UBC Physics Circle problems

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1 Evil subatomic twins

Problem. In 1928, Paul Dirac made a startling prediction: the electron has an evil twin, the *anti-electron* or *positron*. The positron is the same as the electron in every way except that it has positive charge $q = +e$, rather than negative charge $q = -e$. In fact, *every* fundamental particle has an evil, charge-flipped twin; the evil twins are collectively called *antimatter*.[1]

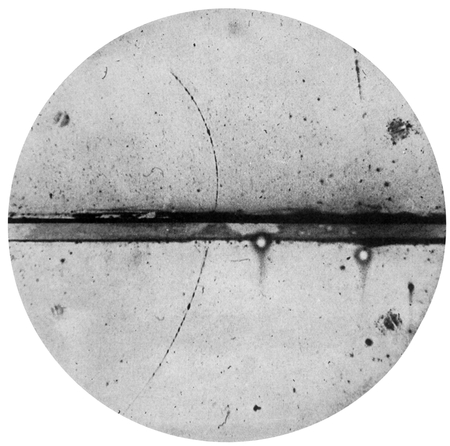


Figure 1: The mysterious trail in Carl Anderson's cloud chamber.

Experimentalist Carl Anderson was able to verify Dirac's prediction using a *cloud chamber*,[2] a vessel filled with alcohol vapour which is visibly ionised when charged particles (usually arriving from space) pass through it. In August 1932, Anderson observed the mysterious track shown above. Your job is to work out what left it!

1. A magnetic field $B = 1.7\,\mathrm{T}$ points into the page in the image above. Suppose that a particle of charge q and mass m moves in the plane of the picture with velocity v. Show that it will move in a circle of radius $R = mv/Bq$, and relate the sign of the charge to the motion.

[1] You may think it is a unfair to call antimatter "evil", but if you met your antimatter twin, hugging them would be extremely deadly! You would annihilate each other, releasing the same amount of energy as a large nuclear bomb.

[2] Cloud chambers are the modest ancestor of particle physics juggernauts like the Large Hadron Collider (LHC). Unlike the LHC, you can build a cloud chamber in your backyard!

2. The thick line in the middle of the photograph is a lead plate, and particles colliding with it will slow down. Using this fact, along with part (1), explain why the track in the image above must be due to a positively charged particle.

3. The width of the ionisation trail depends on what type of particle is travelling through the chamber and how fast it goes. The amount of ionisation in the picture above is consistent with an electron, but also an energetic proton, with momentum

$$p_{\mathrm{p}} \sim 10^{-16}\,\frac{\mathrm{kg}\cdot\mathrm{m}}{\mathrm{s}}.$$

Can you rule the proton out?

Solution.

1. The Lorentz force law tells us that the particle is subject to a constant force of magnitude $F = Bqv > 0$. The force will be normal to the direction of motion, acting centripetally and causing the particle to move in a circle. To find the radius, we use $a = v^2/R$:

$$a = \frac{F}{m} = \frac{Bqv}{m} = \frac{v^2}{R} \quad \Longrightarrow \quad R = \frac{mv}{Bq}.$$

Finally, by the right-hand rule, a positively charged particle will experience a force to its left, causing it to move around the circle anticlockwise (seen from above); similarly, a negatively charged particle will move clockwise.

2. From the previous question, the particle's radius of curvature will get smaller as it slows down. This tells us the particle in the image is moving from bottom to top. (Being able to tell which the particle is going is why Anderson added the plate!) Since its path curves in the anticlockwise sense, it must be positively charged.

3. The radius of the track is comparable to the radius of the chamber, $r \approx 0.1\,\mathrm{m}$. This leads to momentum

$$p = mv = BqR = 1.7 \times 0.1 \times (1.6 \times 10^{-19})\,\frac{\mathrm{kg}\cdot\mathrm{m}}{\mathrm{s}} \sim 10^{-20}\,\frac{\mathrm{kg}\cdot\mathrm{m}}{\mathrm{s}}.$$

This is considerably smaller than the momentum a proton would need to create the trail seen in the photograph. This only leaves one option: it is the positron, the positively charged evil twin of the electron!

2 Getting a lift into space

Problem. A space elevator is a giant cable suspended between the earth and a counterweight at the other end, orbiting the earth. Both the cable and counterweight are fixed in the rotating reference frame of the earth, and can be used to efficien tly transport objects from the surface into orbit, and also as a launchpad for rockets or satellites. Space elevators would completely revolutionise our access to space, and make large-scale projects like interplanetary travel to Mars a possibility.

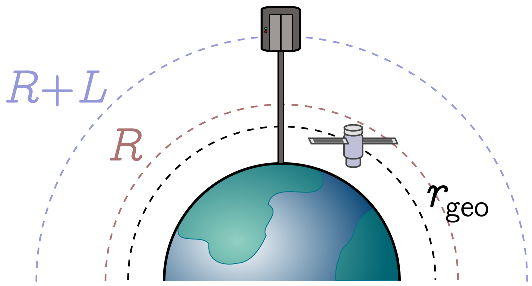


Figure 2: A satellite in geostationary orbit at radius r_{geo}. A space elevator connects a counterweight in low orbit to the surface via a cable of length $2L$. The cable's centre of mass lies at radius R, above r_{geo}.

1. To begin with, forget the cable, and consider a *geostationary* satellite orbiting at a fixed location over the equator.
    - Determine the radius r_{geo} of a geostationary orbit in terms of the mass of the earth M and angular frequency ω about its axis.
    - Confirm that r_{geo} obeys Kepler's third law, i.e. the square of the orbital period is proportional to the cube of the radius.
2. To make the space elevator, we now attach a cable to the satellite. The satellite acts as a counterweight, pulling the cable taut, but needs to move into a higher orbit in order to balance the cable tension. Provided this orbit is high enough, the space elevator will double as a rocket launchpad. Show that objects released from the elevator at $r_{\text{esc}} = 2^{1/3} r_{\text{geo}}$ will be launched into deep space.

    Hint: To launch a rocket, it needs to be travelling at escape velocity. This is the speed needed to leave the earth's gravity well with no gas left in the tank, i.e. the total energy (kinetic plus potential) vanishes.
3. The dynamics of the elevator itself are complicated, so we will consider a simplified model where the cable is treated as a rigid rod of length $2L$, with all of its mass concentrated at the centre, radius R. The counterweight is therefore at radius $R + L$.
    - Find the exact relationship between L, R, and the earth's mass M and rotational period ω.

- Assuming $L \ll R$, show that the rod's centre of mass is further out than the geostationary radius r_{geo}. This somewhat counterintuitive result also holds for real space elevator designs! You may use the fact that, for $x \ll 1$,

$$\frac{1}{1+x} \approx 1 - x.$$

Solution.

1. The gravitational acceleration is given by Newton's law of gravitation:
$$a = \frac{GM}{r_{\text{geo}}^2}.$$
The centrifugal acceleration is
$$a = \frac{v^2}{r_{\text{geo}}} = \omega^2 r.$$
Equating the two, we find
$$r_{\text{geo}}^3 = \frac{GM}{\omega^2}.$$
Since $\omega \propto 1/T$, where T is the period of the orbit, Kepler's third law is obeyed.

2. Since the whole elevator is geostationary, it rotates with angular frequency ω. At radius r_{esc}, the speed is $v = \omega r_{\text{esc}}$. We recall that the gravitational potential is $U = -GMm/r$. Finally, we can determine r_{esc} by demanding that the total energy vanish:
$$E = U + K = m\left(\frac{1}{2}\omega^2 r_{\text{esc}}^2 - \frac{GM}{r_{\text{esc}}}\right) = 0 \quad \Longrightarrow \quad r_{\text{esc}}^3 = \frac{2GM}{\omega^2} = 2r_{\text{geo}}^3.$$

3. Treat the rod as concentrated at its centre of mass at radius R. In order for the rod and the satellite to have the same angular velocity, we require the forces in the rotating reference frame to balance:
$$\omega^2[(R-L) + (R+L)] = 2R\omega^2 = GM\left[\frac{1}{(R-L)^2} + \frac{1}{(R+L)^2}\right] = \frac{2GM(R^2+L^2)}{(R^2-L^2)^2}.$$
Rearranging, we find that
$$\frac{GM}{\omega^2} = \frac{R(R^2-L^2)^2}{(R^2+L^2)}.$$
If $L \ll R$, then $(L/R)^2 \ll 1$ and hence
$$\frac{1}{R^2+L^2} = \frac{1}{R^2(1+L^2/R^2)} \approx \frac{1}{R^2}\left(1 - \frac{L^2}{R^2}\right),$$
using our approximation $1/(1+x) \approx 1-x$. It follows that
$$\frac{GM}{\omega^2} \approx \frac{1}{R}(R^2 - 2L^2)(R^2 - L^2) \approx R^3 - 3RL^2.$$
Comparing to the radius of the geostationary orbit, we find
$$r_{\text{geo}}^3 \approx R^3 - 3RL^2,$$
which implies that $r_{\text{geo}} < R$.

3 Sticks, stones and starlight

Problem. Physics is built on the interplay of idea and experiment. In some cases, very simple measurements, combined with deep ideas, can reveal surprising facts about the world around us. We give three examples: the size of the earth, the mass of the sun, and the age of the universe.

1. Eratosthenes (276–195 BC) was head librarian at the magnificent Library of Alexandria, and one of the great thinkers of the ancient world. In a blow to civilisation, the library was destroyed by invading Roman emperors, and most of Eratosthenes' work along with it. Thankfully, his elegant method for calculating the size of the earth, using only the shadow of a stick, survives.

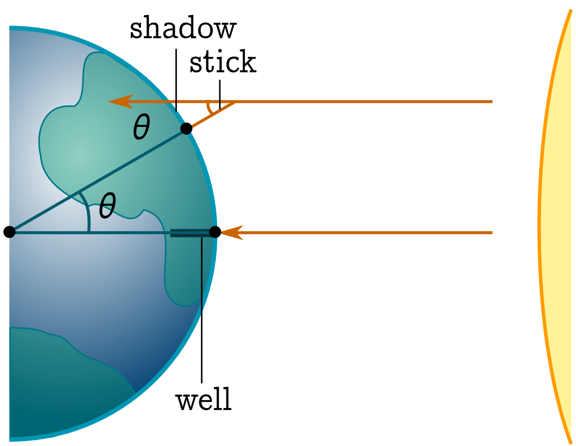


Figure 3: Eratosthenes measures the world with a stick. Figure not to scale.

   On the summer solstice, locals in the Egyptian city of Syene noticed that, at noon, the sun hit the bottom of a deep well. Eratosthenes inferred that the sun must be directly overhead. He also knew from Egyptian surveyors that Syene was roughly 5000 stadia ($\approx 850\,\mathrm{km}$) away from Alexandria. Eratosthenes performed a single experiment. At noon on the summer solstice, he measured the shadow of a vertical rod in Alexandria. He found it was roughly $1/8$ of the length of the rod. Estimate the radius of the earth.

2. In 1666, Cambridge University was closed due to an outbreak of the plague, and a young Isaac Newton was forced to return to his hometown of Grantham. During this unexpected holiday, Newton was inspired by a falling apple[3] to formulate his law of gravitation:

$$F_{\text{grav}} = \frac{GMm}{r^2}.$$

[3] Newton never mentions the apple in his own writings, but his first biographer, William Stukeley, reports this conversation with Newton: "Amid other discourse, he told me, he was just in the same situation, as when formerly the notion of gravitation came into his mind. Why should that apple always descend perpendicularly to the ground, thought he to himself; occasion'd by the fall of an apple, as he sat in contemplative mood. Why should it not go sideways, or upwards? But constantly to the Earth's centre? Assuredly the reason is, that the Earth draws it. There must be a drawing power in matter. And the sum of the drawing power in the matter of the Earth must be in the Earth's centre, not in any side of the Earth."

Use this law, and the length of the year, to estimate the mass of the sun $M_\odot$. You may also use the fact that light takes 8 minutes to arrive from the sun. Some useful physical constants:

$$G = 6.67 \times 10^{-11}\,\frac{\mathrm{N}\cdot\mathrm{m}^2}{\mathrm{kg}^2}, \quad c = 3\times 10^8\,\mathrm{m/s}.$$

3. If we point a telescope at random at the night sky, we discover something surprising: faraway galaxies and stars are all moving away from us.[4] For instance, the Virgo cluster is around 55 million light years away, and receding at a speed of $1200\,\mathrm{km/s}$. Even more surprising, the speed v of any object is proportional to its distance d from the earth:

   $$v = H_0 d.$$

   The number H_0 is called the *Hubble constant*, and the relation between velocity and distance *Hubble's law*.[5]

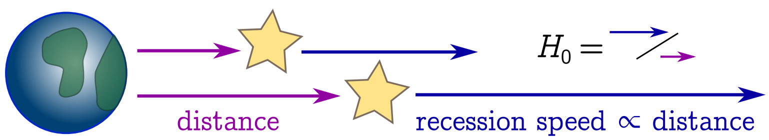
   

   Figure 4: Hubble's law states that recession speed is proportional to distance.

   By running time backwards, explain why you expect a Big Bang where everything is located at the same point, and from the Virgo cluster, estimate the age of the universe.

Solution.

1. We assume the earth is spherical, and the sun far enough away that the arriving rays are parallel. The angle the sunlight makes with the stick is

   $$\theta = \tan^{-1}\left(\frac{\text{shadow}}{\text{rod}}\right) \approx \tan^{-1}\left(\frac{1}{8}\right) \approx 7.1^\circ.$$

   Here, we are making the reasonable assumption that the shadow is small enough to be modelled as a straight line. From the diagram, we see that the angle the sunlight makes with the rod is the angle subtended by the arc of the great circle joining Alexandria and Syene. Since the the length of that great circle is the circumference of the earth, we have

   $$\frac{\tan^{-1}\left(\frac{1}{8}\right)}{2\pi} \approx \frac{850\,\mathrm{km}}{\text{circumference}}.$$

[4] We know what frequencies of light stars like to emit. These frequencies are *Doppler-shifted*, or stretched, if the stars in a galaxy are moving away from us, allowing us to determine the speed of recession. Distance is a bit harder to work out, with different methods needed for different distance scales.

[5] "Constant" is a bit misleading, since it changes over time, and due to gravitational interactions, nearby objects do not obey it. Luckily, we can ignore these subtleties for the purposes of a simple estimate! Although the constant and law bear the name of astronomer Edwin Hubble, credit should also go to theorists Alexander Friedmann and Georges Lemaître, and astronomer Vesto Slipher.

We rearrange and divide the circumference by 2π to estimate the radius:

$$\text{radius} = \frac{\text{circumference}}{2\pi} \approx \frac{850\,\text{km}}{\tan^{-1}\left(\frac{1}{8}\right)} \approx 6800\,\text{km}.$$

The actual value is around $6300\,\text{km}$. Not a bad estimate from the shadow of a stick!

2. Since the sun is much heavier than the earth, we can treat it as fixed. The earth rotates with speed $v = 2\pi r/T$, where r is the distance from the sun and T the period of rotation. The centripetal acceleration is

$$a = \frac{v^2}{r} = \frac{4\pi^2 r}{T^2}.$$

But by Newton's second law, $F = ma$, and the law of gravitation, we also have

$$a = \frac{F_{\text{grav}}}{m_{\text{earth}}} = \frac{GM_\odot}{r^2}.$$

Equating the two expressions for a and isolating $M_\odot$, we find

$$M_\odot = \frac{4\pi^2 r^3}{GT^2}.$$

To evaluate this, we need to know T and r. Of course, T is simply the length of a year:

$$T \approx 365 \times 24 \times 60^2\,\text{s} \approx 3.15 \times 10^7\,\text{s}.$$

The distance r from the earth to the sun is around 8 light minutes, as suggested in the hint:

$$r \approx 8 \times 60 \times 3 \times 10^8\,\text{m} = 1.44 \times 10^{11}\,\text{m}.$$

Plugging these values of r and T into the expression for $M_\odot$, we find

$$M_\odot \approx \frac{4\pi^2(1.44 \times 10^{11})^3}{(6.67 \times 10^{-11})(3.15 \times 10^7)^2}\,\text{kg} \approx 1.78 \times 10^{30}\,\text{kg}.$$

The correct value is $M_\odot = 1.99 \times 10^{30}\,\text{kg}$, so once again we are pretty close!

3. Let's run time backwards until a faraway object collides with us. If the distance is d, and the velocity v, then by Hubble's law the time needed to hit us is

$$t_{\text{collision}} = \frac{d}{v} = \frac{1}{H_0}.$$

Since this is the same for any object, it suggests that a time $t_{\text{collision}}$, every object in the universe was in the same place. This must be the Big Bang! The age of the universe is then $t_{\text{collision}}$, which we can estimate from the Virgo cluster as

$$t_{\text{collision}} = \frac{d}{v} = \frac{53 \times 10^6 \times (3 \times 10^8\,\text{m/s})}{1.2 \times 10^6\,\text{m/s}}\,\text{years} \approx 13.75 \times 10^9\,\text{years}.$$

We guess the universe is about 13.75 billion years old. The current best estimate is 13.80 billion years!

4 Colliding black holes and LIGO

Problem. When a star runs out of nuclear fuel, it can collapse under its own weight to form a black hole: a region where gravity is so strong that even light is trapped. Black holes were predicted in 1915, but it took until 2015, 100 years later, for the Laser Interferometer Gravitational-wave Observatory (LIGO) to observe them directly. When two black holes collide, they emit a characteristic "chirp" of *gravitational waves* (loosely speaking, ripples in spacetime), and through an extraordinary combination of precision physics and engineering, LIGO was able to hear this chirp billions of light years away.

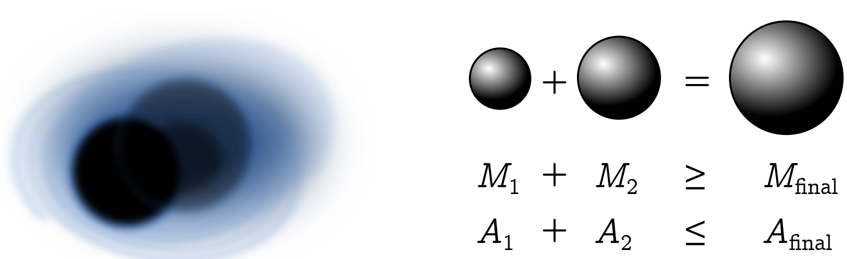


Figure 5: On the left, a cartoon of a black hole merger. On the right, inequalities obeyed by mergers: the mass of the final black hole can decrease when energy is lost (e.g. to gravitational waves), but the area always increases.

1. An infinitely dense point particle of mass M will be shrouded by a black hole. Using dimensional analysis, argue that this black hole has surface area

$$A = \left(\frac{\eta G^2}{c^4}\right) M^2$$

for some constant η.

2. One of Stephen Hawking's famous discoveries is the *area theorem*: the total surface area of any system of black holes increases with time.[6] Using the area theorem, and the result of part (1), show that two colliding black holes can lose at most 29% of their energy to gravitational waves. (Note that to find this upper bound, you need to consider varying the mass of the colliding black holes, and to assume that any lost mass is converted into gravitational waves.)

3. LIGO detected a signal from two black holes smashing into each other 1.5 billion light years away. Their masses were $M_1 = 30M_\odot$ and $M_2 = 35M_\odot$, where $M_\odot \approx 2 \times 10^{30}\,\text{kg}$ is the mass of the sun, and the signal lasted for 0.2 seconds. Assuming the maximum amount of energy is converted into gravitational waves, calculate the average power P_{BH} emitted during the collision. Compare this to the power output of all the stars in the universe, $P_{\text{stars}} \sim 10^{49}\,\text{W}$.

[6] This theorem is actually violated by quantum mechanics, but for large black holes, the violations are small enough to be ignored.

Solution.

1. A black hole, by definition, is a gravitational trap for light. It will therefore involve Newton's constant G, which is related to the strength of gravity, and the speed of light c. The mass of the particle is also relevant, since we expect a heavier particle to correspond to a larger black hole. We denote the units of a quantity by square brackets, $[\cdot]$. Obviously, $[M] = \text{mass}$ and $[c] = \text{distance/time}$. From Newton's law of gravitation,
$$F = \frac{GMm}{r^2} \quad \Longrightarrow \quad [G] = \frac{[F][r]^2}{[M]^2} = \frac{\text{length}^3}{\text{time}^2 \cdot \text{mass}},$$
where we used
$$[F] = [ma] = \text{mass} \cdot \frac{\text{length}}{\text{time}^2}.$$
Area has the units of length^2. We can systematically analyse the units using simultaneous equations, but here is a shortcut: time doesn't appear in the final answer, so we must combine G and c as G/c^2, which has units
$$[Gc^{-2}] = \frac{\text{length}}{\text{mass}}.$$
To get something with units length^2, we must square this and multiply by M^2. It follows that, up to some dimensionless constant η, the area of the black hole is
$$A = \left(\frac{\eta G^2}{c^4}\right) M^2.$$

2. Consider two black holes of mass M_1, M_2. The initial and final area are
$$A_{\text{init}} = A_1 + A_2 = \frac{\eta G^2}{c^4}(M_1^2 + M_2^2), \quad A_{\text{final}} = \left(\frac{\eta G^2}{c^4}\right) M_{\text{final}}^2.$$
If $A_{\text{init}} = A_{\text{final}}$, we have maximal loss of mass; if $M_{\text{final}} = M_1 + M_2$, we minimise the mass loss. The percentage of mass lost will depend on the mass of the black holes, but to place an upper bound, we want to choose the masses to maximise the fraction of mass lost. The simplest way to proceed is to instead look at the difference of *squared* masses,
$$\Delta M^2 = M_{\text{final}}^2 - M_1^2 - M_2^2 = (M_1 + M_2^2)^2 - M_1^2 - M_2^2 = 2M_1M_2.$$
Since we only care about the fraction lost, we can require a total initial mass $M = M_1 + M_2$ for fixed M, and now try to choose M_1, M_2 to maximise the square of mass lost:
$$\Delta M^2 = 2M_1M_2 = 2M_1(M - M_1).$$
This is just a quadratic in M_1, with roots at $M_1 = 0$ and $M_1 = M$. The *maximum* will be precisely in between, at $M_1 = M/2$. Of course, maximising the square of lost mass should be the same as maximising the lost mass itself, so we obtain an upper bound on mass loss in any black hole collision by setting $M_1 = M_2$, with a fractional loss
$$1 - \frac{M_{\text{final}}}{M_1 + M_2} = 1 - \frac{\sqrt{M_1^2 + M_1^2}}{M_1 + M_1} = 1 - \frac{\sqrt{2}}{2} \approx 0.29.$$
Since the mass can be converted into gravitational waves, we have the 29% bound we were looking for!

3. From the last question, we know that we maximise the energy converted into gravitational waves when the total area doesn't change,

$$A_{\text{final}} = A_1 + A_2 = \frac{\eta G^2}{c^4}(M_1^2 + M_2^2) = \left(\frac{\eta G^2}{c^4}\right) M_{\text{final}}^2.$$

   This corresponds to a loss of mass

$$\Delta M = M_1 + M_2 - M_{\text{final}} = M_1 + M_2 - \sqrt{M_1^2 + M_2^2} \approx 18.9\, M_\odot.$$

   We can convert this to energy using the most famous formula in physics, $E = mc^2$. To find the average power P, we divide by the duration of the signal $t = 0.2\,\text{s}$. We find

$$P_{\text{BH}} = \frac{E}{t} = \frac{\Delta M c^2}{t} = \frac{18.9 \cdot 2 \cdot 10^{30}(3 \times 10^8)^2}{0.2}\,\text{W} \approx 1.7 \times 10^{49}\,\text{W}.$$

   Since $P_{\text{BH}} > P_{\text{stars}}$, we see that for a brief moment, colliding black holes can outshine all the stars in the universe![7]

[7]This suggests that black hole mergers should be easy to see, but the analogy to starlight is misleading. Light likes to interact with things and can be easily absorbed, e.g. by the rods and cones in your eye, or the CCDs in a digital camera. In contrast, gravitational waves simply pass through matter, wobbling things a little as they go by. This wobbling is very subtle; so subtle, in fact, that an isolated observer can never detect it! But if we *very carefully* compare the paths of two photons going in different directions, we can discern the wobbling. This is why LIGO has two giant arms at right angles: one for each photon path.

5 Donuts and wobbly orbits

Problem. Take a square of unit length. By folding twice and gluing (see below), you can form a donut. Particles confined to the donut don't know it's curved; it looks like normal space to them, except that if they go too far to the left, they will reappear on the right, and similarly for the top and bottom. Put a different way, the blue lines to the left and right are identified, and similarly for the red lines. This is just like the video game *Portal*!

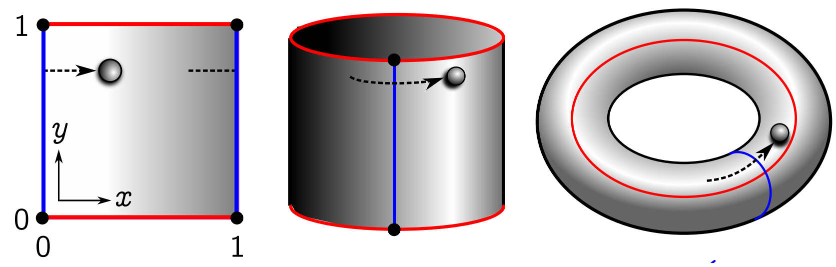


Figure 6: Folding and gluing a square to get a donut.

1. Suppose we have two particles, and shoot them out from the origin at $t = 0$. One particle travels vertically in the y direction with speed v_y, and the other travels in the x direction with speed v_x. Will they ever collide? If so, at what time will the first collision occur?

2. Now consider a *single* particle with velocity vector $\mathbf{v} = (v_x, v_y)$. Show that the particle will never visit the same location on the donut twice if the slope of its path cannot be written as a fraction of whole numbers. Such a non-repeating path is called *non-periodic*.

The earth orbits the sun, but feels a slight attraction to other planets, in particular the gas giant Jupiter. This attraction will deform the circular[8] orbit of the earth onto the surface of a donut, travelling like the particle in question (2). Sometimes, these small changes can accumulate over time until the planet flies off into space! This is obviously something we want to avoid. There is a deep mathematical result[9] which states that the orbit on the donut will be stable provided it is non-periodic; periodic donut orbits, on the other hand, will reinforce themselves over time, like someone pushing a swing in sync with its natural rhythym, which can lead to instabilities.

3. Regarding the x-direction as the circular direction around the sun, and y as the direction of the wobbling due to Jupiter, it turns out that

$$\frac{v_y}{v_x} = \frac{T_{\text{Jupiter}}}{T_{\text{earth}}}.$$

   If the relative size of orbits is $R_{\text{Jupiter}} = 5R_{\text{Earth}}$, will the earth remain in a stable donut-shaped orbit? *Hint:* You may use the fact that $\sqrt{125}$ cannot be written as a fraction.

[8] In fact, the orbit is slightly stretched along one direction to form an ellipse, but we will ignore this point. One complication at a time!

[9] Called the *KAM theorem* after Kolmogorov, Arnol'd and Moser.

Solution.

1. Since the first particle travels on the red line (y-axis) and the second particle travels on the blue line (x-axis), they will only collide if they both return to the origin at the same time. But this means that both must travel an *integer* distance in the same time, so for some natural numbers m_x, m_y, and some time t,

   $$v_x t = m_x, \quad v_y t = m_y.$$

   Dividing one equation by the other, we find that the ratio of velocities must be a fraction:

   $$\frac{v_x}{v_y} = \frac{m_x}{m_y}.$$

   If m_x, m_y have no common denominators, then the first time the particles coincide for $t > 0$ is when $v_x t = m_x$ and $v_y t = m_y$, so $t = v_x/m_x = v_y/m_y$. If the ratio of velocities is not a fraction, they can never collide.

2. This is just the first problem in disguise! The two particles get associated to the x and y coordinates of the single particle. To begin with, suppose the particle starts at the origin at $t = 0$. Let's look for conditions which stop it from returning there. From the first problem, it will never return to the origin as long as v_x/v_y is *irrational*. But there is nothing special about the origin; the same reasoning shows that if the ratio of velocity components is irrational, it will never return to any position it occupies.[10]

3. Kepler's third law states that the radius of an orbit R and the period T (i.e. the length of the year on the planet) are related by

   $$T^2 = \alpha R^3$$

   for some constant α which is the same for all planets. Thus,

   $$\frac{T_{\text{Jupiter}}}{T_{\text{earth}}} = \frac{\sqrt{\alpha} R^{3/2}_{\text{Jupiter}}}{\sqrt{\alpha} R^{3/2}_{\text{earth}}} = 5^{3/2} = \sqrt{125}.$$

   Since this cannot be expressed as a fraction, the results of part (2) show that the orbit is non-periodic. This means that the earth should stay in a stable donut orbit forever![11]

[10]Something even more remarkable is true: the one-dimensional trajectory of the particle manages to fill in most of the the two-dimensional surface of the donut! (It visits everywhere except a miniscule subset of area zero.)

[11]In fact, Jupiter's orbit is only approximately five times larger. But it remains true that a Jupiter year is some irrational number of earth years, which is the key to the stability of the earth's orbit.

6 Butterflies in binary

Problem. Pick a number x between 0 and 1, then double it. The number $2x$ is between 0 and 2, but if we lop off the whole number part, we are left with a number we will call x^* between 0 and 1. We illustrate the process below.

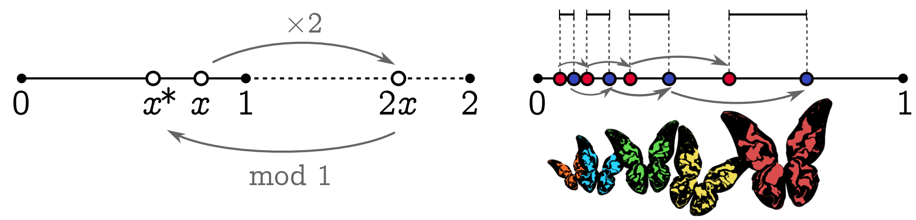


Figure 7: On the left, doubling and taking the fractional part. On the right, the exponential growth of a small error, aka the Butterfly Effect.

If we continue multiplying by two and keeping the fractional part, we get a whole sequence of numbers between 0 and 1:

$$x_0 = x, \quad x_1 = x^*, \quad x_2 = x^{**}, \dots$$

We can view x_t as a particle jumping about on the interval $[0,1]$ with discrete time steps t. Surprisingly, this jumping particle system is chaotic. We will explore what this means below![12]

1. Write x as an expansion in binary digits $b \in \{0,1\}$ rather than decimal,

$$x = 0.b_1 b_2 b_3 \dots .$$

   Show that

$$2x = b_1.b_2 b_3 b_4 \dots, \quad \text{and} \quad x^* = 0.b_2 b_3 b_4 \dots .$$

   In other words, $x \to x^*$ throws away b_1 and shifts the binary expansion to the left.

2. Consider two starting positions,

$$\begin{aligned} x &= 0.b_1 b_2 \dots b_n \dots \\ x' &= 0.b_1 b_2 \dots b_n' \dots \end{aligned}$$

   where $b_n \neq b_n'$ but otherwise the digits in the binary expansion are the same. The *initial error* is the distance between x and x':

$$\Delta x = |x - x'| = \frac{1}{2^n}.$$

   Show that if we start two particles at x and x' and let them jump around, the error grows *exponentially*:

$$\Delta x_t = |x_t - x_t'| = 2^t \Delta x.$$

[12]This problem gratefully on loan from Steve Shenker.

This exponential growth of small errors is the definition of chaos![13] It makes it very hard to predict the future behaviour of the system.

3. We can measure how chaotic a system is by how quickly errors grow. By definition, in a chaotic system errors grow exponentially,[14] with

$$\Delta x_t = e^{\lambda t}\Delta x.$$

The number λ is the called the *Lyapunov exponent*. What is the Lyapunov exponent for our system of jumping particles? By modifying the example, show that we can make the Lyapunov exponent arbitrarily large.

4. Suppose I flip a fair coin an infinite number of times, and convert the heads and tails into a binary sequence. Any *finite sequence* of 1s and 0s is bound to occur at some point in the infinite sequence, by the laws of probability. An infinite sequence with this property is called *normal*.[15]

Let x be a normal binary sequence, and y any number in $[0,1]$. Argue briefly that a chaotically hopping particle with initial position x will jump *arbitrarily close* to y. In some sense, chaos in a confined space lets us visit every point in $[0,1]$!

Solution.

1. A number in binary is a sum of binary powers:

$$x = \frac{b_1}{2^1} + \frac{b_2}{2^2} + \frac{b_3}{2^3} + \cdots = 0.b_1b_2b_3\cdots.$$

Multiplying the RHS by two gives

$$2x = \frac{b_1}{2^0} + \frac{b_2}{2^1} + \frac{b_3}{2^2} + \cdots = b_1.b_2b_3b_4\cdots$$

Throwing away the whole number part $b_1/2^0 = b_1$, we have

$$x^* = \frac{b_2}{2^1} + \frac{b_3}{2^2} + \cdots = 0.b_2b_3b_4\cdots.$$

2. The binary expansion of the difference between x and x' has a 1 in the n position:

$$\Delta x = |x - x'| = 0.00\ldots1\ldots = \frac{1}{2^n}.$$

[13] This is also called *sensitivity to initial conditions* or the *Butterfly Effect*. We can imagine a butterfly flapping its wings in Bombay as a small change to initial conditions. Since the weather is highly chaotic, this small change can get amplified into a hurricane in Kansas!

[14] At least initially. In this case, the error due to flipping a single bit in the binary expansion will disappear once the offending digit has been truncated. But the *initial* errow growth is what defines chaos.

[15] This is a consequence of a deep result called the *Borel-Cantelli lemma*. Roughly, it means that if you give monkeys typewriters, they will eventually type Hamlet. Curiously, it is not only random sequences which are normal. The digits of π in binary are also thought to have this property!

This difference changes with time in the same way as x and x' themselves: shift the binary expansion to the left and throw away any whole number part. So with each time step, the 1 just shifts to the left, until it itself is thrown away. Until this happens, the error is doubling with each time step:

$$\Delta x_t = |x_t - x'_t| = \frac{1}{2^{n-t}} = \frac{2^t}{2^n} = 2^t \Delta x.$$

3. To find the Lyapunov exponent for our jumping particles, we just need to match

$$e^{\lambda t} = 2^t \quad \Longrightarrow \quad \lambda = \log 2.$$

   To change the Lyapunov exponent, consider a different base, say 10 for illustrative purposes. If we multiply by 10 and keep the fractional part, we can repeat the analysis almost word for word, but get $\lambda = \log 10$ at the end! So we see what we need to do: just make the base some large number N, and update the particle by multiplying by N and keeping the fractional part. This gives a Lyapunov exponent of $\lambda = \log N$, which we can make as large as we like since $\log N$ is unbounded.

4. The basic idea here is that we can pick some approximation of y, call it y_{approx}, corresponding to a finite sequence of binary digits. This can be made arbitrarily close to y. Since x is normal, it contains y_{approx} somewhere in the binary expansion, and after we shift enough digits, we will eventually end up with y_{approx} at the front:

$$x(t) = 0.y_{\text{approx}} x_{\text{rest}}$$

   where x_{rest} are the remaining digits. This is very close to y_{approx} (any errors are smaller than the precision we have used to approximate y), and by considering longer and longer rational approximations y_{approx}, we can get the particle to jump as close as we like!

7 The quantum Hall effect

Problem. Suppose we have a conductor made of a long, flat plate, with an electric field E_x running along its length. A magnetic field B points out of the conductor.

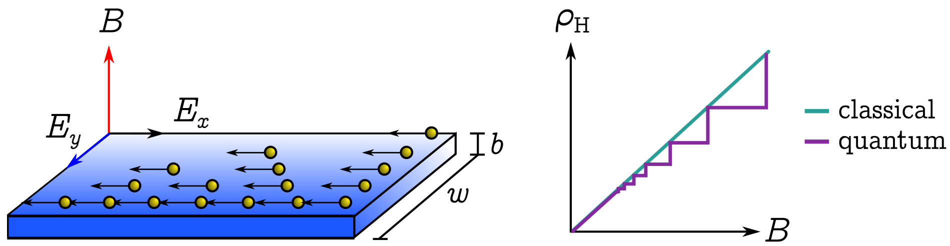


Figure 8: *Left*. A conductor with longitudinal electric field E_x and negative charge carriers. A magnetic field B points out of the conductor, pushing the carriers towards us. A transverse field builds up. *Right*. Classical vs quantum Hall resistivity as we vary the magnetic field.

Charges moving through the conductor are pushed to one side by the magnetic field, until the charge imbalance generates a *transverse field* E_y large enough to counteract the magnetic force. This phenomenon is called the *Hall effect*, and the corresponding voltage across the conductor the *Hall voltage* V_{H}.

1. Show that the current I in the x direction is related to the cross-sectional area A of the conductor
$$I = Anqv_x,$$
where v_x is the velocity in the x direction, q is the charge of the carriers, and n is the number of carriers per unit volume.

2. The transverse field and Hall voltage are related by $E_y = V_{\mathrm{H}}/w$, where w is the width of the conductor. Argue that
$$V_{\mathrm{H}} = \frac{IB}{nbq},$$
where b is the height of the conductor.

3. Edwin Hall discovered the effect in 1879, before anyone knew about the electron. How could he tell that charge carriers in metal were negative?

The Hall voltage depends the size and shape of the conductor. The *Hall resistivity* ρ_{H} is a new quantity we define which depends only on B and microscopic properties of the material:

$$\rho_{\mathrm{H}} = \frac{V_{\mathrm{H}}}{Ib} = \frac{B}{nq}.$$

It seems that by tuning the magnetic field, we could make the Hall resistivity anything we like. But if we switch on quantum mechanics, only certain values of ρ_{H} are allowed. This is called the *quantum Hall effect*. Note that the strength of quantum mechanical effects is governed by *Planck's constant*, $h = 6.63 \times 10^{-34}\ \mathrm{J \cdot s}$.

4. Using dimensional analysis, combine h, B and q to find the *magnetic length* ℓ_B of the system. This characterises the effective strength of the magnetic field, which does not involve n.

5. In a hydrogen atom, we view electrons as tiny waves, and restrict the allowed orbits around the nucleus by asking these waves to be periodic. We can make a similar restriction here, and ask the orbits of the individual charge carriers, encoded in the density n, to fit neatly into a box with side length ℓ_B. More precisely, we demand that[16]

$$n\ell_B^2 = k$$

for a positive integer $k \in \mathbb{N}$.[17] Show that, in consequence, the Hall resistivity takes discrete values

$$\rho_{\mathrm{H}} = \frac{h}{q^2 k}.$$

Solution.

1. The current I is the rate at which charge flows through a cross-section of the conductor. Concretely, if ΔQ moves through the cross-section in time Δt, then $I = \Delta Q/\Delta t$. Take a cross-section slice of the conductor at some fixed time. The charges in the slice move at speed v_x, so in time Δt, they drag out a volume $\Delta V = Av_x\Delta t$. All the charge in this volume will move through a fixed cross-section over time Δt, so we just need to find the charge in the volume ΔV. This is simply the volume multiplied by the charge and density of the carriers:

$$\Delta Q = nq\Delta V = Anqv_x\Delta t.$$

Thus, we have the current:

$$I = \frac{\Delta Q}{\Delta t} = Anqv_x.$$

2. From the Lorentz force law, the force on charges in the y direction is

$$F_y = qE_y - qv_xB.$$

The transverse field is defined as the field needed to balance the magnetic force, so that

$$F_y = qE_y - qv_xB = 0 \quad \Longrightarrow \quad E_y = v_xB.$$

Using $V_{\mathrm{H}} = wE_y$ and the results of part (1), we find that

$$V_{\mathrm{H}} = wE_y = wv_xB = \frac{IwB}{Anq} = \frac{IB}{bnq},$$

using $A = bw$.

[16] There is a dimensional problem with $n\ell_B^2 = k$ that needs to be fixed!

[17] Instead of electron orbitals, the allowed "frequencies" in the presence of a magnetic field are called *Landau levels*, and quantisation of $n\ell_B^2$ is equivalent to restricting to full level Landau levels. It is not obvious why this should happen, even if we take quantum mechanics into account; for instance, we can have atoms with partially filled orbitals. The answer is rather deep, and crucially involves *impurities* in the sample. This is the starting point for the theory of *topological materials*, but this is beyond the scope of the question!

3. The sign of the Hall voltage depends on the sign of the charge q. All Hall needed to do was measure the voltage across the conductor to find the sign of the charge carriers! Choosing $B > 0$, Hall observed that $V_{\mathrm{H}} < 0$. It follows that $q < 0$, i.e. the charge carriers are negative.

4. Denote units of mass, length and time by M, L, T respectively. Using $K = mv^2/2$ to get the units of energy, we have

$$[h] = [\text{energy}][\text{time}] = \frac{ML^2}{T^2} \cdot T = \frac{ML^2}{T}.$$

   On the other hand, from the Lorentz force law and $F = ma$, the units of magnetic field are

$$[Bq] = \frac{[\text{force}]}{[\text{velocity}]} = \frac{ML/T^2}{L/T} = \frac{M}{T}.$$

   Thus, dividing h by Bq gives something with units length squared. Taking the square root, we obtain the magnetic length:

$$\ell_B = \sqrt{\frac{h}{Bq}}.$$

5. We can rewrite the Hall resistivity in terms of the magnetic length:

$$\rho_{\mathrm{H}} = \frac{B}{nq} = \frac{h}{nq^2} \cdot \frac{Bq}{h} = \frac{h}{q^2} \cdot \frac{1}{n\ell_B^2}.$$

   Assuming $n\ell_B^2$ is a positive integer $k \in \mathbb{N}$, we find discrete values for the Hall resistivity as required:

$$\rho_{\mathrm{H}} = \frac{h}{q^2}\frac{1}{k}.$$

8 Turbulence in a tea cup

Problem. Stir a cup of coffee vigorously enough, and the fluid will begin to mix in a chaotic or *turbulent* way. Unlike the steady flow of water through a pipe, the behaviour of turbulent fluids is unpredictable and poorly understood. However, for many purposes, we can do suprisingly well by modelling a turbulent fluid as a collection of (three-dimensional) eddies of different sizes, with larger eddies feeding into smaller ones and losing energy in the process.

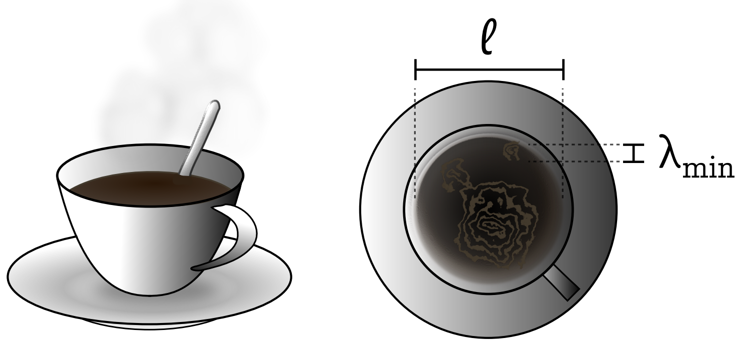


Figure 9: A well-stirred cup of coffee. On the right, a large eddy (size $\sim \ell$) and the smallest eddy (size λ_{min}) are depicted.

Suppose our cup of coffee has characteristic length ℓ, and the coffee has density ρ. When it is turbulently mixed, the largest eddies will be a similar size to the cup, order ℓ, and experience fluctuations in velocity of size Δv due to interaction with other eddies. The fluid also has internal drag[18] or *viscosity* η, with units $\mathrm{N} \cdot \mathrm{s/m^2}$.

1. Let ϵ be the rate at which kinetic energy dissipates per unit mass due to eddies. Observation shows that this energy loss is independent of the fluid's viscosity. Argue on dimensional grounds that
$$\epsilon \approx \frac{(\Delta v)^3}{\ell}.$$
Why doesn't the density ρ appear?

2. Kinetic energy can also be lost due to internal friction. Argue that the time scale for this dissipation due to viscosity is
$$\tau_{\text{drag}} \approx \frac{\ell^2 \rho}{\mu}.$$

3. Using the previous two questions, show that eddy losses[19] dominate viscosity losses provided
$$\frac{\ell \rho \Delta v}{\mu} \gg 1.$$

[18] More precisely, viscosity is the resistance to *shear flows*. A simple way to create shear flow is by moving a large plate along the surface of a stationary fluid. Experiments show that the friction per unit area of plate is proportional to the speed we move it, and inversely proportional to the height; the proportionality constant at unit height is the viscosity. Since layers of fluid also generate shear flows, viscosity creates internal friction.

[19] Since ϵ depends on $\ell, \Delta v$, you need not consider it when finding the time scale for eddy losses.

The quantity on the left is called the *Reynolds number*, $\mathrm{Re} = \ell\rho\Delta v/mu$. In fact, one *definition* of turbulence is fluid flow where the Reynolds number is high.

4. So far, we have focused on the largest eddies. These feed energy into smaller eddies of size λ and velocity uncertainty Δv_λ, which have an associated *eddy Reynolds number*,

$$\mathrm{Re}_\lambda = \frac{\lambda\rho\Delta v_\lambda}{\mu}.$$

When the eddy Reynolds number is less than 1, eddies of the corresponding size are prevented from forming by viscosity.[20] Surprisingly, the rate of energy dissipation per unit mass in these smaller eddies is ϵ, the same as the larger eddies.[21] Show from dimensional analysis that the minimum eddy size is roughly

$$\lambda_{\min} \approx \left(\frac{\mu^3}{\epsilon\rho^3}\right)^{1/4}.$$

5. If a cup of coffee is stirred violently to Reynolds number $\mathrm{Re} \approx 10^4$, estimate the size of the smallest eddies in the cup.

Solution.

1. Let $[\cdot]$ denote the dimensions of a physical quantity, and M, L, T mass, length and time respectively. Then energy per unit mass per unit time has dimension

$$[\epsilon] = \frac{\text{energy}}{MT} = \frac{M(L/T)^2}{MT} = \frac{L^2}{T^3},$$

where we can remember the dimension for energy using kinetic energy, $K = mv^2/2$. (The dimension does not depend on what form of energy we look at.) The dimensions for the remaining physical quantities are easier:

$$[\ell] = L, \quad [\rho] = \frac{M}{L^3}, \quad [\Delta v] = \frac{L}{T}.$$

Since mass does not appear in $[\epsilon]$, and the viscosity is not involved in this type of dissipation, the density ρ cannot appear since there is nothing besides μ to cancel the mass units. We can easily combine ℓ and Δv to get something with the correct dimension, and deduce an approximate relationship between $\epsilon, \Delta v$ and ℓ:

$$\frac{[(\Delta v)^3]}{[\ell]} = \frac{L^3}{LT^3} = [\epsilon] \quad \Longrightarrow \quad \epsilon \approx \frac{(\Delta v)^3}{\ell}.$$

[20] Lewis Fry Richardson not only invented the eddy model, but this brilliant mnemonic couplet: "Big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity."

[21] This is not at all obvious, but roughly, follows because we can fit more small eddies in the container. Intriguingly, this makes the turbulent fluid like a *fractal*: the structure of eddies repeats itself as we zoom in, until viscosity begins to play a role. At infinite Reynolds number, it really is a fractal!

2. Viscosity has dimensions

$$[\mu] = \frac{[\mathrm{N}][\mathrm{s}]}{[\mathrm{m}^2]} = \frac{MLT}{T^2L^2} = \frac{M}{LT}.$$

We can combine with ρ and ℓ to get something with the dimensions of time; Δv is not involved since friction is independent of the eddies. The unique combination with the right units is

$$\frac{[\ell^2\rho]}{[\mu]} = \frac{L^2 \cdot M \cdot LT}{L^3 \cdot M} = T \quad \Longrightarrow \quad \tau_{\mathrm{drag}} = \frac{\ell^2\rho}{\mu}.$$

3. Returning to eddy losses, its easy to cook up a time scale from the basic physical quantities ℓ and Δv:

$$\tau_{\mathrm{eddy}} \approx \frac{\ell}{\Delta v}.$$

In order for dissipation of energy by the eddies to dominate, we require $\tau_{\mathrm{eddy}} \ll \tau_{\mathrm{drag}}$, that is, energy is much more quickly dissipated by the eddies than by friction. Comparing the two expressions, we find

$$\frac{\ell}{\Delta v} \ll \frac{\ell^2\rho}{\mu} \quad \Longrightarrow \quad \frac{\ell\rho\Delta v}{\mu} = \mathrm{Re} \gg 1.$$

4. By assumption, the rate of energy dissipation ϵ is the same for all eddies, so the reasoning in part (1) gives $\epsilon \approx (\Delta v_\lambda)^3/\lambda$. Rearranging, we have $\Delta v_\lambda \approx (\epsilon\lambda)^{1/3}$. We now set $\mathrm{Re}_\lambda = 1$ and solve for the minimum eddy size λ_{min}:

$$1 = \mathrm{Re}_\lambda = \frac{\lambda\rho\Delta v_\lambda}{\mu} \approx \frac{\lambda^{4/3}\epsilon^{1/3}\rho}{\mu} \quad \Longrightarrow \quad \lambda_{\mathrm{min}} \approx \left(\frac{\mu}{\epsilon^{1/3}\rho}\right)^{3/4} = \left(\frac{\mu^3}{\epsilon\rho^3}\right)^{1/4}.$$

5. There is a cute shortcut here. First, the previous question tells us how Re_λ scales with λ:

$$\mathrm{Re}_\lambda \approx \frac{\epsilon^{1/3}\rho\lambda^{4/3}}{\mu} = \alpha\lambda^{4/3},$$

where α is a constant independent of λ. But the Reynolds number is simply the eddy Reynolds number for $\lambda = \ell$, $\mathrm{Re} = \mathrm{Re}_\ell$, and the eddy Reynolds number is unity for the smallest eddies. Hence,

$$\mathrm{Re}_{\lambda_{\mathrm{min}}} \approx \alpha\lambda_{\mathrm{min}}^{4/3} = 1, \quad \mathrm{Re} \approx \alpha\ell^{4/3} \quad \Longrightarrow \quad \lambda_{\mathrm{min}}^{4/3} \approx \frac{\ell^{4/3}}{\mathrm{Re}}.$$

For our turbulent coffee, $\ell \approx 10\,\mathrm{cm}$ and $\mathrm{Re} \approx 10^4$, so we estimate a minimum eddy size

$$\lambda_{\mathrm{min}} \approx \frac{\ell}{\mathrm{Re}^{3/4}} \approx \frac{10\,\mathrm{cm}}{10^3} = 0.1\,\mathrm{mm}.$$

9 The Casimir effect

Problem. Suppose we stretch a string of length L between two fixed points. The string can oscillate sinusoidally in *harmonics*, the first few of which are sketched on the left below. Remarkably, by considering that harmonics of *space itself*, we can show that empty vacuum likes to push metal plates together!

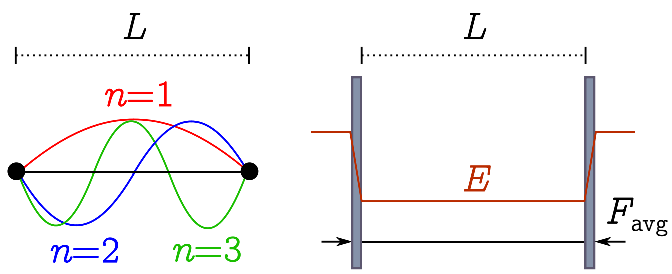


Figure 10: *Left.* Harmonics of a classical string. *Right.* Casimir effect on plates in a vacuum.

1. Show that harmonics on the string have wavelength

$$\lambda_n = \frac{L}{2n}, \quad n = 1, 2, 3, \ldots.$$

2. A classical string can vibrate with some combination of harmonics, including *no harmonics* when the string is at rest. In this case, the string has no energy. A *quantum* string is a little different: even if a harmonic is not active, there is an associated *zero-point energy*:

$$E_n^0 = \frac{\alpha}{\lambda_n},$$

where α is a constant of proportionality. This is related to *Heisenberg's uncertainty principle*, which states that we cannot know both the position and momentum of the string with absolute certainty. Let's calculate the zero-point energy of a quantum string.

Sum up the zero-point energies for each harmonic to find the energy of an unexcited quantum string. Use the infamous result[22] that

$$1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}.$$

3. Classical strings can be found everywhere, but where do we find quantum strings? One answer is *space itself*. Instead of stretching a string between anchors, set two lead plates a distance L apart. (Treat space as one-dimensional for simplicity.) The harmonics are no longer wobbling modes of the string, but *electromagnetic waves*. Outside the plates is empty space, stretching away infinitely; it has zero energy.[23]

[22] There are various ways of showing this; a popular treatment is given in this Numberphile video.

[23] We can model the edge of space with lead plates infinitely far away. Since $L \to \infty$, $E_n^0 \to 0$ and the energy does indeed disappear.

Suppose that the lead plates have thickness ℓ. Show that the plates are pushed together, with each subject to an average force

$$F_{\text{avg}} = \frac{\alpha}{6\ell L}.$$

For completeness (although you won't need it), the constant is

$$\alpha = hc,$$

where $h \approx 6.6 \times 10^{-34}\,\mathrm{J\,s^{-1}}$ is *Planck's constant* and $c \approx 3 \times 10^{8}\,\mathrm{m\,s^{-1}}$ is the speed of light. The remarkable fact that the vacuum can exert pressure on parallel metal plates is called the *Casimir effect*. It can be experimentally detected!

Solution.

1. From the picture, we see that λ is an allowed wavelength if L is a multiple of $\lambda/2$. More precisely,
$$L = \frac{n\lambda}{2} \quad\Longrightarrow\quad \lambda_n = \frac{L}{2n}.$$
2. The total rest energy of the quantum string is
$$E^0 = \frac{\alpha}{\lambda_1} + \frac{\alpha}{\lambda_2} + \frac{\alpha}{\lambda_3} + \cdots = \frac{2\alpha}{L}(1 + 2 + 3 + \cdots) = -\frac{2\alpha}{L} \cdot \frac{1}{12} = -\frac{\alpha}{6L}.$$
3. A jump in energy ΔE in energy over a distance Δx leads to an average force
$$F_{\text{avg}} = -\frac{\Delta E}{\Delta x}.$$
In this case, the distance over which the energy drops is the thickness of the plates, $\Delta x = \ell$, while the change in energy (as we move into the area between plates) is
$$\Delta E = E_{\text{plates}} - E_{\text{vacuum}} = E_{\text{plates}} = \frac{\alpha}{6L},$$
since the energy for the electromagnetic waves between plates takes the same form as harmonics in the stretched string. Thus, the average force on each plate is
$$F_{\text{avg}} = -\frac{E^0}{\ell} = \frac{\alpha}{6\ell L}.$$
This is positive, hence directed *towards* the region between plates. This means the plates are squeezed together!

10 Hard drives and black holes

Problem. Black holes are perhaps the most mysterious objects in the universe. For one, things fall in and never come out again. An apparently featureless black hole could conceal an elephant, the works of Shakespeare, or even another universe! Suppose we wanted to describe all the possible objects that could have fallen into the black hole, but using *binary digits* (bits) 0 and 1, the language of computers. With one bit, we can describe two things, corresponding to 0 and 1; with two bits, we can describe *four* things, corresponding to $00, 01, 10, 11$. Continuing this pattern, with n bits we can describe 2^n things, corresponding to the 2^n sequences of n binary digits. The total number of bits needed to describe all the possibilities, for a given black hole, is called the *entropy* S. Since information is also stored in bits, we can (loosely) equate entropy and information!

We would expect that a large black hole can conceal more than a small black hole, and will therefore have a larger entropy. The *area law*, discovered by Stephen Hawking and Jacob Bekenstein, shows that this is true, with the entropy of the black hole proportional to its *surface area* A:

$$S = \frac{A}{A_0},$$

where $A_0 \approx 10^{-69}\,\mathrm{m}^2$ is a basic unit of area. We can view the black hole surface as a sort of screen, made up of binary pixels of area A_0.

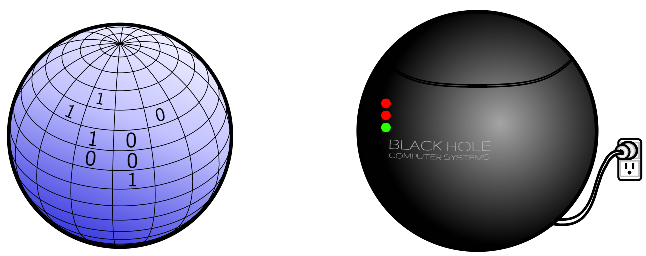


Figure 11: *Left*. The area law, viewed as pixels on the black hole surface. *Right*. A spherical hard drive.

The *Second Law of Thermodynamics* states that the total entropy of a closed system always increases.[24] Combining the area law and the Second Law leads to a surprising conclusion: black holes have the *highest* entropy density of any object in the universe. They are the best hard drives around![25]

1. To get a sense of scale, calculate how many gigabytes of entropy can be stored in a black hole of radius $1\,\mathrm{mm}$. Note that

$$1\,\mathrm{GB} = 10^9\,\mathrm{B} = 8 \times 10^9\,\mathrm{bits}.$$

[24] The entropy of a black hole is the number of bits needed to describe all the things that could have fallen in. The entropy of an ordinary object, like a box of gas, is the number of bits needed to describe all the different *microscopic* configurations which are indistinguishable to a macroscopic experimentalist, i.e. which look like the same box of gas. The function of entropy, in both cases, is to count the number of configurations which look the same!

[25] At least when it comes to information storage density. *Extracting* information is much harder!

Compare this to the total data storage on all the computers in the world, which is roughly

$$3 \times 10^9 \,\text{GB}.$$

2. Consider a sphere of ordinary matter of surface area A. Suppose this sphere has more entropy than a black hole,

$$S' > S_{\text{BH}} = \frac{A}{A_0}.$$

Argue that this violates the Second Law. You may assume that as soon as a system of area A reaches the mass M_A of the corresponding black hole, it immediately collapses to form said black hole.

3. Calculate the optimal information density in a spherical hard drive of radius r.

4. Suppose that the speed at which operations can be performed in a hard drive is proportional to the density of information storage. (This is reasonable, since data which is spread out takes more time to bring together for computations.) Explain why huge (spherical) computers are necessarily slow.

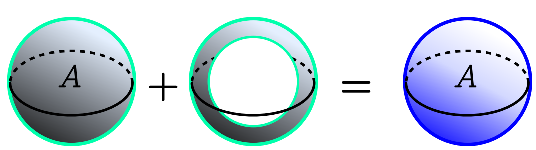
Solution.

1. We calculate the entropy from the area law, and convert the answer from bits to GB:

$$\begin{aligned} S &= \frac{A}{A_0} \,\text{bits} \\ &\approx \frac{4\pi(0.01)^2 \,\text{m}^2}{10^{-69}\,\text{m}^2} \,\text{bits} \\ &\approx 1.25 \times 10^{66} \,\text{bits} \\ &\approx \frac{1.25 \times 10^{65}}{8 \times 10^9} \,\text{GB} \approx 1.6 \times 10^{55} \,\text{GB}. \end{aligned}$$

A tiny black hole contains more information than all the world's computers, by an unimaginably large factor $\sim 10^{46}$!

2. First, note that the mass of the sphere M must be smaller than the mass of the corresponding black hole M_A, otherwise it would have already collapsed! We can therefore add a spherical shell of matter, mass $M_A - M$, causing the spherical object to form a new black hole. Schematically, we are performing the following "sum":



The shell of matter has its own entropy S'', so the total entropy of system before collapse is larger than the black hole entropy:

$$S' + S'' > S' > S_{\text{BH}}.$$

However, after the collapse, the entropy is just the black hole entropy S_{BH}. So we seem to have reduced the total entropy! This violates the Second Law of Thermodynamics. Our assumption, that $S' > S_{\mathrm{BH}}$, must have been incorrect. We learn that black holes are the best spherical hard drives in existence!

3. Black holes have maximum entropy density. Using the area law, the entropy density of a black hole of radius r is
$$\frac{S}{V} = \frac{4\pi r^2}{A_0 4\pi r^3/r} = \frac{3}{A_0 r}.$$

4. The previous result shows that, as a spherical hard drive gets large, the *maximum* information density gets very low. Since this is a maximum, density and hence processing speed is low in *any* large hard drive.