Perturbative Jordan-frame inflation

David Wakeham (MSc)
Supervisor: Ray Volkas
University of Melbourne

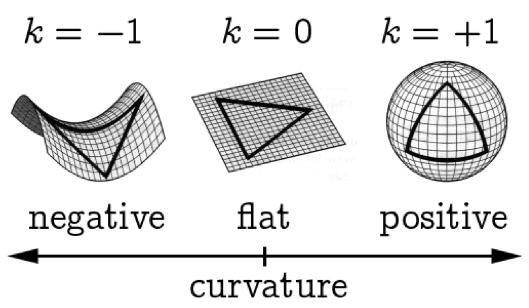
Introduction

Inflation is a period of **accelerated cosmic growth** in the early universe. It solves tuning problems with the Big Bang model and makes quantitative predictions about late-time structure. Inflation can be generated by a single scalar field called the **inflaton**.

We **directly couple** the inflaton and gravity, and trial a perturbative approach to inflation in the directly coupled scalar-tensor theory (**Jordan frame**) rather than a conformally rescaled theory with no direct coupling (**Einstein frame**).

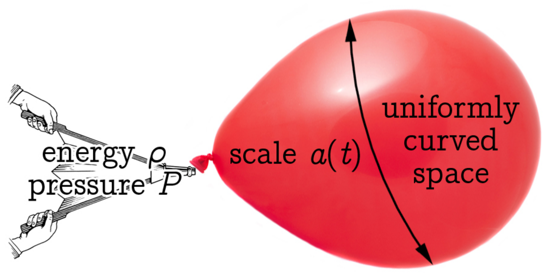
Big Bang cosmology

The **Cosmological Principle** states that, at large scales, the universe looks the same everywhere and in all directions. This implies **Hubble's recession law** (speed of recession $\propto$ distance) and a **uniform spatial geometry**. There are three such spatial geometries: a positively curved **sphere**, negatively curved **saddle** or **flat space**:



$$ds^2 = dt^2 - a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right] \qquad \text{(M)}$$

Einstein's equations imply that the uniform geometry expands in response to large-scale fluids (**energy** ρ and **pressure** P). Scaling is tracked by the **scale factor** $a(t)$, which evolves according to the **Friedmann equations**:



$$H^2 \equiv \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \qquad \text{(F1)}$$

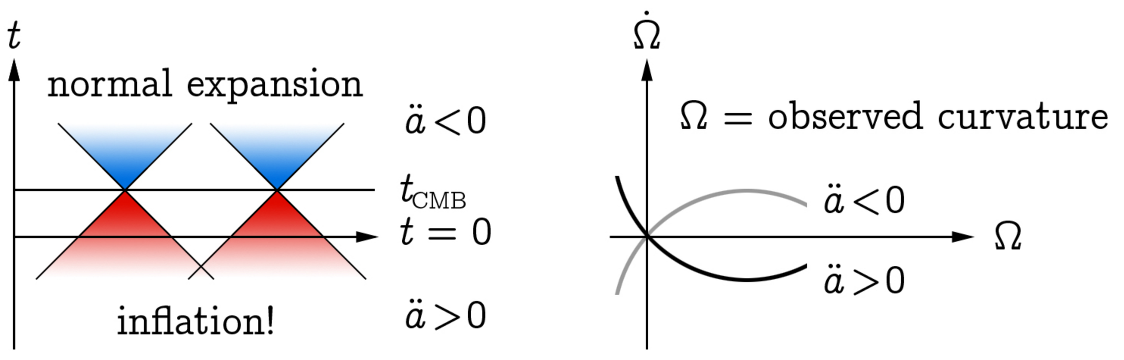
$$\frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3P) \qquad \text{(F2)}$$

Note we have defined the **Hubble parameter** $H \equiv \dot a/a$, the fractional rate of change of the scale factor. H is closely connected to causality: the **horizon radius** $r_H \equiv aH$ is (roughly speaking) the maximal size of an interacting patch of space.

Inflation to the rescue

The model of a uniform geometry "blown up" by large-scale fluids has two serious problems:

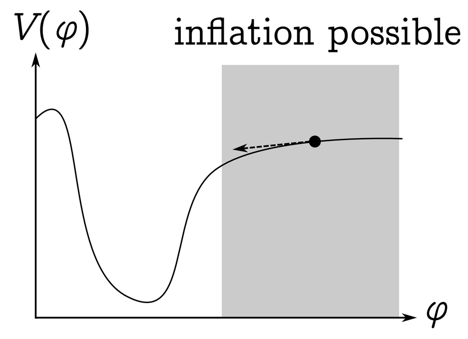
- Space is extremely **flat**, with $\Omega \simeq 0$ where Ω is the energy density parameter (think of it as observed curvature). But $\Omega = 0$ is an **unstable** fixed point of cosmological evolution!
- For conventional stress-energy sources, spacetime splits into many **disjoint patches** ($\dot r_H > 0$), so uniformity (e.g. of the CMB) requires **fine-tuning**.

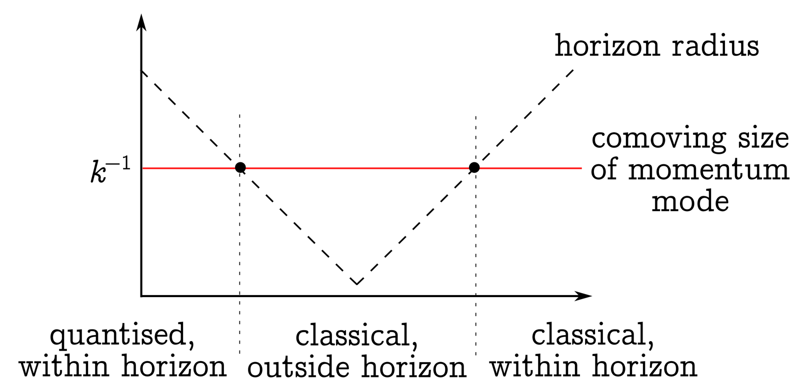


The solution to both problems is an initial period of **inflation**, where $\dot r_H < 0$ or equivalently $\ddot a > 0$. This turns $\Omega = 0$ into a **stable** fixed point (explaining observed flatness) and allows patches to interact early on (explaining observed uniformity).

Slow-roll and spectra

The **inflaton** is a hypothetical scalar ϕ which creates inflation ($\ddot a > 0$) when its potential $V(\phi)$ is greater than its kinetic energy $K(\phi) = \dot\phi^2/2$. For $V \gg K$, we get **slow-roll inflation** on flat patches ($|\dot H| \ll H^2$) of the potential curve:





The growth of large-scale structure (e.g. anisotropies in the CMB) is governed by **quantum ripples** of the inflaton ϕ. Momentum modes ϕ_k start life as quantum operators, with length scale $k^{-1} < r_H$. As r_H shrinks, modes cross the horizon and "freeze" into classical stochastic fields. After inflation, r_H grows and the modes reenter the horizon, seeding large-scale structure. In addition to inflaton (scalar) ripples, inflation generates weak **gravitational waves** (tensor). Both scalar and tensor ripples obey power law distributions, with **spectral indices** n_s and n_t respectively. These are constrained by observation, with $n_s \simeq 0.97$. Another observable is the ratio of tensor to scalar **amplitudes**, $r \simeq 0.11$.

Scalar-tensor gravity

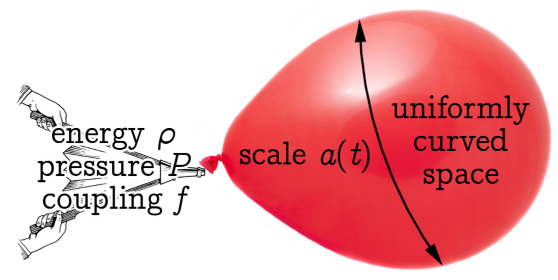
The inflaton Lagrangian in general relativity is

$$\mathcal{L}_{\mathrm{GR}} = \overbrace{\sqrt{-g}}^{\substack{\text{volume element}\\ \text{normalisation}}}\left[\overbrace{(16\pi G_N)^{-1}\mathcal{R}}^{\text{Einstein-Hilbert term}} + \overbrace{T(\phi) - V(\phi)}^{\text{scalar terms}}\right].$$

We consider **scalar-tensor gravity**, where the constant $(16\pi G_N)^{-1}$ is replaced by a function of the inflaton field, $f(\phi)$. This enforces a direct scalar-tensor interaction:

$$\mathcal{L}_{\mathrm{ST}} = \sqrt{-g}\left[\overbrace{f(\phi)\mathcal{R}}^{\text{interaction}} + T(\phi) - V(\phi)\right].$$

It is possible to physically motivate direct couplings, but for our purposes, we view it as a model-building strategy. Once again restricting the metric to (M) via the Cosmological Principle, we obtain scalar-tensor analogues of the Friedmann equations (F1) and (F2):



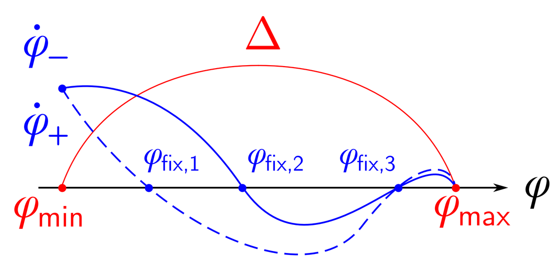
$$6(H^2 f + H\dot f) = \frac{1}{2}\dot\phi^2 + V \qquad \text{(ST1)}$$

$$H\dot f - 2f\dot H = \frac{1}{2}\dot\phi^2 + \ddot f. \qquad \text{(ST2)}$$

These equations couple scalar (ϕ) and metric (H) degrees of freedom. Rescaling the metric conformally $g_{\mu\nu} \to \omega^2 g_{\mu\nu}$ can eliminate the coupling, but may lead to a physically inequivalent frame [3]. We thus look for regimes where equations (ST1)–(ST2) are tractable.

De Sitter inflation

We defined inflation as a period where $\ddot a > 0$. This is equivalent to slow fractional change in H over the Hubble time H^{-1}, or $|\dot H/H^2| < 1$. We thus start by considering $\dot H = 0$, or **de Sitter inflation**, since a spacetime with constant H is de Sitter. In this case, (ST1) and (ST2) imply a quadratic equation for $\dot\phi$. For some $A(\phi)$, $B(\phi)$, and discriminant $\Delta(\phi)$:



$$\dot\phi_\pm = \frac{1}{2A(\phi)}\left[-B(\phi) \pm \sqrt{\Delta(\phi)}\right]. \qquad \text{(dS)}$$

On the left, we sketch the dynamics: **consistent** de Sitter inflation is only possible where $\Delta(\phi) > 0$. There are two branches $\dot\phi_\pm$, which share fixed points $\dot\phi_\pm(\phi_{\mathrm{fix}}) = 0$. Consistency imposes a general constraint on the choice of f, V, and field values assumed during inflation.

Perturbing de Sitter and quantum effects

This is neat, but unrealistic. To introduce slow time variation into H, we **perturbatively expand** around de Sitter inflation:

$$H(t) = H_0 + \lambda H_1(t) + \ldots, \qquad \phi(t) = \phi_0(t) + \lambda\phi_1(t) + \ldots$$

The zeroth-order terms H_0 and ϕ_0 are governed by (dS). To determine first-order corrections, we plug the perturbations into (ST1) and (ST2), arriving at the matrix DE

$$\dot{\mathbf{x}}_1(t) = M(t)\mathbf{x}_1(t), \qquad \mathbf{x}_1(t) \equiv [\phi_1(t), H_1(t)]^{\mathrm{T}},$$

where $M(t)$ is a matrix depending on H_0, ϕ_0, f, V. A decidedly more elaborate calculation [4] extracts the observables n_s, n_t and r for given model parameters.

Future directions

To match observational constraints, we need to choose appropriate model parameters. We are currently exploring a **top down approach**: impose good spectral behaviour, which determines functional forms for $\dot\phi_0(t)$ and $f(t)$; integrate and invert $\dot\phi_0(t)$ to find $\dot\phi_0(\phi_0)$ and $f(\phi_0)$; finally, by comparison to (dS), deduce the potential $V(\phi_0)$.

The perturbative approach makes model-independent predictions, with interesting connections between n_t, r and f. For instance, for $f \gtrsim G_N^{-1}$, it implies $|n_t| \sim 0.1$. Whether this (or other considerations) rules out the class of perturbative Jordan-frame models is being investigated. However, the hope is to provide a new and flexible model-building resource for cosmology and particle physics.

References

1. Daniel Baumann (2009). *The Physics of Inflation*, TASI lectures.
2. Fedor Bezrukov, Mikhail Shaposhnikov (2008). *Phys. Lett. B*, 659(3).
3. Valerio Faraoni (2004). *Cosmology in Scalar-Tensor Gravity*, Kluwer Academic Publishers.
4. Jai-chan Hwang (1994). *Class. Quant. Grav.*, 11: 2305-2316.