UBC Physics Circle problems

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Abstract

This is a subset of questions I've written for the UBC Physics Circle between November 2018 and December 2019. None of them require calculus, but they do assume some problem-solving maturity and a strong background in high school physics and maths. All material here is original, though I of course draw on a variety of inspirations and sources; specific references are listed where possible. Feel free to use problems in a classroom setting, but please cite the author. For solutions, please see the UBC Physics Circle website or contact me at david.a.wakeham@gmail.com.

Contents

1 Dimensional analysis and Fermi problems

1.1 Tsunamis and shallow water

Ocean waves behaves rather differently in deep and shallow water. From dimensional analysis, we can learn a little about these differences, and deduce that waves increase in height as they approach the shore. This phenomenon, called *shoaling*, is responsible for tsunamis.

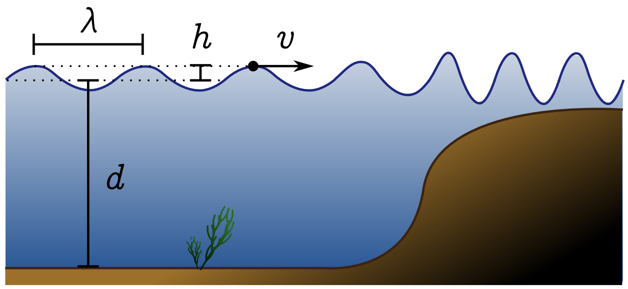

Figure 1: Ocean waves. As the water gets shallower, the waves increase in height.

1. Let λ denote the wavelength of an ocean wave and d the depth of the water. Typically, both are much larger than the height h of the wave, so we can ignore it for the time being. Argue from dimensional analysis that in the *deep water limit* $\lambda \ll d$, the velocity of the wave is proportional to the square root of the wavelength:

$$v \approx \sqrt{g\lambda}.$$

   In the *shallow water limit* $\lambda \gg d$, explain why you expect

$$v \approx \sqrt{gd}.$$

2. Ocean waves can be generated by oscillations beneath the ocean floor. For a source of frequency f, what is the wavelength of the corresponding wave in shallow water? Estimate the wavelength if the source is an earthquake of period $T = 20\,\mathrm{min}$ at depth $d = 4\,\mathrm{km}$, and check your answer is consistent with the shallow water limit.

3. Consider an ocean wave of height h and width w. The energy E carried by a single "cycle" of the wave equals the volume V of water above the mean water level d, multiplied by the gravitational energy density ϵ. By performing a dimensional analysis on each term separately, argue that the total energy in a cycle is approximately

$$E \approx V\epsilon \approx \rho g \lambda w h^2,$$

   where $\rho \approx 10^3\,\mathrm{kg\,m^{-3}}$ is the density of water and g the gravitational acceleration.

4. Energy in waves is generally *conserved*: the factor E is constant, even as the wavelength λ and height h of the wave change. (We will ignore spreading of the wave.) By applying energy conservation to shallow waves, deduce *Green's law*:

$$h \propto \frac{1}{d^{1/4}}.$$

The increase in height is called *shoaling*. The relation breaks down near shore when the depth d becomes comparable to the height h.

5. Our earthquake from earlier creates a tsunami of height $h_0 = 0.5\,\mathrm{m}$. What is the height, speed, and power per unit width of the tsunami close to the shore? (By "close to the shore", we mean at $h \approx d$ where Green's law breaks down.) You may assume the shallow water equation holds.[1]

1.2 Einstein rings

According to *general relativity*, Einstein's theory of gravity, massive objects curve space itself. Even *massless* particles like light rays will be deflected as they try to find the shortest path between A and B. This effect is called *gravitational lensing*, since a heavy body acts like a lens.

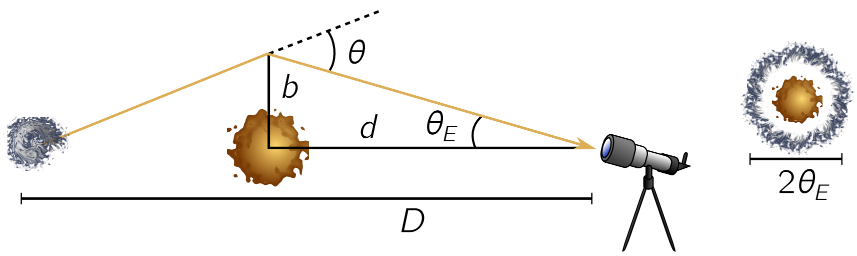

Figure 2: Einstein ring, gravitationally lensed by a large star.

1. Suppose that a light ray passes a spherical star of mass m and radius R, a distance $b > R$ from the centre. The *angle of deflection* θ is dimensionless (in radians). Using dimensional analysis, argue that it takes the form

$$\theta = c_0 + c_1 x + c_2 x^2 + \cdots$$

for dimensionless constants $c_0, c_1, c_2, \ldots$, and

$$x = \frac{Gm}{bc^2}.$$

The speed of light is $c = 3 \times 10^8$ m/s and Newton's constant is $G = 6.7 \times 10^{-11}$ m^3/kg s^2.

Hint. You may assume the wavelength of light isn't relevant, since only the mass $m = 0$ determines its path in spacetime. Why isn't the radius of the star relevant?

2. By considering the limit where the star disappears altogether, $m \to 0$, explain why $c_0 = 0$.

3. Using parts (1) and (2), argue that for $Gm \ll bc^2$,

$$\theta \sim \frac{Gm}{bc^2}.$$

As usual, the $\sim$ includes the unknown constant c_1.

[1] It doesn't quite. We actually need to use the full formula for speed, $v = \sqrt{(g\lambda/2\pi)\tanh(2\pi d/\lambda)^{-1}}$, if we want to make an accurate estimate. But here, the shallow water equation will suffice to get the correct order of magnitude.

Imagine that a star lies directly between a galaxy and a telescope on earth.The galaxy is a distance D away from the earth, and the star a distance d. Define the angle θ_E and deflection angle θ as in Fig. 1.2.

4. Assuming the angles are small, argue that

$$b \approx \theta_E d, \quad \theta_E D \approx \theta(D-d).$$

5. Combining the identities in (4) with (3), deduce that

$$\theta_E \sim \sqrt{\frac{Gm(D-d)}{c^2 Dd}}.$$

   You can repeat this argument, rotating in a circle around the line formed by the galaxy, star and observer on earth. We learn that the galaxy will appear as a ring, called an *Einstein ring*, of (angular) *Einstein radius* θ_E.

6. Explain why we don't observe Einstein rings around the sun. The sun has mass $m_\odot = 2 \times 10^{30}\,\mathrm{kg}$, radius $R_\odot = 7 \times 10^8\,\mathrm{m}$, and is $d = 150 \times 10^9$ m from earth.

1.3 Turbulence in a tea cup

Stir a cup of coffee vigorously enough, and the fluid will begin to mix in a chaotic or *turbulent* way. Unlike the steady flow of water through a pipe, the behaviour of turbulent fluids is unpredictable and poorly understood. However, for many purposes, we can do suprisingly well by modelling a turbulent fluid as a collection of (three-dimensional) eddies of different sizes, with larger eddies feeding into smaller ones and losing energy in the process.

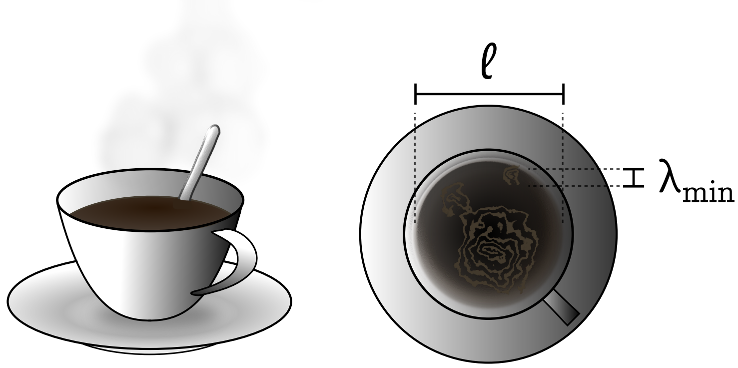

Figure 3: A well-stirred cup of coffee. On the right, a large eddy (size $\sim \ell$) and the smallest eddy (size λ_{min}) are depicted.

Suppose our cup of coffee has characteristic length ℓ, and the coffee has density ρ. When it is turbulently mixed, the largest eddies will be a similar size to the cup, order ℓ, and experience fluctuations in velocity of size Δv due to interaction with other eddies. The fluid also has internal drag[2] or *viscosity* η, with units $\mathrm{N}\cdot\mathrm{s}/\mathrm{m}^2$.

1. Let ϵ be the rate at which kinetic energy dissipates per unit mass due to eddies. Observation shows that this energy loss is independent of the fluid's viscosity. Argue on dimensional grounds that
$$\epsilon \approx \frac{(\Delta v)^3}{\ell}.$$
Why doesn't the density ρ appear?

2. Kinetic energy can also be lost due to internal friction. Argue that the time scale for this dissipation due to viscosity is
$$\tau_{\mathrm{drag}} \approx \frac{\ell^2 \rho}{\mu}.$$

3. Using the previous two questions, show that eddy losses[3] dominate viscosity losses provided
$$\frac{\ell\rho\Delta v}{\mu} \gg 1.$$
The quantity on the left is called the *Reynolds number*, $\mathrm{Re} = \ell\rho\Delta v/mu$. In fact, one *definition* of turbulence is fluid flow where the Reynolds number is high.

4. So far, we have focused on the largest eddies. These feed energy into smaller eddies of size λ and velocity uncertainty Δv_λ, which have an associated *eddy Reynolds number*,
$$\mathrm{Re}_\lambda = \frac{\lambda\rho\Delta v_\lambda}{\mu}.$$
When the eddy Reynolds number is less than 1, eddies of the corresponding size are prevented from forming by viscosity.[4] Surprisingly, the rate of energy dissipation per unit mass in these smaller eddies is ϵ, the same as the larger eddies.[5] Show from dimensional analysis that the minimum eddy size is roughly
$$\lambda_{\min} \approx \left(\frac{\mu^3}{\epsilon\rho^3}\right)^{1/4}.$$

5. If a cup of coffee is stirred violently to Reynolds number $\mathrm{Re} \approx 10^4$, estimate the size of the smallest eddies in the cup.

[2] More precisely, viscosity is the resistance to *shear flows*. A simple way to create shear flow is by moving a large plate along the surface of a stationary fluid. Experiments show that the friction per unit area of plate is proportional to the speed we move it, and inversely proportional to the height; the proportionality constant at unit height is the viscosity. Since layers of fluid also generate shear flows, viscosity creates internal friction.

[3] Since ϵ depends on $\ell, \Delta v$, you need not consider it when finding the time scale for eddy losses.

[4] Lewis Fry Richardson not only invented the eddy model, but this brilliant mnemonic couplet: "Big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity."

[5] This is not at all obvious, but roughly, follows because we can fit more small eddies in the container. Intriguingly, this makes the turbulent fluid like a *fractal*: the structure of eddies repeats itself as we zoom in, until viscosity begins to play a role. At infinite Reynolds number, it really is a fractal!

Figure 4: Our cast of characters.

1.4 A Fermi free-for-all

Order of magnitude approximations, or *Fermi estimates*, are a fun and surprisingly powerful approach to solving problems. Here, we offer a medley of Fermi problems, ranging from Starbucks to stars to sneezes. There are a few techniques you may find useful:

- taking the *geometric mean* $\sqrt{UL}$ of upper and lower guesses U and L for a quantity;
- *factorising* your answer into a string of subestimates with intermediate units; and
- using *dimensional analysis* and *simple physics*.[6]

You may also need data about the world (supplied) and common sense (not included).

1. We start with big numbers, and answer an age old question: are there more stars in the sky, or grains of sand? And what about atoms in a single grain?

   (a) **Stars.** How many stars are there in the observable universe?

   Data. Astronomers count roughly 100 billion galaxies. Small dwarf galaxies have on the order of 100 million stars, while massive elliptical galaxies can have in excess of 10 trillion stars.

   (b) **Sand.** Estimate the number of grains of sand on all the beaches of the world.

   Data. Sand particles range in size from 0.0625 mm to 2 mm. The earth has 620,000 km of coastline.

   (c) **Atoms.** How many atoms are in an average grain of sand? Compare to the two previous numbers, and comment on your result.

   Data. Sand is made out of silicon dioxide SiO_2, with molar mass 60 g. Avogadro's constant is $N_A = 6 \times 10^{23}$.

[6] "Simple physics" means to solve a caricature of the problem, where you ignore everything but the most important mechanism.

2. Next, we add some physics into the mix.

   (a) **Raindrops.**

      i. Using dimensional analysis, estimate the size of a raindrop.

      ii. As they bang into each other, raindrops resonate like the head of a drum, since they are under tension. What is the approximate frequency of this resonance?

   Data. The surface tension of water is $\sigma = 0.07$ N/m, with dimensions $[\sigma] = M/T^2$.[7] Surface tension wants to make the raindrop small; gravity wants to spread it out.

   (b) **Sneezes.** Here's a sillier one.

      i. How much force is released in the average sneeze? No dimensional analysis required, just regular Newtonian mechanics.

      ii. How many people are required to sneeze a 1 kg cubesat[8] into space?

   Data. The lung capacity of an adult is around 5 L, and sneezes are emitted with a final velocity of roughly 50 m/s. Launch velocity at the earth's surface is 11 km/s.

3. We end with some harder "real life" Fermi estimates.

   (a) **Hungarian GDP.** Guess the size of Hungary's economy, measured by GDP.[9]

   Data. Canada's GDP is 1.6 trillion USD. India's GDP is 2.6 trillion USD.

   (b) **Starbucks.** Estimate the number of Starbucks stores in Seattle.

   Data. Seattle city has a population of around 700,000.

[7] Concretely, if I try and cut water with a knife, there is a resistance of 70 mN per metre of knife.

[8] A cubesat is a small, cubical satellite.

[9] *Gross domestic product.* This is the total monetary worth of all goods and services produced in a year, conventionally reported in US dollars.

2 The binomial approximation

Algebra is all about solving problems exactly. The prime example is the quadratic formula, which tells us how to solve any quadratic equation exactly. There is also a cubic formula, for degree 3 equations, and a quartic formula for degree 4 (which is about a page long). But that's it! Famously, there is no simple equation, using roots of coefficients, which solves quintic or higher equations. Algebra has its limits.

In physics and real life, we often need to solve problems where algebra cannot help. Instead of exact answers, we must settle for approximations. But exactness is usually overkill anyway! In this problem, we'll go beyond algebra and explore the *binomial approximation*. We're going to show that, when x is much smaller than 1, $x \ll 1$, then

$$(1+x)^\alpha \approx 1 + \alpha x,$$

when α is *any real number*. This all sounds rather abstract, but in the following problems you can use it to do some real physics!

1. **Coronal mass ejections.** The earth orbits the sun at a distance of around 149.6 million km. The sun's mass is $M = 1.99 \times 10^{30}$ kg. During a *coronal mass ejection (CME)*, huge coils of plasma are released from the sun, causing electromagnetic disruptions on earth. The average mass of a CME is $m = 2 \times 10^{12}$ kg. Assuming the earth's orbital period is fixed, approximately how much closer does it move to the sun?

   Hint. Kepler's third law.

2. **Einstein's energy.** Einstein's theory of special relativity tells us that the total energy of moving object, mass m and speed v, is

   $$E = \gamma mc^2, \quad \text{where} \quad \gamma = [1 - (v/c)^2]^{-1/2}.$$

   Show that when the velocity is much slower than the speed of light, $v \ll c$, the energy can be approximated as

   $$E \approx \frac{1}{2}mv^2 + mc^2.$$

   The first term is just the Newtonian kinetic energy $E_{\text{kin}} = mv^2/2$, while the second is Einstein's famous formula for the mass-energy of an object, $E_{\text{mass}} = mc^2$.

3. **Sunset on a mountain.**

   (a) Show that an observer a distance h above the ground can see a distance

   $$d \approx \sqrt{2hR}\left(1 + \frac{h}{4R}\right).$$

   where $R = 6.38 \times 10^3$ km is the radius of the earth. If $h \ll R$, we can simply set $d \approx \sqrt{2hR}$.

   (b) The summit of Grouse Mountain has an elevation of 1231 m. How much later does the sun set at the top of the mountain, compared to sea level? Ignore complications due to latitude.

For the mathematically curious, here a step-by-step guide to proving binomial approximation using simple methods.

4. **Proving the binomial approximation.**

   (a) Let n be a positive whole number. Expand using basic algebra and show that

   $$(1+x)^n = 1 + nx + O(x^2),$$

   where $O(x^2)$ stands for terms multiplied by x^2 and higher powers.

   Hint. It is helpful to think of the n in nx as the *number of ways* you can obtain x

   (b) Remember that $y^{1/n}$ is the nth root of y, in other words, the number which raised to the nth power gives y. Suppose that[10] for some number β,

   $$(1+x)^{1/n} = 1 + \beta x + O(x^2).$$

   Explain why $\beta = 1/n$.

   (c) Combining the previous two problems, argue that for any positive *rational* number $q = a/b$, we have

   $$(1+x)^q = 1 + qx + O(x^2).$$

   Since *irrational* numbers can be approximated arbitrarily well by rational numbers (with decimals for instance), the binomial approximation for any real $\alpha > 0$ follows.

   (d) Finally, we need to think about negative powers. Recall the geometric series:[11]

   $$1 + r + r^2 + \cdots = \frac{1}{1-r}.$$

   Let $\alpha > 0$ be a positive real number, and consider $(1+x)^{-\alpha} = [1/(1+x)]^\alpha$. Using the geometric series, argue that

   $$(1+x)^{-\alpha} = 1 - \alpha x + O(x^2).$$

   With positive and negative α done, that covers all real numbers!

   (e) *Bonus.* You can push these methods further. Repeating the steps above (with slightly more algebra), show that

   $$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + O(x^3).$$

[10] You might wonder if the first term has to be a 1. To see that it does, we can just set $x = 0$. The LHS is then $1^{1/n} = 1$ and so the RHS has to be 1 as well.

[11] Here is a quick proof. Let $s = 1 + r + r^2 + \cdots$. Then $s = 1 + rs$. Solving gives $s = 1/(1-r)$.

3 Motion

3.1 Gone fishin'

After a day hard at work on kinematics, Emmy decides to take a break from physics and go fishing in nearby Lake Lagrange. But there is no escape! As she prepares to cast her lure, she realises she has an interesting ballistics problem on her hands.

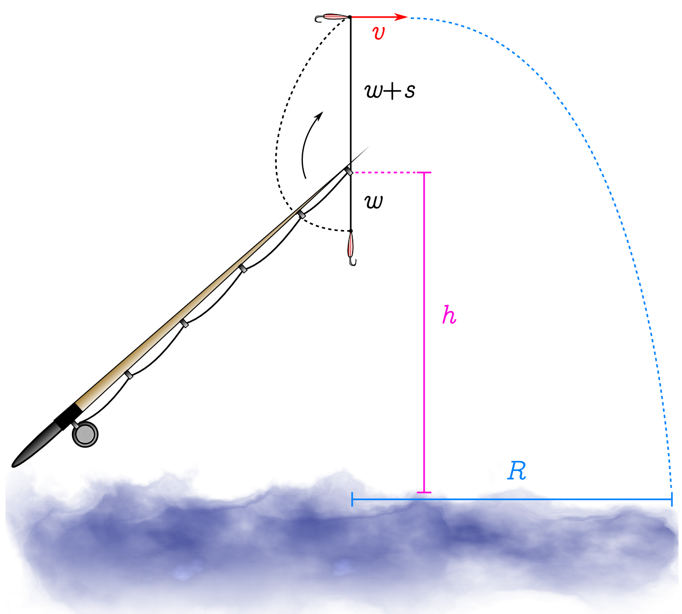

Figure 5: Emmy's unconventional method for casting a lure.

1. The top of her rod is a distance h above the water, and the lure (mass m) hangs on a length of fishing wire w. To cast, Emmy will swing the lure $180°$ around the end of the rod and release at the highest point, where the velocity has no vertical component. Assuming she can impart angular momentum $L = mvw$ to the lure, calculate the range R in terms of h, w, L and m. You can ignore the effect of gravity during the swinging phase.

2. As Emmy swings through, she can introduce some additional slack s into the wire. Assuming conservation of angular momentum, this will slow the lure but raise the release point. Find the range R in terms of the parameters w, s, h, L, and determine the amount of slack s that maximises the casting distance. Again, ignore the effect of gravity.

   Hint. Try maximising the *square* of the range.

3. Now include gravity in the swinging phase, and calculate the range as a function of s. Determine the optimum s, and find a condition on h, w, L, g, m which ensures $s > 0$.

   Hint. Complete the square in R^2.

3.2 Snowballing

Recall the formula for *impulse*, stating that a force applied over time will lead to a change in momentum:

$$F_{\text{avg}}\Delta t = \Delta p.$$

The force can change, with F_{avg} denoting the *average* over the interval. We can approximate the average force as the arithmetic average of the applied force at the start and the end of the interval:

$$F_{\text{avg}} \approx \frac{F_{\text{start}} + F_{\text{end}}}{2}.$$

The impulse formula works for an object with *changing mass*. In fact, we can view it as the

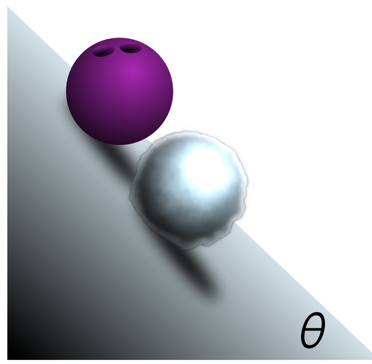

Figure 6: A snowball and a bowling ball racing down a mountainside.

most general statement of Newton's second law! For an object with constant mass m, we can write

$$F = ma = m\frac{\Delta v}{\Delta t} = \frac{\Delta(mv)}{\Delta t} = \frac{\Delta p}{\Delta t}.$$

The rightmost expression is precisely the impulse formula, and works perfectly well for an object with changing mass.

As an example, consider a snowball of initial mass m_0 and radius r_0 sitting on top of a mountain. A northerly begins to blow, dislodging the snowball and causing it to roll down the mountainside and accumulate snow. We will assume the snowball has uniform density ρ and the slope is constant.

1. First, suppose that the snowball is frozen solid with constant mass. If the mountain slopes at an angle θ to the horizontal, what is the snowball's linear accleration a_0?

   Hint. The moment of inertia of the snowball is $I = (2/5)m_0r_0^2$.

2. Now suppose that in a short time increment Δt, the snowball picks up a mass Δm. Show that the acceleration over the interval $a = \Delta v/\Delta t$ is related to the acceleration a_0 of the constant mass snowball by

$$(m + \Delta m)a = \left(m + \frac{1}{2}\Delta m\right) a_0 - v\frac{\Delta m}{\Delta t}.$$

3. Taking the limit of a very small time interval Δt, argue that

$$a = a_0 - \frac{v}{m}\frac{\Delta m}{\Delta t}.$$

4. A bowling ball (constant mass and density) races the snowball down the mountainside. If the snowball is gathering snow, which arrives at the bottom of the slope first? If the sun comes out, and the snowball melts as it travels down the slope instead of getting heavier, what happens then?

5. Suppose that the rate of accumulation $\Delta m/\Delta t > 0$ is proportional to the *surface area* of the snowball, but inversely proportional to the speed.[12] What is the acceleration after the snowball has been rolling for a very long time?

3.3 Evel Knievel and the crocodile pit

Evel Knievel rides his stunt motorcycle over a semicircular ramp of radius R. He is planning to use this ramp to shoot his motorbike over a pit of ravenous Alabama crocodiles, of length L, immediately after the ramp. His motorcycle can achieve a maximum speed of v, and for simplicity, we assume Knievel can accelerate to this speed instantaneously and at will.

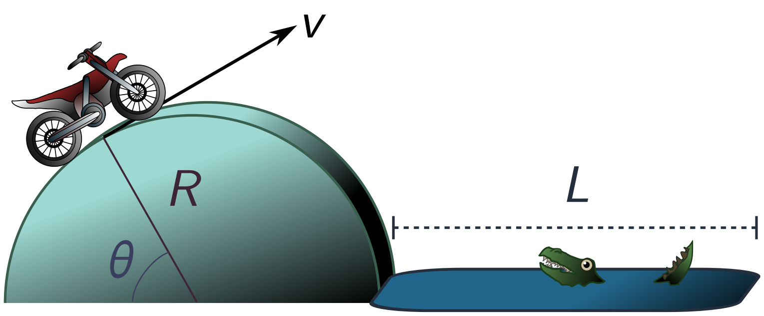

Figure 7: Evil Knievel jumping over a pit of crocodiles.

1. Label the angle from the horizontal by θ. What condition must v satisfy to launch Knievel at an angle θ?

2. Show that if Knievel launches at angle θ, his airtime is

$$t = \frac{1}{g}\left[vc_\theta + \sqrt{v^2c_\theta^2 + 2gRs_\theta}\right],$$

where $s_\theta = \sin\theta$ and $c_\theta = \cos\theta$.

3. Deduce that after launching at θ, his range over the crocodile pit is

$$r = \frac{v^2 s_\theta}{g}\left[c_\theta + \sqrt{c_\theta^2 + \frac{2gRs_\theta}{v^2}}\right] - R(1 + c_\theta).$$

[12] A rolling stone gathers no moss.

4. The range is a very unpleasant function to optimise. Instead, let's study a special case. Suppose that Knievel launches horizontally at the top of the ramp with $\theta = \pi/2$. What does v need to be to clear the crocodile pit?

5. For $\theta = \pi/2$, use part (1) to demonstrate that he will *automatically* clear the pit provided

$$(\sqrt{2} - 1)R > L.$$

3.4 A short voyage to Alpha Centauri

The *Bussard ramjet* is a hypothetical spaceship fuelled by the stray matter between stars, aka the *interstellar medium (ISM)*. Hydrogen atoms are collected by a giant scoop at the front called a *ram*, then funneled into a fusion reactor where they join together to form helium. Fusion imparts energy to the helium nuclei, and they are released as exhaust, pushing the rocket forward according to Newton's third law.

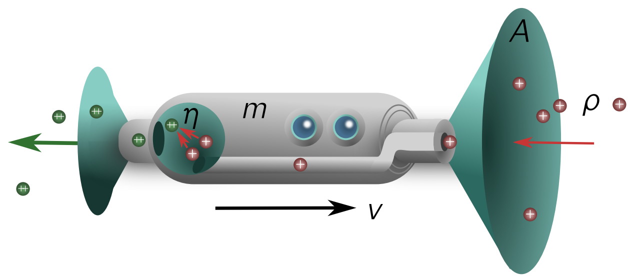

Figure 8: Hydrogen (red) are scooped up by the ram and fused into helium (green). Also labelled are the ramjet mass m and velocity v, scoop area A, hydrogen density ρ, and fusion efficiency η.

1. Let m be the ramjet mass, a the acceleration and v velocity. Show that the power supplied by the engine is

$$P = mav.$$

2. Suppose the scoop has cross-section A. If the ISM has a constant mass density of hydrogen atoms ρ, and can convert a fraction η of the hydrogen's mass-energy into the kinetic energy of fused helium atoms, argue that

$$P = Av\eta\rho c^2.$$

Hint. Recall Einstein's most famous formula, $E = mc^2$.

3. Deduce that

$$a = \frac{A\eta\rho c^2}{m}.$$

In particular, note that the acceleration is constant!

4. The ISM has roughly $n \approx 1$ hydrogen atom per cm^3, each with mass $m_{\mathrm{H}} \approx 1.7 \times 10^{-27}\,\mathrm{kg}$. The conversion efficiency for the fusion reaction is $\eta \approx 0.75\%$. The nearest solar system,

Alpha Centauri, is about 4.4 light years away. What is the average mass density of the scoop, m/A, required to reach Alpha Centauri at constant acceleration in the pilot's lifetime? Comment on any engineering challenges this presents. Recall that $c \approx 3 \times 10^8\,\mathrm{m/s}$.

Bonus. At high speeds, *special relativity* rather than Newtonian mechanics holds sway. The *total relativistic energy* of the ramjet, including its kinetic energy, is

$$E = \gamma mc^2, \quad \text{where } \gamma = \frac{1}{\sqrt{1-(v/c)^2}}.$$

Let's check whether relativity spoils our analysis of the trip to Alpha Centauri.

5. After the ramjet has travelled a distance d, show it has acquired additional energy

$$\Delta E = Ad\eta\rho c^2.$$

6. Computing the total relativistic energy two ways, conclude that the velocity is

$$\frac{v}{c} = \sqrt{\frac{1+2x}{(1+x)^2}}, \quad x = \frac{m}{Ad\rho\eta}.$$

Check that as the distance $d \to \infty$, $v \to c$. In other words, the maximum velocity is the speed of light!

7. Consider the ramjet from question 4. Explain why, for the threshold value of m/A, relativistic effects are negligible for the trip to Alpha Centauri.

Hint. If $\gamma \approx 1$, then Newtonian mechanics is a good approximation.

On a longer trip, special relativity does become relevant, and for $\gamma \gg 1$, extreme time dilation occurs. Clocks on board ticking slower than clocks on earth (or more importantly, clocks fixed with respect to the expanding universe) by a factor of γ. For an entertaining sci-fi novel about the wonders of relativistic ramjets, check out *Tau Zero* by Poul Anderson!

3.5 Black holes at the LHC

The Large Hadron Collider (LHC) is the most powerful particle collider in the world. It accelerates two beams of protons to near the speed of light, then smashes them together to produce a slew of *new particles* according to the laws of quantum mechanics.

We will consider a simple case where the protons collide and turn into two new particles. In this process, the total energy and momentum are conserved, but since the protons travel at close to the speed of light, we must use *relativistic* definitions of energy and momentum. For a particle of mass m travelling at speed v, the *relativistic energy* is

$$E = \gamma mc^2 = \frac{1}{\sqrt{1-(v/c)^2}} mc^2,$$

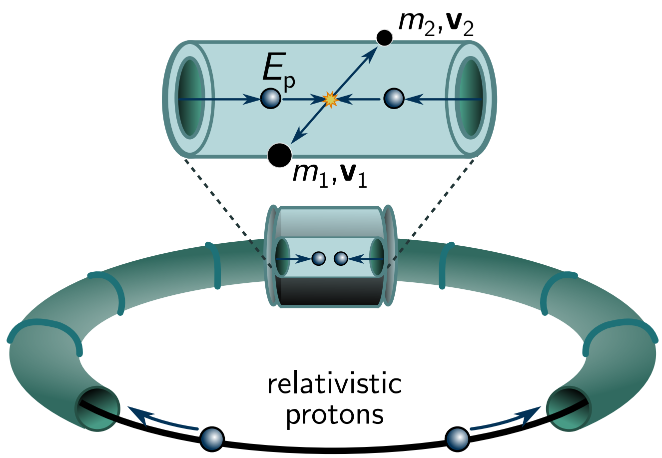

Figure 9: Black holes formed from colliding protons at the LHC.

where $c = 3 \times 10^8\,\mathrm{m/s}$ is the speed of light and $\gamma = [1-(v/c)^2]^{-1/2}$ is the *Lorentz factor*.[13] The *relativistic momentum* is $\mathbf{p} = \gamma m \mathbf{v}$. The proton mass is m_p, so two countermoving protons with opposite velocities $\pm\mathbf{v}$ have energies $E_\mathrm{p} = \gamma m_\mathrm{p} c^2$ and momenta $\pm\mathbf{p} = \gamma m_\mathrm{p}\mathbf{v}$. The total energy and momentum is then

$$E_{\mathrm{tot}} = 2E_\mathrm{p}, \quad \mathbf{p}_{\mathrm{tot}} = \mathbf{0}.$$

Our first task is to see what conservation implies about the particles produced in the collision. We will ignore the *charge* of the protons, though this is also important.

1. Suppose that the protons decay into particles of mass m_1, m_2 travelling with velocity $\mathbf{v}_1, \mathbf{v}_2$. Using conservation of relativistic momentum, show that $\mathbf{v}_1 = -\alpha\mathbf{v}_2$ for some constant $\alpha > 0$.

2. Solve for $\gamma_1 m_1$ and $\gamma_2 m_2$ in terms of $E_\mathbf{p}$ and α. You should find that

$$\gamma_1 m_1 = \left(\frac{2c^{-2}}{1+\alpha}\right) E_\mathbf{p}, \quad \gamma_2 m_2 = \left(\frac{2c^{-2}\alpha}{1+\alpha}\right) E_\mathbf{p}.$$

   If we swap $\alpha \leftrightarrow \alpha^{-1}$, we swap particles $1 \leftrightarrow 2$. From now, we simply focus on particle 1.

3. In the next section, we will study a particle which *decays* in time $\tau \propto \gamma m^3$. If the first particle decays, the *distance* it travels is proportional to

$$\delta_1 = \gamma_1 v_1 m_1^3.$$

   Prove that

$$\delta_1 = \frac{8v_1[1-(v_1/c)^2]}{c^8(1+\alpha)^3} E_\mathrm{p}^3 < 8c^{-7}E_\mathrm{p}^3.$$

[13] For a stationary particle, we have $E = mc^2$ as a special case.

Before the LHC turned on, a group of concerned citizens argued that collisions might create *black holes*. Perhaps these would escape the detector and destroy the earth! Let's put the first part of this alarming claim to the test. We will need some facts about black holes. The radius r of a spherical black hole is related to its mass m by

$$r = \mu m, \quad \mu = 1.5 \times 10^{-27}\,\mathrm{m} \cdot \mathrm{kg}^{-1}.$$

Black holes also radiate light, a highly counterintuitive result discovered by Stephen Hawking. The rate P they lose energy to radiation is governed by the *Stefan-Boltzmann law*,

$$P = \sigma A T^4 = 4\pi\sigma r^2 T^4, \quad \sigma = 5.67 \times 10^{-8}\,\mathrm{J} \cdot \mathrm{m}^{-2} \cdot \mathrm{s}^{-1} \cdot \mathrm{K}^{-4},$$

where A is the surface area of the black hole. Finally, the temperature of the black hole is

$$T = \frac{\kappa}{m}, \quad \kappa = 1.23 \times 10^{23}\,\mathrm{kg} \cdot \mathrm{K},$$

where m is the mass.

4. Make an order-of-magnitude estimate of the time t_{evap} it takes for a stationary black hole of mass m to evaporate. Give your answer in terms of the constants μ, κ, σ.

5. Let's imagine that two protons at the LHC collide and form two black holes, with masses m_1, m_2 and velocities $\mathbf{v}_1, \mathbf{v}_2$. Since they are moving, they will undergo *time dilation*, with a decay time
$$\tau = \gamma t_{\mathrm{evap}}.$$
Using the results of question (3), show that the distance d travelled by the black hole before it decays is roughly upper bounded by
$$d = v_1 \tau \lesssim \frac{2E_{\mathrm{p}}^3}{\pi\sigma c^5 \mu^2 \kappa^4}.$$

6. For a black hole formed in the LHC to imperil the world, it must escape from the detector chamber, since this is mostly vacuum and cannot "feed" the black hole. Detector chambers at the LHC are around $10\,\mathrm{m}$ in width. The LHC can accelerate electrons to energies
$$E_{\mathrm{p}} = 6.5\,\mathrm{TeV} \approx 10^{-6}\,\mathrm{J}.$$
Does the black hole pose a threat to civilisation? If not, roughly how much more energy must we give to the protons so the resulting black holes could escape?

4 Gravity

4.1 Colliding black holes and LIGO

When a star runs out of nuclear fuel, it can collapse under its own weight to form a black hole: a region where gravity is so strong that even light is trapped. Black holes were predicted in 1915, but it took until 2015, 100 years later, for the Laser Interferometer Gravitational-wave Observatory (LIGO) to observe them directly. When two black holes collide, they emit a characteristic "chirp" of *gravitational waves* (loosely speaking, ripples in spacetime), and through an extraordinary combination of precision physics and engineering, LIGO was able to hear this chirp billions of light years away.

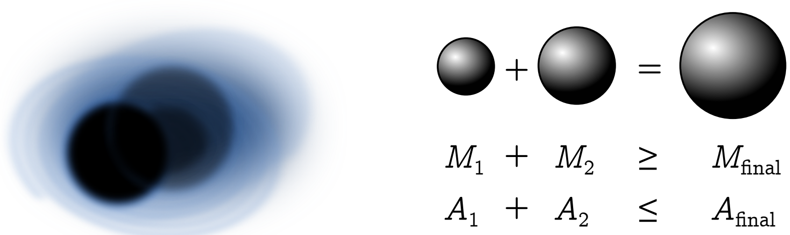

Figure 10: On the left, a cartoon of a black hole merger. On the right, inequalities obeyed by mergers: the mass of the final black hole can decrease when energy is lost (e.g. to gravitational waves), but the area always increases.

1. An infinitely dense point particle of mass M will be shrouded by a black hole. Using dimensional analysis, argue that this black hole has surface area
$$A = \left(\frac{\eta G^2}{c^4}\right) M^2$$
for some dimensionless constant η.

2. One of Stephen Hawking's famous discoveries is the *area theorem*: the total surface area of any system of black holes increases with time.[14] Using the area theorem, and the result of part (1), show that two colliding black holes can lose at most 29% of their energy to gravitational waves. (Note that to find this upper bound, you need to consider varying the mass of the colliding black holes, and to assume that any lost mass is converted into gravitational waves.)

3. LIGO detected a signal from two black holes smashing into each other 1.5 billion light years away. Their masses were $M_1 = 30M_\odot$ and $M_2 = 35M_\odot$, where $M_\odot \approx 2 \times 10^{30}\,\text{kg}$ is the mass of the sun, and the signal lasted for 0.2 seconds. Assuming the maximum amount of energy is converted into gravitational waves, calculate the average power P_{BH} emitted during the collision. Compare this to the power output of all the stars in the universe, $P_{\text{stars}} \sim 10^{49}\,\text{W}$.

[14] This theorem is actually violated by quantum mechanics, but for large black holes, the violations are small enough to be ignored.

4.2 Getting a lift into space

A *space elevator* is a giant cable suspended between the earth and an orbiting counterweight. Both the cable and counterweight are fixed in the rotating reference frame of the earth. The elevator can be used to efficiently transport objects from the surface into orbit, but also as a cheap launchpad.

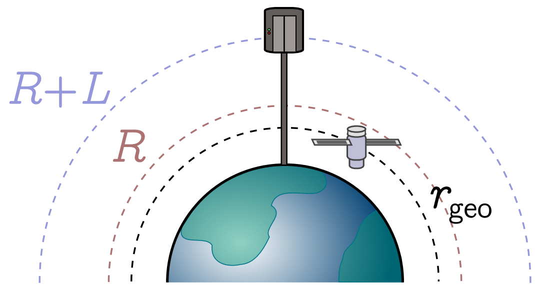

Figure 11: A satellite in geostationary orbit at radius r_{geo}. A space elevator connects a counterweight in low orbit to the surface via a cable of length $2L$. The cable's centre of mass lies at radius R, above r_{geo}.

1. To begin with, forget the cable, and consider a *geostationary* satellite orbiting at a fixed location over the equator.
   - Determine the radius r_{geo} of a geostationary orbit in terms of the mass of the earth M and angular frequency ω about its axis.
   - Confirm that r_{geo} obeys Kepler's third law, i.e. the square of the orbital period is proportional to the cube of the radius.
2. To make the space elevator, we now attach a cable to the satellite. The satellite acts as a counterweight, pulling the cable taut, but needs to move into a higher orbit in order to balance the cable tension. Provided this orbit is high enough, the space elevator will double as a rocket launchpad. Show that objects released from the elevator at $r_{\text{esc}} = 2^{1/3}r_{\text{geo}}$ will be launched into deep space.
3. The dynamics of the elevator itself are complicated, so we will consider a simplified model where the cable is treated as a rigid rod of length $2L$, with all of its mass concentrated at the centre, radius R. The counterweight is therefore at radius $R + L$.
   - Find the exact relationship between L, R, and the earth's mass M and rotational period ω.
   - Assuming $L \ll R$, show that the rod's centre of mass is further out than the geostationary radius r_{geo}. This somewhat counterintuitive result also holds for real space elevator designs! You may use the fact that, for $x \ll 1$,

$$\frac{1}{1+x} \approx 1 - x.$$

4.3 Hubble's law and dark energy

If we point a telescope at random in the night sky, we discover something surprising: faraway galaxies and stars are all moving away from us.[15] Even more surprising, the speed v of any object is proportional to its distance d from the earth, with

$$v = Hd.$$

The parameter H is called the *Hubble constant* (though it can in fact change), and the relation between velocity and distance is called *Hubble's law*.

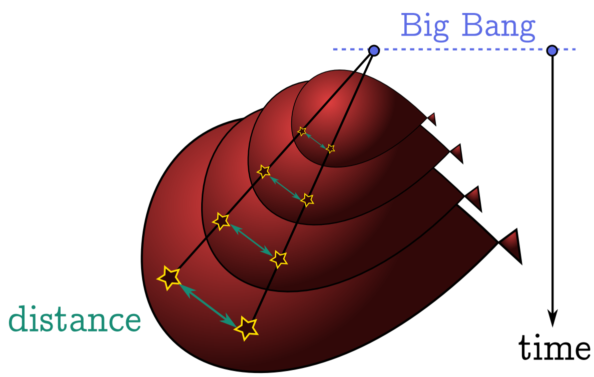

Figure 12: The cosmic balloon, inflated by dark energy.

A simple analogy helps illustrate. Imagine the universe as a balloon, with objects (like the stars in the image above) in a fixed position on the balloon "skin". Both the distance and relative velocity of any two objects will be proportional to the size of the balloon, and hence each other. The constant of proportionality is H.

1. The universe is expanding. Explain why Hubble's law implies that it does so at an *accelerating* rate.

2. The Virgo cluster is around 55 million light years away and receding at a speed of $1200\,\mathrm{km\,s^{-1}}$. By running time backwards, explain why you expect a Big Bang where everything is located at the same point. From the Virgo cluster and Hubble's law, estimate the age of the universe.

3. Since gravity is an attractive force, the continual expansion of the universe is somewhat mysterious. Why doesn't all the mass collapse back in on itself? The answer to this question is *dark energy*. Although we're not entirely sure what dark energy is, we can model it as an energy density ρ due to empty space itself. This energy does not change with time, since the vacuum always looks the same.

[15] How? Well, we know what frequencies of light stars like to emit since they are made of chemicals we find on earth. These frequencies are *Doppler-shifted*, or stretched, if the stars in a galaxy are moving away from us, allowing us to determine the speed of recession. Distance is a bit harder to work out, with different methods needed for different distance scales.

The state-of-the-art description of gravity is Einstein's theory of *general relativity*. For our purposes, all we need to know is that gravitational effects are governed by Newton's constant G and the speed of light c, where

$$G = 6.67 \times 10^{-11}\,\mathrm{N}\cdot\mathrm{m}^2\cdot\mathrm{kg}^{-2}, \quad c = 3\times 10^8\,\mathrm{m\,s^{-1}}.$$

Using dimensional analysis, argue that Hubble's constant is related to dark energy by

$$H^2 = \frac{\eta G\rho}{c^2}$$

for some (dimensionless) number η. This is the *Friedmann equation*.

4. Assuming that $\eta \sim 1$, estimate the dark energy density of the universe.

4.4 Rubber band cosmology

One of the stranger facts about the universe is that it gets bigger with time. To help us wrap our heads around this idea, let's model space as an elastic band. At one end is Alice, an intrepid ant with a birthday invite for Bob, who lives on the other side. Alice decides to cross the rubber band to deliver her invitation, but as she starts to walk, the band begins to stretch (perhaps pulled by a curious entomologist). Assume that the rubber band, like space itself, always *increases* length with time.

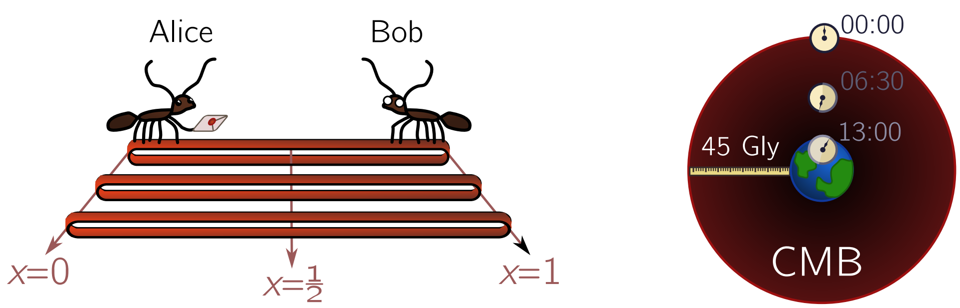

Figure 13: *Left.* Alice and Bob at either end of an expanding rubber band universe. *Right.* The CMB, which using a ruler is 45 billion light years away, but using a giga-light year clock, is only 13 hours old.

On a static rubber band, Alice's steps have fixed length ℓ. Her fitbit counts her steps as she walks across, so after n steps, she feels she has travelled a distance $L_\mathrm{A} = n\ell$. During this process, the rubber band expands from some initial length L_0 to final length L_B. From Bob's perspective, Alice walks a distance L_B.

1. (a) Show that Bob always overestimates how far Alice has travelled, $L_\mathrm{B} \geq L_\mathrm{A}$.
   Hint. Consider what happens to Alice's steps with time.

   (b) Using similar reasoning, argue that $L_\mathrm{A} \geq L_0$.

The elastic band explains some confusing features of our universe. If you point a telescope at the night sky in any direction, you see the same thing: a faint, uniform glow, with light peaking at a wavelength of around 1 mm. This is in the microwave range, so we call it the *Cosmic Microwave Background (CMB)*. This is the oldest light in the universe.

2. (a) The light from the CMB is around 13 billion years old (only slightly younger than the universe itself) but appears to come from a source 45 billion light years away. If light travels at fixed speed c, how can it travel 45 billion light years in 13 billion years, without exceeding this fixed speed?

   (b) Can you put a bound on the size of the universe when the CMB was formed?

Label points along the rubber band with the fraction of its length they represent, setting $x = 0$ at Alice's end and $x = 1$ at Bob's end.[16]

3. (a) Suppose Alice takes a step of length ℓ every second. In between, the rubber band expands by a factor α. Call this the *expansion*. If she walks forever, argue that she reaches x coordinate
$$x_\infty = \min\left\{1, \frac{\ell}{L_0}\left(\frac{\alpha}{\alpha - 1}\right)\right\}.$$
   Hint. You will need the geometric series, $1 + r + r^2 + r^3 + \cdots = 1/(1 - r)$.

   (b) Confirm that if the band expands slowly enough, Alice can always reach Bob.

   (c) Conversely, for step size $\ell < L_0$, prove that if the universe expands quickly enough Alice may *never* reach Bob. In physics, we call this a *cosmological horizon*: space is expanding too fast for messages to reach Bob from some parts of the rubber band. The region from which Bob can receive messages is called his *Hubble patch*.[17]

   (d) If $x_\infty < 1$, verify that Bob cannot receive birthday invites if Alice starts at
$$x < 1 - x_\infty.$$

4. The expansion $\alpha(t)$ changes depending on the dominant form of energy in the universe. We are at a turning point where *dark energy* will cause $\alpha(t)$ to increase without bound over time. What consequences will this have for astronomy?

Bonus. The CMB is spectacularly uniform, with the same radiation profile (up to minuscule variations) emerging from each point on the celestial sphere.[18] Currently, parts of the CMB separated by more than 2° on the celestial sphere have non-overlapping Hubble patches, so they cannot send messages to each other. Astronomical observations also show that, for the history of the observable universe, the expansion $\alpha(t)$ has been *decreasing* with time.

[16] We can imagine drawing marks on the rubber band and labelling them with x values. These will stretch out with the band, so they are called *co-moving* coordinates (though *co-stretching* might be more accurate).

[17] Note that this is generally different from the region Bob can reach with his own birthday invites.

[18] This is a fancy name for the big sphere the night sky appears on. The Greeks thought this was a physical sphere! For us, it is an imaginary surface that incoming light is projected onto.

5. (a) Explain why decreasing $\alpha(t)$, and the existence of horizons for different parts of the CMB, make the uniformity hard to explain. This is called the *horizon problem*, and it puzzled cosmologists for 20 years.

   (b) Solve the horizon problem! More specifically, describe how $\alpha(t)$ must change with time to explain the uniformity of the CMB. A subtle but important point is that the history of the observable universe starts some finite time *after* the Big Bang.

5 Random walks

5.1 The drunkard's dance

Walk in a straight line, and time translates linearly into distance:

$$d = vt.$$

But what if you take tiny steps of size ℓ at random? For a *random walk*, the distance scales as the *square root* of time:

$$d = \sqrt{(\ell v)t},$$

where v is the speed at which you execute individual steps. This basic fact underlies a whole host of phenomena, from molecular motion to DNA packing to bacteria searching for food. We will use basic probability theory to see where this square root scaling comes from.

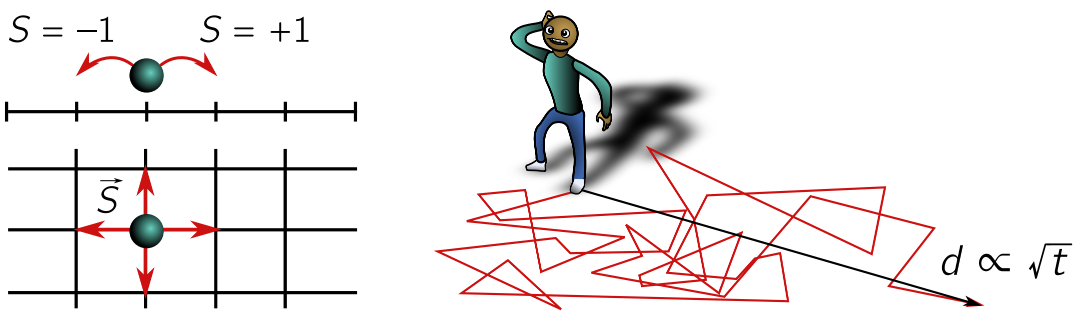

Figure 14: *Left.* A particle hops about randomly on the number line or a higher-dimensional lattice. *Right.* A drunkard randomly walking home.

Consider a particle moving around randomly on the number line. At each time step, it can move forward $(+1)$ or backward (-1) randomly, and with equal probability. We can view steps as *random variables*[19] S, and write

$$\Pr[S = +1] = \Pr[S = -1] = \frac{1}{2}.$$

If you wanted to simulate this walk, you could just flip a coin!Suppose we have a function $f(S)$ of the step, for instance $S^2 + S$ or $-S$. Then the *average value* of the function is just the sum of values, weighted by the probability of that value occurring:

$$\langle f(S)\rangle = \frac{1}{2} \times f(-1) + \frac{1}{2} \times f(+1) = \frac{1}{2}[f(-1) + f(+1)].$$

If $f(S) = g(S) + h(S)$ is a sum of functions, the average of f is the sum of averages of g and h:

$$\langle f(S)\rangle = \frac{1}{2}[f(-1) + f(+1)] = \frac{1}{2}[g(-1) + g(+1)] + \frac{1}{2}[h(-1) + h(+1)] = \langle g(S)\rangle + \langle h(S)\rangle.$$

[19] "Random variable" is fancy math terminology for "randomly generated number". Throwing a die or tossing a coin are simple examples.

More generally, if we have different random steps $S_1, S_2, \ldots, S_n$ (these could represent the first n steps taken by the particle), then the average value of a function $f(S_1, S_2, \ldots, S_n)$ is the sum of all possible values, weighted by their probability of occurring:

$$\langle f(S_1, S_2, \ldots, S_n)\rangle = \frac{1}{2^n}\overbrace{[f(-1,\ldots,-1) + f(-1,\ldots,+1)\cdots f(+1,\ldots,+1)]}^{\text{all } 2^n \text{ possible values for } S_1=\pm1, S_2=\pm1,\ldots,S_n=\pm1}.$$

Two simple functions of n consecutive steps are the *displacement*

$$\Delta_n = S_1 + S_2 + \cdots + S_n$$

and *displacement squared*:

$$d_n^2 = (S_1 + S_2 + \cdots + S_n)^2 = \left(S_1^2 + S_2^2 + \cdots S_n^2\right) + 2\overbrace{(S_1S_2 + S_1S_3 + \cdots + S_{n-1}S_n)}^{\text{products of distinct steps } S_iS_j,\, i\neq j}.$$

As we will see, these two measures of distance travelled behave very differently on average!

1. Show that $\langle S\rangle = 0$ and $\langle S^2\rangle = 1$.

2. Consider two independent steps S, S'. Explain why

$$\langle SS'\rangle = 0.$$

3. Let's scale the number line so that the particle steps forward or backward a distance ℓ. Using the two previous questions, prove that

$$\langle \Delta_n\rangle = 0, \quad \langle d_n^2\rangle = n\ell^2.$$

4. **Bonus.** Now consider a random walk in three-dimensional space[20] with variable step size. Mathematically, each step $\vec{S}$ is a vector which is unbiased in direction, and has some average length ℓ:

$$\langle \vec{S}\rangle = 0, \quad \langle \vec{S}\cdot\vec{S}\rangle = \langle S^2\rangle = \ell^2.$$

Assuming distinct steps are independent $\langle \vec{S}\cdot\vec{S}'\rangle = 0$, argue that the average *vector* displacement and length squared are

$$\langle \vec{\Delta}_n\rangle = 0, \quad \langle d_n^2\rangle = n\ell^2.$$

5. Instead of a particle, let's consider the classic example of a drunkard. At time $t = 0$, the drunkard leaves their tavern of choice and commences a random walk homeward. Suppose that on each random step, the drunkard moves with speed v and average step size ℓ. Argue that after time t, the drunkard's distance from where they start is

$$d = \sqrt{(\ell v)t}.$$

Since Problems 3 and 4 give the same result, this is true in any number of dimensions!

[20] In fact, the result you are about to prove holds in *any* number of dimensions!

5.2 A walk in the sun

When a particle jiggles around at random, the distance it travels scales as the *square root* of time. More specifically, the average separation from the origin of a random walk is[21]

$$d = \sqrt{(\ell v)t},$$

where v is the velocity of the walker, t is the elapsed time, and ℓ is the average length of a step. In the context of molecular jiggling, ℓ is the *mean free path*, or distance between collisons. We will explore a couple of applications of this "law of jiggling".

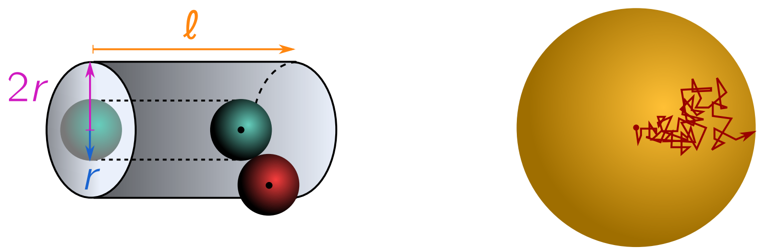

Figure 15: *Left.* The collision cylinder for a particle of radius r. The mean free path is defined so that, on average, the centre of one other particle (red) is contained in the cylinder and collides with the blue particle. *Right.* A photon escaping the sun.

We often model particles as spheres of radius r. They will collide if the centres come within a distance r of each other, since then the boundaries of the spheres are guaranteed to meet. We therefore consider the "collision cylinder" of radius $2r$ swept out by a particle.

1. Argue that, if the number density of jiggling particles (particles per unit volume) is n, then the mean free path is
$$\ell = \frac{1}{4\pi r^2 n}.$$
Hint. A collision occurs as soon as a second particle enters the collision cylinder.

2. The *ideal gas law* tells us that, in a dilute gas, number density depends on temperature and pressure:
$$n = \frac{N}{V} = \frac{P}{k_B T}, \quad k_B = 1.4 \times 10^{-23}\, \frac{\mathrm{m}^2 \cdot \mathrm{kg}}{\mathrm{s}^2 \cdot \mathrm{K}},$$
where k_B is *Boltzmann's constant*. The average air molecule has size $r = 4 \times 10^{-10}$ m. Using this data, estimate the density of air molecules around you, and hence the average distance between collisions.

3. Your favourite science-metal band, "Hordes of Sagan", is touring Canada and playing at the PNE. At the concert, you make your way to the centre of the mosh pit, a disk of radius $R = 5$ m with around 250 people crammed into it. If you bounce around at walking speed, how long will it take before random collisions eject you from the pit?

   Hint. You will need to adapt the answer to Problem 1 to two dimensions.

[21] See Problem 5.1 for a derivation. It holds in any number of dimensions.

Sometimes, the jiggling process is more interesting than particles colliding like billiard balls. The interior of the sun is a dense, roiling plasma of hydrogen nuclei. A photon, just minding its own business and trying to reach the earth, will collide with nuclei, get absorbed, then re-emitted, many times before it reaches the surface and turns into sunshine.

4. The photon also has a "collision cylinder" with an effective cross-section σ, but rather than being due to the physical size of the particle, this is called a *scattering* cross-section and measures the likelihood of interacting with matter. The cross-section for an average photon is
$$\sigma \approx 6 \times 10^{-29} \text{ m}^2.$$
The number density of hydrogen nuclei is roughly
$$n = 5 \times 10^{32} \text{ m}^{-3}.$$
Use these to calculate the mean free path of a photon.[22]

5. The radius of the sun is around $R_\odot = 7 \times 10^8$ m. Between interactions, photons travel at the speed of light, $c = 3 \times 10^8$ m/s, and the time between absorption and emission is negligible. How long does it take a photon to escape from the centre of the sun?[23]

5.3 Packing polymers

Like IKEA furniture, humans comes equipped with their own building instructions. But instead of words and pictures, our instructions are expressed using *protein*. Proteins are paired up and woven together into a gigantic polymer called DNA, with a copy stored in the nucleus of every cell. We have two DNA polymers per cell (one from each parent), and each polymer consists of around 3 billion coupled amino acids, or *base pairs* (bp).

1. (a) A base pair has length $\ell_{\text{bp}} = 0.34 \times 10^{-9}$ m $= 0.34$ nm. What is the total length of DNA in a human cell?
   (b) The cell nucleus has diameter $d_{\text{nuc}} \approx 10^{-5}$ m, while the DNA polymer has radius $r_{\text{DNA}} \approx 1$ nm. Calculate the *packing fraction* f, the fraction of the nucleus occupied by DNA. You should find that the nucleus has plenty of room.

Although there is room for DNA, packing it into the nucleus requires a staggering amount of compression. To see why, let's scale everything up by a factor of a million. The DNA strand is now the thickness of cooked spaghetti and stretches from Vancouver to Winnipeg, while the nucleus is a sack 10 m across. We need to stuff the spaghetti into the sack!

The naive approach is to simply feed the spaghetti into the sack. As it enters, it will curl into a random configuration; we can ignore gravity if, like the nucleus, the sack is filled with a gel-like substance called *nucleoplasm* which keeps everything afloat. But there is an important biological constraint: the spaghetti can't touch the sides! If it does, it can get corrupted by chemical interactions and become unreadable.

[22] The number density changes in different regions of the sun, and the cross-section depends on photon energy. The numbers quoted here are *averages*. We are taking averages everywhere!

[23] In fact, photons are *destroyed* when absorbed, with a fresh one minted in the emission process. But we can still chain together the absorption and emission events and ask how long it takes for this process to reach the surface.

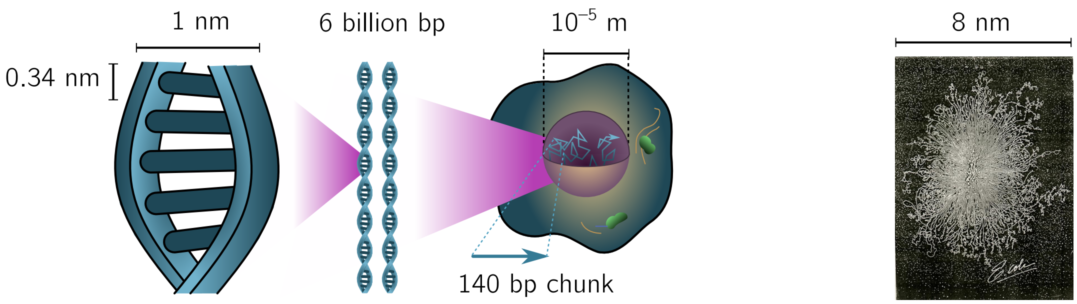

Figure 16: *Left.* Base pairs, woven into DNA strands and stuffed into the nucleus. *Right.* A ruptured *E. coli* bacterium.

2. (a) The curling can be modelled by a *random walk*.[24] If we break the strand up into small, approximately straight chunks of length ℓ, the average radius of a chain of N chunks is

$$r = \ell\sqrt{N}.$$

The "chunk length" of DNA is $\ell = 140$ bp. Show that the ball of spaghetti won't fit in a 10 m sack (equivalently, DNA can't fit in the nucleus) without touching the sides.

(b) Evolution could have made the nucleus bigger to accomodate a random walk of DNA. Can you think of a biological reason this solution isn't preferred?

(c) If a cell ruptures, genetic material can spill out, as in the image of the *E. coli* bacterium above. In this case, the DNA configuration is a random walk! Use this fact (with chunk length $\ell = 140$ bp) to estimate the number of base pairs in *E. coli* DNA.

Clearly, we need to do something else. In fact, the first step in DNA compression is the same method we use to get spaghetti into our mouths: twirl it on a fork! Instead of forks, DNA twirls around small cylindrical proteins called *histones*, and the twirled bundles *nucleosomes*.

3. (a) Assume the DNA is thickened into a random walk of adjacent nucleosomes, each containing around 140 bp of genetic material. What volume should the histone have in order for the DNA to fit into the nucleus? How many times does DNA wind around the histone? Treat it as a cylinder with radius equal to height.

(b) **Bonus.** The previous estimate is misleadingly close. Twirling into nucleosomes is only the first step in DNA compression. The *fractal globule* hypothesis suggests that DNA is twirled by successively bigger forks until you get a single giant twirl![25]

Consider a sphere of DNA of radius $r(L)$, where L is the length of DNA contained inside. How should the radius scale with L if the fractal globule hypothesis is correct? Contrast with random walk behaviour. Biologically speaking, why are random walks at any scale unfavourable?

[24] See Problem 5.1 for more discussion of random walks.

[25] This is called the "fractal globule" model since it assumes DNA is roughly *fractal* or *self-similar*: it looks the same at different magnifications, corresponding to different fork sizes. This property allows one to construct *space-filling curves*, where a 1D line, by virtue of its fractal complexity, can fill 3D space.

6 Particles, strings and springs

6.1 Quantum strings and vacuums

Suppose we stretch a string of length L between two fixed points. The string can oscillate sinusoidally in *harmonics*, the first few of which are sketched on the left below. Remarkably, by considering that harmonics of *space itself*, we can show that empty vacuum likes to push metal plates together!

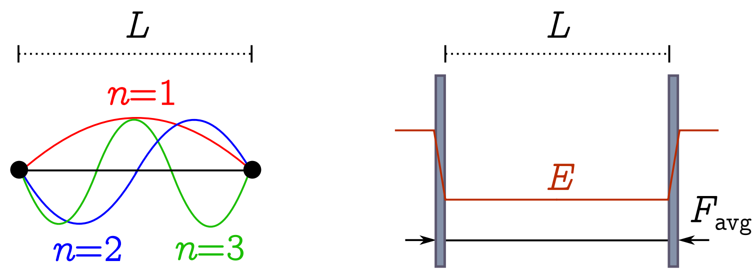

Figure 17: *Left.* Harmonics of a classical string. *Right.* Casimir effect on plates in a vacuum.

1. Show that harmonics on the string have wavelength

$$\lambda_n = \frac{2L}{n}, \quad n = 1, 2, 3, \ldots.$$

2. A classical string can vibrate with some combination of harmonics, including *no harmonics* when the string is at rest. In this case, the string has no energy. A *quantum* string is a little different: even if a harmonic is not active, there is an associated *zero-point energy*:

$$E_{0n} = \frac{\alpha}{\lambda_n},$$

   where α is a constant of proportionality. This is related to *Heisenberg's uncertainty principle*, which states that we cannot know both the position and momentum of the string with absolute certainty. Let's calculate the zero-point energy of a quantum string.

   Sum up the zero-point energies for each harmonic to find the energy of an unexcited quantum string. Use the infamous result[26] that

$$1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}.$$

3. Classical strings can be found everywhere, but where do we find quantum strings? One answer is *space itself*. Instead of stretching a string between anchors, set two lead plates a distance L apart. (Pretend that it vibrates in a plane, as in the picture above.)

[26] There are various ways of showing this, but the basic idea is that very large numbers in this sum correspond to high frequencies which would break the string if we tried to excite them. So we have to throw most of these large numbers away, i.e. subtract them from our running tally. In the process, we overcorrect and get a slightly negative result!

The harmonics are no longer wobbling modes of the string, but *electromagnetic waves*. Outside the plates is empty space, stretching away infinitely; it has zero energy.[27]

Suppose that the lead plates have thickness ℓ. Show that the plates are pushed together, with each subject to an average force

$$F_{\text{avg}} = \frac{\alpha}{24\ell L}.$$

The remarkable fact that the vacuum can exert pressure on parallel metal plates is called the *Casimir effect*. Although weak, it can be experimentally detected!

Bonus. These methods can also be applied to *string theory*. String theory posits that everything in the universe is made out of tiny vibrating strings. Different subatomic particles, like electrons and photons, correspond to the different ways that the string can vibrate. We will learn that string theory requires 25 spatial dimensions![28]

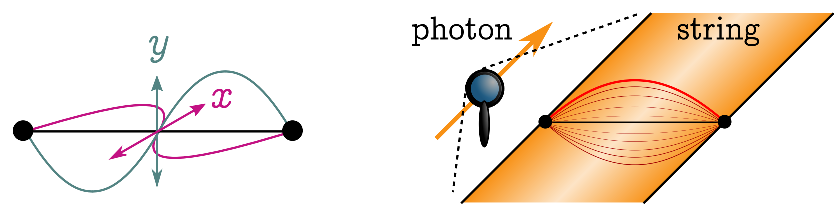

Figure 18: *Left.* Strings vibrating in different independent directions. *Right.* If we zoom in on a photon, we get a string with a single excited harmonic.

When we treat the string as a quantum object, each independent direction gets independent harmonics. Put a different way, we can split the string into $D-1$ independent strings wobbling in two dimensions, labelled by $i = 1, 2, \ldots, D-1$. To get different fundamental particles, we need to be able to *excite* harmonics. It turns out that, according to quantum theory, they have discrete *energy levels*, separated by "quantum leaps" in energy:

$$E^i_{mn} = \frac{\alpha}{\lambda_n}(1 + 2m), \quad m = 0, 1, 2, \ldots.$$

The superscript i denotes the direction the harmonic wobbles; the subscript n refers to the harmonic, while m refers to how excited that harmonic is. To find the total energy of the string, we just add up the energy of each harmonic.

4. The string can vibrate in any direction perpendicular to the string. In three spatial dimensions, there are two perpendicular directions for the string to vibrate (labelled by x and y above). Explain why, for D spatial dimensions, the string can vibrate in $D-1$ independent directions.

5. Suppose that we excite a first harmonic ($n = 1$) in some direction to its lowest excited state ($m = 1$). A string vibrating this way looks like a *photon* from far away, i.e. a particle of light. Use the fact that the photon *has zero mass* to deduce that $D = 25$.

[27]We can model the edge of space with lead plates infinitely far away. Since $L \to \infty$, $E^0_n \to 0$ and the energy does indeed disappear.

[28]Since we only see three dimensions, the remaining 22 must somehow be "curled up" and hidden from view.

6.2 Evil subatomic twins

In 1928, Paul Dirac made a startling prediction: the electron has an evil twin, the *anti-electron* or *positron*. The positron is the same as the electron in every way except that it has positive charge $q = +e$, rather than negative charge $q = -e$. In fact, *every* fundamental particle has an evil, charge-flipped twin; the evil twins are collectively called *antimatter*.[29]

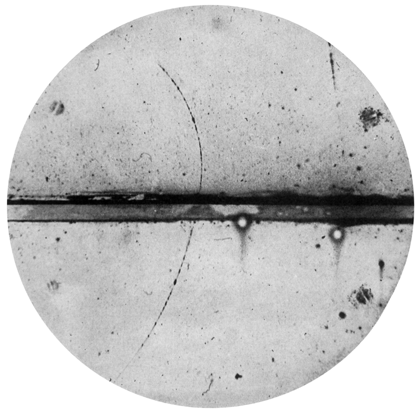

Figure 19: The mysterious trail in Carl Anderson's cloud chamber.

Experimentalist Carl Anderson was able to verify Dirac's prediction using a *cloud chamber*,[30] a vessel filled with alcohol vapour which is visibly ionised when charged particles (usually arriving from space) pass through it. In August 1932, Anderson observed the mysterious track shown above. Your job is to work out what left it!

1. A magnetic field $B = 1.7\,\mathrm{T}$ points into the page in the image above. Suppose that a particle of charge q and mass m moves in the plane of the picture with velocity v. Show that it will move in a circle of radius $R = mv/Bq$, and relate the sign of the charge to the motion.

2. The thick line in the middle of the photograph is a lead plate, and particles colliding with it will slow down. Using this fact, along with part (1), explain why the track in the image above must be due to a positively charged particle.

3. The width of the ionisation trail depends on what type of particle is travelling through the chamber and how fast it goes. The amount of ionisation in the picture above is consistent with an electron, but also an energetic proton, with momentum

$$p_{\mathrm{p}} \sim 10^{-16}\,\frac{\mathrm{kg}\cdot\mathrm{m}}{\mathrm{s}}.$$

   Can you rule the proton out?

[29] You may think it is a unfair to call antimatter "evil", but if you met your antimatter twin, hugging them would be extremely deadly! You would annihilate each other, releasing the same amount of energy as a large nuclear bomb.

[30] Cloud chambers are the modest ancestor of particle physics juggernauts like the Large Hadron Collider (LHC). Unlike the LHC, you can build a cloud chamber in your backyard!

6.3 Gravitational postal service

Mega-corporation Mammonzon drills a hole through the centre of the earth and sets up an antipodal delivery service, dropping packages directly through to the other side of the world. Your job, as a new Mammonzon employee, is to determine package delivery times! You soon realise that there is a complication: the strength of gravity changes as the package moves through the tunnel. To help out, your boss recommends Newton's *Principa Mathematica*, which provides a marvellous result called the Sphere Theorem:

- an object outside a spherical body (of constant density) is gravitationally attracted to it *as if all the mass were concentrated at the centre*;[31]
- an object inside a spherical shell *feels no gravitational attraction to the shell*.

We can use the Sphere Theorem, and a surprising analogy to springs, to work out the package transit time.

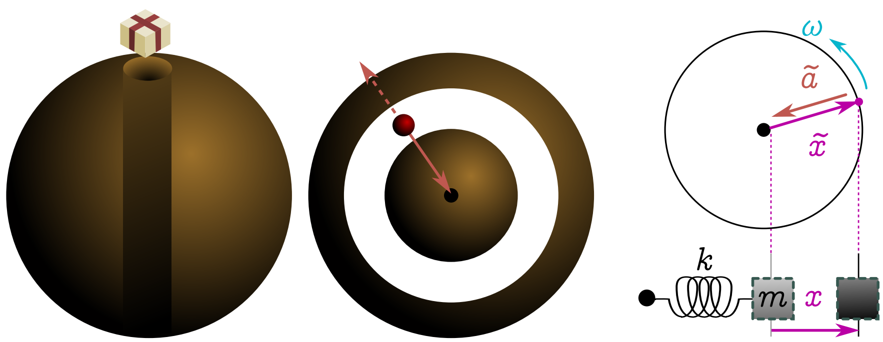

Figure 20: *Left*. A package travelling through the hole in the middle of the earth. *Middle*. The sphere theorem; the red mass feels no attraction to the shell, and attraction to the inner sphere as if all the mass were concentrated at the centre. *Right*. Phasor approach to solving the spring-mass problem.

1. Let r denote the radial distance from the centre of the earth. From the Sphere Theorem, show that a package at position r is subject to a gravitational force

$$F = \left(\frac{mg}{R}\right) r$$

directed towards the centre, where R is the radius of the earth and g the gravitational acceleration at the surface.

2. The force on the package is proportional to the distance from the centre. This is just like a spring! Let's understand springs first, then return to the delivery problem. If we attach a mass m to a spring of stiffness k, and pull the mass a distance x away from the equilibrium position, there is a restoring force

$$F = -kx.$$

[31]This explains why we always just treat planets as point masses in gravity problems.

If we displace the mass and let it go, the result is *simple harmonic motion*, where the mass just oscillates back and forth. To understand this motion, we can use the *phasor trick*. The basic idea is to upgrade x to a complex variable $\tilde{x} = re^{i\omega t}$ in uniform circular motion on the complex plane. Treating the acceleration $\tilde{a}$ and position $\tilde{x}$ as phasors, show that the phasor satisfies

$$\tilde{a} = -\omega^2 \tilde{x}.$$

Just so you know, you don't need any calculus!

3. Conclude that the phasor satisfies a spring equation for

$$k = \omega^2 m.$$

4. We must return to the harsh realities of the real line. To pluck out a real component of the phasor, we can use *Euler's formula*:

$$e^{i\omega t} = \cos(\omega t) + i \sin(\omega t).$$

By taking the real part of the phasor solution, show that a mass m oscillates on a spring of stiffness k according to

$$x(t) = x(0)\cos(\omega t), \quad \omega = \sqrt{\frac{k}{m}},$$

where it is released from rest at $x(0)$.

5. Using questions (1) and (4), argue that the package reaches the other side of the world in time

$$t_{\text{delivery}} = \pi \sqrt{\frac{R}{g}}.$$

7 Randomness and entropy

7.1 Donuts and wobbly orbits

Take a square of unit length. By folding twice and gluing (see below), you can form a donut. Particles confined to the donut don't know it's curved; it looks like normal space to them, except that if they go too far to the left, they will reappear on the right, and similarly for the top and bottom. Put a different way, the blue lines to the left and right are identified, and similarly for the red lines. This is just like the video game *Portal*!

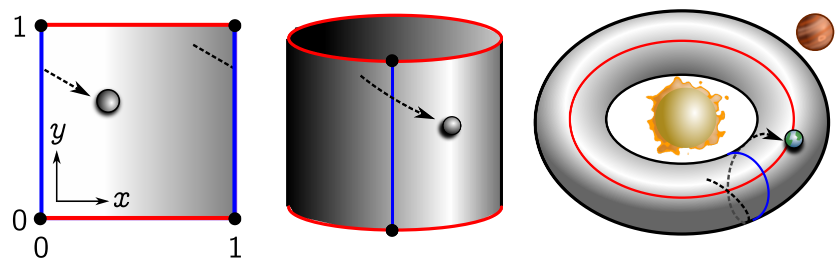

Figure 21: Folding and gluing a square to get a donut. The earth has a wobbly donut orbit (highly exaggerated) due to its attraction to Jupiter.

1. Suppose we have two particles, and shoot them out from the origin at $t = 0$. One particle travels vertically in the y direction with speed v_y, and the other travels in the x direction with speed v_x. Will they ever collide? If so, at what time will the first collision occur?

2. Now consider a *single* particle with velocity vector $\mathbf{v} = (v_x, v_y)$. Show that the particle will never visit the same location on the donut twice if the slope of its path cannot be written as a fraction of whole numbers. Such a non-repeating path is called *non-periodic*.

The earth orbits the sun, but feels a slight attraction to other planets, in particular the gas giant Jupiter. This attraction will deform the circular[32] orbit of the earth onto the surface of a donut, travelling like the particle in question (2). Sometimes, these small changes can accumulate over time until the planet flies off into space! This is obviously something we want to avoid. There is a deep mathematical result[33] which states that the orbit on the donut will be stable provided it is non-periodic. Periodic donut orbits, on the other hand, will reinforce themselves over time and create instabilities. This is like pushing a swing in sync with its natural rhythm: eventually, the occupant of the swing will fly off into space as well!

3. Regarding the x-direction as the circular direction around the sun, and y as the direction of the wobbling due to Jupiter, it turns out that

$$\frac{v_y}{v_x} = \frac{T_{\text{Jupiter}}}{T_{\text{earth}}}.$$

If the relative size of orbits is $R_{\text{Jupiter}} = 5R_{\text{Earth}}$, will the earth remain in a stable donut-shaped orbit? *Hint:* You may use the fact that $\sqrt{125}$ cannot be written as a fraction.

[32] In fact, the orbit is slightly stretched along one direction to form an ellipse, but we will ignore this point. One complication at a time!

[33] Called the *KAM theorem* after Kolmogorov, Arnol'd and Moser.

7.2 Butterflies in binary

Pick a number x between 0 and 1, then double it. The number $2x$ is between 0 and 2, but if we lop off the whole number part, we are left with a number we will call x^* between 0 and 1. We illustrate the process below.

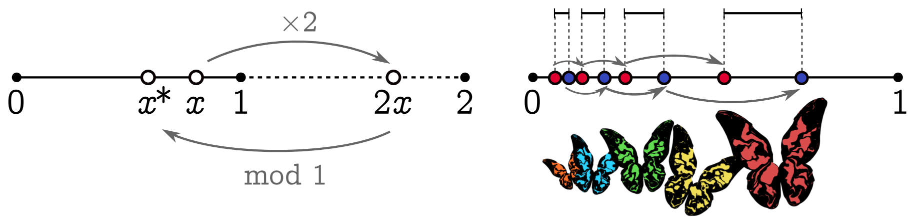

Figure 22: On the left, doubling and taking the fractional part. On the right, the exponential growth of a small error, aka the Butterfly Effect.

If we continue multiplying by two and keeping the fractional part, we get a whole sequence of numbers between 0 and 1:

$$x_0 = x, \quad x_1 = x^*, \quad x_2 = x^{**}, \dots$$

We can view x_t as a particle jumping about on the interval $[0, 1]$ with discrete time steps t. Surprisingly, this jumping particle system is chaotic. We will explore what this means below![34]

1. Write x as an expansion in binary digits $b \in \{0, 1\}$ rather than decimal,

   $$x = 0.b_1b_2b_3 \dots.$$

   Show that

   $$2x = b_1.b_2b_3b_4 \dots, \quad \text{and} \quad x^* = 0.b_2b_3b_4 \dots.$$

   In other words, $x \to x^*$ throws away b_1 and shifts the binary expansion to the left.

2. Consider two starting positions,

   $$x = 0.b_1b_2 \dots b_n \dots$$
   $$x' = 0.b_1b_2 \dots b_n' \dots$$

   where $b_n \neq b_n'$ but otherwise the digits in the binary expansion are the same. The *initial error* is the distance between x and x':

   $$\Delta x = |x - x'| = \frac{1}{2^n}.$$

   Show that if we start two particles at x and x' and let them jump around, the error grows *exponentially*:

   $$\Delta x_t = |x_t - x_t'| = 2^t \Delta x.$$

[34] This problem gratefully "borrowed" from Steve Shenker.

This exponential growth of small errors is the definition of chaos![35] It makes it very hard to predict the future behaviour of the system.

3. We can measure how chaotic a system is by how quickly errors grow. By definition, in a chaotic system errors grow exponentially,[36] with

$$\Delta x_t = e^{\lambda t}\Delta x.$$

The number λ is the called the *Lyapunov exponent*. What is the Lyapunov exponent for our system of jumping particles? By modifying the example, show that we can make the Lyapunov exponent arbitrarily large.

4. Suppose I flip a fair coin an infinite number of times, and convert the heads and tails into a binary sequence. Any *finite sequence* of 1s and 0s is bound to occur at some point in the infinite sequence, by the laws of probability. An infinite sequence with this property is called *normal*.[37]

   Let x be a normal binary sequence, and y any number in $[0, 1]$. Argue briefly that a chaotically hopping particle with initial position x will jump *arbitrarily close* to y. In some sense, chaos in a confined space lets us visit every point in $[0, 1]$!

7.3 Black hole hard drives

Black holes are perhaps the most mysterious objects in the universe. For one, things fall in and never come out again. An apparently featureless black hole could conceal an elephant, the works of Shakespeare, or even another universe! Suppose we wanted to describe all the possible objects that could have fallen into the black hole, but using *binary digits* (bits) 0 and 1, the language of computers. With one bit, we can describe two things, corresponding to 0 and 1; with two bits, we can describe *four* things, corresponding to $00, 01, 10, 11$. Continuing this pattern, with n bits we can describe 2^n things, corresponding to the 2^n sequences of n binary digits. The total number of bits needed to describe all the possibilities, for a given black hole, is called the *entropy* S. Since information is also stored in bits, we can (loosely) equate entropy and information!

We would expect that a large black hole can conceal more than a small black hole, and will therefore have a larger entropy. The *area law*, discovered by Stephen Hawking and Jacob Bekenstein, shows that this is true, with the entropy of the black hole proportional to its *surface area* A:

$$S = \frac{A}{A_0},$$

[35] This is also called *sensitivity to initial conditions* or the *Butterfly Effect*. We can imagine a butterfly flapping its wings in Bombay as a small change to initial conditions. Since the weather is highly chaotic, this small change can get amplified into a hurricane in Kansas!

[36] At least initially. In this case, the error due to flipping a single bit in the binary expansion will disappear once the offending digit has been truncated. But the *initial* errow growth is what defines chaos.

[37] This is a consequence of a deep result called the *Borel-Cantelli lemma*. Roughly, it means that if you give monkeys typewriters, they will eventually type Hamlet. Curiously, it is not only random sequences which are normal. The digits of π in binary are also thought to have this property!

where $A_0 \approx 10^{-69}\,\mathrm{m}^2$ is a basic unit of area. We can view the black hole surface as a sort of screen, made up of binary pixels of area A_0.

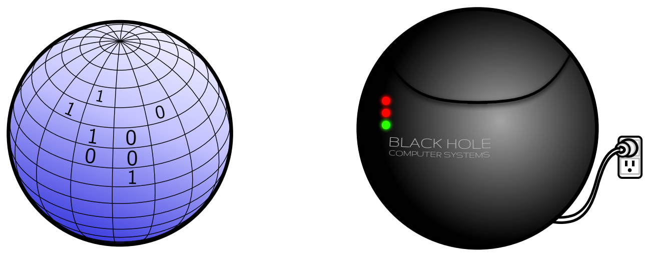

Figure 23: *Left.* The area law, viewed as pixels on the black hole surface. *Right.* A spherical hard drive.

The *Second Law of Thermodynamics* states that the total entropy of a closed system always increases.[38] Combining the area law and the Second Law leads to a surprising conclusion: black holes have the *highest* entropy density of any object in the universe. They are the best hard drives around![39]

1. To get a sense of scale, calculate how many gigabytes of entropy can be stored in a black hole the size of a proton, radius $\sim 10^{-15}\,\mathrm{m}$. Note that

$$1\,\mathrm{GB} = 10^9\,\mathrm{B} = 8 \times 10^9\,\mathrm{bits}.$$

   Compare this to the total data storage on all the computers in the world, which is approximately $1.5 \times 10^{12}\,\mathrm{GB}$.

2. Consider a sphere of ordinary matter of surface area A. Suppose this sphere has more entropy than a black hole,

$$S' > S_{\mathrm{BH}} = \frac{A}{A_0}.$$

   Argue that this violates the Second Law. You may assume that as soon as a system of area A reaches the mass M_A of the corresponding black hole, it immediately collapses to form said black hole. *Hint.* How could you force it to collapse?

3. Calculate the optimal information density in a spherical hard drive of radius r.

4. Suppose that the speed at which operations can be performed in a hard drive is proportional to the density of information storage. (This is reasonable, since data which is spread out takes more time to bring together for computations.) Explain why huge (spherical) computers are necessarily slow.

[38] The entropy of a black hole is the number of bits needed to describe all the things that could have fallen in. The entropy of an ordinary object, like a box of gas, is the number of bits needed to describe all the different *microscopic* configurations which are indistinguishable to a macroscopic experimentalist, i.e. which look like the same box of gas. The function of entropy, in both cases, is to count the number of configurations which look the same!

[39] At least when it comes to information storage density. *Extracting* information is much harder!