Problems in Real Analysis

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Curry's Paradox

We start with a simple identity due to philosopher Charles Peirce (1839–1914). An odd consequence is *Curry's paradox*, discovered by logician Haskell Curry (1900–1982). Like Russell's paradox (discussed in lectures), the paradox arises when we allow *self-reference*.

1. Show that Peirce's law,
$$((p \implies q) \implies p) \implies p$$
is a tautology via truth tables.
2. Suppose that $(p \implies q) \Longleftrightarrow p$ is true. Use part (a) and modus ponens to deduce q.
3. Argue informally that, for any proposition q, the self-referential sentence
$$p = \text{“If } p \text{ is true, then it implies } q\text{”}$$
satisfies $(p \implies q) \Longleftrightarrow p$.
4. Combine (b) and (c) to conclude that anything is true. What has gone wrong here?

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The Terrible Dynasties

Sets A and B are said to have the same *cardinality* if there exists a bijection (one-to-one, onto function) $f : A \to B$. Cardinality lets us think about the size of *infinite* sets.

1. For an infinite set X, consider a map $f : X \to \mathcal{P}(X)$. Show that f cannot be onto by considering the subset $R = \{x \in X : x \notin f(x)\}$. This means that sets are always "smaller" than their power sets. This result was proved by the founder of set theory, Georg Cantor (1845–1918). Hint: This is very similar to Russell's paradox.
2. Let $\aleph_0$ denote $|\mathbb{N}|$, the cardinality of the natural numbers. We call any cardinal of an infinite set an *infinite cardinal*; if you like, it is a "type of infinity". Let
$$\aleph_{n+1} \equiv |\mathcal{P}(A_n)|,$$
where A_n is a set with cardinality $\aleph_n$. Using part (a), argue that there is a tower of ever-bigger infinite cardinals
$$\aleph_0, \aleph_1, \aleph_2, \ldots$$
In other words, there is an infinite number of different infinities!

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Models and Non-implication

Suppose that we have a binary operation $\otimes$ ("bizarro" multiplication), which could have the following properties:

$$x \otimes (y \otimes z) = (x \otimes y) \otimes z \tag{A}$$

$$x \otimes x = x \tag{B}$$

$$(x \otimes y) \otimes z = x \otimes z. \tag{C}$$

If we want to show that some mathematical statements $A_1, A_2, \ldots, A_n$ (such as axioms) do not imply some other statement B, we need only find a single *model* of the situation where $A_1, A_2, \ldots, A_n$ are true but B is false. Truth tables are a special case of this, where we show a statement is *not* a tautology by finding a single assignment of truth values (a "model") which makes it false.

1. Show that if $x \otimes y \equiv \max(x, y)$, then $\otimes$ satisfies (A) and (B) but not (C).
2. Find a binary operator which which satisfies (A) and (C) but not (B).

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Zippers and Hypercubes

Consider a real number in the unit interval, $x \in [0, 1]$. We can expand x as an infinite decimal

$$x = 0.d_1 d_2 d_3 \ldots, \quad d_i \in \{0, 1, \ldots, 9\}.$$

Thus, a real number between 0 and 1 can be represented as an infinite sequence of digits.

1. Are these digit sequence representations unique? If not, can we adopt conventions to make them unique?
2. Find a procedure to "smush" two digit sequences together to form a third sequence. Your procedure should be reversible, that is, you should be able to "un-smush" a digit sequence to uniquely recover the two digit sequences which were smushed to make it.
3. Use your answer to (b) to find a correspondence between the unit interval $[0, 1]$ and the unit square $[0, 1]^2 = [0, 1] \times [0, 1]$.
4. Extend the procedure from (b) to n digit sequences, and therefore deduce a correspondence between the unit interval [0, 1] and the n-cube

   $$[0, 1]^n = \overbrace{[0, 1] \times \cdots \times [0, 1]}^{n \text{ times}}.$$

   Remarkably, this shows that the unit interval is the *same size* (in the sense of set theory) as the unit hypercube in n dimensions!

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Taming the Tails

A *summation machine* is an operator $\mathcal{S}$ which takes a sequence of real numbers and either (a) produces out a real number, or (b) gives up. We write the result of applying the machine to a sequence $\{a_1, a_2, a_3, \ldots\}$ as

$$\mathcal{S}\left[\sum_{n=1}^{\infty} a_n\right].$$

In the first case, we interpret the number it spits out as the result of adding all the numbers up, and say the series $\sum_n a_n$ *converges according to* $\mathcal{S}$. In the second, we say the series is *divergent* according to that procedure. In order to get a sensible addition operator $\mathcal{S}$, we impose two additional contraints:

$$\mathcal{S}\left[\sum_{n=1}^{\infty} a_n\right] = a_1 + \mathcal{S}\left[\sum_{n=2}^{\infty} a_n\right] \qquad \text{(additivity)}$$

$$\mathcal{S}\left[\alpha \sum_{n=1}^{\infty} a_n + \beta \sum_{n=1}^{\infty} b_n\right] = \alpha\mathcal{S}\left[\sum_{n=1}^{\infty} a_n\right] + \beta\mathcal{S}\left[\sum_{n=1}^{\infty} b_n\right]. \qquad \text{(linearity)}$$

1. Using additivity and linearity, show that if the following series converge according to $\mathcal{S}$, they must take specific values:

   (a) *Grandi's series:*

   $$\mathcal{S}[1 - 1 + 1 - 1 + \cdots] = \mathcal{S}\left[\sum_{n=0}^{\infty} (-1)^n\right] = \frac{1}{2}.$$

   (b) *Alternating natural numbers:*

   $$\mathcal{S}[1 - 2 + 3 - 4 + \cdots] = \mathcal{S}\left[\sum_{n=1}^{\infty} (-1)^n n\right] = \frac{1}{4}.$$

   Hint: Use (a) and additivity.

   (c) *Natural numbers:*

   $$\mathcal{S}[1 + 2 + 3 + 4 + \cdots] = \mathcal{S}\left[\sum_{n=1}^{\infty} n\right] = -\frac{1}{12}.$$

   Hint: Use (b), linearity, and $L - 4L = -3L$, where L is the limit.

2. The *Cesàro sum* (Ernesto Cesàro, 1859–1906) is the limit of the average of the first N partial sums:

   $$\mathcal{C}\left[\sum_{n=1}^{\infty} a_n\right] = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \sum_{n=1}^{n} a_n = \frac{1}{N} \sum_{k=1}^{N} S_k.$$

   Check this is a summation machine, and verify that Grandi's series converges.

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Simple Polylogarithms

We will investigate the (non-examinable) series

$$a_m(n) = \sum_{k=1}^{\infty} \frac{k^n}{m^k}.$$

(a) Show using an appropriate test that $a_m(n)$ converges for any $n \in \mathbb{N} \cup \{0\}$ and $|m| > 1$.

(b) What is $a_m(0)$? Your answer will depend on m.

(c) Show that

$$a_m(n) = \frac{1}{m} + \sum_{k=1}^{\infty} \frac{(k+1)^n}{m^{k+1}} = \frac{1}{m}\left(1 + \sum_{k=1}^{\infty} \frac{(k+1)^n}{m^k}\right).$$

(d) Recall the *binomial theorem*

$$(k+1)^n = \sum_{j=0}^{n} \binom{n}{j} k^j.$$

Using this identity, prove that

$$a_m(n) = \frac{1}{m-1}\left[1 + \sum_{j=0}^{n-1} \binom{n}{j} a_m(j)\right].$$

HINT: You are allowed to swap the order of an infinite summation $\sum_{k=1}^{\infty}$ and a finite summation $\sum_{j=0}^{n}$.

(e) We can calculate $a_m(0)$ using the results of (b). Using the identity in part (d), we can iteratively calculate any $a_m(n)$ we like! Put theory into practice, and explicitly evaluate the series

$$\sum_{k=1}^{\infty} \frac{1}{2^k}, \quad \sum_{k=1}^{\infty} \frac{k}{2^k}, \quad \sum_{k=1}^{\infty} \frac{k^2}{2^k}, \quad \sum_{k=1}^{\infty} \frac{k^3}{2^k}.$$

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Pi production

Power series and Taylors theorem give us a powerful machine for representing functions and constants. For instance, using the Taylor series for tangent (and Abels theorem since we evaluated at an endpoint), we found that

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^n}{2n+1} + \cdots$$

In this problem, we prove a curious infinite product identity for π. For $n = 0, 1, 2, \ldots$, let

$$I(n) = \int_0^\pi dx\, \sin^n(x).$$

1. Show that $I(0) = \pi$ and $I(1) = 2$. Using $sin^2 x + \cos^2 x = 1$, and integration by parts, deduce that for $n \geq 2$,
$$I(n) = \frac{n-1}{n} I(n-2).$$

2. Use induction and (1) to prove that
$$I(2n) = \pi \prod_{k=1}^{n} \frac{2k-1}{2k}, \quad I(2n+1) = 2 \prod_{k=1}^{n} \frac{2k}{2k+1}.$$

3. By comparing integrands, show that $I(2n+1) \leq I(2n) \leq I(2n-1)$. Divide through by $I(2n+1)$ and use (1),
$$\lim_{n\to\infty} \frac{I(2n)}{I(2n+1)} = 1.$$

4. Rewriting the limit in (3), obtain the final result:
$$\frac{\pi}{2} = \lim_{n\to\infty} \prod_{k=1}^{n} \left(\frac{2k}{2k-1} \cdot \frac{2k}{2k+1} \right) = \left(\frac{2}{1} \cdot \frac{2}{3} \right) \left(\frac{4}{3} \cdot \frac{4}{5} \right) \left(\frac{6}{5} \cdot \frac{6}{7} \right) \cdots$$

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Fourier from Power Series

Adding *negative* powers x^{-n} to a power series yields what is called a *Laurent series*. These converge on an *annulus* rather than a disc. Laurent series are important in complex analysis, where instead of real x, we have a function F of a complex variable $z \in \mathbb{C}$:

$$F(z) = \sum_{k \in \mathbb{Z}} a_k z^k.$$

We can use these to derive Fourier series.

1. We can restrict z to the unit circle in $\mathbb{C}$ via $z = e^{i\theta}$. Let $f(\theta) = F\left(e^{i\theta}\right)$. Argue that the functin f is periodic with period 2π, and give a Laurent series for $f(\theta)$.

2. Integrate $f(\theta)e^{-i\ell\theta}$ for $\ell \in \mathbb{Z}$, $\theta \in [0, 2\pi)$. Use this to give an integral expression for a_k in terms of $f(\theta)$.

   Hint: You may interchange integration and summation.

3. Suppose that $f(\theta)$ is real. Using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, and writing $a_k = b_k + ic_k$, show that

   $$f(\theta) = \frac{1}{2}B_0 + \sum_{n=1}^{\infty} B_n \cos(n\theta) + C_n \sin(n\theta),$$

   where $B_n = \frac{1}{2}(b_n + b_{-n})$ and $C_n = \frac{1}{2}(c_{-n} - c_n)$.

4. Convert your answer from (b) into an integral for B_n and C_n in terms of $f(\theta)$.

To complete our derivation, we still need to prove that (1) any periodic real function f has a suitable F, and (2) that F has a Laurent series which converges on the unit circle in $\mathbb{C}$. You will need to wait for your complex analysis course!