UBC Physics Circle problems

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1 Evil subatomic twins

Problem. In 1928, Paul Dirac made a startling prediction: the electron has an evil twin, the *anti-electron* or *positron*. The positron is the same as the electron in every way except that it has positive charge $q = +e$, rather than negative charge $q = -e$. In fact, *every* fundamental particle has an evil, charge-flipped twin; the evil twins are collectively called *antimatter*.[1]

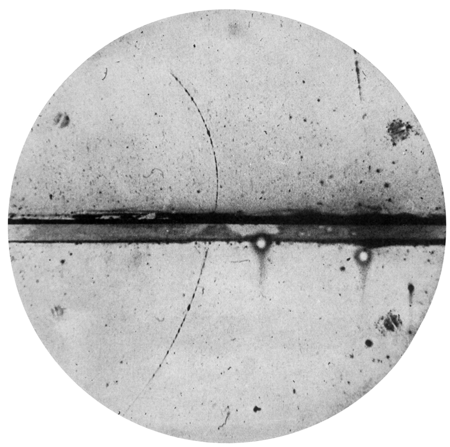

Figure 1: The mysterious trail in Carl Anderson's cloud chamber.

Experimentalist Carl Anderson was able to verify Dirac's prediction using a *cloud chamber*,[2] a vessel filled with alcohol vapour which is visibly ionised when charged particles (usually arriving from space) pass through it. In August 1932, Anderson observed the mysterious track shown above. Your job is to work out what left it!

1. A magnetic field $B = 1.7\,\mathrm{T}$ points into the page in the image above. Suppose that a particle of charge q and mass m moves in the plane of the picture with velocity v. Show that it will move in a circle of radius $R = mv/Bq$, and relate the sign of the charge to the motion.

[1] You may think it is a unfair to call antimatter "evil", but if you met your antimatter twin, hugging them would be extremely deadly! You would annihilate each other, releasing the same amount of energy as a large nuclear bomb.

[2] Cloud chambers are the modest ancestor of particle physics juggernauts like the Large Hadron Collider (LHC). Unlike the LHC, you can build a cloud chamber in your backyard!

2. The thick line in the middle of the photograph is a lead plate, and particles colliding with it will slow down. Using this fact, along with part (1), explain why the track in the image above must be due to a positively charged particle.

3. The width of the ionisation trail depends on what type of particle is travelling through the chamber and how fast it goes. The amount of ionisation in the picture above is consistent with an electron, but also an energetic proton, with momentum

$$p_{\mathrm{p}} \sim 10^{-16}\,\frac{\mathrm{kg}\cdot\mathrm{m}}{\mathrm{s}}.$$

Can you rule the proton out?

2 Elevator into space

Problem. A space elevator is a giant cable suspended between the earth and a counterweight at the other end, orbiting the earth. Both the cable and counterweight are fixed in the rotating reference frame of the earth, and can be used to efficienctly transport objects from the surface into orbit, and also as a launchpad for rockets or satellites. Space elevators would completely revolutionise our access to space, and make large-scale projects like interplanetary travel to Mars a possibility.

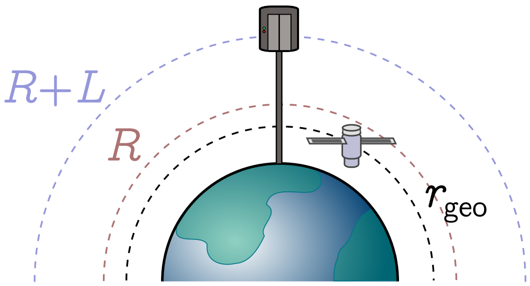

Figure 2: A satellite in geostationary orbit at radius r_{geo}. A space elevator connects a counterweight in low orbit to the surface via a cable of length $2L$. The cable's centre of mass lies at radius R, above r_{geo}.

1. To begin with, forget the cable, and consider a *geostationary* satellite orbiting at a fixed location over the equator.

   - Determine the radius r_{geo} of a geostationary orbit in terms of the mass of the earth M and angular frequency ω about its axis.
   - Confirm that r_{geo} obeys Kepler's third law, i.e. the square of the orbital period is proportional to the cube of the radius.

2. To make the space elevator, we now attach a cable to the satellite. The satellite acts as a counterweight, pulling the cable taut, but needs to move into a higher orbit in order to balance the cable tension. Provided this orbit is high enough, the space elevator will double as a rocket launchpad. Show that objects released from the elevator at $r_{\text{esc}} = 2^{1/3} r_{\text{geo}}$ will be launched into deep space.

   Hint: To launch a rocket, it needs to be travelling at escape velocity. This is the speed needed to leave the earth's gravity well with no gas left in the tank, i.e. the total energy (kinetic plus potential) vanishes.

3. The dynamics of the elevator itself are complicated, so we will consider a simplified model where the cable is treated as a rigid rod of length $2L$, with all of its mass concentrated at the centre, radius R. The counterweight is therefore at radius $R + L$.

   - Find the exact relationship between L, R, and the earth's mass M and rotational period ω.

- Assuming $L \ll R$, show that the rod's centre of mass is further out than the geostationary radius $r_{\rm geo}$. This somewhat counterintuitive result also holds for real space elevator designs! You may use the fact that, for $x \ll 1$,

$$\frac{1}{1+x} \approx 1 - x.$$

3 Simple measurements and deep ideas

Problem. Physics is built on the interplay of idea and experiment. In some cases, very simple measurements, combined with deep ideas, can reveal surprising facts about the world around us. We give three examples: the size of the earth, the mass of the sun, and the age of the universe.

1. Eratosthenes (276–195 BC) was head librarian at the magnificent Library of Alexandria, and one of the great thinkers of the ancient world. In a blow to civilisation, the library was destroyed, and most of Eratosthenes' work along with it. Thankfully, his elegant method for calculating the size of the earth, using only the shadow of a stick, survives.

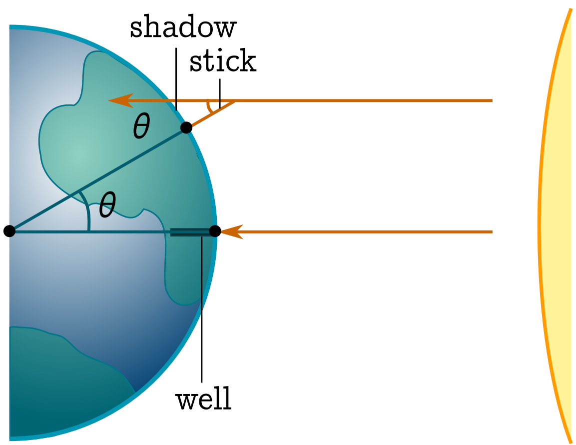

Figure 3: Eratosthenes measures the world with a stick. Figure not to scale.

   On the summer solstice, locals in the Egyptian city of Syene noticed that, at noon, the sun hit the bottom of a deep well. Eratosthenes inferred that the sun must be directly overhead. He also knew from Egyptian surveyors that Syene was roughly 5000 stadia ($\approx 850\,\mathrm{km}$) away from Alexandria. Eratosthenes performed a single experiment. At noon on the summer solstice, he measured the shadow of a vertical rod in Alexandria. He found it was roughly $1/8$ of the length of the rod. From this data, estimate the radius of the earth.

2. In 1666, Cambridge University was closed due to an outbreak of the plague, and a young Isaac Newton was forced to return to his hometown of Grantham. During this unexpected holiday, Newton was inspired by a falling apple[3] to formulate his law of gravitation:

$$F_{\text{grav}} = \frac{GMm}{r^2}.$$

[3] Newton never mentions the apple in his own writings, but his first biographer, William Stukeley, reports this conversation with Newton: "Amid other discourse, he told me, he was just in the same situation, as when formerly the notion of gravitation came into his mind. Why should that apple always descend perpendicularly to the ground, thought he to himself; occasion'd by the fall of an apple, as he sat in contemplative mood. Why should it not go sideways, or upwards? But constantly to the Earth's centre? Assuredly the reason is, that the Earth draws it. There must be a drawing power in matter. And the sum of the drawing power in the matter of the Earth must be in the Earth's centre, not in any side of the Earth."

Use this law, and the length of the year, to estimate the mass of the sun $M_\odot$. You may also use the fact that light takes 8 minutes to arrive from the sun. Some useful physical constants:

$$G = 6.67 \times 10^{-11}\,\frac{\mathrm{N}\cdot\mathrm{m}^2}{\mathrm{kg}^2}, \quad c = 3\times 10^8\,\mathrm{m/s}.$$

3. If we point a telescope at random at the night sky, we discover something surprising: faraway galaxies and stars are all moving away from us.[4] For instance, the Virgo cluster is around 55 million light years away, and receding at a speed of $1200\,\mathrm{km/s}$. Even more surprising, the speed v of any object is proportional to its distance d from the earth:

$$v = H_0 d.$$

The number H_0 is called the *Hubble constant*, and the relation between velocity and distance *Hubble's law*.[5]

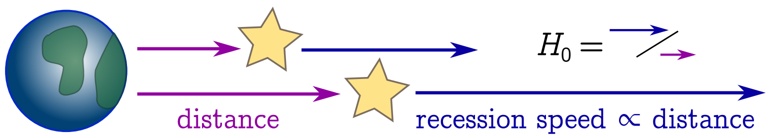

Figure 4: Hubble's law: recession speed is proportional to distance.

By running time backwards, explain why you expect a Big Bang where everything is located at the same point, and from the Virgo cluster, estimate the age of the universe.

[4]We know what frequencies of light stars like to emit. These frequencies are *Doppler-shifted*, or stretched, if the stars in a galaxy are moving away from us, allowing us to determine the speed of recession. Distance is a bit harder to work out, with different methods for different distance scales.

[5]"Constant" is a bit misleading, since it changes over time, and due to gravitational interactions, nearby objects do not obey it. Luckily, we can ignore these subtleties for the purposes of a simple estimate! Although the constant and law bear the name of astronomer Edwin Hubble, credit should also go to theorists Alexander Friedmann and Georges Lemaître, and astronomer Vesto Slipher.

4 Colliding black holes

Problem. When a star runs out of nuclear fuel, it can collapse under its own weight to form a black hole: a region where gravity is so strong that even light is trapped. Black holes were predicted in 1915, but it took until 2015, 100 years later, for the Laser Interferometer Gravitational-wave Observatory (LIGO) to observe them directly. When two black holes collide, they emit a characteristic "chirp" of *gravitational waves* (loosely speaking, ripples in spacetime), and through an extraordinary combination of precision physics and engineering, LIGO was able to hear this chirp billions of light years away.

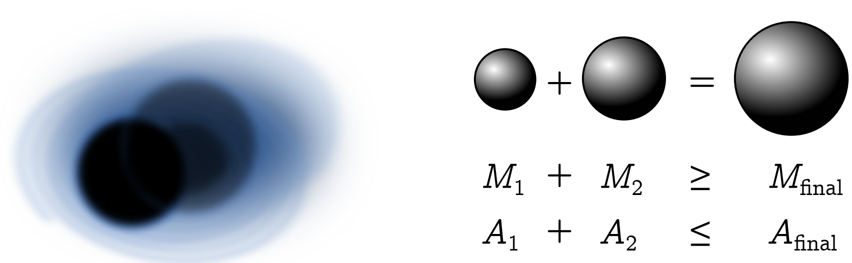

Figure 5: On the left, a cartoon of a black hole merger. On the right, inequalities obeyed by mergers: the mass of the final black hole can decrease when energy is lost (e.g. to gravitational waves), but the area always increases.

1. An infinitely dense point particle of mass M will be shrouded by a black hole. Using dimensional analysis, argue that this black hole has surface area

$$A = \left(\frac{\eta G^2}{c^4}\right) M^2$$

   for some constant η.

2. One of Stephen Hawking's famous discoveries is the *area theorem*: the total surface area of any system of black holes increases with time.[6] Using the area theorem, and the result of part (1), show that two colliding black holes can lose at most 29% of their energy to gravitational waves. (Note that to find this upper bound, you need to consider varying the mass of the colliding black holes, and to assume that any lost mass is converted into gravitational waves.)

3. LIGO detected a signal from two black holes smashing into each other 1.5 billion light years away. Their masses were $M_1 = 30M_\odot$ and $M_2 = 35M_\odot$, where $M_\odot \approx 2 \times 10^{30}\,\text{kg}$ is the mass of the sun, and the signal lasted for 0.2 seconds. Assuming the maximum amount of energy is converted into gravitational waves, calculate the average power P_{BH} emitted during the collision. Compare this to the power output of all the stars in the universe, $P_{\text{stars}} \sim 10^{49}\,\text{W}$.

[6] This theorem is actually violated by quantum mechanics, but for large black holes, the violations are small enough to be ignored.

5 Traversing the donut

Problem. Take a square of unit length. By folding twice and gluing (see below), you can form a donut. Particles confined to the donut don't know it's curved; it looks like normal space to them, except that if they go too far to the left, they will reappear on the right, and similarly for the top and bottom. Put a different way, the blue lines to the left and right are identified, and similarly for the red lines.

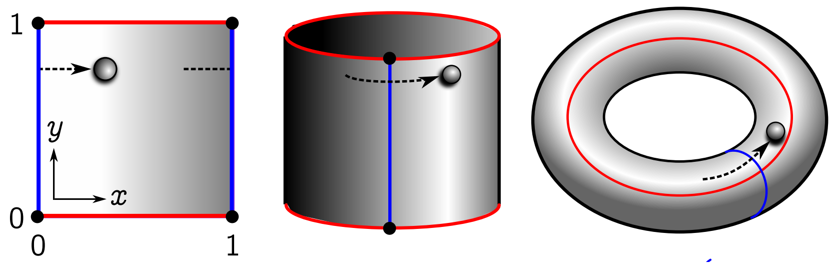

Figure 6: Folding and gluing a square to get a donut

1. Suppose we have two particles, and shoot them out from the origin at $t = 0$. One particle travels vertically in the y direction with speed v_y, and the other travels in the x direction with speed v_x. Will they ever collide? If so, at what time will the first collision occur?

2. Now consider a *single* particle with velocity vector $\mathbf{v} = (v_x, v_y)$. Find a condition on $\mathbf{v}$ so that the particle will never visit the same location on the donut twice.

6 Turbulence in a tea cup

Problem. Stir a cup of coffee vigorously enough, and the fluid will begin to mix in a chaotic or *turbulent* way. Unlike the steady flow of water through a pipe, the behaviour of turbulent fluids is unpredictable and poorly understood. However, for many purposes, we can do suprisingly well by modelling a turbulent fluid as a collection of (three-dimensional) eddies of different sizes, with larger eddies feeding into smaller ones and losing energy in the process.

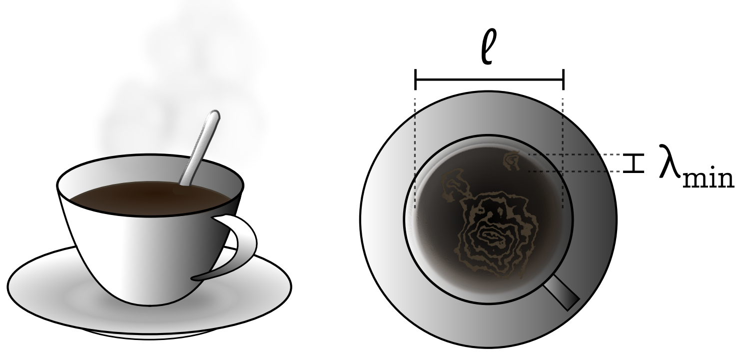

Figure 7: A well-stirred cup of coffee. On the right, a large eddy (size $\sim \ell$) and the smallest eddy (size $\lambda_{\min}$) are depicted.

Suppose our cup of coffee has characteristic length ℓ, and the coffee has density ρ. When it is turbulently mixed, the largest eddies will be a similar size to the cup, order ℓ, and experience fluctuations in velocity of size Δv due to interaction with other eddies. The fluid also has internal drag[7] or *viscosity* η, with units $\mathrm{N} \cdot \mathrm{s}/\mathrm{m}^2$.

1. Let ϵ be the rate at which kinetic energy dissipates per unit mass due to eddies. Observation shows that this energy loss is independent of the fluid's viscosity. Argue on dimensional grounds that
$$\epsilon \approx \frac{(\Delta v)^3}{\ell}.$$
Why doesn't the density ρ appear?

2. Kinetic energy can also be lost due to internal friction. Argue that the time scale for this dissipation due to viscosity is
$$\tau_{\text{drag}} \approx \frac{\ell^2 \rho}{\mu}.$$

3. Using the previous two questions, show that eddy losses[8] dominate viscosity losses provided
$$\frac{\ell \rho \Delta v}{\mu} \gg 1.$$

[7] More precisely, viscosity is the resistance to *shear flows*. A simple way to create shear flow is by moving a large plate along the surface of a stationary fluid. Experiments show that the friction per unit area of plate is proportional to the speed we move it, and inversely proportional to the height; the proportionality constant at unit height is the viscosity. Since layers of fluid also generate shear flows, viscosity creates internal friction.

[8] Since ϵ depends on $\ell, \Delta v$, you need not consider it when finding the time scale for eddy losses.

The quantity on the left is called the *Reynolds number*, $\mathrm{Re} = \ell\rho\Delta v/mu$. In fact, one *definition* of turbulence is fluid flow where the Reynolds number is high.

4. So far, we have focused on the largest eddies. These feed energy into smaller eddies of size λ and velocity uncertainty Δv_λ, which have an associated *eddy Reynolds number*,

$$\mathrm{Re}_\lambda = \frac{\lambda\rho\Delta v_\lambda}{\mu}.$$

When the eddy Reynolds number is less than 1, eddies of the corresponding size are prevented from forming by viscosity.[9] Surprisingly, the rate of energy dissipation per unit mass in these smaller eddies is ϵ, the same as the larger eddies.[10] Show from dimensional analysis that the minimum eddy size is roughly

$$\lambda_{\mathrm{min}} \approx \left(\frac{\mu^3}{\epsilon\rho^3}\right)^{1/4}.$$

5. If a cup of coffee is stirred violently to Reynolds number $\mathrm{Re} \approx 10^4$, estimate the size of the smallest eddies in the cup.

[9]Lewis Fry Richardson not only invented the eddy model, but this brilliant mnemonic couplet: "Big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity."

[10]This is not at all obvious, but roughly, follows because we can fit more small eddies in the container. Intriguingly, this makes the turbulent fluid like a *fractal*: the structure of eddies repeats itself as we zoom in, until viscosity begins to play a role. At infinite Reynolds number, it really is a fractal!