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# Scalar-tensor inflation in the Jordan frame

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**Abstract**

Inflation is a burst of rapid expansion in the early universe. It is the favoured explanation for the uniformity and flatness of space at large scales, and the existence of inhomogeneous structure at small scales. The force driving inflation is unknown, but the simplest possibility is a scalar field. The Standard Model has a unique scalar field, and hence, a unique inflaton candidate: the Higgs boson.

Inflation from a Higgs boson minimally coupled to gravity is not compatible with observation. To obtain viable inflation, we can instead couple the Higgs non-minimally. Non-minimal couplings tie the curvature of spacetime to the value of the Higgs field, violating the equivalence principle and complicating the gravitational dynamics. Usually, these complications are moved from the gravity sector to the scalar sector by a local rescaling of the theory. The result is a different scalar field governed by general relativity. The original scalar-tensor theory is called the *Jordan frame*, and the rescaled theory the *Einstein frame*.

For some aspects of inflation, the Jordan and Einstein frame are not equivalent since the scalar fields are different. Inspired by the success of Higgs-based models but eschewing rescaling, we develop a new framework for scalar-tensor inflation in the Jordan frame. This includes classical aspects needed for large-scale structure, and quantum corrections needed for small-scale inhomogeneities. We finish by exploring some toy models of inflation, and discuss preliminary efforts to build a more realistic model.

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Last but not least, I am grateful to the University of Melbourne, the School of Physics, and the Centre of Excellence for Particle Physics at the Terascale for financial and organisational support during my studies.

## *Statement of Contribution*

Chapters 2, 3 and §4.3 are an original review of the literature; §3.3 contains some original calculations and new ways of presenting the material. Chapters 4 and 5 (excepting §4.3) comprise original work by the author.

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Chapter 1

# *Introduction*

Zooming out far enough, the universe is *homogeneous* and *isotropic*—there are no preferred locations or directions in space. Philosophically, this is a satisfying extension of the Copernican principle to cosmic scales. Physically, this has testable consequences like the Hubble recession law

$$\text{recession speed} \propto \text{distance},$$

and forms the starting point for *cosmology*, the quantitative treatment of space and matter at large scales. Our evidence for homogeneity and isotropy comes from looking at the sky. Most spectacularly, in 1964, blackbody radiation at $T \approx 2.7$ K was discovered streaming in isotropically from all directions [37]. Since it peaks in the microwave, this radiation was christened the Cosmic Microwave Background or CMB (Fig. 1.1). It is an afterimage of the oldest light in the universe as it decouples from the cosmic soup.

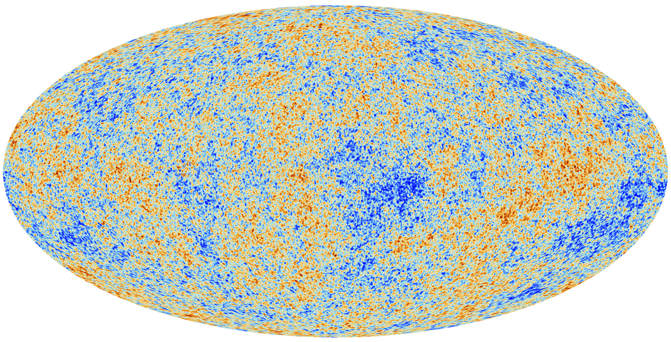


Figure 1.1: The CMB [2]. Mean temperature $T \approx 2.7$ K, fluctuations $\Delta T/T \sim 10^{-4}$.

Despite its successes, there are problems with this basic picture. First of all, why is the universe uniform? The CMB, for instance, splits into $\sim 10^4$ patches which have never interacted but are almost at the same temperature. Something of a conspiracy—*fine-tuning* in technical parlance—is needed to explain why. This is called the *horizon problem*. The *flatness problem* is a similar apparent fine-tuning. Observations show that the universe is almost flat (that is, Euclidean) at each fixed moment of cosmological time. Add a little more matter to the universe and it curls into a ball; take some away, and it deforms into a saddle. We appear to live in the tiny (and unstable) part of cosmic parameter space corresponding to extreme flatness.

In the 1980s, cosmologists came up with an elegant fix for these problems: add a phase of rapid growth called *inflation* just after the Big Bang ([3], [18], [28], [32], [39], [40]). This lets patches of the CMB speak at early times, and flattens out the universe, solving both tuning problems at once. The energy for inflation is supplied by a field called the *inflaton*. Treating the inflaton quantum-mechanically leads to little lumps of quantum uncertainty on the uniform background. These later grow into *classical* lumps like stars, galaxies, anisotropies in the CMB (Fig. 1.1) and other interesting departures from uniformity. So, although inflation was designed to explain large-scale regularity, it includes small-scale irregularity as a quantum side effect. Chapter 2 provides more detail about the basic Big Bang model, its shortcomings, and how inflation solves them.

Inflation is a powerful and elegant paradigm, but is saddled with the inflaton field. The term "inflaton" is little more than a placeholder for a particle we think exists, but have no way of observing directly. The simplest possibility for the inflaton is a *scalar* field, with no spin, charge, or internal structure. There is a unique scalar field in the Standard Model, and hence a unique inflaton candidate among known particles: the *Higgs boson*. The Higgs was predicted in 1964 ([15], [17], [19]) and discovered at the Large Hadron Collider in 2012 ([1], [11]). It gives mass to the fermions and weak force carriers of the Standard Model via the *Higgs mechanism*.

Can we simply plonk the Higgs into the inflaton field and get realistic inflation? A simple analysis (performed in §2.4) shows that, if general relativity holds, the universe we see today *cannot* be inflated by a Standard Model Higgs. One could abandon the idea of Higgs as inflaton; a sneaky alternative is to tweak gravity. The basic strategy is to modify general relativity at early times by coupling the Higgs *directly* or *non-minimally* to spacetime curvature. This is called a *scalar-tensor* theory of gravity; such theories pop up elsewhere[1] with non-cosmological motivations. A viable model of Higgs inflation along these lines was proposed by Bezrukov and Shaposhnikov [7]. In their model, the Higgs mechanism also provides a graceful transition from scalar-tensor to normal gravity when inflation ends. Scalar-tensor gravity and Higgs inflation are outlined in Chapter 3.

Mathematically speaking, general relativity is formidable set of coupled, nonlinear equations; by throwing another field into the mix, scalar-tensor gravity is even harder. The scalar field acts as local warping factor, expanding and contracting space via the non-minimal coupling. To make life easier, it is conventional to transfer the local warping of space into a local warping of the inflaton via a *conformal transformation*; this restores general relativity at the cost of distorting the scalar field. The theory with locally warped space is called the *Jordan frame*, and with the locally warped scalar the *Einstein frame*. There is a large literature (see [16]) on the physical equivalence of the two frames, but a key point is that scalar fields in different frames are *different* though closely related. For this reason, classical and quantum regimes of physical interest in one frame may not translate (easily or at all) into the other. We discuss these subtleties in Chapter 3.

In contrast to conventional Higgs inflation, this thesis takes the Jordan frame as physically basic. Moreover, we remain agnostic about the identity of the inflaton. With these two provisos in mind, we develop a new, perturbative framework for scalar-tensor inflation in the Jordan frame. Both classical (spatially homogeneous) and quantum (inhomogeneous) aspects are discussed in Chapter 4. In Chapter 5, we apply this framework to some toy models of inflation. More work is required to produce observationally viable scalar-tensor inflation, and in §5.4 we outline preliminary efforts in this direction. Our results indicate that the Jordan frame approach has interesting phenomenological possibilities, and gives a new avenue into the earliest and most dramatic chapter of the cosmic narrative.

1. For instance, QFT on a curved background [8].

Chapter 2

# *Cosmology and inflation*

## 2.1 The Big Bang model

We start with the rudiments of the Big Bang model, namely, the FRW metric and Friedmann equations. We assume some familiarity with general relativity.

### The FRW metric

At cosmological scales, we assume that the universe is homogenous and isotropic. Imposing these large-scale symmetries yields a family of metrics jointly referred to as the *Friedmann-Robertson-Walker (FRW) metric*. Writing the spatial part of the metric both in general coordinates $\gamma_{ij}$ and more conventional polar coordinates,

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}dx^i dx^j = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right], \tag{2.1}$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$. Here, $a(t)$ is a time-dependent *scale factor* capturing the expansion of space, also called the *Hubble flow*. The parameter $k \in \{\pm 1, 0\}$ represents the spatial curvature, which we discuss in the next paragraph. Using rescaling arguments, we can set $a(t_{\text{now}}) \equiv 1$, where $t_{\text{now}}$ is the present time. By convention, at $t = 0$ we set $a = 0$ to represent the Big Bang singularity.[1]

At fixed cosmological time $t$, we slice out a hypersurface $\Sigma_t$ of constant curvature. There are three such geometries up to rescaling: $k = +1$ corresponds to a 3-sphere (uniform positive curvature), $k = 0$ corresponds to Euclidean 3-space (uniformly flat), and $k = -1$ corresponds to hyperbolic 3-space (uniform negative curvature). These are also called *closed*, *flat*, and *open* universes respectively. Note that $k$ is constant.

It will prove useful below to define new time and radial coordinates,

$$d\tau \equiv \frac{dt}{a(t)}, \quad d\chi \equiv \frac{dr}{\sqrt{1-kr^2}}. \tag{2.2}$$

The parameter $\tau$ is *conformal time*. With these changes, the FRW metric becomes

$$ds^2 = a(t)^2\left[-d\tau^2 + d\chi^2 + S_k^2(\chi)d\Omega^2\right], \tag{2.3}$$

with $S_{+1}(\chi) \equiv \sin\chi$, $S_{-1}(\chi) \equiv \sinh\chi$ and $S_0(\chi) \equiv \chi$. From (2.1), the Christoffel symbols are

$$\Gamma^0_{ij} = a\dot{a}\gamma_{ij}, \quad \Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^i_j, \quad \Gamma^i_{jk} = \frac{1}{2}\gamma^{il}(\partial_j\gamma_{kl} + \partial_k\gamma_{jl} - \partial_l\gamma_{jk}). \tag{2.4}$$

1. You may wonder if this is necessary. In fact, under relatively weak assumptions, a classical singularity is guaranteed to exist by the Hawking-Penrose singularity theorems [41].

The Ricci tensor and scalar can then be calculated:

$$\mathcal{R}_{00} = -3\frac{\ddot{a}}{a}, \quad \mathcal{R}_{ij} = \left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2}\right] g_{ij}, \quad \mathcal{R} = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right]. \tag{2.5}$$

Hence, the nonzero components of the Einstein tensor $\mathcal{G}_{\mu\nu} \equiv \mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu}$ are

$$\mathcal{G}_{00} = 3\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right], \quad \mathcal{G}_{ij} = -\left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right] g_{ij}. \tag{2.6}$$

We will need these below.

### Einstein and Friedmann equations

Now we consider the large-scale matter content of the universe. The most general homogeneous, isotropic stress-energy tensor is a *perfect fluid* described by

$$\mathcal{T}_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}, \tag{2.7}$$

where $u^\mu = \hat{e}_{(0)}$ is the 4-velocity of the fluid, $\rho(t)$ is the *energy density* and $P(t)$ is the *pressure.*[2] Using Einstein's field equations $\mathcal{G}_{\mu\nu} = 8\pi G\mathcal{T}_{\mu\nu}$ and the trivial identities $\mathcal{T}_{00} = \rho$, $\mathcal{T}^\mu{}_\mu = \rho - 3P$, and $\mathcal{G}^\mu{}_\mu = -\mathcal{R}$, we obtain the *Friedmann equations*:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P). \tag{2.8}$$

If the ratio $w \equiv P/\rho$ is constant, it defines an *equation of state.*[3] For a single fluid with constant $w$, the solution of equations (2.8) for $k = 0$ is $\rho \propto a^{-3(1+w)}$.

Define the *Hubble parameter* $H(t)$ and the *conformal Hubble parameter* $\mathcal{H}(t)$ by

$$H(t) \equiv \frac{\dot{a}}{a}, \quad \mathcal{H}(t) \equiv \frac{a'}{a}. \tag{2.9}$$

Here and elsewhere, $f' \equiv \partial_\tau f$ indicates the conformal time derivative. Using the relation $\dot{H} + H^2 = \ddot{a}/a$ and defining the *reduced Planck mass*

$$M_{\rm pl} \equiv 1/\sqrt{8\pi G} \simeq 2.4 \times 10^{18}\,\text{GeV}, \tag{2.10}$$

we can rewrite the Friedmann equations in a more apposite form:

$$H^2 = \frac{\rho}{3M_{\rm pl}^2} - \frac{k}{a^2} \tag{2.11}$$

$$\dot{H} + H^2 = -\frac{1}{6M_{\rm pl}^2}(\rho + 3P). \tag{2.12}$$

The first Friedmann equation relates the Hubble parameter to the energy density and spatial curvature. For $k = 0$ and a single fluid with constant $w \neq -1$, we get

$$H \propto a^{-3(1+w)/2} \quad \Longrightarrow \quad a(t) \propto t^{2/3(1+w)}. \tag{2.13}$$

We will also need the *critical energy density* $\rho_{\rm crit}(t)$ and *density parameter* $\Omega(t)$:

$$\rho_{\rm crit} \equiv 3M_{\rm pl}^2 H^2, \quad \Omega(t) \equiv \frac{\rho}{\rho_{\rm crit}}. \tag{2.14}$$

2. Isotropy eliminates shear terms and homogeneity eliminates spatial dependence.
3. From now on, "constant $w$" means "constant equation of state $w = P/\rho$".

## 2.2 Tuning problems and inflation

The Big Bang model of §2.1 does a good job of describing large-scale dynamics. However, as we now show, it suffers from tuning problems and serious explanatory gaps. We will see how these problems are cured by an initial period of rapid cosmic growth.

### Horizon and Flatness Problems

From (2.11) and (2.14), curvature and density are related by $k = (aH)^2(\Omega - 1)$. Although $k$ is constant, $\Omega(t)$ *is* a function of time. The *flatness problem* is simply that $\Omega(t_{\text{now}})$ is observed to be very close to 1, so the universe is nearly flat (see [13]). For a fluid with constant $w$, (2.13) gives

$$\frac{d\Omega}{d\ln a} = (1 + 3w)\Omega(\Omega - 1). \tag{2.15}$$

For familiar fluids such as dust ($w = 0$) and radiation ($w = 1/3$), $1 + 3w > 0$. This is called the *strong energy condition* (SEC). For a universe dominated by such a fluid, $\Omega = 1$ is an *unstable* fixed point of cosmological evolution. Thus, the early universe must be extremely flat to explain current observations, leading to a fine-tuning problem.

The *horizon problem* is a conflict between large-scale uniformity and causality. To look at causality, consider the light cone of an event at time $t$, traced out by outgoing and incoming photons. Thanks to isotropy, we can focus on light rays moving in the radial direction. Thus,

$$ds^2 = a^2(d\chi^2 - d\tau^2) = 0,$$

and the width of the light cone stretching back to the big bang (the *particle horizon*) is

$$\Delta\chi(t) = \Delta\tau(t) = \int_0^t \frac{dt}{a} = \int_{-\infty}^{\ln a(t)} \frac{d\ln a}{aH} \equiv \int_{-\infty}^{\ln a(t)} r_H\, dN, \tag{2.16}$$

where $r_H \equiv (aH)^{-1}$ is the *comoving Hubble radius* and $dN \equiv d\ln a = H\, dt$ counts the logarithmic change in scale factor, or number of *e-folds*, $N$. For a fluid with constant $w$ satisfying the SEC, $r_H = H_0^{-1}a^{(1+3w)/2}$ and hence

$$\Delta\chi(t) = H_0^{-1}\int_{-\infty}^{\ln a(t)} e^{N(1+3w)/2}\, dN = \frac{2}{1+3w} r_H(t) \propto t^{(1+3w)/(3+3w)}. \tag{2.17}$$

This increases with time and leads to disjoint particle horizons. In other words, the uniformity of the universe at later times (exemplified by the near-isotropic CMB) requires extreme fine-tuning of the patches which have never interacted.

### Introducing Inflation

The simplest way to solve the horizon problem is to postulate an early phase of *inflation*. By definition, this is a period of decreasing $r_H$, or equivalently, accelerating expansion:

$$\dot{r}_H = -\frac{\ddot{a}}{\dot{a}^2} < 0 \quad \Longleftrightarrow \quad \ddot{a} > 0. \tag{2.18}$$

This lets patches talk early on. For a single fluid with constant $w$, we can easily implement inflation by breaking the SEC. As a welcome byproduct, we see from (2.15) that breaking the SEC turns $\Omega = 1$ into a *stable* fixed point and therefore solves the flatness problem!

Another way of writing $\dot{r}_H < 0$ is

$$\frac{d}{dt}(aH)^{-1} = -\left(\frac{1}{a} + \frac{\dot{H}}{aH^2}\right) \equiv -\frac{1}{a}(1-\epsilon) < 0, \tag{2.19}$$

where $\epsilon$ is the fractional change in $H$ per $e$-fold,

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = -\frac{d\ln H}{dN}. \tag{2.20}$$

Since inflation occurs for $\epsilon < 1$, the phase is bookended by boundary conditions $\epsilon(t_i) = \epsilon(t_f) = 1$ and $\epsilon < 1$ in between. A minimal constraint on the length of inflation is that the observable universe fit into the Hubble sphere at $t_i$ so the patches can indeed interact; this requires at least $\sim 60$ $e$-folds.[4] We can relate the length of inflation to the fractional change in $\epsilon$ per $e$-fold,

$$\eta \equiv \frac{d\ln\epsilon}{d\ln N} = \frac{\dot{\epsilon}}{H\epsilon}. \tag{2.21}$$

Together, $\epsilon$ and $\eta$ are called the *Hubble slow-roll parameters* [27].

### Single-field slow-roll approximation

To create inflation, the simplest option is to use a single real scalar field $\phi$ called the *inflaton*.[5] We start with the classical picture of what happens. From classical field theory, the inflaton stress-energy is

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\right]. \tag{2.22}$$

For a spatially uniform field ($\nabla\phi = \mathbf{0}$), we find energy and pressure

$$\rho_\phi = T^0{}_0 = \frac{1}{2}\dot{\phi}^2 + V, \quad P_\phi = -\frac{1}{3}T^i{}_i = \frac{1}{2}\dot{\phi}^2 - V. \tag{2.23}$$

Differentiating (2.11), and substituting $\rho_\phi, P_\phi$ into (2.12),

$$\dot{\phi}\ddot{\phi} + V'\dot{\phi} = 6M_{\rm pl}^2\dot{H}H, \quad \dot{H} = -\frac{1}{2M_{\rm pl}^2}(P_\phi + \rho_\phi) = -\frac{1}{2M_{\rm pl}^2}\dot{\phi}^2. \tag{2.24}$$

Inserting the second into the first and dividing by $\dot{\phi}$, we obtain a *Klein-Gordon equation*

$$\Box\phi + V' = \ddot{\phi} + 3H\dot{\phi} + V' = 0, \tag{2.25}$$

where the first equality follows from the FRW Christoffel symbols (2.4).

Using (2.11) and (2.24), we can relate $\epsilon$ to the inflaton field $\phi$ via

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{\dot{\phi}^2}{2M_{\rm pl}^2H^2} = \frac{3\dot{\phi}^2}{2\rho_\phi}. \tag{2.26}$$

4. For a nice back-of-the-envelope calculation, see [5].

5. In fact, a scalar inflaton arises *generically* from the effective theory of inflation. When we spontaneously break the time-translation symmetry of a pure de Sitter phase, the inflaton is the associated Goldstone boson. For more details, see [4].

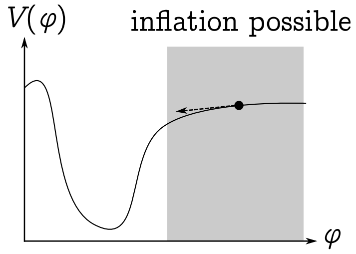



Figure 2.1: Slow roll on a flat patch.

Hence, $\epsilon < 1$ means the kinetic energy is less than the potential energy. For this to remain true during the course of inflation, the acceleration per Hubble time must be small, so

$$\delta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}}, \quad |\delta| < 1. \tag{2.27}$$

We can simplify the equations of motion by making the *slow roll approximation* $\epsilon, |\delta| \ll 1$. This simplies the Klein-Gordon equation to a single *Hubble friction* term:

$$3H\dot{\phi} \simeq -V'. \tag{2.28}$$

See Fig 2.1 for a cartoon. Using the slow-roll approximation, it follows that

$$1 \gg \epsilon = \frac{3\dot{\phi}^2}{2\rho_\phi} \simeq \frac{1}{2}M_{\rm pl}^2\left(\frac{V'}{V}\right)^2 \equiv \epsilon_{\rm P} \tag{2.29}$$

$$1 \gg |\delta + \epsilon| = \left|\frac{H\ddot{\phi} + \dot{H}\dot{\phi}}{H^2\dot{\phi}}\right| \simeq M_{\rm pl}^2\frac{|V''|}{V} \equiv |\eta_{\rm P}|. \tag{2.30}$$

The *potential slow-roll parameters* $\epsilon_{\rm P}$ and $\eta_{\rm P}$ are defined exactly in terms of the inflaton potential $V$. For future reference, $\eta_{\rm P}$ is related to the Hubble slow-roll parameters by

$$\eta = \frac{\dot{\epsilon}}{H\epsilon} = -4\left(\frac{\dot{H}}{H^2}\right) - 2\left(\frac{H\ddot{\phi} + \dot{H}\dot{\phi}}{H^2\dot{\phi}}\right) \approx 4\epsilon - 2\eta_{\rm P}. \tag{2.31}$$

We distinguish the *Hubble slow-roll approximation*, which assumes $\epsilon$ and $\eta$ are small but is agnostic about the source of inflation, from the *potential slow-roll approximation*, which requires small $\epsilon_{\rm P}$ and $\eta_{\rm P}$ and assumes single-field slow-roll. See [27] for further discussion.

We list some expressions for $N$ during single-field inflation. Using (2.26),

$$N = \int_{t_i}^{t_f} H\, dt = \int_{\phi_f}^{\phi_i} d\phi\, \frac{H}{\dot{\phi}} = \frac{1}{M_{\rm pl}}\int_{\phi_f}^{\phi_i} \frac{d\phi}{\sqrt{2\epsilon}}. \tag{2.32}$$

The last expression will prove particularly convenient in §2.4.

## 2.3 Quantum corrections and classical spectra

The FRW metric and uniform inflaton field receive perturbations from various sources.[6] If we quantise these perturbations, something remarkable happens: we get a mechanism which explains the origin and hierarchy of large-scale structure in the late universe. We only give a brief precis here; more details can be found in [5], [13], or [31]. We will do expository penance in §4.3.

6. The fact that these corrections can be treated perturbatively is a fine-tuning problem for inflation!

**Quantum fields in de Sitter space**

The full field $\phi(t, \mathbf{x})$ consists of a uniform, space-averaged background $\bar{\phi}(t)$ (which we previously denoted by $\phi$) and an inhomogeneous correction:

$$\phi(t, \mathbf{x}) \equiv \bar{\phi}(t) + \delta\phi(t, \mathbf{x}) \equiv \bar{\phi}(t) + \frac{\vartheta(t, \mathbf{x})}{a(t)}. \tag{2.33}$$

The *comoving perturbation* $\vartheta$ is the more convenient variable for quantisation, since it is going with the (Hubble) flow.

The spatial Fourier components of the perturbation are denoted $\delta\phi_{\mathbf{k}}$ or $\vartheta_{\mathbf{k}}$. Expanding the perturbed action out to second order and setting the variation to zero, we obtain the *Mukhanov-Sasaki equation*:

$$\vartheta''_{\mathbf{k}} + \left(k^2 - \frac{a''}{a}\right)\vartheta_{\mathbf{k}} = 0. \tag{2.34}$$

To first order, we can treat space as pure de Sitter ($H$ constant) rather than quasi-de Sitter ($H$ slowly changing), in which case $a''/a \simeq 2/\tau^2$ at early times. Writing a Lagrangian which reproduces the equation of motion (2.34), Legendre transforming, and canonically quantising the Hamiltonian leads to the mode expansion

$$\hat{\vartheta}(\tau, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} \left[\vartheta_k(\tau)\hat{a}_{\mathbf{k}} + \vartheta_k^*(\tau)\hat{a}^\dagger_{\mathbf{k}}\right]. \tag{2.35}$$

Here, $\vartheta_k, \vartheta_k^*$ are independent classical solutions of (2.34), while $\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}}$ are ladder operators for perturbation quanta. Provided we normalise the $\vartheta_k, \vartheta_k^*$ appropriately, the ladder operators satisfy the usual commutation relations

$$[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = \delta^{(3)}(\mathbf{k} + \mathbf{k}').$$

The expectation $\langle\hat{\vartheta}\rangle$ vanishes since the ladder operators kill the vacuum on either side. A short calculation shows that the quantum uncertainty as a function of $\tau$ is

$$\langle|\hat{\vartheta}|^2\rangle = \int \frac{d^3k}{(2\pi)^3}|\vartheta_k(\tau)|^2 = \int d\ln k \frac{k^3}{2\pi^2}|\vartheta_k(\tau)|^2 \equiv \int d\ln k\, \Delta^2_\vartheta(k, \tau), \tag{2.36}$$

where we have defined the *power spectrum* $\Delta^2_\vartheta(k, \tau) \equiv (k^3/2\pi^2)|\vartheta_k(\tau)|^2$.

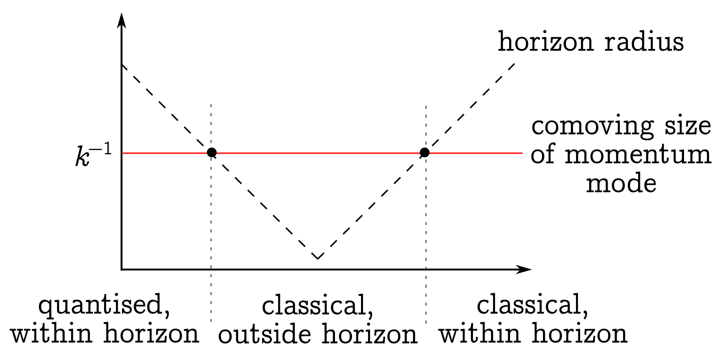



Figure 2.2: Inflation and comoving scale.

Recall that all this is happening inside the comoving Hubble horizon. When modes grow large enough, they exit the horizon and quantisation breaks down (Fig. 2.2). Luckily, we can transfer the fluctuations in $\vartheta$ (or $\delta\phi$) to a quantity conserved on superhorizon scales: the *comoving curvature perturbation* $\zeta$. This sources the curvature of 3-slices

$$\nabla^2\zeta \equiv \frac{a^2}{4}{}^{(3)}\mathcal{R} \tag{2.37}$$

and is related to $\delta\phi$ in *spatially flat* gauge (see [13] or [5]) by $\zeta = -(aH/\bar{\phi}')\delta\phi$. Hence,

$$\langle|\zeta_k|^2\rangle = \left(\frac{H}{\bar{\phi}'}\right)^2 \langle|\vartheta_k|^2\rangle. \tag{2.38}$$

The conservation of $\zeta$ on superhorizon scales was first proved by Weinberg [42].

### Spectral indices

At horizon crossing, $k = aH$, and an analysis of the classical solutions to Mukhanov-Sasaki shows that $\Delta_{\delta\phi}(aH) \approx (H/2\pi)^2|_{k=aH}$. Thus, using (2.26) and (2.38), we get

$$\Delta^2_\zeta(k) \equiv \left(\frac{aH}{\bar{\phi}'}\right)^2 \Delta_{\delta\phi}(k)^2\bigg|_{k=aH} = \left(\frac{H}{\dot{\bar{\phi}}}\right)^2 \Delta_{\delta\phi}(k)^2\bigg|_{k=aH} = \frac{1}{8\pi^2\epsilon}\frac{H^2}{M_{\rm pl}^2}\bigg|_{k=aH}. \tag{2.39}$$

A *scale-invariant* power spectrum $\Delta^2_\zeta$ is independent of $k$. Near a reference scale or *pivot* $k_*$, we expect close to scale-invariant power law dependence of the form:

$$\Delta^2_\zeta(k) \equiv A_s\left(\frac{k}{k_*}\right)^{n_s-1}. \tag{2.40}$$

I'll sketch a proof of near scale-invariance below. Here, $n_s \equiv 1 + d\ln\Delta^2_\zeta/d\ln k$ is the *scalar spectral index*. For $k_* = 0.05\ \mathrm{Mpc}^{-1}$, measurements of the CMB [2] show that

$$A_s = (2.142 \pm 0.049) \times 10^{-9}, \quad n_s = 0.967 \pm 0.004. \tag{2.41}$$

A similar analysis can be performed for *tensor perturbations*, that is, gravitational waves, in terms of the two polarisation modes $+, \times$. This yields a power-law spectrum $\Delta_t(k)$ for primordial gravitational waves:

$$\Delta^2_t(k) = \frac{2}{\pi^2}\frac{H^2}{M_{\rm pl}^2}\bigg|_{k=aH} \equiv A_t\left(\frac{k}{k_*}\right)^{n_t}. \tag{2.42}$$

For historical reasons, the tensor spectral index $n_t$ is defined without the offest. The ratio of spectral amplitudes is called the *tensor-to-scalar ratio*. We can approximate it using (2.40) and (2.42). We also quote an upper bound [2] from non-observation of primordial gravitational waves (see below for more detail):

$$r \equiv \frac{A_t}{A_s} \approx 16\epsilon, \quad r \lesssim 0.113. \tag{2.43}$$

Finally, let's see why power law behaviour is expected and how the spectral indices encode inflationary parameters. Using (2.19), (2.21), and $\ln k = N + \ln H$ at horizon crossing, to first-order in the Hubble slow-roll parameters we have

$$\begin{aligned} n_s - 1 &= \frac{d\ln\Delta_\zeta^2}{dN}\frac{dN}{d\ln k} \\ &= \left(2\frac{d\ln H}{dN} - \frac{d\ln\epsilon}{dN}\right)\left(1 + \frac{d\ln H}{dN}\right)^{-1} \\ &= (-2\epsilon - \eta)(1+\epsilon) + \mathcal{O}(\epsilon^2) \\ &= -2\epsilon - \eta + \mathcal{O}(\epsilon^2). \end{aligned} \tag{2.44}$$

The same argument (omitting $\eta$) shows that $n_t = -2\epsilon + \mathcal{O}(\epsilon^2)$. Note that $r \simeq -8n_t$; this is a nice consistency check for any putative model of inflation. We conclude that near scale-invariant spectra result for small, roughly constant $\epsilon$ and $\eta$.

### Observations

A stochastic background of gravitational waves is one of the most robust predictions of inflation, with the amplitude tied directly to the energy scale of inflation. In fact, from (2.11) we see that $H^2 \sim V/M_{\text{pl}}^2$ during slow-roll, and we can rewrite $A_t$ in terms of $A_s$ as

$$V^{1/4} \sim (rA_s)^{1/4}M_{\text{pl}} \sim \left(\frac{r}{0.01}\right)^{1/4} \cdot 10^{16}\,\text{GeV}. \tag{2.45}$$

There are two possible observational signatures: $B$-mode polarisation of the CMB and direct detection of primordial gravitational waves. Neither has been observed.

The CMB is polarised by electron-photon collisions just before the photons decouple. The polarisation can be split into irrotational $E$-modes and incompressible $B$-modes. Scalar perturbations create $E$-modes, while tensor perturbations create both modes, so the detection of $B$-modes would be a "smoking gun" for inflation. Current instruments (such as Keck-BICEP3) can measure $r \gtrsim 0.01$, so in order to detect $B$-modes, (2.45) requires an energy scale $\gtrsim 10^{16}$ GeV. This is ignoring the rather considerable dust foreground, which tripped up BICEP2 in 2014. For direct detection at a frequency of $f = 0.1$ Hz, an upper bound on the strain signal is $h \sim 5 \cdot 10^{-25}$ [38]. This is just beyond the reach of the next generation of gravitational wave detectors [30].

There is a less famous "smoking gun" for inflation which *has* been observed: the phase coherence of Fourier modes for the scalar perturbation. Observationally, this follows from well-defined peaks and troughs in the CMB angular power spectrum, and indicates that some primordial mechanism (like inflation) coordinated the phases. Moreover, the measured cross-correlation between temperature and polarisation anisotropies rules out many alternative explanations. See [14] for a very readable account.

## 2.4 A first pass at Higgs inflation

Let's apply our results for single-field slow-roll to the Standard Model Higgs and see what goes wrong. The Higgs potential, introduced at somewhat greater length in §3.2, is

$$V(\phi) = \frac{1}{4}\lambda(\phi^2 - v^2)^2 \equiv \frac{1}{4}\lambda(\Delta\phi^2)^2$$

where $v \simeq 246\,\text{GeV}$, $\lambda \simeq 0.13$, and $m_H = \sqrt{2\lambda}v \simeq 126\,\text{GeV}$. The potential is pictured below. There are two regions that look promising for slow-roll: on top of the hill at $\phi = 0$

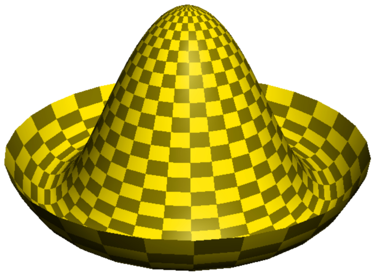


Figure 2.3: The famous "Mexican hat" potential.

and the large field region $\phi \gg v$. We first calculate the potential slow-roll parameters:

$$\epsilon_{\text{P}} = \frac{1}{2}M_{\text{pl}}^2\left(\frac{V'}{V}\right)^2 = 8M_{\text{pl}}^2\left(\frac{\phi}{\Delta\phi^2}\right)^2, \quad \eta_{\text{P}} = M_{\text{pl}}^2\frac{V''}{V} = 4M_{\text{pl}}^2\frac{3\phi^2 - v^2}{(\Delta\phi^2)^2}.$$

Near $\phi = 0$, $v \gg \phi$ and hence

$$\epsilon_{\text{P}} \simeq 8M_{\text{pl}}^2\left(\frac{\phi}{v^2}\right)^2, \quad \eta_{\text{P}} \simeq \frac{4M_{\text{pl}}^2}{v^2}.$$

Since $M_{\text{pl}} \sim 10^{18}$ GeV $\gg v$, $\eta_{\text{P}}$ is huge near the hill and slow-roll is impossible.

On the outer reaches of the brim, $\phi \gg v$ and hence

$$\epsilon_{\text{P}} \simeq \frac{8M_{\text{pl}}^2}{\phi^2} = \frac{2}{3}\cdot\frac{12M_{\text{pl}}^2}{\phi^2} \simeq \frac{2}{3}\eta_{\text{P}}.$$

We will get slow-roll inflation for *trans-Planckian* field values $\phi \gtrsim M_{\text{pl}}$. The number of $e$-folds $N(\phi_i)$ from some initial field value $\phi_i$ to the end of inflation $\phi_f$ is

$$N(\phi_i) = -\int_{\phi_i}^{\phi_f}\frac{d\phi}{\sqrt{2M_{\text{pl}}^2\epsilon_{\text{P}}}} = -\frac{1}{4M_{\text{pl}}^2}\int_{\phi_i}^{\phi_f} d\phi\,\phi \simeq \frac{\phi_i^2}{8M_{\text{pl}}^2},$$

where we have used (2.26), (2.32), $\epsilon \simeq \epsilon_{\text{P}}$, and assumed $\phi_i \gg \phi_f$. Imposing $N(\phi_i) \simeq 60$ to get the requisite number of $e$-folds, we find that $\phi_i = \sqrt{8N}M_{\text{pl}} \simeq 22M_{\text{pl}}$. Using (2.39), our expression for $\epsilon_{\text{P}}$, and the Friedmann equation $H^2 \simeq V/3M_{\text{pl}}^2 \simeq \lambda\phi^4/12M_{\text{pl}}^2$, the amplitude of scalar fluctuations is

$$A_s \sim \frac{1}{8\pi^2\epsilon}\frac{H^2}{M_{\text{pl}}^2} \simeq \frac{\lambda\phi_i^6}{768\pi^2M_{\text{pl}}^6} \simeq \frac{\lambda 8N^3}{12\pi^2} \sim 10^3.$$

We have exploited the weak $k$-dependence to ignore the horizon crossing condition.

These fluctuations are considerably more violent than the observed amplitude $A_s \sim 10^{-9}$. This means that naive Higgs inflation is observationally unviable. In terms of the Higgs parameters, we need to dial down the self-coupling constant to $\lambda \sim 10^{-13}$, or keeping $v$ fixed, the mass to $m_H \sim 10^{-4}\,\text{GeV}$. It will evidently take more work to identify the Standard Model Higgs with the inflaton.

Chapter 3

# *Modified gravity and Higgs inflation*

## 3.1 Scalar-tensor gravity

We've just seen that the Standard Model Higgs doesn't inflate the universe correctly—the bulge of the Mexican hat isn't flat enough to support slow roll, and the brim isn't flat enough to match the observed amplitude of scalar fluctuations.

We could give up on the Higgs and look for other inflaton candidates. Another strategy is to try coupling the Higgs *directly* to the gravitational background. From the Higgs mechanism, we can dynamically recover general relativity when inflation ends, forming a bridge from modified gravity during inflation to the standard Big Bang model of cosmology. Theories with direct scalar-gravity couplings come under the umbrella of *scalar-tensor gravity*, which we now introduce. For a comprehensive review, see [16].

### The Palatini identity

We can derive the equations of motion for scalar-tensor gravity from an action principle, but to do so, we require the *Palatini identity*. At a given point in spacetime, choose normal coordinates so that the Christoffel symbols (but not their derivatives) vanish. Then the Ricci tensor is $\mathcal{R}_{\mu\nu} = 2\partial_{[\rho}\Gamma^{\rho}_{\nu]\mu}$, and hence, we have the Palatini identity:

$$\delta\mathcal{R}_{\mu\nu} = 2\partial_{[\rho}\delta\Gamma^{\rho}_{\nu]\mu} = 2\nabla_{[\rho}\delta\Gamma^{\rho}_{\nu]\mu}. \tag{3.1}$$

We can view $\delta\Gamma^{\alpha}_{\alpha\nu}$ as an infinitesimal difference of two Christoffel symbols, hence a tensor (see [41] for a proof). It follows that (3.1) is a tensor equation independent of coordinates.

Using $\delta(g_{\mu\nu}g^{\mu\nu}) = 0$, the identity[1] $g^{-1}\delta g = g^{\alpha\beta}\delta g_{\alpha\beta}$, and the chain rule, we find the variational derivative of $\sqrt{-g}$:

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g = -\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta}. \tag{3.2}$$

Now combining (3.1), (3.2) and $\nabla g = 0$,

$$\delta(\sqrt{-g}\mathcal{R}) = 2\nabla_{[\rho}\sqrt{-g}g^{\mu\nu}\delta\Gamma^{\rho}_{\nu]\mu} + \sqrt{-g}\delta g^{\mu\nu}\mathcal{G}_{\mu\nu}. \tag{3.3}$$

Similarly, the variation of the Christoffel symbol is

$$\delta\Gamma^{\sigma}_{\mu\nu} = -\frac{1}{2}\big[g_{\lambda\mu}\nabla_{\nu}(\delta g^{\lambda\sigma}) + g_{\lambda\nu}\nabla_{\mu}(\delta g^{\lambda\sigma}) - g_{\mu\alpha}g_{\nu\beta}\nabla^{\sigma}(\delta g^{\alpha\beta})\big]. \tag{3.4}$$

Applying (3.4) to (3.1), contracting with $g^{\mu\nu}$ and using metric compatibility,

$$g^{\mu\nu}\delta\mathcal{R}_{\mu\nu} = \nabla_{\sigma}\big[g_{\mu\nu}\nabla^{\sigma}(\delta g^{\mu\nu}) - \nabla_{\rho}(\delta g^{\sigma\rho})\big]. \tag{3.5}$$

We will use (3.3) and (3.5) to vary the scalar-tensor action below.

1. This follows from the matrix identity $\ln(\det A) = \mathrm{tr}(\ln A)$ with $A = (g_{\mu\nu})$. To prove this, just choose an upper triangular basis for $A$.

### Direct scalar-tensor couplings

Einstein's equations can be derived from the *Einstein-Hilbert action*:

$$S_{\rm EH} \equiv \int d^4x\sqrt{-g}\,\frac{1}{2}M_{\rm pl}^2\mathcal{R}. \tag{3.6}$$

Varying with respect to $g^{\mu\nu}$ gives Einstein's equation in a vacuum, $\mathcal{G}_{\mu\nu} = 0$; adding the matter action $S_{\rm M}$ and varying gives the full Einstein equations $M_{\rm pl}^2\mathcal{G}_{\mu\nu} = \mathcal{T}_{\mu\nu}$.

To directly couple a scalar $\phi$ to gravity, we just replace the constant $M_{\rm pl}^2/2$ in (3.6) with a function $f(\phi)$. We also need the action for $\phi$. Thus, we are led to define:

$$S_{f\mathcal{R}} \equiv \int d^4x\sqrt{-g}f(\phi)\mathcal{R}. \tag{3.7}$$

$$S_\phi \equiv \int d^4x\sqrt{-g}\Big[-\frac{1}{2}\omega(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\Big]. \tag{3.8}$$

The total action $S_{\rm ST} \equiv S_{fR} + S_\phi + S_{\rm M}$ gives a *scalar-tensor theory*. Using (3.3), (3.5), and integration by parts:

$$\delta S_{f\mathcal{R}} = \int d^4x\sqrt{-g}\Big[f'\mathcal{R}\delta\phi + \Big(f\mathcal{G}_{\mu\nu} + g_{\mu\nu}\Box f - \nabla_\mu\nabla_\nu f\Big)\delta g^{\mu\nu}\Big]. \tag{3.9}$$

In this section, primes denote derivatives with respect to $\phi$. Now for $S_\phi$:

$$\delta S_\phi = -\int d^4x\sqrt{-g}\Big[\frac{1}{2}\mathcal{T}^{(\phi)}_{\mu\nu}\delta g^{\mu\nu} + \Big(-\frac{1}{2}g^{\mu\nu}\omega'\partial_\mu\phi\partial_\nu\phi - \omega\Box\phi + V'\Big)\delta\phi\Big]. \tag{3.10}$$

As usual, for component $I$ of the action we define

$$\mathcal{T}^{(I)}_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_I}{\delta g^{\mu\nu}} \tag{3.11}$$

and hence

$$\mathcal{T}^{(\phi)}_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} = \omega\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\omega\partial^\lambda\phi\partial_\lambda\phi - Vg_{\mu\nu}.$$

For simplicity, we assume $\phi$ does not appear in $S_{\rm M}$. Applying Euler-Lagrange to $S$, (3.9) and (3.10) yield scalar-tensor analogues for Einstein's equations:

$$f\mathcal{G}_{\mu\nu} + g_{\mu\nu}\Box f - \nabla_\mu\nabla_\nu f - \frac{1}{2}\mathcal{T}_{\mu\nu} = 0 \tag{3.12}$$

$$f'\mathcal{R} + \frac{1}{2}\omega'(\partial\phi)^2 + \omega\Box\phi - V' = 0, \tag{3.13}$$

where $\mathcal{T}_{\mu\nu} \equiv \mathcal{T}^{(\phi)}_{\mu\nu} + \mathcal{T}^{(\rm M)}_{\mu\nu}$. Setting $f(\phi) = M_{\rm pl}^2/2$ in (3.12), we indeed recover Einstein's equations. For well-behaved (monotonic) $\omega$, we can perform a field redefinition, setting $\omega = 1$ to produce a canonical kinetic term. We assume this is always possible, and henceforth deal with the simpler equations:

$$f\mathcal{G}_{\mu\nu} + g_{\mu\nu}\Box f - \nabla_\mu\nabla_\nu f - \frac{1}{2}\mathcal{T}_{\mu\nu} = 0 \tag{3.14}$$

$$\Box\phi + f'\mathcal{R} - V' = 0. \tag{3.15}$$

So far, we have not mentioned the cosmological constant $\Lambda$. There are two reasonable choices of coefficient: $\sqrt{-g}f(\phi)$ or $\sqrt{-g}$. Both can be easily incorporated into $S_\phi$ as terms in the scalar potential $V(\phi)$. In §5.2, we will use it to mean the latter.

### Scalar-tensor inflation

To see what inflation looks like in scalar-tensor gravity, we substitute the FRW metric (2.1) into our equations and use (2.4)–(2.6). It is worth remembering that $g_{\mu\nu}$ still just describes the geometry of spacetime, which or may not have certain symmetries; the difference from general relativity only shows up when we solve for the matter content.

We set $k = 0$ since we still expect inflationary flattening. We also assume $\phi$ is isotropic and dominates the stress energy, letting us ignore $S_{\mathrm{M}}$. From the chain rule,

$$\nabla_\mu \nabla_\nu f = f'' \partial_\mu \phi \partial_\nu \phi + f' \nabla_\mu \nabla_\nu \phi, \tag{3.16}$$

and using isotropy,

$$(\partial\phi)^2 = -\dot{\phi}^2, \quad \nabla_\mu \nabla_\nu \phi = \partial_\mu \partial_\nu \phi - \Gamma^0_{\mu\nu}\dot{\phi}. \tag{3.17}$$

From (3.15) and $\mathcal{R} = 6(\dot{H} + 2H^2)$, we obtain the Klein-Gordon equation

$$\Box\phi - V' + 6f'(\dot{H} + 2H^2) = 0. \tag{3.18}$$

We can take the $(0, 0)$ and $(i, j)$ component of (3.14), and simplify with (3.16) and (3.17). We can also simplify the $(i, j)$ component using the $(0, 0)$ component. These manipulations yield the scalar-tensor Friedmann equations governing inflation:

$$6(H^2 f + H\dot{f}) = \frac{1}{2}\dot{\phi}^2 + V \tag{3.19}$$

$$H\dot{f} - 2f\dot{H} - \ddot{f} = \frac{1}{2}\dot{\phi}^2. \tag{3.20}$$

### Dynamical gravity

During slow-roll inflation, $\phi$ trickles down the potential curve, corresponding to small fractional change in $H$. Typically, inflation ends when $\phi$ falls into a potential well, with oscillations around the minimum allowing energy to be transferred from the inflaton into Standard Model degrees of freedom. This phenomenon is called *reheating*, and it is beyond our current scope, though it will make a brief appearance in §5.3). For a pedagogical introduction to reheating, see [4].

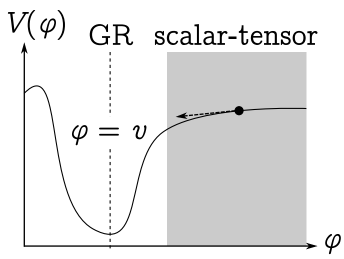



Figure 3.1: Dynamically induced general relativity.

In scalar-tensor gravity, falling into the minimum plays another role: spontaneously inducing general relativity (Fig. 3.1). More precisely, if $\phi \to v$, scalar-tensor gravity dynamically induces general relativity provided

$$f(v) = \frac{1}{2}M_{\mathrm{pl}}^2.$$

We will treat this as a phenomenological constraint in subsequent chapters. Note that $v$ may not be a minimum of $V(\phi)$; see §3.3 for more discussion of how $v$ is determined.

## 3.2 Einstein frame Higgs inflation

With scalar-tensor gravity at our disposal, we can interpret $\phi$ as the Standard Model Higgs boson and implement a new strategy for Higgs inflation.

### The Higgs boson in brief

Let's take a short break from gravity and talk about particle physics. Recall the electroweak sector of the standard model, with SU(2) gauge symmetry for the left-handed fermions and weak hypercharge $Y$. The combined gauge group is

$$\mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y.$$

To get gauge-invariant mass terms for the fermions and gauge bosons $W^\pm$ and $Z$, we introduce a complex *Higgs doublet*, with gauge charges and Lagrangian as follows:

$$\Phi = \begin{bmatrix} \varphi^+ \\ \varphi^0 \end{bmatrix} \sim (2,1), \quad \mathcal{L}(\Phi) \equiv |D_\mu \Phi|^2 - V(\Phi), \quad V(\Phi) \equiv \lambda(|\Phi|^2 - v^2)^2. \tag{3.21}$$

The potential has a degenerate set of minima with $|\Phi| = v$, and hence, in the ground state (or vacuum) the field has a nonzero *vacuum expectation value* (vev). Using a U(2) gauge transformation, we can put the doublet in the form

$$\Phi' = \begin{bmatrix} 0 \\ (v+\phi)/\sqrt{2} \end{bmatrix}, \quad \langle 0|\Phi'|0\rangle = \begin{bmatrix} 0 \\ v/\sqrt{2} \end{bmatrix},$$

where $\phi$ is a real scalar called the *Higgs boson*. The gauge transformations cleverly shifts the other 3 degrees of freedom of $\Phi$ into the longitudinal polarisations of $W^\pm$, $Z$, and thus endows them with mass. The factor of $\sqrt{2}$ in the vev leads to

$$V(\phi) = \frac{1}{4}\lambda(\phi^2 - v^2)^2 \equiv \frac{1}{4}\lambda(\Delta\phi^2)^2, \tag{3.22}$$

with minima at $\phi = \pm v$. The Higgs field parameters [35] are

$$m_H = \sqrt{2\lambda}v = 125.7 \pm 0.4\,\mathrm{GeV}, \quad v \simeq 246\,\mathrm{GeV}, \quad \lambda \simeq 0.13. \tag{3.23}$$

During inflation, we will ignore Standard Model interactions with other particles.

### Einstein frame Higgs inflation

Einstein frame Higgs inflation was proposed by Bezrukov and Shaposhnikov ([6], [7]). Using scalar-tensor gravity and identifying $\phi$ with the Standard Model Higgs, the potential is fixed as (3.22). The scalar-tensor coupling is chosen to be quadratic:

$$f(\phi) \equiv \frac{1}{2}[M_{\mathrm{pl}}^2 + \xi(\phi^2 - v^2)] \equiv \frac{1}{2}(M^2 + \xi\phi^2) \tag{3.24}$$

for a dimensionless parameter $\xi$. By design, this obeys the induced gravity constraint $f(\pm v) = M_{\mathrm{pl}}^2/2$ for the Higgs vev $v$. If $\xi v^2 \ll M_{\mathrm{pl}}^2$, we have $M \simeq M_{\mathrm{pl}}$.

To simplify the scalar-tensor theory, it is customary to perform a local scale transformation or *Weyl rescaling*, first proposed in this context by Dicke [12]:

$$g_{\mu\nu} \to \bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad \Omega^2(\phi) \equiv 2M_{\rm pl}^{-2} f(\phi). \tag{3.25}$$

Under a Weyl rescaling, the terms $f$ and $V$ are unchanged, but $\mathcal{R}$ transforms as[2]

$$\mathcal{R} = \Omega^2 \bar{\mathcal{R}} + 6\Omega\bar{\Box}\Omega - 12(\bar{\nabla}\Omega)^2. \tag{3.26}$$

The original metric is called the *Jordan frame* and the Weyl transformed metric the *Einstein frame*. (The term "frame" is conventional usage although a misnomer.) Using (3.25) and (3.26),

$$\begin{aligned} S &= \int d^4x\sqrt{-g}\left[f(\phi)\mathcal{R} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] \\ &= \int d^4x\sqrt{-\bar{g}}\,\Omega^{-4}\left[f(\phi)\Big(\Omega^2\bar{\mathcal{R}} + 6\Omega\bar{\Box}\Omega - 12(\bar{\nabla}\Omega)^2\Big) - \frac{1}{2}\Omega^2\bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] \\ &= \int d^4x\sqrt{-\bar{g}}\left[\frac{1}{2}M_{\rm pl}^2\bar{\mathcal{R}} - \frac{1}{2}\Omega^{-2}\Big(1 + 6M_{\rm pl}^2(\Omega')^2\Big)\bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \Omega^{-4}V(\phi)\right]. \end{aligned} \tag{3.27}$$

To obtain the last line, we integrated by parts and used metric compatibility.

The goal of the Weyl rescaling is the first term: the non-minimal coupling has been replaced by the Einstein-Hilbert action (3.6) of general relativity. We can make things neater by defining a new scalar field $\chi$:

$$\frac{d\chi}{d\phi} \equiv \Omega^{-2}\sqrt{\Omega^2 + \frac{3}{2}M_{\rm pl}^2[(\Omega^2)']^2} = \Omega^{-1}\sqrt{1 + 6M_{\rm pl}^2(\Omega')^2}. \tag{3.28}$$

Inserting this into (3.27), we end up with the Einstein frame action

$$S = \int d^4x\sqrt{-\bar{g}}\left[\frac{1}{2}M_{\rm pl}^2\bar{\mathcal{R}} - \frac{1}{2}\bar{g}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - U(\chi)\right], \tag{3.29}$$

where $U(\chi) \equiv \Omega^{-4}V(\phi(\chi))$. Formally, this is just the action of a scalar field in general relativity! We now tune $\xi$ and see if the new field $\chi$ can be made to viably slow-roll.

In the Einstein frame, $\Omega \simeq 1$ near $\phi = 0$, whatever value of $\xi$ we choose. Thus, we can't tweak $\xi$ to get slow-roll on top of the hat. However, we can try to fix the large-field behaviour, assuming $\xi \gg 1$. In the regime $\phi^2 \gg M_{\rm pl}^2/\xi \gg v^2$, we have $\Omega \simeq \sqrt{\xi}\phi/M_{\rm pl}$ and hence

$$\begin{aligned} \frac{d\chi}{d\phi} &= \Omega^{-1}\sqrt{1 + 6M_{\rm pl}^2(\Omega')^2} \simeq \frac{\sqrt{6}M_{\rm pl}}{\phi} \\ \implies \quad \phi(\chi) &\simeq \frac{M_{\rm pl}}{\sqrt{\xi}}\exp\left(\frac{\chi}{\sqrt{6}M_{\rm pl}}\right). \end{aligned} \tag{3.30}$$

The factor $1/\sqrt{\xi}$ parameterises $\xi\phi^2$ as exponentially larger than $M_{\rm pl}$. With this relation in hand, the Einstein frame potential is

$$U(\chi) = \Omega^{-4}V(\phi(\chi)) \simeq \frac{\lambda M_{\rm pl}^4}{4(M_{\rm pl}^2 + \xi\phi^2)^2}\phi^4 \simeq \frac{\lambda M_{\rm pl}^4}{4\xi^2}\left(1 - 2\exp\left(\frac{-2\chi}{\sqrt{6}M_{\rm pl}}\right)\right). \tag{3.31}$$

2. See [10] or [41] for a proof.

Thus, $U(\chi)$ is exponentially flattened in the large-field region, fixing the problem with naive Higgs inflation (§2.4). A somewhat gratuitous visualisation of this process is given in Fig. 3.2.

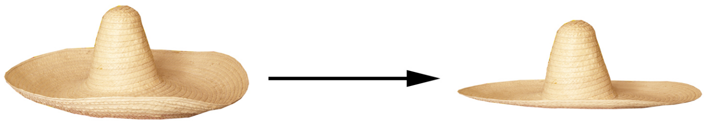


Figure 3.2: Large $\xi$ flattens the outer brim of Mexican hat.

To constrain $\xi$ more precisely, we first need the slow-roll parameters (2.29) and (2.30):

$$\epsilon_{\mathrm{P}} \equiv \frac{1}{2} M_{\mathrm{pl}}^2 \left(\frac{dU/d\chi}{U}\right)^2 \simeq \frac{4}{3} \exp\left(\frac{-4\chi}{\sqrt{6} M_{\mathrm{pl}}}\right) \tag{3.32}$$

$$\eta_{\mathrm{P}} \equiv M_{\mathrm{pl}}^2 \frac{1}{U} \frac{d^2U}{d\chi^2} \simeq -\frac{4}{3} \exp\left(\frac{-2\chi}{\sqrt{6} M_{\mathrm{pl}}}\right) = -\sqrt{\frac{4\epsilon_{\mathrm{P}}}{3}}. \tag{3.33}$$

To find the number of $e$-folds to the end of inflation, we integrate $dN = d\ln a$ between the field value bookends $\phi_i$, $\phi_f$ with $\epsilon_{\mathrm{P}}(\phi_{i,f}) \approx \epsilon(\phi_{i,f}) = 1$. The starting value will turn out to be irrelevant, and we can easily solve for the end of inflation from (3.32):

$$1 = \frac{4}{3} \exp\left(\frac{-4\chi(\phi_f)}{\sqrt{6} M_{\mathrm{pl}}}\right) = \frac{4M_{\mathrm{pl}}^4}{3\xi^2} \phi_f^{-4} \quad \Longrightarrow \quad \phi_f = \left(\frac{4}{3}\right)^{1/4} \frac{M_{\mathrm{pl}}}{\sqrt{\xi}}. \tag{3.34}$$

In the large field regime, $\epsilon_{\mathrm{P}} \ll \eta_{\mathrm{P}}$ and (2.31) implies $\eta \approx -2\eta_{\mathrm{P}}$. Hence, by (2.32), (3.32) and (3.33),

$$\begin{aligned} N &= \int_{\chi_f}^{\chi_i} \frac{d\chi}{M_{\mathrm{pl}} \sqrt{2\epsilon_{\mathrm{P}}}} \\ &= \int_{\chi_i}^{\chi_f} \frac{3}{2\sqrt{6} M_{\mathrm{pl}}} \exp\left(\frac{2\chi}{\sqrt{6} M_{\mathrm{pl}}}\right) d\chi \\ &\simeq \frac{3}{4} \exp\left(\frac{2\chi_i}{\sqrt{6} M_{\mathrm{pl}}}\right) = -\eta_{\mathrm{P}}^{-1} \end{aligned} \tag{3.35}$$

where $\chi_{i,f} \equiv \chi(\phi_{i,f})$ and we assume $\chi_i \gg \chi_f$. The coupling parameter $\xi$ will follow from the relationship between $N$ and $\eta_{\mathrm{P}}$. Plugging (3.33), (3.34) and (3.35) into the scalar amplitude (2.39),

$$A_s \simeq \frac{1}{8\pi^2 \epsilon_{\mathrm{P}}} \frac{H^2(\phi_f)}{M_{\mathrm{pl}}^2} = \frac{1}{6\pi^2 \eta_{\mathrm{P}}^2} \frac{V(\phi_f)}{3M_{\mathrm{pl}}^4} = \frac{\lambda N^2}{72\pi^2} \frac{\phi_f^4}{M_{\mathrm{pl}}^4} \approx \frac{\lambda N^2}{54\pi^2 \xi^2}. \tag{3.36}$$

Since the scalar amplitude $A_s \approx 2.142 \times 10^{-9}$ and $N \sim 60$, we rearrange to obtain

$$\xi \simeq \left[\frac{\lambda N^2}{54\pi^2 A_s}\right]^{1/2} \sim 55000\sqrt{\lambda}. \tag{3.37}$$

Similarly, from (3.35) and the fact that $\eta_{\mathrm{P}} \ll \epsilon_{\mathrm{P}}$, we obtain

$$n_s \simeq 1 - 2\eta_{\mathrm{P}} \simeq 1 - \frac{2}{N} \simeq 0.97, \quad r = 16\epsilon_{\mathrm{P}} \simeq \frac{12}{N^2} \simeq 0.003. \tag{3.38}$$

These are consistent with measurements (2.41) and (2.43).

## 3.3 Jordan and Einstein frames

The Einstein frame approach agrees with the basic phenomenology of inflation, but comes at the cost of a large coupling constant $\xi \gg 1$ and a Weyl rescaling. With large couplings, one must guard against radiative corrections, either by fine-tuning, new fields at high energies, or other tricks from the toolbox of effective field theory.

For our purposes, the more serious problem is that the Higgs field is defined in the Jordan frame, but slow-roll calculations are performed in the Einstein frame. The Einstein frame scalar $\chi$ is *not* the Higgs field $\phi$. Rather, it is a reparameterisation of $\phi$ defined rather formally to clean up terms left over from the Weyl rescaling. Assuming the Jordan frame is physical, we ought to check what Einstein frame slow-roll means for $\phi$. As a rule, authors neglect this check because they view the Einstein frame, rather than the Jordan frame, as the physical metric, and for some questions both metrics give the same answers.

#### Rigid versus wobbly units

Let's pause for a moment and consider the relation between the Jordan and Einstein frame philosophically. The Weyl transformation taking the Jordan frame metric $g_{\mu\nu}$ to the Einstein frame metric $\bar{g}_{\mu\nu}$ is neither a physical symmetry (except in the special case of *scale-invariant* theories) or a change of coordinates. In fact, Dicke [12] proposed we think of the transformation as keeping the coordinates fixed (they label events) but changing the length scale in a "wobbly", position-dependent way. This is a gauge transformation as originally envisaged by Weyl [43].

To picture what is going on, imagine spacetime as a torus equipped with a coordinate lattice (Fig. 3.3). The Jordan frame is a regular torus with uniformly spaced lattice points, but nontrivial dynamics; the Einstein frame is a deformation of the torus where the lattice is no longer uniform, but the dynamics of fields propagating on the torus simplify.

If one allows wobbly units, the Jordan and Einstein frame are indeed equivalent. But in the Einstein frame, "constants" with nonzero mass dimension will become position-dependent. The field equations may be Einstein's, but this is not general relativity as we usually interpret it; typically we use a *rigid* system of units. There are interesting physical and philosophical arguments in favour of wobbly units, but they are outside our ambit. If we stick to general relativity with rigid units, the Jordan and Einstein frames are no longer equivalent.

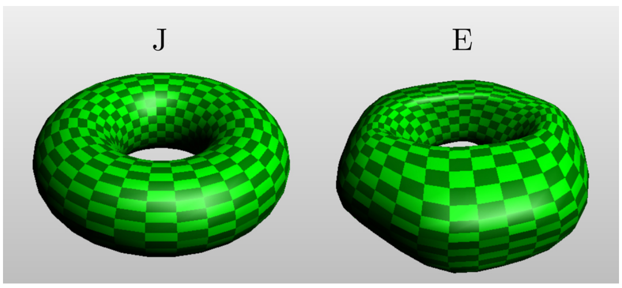



Figure 3.3: Jordan frame with rigid units or Einstein frame with wobbly units.

$$\begin{tikzcd} \mathrm{J} \arrow[rr, "\Omega"] \arrow[dr, "X_J"'] & & \mathrm{E} \arrow[dl, "X_{\mathrm{E}}"] \\ & \mathrm{J} \sim \mathrm{E} & \end{tikzcd} \qquad \begin{tikzcd} \mathrm{J} \arrow[r, "\Omega"] \arrow[d, "X_{\mathrm{J}}"'] & \mathrm{E} \arrow[d, "X_{\mathrm{E}}"] \\ \mathrm{J} & \mathrm{E} \arrow[l, "\Omega^{-1}"'] \end{tikzcd}$$

Figure 3.4: Different approaches to performing a calculation $X$.

Another problem arises when considering global causal properties. The flat FRW metric $g_{\mu\nu} = a^2\eta_{\mu\nu}$ with conformal time (2.1) only differs from Minkowski space by a conformal rescaling, but interpreted in the usual way, they are physically very different. FRW has particle horizons, for instance, while Minkowski does not.

So far, we have only considered problems at the classical level. If we start trying to do *particle physics* with our scalars, we expect mass scales, the precise form of the potential, etc, to have important ramifications for the governing effective field theory. It is not clear that the quantum behaviour of $\phi$ and $\chi$ should be regarded as interchangeable, even if we do embrace wobbly units. Since the spectrum is generated by quantum effects, this is a strong reason to be cautious about identifying frames.

### Relation between frames

If the frames are not equivalent, we have two issues to address. First of all, which is the physical metric? There are situations where the energy density of $\phi$ in the Jordan frame is not positive definite. This leads to unphysical consequences like negative energy gravitational radiation from stellar binaries and kinetic ghosts when quantising [16]. Prima facie, these do not appear to be issues during inflation. In fact, as previously advertised, our working hypothesis will be that the Jordan frame is physical, since the scalar field (Higgs or otherwise) is originally defined there.

The second issue is how to exploit the Einstein frame as a calculational device. One approach is to determine which questions are conformally invariant, and can therefore be calculated in the Einstein frame. This corresponds to the first diagram in Fig. 3.4.The existence of conformal invariants is a much weaker notion than the equivalence of the Jordan and Einstein frame. A less ambitious approach is to use a translation dictionary to go back and forth between frames. In other words, we take a problem $X_J$ in the Jordan frame, translate it into an Einstein frame problem $X_{\mathrm{E}}$, solve it, then translate the answer back into the Jordan frame. This should give the same result as doing the calculation $X_{\mathrm{J}}$ in the Jordan frame. This equivalence is captured by the second diagram in Fig. 3.4.

### Einstein frame slow-roll in the Jordan frame

Having argued that the Jordan and Einstein frame are not equivalent, and that it is sensible to take the Jordan frame as physical, we should see what Einstein frame slow-roll looks like in the Jordan frame.

First, we generalise the dictionary set up in §3.2 for Higgs inflation. Writing the

Einstein frame (barred) quantities explicitly in terms of $f$,

$$g_{\mu\nu} \to \bar{g}_{\mu\nu} \equiv \Omega^2 g_{\mu\nu} \tag{3.39}$$

$$\phi \to \chi(\phi) \equiv \int^{\phi} \frac{d\chi}{d\phi'}\, d\phi' \tag{3.40}$$

$$V(\phi) \to U(\chi) \equiv \Omega^{-4} V(\phi(\chi)) \tag{3.41}$$

where

$$\Omega^2(\phi) \equiv 2M_{\rm pl}^{-2} f(\phi), \quad \frac{d\chi}{d\phi} \equiv \Omega^{-1}\left[1 + 3f^{-1}(f')^2\right]^{1/2} \equiv \Omega^{-1} G. \tag{3.42}$$

Since distances are scaled by $\Omega$, in the Einstein frame

$$a \to \bar{a} = \Omega a, \quad dt \to d\bar{t} = \Omega\, dt. \tag{3.43}$$

Note that we have opted to attach $\Omega$ to the scale factor rather than the comoving coordinates $x^i$. Overdots will indicate derivatives with respect to $t$ (Jordan frame) not $\bar{t}$. Let $\bar{H}$ denote the Hubble parameter in the Einstein frame, where general relativity reigns. Then

$$\bar{H} = \frac{d \ln \bar{a}}{d\bar{t}} = \frac{d \ln(\Omega a)}{\Omega\, dt} = \frac{\dot{\Omega}}{\Omega^2} + \frac{H}{\Omega} = \frac{1}{\Omega}\left[H + \left(\frac{f'}{f}\right)\dot{\phi}\right]. \tag{3.44}$$

We can now check what the Einstein frame (potential) slow-roll regime looks like in the Jordan frame. Setting $k = 0$ and $\rho \simeq U \gg (d\chi/d\bar{t})^2$, (2.11) and (2.28) give

$$\bar{H}^2 = \frac{U}{3M_{\rm pl}^2}, \quad 3\bar{H}\left(\frac{d\chi}{d\bar{t}}\right) = -\frac{dU}{d\chi}. \tag{3.45}$$

Slow-roll inflation also requires the slow-roll parameters $\epsilon \ll 1$, $|\delta| \ll 1$. Referring back to §2.2, this implies

$$\left(\frac{d\chi}{d\bar{t}}\right)^2 \ll U, \quad \frac{d^2\chi}{d\bar{t}^2} \ll \bar{H}\left(\frac{d\chi}{d\bar{t}}\right). \tag{3.46}$$

In the Jordan frame, the first condition is equivalent to

$$V \gg \Omega^4 \left(\frac{d\chi}{d\bar{t}}\right)^2 = G^2 \dot{\phi}^2. \tag{3.47}$$

For $f > 0$, $G \geq 1$ and hence (3.47) implies $V \gg \dot{\phi}^2$, a classical slow-roll constraint.

The second condition in (3.46) is more complicated. The LHS can be written

$$\frac{d^2\chi}{d\bar{t}^2} = \frac{1}{\Omega}\frac{d}{dt}\left(\frac{\dot{\chi}}{\Omega}\right) = \frac{1}{\Omega}\frac{d}{dt}\left(\frac{G}{\Omega^2}\dot{\phi}\right) = \left[\frac{\ddot{\phi}}{\dot{\phi}} + \frac{9f'}{2G^2}\frac{2ff'' - (f')^2}{f^2} - \frac{f'}{f}\right]\frac{G}{\Omega^3}\dot{\phi}. \tag{3.48}$$

Using (3.44), the RHS of the second condition in (3.46) is

$$\bar{H}\left(\frac{d\chi}{d\bar{t}}\right) = \left[H + \left(\frac{f'}{f}\right)\dot{\phi}\right]\frac{G}{\Omega^3}\dot{\phi}. \tag{3.49}$$

Diving both sides by $\Omega^{-3}G$ and tidying up, we get

$$\ddot{\phi} + \frac{9\dot{f}}{2G^2}\frac{2ff'' - (f')^2}{f^2} - \frac{\dot{f}}{f} \ll H\dot{\phi} + \left(\frac{\dot{f}}{f}\right)\dot{\phi}. \tag{3.50}$$

In general, this complicated condition does not correspond to slow roll in the Jordan frame. For instance, taking the Higgs inflation coupling (3.24), the second term on the LHS of (3.50) is a mess:

$$\frac{9\dot{f}}{2G^2}\frac{2ff'' - (f')^2}{f^2} = \frac{18\xi^2\phi\dot{\phi}}{M^2 + \xi\phi^2}\left[\frac{M^2 - \xi\phi^2}{M^2 + \xi(1+6\xi)\phi^2}\right].$$

We would have to work hard to make something sensible of this. As we might have guessed, slow roll is not a parameterisation-invariant notion.

Following the first approach in Fig. 3.4, we could look for combinations of $f$ and $V$ such that Einstein frame slow-roll conditions translate into Jordan frame slow-roll. Instead, we will explore the broader range of possibilities that appear if we perform our analysis in the Jordan frame from the outset.

### Minima and induced gravity

We finish by discussing the induced gravity constraint in the Jordan frame. In scalar-tensor gravity, the long-term behaviour of $\phi$ involves both the potential $V(\phi)$ and coupling $f(\phi)$. These are packaged together into the Einstein frame potential, so it is easier to look for minima of $U(\chi)$, noting that if $\chi \to u$, then $\phi(\chi) \to \phi(u)$.[3] To minimise $U$, we require $\partial_\chi U = 0$. Using (3.41) and (3.42), this implies

$$\begin{aligned} 0 = \frac{dU}{d\chi} = \frac{d\phi}{d\chi}\left(\frac{V}{\Omega^2}\right)' &= \frac{\Omega M_{\text{pl}}^4}{4G}\left(\frac{V}{f^2}\right)' \\ &= \frac{M_{\text{pl}}^3}{2\sqrt{2}Gf^{5/2}}\big(fV' - 2f'V\big) \\ \implies \quad fV' &= 2Vf'. \end{aligned} \tag{3.51}$$

Although we do not need both sides of (3.51) to vanish, this is the simplest way to ensure the identity holds. Setting $f = 0$ can lead to singular behaviour in $U$, so instead we want

$$V'(v) = 0, \quad \text{and} \quad V(v) = 0 \text{ or } f'(v) = 0. \tag{3.52}$$

For instance, values of $\phi$ which minimise $V$ and maximise $f$ will also minimise $U$. If $\phi \to v$ as inflation ends, then general relativity is spontaneously induced provided

$$f(v) = \frac{1}{2}M_{\text{pl}}^2, \tag{3.53}$$

as discussed in Chapter 3. Together, (3.52) and (3.53) form model-building constraints we will exploit in the following chapters.

3. In a given model, it is possible that $\phi$ *never* settles down to a fixed value $v$. However, the Einstein frame condition seems like a reasonable *necessary*, if not sufficient, condition.

Chapter 4

# *Jordan frame inflation*

## 4.1 De Sitter inflation

If the Jordan frame is physical, we need a different approach from §3.2. Using equations (3.18), (3.19) and (3.20), we can set up inflationary cosmology in the Jordan frame. The equations are hard, so we do what physicists usually do when confronted with overwhelming mathematical odds: try perturbation theory.

By definition, inflation involves a slowly varying Hubble constant. Since constant $H$ describes de Sitter space, inflation is often called a *quasi-de Sitter* phase of expansion. This motivates the use of *exact* de Sitter evolution as the basis for our perturbation theory. In this section, we discuss constraints on $f$ and $V$ from pure de Sitter evolution, and leave first-order corrections to §4.2.

### A useful inequality

For constant $H$, the Friedmann equations (3.19)–(3.20) reduce to

$$6(H^2 f + H\dot{f}) = \frac{1}{2}\dot{\phi}^2 + V \tag{4.1}$$

$$H\dot{f} = \frac{1}{2}\dot{\phi}^2 + \ddot{f}. \tag{4.2}$$

The Ricci scalar and Hubble parameter satisfy $\mathcal{R} = 12H^2 \geq 0$. Together with (3.18), (4.1) and (4.2) yield

$$\begin{aligned} 0 &= [12f'' + 5]\dot{\phi}^2 - 36Hf'\dot{\phi} + [\mathcal{R}f - 2V + 12(f')^2\mathcal{R} - 12f'V'] &(4.3)\\ &\equiv A\dot{\phi}^2 + B\dot{\phi} + C. &(4.4)\end{aligned}$$

This is a quadratic equation in $\dot{\phi}$, with solutions

$$\dot{\phi}_{\pm} = \frac{1}{2A}\Big[ - B \pm \sqrt{B^2 - 4AC}\Big]. \tag{4.5}$$

The solutions are real provided the discriminant $\Delta \equiv B^2 - 4AC \geq 0$. There are two branches, $\dot{\phi}_{\pm}$, which we can think of as distinct functions of $\phi$. For analytic work, it is useful to isolate the $V$ dependence, with $\Delta \geq 0$ becoming

$$\mathcal{V} \equiv \big(V + 6f'V'\big)\big(12f'' + 5\big) \geq \frac{1}{2}\mathcal{R}\Big[\big(f + 12(f')^2\big)\big(12f'' + 5\big) - 27(f')^2\Big] \equiv \mathcal{R}F. \tag{4.6}$$

In words, we require the modified potential $\mathcal{V}$, depending on $f$ and $V$, to lie above a curve $\mathcal{R}F$ which depends on $f$ and $H$. We call equation (4.6), or equivalently $\Delta \geq 0$, the

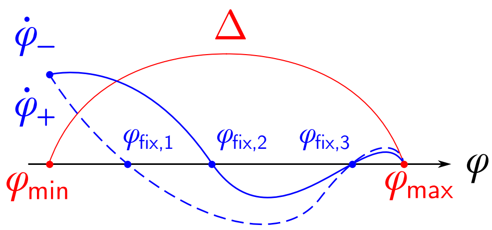



Figure 4.1: An example. The de Sitter consistent region $\Delta > 0$ (red) lies between $\phi_{\rm min}$ and $\phi_{\rm max}$. Both branches $\dot{\phi}_\pm$ are shown (blue). There are three roots of $C$, $\phi_{{\rm fix},i}$ for $i = 1, 2, 3$, with $B(\phi_{\rm fix,1}) > 0$, $B(\phi_{\rm fix,2}) < 0$ and $B(\phi_{\rm fix,3}) = 0$.

*de Sitter consistency constraint*, since it tells us when de Sitter evolution is consistent. It governs the values $\phi$ can assume during de Sitter inflation, with regions of de Sitter consistency satisfying $\Delta(\phi) \geq 0$, and bounded by points where $\Delta = 0$.

### Consistency

How can we guarantee de Sitter consistent evolution for all $t$? The most stringent condition is $\Delta(\phi) \geq 0$ for all field values $\phi$. We call this *global de Sitter consistency*, since $\dot{\phi}$ will be well-defined whatever $\phi$ does. A weaker notion of consistency involves finding de Sitter consistent *regions*, where $\Delta \geq 0$, and fixed points which control the local dynamics of $\phi$. We will obtain consistent evolution if $\phi$ starts in one of these regions and never leaves it. Let's examine this possibility in a bit more detail.

Suppose a consistent region has lower and upper boundaries $\phi_{\rm min}$ and $\phi_{\rm max}$, satisfying $\Delta(\phi_{\rm max}) = \Delta(\phi_{\rm min}) = 0$ and $\Delta \geq 0$ in between. If the field $\phi$ approaches either boundary, it will leave the region (and consistency will break down) unless there is a *fixed point* to prevent it leaking out:

$$\dot{\phi}_\pm \Big|_{\phi=\phi_{\rm min/max}} = -\frac{B}{2A} = \frac{18Hf'}{12f''+5} = 0. \tag{4.7}$$

There may also be fixed points on the *interior* of a consistent region. From (4.4), these will occur at roots of

$$C = \mathcal{R}f - 2V + 12(f')^2\mathcal{R} - 12f'V' = \dot{\phi}_+\dot{\phi}_-. \tag{4.8}$$

If $\phi_{\rm fix}$ is a root of $C$, from (4.5) its fixed point properties depend on the sign of $B(\phi_{\rm fix})$: if $B(\phi_{\rm fix}) = 0$, $\phi_{\rm fix}$ is a fixed point of for both branches, while if $B(\phi_{\rm fix})$ is positive (negative), $\phi_{\rm fix}$ is a fixed point for the positive (negative) branch. Since $B = -36Hf'$ with $H$ constant, the sign of $B$ depends on whether $f$ is increasing ($B < 0$), decreasing ($B > 0$), or stationary ($B = 0$). Finally, the stability of fixed points depends on the behaviour of $C$ in the neighbourhood of $\phi_{\rm fix}$; this require more information about $C$ to determine. We provide a cartoon example in Fig. 4.1.

For inflationary model-building, we note that de Sitter consistent evolution for *all* time is unnecessary. All we really need is consistency over the course of inflation. To implement this, we start $\phi$ in a consistent patch and simply ensure (in general, numerically) that it doesn't leak out of the patch before inflation ends. To see how inflation actually arises, however, we need to go to the next order in perturbation theory.

## 4.2 Perturbing de Sitter

We now consider perturbative corrections to de Sitter evolution. We expand $H$ and $\phi$ as

$$H = \sum_{n=0}^{\infty} \lambda^n H_n, \quad \phi = \sum_{n=0}^{\infty} \lambda^n \phi_n, \tag{4.9}$$

where $H_0$ is constant, and $\lambda$ is a formal variable to distinguish order in perturbation theory. For later convenience, we write down the first few terms in the perturbation expansions of $\mathcal{R}$ and $\Box\phi$. To second order, the Ricci scalar is

$$\begin{aligned}\mathcal{R} &= 6(\dot{H} + 2H^2) \equiv \sum_{n=0}^{\infty} \lambda^n \mathcal{R}_n \\ &= 12H_0^2 + 6\lambda(\dot{H}_1 + 4H_0H_1) + 6\lambda^2(\dot{H}_2 + 2H_1^2 + 4H_0H_2) + \mathcal{O}(\lambda^3).\end{aligned} \tag{4.10}$$

We also have

$$\begin{aligned}-\Box\phi = \ddot{\phi} + 3H\dot{\phi} &= [\ddot{\phi}_0 + 3H_0\dot{\phi}_0] + \lambda[\ddot{\phi}_1 + 3(H_0\dot{\phi}_1 + H_1\dot{\phi}_0)] \\ &\quad + \lambda^2[\ddot{\phi}_2 + 3(H_0\dot{\phi}_2 + H_1\dot{\phi}_1 + H_2\dot{\phi}_0)] + \mathcal{O}(\lambda^3)\end{aligned} \tag{4.11}$$

The perturbative expansions of $f$ and $V$ are simply Taylor series around $\phi_0$:

$$\begin{aligned}f(\phi) &= \sum_{n=0}^{\infty} \frac{1}{n!}(\phi - \phi_0)^n f^{(n)}(\phi_0) \equiv \sum_{n=0}^{\infty} \lambda^n f_n(\phi) \\ &= f(\phi_0) + \lambda\phi_1 f'(\phi_0) + \lambda^2\left[\phi_2 f'(\phi_0) + \frac{1}{2}\phi_1^2 f''(\phi_0)\right] + \mathcal{O}(\lambda^3),\end{aligned} \tag{4.12}$$

$$\begin{aligned}V(\phi) &= \sum_{n=0}^{\infty} \frac{1}{n!}(\phi - \phi_0)^n V^{(n)}(\phi_0) \equiv \sum_{n=0}^{\infty} \lambda^n V_n(\phi) \\ &= V(\phi_0) + \lambda\phi_1 V'(\phi_0) + \lambda^2\left[\phi_2 V'(\phi_0) + \frac{1}{2}\phi_1^2 V''(\phi_0)\right] + \mathcal{O}(\lambda^3).\end{aligned} \tag{4.13}$$

The expansion for derivatives is analogous.

### Inflation and $\epsilon$

Recall that inflation occurs while the Hubble slow-roll parameter $\epsilon < 1$. Using our perturbation series,

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = -\frac{1}{H_0^2}\left[\lambda\dot{H}_1 + \lambda^2\left(\dot{H}_2 - \frac{2H_1\dot{H}_1}{H_0}\right)\right] + \mathcal{O}(\lambda^3). \tag{4.14}$$

Setting $\lambda = 1$, the first order approximation is valid provided

$$\dot{H}_2 \ll \dot{H}_1, \quad H_1 \ll H_0.$$

The condition for inflation is simply $|\dot{H}_1| < H_0^2$. Classical inflationary properties like the number of $e$-folds are therefore determined by $H_0$, $H_1(t)$, and $\dot{H}_1(t)$. For our analysis to work, we require the de Sitter consistency condition (4.6) to hold for the values $\phi_0$ assumes during inflation, though the analysis is simpler if we make the stronger consistency assumptions of §4.1. See Chapter 5 for examples.

### First order equations

Using (4.10) and (4.11), the scalar-tensor equations (3.18)–(3.20) at first order are

$$(12f''H_0^2 - V')\phi_1 - 3H_0\dot{\phi}_1 + 3\big[2f'\dot{H}_1 + H_1(8f'H_0 - \dot{\phi}_0)\big] = \ddot{\phi}_1 \tag{4.15}$$

$$(6H_0\dot{\phi}_0 f'' + 6H_0^2 f' - V')\phi_1 + (6f'H_0 - \dot{\phi}_0)\dot{\phi}_1 + 6(2H_0 f + f'\dot{\phi}_0)H_1 = 0 \tag{4.16}$$

$$(f''H_0\dot{\phi}_0 - f'''\dot{\phi}_0^2 - f''\ddot{\phi}_0)\phi_1 + (f'H_0 - 2f''\dot{\phi}_0 - \dot{\phi}_0)\dot{\phi}_1 + (f'H_1\dot{\phi}_0 - 2f\dot{H}_1) = f'\ddot{\phi}_1. \tag{4.17}$$

The functions $f$ and $V$, along with their derivatives, are evaluated at $\phi_0$ as per the expansions (4.12), (4.13), but we suppress the notation $f_0, V_0$ here and below in order to minimise clutter. We can eliminate $\ddot{\phi}_1$ in (4.17) using (4.15):

$$\begin{aligned} 0 = {} & (f''H_0\dot{\phi}_0 - f'''\dot{\phi}_0^2 - f''\ddot{\phi}_0 - 12f'f''H_0^2 + f'V')\phi_1 \\ & + (4f'H_0 - 2f''\dot{\phi}_0 - \dot{\phi}_0)\dot{\phi}_1 \\ & + 4(f'\dot{\phi}_0 - 6(f')^2H_0)H_1 - 2(f + 3(f')^2)\dot{H}_1. \end{aligned} \tag{4.18}$$

Equation (4.16) gives us a similar relation directly. We can rearrange these two equations into coupled ODEs:

$$\dot{\phi}_1 = -\frac{(6H_0\dot{\phi}_0 f'' + 6H_0^2 f' - V')}{(6f'H_0 - \dot{\phi}_0)}\phi_1 - \frac{6(2H_0 f + f'\dot{\phi}_0)}{(6f'H_0 - \dot{\phi}_0)}H_1 \tag{4.19}$$

$$\begin{aligned} \dot{H}_1 = {} & \Bigg[\frac{(f''H_0\dot{\phi}_0 - f'''\dot{\phi}_0^2 - f''\ddot{\phi}_0 - 12f'f''H_0^2 + f'V')}{2(f + 3(f')^2)} \\ & - \frac{(4f'H_0 - 2f''\dot{\phi}_0 - \dot{\phi}_0)(6H_0\dot{\phi}_0 f'' + 6H_0^2 f' - V')}{2(f + 3(f')^2)}\Bigg]\phi_1 \\ & - \frac{3(2H_0 f + f'\dot{\phi}_0)(4f'H_0 - 2f''\dot{\phi}_0 - \dot{\phi}_0) + 2(f'\dot{\phi}_0 - 6(f')^2H_0)}{(f + 3(f')^2)}H_1. \end{aligned} \tag{4.20}$$

Recall that the quantities $\phi_0$, $H_0$, $f^{(n)}(\phi_0)$ and $V^{(n)}(\phi_0)$ are all determined as functions of time by the zeroth order de Sitter equations. Schematically, (4.19) and (4.20) can be written

$$\dot{\mathbf{x}}_1(t) = M_1(t)\mathbf{x}_1(t), \tag{4.21}$$

where

$$\mathbf{x}_1(t) \equiv \begin{bmatrix} \phi_1(t) \\ H_1(t) \end{bmatrix} \tag{4.22}$$

and $M_1(t)$ is a matrix function depending on zeroth order quantities. This has the solution

$$\mathbf{x}_1(t) = \exp\left[\int_{t_i}^{t} M_1(t')\,dt'\right]\mathbf{x}_1(t_i), \tag{4.23}$$

which we can evaluate using symbolic or numerical methods.

### Calculating the $e$-folds

The goal is now to evolve $\epsilon \simeq -\dot{H}_1/H_0^2$ between times $t_i, t_f$ where $\epsilon(t_{i,f}) = 1$, with $\epsilon < 1$ in betweem, and count the $e$-folds along the way. Since $dN = H\,dt$,

$$N = \int_{t_i}^{t_f} H dt \simeq H_0\Delta t + \int_{t_i}^{t_f} H_1 dt. \tag{4.24}$$

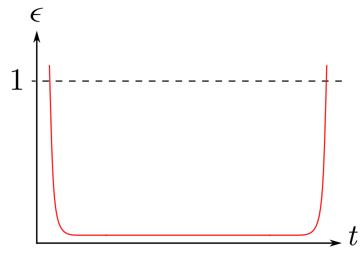



Figure 4.2: The slow-roll parameter $\epsilon$ is unity at the start and end of inflation, but for a well-behaved spectrum, must be small and approximately constant in between.

We see from the first term that $H_0$ directly contributes the number of $e$-folds, but also indirectly via the $H_0$ dependence of $H_1$ and $t_i, t_f$. Since $H_1 \ll H_0$ is built into our perturbation expansion, we have $N \sim H_0 \Delta t$, with the number of $e$-folds approximately proportional to the length of inflation.

### Initial conditions

Picking initial conditions for inflation is in general a tricky business, and Jordan frame inflation is no exception. From (4.23), we see that the vector $\mathbf{x}_1(t)$ (containing $H_1(t)$ and $\phi_1(t)$) is proportional to the initial condition $\mathbf{x}_1(t_i)$. Hence, $\epsilon$ is also proportional to $\mathbf{x}_1(t_i)$, and we can directly control the size of $\epsilon$ by scaling $H_1(t_i)$ and $\phi_1(t_i)$.

On the one hand, we would like to generate inflation with the correct number of $e$-folds as per (4.24). We can try scaling the initial conditions to achieve this. On the other hand, for our perturbation expansion to be valid, we must keep the first order corrections small. Moreover, as we will see below, $\epsilon$ must be small and roughly constant over the course of inflation to generate a well-behaved spectrum. Schematically, $\epsilon$ has to look something like Fig. 4.2, and in a realistic theory of inflation we will have less control over the number of $e$-folds than we might have initially thought.

## 4.3 Jordan frame power spectrum

In the previous section, we dealt with small corrections to de Sitter expansion in *time*. We now add *spatial* perturbations to the picture and quantise to obtain the power spectrum in the Jordan frame. Broadly speaking, this mirrors the quantisation process in §2.3, but the fine print increases substantially. We do not have time to develop cosmological perturbation theory here, but assume previous exposure and refer the interested reader to [5] or [13]. This section follows [16], [23], [24] and [34].

In the notation of §2.3,

$$\phi(t, \mathbf{x}) \equiv \bar{\phi}(t) + \delta\phi(t, \mathbf{x}). \tag{4.25}$$

We will drop the bar from now on unless it leads to ambiguity. It will prove advantageous to use a gauge-invariant relative of $\delta\phi$:

$$\delta\phi_\varphi \equiv \delta\phi - \frac{\dot{\phi}}{H}\varphi \equiv -\frac{\dot{\phi}}{H}\varphi_{\delta\phi}, \tag{4.26}$$

where $\varphi$ is the metric perturbation defined by $g_{ij} = a^2\delta_{ij}(1 + 2\varphi)$, $|\varphi| \ll 1$.

**Expanding the action**

As in general relativity, we write out the action for $\phi$, expand to second order in the perturbation $\delta\phi_\varphi$, and set the corresponding variation to zero. Recalling (3.7)–(3.8), the scalar-tensor action for $\phi$ is

$$S = \int d^4x\sqrt{-g}\Big[f(\phi)\mathcal{R} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\Big].$$

It was shown by Hwang [23] that at second order in $\delta\phi_\varphi$, this becomes

$$S^{(2)} \equiv \int dV\mathcal{L}^{(2)} = \frac{1}{2}\int d^4x\, a^3 Z\Big\{\delta\dot{\phi}_\varphi^2 - \frac{1}{a^2}g^{ij}\partial_i(\delta\phi_\varphi)\partial_j(\delta\phi_\varphi) + \frac{1}{a^3 Z}\frac{H}{\dot{\phi}}\frac{\partial}{\partial t}\left[a^3 Z\frac{\partial}{\partial t}\left(\frac{\dot{\phi}}{H}\right)\right]\delta\phi_\varphi^2\Big\} \tag{4.27}$$

where

$$Z \equiv \frac{1+3f^{-1}(f')^2}{[1+(f'\dot{\phi}/2Hf)]^2}. \tag{4.28}$$

A full derivation would take us too far afield, but the basic strategy is to go to the Einstein frame, write the corresponding second-order term for $\delta\chi_\varphi$ (the Einstein frame scalar associated with $\delta\phi_\varphi$), then convert back to $\delta\phi_\varphi$. The Einstein frame part of the calculation is still nontrivial; Hwang [21] gives a reasonably clear treatment.

Integrating the $g^{ij}$ term by parts, applying Euler-Lagrange, and multiplying by $a^3 Z$, we obtain the equation of motion

$$\delta\ddot{\phi}_\varphi + \frac{1}{a^3 Z}\frac{\partial(a^3 Z)}{\partial t}\delta\dot{\phi}_\varphi - \left\{\frac{1}{a^2}\nabla^2 + \frac{1}{a^3 Z}\frac{H}{\dot{\phi}}\frac{\partial}{\partial t}\left[a^3 Z\frac{\partial}{\partial t}\left(\frac{\dot{\phi}}{H}\right)\right]\right\}\delta\phi_\varphi = 0. \tag{4.29}$$

There is a special case we can solve immediately and that is relevant to the spectrum: the *large scale limit* $k|\tau| \ll 1$ (in momentum space). In this case, we can drop the $\nabla^2$ term, and the general solution (as can be directly verified) is

$$\delta\phi_\varphi(t,\mathbf{x}) = -\frac{\dot{\phi}}{H}\left[C(\mathbf{x}) - D(\mathbf{x})\int^t \frac{H^2}{a^3 Z\dot{\phi}}\,dt'\right]. \tag{4.30}$$

The second term is called the *decaying mode*, since it decays as $a$ grows; the first is the *growing mode*. Ignoring the decaying mode, we see from (4.26) that $C(\mathbf{x}) = \varphi_{\delta\phi}(\mathbf{x})$.

We expect that a change of variable (analogous to comoving variables in §2.3) will eliminate the $\delta\dot{\phi}_\varphi$ term. Indeed, substituting the auxiliary variables

$$z(t) \equiv \frac{a\dot{\phi}}{H}\sqrt{Z}, \quad \vartheta(t,\mathbf{x}) \equiv \frac{zH}{\dot{\phi}}\delta\phi_\varphi = a\sqrt{Z}\,\delta\phi_\varphi, \tag{4.31}$$

into (4.29) and multiplying by $a^3\sqrt{Z}$, we obtain (after some algebra) a Mukhanov-Sasaki equation in Fourier space:

$$\partial_\tau^2\vartheta_{\mathbf{k}} + \left(k^2 - \frac{\partial_\tau^2 z}{z}\right)\vartheta_{\mathbf{k}} = 0. \tag{4.32}$$

This only differs from (2.34) by a replacement $\partial_\tau^2 a/a \to \partial_\tau^2 z/z$.

### Quantisation

We quantise by promoting the gauge-invariant perturbation to a Heisenberg operator:

$$\delta\phi_\varphi \to \delta\hat{\phi}_\varphi \equiv \delta\hat{\phi} - \frac{\dot{\phi}}{H}\hat{\varphi}, \quad \hat{\vartheta} \equiv a\sqrt{Z}\delta\hat{\phi}_\varphi. \tag{4.33}$$

Note that in §2.3, the metric corrections were treated classically, but here we quantise $\varphi$ since it is part of the gauge-invariant perturbation $\delta\phi_\varphi$. The conjugate momentum is then

$$\delta\hat{\pi}_\varphi \equiv \frac{\partial\mathcal{L}^{(2)}}{\partial(\delta\dot{\hat{\phi}}_\varphi)} = a^3 Z\delta\dot{\hat{\phi}}_\varphi,$$

and the usual equal time commutation relations imply

$$[\delta\hat{\phi}_\varphi(\tau,\mathbf{x}_1), \delta\dot{\hat{\phi}}_\varphi(\tau,\mathbf{x}_2)] = \frac{i}{a^3 Z}\delta^{(3)}(\mathbf{x}_1 - \mathbf{x}_2). \tag{4.34}$$

As usual, we assemble these into a mode expansion

$$\delta\hat{\phi}_\varphi(\tau,\mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}}\left[\delta\phi_{\varphi k}(\tau)\hat{a}_\mathbf{k} + \delta\phi^*_{\varphi k}(\tau)\hat{a}^\dagger_\mathbf{k}\right], \tag{4.35}$$

where the coefficients $\delta\phi_{\varphi k}, \delta\phi^*_{\varphi k}$ satisfy (4.32) by the Heisenberg equations of motion. The relation (4.34) implies ladder operator commutation relations

$$[\hat{a}_{\mathbf{k}_1}, \hat{a}^\dagger_{\mathbf{k}_2}] = c\cdot\delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2), \quad [\hat{a}_{\mathbf{k}_1}, \hat{a}_{\mathbf{k}_2}] = [\hat{a}^\dagger_{\mathbf{k}_1}, \hat{a}^\dagger_{\mathbf{k}_2}] = 0$$

for some constant $c$. We can set $c = 1$ by demanding the coefficients $\delta\phi_{\varphi k}, \delta\phi^*_{\varphi k}$ satisfy a Wronskian condition

$$\mathcal{W}[\delta\phi_{\varphi k}, \delta\phi^*_{\varphi k}] \equiv \delta\phi_{\varphi k}\delta\dot{\phi}^*_{\varphi k} - \delta\phi^*_{\varphi k}\delta\dot{\phi}_{\varphi k} = \frac{i}{a^3 Z}. \tag{4.36}$$

In §2.3, we simplified the Mukhanov-Sasaki equation with the de Sitter approximation $\partial_\tau^2 a/a \simeq 2/\tau^2$. We can make a similar approximation here, and assume that for some constant $m$,

$$\frac{\partial_\tau^2 z}{z} \simeq \frac{m}{\tau^2} \equiv \frac{\nu^2 - \frac{1}{4}}{\tau^2}, \tag{4.37}$$

where $\nu$ will become important shortly. Below, we will explicitly calculate $m$ and the regime in which the approximation is valid. Using (4.37) and the substitutions $s \equiv k|\tau|$ and $\vartheta_k(\tau) \equiv \sqrt{s}J(s)$, equation (4.32) reduces to the Bessel equation

$$s^2\frac{d^2J}{ds^2} + s\frac{dJ}{ds} + \left(s^2 - \nu^2\right)J = 0. \tag{4.38}$$

The two linearly independent solutions are Bessel functions $J_{\pm\nu}$ (§10.2, [36]). It is convenient to instead write the general solution as a linear combination of Hankel functions,

$$\begin{aligned} \vartheta_k(\tau) &= \frac{\sqrt{\pi|\tau|}}{2}\left[c_1(k)H_\nu^{(1)}(k|\tau|) + c_2(k)H_\nu^{(2)}(k|\tau|)\right] \\ \implies \qquad \delta\phi_{\varphi k}(\tau) &= \frac{\sqrt{\pi|\tau|}}{2a\sqrt{Z}}\left[c_1(k)H_\nu^{(1)}(k|\tau|) + c_2(k)H_\nu^{(2)}(k|\tau|)\right]. \end{aligned} \tag{4.39}$$

Using the identities (§10.5, [36])

$$\mathcal{W}[H_\nu^{(1)}(z), H_\nu^{(2)}(z)] = -\frac{4i}{\pi z}, \quad H_\nu^{(1)*} = H_\nu^{(2)},$$

a short calculation shows that (4.36) is maintained provided $|c_2(k)|^2 - |c_1(k)|^2 = 1$.

To determine $c_1(k), c_2(k)$, we go to very early times $k|\tau| \gg 1$. Then (4.32) becomes the equation of a simple harmonic oscillator, $\partial_\tau^2 \vartheta_{\mathbf{k}} = -k^2 \vartheta_{\mathbf{k}}$. The general solution is

$$\delta\phi_{\varphi k}(\tau) = \frac{1}{a\sqrt{2kZ}}\left[c_1(k)e^{ik|\tau|} + c_2(k)e^{-ik|\tau|}\right]$$

where the normalisation and identification of coefficients come from (4.39) and large-argument asymptotics of $H_\nu^{(i)}$ (§10.2, [36]). Apart from the funny normalisation, this corresponds to quantum field theory in flat space, and from (4.35) we can read off the familiar positive frequency solution $c_1(k) = 0, c_2(k) = 1$.

### Scalar spectral index

The scalar spectral index involves modes that cross the horizon during inflation, and are therefore large-scale with $k|\tau| \ll 1$. Using (4.39) and the small argument asymptotics of $H_\nu^{(i)}$ for $\nu \neq 0$ (§10.7, [36]),

$$\delta\phi_{\varphi k}(\tau) \sim \frac{i\sqrt{|\tau|}\Gamma(\nu)}{2a\sqrt{\pi Z}}\left(\frac{k|\tau|}{2}\right)^{-\nu}. \tag{4.40}$$

Recalling the position space expansion (4.35), we see immediately that the expectation $\langle \delta\hat{\phi}_\varphi \rangle$ at any position vanishes. As we expect, the uncertainty does not:

$$\begin{aligned}\langle |\delta\hat{\phi}_\varphi|^2 \rangle &= \int \frac{d^3k\, d^3p}{(2\pi)^3} e^{i(\mathbf{k}+\mathbf{p})\cdot\mathbf{x}} \delta\phi_{\varphi k}(\tau)\delta\phi_{\varphi p}^*(\tau)\langle \hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{p}}^\dagger \rangle \\ &= \int \frac{d^3k}{(2\pi)^3}|\delta\phi_{\varphi k}(\tau)|^2 \equiv \int d\ln k\, \Delta_{\delta\phi}^2(k,\tau),\end{aligned} \tag{4.41}$$

where, just as in §2.3, we have defined the power spectrum $\Delta_{\delta\phi_\varphi}^2(k,\tau) \equiv (k^3/2\pi^2)|\delta\phi_{\varphi k}(\tau)|^2$. Equations (4.40) and (4.41) give

$$\Delta_{\delta\phi_\varphi}^2(k,\tau) = \frac{\Gamma^2(\nu)}{a^2|\tau|^2\pi^3 Z}\left(\frac{k|\tau|}{2}\right)^{3-2\nu}. \tag{4.42}$$

Hence, the scalar spectral index is

$$n_s \equiv 1 + \frac{d\ln\Delta_{\delta\phi_\varphi}^2(k,\tau)}{d\ln k} = 4 - 2\nu. \tag{4.43}$$

### Slow-roll and the spectrum

We still need to relate $\nu$ to Jordan frame slow-roll parameters. From (4.37), our analysis requires

$$\nu^2 \simeq \frac{\tau^2\partial_\tau^2 z}{z} + \frac{1}{4},$$

so we must find an expression for $\partial_\tau^2 z/z$. To accomplish this, we introduce four slow-roll parameters. We also write these to first order in the de Sitter perturbation expansion:

$$\epsilon_1 \equiv -\epsilon = \frac{\lambda \dot{H}_1}{H_0^2} + \mathcal{O}(\lambda^2) \tag{4.44}$$

$$\epsilon_2 \equiv \frac{\ddot{\phi}}{H\dot{\phi}} = \frac{\ddot{\phi}_0}{H_0\dot{\phi}_0} + \frac{1}{H_0\dot{\phi}_0}\left\{\ddot{\phi}_1 - \ddot{\phi}_0\left(\frac{H_1}{H_0} + \frac{\dot{\phi}_1}{\dot{\phi}_0}\right)\right\} + \mathcal{O}(\lambda^2) \tag{4.45}$$

$$\epsilon_3 \equiv \frac{f'\dot{\phi}}{2Hf} = \frac{f_0'\dot{\phi}_0}{2H_0 f_0} + \frac{1}{2H_0 f_0}\left\{f_0'\dot{\phi}_1 + f_1'\dot{\phi}_0 - f_0'\dot{\phi}_0\left(\frac{H_1}{H_0} + \frac{f_1}{f_0}\right)\right\} + \mathcal{O}(\lambda^2) \tag{4.46}$$

$$\begin{aligned}\epsilon_4 \equiv \frac{\alpha'\dot{\phi}}{2H\alpha} &= \frac{3f_0'\dot{\phi}_0}{2f_0H_0}\left[\frac{2f_0f_0'' - (f_0')^2}{f_0 - 3(f_0')^2}\right] \\ &\quad + \frac{1}{2H_0\alpha_0}\left\{\alpha_0'\dot{\phi}_1 + \alpha_1'\dot{\phi}_0 - \alpha_0'\dot{\phi}_0\left(\frac{H_1}{H_0} + \frac{\alpha_1}{\alpha_0}\right)\right\} + \mathcal{O}(\lambda^2),\end{aligned} \tag{4.47}$$

where (recalling (3.42))

$$\alpha \equiv G^2 = 1 + \frac{3(f')^2}{f} \tag{4.48}$$

$$\alpha' = \frac{3f'}{f^2}\left[2ff'' - (f')^2\right] \tag{4.49}$$

$$\alpha_0 \equiv \alpha(f_0), \quad \alpha_1 \equiv \phi_1\alpha_0'. \tag{4.50}$$

Neglecting the derivatives $\dot{\epsilon}_i$, it was shown by Hwang and Noh [34] that to first order in the $\epsilon_i$,

$$\frac{\partial_\tau^2 z}{z} \simeq a^2H^2\big(2 - 2\epsilon_1 + 3\epsilon_2 - 3\epsilon_3 + 3\epsilon_4\big).$$

Using the relation

$$\tau \simeq -\frac{1}{aH}\frac{1}{1+\epsilon_1},$$

this is equivalent to

$$\nu^2 \simeq \frac{\tau^2\partial_\tau^2 z}{z} + \frac{1}{4} \simeq \frac{9}{4}\left[1 - \frac{4}{3}\big(2\epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4\big)\right].$$

Hence,

$$\nu \simeq \frac{3}{2} - \big(2\epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4\big), \tag{4.51}$$

and

$$n_s \simeq 4 - 2\nu \simeq 1 + 2\big(2\epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4\big). \tag{4.52}$$

To summarise, the approach here requires that, during inflation, the $\epsilon_i$ are small ($|\epsilon_i| \ll 1$) and slowly changing ($|\dot{\epsilon}_i| \ll |\epsilon_i|$). We note $\epsilon_1 = \mathcal{O}(\lambda)$ while $\epsilon_i = \mathcal{O}(1)$ for $i = 2, 3, 4$, so the latter are not automatically small when the de Sitter perturbation is valid. We discuss these contraints at greater length in §5.4.

### Tensor perturbations

Having subjected the reader to a tortuous derivation in the scalar case, we will only briefly sketch the analogous results for primordial gravitational waves. The metric perturbation $c_{ij}$ is defined by

$$g_{ij} = \delta_{ij} + 2c_{ij}, \quad |c_{ij}| \ll 1,$$

and can be chosen to be transverse and traceless. As for the scalar perturbation, $c_{ij}$ satisfies a Mukhanov-Sasaki equation after a change of variables.

To quantise, we promote $c_{ij} \to \hat{c}_{ij}$ and Fourier expand:

$$\hat{c}_{ij}(t, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} \sum_{l=+,\times} \left[ h_{lk}\hat{a}_{l\mathbf{k}} + h^*_{lk}\hat{a}^\dagger_{l\mathbf{k}} \right]. \tag{4.53}$$

The mode functions $h_{lk}, h^*_{lk}$ satisfy the classical equation of motion and the ladder operators $\hat{a}_{l\mathbf{k}}, \hat{a}^\dagger_{l\mathbf{k}}$ have canonical commutation relations subject to a Wronskian normalisation condition on the mode functions.

Squint a little, and you will see two scalar perturbations, one for each polarisation mode. This leads to familiar-looking definitions of spectral density and index:

$$\Delta^2_{c_{ij}}(k, \tau) \equiv 2 \sum_l \frac{k^3}{2\pi^2} |h_{l\mathbf{k}}(\tau)|^2, \quad n_t \equiv \frac{d \ln \Delta^2_{c_{ij}}(k, \tau)}{d \ln k}. \tag{4.54}$$

Apart from some numerical factors, the only difference between $\Delta^2_{c_{ij}}(k, \tau)$ and $\Delta^2_{\delta\phi_\varphi}(k, \tau)$ is the substitution $Z \to f$ in (4.42). This simplication occurs because $c_{ij}$ is automatically gauge-invariant. With this substitution, we find

$$n_t = 2(\epsilon_1 - \epsilon_3). \tag{4.55}$$

Hwang and Noh [34] derive the corresponding tensor-to-scalar ratio[1]

$$r = 4\pi \left( -\frac{1}{2} M^2_{\text{pl}} f^{-1} \epsilon_1 + 3\epsilon_3^2 \right). \tag{4.56}$$

Unlike general relativity, there is no consistency relation between $r$ and $n_t$.

The form of $n_t$ is simple enough to extract some physical implications. From (4.44), (4.46), and (4.55), $|\epsilon_1| \ll |\epsilon_3|$ provided the de Sitter perturbation is valid. The behaviour of $n_t$ should therefore determined by $\epsilon_3$. This also eliminates the $1/f$ dependence in the first term. The sign of $\epsilon_3$, and hence $n_t$, depends on how $f$ changes during inflation: if $\dot{f} \simeq f'_0 \dot{\phi}_0 < 0$, then $n_t$ is positive, and vice versa. A positive spectral index (aka *blue tilt*) means more power at the the smaller wavelengths where gravitational wave detectors are sensitive. The vanilla slow-roll model in §2.3 predicts a negative spectral index (aka *red tilt*). Current constraints on $n_t$ are rather loose:

$$n_t = -0.76^{+1.37}_{-0.52},$$

with either sign of the index compatible with observation [20].

1. Note that $r$ here is normalised slightly differently from §2.3, since it is calculated in [34] as the ratio of the dominant *quadrupole anisotropies* rather than full amplitudes.

Chapter 5

# *Scalar-tensor models*

We finish by applying the machinery of Chapter 4 to some concrete models of scalar-tensor inflation. In §§5.1–5.3, we explore toy models based on simple choices for $f$ and $V$. This provides some intuition for the dynamics of scalar-tensor models, and illustrates how the de Sitter approach works in practice. Currently, we lack the numerical firepower to sweep out all of parameter space, and therefore focus on specific parameters for which the models are numerically tractable and give qualitative insights.

Although we can usually tune parameters to get inflation with 60 $e$-folds, it turns out to be impossible to obtain a viable spectrum from the toy models. In §5.4, we outline a "top down" approach which addresses this problem. Unless otherwise specified, plots are in reduced Planck units.

## 5.1 Jordan frame Higgs inflation

In the first instance, scalar-tensor gravity was designed to enable Higgs inflation. Now we look at Higgs inflation in the Jordan frame, with the Higgs potential

$$V(\phi) = \frac{1}{4}\lambda(\Delta\phi^2)^2, \quad \lambda = 0.13, \quad v = 246\,\mathrm{GeV},$$

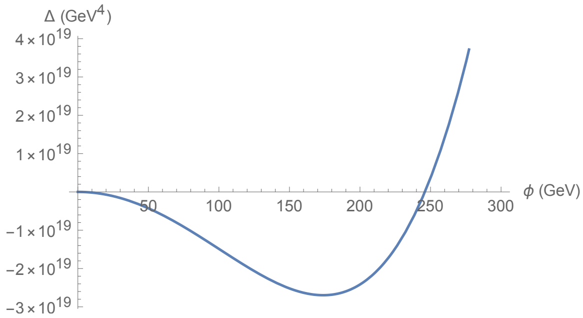
and the Bezrukov-Shaposhnikov coupling

$$f(\phi) = \frac{1}{2}\left[\xi(\phi^2 - v^2) + M_{\mathrm{pl}}^2\right].$$

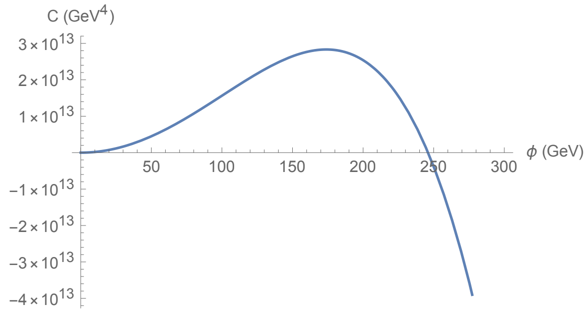
Since $V(\pm v) = V'(\pm v) = 0$, we see that the vevs $\phi = \pm v$ satisfy the induced gravity condition (3.53). In fact, $\phi = \pm v$ will be a minimum of the Einstein frame potential $U \propto V/f^2$ provided $f$ is positive ($2\xi v^2 < M_{\mathrm{pl}}^2$).[1] In contrast to the Einstein frame approach, we do not impose slow-roll conditions on the associated field $\chi$ (3.28). Instead, we stick with $\phi$ in the Jordan frame and use the de Sitter perturbative expansion.

In order to compare to the Bezrukov-Shaposhnikov model, we set $\xi \simeq 55000\sqrt{\lambda}$. The only parameter left to vary is $\mathcal{R}_0$, or equivalently, $H_0$. By symmetry, we can focus on positive field values. First, we determine which regions of $\phi_0$ are de Sitter consistent, and the position of any fixed points. Recall the consistency condition $\Delta(\phi) \geq 0$ from §4.1. There is a large-field, de Sitter consistent region with lower bound $\phi_{\min} \sim v$ and no upper bound; we plot $\Delta$ in Fig. 5.1(a). From (4.7), a boundary fixed point must satisfy $f' = 0$. Since $f'(\phi_{\min}) = 2\xi\phi_{\min} > 0$, $\phi_{\min}$ is *not* a fixed point.

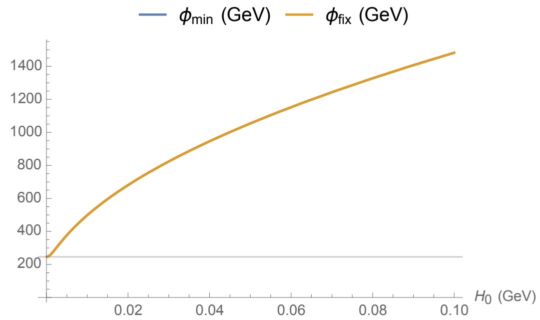
1. Having minima at $\pm v$, rather than simply $v$, is required for the electroweak Higgs mechanism.



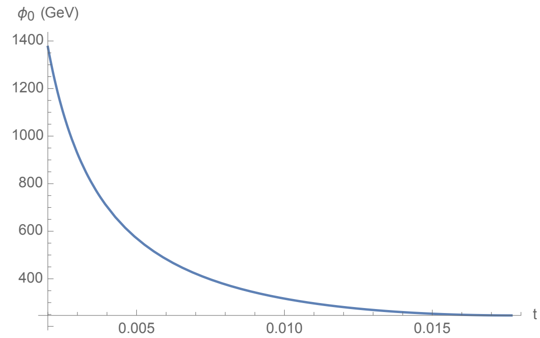

(a) $\Delta(\phi)$ has a root $\phi_{\min} \sim v = 246$ GeV. This bounds a de Sitter consistent region but is *not* a fixed point.




(b) $C(\phi)$ has a root $\phi_{\text{fix}} > \phi_{\min}$ with $\phi_{\text{fix}} \approx \phi_{\min}$. This is a stable fixed point for the negative branch of $\phi_0$.




(c) The lower boundary $\phi_{\min}$ and fixed point $\phi_{\text{fix}}$ are almost equal for small $H_0$, and both approach $v$ as $H_0 \to 0$.




(d) The negative branch of $\phi_0$ rapidly approaches the fixed point $\phi_{\text{fix}}$. For small $H_0$, $\phi_{\text{fix}} \approx v$, so the field value falls into the vev.

Figure 5.1: Pure de Sitter evolution for the Higgs potential and Bezrukov-Shaposhnikov coupling.

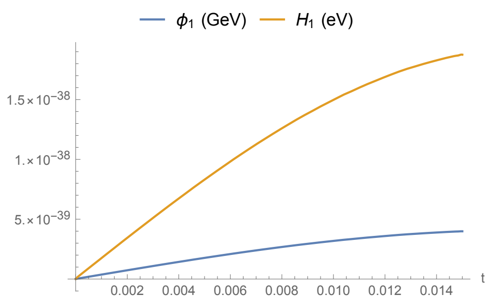
As described in §4.1, fixed points are roots of

$$\begin{aligned} C &= \mathcal{R}_0 f - 2V + 12(f')^2\mathcal{R}_0 - 12f'V' \\ &= \frac{1}{2}\mathcal{R}[M_{\text{pl}}^2 + \xi(\Delta\phi^2)] - \frac{1}{2}\lambda(\Delta\phi^2)^2 + 12\mathcal{R}_0\xi^2\phi^2 - 6\xi\lambda\phi(\Delta\phi^2). \end{aligned} \tag{5.1}$$

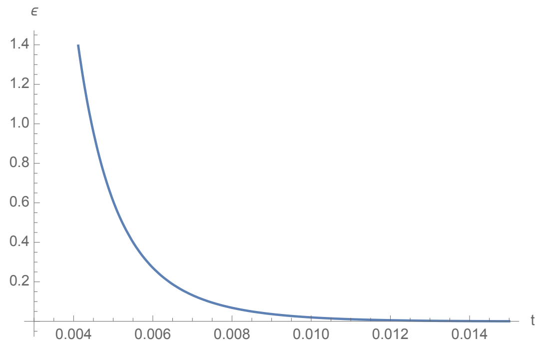
The function $C$ is depicted in 5.1(b), and has a root $\phi_{\text{fix}} > \phi_{\min}$. We plot $\phi_{\min}$ and $\phi_{\text{fix}}$ against $H_0$ in Fig. 5.1(c). For small $H_0$, they are almost (but not quite) equal. The vev $v = 246$ GeV is drawn as a horizontal line, and we can see that, as $H_0 \to 0$, both $\phi_{\min}$ and $\phi_{\text{fix}}$ approach $v$.

The root $\phi_{\text{fix}}$ is a fixed point of the *negative* branch of $\phi_0$, since $f'(\phi_{\text{fix}}) = 2\xi\phi_{\text{fix}} > 0$. To ensure de Sitter consistency, we therefore choose the negative branch. Characteristic behaviour for $\phi_0$ is plotted in Fig. 5.1(d). The initial condition is $\phi_0(0) = 10^5$ GeV, with other parameters to be specified below. The field value $\phi_0$ quickly asymptotes to the fixed point at $\phi_{\text{fix}}$, which for small values of $H_0$ satisfies $\phi_{\text{fix}} \approx v$.

The next step is to compute the first-order corrections $\phi_1$ and $H_1$ using the framework of §4.2. We choose a small value $H_0 = 10^{-20}\,M_{\text{pl}}$ to avoid numerical issues, and initial conditions $\phi_1(0) = 10^{-41}\,M_{\text{pl}}$ and $H_1(0) = 10^{-70}\,M_{\text{pl}}$ to ensure that inflation occurs. The perturbations $\phi_1$ and $H_1$ grow slowly then taper off, as plotted in Fig. 5.2(a). The slow-roll parameter $\epsilon$ in Fig. 5.2(b). Inflation starts around $t = 0.005\,M_{\text{pl}}^{-1}$. As we can see




(a) The first-order corrections $\phi_1$(GeV), $H_1$(eV) taper off. Initial values are used to scale $\epsilon$ so inflation occurs.




(b) Inflation starts at $t \approx 0.005 M_{\rm pl}^{-1}$. Since $\epsilon$ approaches zero, inflation never ends.

Figure 5.2: First-order corrections and the slow-roll parameter for Higgs inflation. The first-order corrections grow slowly and taper off. This is reflected in the slow-roll parameter $\epsilon$, which monotonically decreases. This means inflation starts but never ends.

from the diagram, $\epsilon$ approaches zero and inflation continues forever. Before we draw any hasty conclusions about the viability of Jordan frame Higgs inflation, we need to explore parameter space more fully.

## 5.2 Coupling driven inflation

Our next model has a simple motivation: with a scalar-tensor coupling $f$ to play around with, do we even need a potential? Perhaps we can we start inflation with a cosmological constant and end it using the direct interaction between $\phi$ and gravity. Recall from §3.1 that a cosmological constant is effectively given by a constant potential $V = \Lambda$.

One approach would be to reverse engineer $f$ from a Higgs-like potential in the Einstein frame. Since this leads to inflation in general relativity, we would hope to get inflation in the Jordan frame. The reverse-engineered coupling $f$ obeys

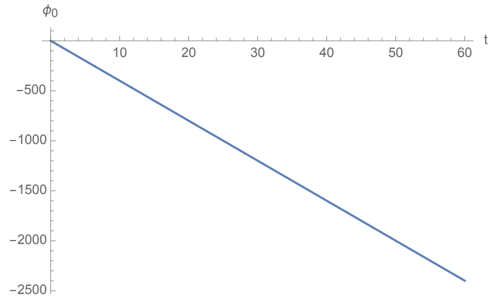
$$U(\chi) = \frac{M_{\rm pl}^4 \Lambda}{4 f^2(\phi(\chi))} = \frac{\lambda}{4}(\Delta\chi^2)^2 \quad \Longrightarrow \quad f^2(\phi(\chi)) = \frac{M_{\rm pl}^4 \Lambda}{\lambda(\Delta\chi^2)^2}.$$

As we know from (3.42), $f$ also appears in the relation between $\chi$ and $\phi$. This means we must extract $f$ from an integro-differential equation. In the spirit of the thesis, we abandon this messy endeavour, and look for a natural Jordan frame candidate instead.

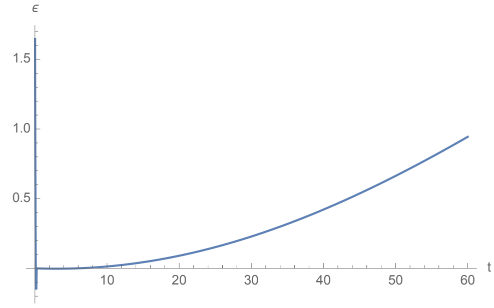
We assume that $\Lambda > 0$, since in general relativity positive vacuum energy leads to de Sitter inflation. The goal now is to tailor $f$ to start and end inflation. We need the Einstein frame potential $U \propto \Lambda/f^2$ to be a minimum for some $\phi = v$. Thus, $v$ should be a *maximum* of $f$, with $f(v) = M_{\rm pl}^2/2$ to induce general relativity. Assuming $f > 0$, the simplest option is a Gaussian

$$f(\phi) = \frac{1}{2} M_{\rm pl}^2 e^{-\beta(\phi - v)^2} \equiv \frac{1}{2} M_{\rm pl}^2 e^{-\beta x^2}, \quad \beta > 0. \tag{5.2}$$

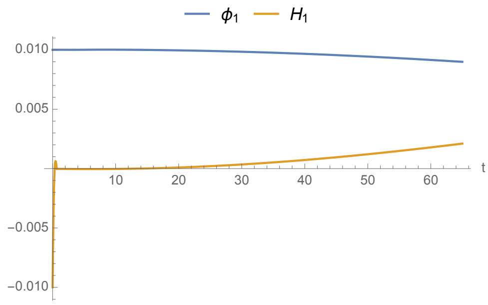
This model has three important parameters: $\beta$, $\mathcal{R}_0$, and $\Lambda$. Unlike Higgs inflation, the value of the minimum $v$ does not affect the dynamics.



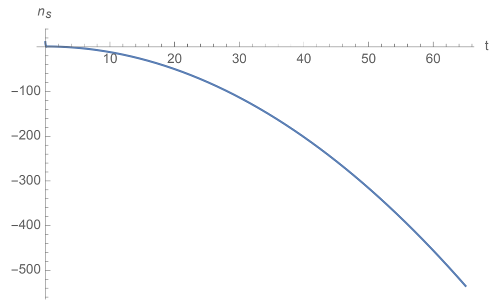

(a) Away from the Gaussian bump in $f$, $\phi_0$ is essentially linear. The negative branch of $\phi_0$ satisfies $\dot{\phi}_0 \simeq -\sqrt{2\Lambda/5}$.




(b) The slow-roll parameter $\epsilon$ dips as $\phi_0$ passes through the Gaussian bump, then slowly increases. Rescalng gives 60 $e$-folds.




(c) The first-order corrections $\phi_1$ and $H_1$. $H_1$ is "reset" to zero whatever the initial value $H_1(0)$, indicating an attractor.




(d) The scalar spectral index $n_s$ is not physically viable. It inherits its bad behaviour from the spectral indices $\epsilon$ and $\epsilon_3$.

Figure 5.3: Inflation from a Gaussian coupling $f(x) = \frac{1}{2}M_{\rm pl}^2 e^{-\beta x^2}$ and cosmological constant $\Lambda > 0$. We obtain inflation and 60 $e$-folds, but the spectral index $n_s$ is ill-behaved.

We now show how global de Sitter consistency (4.6) is used to constrain parameters in the model. First, to simplify the analysis choose $\beta$ such that

$$A(x) = 12f''(x) + 5 = 12M_{\rm pl}^2\beta e^{-\beta x^2}(2\beta x^2 - 1) + 5 > 0, \qquad \text{for all } x. \tag{5.3}$$

Since $A(x)$ is a minimum at $x = 0$, (5.3) holds for $0 < M_{\rm pl}^2\beta < 5/12$. We can then divide both sides of (4.6) by $A$. Hence, $\mathcal{V}/A = V + 6f'V' = \Lambda$ and

$$\frac{\mathcal{R}_0 F(x)}{A(x)} = \frac{1}{4}M_{\rm pl}^2 e^{-\beta x^2}\mathcal{R}_0\left[1 + 6M_{\rm pl}^2\beta^2 x^2 e^{-\beta x^2}\left(4 - \frac{9}{A(x)}\right)\right]. \tag{5.4}$$

It follows that the model is globally consistent if (5.4) is bounded above by $\Lambda$. Noting that $A(x) > 0$, the expression (5.4) is automatically bounded, and by determining the maximum value of $F/A$, global consistency boils down to a constraint on $\beta$, $\Lambda$ and $\mathcal{R}_0$. Numerics show that if $\beta M_{\rm pl}^2 \lesssim 1/10$, (5.4) is maximised at $x = 0$, so in this case we require

$$\Lambda \geq \frac{\mathcal{R}_0 F(0)}{A(0)} = \frac{1}{4}M_{\rm pl}^2\mathcal{R}_0. \tag{5.5}$$

Thus, globally consistent models satisfy $0 < M_{\rm pl}^2\beta < 5/12$ and $4\Lambda \geq M_{\rm pl}^2\mathcal{R}_0$.

The positive branch of $\phi_0$ leads to bad behaviour at first order, so we focus on the negative branch, plotted in Fig. 5.3(a). We note that $\phi_0$ is essentially linear, with

$\dot{\phi}_0 \approx -\sqrt{2\Lambda/5}$. As $\phi_0$ is dragged through the Gaussian bump, there is some transient behaviour in the first-order corrections which can be used to generate inflation, as we will see shortly. We also set $v = 0$ (using translation invariance) and the initial condition $\phi_0(0) = 1\, M_{\rm pl}$. Provided $\phi_0$ passes through the bump, the initial value only controls the onset time of inflation and is irrelevant.

Recalling the discussion surrounding Fig. 4.2, we want a slow-roll parameter $\epsilon$ which "dips" appropriately. The parameters $H_0 = 1\, M_{\rm pl}$, $\beta = 4 \times 10^{-5} M_{\rm pl}^2$ and $\Lambda = 4 \times 10^3 M_{\rm pl}^4$ satisfy the global consistency relation (5.5) and yield a suitable shape, shown in Fig. 5.3(b). $\epsilon$ falls sharply as $\phi_0$ passes through the Gaussian bump (with a rate connected to the width of the bump), then slowly increases. The first-order perturbations are plotted in Fig. 5.3(c). The initial conditions

$$\phi_1(0) = -H_1(0) = 10^{-2} M_{\rm pl}$$

are precisely tuned to obtain the required number of $e$-folds. Finally, we note the presence of an *attractor* in the system, which "resets" $H_1$ to 0 whatever its initial value. More work is required to understand how this is related to the coupling.

The scalar spectral index $n_s$ is shown over the course of inflation in Fig. 5.3(d). It is too large and rapidly changing to make physical sense. While $\epsilon_2$ and $\epsilon_4$ are well-behaved over the course of inflation, $\epsilon$ and $\epsilon_3$ are neither small nor approximately constant. We have another piece of evidence that controlling spectral indices indirectly is hard. In future, we intend to explore the parameter space of this model more thoroughly, and consider other couplings that could drive inflation.

## 5.3 Linearly coupled massive scalar

A constant $f \neq 0$ corresponds to general relativity. In our final toy model, we consider a *linear* coupling, in some sense the simplest nontrivial possibility. Any linear function $f(\phi)$ has a root $\phi = \phi^*$. Unless $V$ has a matching zero, the Einstein potential $U \propto V/f^2$ blows up at $\phi^*$. This will act as a *wall* or *well* in field space, depending on which side you approach from.

If $v$ is the target asymptotic value of $\phi$, induced gravity requires

$$f(\phi) = \alpha(\phi - v) + \frac{1}{2}M_{\rm pl}^2 = \alpha x + \frac{1}{2}M_{\rm pl}^2, \tag{5.6}$$

where $x \equiv \phi - v$ as in the previous section. The wall/well is then located at

$$\phi^* = v - \frac{1}{2\alpha}M_{\rm pl}^2.$$

We assume that $\alpha > 0$, and restrict our attention to $\phi > \phi^*$.

Now we look at possibilities for $V$. The analysis in §5.2 implies that a cosmological constant and linear coupling are not globally de Sitter consistent. A linear potential $V = \kappa x + \Lambda$, on the other hand, yields the inequality:

$$\mathcal{V} = 5(\kappa x + \Lambda + 6\alpha\kappa) \geq \mathcal{R}_0 F = \frac{1}{2}\mathcal{R}_0\alpha\left[5x + \frac{5}{2\alpha}M_{\rm pl}^2 + 33\alpha\right]. \tag{5.7}$$

Since both sides are linear, the inequality holds for identical gradients and a smaller constant term on the RHS:

$$2\kappa = \mathcal{R}_0\alpha, \quad 5(\Lambda + 6\alpha\kappa) \geq \frac{1}{2}\mathcal{R}_0\alpha\left(\frac{5}{2\alpha}M_{\rm pl}^2 + \frac{33}{5}\alpha\right).$$

In this case, both $\Delta$ and $\dot{\phi}$ are constants. We do not consider this model any further.

A more interesting choice is a *harmonic* potential $V(\phi) = \kappa(\phi - v)^2$, with $\kappa > 0$. The constant $\kappa$ can be related to the mass of the scalar $\phi$ via $m_\phi = \sqrt{2\kappa}$. It is easy to show that $\phi = v$ is a minimum of $U = M_{\rm pl}^4 V_2(\phi)/4f(\phi)^2$. We have

$$\mathcal{V} = \left(V_2 + 6f'V_2'\right)\left(12f'' + 5\right) = 5\kappa x(x + 12\alpha), \tag{5.8}$$

Since $f$ is unchanged, $F$ is still given by the RHS of (5.7). Setting

$$S \equiv \frac{\mathcal{R}_0\alpha}{10\kappa}, \quad T \equiv S\left[\frac{5}{2\alpha}M_{\rm pl}^2 + 33\alpha\right], \tag{5.9}$$

the inequality $\mathcal{V} \geq \mathcal{R}_0 F$ becomes

$$x^2 + x(\alpha - S) - ST \geq 0. \tag{5.10}$$

The discriminant $(\alpha-S)^2+4ST$ of (5.10) is always positive, so there are two consistent regions. The negative region satisfies $x \leq x_{\rm left}$, while the positive one satisfies $x \geq x_{\rm right}$, where

$$x_{\rm left/right} = \frac{1}{2}\left[S - \alpha \pm \sqrt{(S-\alpha)^2 + 4ST}\right]. \tag{5.11}$$

At the boundaries, $\dot{\phi}_0\big|_{x=x_{\rm left/right}} = 1.8H_0\alpha$ is nonzero, so neither is a fixed point. However, in this case, the function $C$ from (4.8) is

$$C = -2\kappa x^2 + \alpha\left(\mathcal{R}_0 - 24\kappa\right)x + \left(\frac{1}{2}\mathcal{R}_0 M_{\rm pl}^2 + 12\mathcal{R}_0\alpha^2\right), \tag{5.12}$$

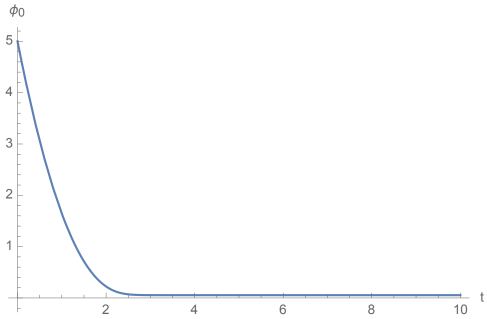
and has roots at

$$x_{{\rm fix},\pm} = \frac{1}{4\kappa}\left[\alpha(\mathcal{R}_0 - 24\kappa) \pm \sqrt{\alpha^2(\mathcal{R}_0 - 24\kappa)^2 + 4\kappa\mathcal{R}_0(M_{\rm pl}^2 + 24\alpha^2)}\right]. \tag{5.13}$$

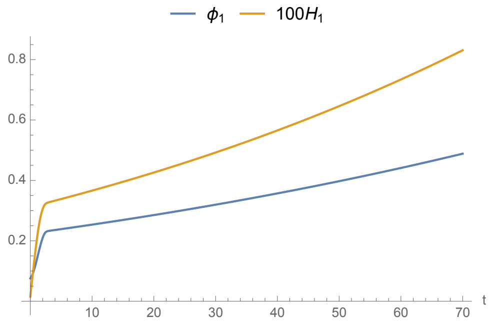
Since $f' = \alpha > 0$, the roots $x_{{\rm fix},\pm}$ are fixed points for the *negative* branch of $\phi_0$. Moreover, it is straightforwad to see they are *attractive* fixed points. For instance, as $x$ increases through $x_{{\rm fix},+}$, $C$ goes from positive to negative; recalling (4.5), we see that the negative branch of $\dot{\phi}_0$ goes from positive to negative. So $x_{{\rm fix},+}$ is indeed attractive.

### Exponential growth and oscillations

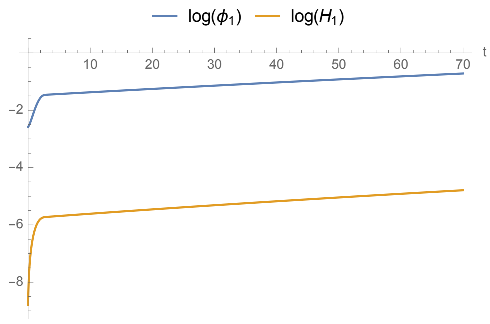
We now set $v = 0$ and transfer the labelling of fixed points from $x$ to $\phi$. Starting the negative branch of $\phi_0$ at any value in the positive de Sitter patch, it quickly converges to the attractive fixed point at $\phi_{{\rm fix},+}$. Typical behaviour is shown in Fig. 5.4(a). The behaviour of $H_1$ and $\phi_1$ depends on $\alpha, \kappa$ and $H_0$, exhibiting both exponential growth and oscillations. Since we have an oscillator potential, this is somewhat unsurprising.



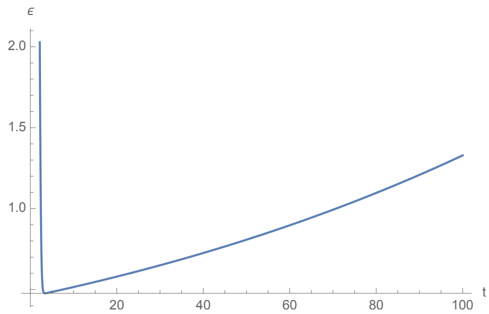

(a) The negative branch of $\phi_0$ converges to the fixed point at $\phi_{\text{fix},+} \approx 0.05$.




(b) After $\phi_0$ converges to $\phi_{\text{fix},+}$, the first-order corrections $H_1, \phi_1$ grow exponentially.




(c) Log plots of $\phi_1, H_1$ confirm the perturbations are proportional to $e^{\varsigma t}$, for $\varsigma \approx 0.011\, M_{\text{pl}}$.




(d) The slow-roll parameter $\epsilon$ dips sharply then slowly increases, allowing us to tune the number of $e$-folds via $\phi_1(0)$ and $H_1(0)$.

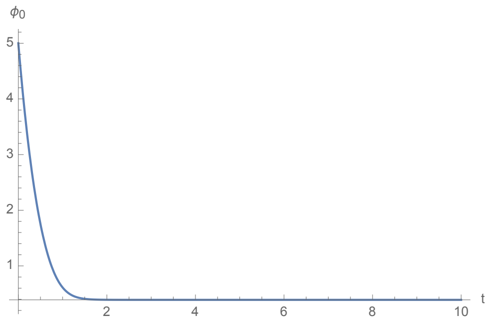
Figure 5.4: Exponentially growing first-order corrections to the linearly coupled massive scalar. Inflation and 60 $e$-folds are possible due to the shape of $\epsilon$, but the resulting spectral indices are unphysical.

We illustrate exponential growth in Fig. 5.4(b) for parameters $\alpha = 9\, M_{\text{pl}}$, $\kappa = 0.1\, M_{\text{pl}}^2$, and $H_0 = 0.1\, M_{\text{pl}}$, where $\phi_{\text{fix},+} \approx 0.05$. After $\phi_0$ converges to $\phi_{\text{fix},+}$, the corrections grow exponentially, with $\phi_1, H_1 \propto e^{\varsigma t}$ for $\varsigma \sim 0.011\, M_{\text{pl}}$. This is confirmed by the log plot in 5.4(c). In this scenario, the shape of $\epsilon$ allows us to tune the first-order initial conditions to get 60 $e$-folds, with

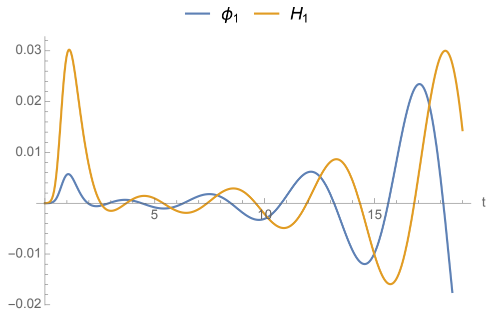
$$\phi_1(0)/\phi_0(0) = H_1(0)/H_0 = 1.5 \times 10^{-2}.$$

However, $\epsilon$ is neither small nor roughly constant over inflation, and other spectral indices are also ill-behaved.

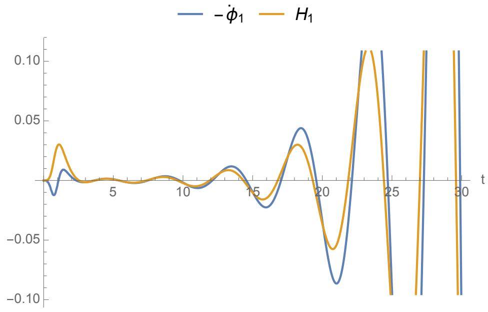
We illustrate first-order oscillations in Fig. 5.5(b). The parameters used for these results are $\alpha = 1\, M_{\text{pl}}$, $\kappa = 4\, M_{\text{pl}}^2$, and $H_0 = 0.5\, M_{\text{pl}}$. As before, the negative branch $\phi_0$ converges to the fixed point at $\phi_{\text{fix},+} \approx 0.39$. After $\phi_0$ has hit the bottom of the well, $\phi_1$ and $H_1$ begin to oscillate, with the envelope increasing exponentially, In other words, we have some sort of *resonance* phenomenon. This behaviour is largely independent of the initial conditions, but in the graphs we have $\phi_1(0) = 10^{-6}\phi_0(0)$ and $H_1(0) = 10^{-6}H_0$. The corrections $\phi_1$ and $H_1$ are out of phase; in fact, during resonance, $\dot{\phi}_1 \simeq -H_1$, as shown in 5.5(c). This means that $\epsilon$ and other spectral indices oscillate; we have no hope of good spectral behaviour in this case. Once again, the spectral indices are a sticking point.



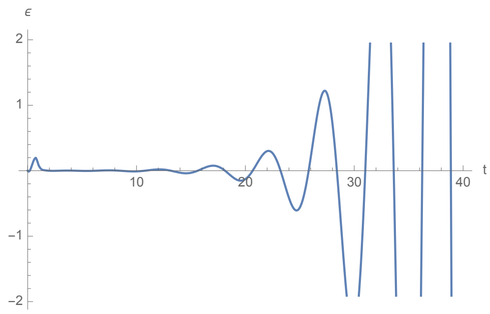

(a) The negative branch of $\phi_0$ converges to the fixed point at $\phi_{\text{fix},+} \approx 0.39$.




(b) After $\phi_0$ converges to $\phi_{\text{fix},+}$, the first-order corrections $H_1, \phi_1$ appear to oscillate with an exponentially growing envelope.




(c) A plot show $H_1 \approx -\dot{\phi}_1$, confirming the two are out of phase.




(d) Due to resonant oscillations in $H_1$, the slow-roll parameter $\epsilon$ also oscillates.

Figure 5.5: Resonant oscillations in the linearly coupled massive scalar. Although not a viable model of inflation, it resembles a simple version of *reheating*.

While not itself a viable model, the linearly coupled massive scalar provides some dynamical insight into scalar-tensor inflation. In particular, the resonant oscillations for the second set of parameters provides an intriguing parallel to *reheating*. This simple arrangement may yet have a part to play in model-building ventures.

## 5.4 The top down approach

We recall the scalar-tensor spectral indices (4.44)–(4.47), given by $\epsilon_1 = -\epsilon$, $\epsilon_2 = -\delta$, and

$$\epsilon_3 \equiv \frac{f'\dot{\phi}}{2Hf} = \frac{f_0'\dot{\phi}_0}{2H_0 f_0} + \mathcal{O}(\lambda) \tag{5.14}$$

$$\epsilon_4 \equiv \frac{\alpha'\dot{\phi}}{2H\alpha} = \frac{3f_0'\dot{\phi}_0}{2f_0 H_0}\left[\frac{2f_0 f_0'' - (f_0')^2}{f_0 - 3(f_0')^2}\right] + \mathcal{O}(\lambda), \tag{5.15}$$

where $\alpha = G^2$ as per (3.42) and (4.48). As discussed in §4.3, a well-behaved spectrum requires that the $\epsilon_i$ are small ($|\epsilon_i| \ll 1$) and slowly changing ($|\dot{\epsilon}_i| \ll |\epsilon_i|$). For $i = 1, 2$, this reproduces the familiar Hubble slow-roll conditions. The constraints for $i = 3, 4$ are new slow-roll conditions for scalar-tensor gravity. From (5.14) and (5.15), we obtain

$$\left|\frac{\dot{f}}{f}\right|, \left|\frac{\dot{\alpha}}{\alpha}\right| \ll H. \tag{5.16}$$

In other words, $f$ and the associated function $\alpha$ must change at a slow fractional rate on the Hubble scale. Moreover, the ratio between the fractional rates and $H$ should also change slowly over the course of inflation. This is close in spirit to the usual slow-roll constraints, but now involve the scalar-tensor coupling $f$ and the auxiliary function $\alpha$.

As we saw with our toy models, satisfying the spectral constraints is hard; simple choices of $f$ and $V$ do not yield good spectral behaviour. We now outline an alternative "top down" approach where we engineer a model to directly satisfy (5.16). As we show below, the tricky part is figuring out how $\epsilon_1$ behaves. In other words, it will be hard to guarantee we get inflation in the first place! This is the trade-off for direct control over the parameters $\epsilon_3$ and $\epsilon_4$.

The basic strategy is to set the zeroth order part of $\epsilon_3$ and $\epsilon_4$ to constants $\epsilon_3^{(0)}, \epsilon_4^{(0)}$ with $|\epsilon_3^{(0)}|, |\epsilon_4^{(0)}| \ll 1$. This simplicity assumption has far-reaching consequences. In the calculations developed here, we assume that $\epsilon_3^{(0)}, \epsilon_4^{(0)}$, and $f > 0$, but the broader program outlined at the end of the section does not depend on these choices. From (5.14), we first obtain

$$
\begin{aligned}
f'(\phi_0(t))\dot{\phi}_0 = \dot{f}(\phi_0(t)) &= 2\epsilon_3^{(0)} H_0 f(\phi_0(t)) \\
\implies \quad f(\phi_0(t)) &= C_f \exp(2\epsilon_3^{(0)} H_0 t)
\end{aligned}
\tag{5.17}
$$

for some constant $C_f$ with mass dimension 2. Similarly, from (5.15) we get

$$
\alpha(\phi_0(t)) = 1 + \frac{3[f'(\phi_0(t))]^2}{f(\phi_0(t))} = C_\alpha \exp(2\epsilon_4^{(0)} H_0 t), \tag{5.18}
$$

where $C_\alpha > 1$ is a dimensionless constant. Combining (5.17) and (5.18),

$$
\begin{aligned}
C_\alpha \exp(2\epsilon_4^{(0)} H_0 t) &= 1 + \frac{3[f'(\phi_0(t))]^2}{f(\phi_0(t))} = 1 + \frac{3[\dot{f}(\phi_0(t))]^2}{f(\phi_0(t))\dot{\phi}_0(t)^2} \\
&= 1 + \frac{12 C_f (\epsilon_3^{(0)} H_0)^2 \exp(2\epsilon_3^{(0)} H_0 t)}{\dot{\phi}_0(t)^2}.
\end{aligned}
\tag{5.19}
$$

We can rearrange (5.19) to find an expression for $\dot{\phi}_0(t)$:

$$
\begin{aligned}
\dot{\phi}_0(t) &= \pm \left[ \frac{12 C_f (\epsilon_3^{(0)} H_0)^2 \exp(2\epsilon_3^{(0)} H_0 t)}{C_\alpha \exp(2\epsilon_4^{(0)} H_0 t) - 1} \right]^{1/2} \\
&= \pm \left[ \frac{K}{C_\alpha - e^{-\beta t}} \right]^{1/2} e^{\gamma t},
\end{aligned}
\tag{5.20}
$$

where $\beta \equiv 2\epsilon_4^{(0)} H_0$, $\gamma \equiv \left(\epsilon_3^{(0)} - \epsilon_4^{(0)}\right) H_0$ and $K \equiv 12 C_f (\epsilon_3^{(0)} H_0)^2$. As usual, there are two branches for $\dot{\phi}_0$ (unless $K = 0$).

The long-term behaviour of $\phi_0(t)$ depends on the relative size of $\epsilon_3^{(0)}$ and $\epsilon_4^{(0)}$. If $\gamma > 0$, $\phi_0$ blows up in the long run, and if $\gamma < 0$, it is asymptotically constant. Finally, if $\gamma = 0$, then $\phi_0$ is asymptotically linear, with

$$
\dot{\phi}_0(t) \to \pm \sqrt{\frac{K}{C_\alpha}}.
$$

This is enough information to calculate $\epsilon_2$ (defined in (4.45)) at leading order:

$$\begin{aligned}\epsilon_2^{(0)} = \frac{\ddot{\phi}_0}{H_0\dot{\phi}_0} &= -\frac{\beta e^{-\beta t}}{2H_0(C_\alpha - e^{-\beta t})} + \frac{\gamma}{H_0} \\ &= \epsilon_3^{(0)} - \epsilon_4^{(0)} - \frac{\epsilon_4^{(0)}}{C_\alpha e^{\beta t} - 1}.\end{aligned} \tag{5.21}$$

Note that this expression holds for either branch of $\dot{\phi}_0$. Although $\epsilon_2^{(0)}$ is not constant, it quickly decays to $\epsilon_3^{(0)} - \epsilon_4^{(0)}$. Recalling the Jordan frame spectral index (4.52),

$$n_s - 1 \simeq 2\big(2\epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4\big)$$

we see that the last three terms *cancel* at zeroth order in the de Sitter perturbation. Since $\epsilon_1 = \mathcal{O}(\lambda)$, the scalar spectral index $n_s - 1 = \mathcal{O}(\lambda)$, i.e. first order in de Sitter perturbation. This tells us we are on the right track.

Now we come to the tricky part. The first order terms in $\epsilon_i$ (see (4.44)–(4.47)) are constructed from $\phi_1$ and $H_1$, along with zeroth order quantities. To calculate the first-order perturbations, equations (4.19) and (4.20) instruct us to use $f_0(t)$, $H_0$, $\dot{\phi}_0(t)$ and $\ddot{\phi}_0(t)$, the derivatives

$$f_0'(t) = \frac{\dot{f}_0(t)}{\dot{\phi}_0(t)}, \qquad f_0''(t) = \frac{1}{\dot{\phi}_0(t)^2}\big[\ddot{f}_0(t) - f_0'(t)\ddot{\phi}_0(t)\big],$$

and finally, the derivative of the potential $V'$. Given what we already know, we can calculate all of these except $V'$.

We briefly outline the program for determining $V$:

1. First, integrate (5.20) to find $\phi_0(t)$ explicitly. There will be two branches.
2. Write $\dot{\phi}_0$ as a function of $\phi_0$, with no explicit $t$ dependence.
3. Explicitly solve for $f(\phi)$ from (5.20) and our expression for $\phi_0(t)$ from Step 1.
4. Finally, compare $\dot{\phi}_0$ from Step 2 to the corresponding de Sitter expression (4.5). Substituting $f$ (from Step 3) into (4.5) gives a differential equation for $V$.

Actually implementing these steps is hard, but a key point is that $f$ and $V$ are *determined*, up to some free parameters ($\epsilon_3^{(0)}, \epsilon_4^{(0)}, C_f, C_\alpha$ and initial conditions) still in play. Once we have $V$, we can see how the $\epsilon_i$ behave at first order as a function of these parameters. Hopefully, by exploring this parameter space we can find a model of scalar-tensor inflation which satisfies the remaining observational criteria as well as the spectral constraints.

Chapter 6

# *Conclusion*

In nature's infinite book of secrecy
A little can I read.

—*Antony and Cleopatra* (I.2), Shakespeare

We started with a thumbnail review of the Big Bang model, and saw that fine-tuning of curvature and initial conditions are required to explain how the universe looks today. A period of rapid *inflationary* growth saves us from fine-tuning, but at the cost of introducing the enigmatic inflaton field. After setting up the slow-roll paradigm, we tried to identify the inflaton with the Higgs boson, the unique candidate in the Standard Model. Sadly, the Mexican hat potential of the Higgs is too bumpy to viably inflate a universe like ours.

Venturing out of the comfort zone of minimal coupling and general relativity, new possibilities opened up. We connected the Higgs directly to gravity, performed a local rescaling, and ended up with a new field, related to the Higgs but governed by general relativity. The new scalar had a Mexican hat potential we could manipulate in the large-field region; after wiggling the brim for a while, we were able to obtain slow-roll inflation which matches observational cosmology. This approach had the additional feature that, after inflation, general relativity was restored by the same symmetry-breaking mechanism that gives fundamental particles mass.

Introducing scalar-tensor interactions is a game-changer, but perhaps we were too quick to identify the Higgs with the rescaled field. The rescaling took us from a scalar-tensor theory where space is warped by the field value *and* curved by stress-energy (the Jordan frame) to general relativity, where space is curved but warping is transferred from space to the field (the Einstein frame). By carefully considering the rescaling dictionary, we argued that the original and warped scalar field are not identical for the purposes of inflation. Although we can translate back and forth between frames, it seems reasonable to assume that the frames are distinct and develop tools for inflation in the Jordan frame. While we were at it, we decided to generalise from the Higgs to an arbitrary scalar field.

By definition, inflation is a period where the Hubble parameter changes slowly. We therefore set up a perturbation expansion around a constant Hubble parameter, or pure de Sitter evolution. This lead to zeroth order constraints on the scalar-tensor coupling and potential; turning the crank of perturbation theory, we obtained coupled equations for the first-order perturbations, and in principle could ascend the perturbative ladder as high as we like. This took care of the classical side of things. On the quantum side, we needed to recycle the Jordan frame calculations of Hwang and Noh to figure out how lumpy, large-scale structures evolve during inflation.

Putting it all together, what sort of universe can we build? To find out, we applied our first-order perturbation theory to some toy models. The jury is still out on Higgs inflation in the Jordan frame, but we saw nontrivial behaviour for inflation driven by a Gaussian coupling, and more intriguingly, a linearly coupled massive scalar. Although these models hint at interesting dynamical possibilities, none met the observational and mathematical constraints on spectral behaviour. We finished with preliminary efforts to construct a model which directly meets these requirements.

There are a number of loose ends and open questions for future work. To test the validity of our perturbative approximations, we need to learn more about the higher order terms. We must also improve the numerical methods used to explore parameter space for specific models, such as Higgs inflation and the massive scalar. For generating phenomenologically viable scalar-tensor inflation, the top down program is promising and needs to be brought to fruition.

Inflation is now an essential part of the cosmic story. Despite its importance, we still know very little about what sort of physics sustains rapid growth for a short time—the controlled explosion that fixes the Big Bang model. But with the single-field slow-roll paradigm, the Bezrukov-Shaposhnikov model, and other results in the same vein, inflation has become a forum for particle physics and cosmology to interact fruitfully and address these questions. The Jordan frame approach is a small contribution to this enterprise. I hope that, in time, it will provide new ways for cosmology and particle physics to enrich each other, and teach us about the cosmos in the process.

# *Bibliography*


[1] Georges Aad et al. Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC. *Phys. Lett. B*, 716:1–29, 2012.

[2] P. A. R. Ade et al. Planck 2015 results. XIII. Cosmological parameters. 2015.

[3] Andreas Albrecht and Paul J. Steinhardt. Cosmology for grand unified theories with radiatively induced symmetry breaking. *Phys. Rev. Lett.*, 48:1220–1223, Apr 1982.

[4] Daniel Baumann. The physics of Inflation, 2012. University of Cambridge graduate lecture course, based on lectures for TASI 2009.

[5] Daniel Baumann. Cosmology (University of Cambridge, Part III lecture notes), 2013.

[6] Fedor Bezrukov. The Higgs field as an inflaton. *Class. Quantum Grav.*, 21(31), 2013.

[7] Fedor Bezrukov and Mikhail Shaposhnikov. The Standard Model Higgs boson as the inflaton. *Phys. Lett. B*, 659(3), 2008.

[8] N. D. Birrell and P. C. W. Davies. *Quantum Fields in Curved Space.* Cambridge University Press, 1982.

[9] Carl H. Brans and Robert H. Dicke. Mach's principle and a relativistic theory of gravitation. *Phys. Rev.*, 3(124), 1961.

[10] Sean Carroll. *Spacetime and Geometry: An Introduction to General Relativity.* Addison-Wesley, 2004.

[11] Serguei Chatrchyan et al. Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC. *Phys. Lett. B*, 716:30–61, 2012.

[12] Robert H. Dicke. Mach's principle and invariance under transformation of units. *Phys. Rev.*, 2163(125), 1962.

[13] S. Dodelson. *Modern Cosmology.* Academic Press, 2003.

[14] Scott Dodelson. Coherent phase argument for inflation. *AIP Conf. Proc.*, 689: 184–196, 2003.

[15] F. Englert and R. Brout. Broken Symmetry and the Mass of Gauge Vector Mesons. *Physical Review Letters*, 13:321–323, August 1964.

[16] Valerio Faraoni. *Cosmology in Scalar-Tensor Gravity.* Kluwer Academic Publishers, 2004.

[17] G. S. Guralnik, C. R. Hagen, and T. W. B. Kibble. Global conservation laws and massless particles. *Phys. Rev. Lett.*, 13:585–587, Nov 1964.

[18] Alan H. Guth. Inflationary universe: A possible solution to the horizon and flatness problems. *Phys. Rev. D*, 23:347–356, Jan 1981.

[19] P. W. Higgs. Broken Symmetries and the Masses of Gauge Bosons. *Physical Review Letters*, 13:508–509, October 1964.

[20] Qing-Guo Huang and Sai Wang. No evidence for the blue-tilted power spectrum of relic gravitational waves. *JCAP*, 1506(06):021, 2015.

[21] J.-c. Hwang. Perturbative semiclassical approximation in the uniform curvature gauge. *Class. Quant. Grav.*, 11:2305–2316, September 1994.

[22] Jai-chan Hwang. Perturbations of the Robertson-Walker space - Multicomponent sources and generalized gravity. *Astrophys. J.*, 375:443–462, 1991.

[23] Jai-chan Hwang. Unified analysis of cosmological perturbations in generalized gravity. *Phys. Rev. D*, 53:762–765, 1996.

[24] Jai-chan Hwang. Quantum fluctuations of cosmological perturbations in generalized gravity. *Class. Quant. Grav.*, 14:3327–3336, 1997.

[25] David I. Kaiser. Frame independent calculation of spectral indices from inflation. *Submitted to Phys. Lett. B*, 1995.

[26] David I. Kaiser. Primordial spectral indices from generalized einstein theories. *Phys. Rev. D*, 52:4295–4306, Oct 1995.

[27] Andrew R. Liddle, Paul Parsons, and John D. Barrow. Formalizing the slow-roll approximation in inflation. *Phys. Rev. D*, 50:7222–7232, Dec 1994.

[28] A. D. Linde. A new inflationary universe scenario: A possible solution of the horizon, flatness, homogeneity, isotropy and primordial monopole problems. *Phys. Lett. B*, 108:389–393, February 1982.

[29] A. D. Linde. Chaotic inflation. *Phys. Lett. B*, 129:177–181, September 1983.

[30] C. J. Moore, R. H. Cole, and C. P. L. Berry. Gravitational-wave sensitivity curves. *Class. Quant. Grav.*, 32(1):015014, 2015.

[31] V. Mukhanov and S. Winitzki. *Quantum Effects in Gravity.* Cambridge University Press, 2007.

[32] V. F. Mukhanov and G. V. Chibisov. Quantum fluctuations and a nonsingular universe. *JETP Lett.*, 33:532, May 1981.

[33] Mikio Nakahara. *Geometry, Topology, and Physics*. Taylor and Francis Group, 2nd edition, 2003.

[34] Hyerim Noh and Jai-chan Hwang. Inflationary spectra in generalized gravity: Unified forms. *Phys. Lett. B*, 515:231–237, 2001.

[35] K. A. Olive et al. Review of Particle Physics. *Chin. Phys.*, C38, 2014.

[36] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, editors. *NIST Handbook of Mathematical Functions*. Cambridge University Press, New York, NY, 2010.

[37] A. A. Penzias and R. W. Wilson. A Measurement of Excess Antenna Temperature at 4080 Mc/s. *Astrophysics Journal*, 142:419–421, July 1965.

[38] Tristan L. Smith, Marc Kamionkowski, and Asantha Cooray. The inflationary gravitational-wave background and measurements of the scalar spectral index. *Phys. Rev. D*, 78:083525, 2008.

[39] Alexei A. Starobinsky. Spectrum of relict gravitational radiation and the early state of the universe. *JETP Lett.*, 30:682–685, 1979.

[40] Alexei A. Starobinsky. A New Type of Isotropic Cosmological Models Without Singularity. *Phys. Lett. B*, 91:99–102, 1980.

[41] Robert Wald. *General Relativity*. University of Chicago Press, 1984.

[42] Steven Weinberg. Adiabatic modes in cosmology. *Phys. Rev. D*, 67:123504, 2003.

[43] Hermann Weyl, editor. *Space-Time-Matter*. Dover, New York, 1950. English translation of the 4th edition (1921) of *Raum-Zeit-Materie* by Henry L. Brose.