... marginal normal distribution.

- Conditional normal distribution

$a^T b = b^T . a$

$\overset{3\times 2}{[\quad]}\overset{2\times 3}{[\quad]} \qquad \overset{3\times 2}{[}$

$$E[(x - E[x])(x - E[x])^T]$$

$$= E[(x - E[x])(x^T - E[x]^T)]$$

$$= E(xx^T - E[x]x^T - E[x]^T x - \underbrace{E[x]E[x]^T}_{})$$

$$E(xx^T) \underbrace{- \mu x^T - \mu^T x}_{??} \qquad \cancel{-\mu\mu^T - 1\Sigma}$$

$$\cancel{\mu\mu^T} + \Sigma \qquad - \cancel{\mu\mu^T}$$

1 Gaussian Distribution (Multivariate)

$$N(x|\mu, \Sigma) = \frac{1}{(2\pi)^{D/2}} \frac{1}{|\Sigma|^{1/2}} e^{\{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\}}$$

- x have D-dimension (vector)
- μ is D-dimension MEAN vector
- Σ is $D \times D$ covariance matrix $\rightarrow |\Sigma|$ is determinant

VD $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \rightarrow D = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Đối chiếu ① và ②

① $= -\frac{1}{2}\left(x_a^T \Lambda_{aa}(x_a - \mu_a) - \mu_a^T \Lambda_{aa}(x_a - \mu_a)\right)$

$-\frac{1}{2}\left(x_a^T \Lambda_{ab}(x_b - \mu_b) - \mu_a^T \Lambda_{ab}(x_b - \mu_b)\right)$

...

$= -\frac{1}{2} x_a^T \Lambda_{aa} x_a + \frac{1}{2} x_a^T \Lambda_{aa}\mu_a + \frac{1}{2}\mu_a^T \Lambda_{aa} x_a$

$- \frac{1}{2}\cancel{\mu_a^T} \Lambda_{aa}\mu_a - \frac{1}{2} x_a^T \Lambda_{ab} x_b + \frac{1}{2} x_a^T \Lambda_{ab}\mu_b$

$+ \frac{1}{2}\mu_a^T \Lambda_{ab} x_b - \frac{1}{2}\mu_a^T \Lambda_{ab}\mu_b + \dots$

$= -\frac{1}{2} x_a^T \Lambda_{aa} x_a + \frac{1}{2} x_a^T \{\Lambda_{aa}\mu_a - \Lambda_{ab}(x_b - \mu_b)\}$

$+ \dots$

② $= -\frac{1}{2} x^T \Sigma^{-1} x + x^T \Sigma^{-1}\mu + const$

$\Rightarrow \boxed{\Sigma_{a|b} = \Lambda_{aa}^{-1}}$

$\mu_{a|b} = \Sigma_{a|b}\{\Lambda_{aa}\mu_a - \Lambda_{ab}(x_b - \mu_b)\}$

$\mu_{a|b} = \Lambda_{aa}^{-1}\Lambda_{aa}\mu_a - \Lambda_{aa}^{-1}\Lambda_{ab}(x_b - \mu_b)$

$\Rightarrow \boxed{\mu_{a|b} = \mu_a - \Lambda_{aa}^{-1}\Lambda_{ab}(x_b - \mu_b)}$

KLONG

The functional dependence of the Gaussian on X

$$\Delta^2 = (x-\mu)^T \Sigma^{-1} (x-\mu) \quad \boxed{2.44}$$

quadratic form

Mahalanobis distance from μ to x

[when Σ is the identity matrix => Euclidean distance

Note: Σ is symmetric (any antisymmetric component would disappear from exponent)

+ Proof normalization

COVARIANCE MATRIX:

$$\Sigma u_i = \lambda_i u_i$$

eigen vectors

eigen values

$$\Rightarrow u_i^T u_j = I_{ij} \quad \left[\begin{array}{ll} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{array}\right.$$

VD.

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \begin{pmatrix} u_1 & u_2 & \cdots & u_n \end{pmatrix} = \begin{pmatrix} u_{11}=1 & 0 & \\ 0 & u_{22}=1 & \\ 0 & 0 & \ddots \end{pmatrix}$$

$$\Rightarrow \Sigma = \sum_{i=1}^{D} \Lambda_i u_i u_i^T = \begin{pmatrix} \Lambda_1 & & \\ \vdots & \Lambda_2 & \\ \vdots & \cdot & \Lambda_n \end{pmatrix}$$

??

$$\Rightarrow \Sigma^{-1} = \sum_{i=1}^{D} \frac{1}{\lambda_i} u_i u_i^T = \begin{pmatrix} \frac{1}{\Lambda_1} & \cdot & \cdot \\ \vdots & 1/\Lambda_2 & \cdot \\ \vdots & \cdot & 1/\Lambda_n \end{pmatrix}$$

Applying for $\boxed{2.44}$

$$\Delta^2 = \underbrace{(x-\mu)^T}_{1\times D} \sum_{i=1}^{D} \frac{1}{\Lambda_i} \underbrace{u_i}_{D\times 1} \underbrace{u_i^T}_{1\times D} \underbrace{(x-\mu)}_{D\times 1}$$

$$y_i = u_i^T (x-\mu)$$

$$\Delta^2 = \sum_{i=1}^{D} \frac{y_i^2}{\Lambda_i}$$

---> y_i là toạ độ mới đc tạo ra bởi vector u_i (eigen vectors)

the original vector x_i

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2 Conditional Gaussian Distributions.

We partition x into two disjoint subset x_a and x_b

C_1 (precision matrix)

$$x = \begin{pmatrix} x_a \\ x_b \end{pmatrix}$$

$$\mu = \begin{pmatrix} \mu_a \\ \mu_b \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{pmatrix}$$

Σ_{aa} and Σ_{bb} are symmetrics

$\Sigma_{ba} = \Sigma_{ab}^T$

Inverse the covariance matrix. $\Lambda \equiv \Sigma^{-1}$

$$\Lambda = \begin{pmatrix} \Lambda_{aa} & \Lambda_{ab} \\ \Lambda_{ba} & \Lambda_{bb} \end{pmatrix}$$

$\Lambda_{ab} = \Lambda_{ba}^T$

Λ_{aa} and Λ_{bb} are still symmetrics

Phân ưu (Quadratic form.)

$$-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu) = -\frac{1}{2}(x_a-\mu_a)^T\Lambda_{aa}(x_a-\mu_a)$$

$$-\frac{1}{2}(x_a-\mu_a)^T\Lambda_{ab}(x_b-\mu_b) - \frac{1}{2}(x_b-\mu_b)^T\Lambda_{ba}(x_a-\mu_a)$$

$$-\frac{1}{2}(x_b-\mu_b)^T\Lambda_{bb}(x_b-\mu_b) \quad ??? \quad (1)$$

(Vì $e^a \cdot e^b = e^{a+b}$)

$\Rightarrow p(x_a|x_b)$ will be gaussian (giống form)

$$N(x|\mu,\Sigma) = -\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)$$

$$= -\frac{1}{2}\left(x^T\Sigma^{-1}(x-\mu) - \mu^T\Sigma^{-1}(x-\mu)\right)$$

$$= -\frac{1}{2}\left(x^T\Sigma^{-1}x - x^T\Sigma^{-1}\mu - \mu^T\Sigma^{-1}x + \mu^T\Sigma^{-1}\mu\right)$$

(2)

$$= -\frac{1}{2}(x^T\Sigma^{-1}x + x^T\Sigma^{-1}\mu \underbrace{- \frac{1}{2}\mu^T\Sigma^{-1}\mu}_{\text{Const}}$$

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Δ^2 tạo thành ellipsoids (center at μ and their axes oriented along u_i)

Có $y = U(x - \mu)$

matrix whose rows is u_i^T $\quad U = \begin{pmatrix} u_1 & u_2 & \cdots & u_n \end{pmatrix}$

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

chuyển hệ x sang y $\quad J_{ij} = \dfrac{\partial x_i}{\partial y_j} = U_{ji}$ (element of U^T)

Có U is orthonormality

$\Rightarrow |J|^2 = |U^T|^2 = |U^T||U| = |U^T U| = |U| = 1$

Vì $|J| = 1 \Rightarrow$ Determinant $|\Sigma|$ of covariance matrix ??? can be write as

$$|\Sigma|^{1/2} = \prod_{j=1}^{D} \lambda_j^{1/2}$$

tích của các eigen values

~~p(x)~~ =

Biến đổi từ $p(x)$ thành $p(y)$ (Hệ tọa độ y)

$$p(y) = p(x)\,|J| = \prod_{j=1}^{D} \frac{1}{(2\pi\lambda_j)^{1/2}} e^{-\frac{y_j^2}{2\lambda_j}}$$

tích của D independent univariate Gaussian distribution

$$\int p(y)\,dy = \prod_{j=1}^{D} \int_{-\infty}^{\infty} \frac{1}{(2\pi\lambda_j)^{1/2}} e^{-\frac{y_j^2}{2\lambda_j}}\,dy_j = 1$$

$\Rightarrow$ Mulivariate Gaussian is normalization