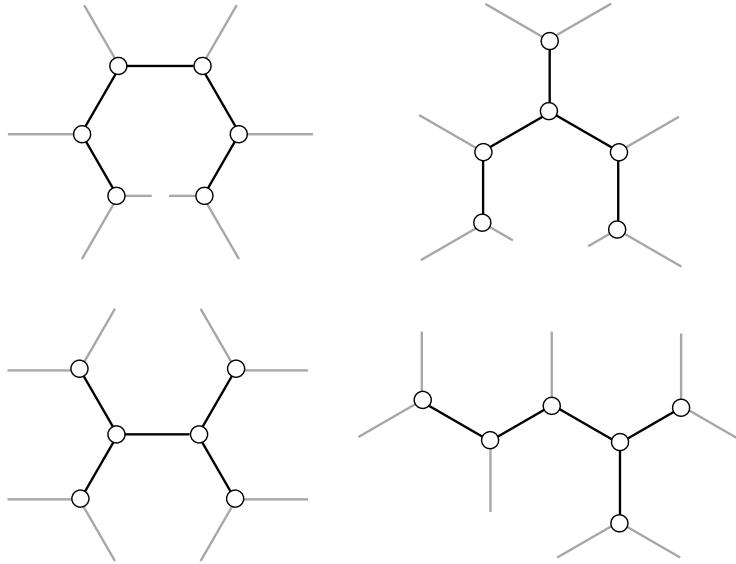


Figure 15: A selection of tinkertoys for $h = 6$.

In fact, as h gets larger, the total number of tinkertoys T_h *explodes* combinatorially, growing exponentially as a function of h . A brute force approach, which simply fiddles with each tinkertoy to see if it can be made to fit the fixed point, will take an exponential amount of time. This is

beginning to seem like a hard problem in general! Counting the total number of tinkertoys is difficult (though see Exercise 3.6 for a physicist's shortcut). To demonstrate this exponential growth, we are instead going to focus on a subset of tinkertoys which are easier to count.

The first problem with counting tinkertoys is that trees in general have complicated structure we don't want to have to worry about. To simplify, we consider only *linear* tinkertoys, whose hubs lie on a line, so that no hub has more than two neighbours, e.g. Fig. 16 (left). The next problem is that even these linear tinkertoys can be rotated by 180° . To avoid double-counting, we need some way of knowing which end is which. A simple method is to start and end with a \vee -shaped segment, as in Fig. 16 (right). If we rotate 180° , the tinkertoy is bookended by \wedge -shaped segments, which is clearly distinct. Not every linear tinkertoy has this form, so we call these special tinkertoys *oriented*.

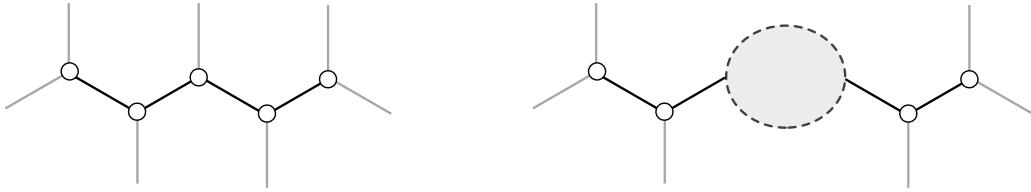


Figure 16: *Left*. A linear tinkertoy. *Right*. An oriented tinkertoy.

With the notion of oriented tinkertoys, we can immediately find an exponentially growing set! There are $h - 1$ edges altogether since the hubs form a tree. We fix four (two at each end) to ensure the tinkertoy is oriented. That leaves $h - 5$ edges within the grey bubble of Fig. 16 (right). As we move along from the leftmost \vee , these edges constitute $h - 5$ turns left or right by 60° before we exit again to hit the final \vee . At each point, either a left or a right turn is allowed, so there are 2^{h-5} possible choices altogether. To make this more transparent, we could label left and right turns with 1s and 0s respectively, so that a tinkertoy is just a sequence of binary digits, as in Fig. 17. Thus, there are an exponential number of oriented tinkertoys.⁷

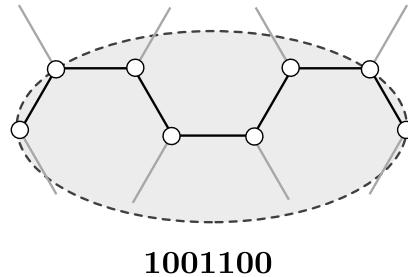


Figure 17: The binary sequence for an oriented tinkertoy.

Exercise 3.6. *Physicist's induction.* \blacktriangle

Calculate the number of tinkertoys T_h from $h = 0$ to $h = 6$. You should be

⁷You might be worried that if we turn too many times, the tinkertoy will collide with itself and no longer be valid. For instance, after six right turns, edges of equal length will form a closed hexagon! But we can always adjust the length of edges to prevent this from happening, so the count remains valid.

able to find the general sequence T_h by searching for these numbers in the [Online Encyclopedia of Integer Sequences](#). At large h , the OEIS informs us that this sequence grows exponentially, with

$$T_h \approx \frac{2^{2h-4}}{\sqrt{\pi} h^{5/2}}.$$

Of course, this doesn't prove that the sequences match up. But this sort of evidence satisfies a physicist, which is why I (fondly) call it "physicist's induction"!

By now, we should be confident that there are many tinkertoys. If we have to consider even a fraction of them at large h , any computer is doomed to failure. For instance, using the counting in Exercise 3.6, suppose a computer can check a billion tinkertoys per second, and wants to design a railway network to connect the ~ 800 cities in Europe with more than 50,000 inhabitants. If it has to check every tinkertoy, it will take an unimaginably long

$$\frac{2^{2 \cdot 800 - 4}}{\sqrt{\pi} 800^{5/2} \cdot 10^9} \text{ s} \approx 10^{456} \text{ years.}$$

No realistic improvements in processing speed will make this computation achievable in the lifetime of the universe. Even a minute fraction will take many cosmic lifetimes to check!

The problem is even worse than advertised. In Exercise 3.4, we saw that a tinkertoy can fit the fixed nodes without being minimal. So if we find a tinkertoy that fits, we still need to keep checking to ensure it is truly the shortest! We can distinguish two related problems here: checking if there is a tinkertoy that fits; and finding the truly minimal network. The first problem is easier in the sense that, once we have a candidate solution, we can check immediately that it's correct. We don't need to keep searching.

There is an area of computer science called *complexity theory* which classifies problems according to how computationally difficult they are. Finding a tinkertoy that fits is in a class of problems called **NP-complete**⁸, which means "hard to solve but easy to check". Since we might need to continue looking, finding shortest networks is probably not easy to check, and certainly at least as hard as finding tinkertoys, so instead it is in a class of problems called **NP-hard** [14]. This means "hard to solve and potentially hard to check". We can summarize these results as follows:

Box 3.1. Complexity I.

Finding tinkertoys that fit is NP-complete. Finding the minimal network is NP-hard.

3.4 Minimal spanning trees

Does this make life impossible for network planners? Not really! It turns out that, while finding the minimal network is difficult, approximating is easy. Life, and near-optimal rail travel, go on. We'll discuss a particularly simple approximation scheme. The basic idea is to generalize Exercise 2.1.

⁸Here, NP stands for "non-deterministic polynomial time". This is a fancy way of saying a computer could make a lucky guess and verify it quickly. For instance, a computer might luckily guess the correct tinkertoy and show that it fits. The **complete** part in **NP-complete** means that tinkertoys are as hard as any other problem in NP.

Recall that, for three cities, the triangle network consists of the two shortest sides of the triangle. Put differently, we draw an edge between each city, and select the two shortest ones, which happen to form a tree which connects all three cities.

For n cities, we now draw an edge between each city, forming what is called the *complete graph* on n nodes. From these edges, we select a subset which form a tree, connecting each city and of minimum total length. This is called a *minimum spanning tree (MST)*, since it “spans” the cities. We illustrate the construction for $n = 4$ in Fig. 18.

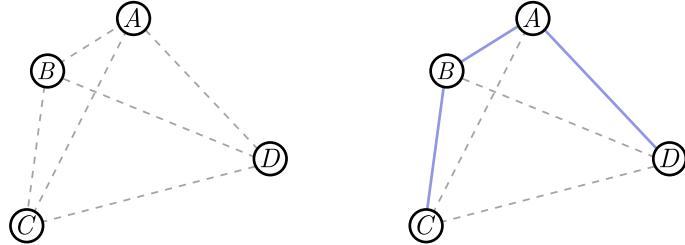


Figure 18: *Left*. The complete graph for four cities. *Right*. The minimum spanning tree.

The usefulness of MSTs depends on whether they are fast to compute and close to optimal. We start with the first question. Unlike tinkertoys, we can directly construct the MST. The procedure is very simple: pick a random vertex, and “grow” the tree by selecting the shortest available edge which extends it. We do this $n - 1$ times until we have a spanning tree. This algorithm was discovered in 1930 by Jarník [20], but subsequently rediscovered by ROBERT PRIM in 1957 [23], so it is called the *Prim-Jarník algorithm*. We perform the algorithm for $n = 4$ in Fig. 19.

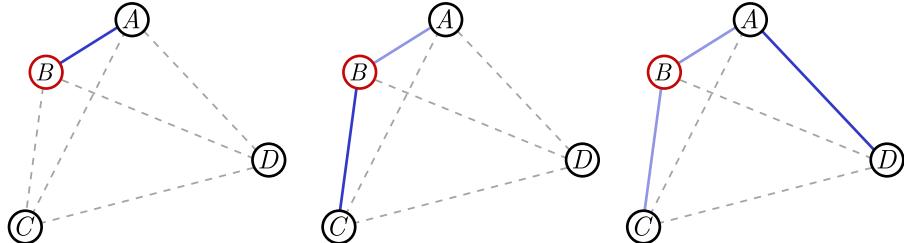


Figure 19: The Prim-Jarník algorithm for $n = 4$. Dark blue edges are added sequentially.

Exercise 3.7. MST is easy.

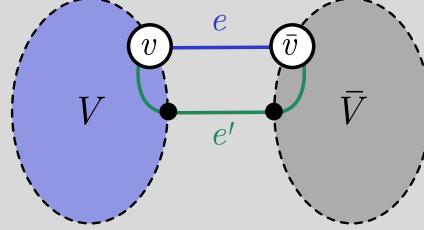
Suppose we have n cities, and we want to find an MST using the Prim-Jarník algorithm. Here, we will give a very lazy bound on the number of steps required.

- (a) Using the handshake lemma (1), show the total number of edges in the complete graph on n cities is $E_{\text{complete}} = n(n + 1)/2$.
- (b) The algorithm has $n - 1$ steps where it adds an edge. For each step, it must consider the available edges. Call this a *substep*. Give a very lazy argument that the total number of substeps for the algorithm is $\leq n^3$.

This is a polynomial function of n , rather than the exponential function of n we found for minimal networks. Finding MSTs is much faster!

Exercise 3.8. *Correctness of Prim-Jarník.* \blacktriangle

Suppose that the Prim-Jarník algorithm produces a tree T which is not minimal, with $T' \neq T$ the genuine MST. Then there must be a step in the construction where we first add an edge e which is not in T' . We will show that the algorithm is *correct* in the sense that this situation cannot occur! There will always be a shorter edge e' it should add instead of e . The setup is shown below.



- (a) Suppose that, before the algorithm adds the “bad edge” e , it spans a set of cities V . The complementary set of cities is \bar{V} . Show that e connects a vertex $v \in V$ to a vertex $\bar{v} \in \bar{V}$.
- (b) Argue that the MST T' has an edge e' connecting V to \bar{V} . *Hint.* Use the fact that there is a path from v to \bar{v} in T' .
- (c) Explain why removing e' from T' , and replacing it with e , results in a *tree*. *Hint.* Show there is still exactly one route between any two nodes.
- (d) From part (c), conclude that the Prim-Jarník algorithm is correct.

Finding MSTs is quick. But is it useful? This depends on the answer to our second question, namely how much longer the MST is than the minimal network. Once again, our simple results on triangles provide some insight. Let’s start with an equilateral triangle of side length L . In Exercise 2.3, you found that the minimal network has length $\sqrt{3}L$. The MST for the equilateral triangle just consists of any two sides, and therefore has length $2L$. The *ratio* of these two lengths is $\rho = 2/\sqrt{3} \approx 1.15$. The MST is about 15% longer than the Steiner tree. This is close enough for many practical purposes. We might wonder, in general, how bad this ratio can get.

Let’s see what happens when we squeeze or stretch the triangle symmetrically. If we squeeze it, like Fig. 20 (left), the MST consists of a long side of length L and a side which shrinks to zero. Similarly, the minimal network consists of two short sides which approach zero, and a long side which approaches L . So the ratio of lengths approaches 1. Similarly, as we stretch the triangle out like Fig. 20 (left), the MST is the shorter two sides of the top, length $2L$, while the hub eventually hits the top vertex, so it coincides with the MST. Once again, the ratio approaches 1.

This hints that the equilateral triangle is the worst-case scenario. In fact, you can show in Exercise 3.10 that this ratio is at most $2/\sqrt{3}$ for any triangle. In GILBERT and POLLAK’s magisterial study [15], they conjecture that this holds for any number of cities! In other words, if ρ is the ratio of the length of the MST to the minimal network, the *Gilbert-Pollak conjecture* states that

$$\rho \leq \frac{2}{\sqrt{3}}. \tag{5}$$

The conjecture remains unproven. The best we can do right now is $\rho \leq 1.21$ [4].

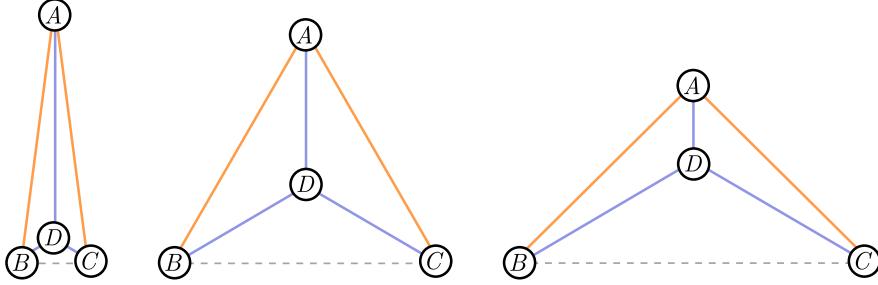


Figure 20: Stretching and squeezing the equilateral triangle.

What if we want to do better than 15%? We can tweak the MST a little to get closer to the optimal network length. One particularly simple method is the *Steiner insertion heuristic* [9], which elegantly combines the MST and our work with triangles. The basic observation is that no edges in a minimal network are separated by less than 120° , since hubs always have edges separated by *exactly* 120° , and edges at fixed nodes must be separated by at least 120° according to Exercise 2.6(a). The idea is to find edges making “small” angles ($< 120^\circ$) and replace them with hubs.

The insertion heuristic works as follows. We first find the MST (using Prim-Jarník for instance), and then search for the pair edges with the smallest angle $< 120^\circ$. If so such angle exists, we are done! If such an angle does exist, the two edges connect a vertex, say A , to vertices B and C , as below in Fig. 21. We introduce a hub for these three vertices, which satisfies the 120° rule. And then we do the whole thing again! We keep going until there are no more small angles to fix.

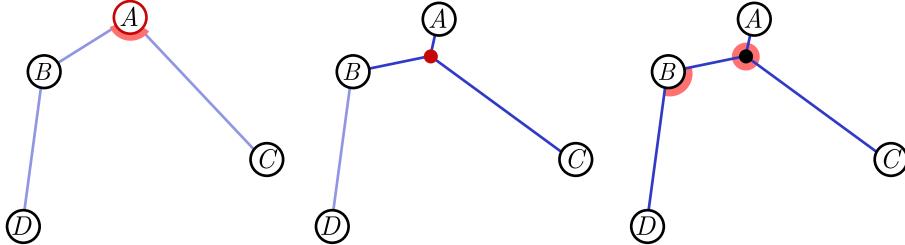


Figure 21: Applying the Steiner insertion heuristic to our MST. First, we find the smallest angle $< 120^\circ$. Then, add a hub. No more small angles, so we’re done!

Although Steiner insertion is quick in practice, it is unknown how close to optimal they are. A very different and less practical method [3] shows that you can actually approximate the minimal network as *closely as you like* in some number of steps at most polynomial in n , where n is the number of cities. For this reason, minimal networks belong to a complexity class called PTAS (“Polynomial Time Approximation Scheme”), the problems which can be easily approximated. We can update our statement about complexity:

Box 3.2. Complexity II.

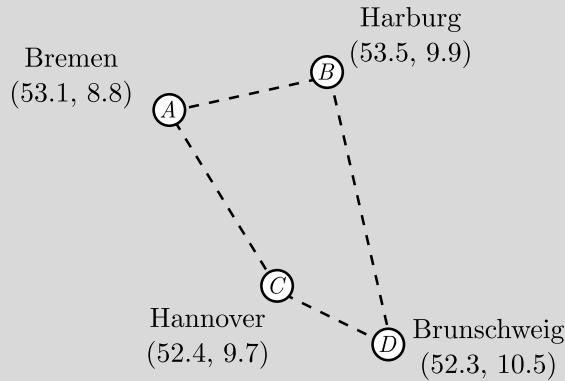
Finding minimal networks is in **NP-hard** but also in **PTAS**.

Exercise 3.9. Trains and trees.

As advertised in §2.4, the mathematician Gauss wanted to connect four cities with a minimal rail network. In an 1836 letter to his friend, the astronomer HEINRICH SCHUHMACHER (1780–1850), Gauss asked:

How does a railway network of minimal length connect the four German cities of Bremen, Harburg, Hannover, and Braunschweig?

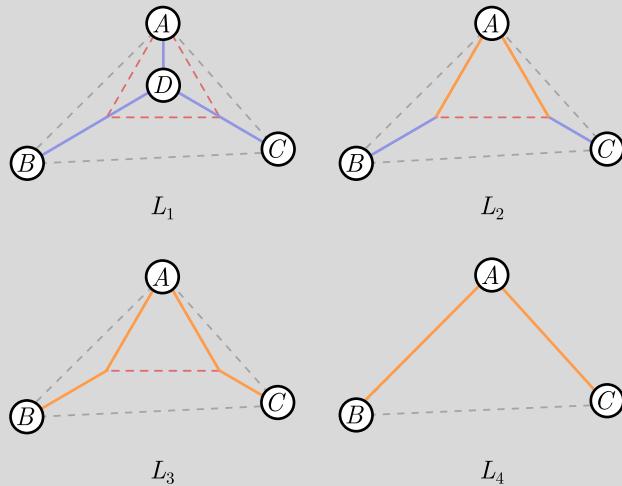
The cities are drawn, along with their GPS coordinates, below:



- (a) Find the minimum spanning tree using the Prim-Jarník algorithm.
- (b) Assuming Gilbert-Pollak, lower bound the length of the minimal network.
- (c) Improve the MST using the Steiner insertion heuristic.

Exercise 3.10. Gilbert-Pollak for triangles. ▲

Below, we give a visual proof of the Gilbert-Pollak conjecture for triangles. The basic idea is that, in an arbitrary triangle with angles $\leq 120^\circ$, we can attach a small equilateral triangle to the largest angle (city A below).



The lengths L_1, L_2, L_3, L_4 are made up of lengths of coloured lines, but blue lines have weight 1, while orange lines have a weight $\sqrt{3}/2$. For instance,

$$L_1 = |DA| + |DB| + |DC|, \quad L_4 = \frac{\sqrt{3}}{2}(|AC| + |BC|).$$

In other words, L_1 is the length of the minimal network, and L_4 is $\sqrt{3}/2$ times the length of the MST.

- (a) Argue that $L_1 \leq L_2 \leq L_3 \leq L_4$.
- (b) Use this (along with the case where some internal angle is $\geq 120^\circ$) to establish the Gilbert-Pollak conjecture for triangles.

4 Bubble networks

Humans are not the only players in the minimization game. Nature also likes to minimize, and has far more power at her disposal than us mere mortals. While we struggle with tinkertoys, Nature seems to minimize networks as effortlessly as a child drops a ball.⁹ Bubbles are formed when a film of liquid separates two volumes of air. Surface tension tries to pull the bubble surface taut in all directions, which results in the *minimization of area*. A soap bubble is spherical, for instance, since a sphere is the smallest surface containing a fixed volume of air, a point we return to below. If we attach two plexiglass plates together and dip them in soapy water, a two-dimensional network of bubbles, or *bubble network*, with vertical walls will form. A single bubble will now be a cylinder, rather than a sphere, as in Fig. 22.

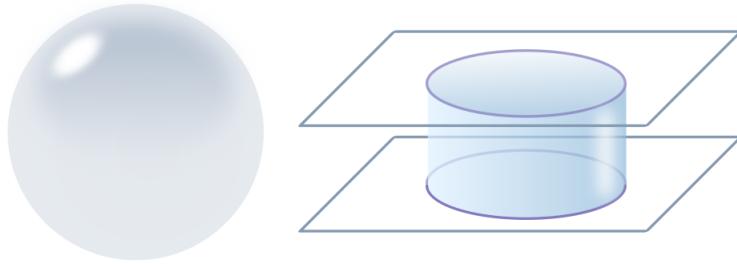


Figure 22: Soap bubbles in three and two dimensions.

The example above shows that bubble networks can form closed loops and hence are not minimal networks in our sense, since we explicitly ruled out loops in §3.1. But the networks are *locally minimal*, in the sense that surface tension will tug on bubble walls, and reconfigure small bubbles, until doing so cannot reduce the area. And area is proportional to the length of the bubble network, viewed from above, so length is locally minimized.

Another important point to note is that when we derived the 120° rule, we only needed to know what the network looked like in the region around a node. The rest of the network was irrelevant! This is a *local* condition. Since bubble networks are locally minimal—they tug on bubble walls until they cannot be reduced further—they will obey the 120° rule. A subtler point is that this remains true even if the bubble walls are curved! Zoom in enough on a node, and curved bubble walls appear straight. Hence, the straight-line reasoning we used in §2.3 applies. So far, we haven’t talked about what bubble networks do *away* from a node. As we will discuss in more detail in §4.4, a wall has two options: it can be straight, or it can curve along the arc of a circle.

4.1 Computing with bubbles

Soap bubbles find local minima by virtue of the laws of physics. Can we somehow “hack” soap bubbles, and turn them into *analogue computers* to find minimal networks? Yes! The key is to give the bubble walls something to hold onto. In a foam, they can only hold onto themselves, and we get the non-crossing bubble networks described above. But if we drill some screws through the

⁹This is also a minimization problem! There is a quantity in physics called the *action* — roughly, the *difference* between kinetic and potential energy — which Nature likes to make as small as possible.

plexiglass, these will act like the cities in our network, and walls can form between the screws.¹⁰ Hubs arise at the junction of three bubble walls. Fig. 23 shows an example for four fixed nodes.

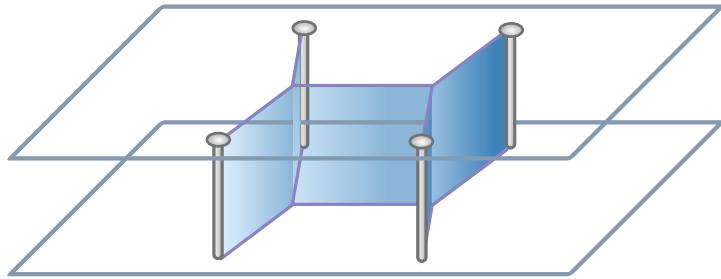


Figure 23: A soap bubble computer for finding minimal networks.

You can use a soap bubble computer to solve Gauss' railway problem.

Exercise 4.1. Trains and soap bubbles.

Build a soap bubble computer and solve the Gauss' railway problem from Exercise 3.9. How does this compare to the results of the Steiner insertion heuristic?

For small networks, the soap bubble method is fast and impressive. It's easy to believe it will quickly converge on the correct answer for *large* networks, with the same smooth inevitability. But *this cannot possibly be true*. To see why, recall that in §3.3, we argued that finding a tinkertoy is NP-complete, and finding the genuine minimal network is NP-hard. Computer scientists believe that neither problem can be solved quickly on a regular digital computer. The set of problems that *can* be solved quickly is called P (for “Polynomial time”), so it is conjectured that $P \neq NP$.¹¹

But our soap bubble setup is an *analogue* computer, built using a fundamentally different physical mechanism from the digital 1s and 0s in your laptop processor. Could it do things quickly that would take a regular computer longer than the age of the universe? Computer scientist SCOTT AARONSON has persuasively argued [1] that the problems in NP-complete and NP-hard cannot be solved quickly by *any* computer, digital or analogue. This is called the *NP Hardness Assumption*. This is simply because many of the hardest problems we know are NP-complete, and if we could solve them, then as Aaronson puts it,

...we would be almost like gods. The NP Hardness Assumption is the belief that such power will be forever beyond our reach.

This means we cannot quickly find the minimal rail network for 800 European cities using soap bubbles, a black hole, human DNA, a quantum computer, or any other conceivable mechanism. No one will ever know what the network looks like.

That raises the question: what do soap bubbles actually compute? As discussed above, the first point is that they *locally minimize* only. It is a lucky dip! The soap bubble will converge on

¹⁰For a reconfigurable soap bubble computer, you can use suction cups with rods between them.

¹¹In other words, just because you can quickly make a lucky guess doesn't mean you can quickly find the answer by honest labor. Although widely believed to be true, proving $P \neq NP$ is extremely difficult, so difficult, in fact, that there is a \$1 million bounty! If you happen to stumble upon a proof, you can claim your prize [here](#).

a tinkertoy which spans the screws, but it may not give the shortest possible network. Since even finding a tinkertoy that works (let alone the minimal one) is NP-complete, it should take longer and longer for the soap bubbles to settle down as we increase the number of screws. Aaronson saw this happening even for a few screws [1].

Although less computationally powerful than sometimes claimed, I am not trying to deflate soap bubbles. They compute many cool things, and we'll see more of these below. Rather, the moral is that physics and computation *interact in interesting ways*. Results about computation lead to physical predictions (see Exercise 4.2 for examples). But physics is not without its computational surprises! The biggest surprise is that computers based on *quantum mechanics* can quickly perform tasks which are hard (though not NP-complete) for a regular digital computer. There's a beautiful story here, but I'll leave that for another bloated compendium.

Exercise 4.2. NP Hardness and the laws of physics.

Here are a few fun ways to solve NP-complete problems:

- (a) Create a time machine, and by sending a computer through it again and again, perform an arbitrary number of computations in finite time.
- (b) Develop a “fractal hypercomputer” which does the first step in 1 second, the second step in 1/2 second, the third step in 1/4 second, etc, so an infinite number of steps takes 2 seconds.
- (c) Store information in infinite precision real numbers, e.g. points on a line, and manipulate them using simple arithmetic [25].

If the NP Hardness Assumption is correct, none of these works! In each case, what do you think this is telling us about the nature of the universe?

4.2 The many faces of networks

While we can use soap bubbles to learn about minimal networks, we can arguably obtain more insight by going in the other direction. What do minimal networks teach us about soap bubbles? In this section, we will specifically be considering the two-dimensional bubble networks sandwiched between plexiglass, but with *no screws*. We'll just let the bubbles do their own thing! In Fig. 24, we show a real two-dimensional soap foam.¹² The colored numbers in the centre of a bubble indicate the number of sides it has. Curiously, the majority of bubbles are hexagonal. Why?

It turns out to follow from the 120° rule, some more graph theory, and a little physics. The main result we will need from graph theory is *Euler's formula*, discovered by the prolific Swiss mathematician LEONHARD EULER (1707–1783) in 1735. It states a relationship between the number of nodes N , edges E , and faces F in a graph, proved below in Exercise 4.3:

$$N - E + F = 2. \tag{6}$$

Importantly, this only holds for *planar graphs*, which are connected and can be drawn without any edges crossing. See Fig. 25. A face is defined as any region enclosed by a loop of edges, including the exterior of the graph.

¹²Based on a photograph by Klaus-Dieter Keller, Wikimedia Commons.

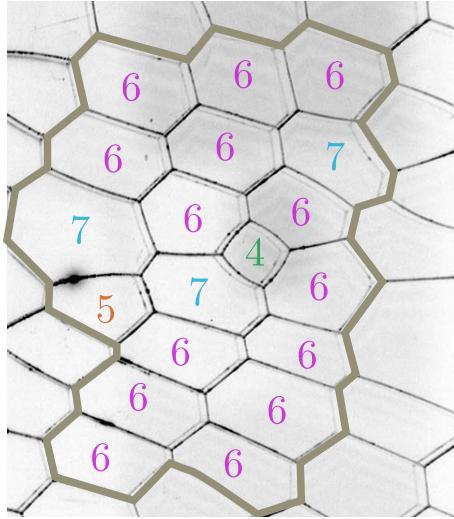


Figure 24: Most cells in a bubble network are hexagonal.

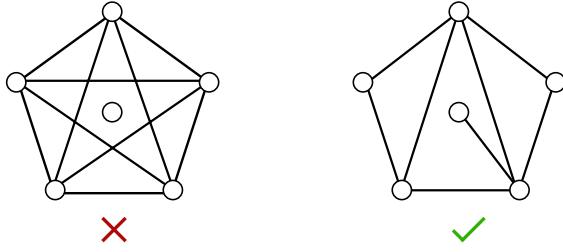


Figure 25: *Left.* A disconnected graph which cannot be drawn without edge crossings. *Right.* A planar graph. Euler's formula holds if we count the region outside the graph as a face.

One way to obtain a planar graph is to take a three-dimensional polyhedron, remove a single face, and flatten what remains onto the plane. This flattening process is shown for the cube in Fig. 26. The removed face becomes the exterior region of the graph, which is why we count it as a face. Let's check that it works. For the cube, we have $F = 6$ faces, $E = 12$ edges, and $N = 8$ corners, with $F - E + N = 2$ just as Euler's formula requires. We can count either using the cube itself, or the flattened graph!

Exercise 4.3. *Euler's formula.* \blacktriangle

Define the *Euler characteristic*

$$\chi = N - E + F.$$

Our goal will be to show $\chi = 2$ for a graph without crossings. First, we will establish Euler's formula for networks made out of triangles. We can then extend this to any graph without crossings. Below we depict stages (a), (b), (c) and (e).

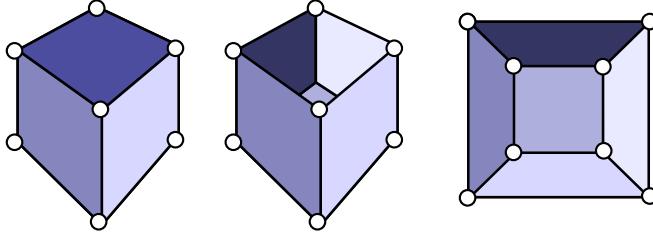
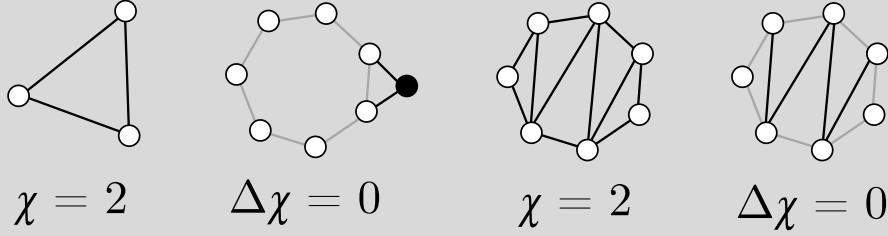


Figure 26: Remove the top of the cube and flatten.



- (a) Show that a lone triangle in the plane obeys Euler's formula.
- (b) Suppose a network obeys Euler's formula. Add a triangle (two edges and a node) to an external edge, and explain why the Euler characteristic doesn't change, $\Delta\chi = 0$. Conclude that the new network obeys Euler's formula.
- (c) Explain why a network composed of triangles obeys Euler's formula.

Now we can generalize to any network without crossings.

- (d) Consider a face, i.e. loop of edges, in such a network. Describe a procedure to add edges so that the face is split into triangles.
- (e) Show that, after your procedure in part (d), $\Delta\chi = 0$.
- (f) Conclude that any network without crossings obeys $\chi = 2$.

When there are no screws, a bubble networks forms between the plexiglass with no fixed nodes. Thus, each node is a hub, and the 120° rule dictates that each hub is a junction of three bubble walls. By the handshake lemma (1), we have $2E = 3N$. Putting this into Euler's formula, we can eliminate N and find a relation between the number of faces and number of edges:

$$N - E + F = \frac{2}{3}E - E + F = 2 \implies 3F - E = 6. \quad (7)$$

It will be useful to treat the external face a little differently. Let F' be the number of *internal* faces, so that $F = F' + 1$. Then (7) becomes $3F' - E = 3$.

Before proceeding, we need two additional properties of our bubble networks. First of all, an edge cannot dangle into the middle of a face. If it did, the vertex at the end of the dangling edge would not have three attached edges, only one, which is impossible by the 120° rule. It follows that every edge straddles two distinct faces.¹³

¹³Counterexamples like an edge cutting across two concentric circles are also ruled out by the 120° rule.

Let F_s denote the number of internal faces with s sides, and let E_b stand for the number of edges of the outer face. The total number of internal faces is

$$F' = F_1 + F_2 + F_3 + \dots . \quad (8)$$

But since each edge is associated with *two* faces, we can also express edges in terms of these numbers:

$$2E = E_b + 1 \cdot F_1 + 2 \cdot F_2 + \dots + s \cdot F_s + \dots . \quad (9)$$

If we plug (8) and (9) into $3F' - E = 3$, we finally get

$$6 + E_b = 6F' - 2E + E_b = (6 - 1) \cdot F_1 + (6 - 2) \cdot F_2 + \dots + (6 - s) \cdot F_s + \dots .$$

We will call the RHS the *hexagonal difference* D_{hex} , since it counts the number of edges which do not belong to a hexagonal face, with a sign depending on whether the face is smaller (+) or larger (-) than a hexagon. So, more simply, we have

$$D_{\text{hex}} = 6 + E_b. \quad (10)$$

The hexagonal difference is 6 more than the number of boundary edges. We give a few simple examples in Fig. 27, with the contribution to D_{hex} indicated in each cell.

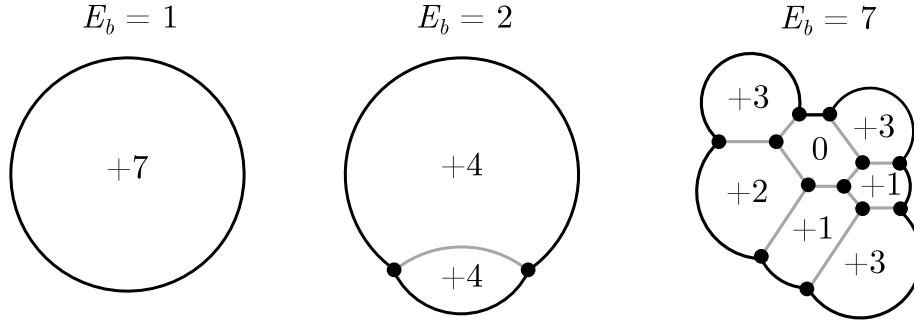


Figure 27: D_{hex} , the sum of numbers in cells, is always 6 more than E_b .

Exercise 4.4. Large and small faces.

We can already use (10) to learn interesting things.

- (a) Explain why $D_{\text{hex}} \geq 6$.
- (b) Deduce that

$$6 + 5 \cdot F_1 + 4 \cdot F_2 + \dots + 1 \cdot F_5 \geq 1 \cdot F_7 + 2 \cdot F_8 + \dots .$$

- (c) Suppose a bubble network has two bubbles with four sides and no other small faces. What is the maximum number of 10-sided bubbles?

In general, once we count the “small” faces F_1, \dots, F_5 , we can constrain the number of “large” faces F_7, F_8, \dots

4.3 Hexagons and honeycomb

It's still not clear why most bubbles are hexagonal. At this point, we need some physical intuition. Suppose the foam has overall size $\sim L$. Assuming bubbles have a typical size independent of L , the number of external edges $E_b \sim L$. The total area of the bubble network should scale as $A \sim L^2$.

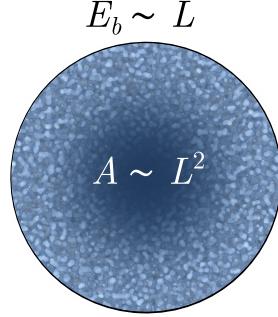


Figure 28: As the foam gets large, the number of outer edges scales as L , and the area as L^2 .

For instance, consider a roughly circular foam of radius L (Fig. 28). If bubbles have average edge length ℓ , independent of L , then $E_b \approx (\pi/\ell)L$, while $A = \pi L^2$.

It follows that, for large L , the hexagonal difference $D_{\text{hex}} = 6 + E_b \sim L$. The “density” of non-hexagonal edges d_{hex} is just the total hexagonal difference divided by the area of the foam. Since area scales as L^2 , the density of non-hexagonal edges scales as

$$d_{\text{hex}} \sim \frac{D_{\text{hex}}}{L^2} \sim \frac{1}{L}. \quad (11)$$

As L becomes larger, edges belonging to non-hexagons become increasingly rare. This explains why a typical cell in a bubble network has six sides, just like Fig. 24.

Exercise 4.5. Bubble blobs.

A *bubble blob* is a set of contiguous bubbles in a bubble network. Let E_o denote the number of edges extending outward from the boundary, and E_i the number extending inward.

- (a) Explain why the difference from hexagonality is now given by

$$D_{\text{hex}} = 6 + E_i - E_o. \quad (12)$$

- (b) Verify that the yellow blob of cells in Fig. 24 satisfies (12).
- (c) Repeat the scaling argument above, and conclude that in a large blob, departures from hexagonality become rare.

You may have wondered if the hexagonality of bubbles is related to the fact that bees build honeycombs in a hexagonal lattice. It is! Bees have a clear evolutionary reason to minimize the amount of wax used. CHARLES DARWIN (1809–1882) discusses the hive-making instinct and its relation to fitness in his *Origin of Species* [6]:

That motive power of the process of natural selection having been economy of wax; that individual swarm that wasted least honey in the secretion of wax, having succeeded best.

For honeycomb walls to be minimal, they must obey the 120° rule. If honeycomb cells are equal in size (which bees might prefer for simplicity of construction), then a natural guess at the optimal arrangement is the *hexagonal lattice*. This is the only regular tessellation satisfying the 120° rule.

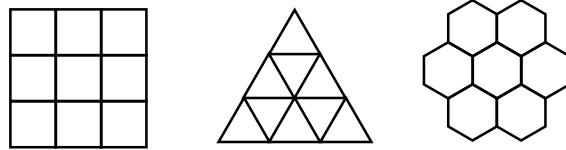
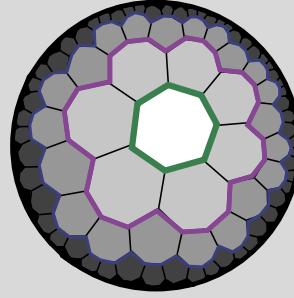


Figure 29: The three regular tessellations of the plane: square, triangle, hexagon.

The *honeycomb conjecture* states that the hexagonal lattice is the *globally* minimal solution among tessellations of the plane with equal cell size. It is hard to verify this guess, since you need to somehow check all possible *irregular* tilings as well as the regular ones. But in 1999, it turned from conjecture into theorem after THOMAS HALES gave a formal proof [17]. In Exercise 4.6, we explore the analogous problem for the strangely curved *hyperbolic* plane.

Exercise 4.6. Hyperbolic honeycomb. ↗

Our scaling argument assumed we were on a regular, Euclidean plane. But we can see what happens if, instead of working on the Euclidean plane, we work on the strangely curved *hyperbolic plane*.



Above, we have tiled the hyperbolic plane with heptagons. Each heptagon has the same area, and sides of equal length, but the curvatures means they must be drawn with different lengths on our flat page!

- (a) Find A (in heptagon units) and E_b for the regions enclosed in (i) green, (ii) purple, (iii) blue. Does the ratio E_b/A appear to be decreasing?
- (b) Argue that, in general, for $n > 1$ “rings” of heptagons,

$$E_b = 4 \cdot 7^n, \quad A = \frac{1}{6}(7^{n+1} - 1) \approx \frac{7}{6} \cdot 7^n.$$

Hint. Use a geometric sum for A .

This shows that on the hyperbolic plane, the scaling $E_b \sim L$, $A \sim L^2$ no longer holds. Instead, the boundary and area *scale the same way*.

- (c) Why does the 120° rule still hold for minimal networks on the hyperbolic plane? *Hint.* What happens when you zoom in on a node?
- (d) Show that for ring n , $E_o - E_i = 2 \cdot 7^n$. Using part (c) and similar reasoning to the plane, conclude that a large number of heptagonal rings,

$$D_{\text{hex}} = 6 - 2 \cdot 7^n \approx -2 \cdot 7^n.$$

- (e) Finally, show that our heptagonal tiling has

$$d_{\text{hex}} = \frac{D_{\text{hex}}}{A} \approx -\frac{12}{7}.$$

- (f) Given Exercise 4.4(a), how can D_{hex} be negative?

The weird properties of hyperbolic space mean that the optimal tessellation depends on the size of the cells. The heptagonal tiling is optimal for the cell size pictured above, among *regular* hyperbolic tilings [7]. The “hyperbolic honeycomb conjecture”—that this is optimal among *all* hyperbolic tessellations with this cell size, including the irregular ones—remains open.¹⁴ Perhaps we should breed some hyperbolic bees, and inspect their honeycomb in a few million years!

4.4 The isoperimetric inequality and bubbletoys

In §4.2–4.3, we studied two-dimensional bubble foams, assuming there were no fixed nodes. But for minimal networks, the fixed nodes *define* the problem we are meant to be solving. An urban planner hands us a map and tells us to design a railway or a telecommunications network or a canal system. We find or approximate a tinkertoy which does so. If a bubble network has no fixed nodes, is it solving a problem at all? Or is it all just froth and horseplay?

The honeycomb theorem suggests a different problem. The bees have no fixed nodes, since they are not trying to connect anything. Instead, they are trying to build cells to store honey, and an infinite number of the same size, if you please. But what if the bees only want six? Or want to vary their serving sizes with cells of different area? In general, we can ask for the *minimal length bubble network* enclosing cells of size A_1, A_2, \dots, A_n .

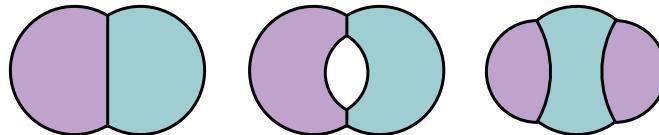


Figure 30: *Left.* The standard double bubble. *Middle.* An air pocket. *Right.* A split bubble.

In the same way that we are allowed to add nodes to minimal networks to decrease length, we

¹⁴A similar statement holds for other cell sizes.

allow air pockets and splitting of bubbles if it reduces length (Fig. 30). We will ask that the bubble network is always connected though. This leads to the following optimization problem.

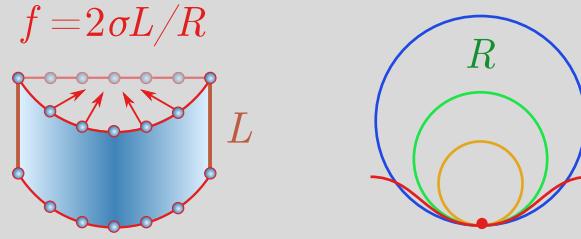
Box 4.1. *The planar bubble configuration problem.*

Find the connected bubble network of smallest perimeter enclosing cells of area A_1, A_2, \dots, A_n , allowing air pockets and split bubbles.

While we derived the 120° rule directly from minimizing length, the other salient property of bubble networks is that walls are straight or arcs of circles. This follows from the physics of surface tension, as you can explore now in Exercise 4.7.

Exercise 4.7. *Young-Laplace I.*

The molecules in a bubble wall are attracted to each other. If you try to *bend* the surface, it strains the bonds, which attempt to restore the unstretched state. The amount of bending at a point can be quantified by finding a circle which fits snugly into the curve at that point (the green circle, below right).



If this snug circle has radius R , we say the bend has *radius of curvature* R . For a bubble wall of height L , the restoring force per unit length of curve is $f = 2\sigma L/R$, where σ is the *surface tension*.

- (a) Show that if there are no other forces acting on the wall, it must be straight.
- (b) Now consider the effects of *pressure* at a point on the wall. If the pressure on one side is P_{out} , and inside is P_{in} , argue the bend will have radius of curvature

$$R = \frac{2\sigma}{\Delta P}, \quad (13)$$

where $\Delta P = P_{\text{out}} - P_{\text{in}}$. This is called the *Young-Laplace law*. Check it is consistent with (a).

- (c) Within a cell, pressure difference equalize very quickly, so it is reasonable to assume pressure is constant on a face of the network. Deduce that bubble walls are either flat or arcs of circles.

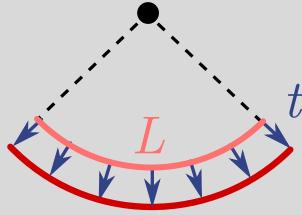
The condition on bubble walls follows from physics rather than the requirement that it locally minimize length. Indeed, some bubble arrangements are *unstable*, with small changes to the bubble

network reducing the length. However, the bubbles we see in Nature tend to be stable (i.e. local minima) and it is tempting to guess that they are sometimes *globally* minimal as well, i.e. they solve the relevant bubble configuration problem. We can view these common configurations as “physical conjectures” about minimal bubble networks.

The simplest such physical conjecture is for a single bubble of area A . The only smooth way to draw a single cell is a circle (Fig. 33 (left)), and when Nature is left to its own devices, bubbles tend to assume this form. You can also check that there is no way to split the single bubble, or introduce air pockets, while maintaining a connected bubble network. Mathematicians as far back as ARCHIMEDES (287–212 BC) have suggested that the circle is the shape of smallest perimeter for a fixed area A , a guess called the *isoperimetric inequality*.¹⁵ This guess wasn’t verified until the 19th century, but the modern proof is simple enough to present in outline. The basic idea is to take a curve which is minimal, add small wobbles, and see what happens.

Exercise 4.8. Stretched arcs.

To start with, we consider wobbling the radius of a single circular arc. Suppose an arc of length L and radius R is part of a curve enclosing some area on the plane. Consider extending the radius by a small amount t , where “small” means much smaller than L .



- (a) Show that the area enclosed changes by

$$\Delta A \approx Lt. \quad (14)$$

- (b) Assuming that the angle subtended by the arc is the same, explain why the length of the arc changes by

$$\Delta L \approx \frac{Lt}{R}. \quad (15)$$

- (c) Check that (b) still makes sense for a straight line.

For both ΔA and ΔL , there are some additional corrections, but these will appear as higher powers of t , starting at t^2 .

In general, we can take a curve on the plane and chop it up into k small pieces of length L_1, L_2, \dots, L_k and constant radius of curvature R_1, R_2, \dots, R_k , setting $R_i = \infty$ for any straight lines.¹⁶ Imagine we wobble the curve by *independently* changing the radii for each segment, adding

¹⁵Saying the circle has the *least perimeter* of all figures of area $A = \pi r^2$ is the same as saying it has *most area* of all figures with the same perimeter $2\pi r$. ‘Isoperimetric’ means ‘same perimeter’.

¹⁶If we wanted to be rigorous, we would actually chop the line up into an *infinite* number of pieces using calculus.

t_1, t_2, \dots, t_k . Using (14), the total change in area is

$$\Delta A = \Delta A_1 + \Delta A_2 + \dots + \Delta A_k = L_1 t_1 + L_2 t_2 + \dots + L_k t_k. \quad (16)$$

From (15), the total change in length is

$$\Delta L = \Delta L_1 + \Delta L_2 + \dots + \Delta L_k = \frac{L_1 t_1}{R_1} + \frac{L_2 t_2}{R_2} + \dots + \frac{L_k t_k}{R_k}. \quad (17)$$

It's clear that if we deform the curve so as to preserve area, then $\Delta A = 0$.

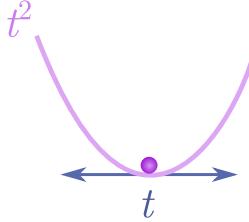


Figure 31: If perimeter is a local minimum, it looks like a quadratic function of the wobble.

But here is the clever part: if the curve is a local minimum of perimeter, then the perimeter looks like a quadratic function of the wobbling, as in Fig. 31. The shape of this minimum turns up when we look at the terms which go like t^2 . But in our approximation, we only keep the terms which go like t , which cannot see those t^2 terms. Thus, in the approximation we have used to compute (17), a perimeter-minimizing curve has $\Delta L = 0$. You can show in the next exercise that, given the forms for ΔA and ΔL , this is only possible if

$$R_1 = R_2 = \dots = R_k = R,$$

i.e. the radius of curvature is constant. Thus, the perimeter-minimizing curve has constant R .

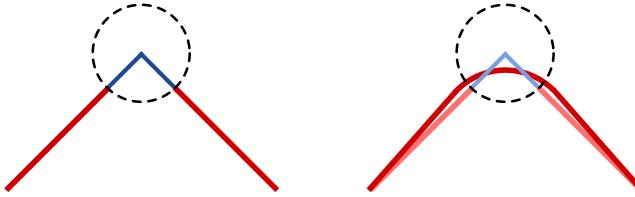


Figure 32: Replacing a kink by a smooth edge which encloses the same area.

Does constant radius of curvature mean we have a circle? Not necessarily. You could join arcs of the same circle with a “kink”. But we can always approximate a kink as closely as we like by a smooth edge which encloses the same area (Fig. 32). This edge will have a *different* radius of curvature, which contradicts our argument! The only smooth, closed curve we can draw, which has the same radius of curvature R at every point, is the circle of radius R itself. This more or less proves the isoperimetric theorem.¹⁷

But the spirit of the argument is the same.

¹⁷Technically, we have only shown that if there is a perimeter-minimizing shape of fixed area, it is a circle. But our approximation strategy can also be turned into a proof that the circle does minimize.

Exercise 4.9. *Constant radius of curvature.*

If we vary a perimeter-minimizing curve, then $\Delta L = 0$. If the wobbles also preserve area, then $\Delta A = 0$. We will show that this implies all the radii R_1, R_2, \dots, R_k are the same.

- (a) Suppose that only t_1 and t_2 are nonzero in (16). Show that $\Delta A = 0$ implies

$$t_1 = -\frac{L_2 t_2}{L_1}.$$

- (b) Now substitute this into (16), and from $\Delta L = 0$, argue $R_1 = R_2$.

- (c) Extend this argument to show that $R_1 = R_2 = R_3 = \dots = R_k$.

So much for a single bubble. The next simplest problems involve two and three bubbles of equal area A . The standard *double bubble* (Fig. 33 (middle)) and *tripple bubble* (Fig. 33 (right)) configurations are drawn below. Since we can now introduce air pockets and splitting, as in Fig. 30 for two bubbles, it is much harder to show these simple arrangements are minimal. The double bubble was only shown to be minimal in 1993 [11], and the triple bubble in 2002 [29]. As far as I know, no other simple bubble configurations are known to be minimal.

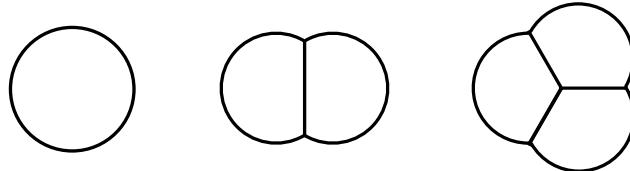


Figure 33: *Left.* A circle. *Middle.* The standard double bubble. *Right.* The standard triple bubble.

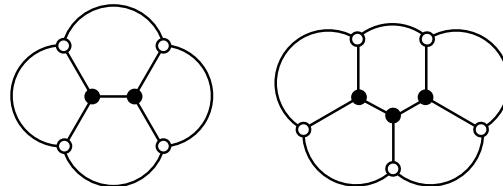


Figure 34: Tinkertoys giving rise to bubbletoys.

Of course, we might say: forget mathematics, and let Nature be our guide. By carrying out simple experiments, we should be able to see which configurations are predicted by physics. Right? Unfortunately, the same combinatorial explosion that plagued soap bubble computers in §3.3 afflicts planar bubble configurations. One argument is simply that every tinkertoy gives rise to a bubble configuration, simply by adding arcs to the outside as illustrated in Fig. 34.¹⁸ I call these *bubbletoys*.

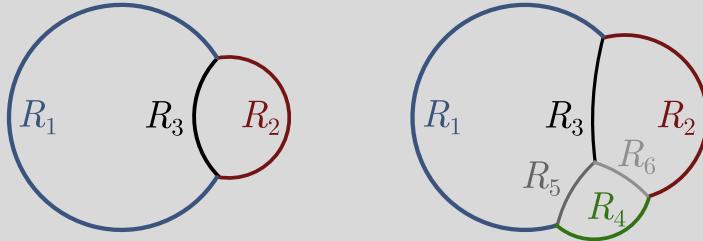
¹⁸ More generally, we will have to bend the inner walls to make sure the pressure difference is balanced by tension. See Exercise 4.10 for more details.

Incidentally, the two pictured bubbletoys are conjectured to be the minimal planar configurations for four and five equal-area bubbles.

This suggests that solving the planar bubble configuration problem is NP-hard, and even finding a bubble configuration which encloses the volumes A_1, A_2, \dots, A_n is NP-complete.¹⁹ Nature will take increasingly long times to converge on her “conjectures”. Either way, we cannot get physics to magically solve our NP-complete problems for us!

Exercise 4.10. *Bubble radii and pressure cocycles.* \blacktriangle

We show the standard double and triple bubble for bubbles of different radii below.



This exercise shows how the bubble radii are simply related by physics! The equivalent derivations from the 120° rule are much messier [16].

- (a) Let’s start with the double bubble. By considering pressure differences as across interfaces, explain why

$$\frac{1}{R_2} = \frac{1}{R_1} + \frac{1}{R_3}. \quad (18)$$

Hint. Use Exercise 4.7(b).

- (b) We can make this observation more general. Consider moving around a loop on a bubble network. Across each interface, there are pressure differences $\Delta P_1, \Delta P_2, \dots, \Delta P_n$. Show that, along the loop,

$$\Delta P_1 + \Delta P_2 + \cdots + \Delta P_n = 0.$$

This is called the *pressure cocycle condition* in the mathematics literature.

- (c) Using the pressure cocycle condition for the triple bubble, calculate that in addition to (18), we have

$$\frac{1}{R_4} = \frac{1}{R_1} + \frac{1}{R_5} = \frac{1}{R_2} + \frac{1}{R_6}.$$

Check that the results of executing a loop around the inner junction are consistent with these relations.

¹⁹ In fact, this argument suggests that a *restricted* version of the problem in Box 4.1, without air pockets or bubble splitting, is very difficult. Including these extra possibilities, there are now an *infinite* number of possibilities for any given set of areas, hence the difficulty of establishing results even for two or three bubbles.

5 Bubbles in three dimensions

So far, we've only considered two-dimensional networks, which arise from sandwiching bubbles between plates of plexiglass. But the real world has three dimensions. Thankfully, our work with two dimensional bubbles is excellent preparation for the real world! Let's start by summarizing what we know about bubble networks. The key result from §2 was the 120° rule. In §4, particularly the last section, we learned that bubble walls are straight or arcs of circles, so that they have constant radius of curvature (Exercise 4.7). We also learned from our treatment of the isoperimetric problem that, if walls are perimeter-minimizing, they do not have “kinks”. We can encode these three insights as three laws for bubble networks:

Box 5.1. Bubble network laws I.

1. *No kinks.* Edges are smooth, i.e. no vertices attached to one or two edges.
2. *Constant curvature.* Edges have constant radius of curvature.
3. *The 120° rule.* Three edges meet at a junction, separated by 120° .

There is another way to motivate the 120° rule that will prove very useful. Given the law of “no kinks”, the smallest number of edges that can meet at a junction is three. Moreover, meeting at angles of 120° is the most *symmetric* way for incoming edges to be separated. Way back in §2.1, we saw that symmetry had an important role to play in minimizing the length of the network on an equilateral triangle, so perhaps it is unsurprising that the two are connected here. We call this the *minsym principle*. It lets us reformulate our network laws in a slightly different way:

Box 5.2. Bubble network laws II.

1. *No kinks.* Edges are smooth, i.e. no vertices attached to one or two edges.
2. *Constant curvature.* Edges have constant radius of curvature.
3. *Minsym.* At a junction, the minimal number of edges meet symmetrically.

5.1 Mean curvature

A bubble network is an arrangement of two-dimensional cells separated by a network of bubble walls which are effectively one-dimensional. But when bubbles have a full three dimensions to roam around in, the cells are three-dimensional volumes separated by two-dimensional walls. It seems like this should be a whole different kettle of fish. But marvellously, the laws governing three-dimensional bubbles are almost exactly the same as the laws governing their two-dimensional counterparts. The three-dimensional laws are called *Plateau's laws*, after the Belgian physicist JOSEPH PLATEAU (1801–1883) who guessed them by assiduously observing bubbles.

“No kinks” seems straightforward: bubble walls are smooth and cannot terminate unless there is something for them to end on, e.g. a bubble blower. But there are subtleties for the remaining two rules. In a bubble network, edges have constant radius of curvature. What is the analogous

statement for surfaces? It turns out there are different notions of curvature for a surface, but for our purposes, the relevant notion is *constant mean curvature*. This requires a little explanation.

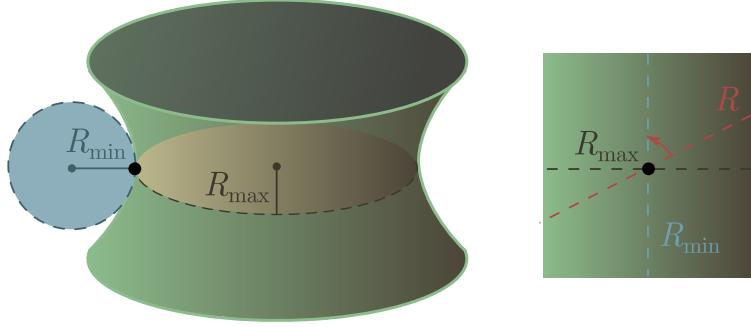


Figure 35: *Left*. A surface, with principal “snug” circles. *Right*. Straight slices through a point.

Suppose we have a two-dimensional surface like the one in Fig. 35 (left). If we take various straight slices through the black point (shown in Fig. 35 (right)) each will give rise to a radius of curvature, i.e. the radius of a circle which fits “snugly” onto the curve at that point, and which is normal to it, i.e. is perpendicular to the surface at that point. As we rotate the red slice in Fig. 35 (right), the radius of curvature R will vary, producing a maximum value R_{\max} and minimum value R_{\min} . The reciprocals $1/R_{\max}$ and $1/R_{\min}$ are called the *principal curvatures*. We picture these on Fig. 35 (left). In this case, we also have a *minus* sign for the minimum curvature, since the circle is on the other side of the surface.

Finally, the *mean curvature* H is the sum of principal curvatures:

$$H = \frac{1}{R_{\max}} + \frac{1}{R_{\min}}. \quad (19)$$

A *constant mean curvature (CMC)* surface is one where the mean curvature H is the same everywhere on the surface. Notice that, as in Fig. 35 (right), it is always the case that the principle curvatures R_{\max} and R_{\min} are measured along orthogonal slices. We call this the *orthogonal circle rule*. To generalize the constant curvature rule from bubble networks, we simply require bubble surfaces to be CMC. You can explore some of the physics behind this in Exercise 5.2.

Exercise 5.1. Spheres are CMC.

Show that a sphere of radius R has constant mean curvature $H = 2/R$. Hint. The slice normal to the sphere at any point is a great circle.

Exercise 5.2. Young-Laplace II. ▲

In Exercise 4.7, we saw the Young-Laplace law (13) for a bubble wall:

$$\Delta P = \frac{2\sigma}{R},$$

for $\Delta P = P_{\text{out}} - P_{\text{in}}$ and R the radius of curvature of the wall.

- (a) Viewing a bubble wall as a surface, argue that $1/R_{\max} = 0$.

- (b) Using the orthogonal circle rule, deduce that $H = 1/R$ is the mean curvature of the wall. Hence, the Young-Laplace law can be written

$$\Delta P = 2\sigma H. \quad (20)$$

This turns out to be the correct form for an arbitrary surface!

- (c) If we dip two identical circular bubble blowers in soap film (red below), the surface that results is typically like the one below left, rather than a cylinder:



Give a qualitative explanation, using (20) and mean curvature.

- (d) Using Exercise 5.1, what is the *smallest* spherical bubble that can form in the atmosphere? Atmospheric pressure is $P = 10^5 \text{ N/m}^2$ and the surface tension of soapy water is $\sigma = 7 \times 10^{-2} \text{ N/m}$. Can a spherical bubble form in space?

5.2 Plateau's laws

The last point that requires clarification is the “minsym” principle. The “no kink” requirement means that we cannot have two walls meeting at an angle, since that would introduce a kink, and if there is no angle, they may as well be counted as part of the same wall. Thus, we must have at least three walls meet at any junction of walls. According to the minsym principle precisely three faces should meet (minimum) separated by 120° (symmetry), as in Fig. 36 (left). In fact, this is exactly what we require to reproduce the 120° rule in a bubble network formed between plates, since there, the walls are vertical but bubble walls nevertheless (Fig. 36 (middle)).

The edge along which three bubble walls meet is called a *Plateau border*. These borders themselves can intersect! Minsym requires us to figure out the minimum number to avoid kinks, and the most symmetric arrangement thereof. Clearly, we need at least three, since otherwise we can arrange a junction of three walls with a kink, as in Fig. 36 (right).

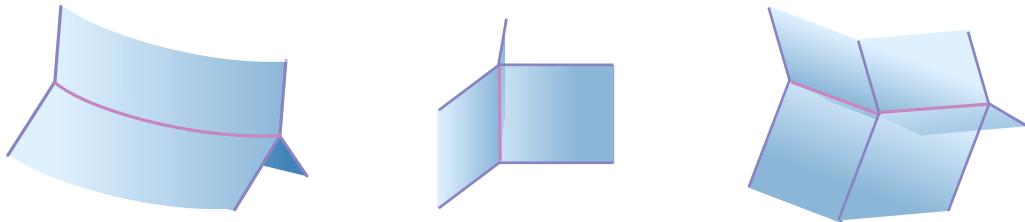


Figure 36: *Left*. Three faces meeting at a border. *Middle*. Vertical bubble walls. *Right*. A kink.

What about three Plateau borders? It's not hard to see that the three sets of three faces cannot

be connected smoothly, simply because we have an odd number of faces! You can check the details in Exercise 5.3. This exercise also shows that it *is* possible to connect the faces smoothly for four sets of Plateau borders. Thus, the minsym principle suggests that *precisely* four Plateau borders should meet in the most symmetric arrangement. Symmetry is maximized by shooting out the Plateau borders *tetrahedrally*, i.e. from the center towards the corners of a regular tetrahedron (Fig. 37). If you like vectors, you can play around with the geometry in Exercise 5.4.

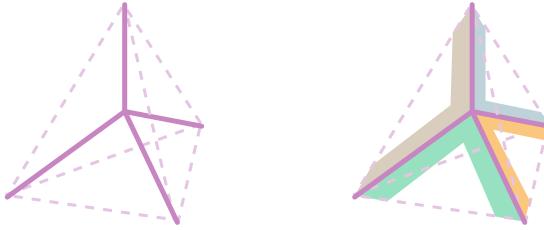


Figure 37: *Left.* Four Plateau borders meeting tetrahedrally. *Right.* Smoothly connected walls.

Having understood mean curvature, and the minsym principle in three dimensions, we are finally in a position to state the laws Plateau discovered in 1873:

Box 5.3. Plateau's laws.

1. *No kinks.* The faces in a soap film are smooth.
2. *Constant curvature.* Any face has constant mean curvature.
3. *Minsym I.* Three faces always meet at a Plateau border, separated by 120° .
4. *Minsym II.* Plateau borders always meet tetrahedrally at a vertex.

These are empirical observations about bubbles. They are consistent with the Young-Laplace law (20) for surface tension, and surface tension tries to pull bubble walls taut. So it seems plausible that the bubbles are solving *some* minimization problem. But which one? Is every configuration satisfying Plateau's laws a local minimum? And does every locally minimal solution satisfy Plateau's laws? We will answer these questions in the next section.

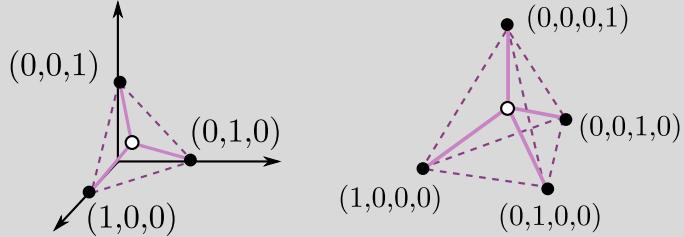
Exercise 5.3. Plateau borders.

Let's check we need four Plateau borders in order to smoothly connect walls. Consider some number of Plateau borders meeting at a vertex. Each border has three associated bubble walls.

- (a) Explain why “no kinks” requires each wall to connect smoothly to another.
- (b) Argue that this is impossible for an odd number of borders meeting at a node.
- (c) Show explicitly it is possible for four Plateau borders to smoothly connect.

Exercise 5.4. Simplices.

The equilateral triangle and the tetrahedron are part of a family of symmetric shapes called *simplices*. We can describe them using vector analysis.



- (a) We can embed the vertices of an equilateral triangle in three dimensions as

$$\Delta_3 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}.$$

Why is this a maximally symmetric arrangement of three points?

- (b) The *center* of the triangle is just the average of the vertices. Show that the vectors from center to vertices have length $\sqrt{2}$ and are given by

$$V_3 = \left\{ \frac{1}{3}(-2, 1, 1), \frac{1}{3}(1, -2, 1), \frac{1}{3}(1, 1, -2) \right\}.$$

- (c) Using the formula

$$\theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} \right),$$

check that the vectors in V_3 make angle $120^\circ = \cos^{-1}(-1/2)$ with each other.

- (d) We can embed the tetrahedron in *four* dimensions as

$$\Delta_4 = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}.$$

Show that the vectors from center to vertex have length $\sqrt{3}$, and make angles

$$\theta = \cos^{-1} \left(-\frac{1}{3} \right) \approx 109.5^\circ.$$

5.3 Bubbles and wireframes

As you might have guessed, bubbles are related to the three-dimensional version of the planar bubble configuration problem outlined in Box 4.1. Basically, instead of enclosing area A_1, A_2, \dots, A_n , we want to enclose volumes V_1, V_2, \dots, V_n , with a bubble film of minimal area. As before, we allow for air pockets and split bubbles. We state the optimization problem as follows:

Box 5.4. *The bubble configuration problem.*

Find the connected bubble film of smallest area enclosing volumes V_1, V_2, \dots, V_n , allowing air pockets and split bubbles.

Surfaces with bubbles of fixed volume, and which locally minimize area, also satisfy Plateau's laws, as mathematician JEAN TAYLOR proved in her 1976 tour-de-force [27]. The converse is not true, since we can find bubbles satisfying Plateau's laws that are not stable (Fig. 39).

The planar bubble configuration problem (Box 4.1) is a special case where we put the foam between plates. This implies that local minima satisfy the bubble network laws (Box 5.1), since these are simply Plateau's laws in the case where bubble walls are vertical and there are no vertices at which Plateau borders intersect. And since solving the planar bubble problem is a special case of the general bubble configuration problem, we know that the general problem must be extremely hard! Physically speaking, we expect that foams will take longer and longer to converge to a minimum if we force them to compute for us.

Even when Nature makes conjectures, they can be bewilderingly hard to prove. The simplest example is a single bubble of volume V . Experience suggests that a lone bubble is always spherical, as in Fig. 38 (left). The corresponding conjecture is that the area-minimizing surface of volume V is a sphere, which is simply the *isoperimetric theorem* in three dimensions.

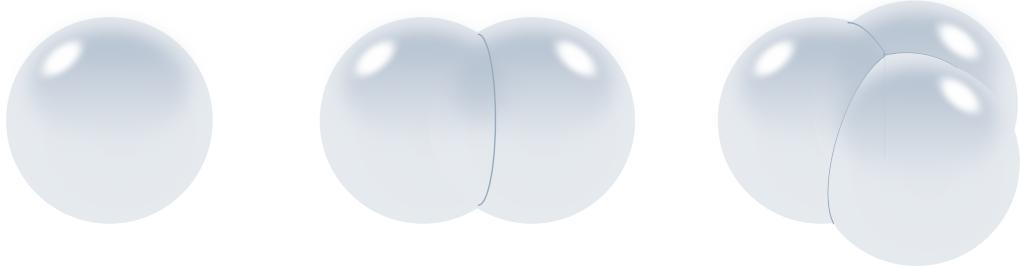


Figure 38: *Left.* A single spherical bubble. *Middle.* The standard double bubble. *Right.* The standard triple bubble.

The proof has two stages. First, one shows that whatever the area-minimizing surface is, it must have constant mean curvature. This is similar to the argument we gave in §4.4 perimeter-minimizing curves have constant radius of curvature.²⁰ In fact, we do exactly the same thing. We split the surface into many small patches of area A_1, A_2, \dots, A_k , with mean curvature H_1, H_2, \dots, H_k . If these areas are “pushed out” a distance t_1, t_2, \dots, t_k normal to the surface, the volume and area change as

$$\begin{aligned}\Delta V &= A_1 t_1 + A_2 t_2 + \cdots + A_k t_k \\ \Delta A &= A_1 t_1 H_1 + A_2 t_2 H_2 + \cdots + A_k t_k H_k.\end{aligned}$$

If the wobbling preserves volume, then $\Delta V = 0$, and if area is locally minimized, then $\Delta A = 0$ as before. We can then repeat our argument word for word to conclude that $H_1 = H_2 = \cdots = H_k$. Mean curvature is the same everywhere, and we have a CMC surface!

²⁰ This result is also one ingredient of the proof that area-minimizers obey Plateau's laws, since the second law states that foams consist of CMC faces.

In the plane, there was exactly one way to have a smooth curve with constant radius of curvature. In three dimensions, there are all sorts of exotic CMC surfaces. But it turns out that the sphere is the *only* CMC surface that enclose a finite volume, as proved by ALEKSANDR ALEKSANDROV (1912–1999) in 1958 [2]. This proves the isoperimetric theorem in three dimensions: the sphere is the surface of smallest area enclosing a given volume V .

The next simplest problem is two bubbles of equal volume V . Again, Nature seems to prefer the “standard double bubble”, with two spheres fused at a single Plateau border (Fig. 38 (middle)) over its non-standard competitors. One of these competitors is the “donut” configuration, where a single bubble is squeezed into the shape of an apple core by a donut-shaped ring around the outside (Fig. 39). This bubble satisfies Plateau’s laws, but turns out to be unstable, and jiggling the donut will cause it to collapse into the standard double bubble [26]. It wasn’t until 2002 that the double bubble configuration was shown to be minimal [19]. Similarly, we often observe the standard triple bubble for three cells of volume V (Fig. 38 (right)). No one knows if this is truly minimal, so it remains the *triple bubble conjecture*.

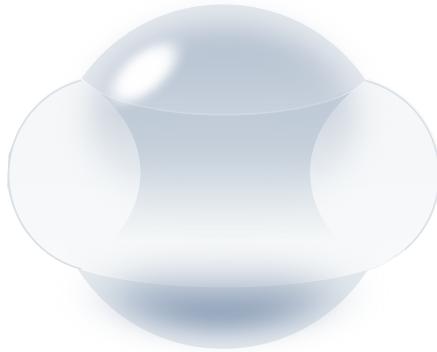


Figure 39: An unstable double bubble, consisting of an apple core wrapped in a donut.

But if there is a three-dimensional version of the bubble configuration problem, it stands to reason that there should also be a three-dimensional version of our original minimal network problem (Box 2.1). In the minimal network problem, we are asked to find a network of minimal length connecting some fixed nodes, i.e. points. If we raise the number of dimensions of the configurable object, going from a one-dimensional network to a two-dimensional bubble film, perhaps we should raise the dimensions of the fixed object, going from points to curves.

Box 5.5. Wireframe problem.

Given some fixed curves C_1, C_2, \dots, C_n in three-dimensional space, find a soap film of minimal area that connects them.

We will call these fixed curves *wireframes*, since physically speaking, we can implement curves with twisted pieces of wire. If we dunk them in soapy water, the soap bubbles can hold onto the wireframe and, in some cases, solve our problem for us! We give two very beautiful examples in Fig. 40: the *catenoid*, the surface which forms between two rings, and the *tesseract* formed when we dip

a wireframe cube. The cube creates a second, slightly puffed out²¹ inner cube, and then connects corresponding corners with Plateau borders.

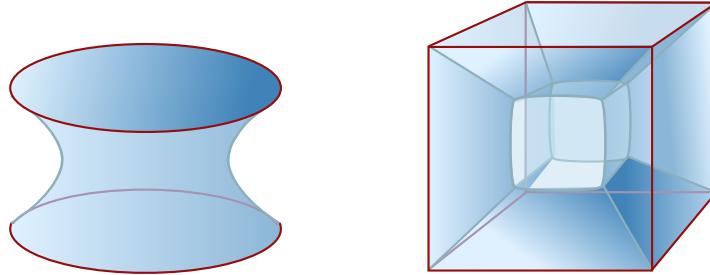


Figure 40: *Left*. The catenoid, a surface with mean curvature zero, which forms between identical wireframe rings. *Right*. The tesseract formed from a wireframe cube.

Exercise 5.5. Tesseract.

Verify that the tesseract has three faces meeting at 120° , and four Plateau borders meeting at each vertex.

Because this is a generalization of the minimal network problem, and we know that problem is difficult, the wireframe problem is also difficult. We can dip some arbitrarily complicated piece of wire into the soap, but when we pull it out, it may take a very long time — longer than the age of the universe in some cases — for the soap bubbles to converge on a stable solution. This is yet another physical prediction!

But it's not even obvious there is a solution at all. If we dunk some random wireframe into the water, the bubble film that connects them must satisfy Plateau's laws, except along the wire itself, in the same way that minimal networks satisfy the 120° rule at a hub but not at a fixed node. But do Plateau's laws always allow a solution? Perhaps we can defeat Nature by giving it some wacky curve it cannot connect with soap film! The intuitive physical argument is that we can simply dip our wireframe in and see what comes out, but as we've just argued, for a complicated enough problem, we will die long before we see the film converge! So this doesn't strike me as a good argument. The question of the *existence* of a solution to the wireframe task is called *Plateau's problem*. In the 1930s, mathematicians JESSE DOUGLAS (1897–1965) and TIBOR RADÓ (1895–1965) independently showed these solutions always exist [8, 24]. Even if takes longer than the lifetime of universe, Nature will eventually get there.

5.4 Space-filling foams

In the last section, we will consider the infinite cell version of the bubble configuration problem (Box 5.4). This is like honeycomb on the plane, but now we wish to partition an infinite number of equal volumes, rather than an infinite number of equal areas. For this partition to minimize the surface area per cell, we know it must satisfy Plateau's laws. In the same way that the 120° had powerful for the structure of bubble networks, the tetrahedral law has similar implications for the

²¹To ensure borders meet tetrahedrally.

structure of three-dimensional foams. To explore these, we first need to extend Euler's formula (6) to include multiple bubbles.

In a finite configuration of soap bubbles, let N denote the number of vertices (joining Plateau borders), E the number of Plateau borders, F the number of bubble faces (which meet at a border), and C the number of enclosed bubble cells. As before, we will count the region outside the bubble configuration as a cell as well. Recall from §4.2 that Euler's formula also applies to polyhedron like the cube. This has only two cells, the inside and the outside. But we can divide up this internal cell into multiple internal cells by adding inner walls.

If there are C cells altogether, there are $C - 1$ internal cells, and $C - 2$ “extra” internal cells compared to a regular polyhedron. For each extra cell, we can remove an internal face so that it forms a single cell with one of its neighbours. This leaves something we can flatten into a planar graph, with $F' = F - (C - 2)$ faces, and hence by Euler's formula

$$N - E + F' = 2.$$

Rearranging gives Euler's “foamula”:

$$N - E + F = 2 + (C - 2) = C. \quad (21)$$

Equation (21) is generally true for a polyhedron with extra internal cells, whether or not it also satisfies Plateau's laws.

Using this formula, along with Plateau's laws, you can show that bubble faces tend to have *less* than six sides. More precisely, if F_{avg} is the average number of faces per bubble cell, E_{avg} the average number of edges around the boundary of a cell, and e_{avg} the average number of edges per face, you can show in Exercise 5.6 that

$$F_{\text{avg}} = \frac{1}{3}E_{\text{avg}} + 2 = \frac{12}{6 - e_{\text{avg}}}. \quad (22)$$

Since F_{avg} is positive, it follows that $e_{\text{avg}} < 6$, i.e tend to be “sub-hexagons”.

Exercise 5.6. Sub-hexagonal faces. \blacktriangle

- (a) From Plateau's fourth law and the handshake lemma (1), argue that $E = 2N$.
- (b) Let F_{avg} denote the average number of faces per cell and E_{avg} the average number of edges per cell. Show that

$$F_{\text{avg}} = \frac{2F}{C}, \quad E_{\text{avg}} = \frac{3E}{C}.$$

Hint. You may assume that, like in a bubble network, a face in a bubble foam always has different cells on either side.

- (c) From (21), deduce the relation between average number of edges and faces:

$$3F_{\text{avg}} - E_{\text{avg}} = 6.$$

This is analogous to the result $3F - E = 6$ for bubble networks.

(d) Let e_{avg} be the average number of edges per face. Derive the relation

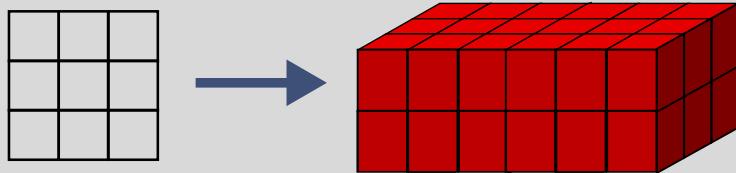
$$F_{\text{avg}} = \frac{12}{6 - e_{\text{avg}}}.$$

Hint. Write E_{avg} in terms of F_{avg} and e_{avg} . Don't forget to handshake!

Since we are discussing *averages*, they continue to make sense even if the foam is infinite! The simplest infinite foams are those in which each bubble is the same, so the bubbles forms a *space-filling tessellation*. This is the three-dimensional version of the plane tessellations we saw earlier in Fig. 29. First, we will rule out a simple scheme based on plane tessellations.

Exercise 5.7. Prisms.

One way to tessellate space is to take a regular tessellation of the plane, then extend it in the perpendicular direction to form a layer of prisms (as below). We can then stack these layers on top of each other to tessellate space.



Explain why no prism-based tessellation satisfies Plateau's laws.

Our next step is to consider the three-dimensional analogue of regular polygons, the *Platonic solids* (Fig. 41). These are polyhedra whose faces are identical regular polygons. Could any of these describe an infinite soap foam? Of these solids, only the cube can tessellate space by itself, but since this is a prism-based tessellation (it is an extruded version of the rectangular tiling of the plane), Exercise 5.7 rules it out. We need to work harder!

But before we move on, we should mention that the Platonic solids have a beautiful alternative interpretations as *regular tessellations of the sphere*. (They are not the only regular tessellations, as you can check in Exercise 5.8.) If we “inflate” the polyhedron into a sphere, the faces will form regular polygons. We show a few of these on the upper row of Fig. 42. And in fact, the solids with three edges meeting at a node—tetrahedron, cube, dodecahedron—obey the 120° rule. If we “flatten” them in the same way we did the cube²² (Fig. 26) we get the bubble networks shown on the bottom row of Fig. 42. The first is the standard triple bubble we encountered in Fig. 33. It is remarkable that this pattern, which so beautifully exhibits the 120° rule, also shows up as a bubble configuration, the tetrahedron, and a spherical tessellation!

Like Exercise 4.6, the optimal tessellation now depends on the cell size. But we can ask if these regular tessellations minimize the total amount of wax required for a spherical honeycomb with given cell size. The “spherical honeycomb conjecture” is proved for the dodecahedron and tetrahedron

²²Technically, we have drawn a very special flattening called the *stereographic projection*. This is what the vertices would look like to an observer positioned on top of the sphere.

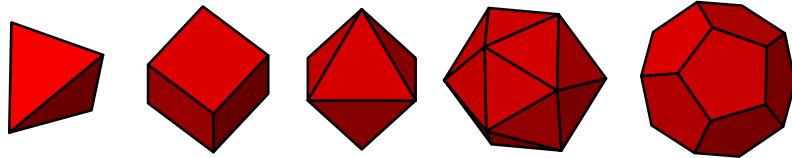


Figure 41: From left to right: tetrahedron, cube, octahedron, icosahedron, dodecahedon.

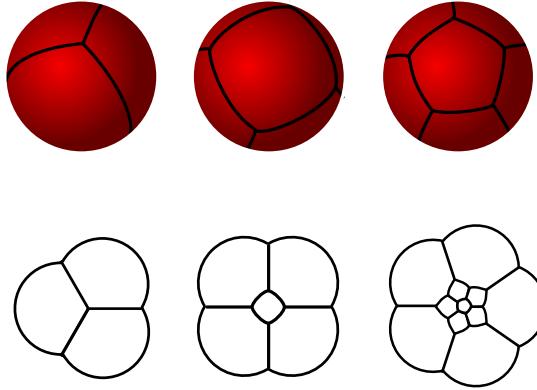


Figure 42: The tetrahedron, cube, and dodecahedron as tessellations of a sphere (above) and bubble networks (below).

[18, 10], but apparently remains a conjecture for the cube. And in case you’re wondering, our hexagonality argument from §4.3 does not apply simply because the sphere does not allow the network to get large!

Exercise 5.8. Platonic solids. \blacktriangleleft

In this exercise, we’ll classify the regular tessellations of the sphere. Recall Euler’s formula $N - E + F = 2$.

- (a) Suppose each face of the tessellation has a edges, and each node joins up with b edges. Show that

$$2E = Ne = Fk.$$

- (b) Using Euler’s formula, deduce that

$$E = \frac{2ab}{2(a+b)-ab}. \quad (23)$$

- (c) Since E is a whole number, so is the RHS of (23). We will find all possible solutions! To begin with, argue that we can interchange the roles of a and b , so a tessellation with a edges per face and b edges per node also gives a tessellation with b edges per face and a edges per node. These tessellations are said to be *dual*.
- (d) If $a = 2$, what are the possible values of b ? Draw the corresponding patterns on the sphere and their duals.

(e) The denominator of (23) must be positive. Show that this implies

$$a < \frac{2b}{b-2},$$

and hence there are no solutions for $b \geq 6$ and $a \geq 3$.

- (f) The numerator in (23) is even, for a, b whole numbers. Argue that, in order for E to be a whole number, at most one of a and b can be odd.
- (g) Finally, using part (f), conclude that the remaining solutions that only five combinations of a and b are allowed for $3 \leq a, b \leq$. Check that each of these gives a Platonic solid.

Let's return to the problem of space-filling foams. None of the Platonic solids creates a tessellation satisfying Plateau's laws, so we must turn to the the "semi-regular" polyhedra: the 13 *Archimedean solids*, whose vertices all look alike, and the 13 *Catalan solids*, whose faces all look alike. The faces of Archimedean solids are regular polygons, while the Catalan solids do not. Rather than describe these all in detail, we skip to the shortlist which can tessellate space:

- the *rhombic dodecahedron*, a Catalan solid with twelve rhombic faces;
- the *truncated octahedron*, an Archimedean solid we get by snipping off an octahedron's corners.

These are shown in Fig. 43, including the "snipping" of a single corner of the octahedron.

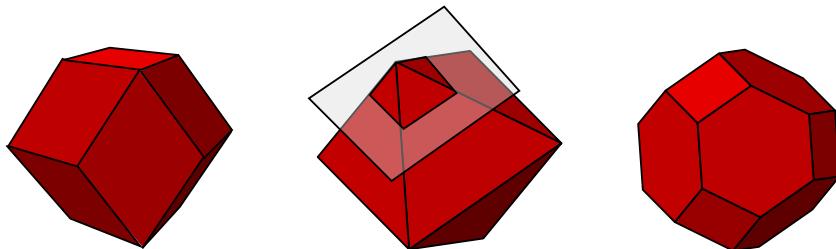


Figure 43: *Left*. The rhombic dodecahedron, with 12 rhombic faces. *Middle*. Snipping off the corner of an octahedron. *Right*. Doing this for all six corners gives the truncated octahedron.

We can take each face of the rhombic dodecahedron and extrude it to form a pyramid, so that each rhombic face is replaced by four triangular faces. The result is called a *stellated rhombic dodecahedron*, with "stellated" meaning "star-like". It is also called *Escher's solid*, since it features in Escher's marvellous lithograph *Waterfall* (Fig. 44). Remarkably, the extrusions interlock in such a way that Escher's solid continues to tessellate space. Perhaps it should be called a "testellation"²³? We can now use (22) to eliminate all but one candidate.

Exercise 5.9. The Kelvin structure.

We now have three candidates for space-filling foam:

- the rhombic dodecahedron, with two rhombic faces;

²³Admittedly, I've included Escher's solid mainly to justify this pun.

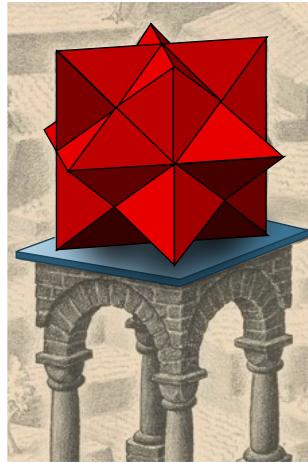


Figure 44: Stellated rhombic dodecahedron. Adapted from *Waterfall* (1961), M. C. Escher.

- Escher's solid, with 48 triangular faces; and
- the truncated octahedron, with eight hexagonal faces and six squares.

Show that only the truncated octahedron satisfies (22).

This truncated octahedron tessellation (Fig. 45) is called the *Kelvin structure*, after physicist WILLIAM THOMSON, 1ST BARON KELVIN (1824–1907), who conjectured it was the most efficient way to separate equal volume cells.²⁴ Kelvin's conjecture, often called the *Kelvin problem*, is the three-dimensional version of the honeycomb conjecture: are there any *irregular*, equal-volume tessellations of space which are more efficient than the Kelvin structure? Put differently, what structure should four-dimensional bees store their honey in?

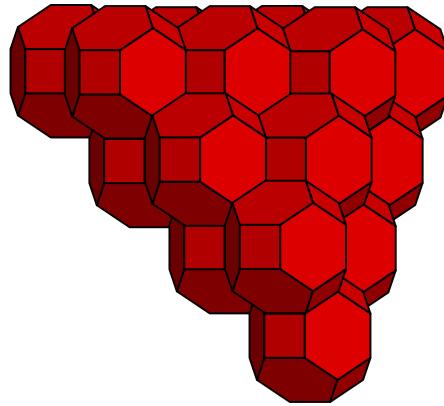


Figure 45: The Kelvin structure, made by tiling truncated octahedra.

Like the hexagonal lattice, the Kelvin structure is the only regular tessellation which is locally

²⁴Note that we need to bend the edges a little to ensure they meet at $\theta \approx 109.5^\circ$, in accord with Plateau's laws.

minimal. But the space of possibilities is much richer in three dimensions than in two, and in 1993, DENIS WEAIRE and ROBERT PHELAN discovered they could beat the Kelvin structure by weaving together two funny-shaped cells of equal volume [28]. The arrangement is called the *Weaire-Phelan structure*, shown in Fig. 46. The *Weaire-Phelan structure* interleaves two different cells:

- an irregular dodecahedron A_0 , with twelve pentagonal faces; and
- a 14-hedron A_2 with two hexagonal and twelve pentagonal faces.

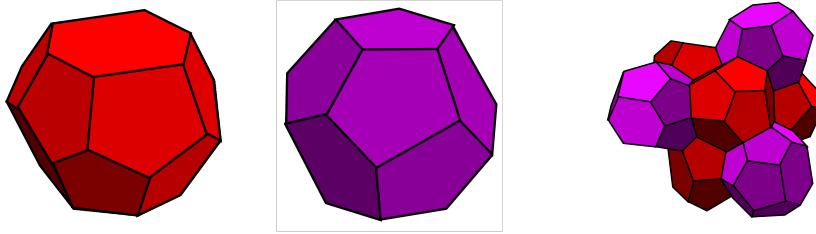


Figure 46: *Left*. The 14-hedron A_2 . *Middle*. The irregular dodecahedron A_0 . *Right*. A chunk of the Weaire-Phelan structure.

While we have extolled the virtues of soap bubble computers, Weaire and Phelan took the amusing approach of *simulating* foams on a regular digital computer. They borrow the *software* of the physical universe to solve a minimisation problem, without using the hardware! The problem with real soap films is that they tend to fall into the local minimum of the Kelvin structure. But Nature computes both fast and slow. While it may not give the optimal answer in any given experiment, it performs many experiments itself. The *structure of matter* is one such set of experiments.²⁵ In 1958, chemists F. C. FRANK and J. S. KASPER studied arrangements of atoms called *tetrahedrally closed-packed (TCP) structures*. One of these TCP structures is the *A15 phase* [12, 13]. This is the arrangement you get if you put an atom at the centre of each solid in the Weaire-Phelan structure! Nature knew about this structure long before its ingenious monkeys.

Above, we introduce the 12- and 14-sided polyhedra A_{12} and A_{14} . It turns out there are *four* polyhedra appearing in the TCP structures studied by Frank and Kasper, pictured below (Fig. 47). Each has twelve pentagonal faces, and either 0, 2, 3 or 4 hexagonal faces, so we label them according to the number of hexagonal faces they have. No one has classified all the combinations that are possible with these TCP polyhedra, though it seems there are an infinite number of possibilities. You can explore the constraints from graph theory in Exercise 5.10. Whether any of these possibilities beats Weaire-Phelan is an open problem! See [26] for further discussion.

Exercise 5.10. *TCP structures.*

- (a) Suppose we can build a tessellation out of the TCP polyhedra A_0, A_2, A_3, A_4 in the ratio

$$a_0 : a_2 : a_3 : a_4.$$

Using (22), what are the constraints on the possible ratios?

- (b) The A15 phase (aka Weaire-Phelan structure) interleaves A_0 and A_2 polyhe-

²⁵Evolution can also be viewed as an elegant (if cruel) sequence of biological trials carried out by the universe.

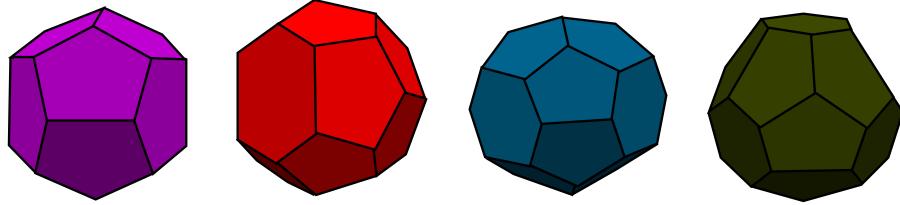


Figure 47: The polyhedra A_0, A_2, A_3 and A_4 appearing in TCP structures.

dra. What must their ratio be?

- (c) The Z phase has A_0, A_2 and A_3 in the ratio $3 : 2 : 2$. Check this is consistent.
- (d) The $C15$ phase interleaves A_0 and A_4 polyhedra only. What is the ratio?
- (e) Finally, we show that we can write *any* ratio for a TCP structure as a combination of A15, C15 and Z ratios.

6 Conclusion

Write something conclusive here.

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