Properties of Networks, Random Graph Models

CS224W: Analysis of Networks

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Network Properties: How to Measure a Network

Plan: Key Network Properties

Degree distribution: $P(k)$

Path length: h

Clustering coefficient: C

Connected components: s

Definitions will be presented for undirected graphs, sometimes we will explicitly mention extensions to directed graphs, and sometimes extensions will be obvious

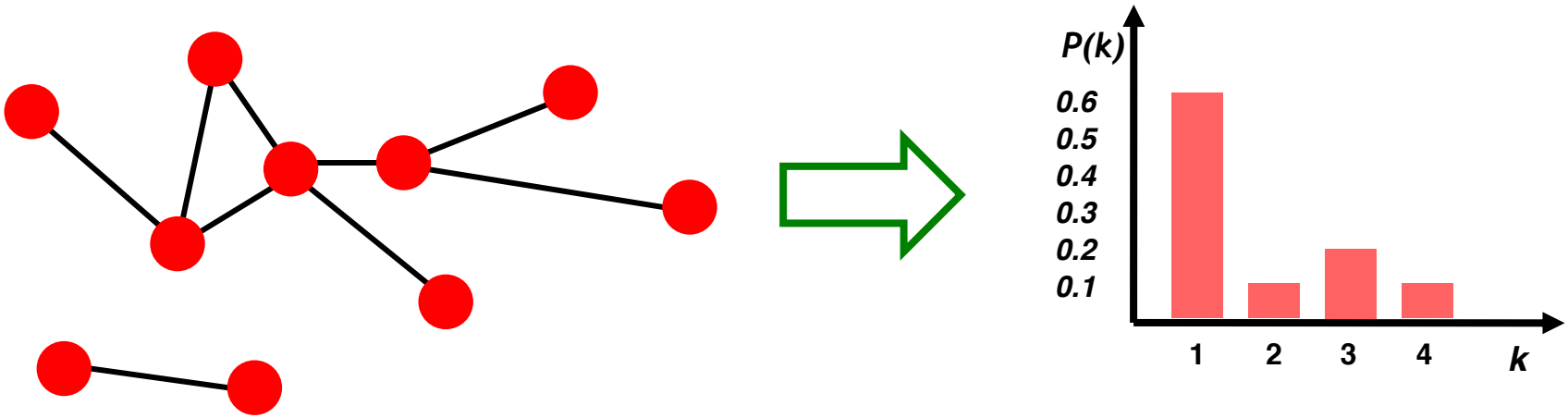
(1) Degree Distribution

- **Degree distribution $P(k)$:** Probability that a randomly chosen node has degree k

  N_k = # nodes with degree k

- Normalized histogram:

  $P(k) = N_k / N$ ➔ **plot**



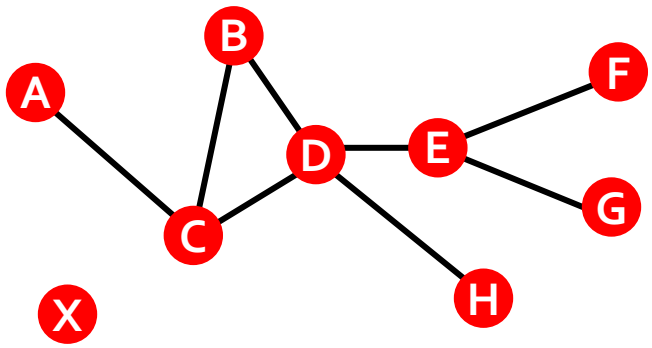
For directed graphs we have separate in- and out-degree distributions.

(2) Paths in a Graph

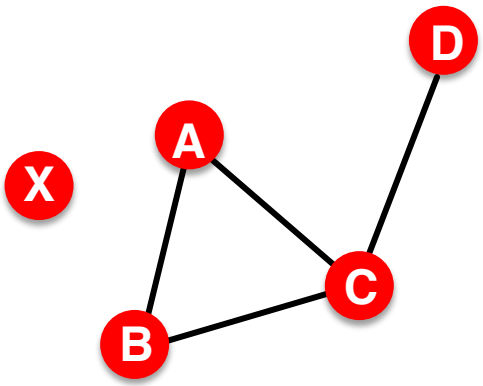
- A ***path*** is a sequence of nodes in which each node is linked to the next one

$$P_n = \{i_0, i_1, i_2, \ldots, i_n\} \qquad P_n = \{(i_0, i_1), (i_1, i_2), (i_2, i_3), \ldots, (i_{n-1}, i_n)\}$$

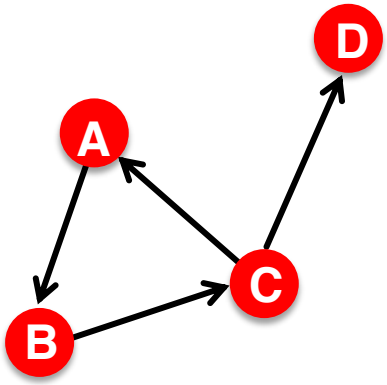
- A path can intersect itself and pass through the same edge multiple times
  - E.g.: ACBDCDEG



Distance in a Graph



$h_{B,D} = 2$
$h_{A,X} = \infty$



$h_{B,C} = 1$, $h_{C,B} = 2$

- **Distance (shortest path, geodesic)** between a pair of nodes is defined as the number of edges along the shortest path connecting the nodes
  - *If the two nodes are not connected, the distance is usually defined as infinite (or zero)
- In **directed graphs,** paths need to follow the direction of the arrows
  - Consequence: Distance is **not symmetric**: $h_{B,C} \neq h_{C,B}$

Network Diameter

- **Diameter:** The maximum (shortest path) distance between any pair of nodes in a graph

- **Average path length** for a connected graph or a strongly connected directed graph

$$\bar{h} = \frac{1}{2E_{\max}} \sum_{i,j \neq i} h_{ij}$$

  - h_{ij} is the distance from node i to node j
  - E_{max} is the max number of edges (total number of node pairs) $= n(n-1)/2$

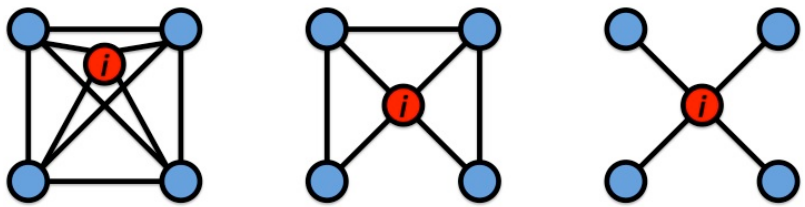
  - Many times we compute the average only over the connected pairs of nodes (that is, we ignore "infinite" length paths)
  - Note that ths measure also applied to (strongly) connected components of a graph

(3) Clustering Coefficient

- **Clustering coefficient (for undirected graphs):**
  - How connected are ***i***'s neighbors to each other?
  - Node ***i*** with degree k_i
  - $C_i \in [0,1]$
  - $C_i = \frac{2e_i}{k_i(k_i-1)}$

where e_i is the number of edges between the neighbors of node i

Note $k_i(k_i-1)$ is max number of edges between the k_i neighbors



$C_i = 1$ $C_i = 1/2$ $C_i = 0$

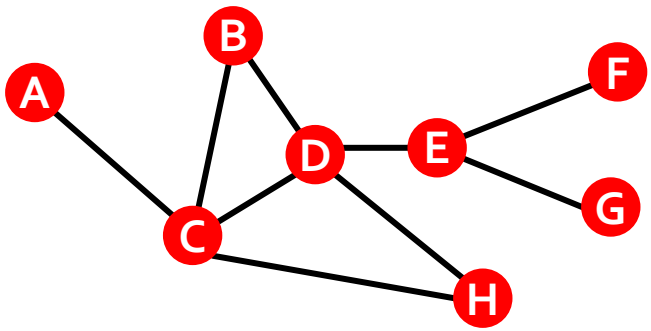
Clustering coefficient is undefined (or defined to be 0) for nodes with degree 0 or 1

- **Average clustering coefficient:** $C = \frac{1}{N}\sum_{i}^{N} C_i$

Clustering Coefficient

- **Clustering coefficient (for undirected graps):**
  - How connected are ***i***'s neighbors to each other?
  - Node ***i*** with degree k_i
  - $C_i = \frac{2e_i}{k_i(k_i - 1)}$

where e_i is the number of edges between the neighbors of node i



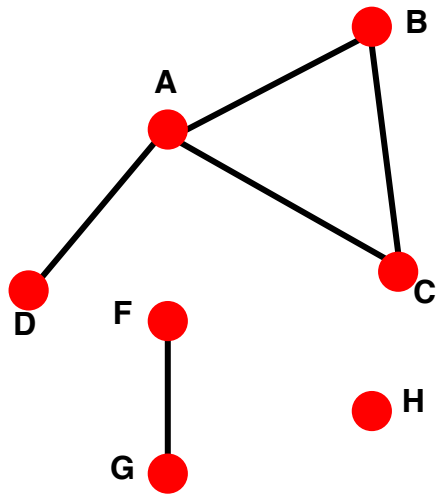
$k_B=2,\ e_B=1,\ C_B=2/2 = 1$

$k_D=4,\ e_D=2,\ C_D=4/12 = 1/3$

Avg. clustering: C=0.33

(4) Connectivity

- **Size of the largest connected component**
  - Largest set where any two vertices can be joined by a path
- **Largest component = Giant component**



How to find connected components:

- Start from random node and perform Breadth First Search (BFS)
- Label the nodes that BFS visits
- If all nodes are visited, the network is connected
- Otherwise find an unvisited node and repeat BFS

Summary: Key Network Properties

Degree distribution: $P(k)$

Path length: h

Clustering coefficient: C

Connected components: s

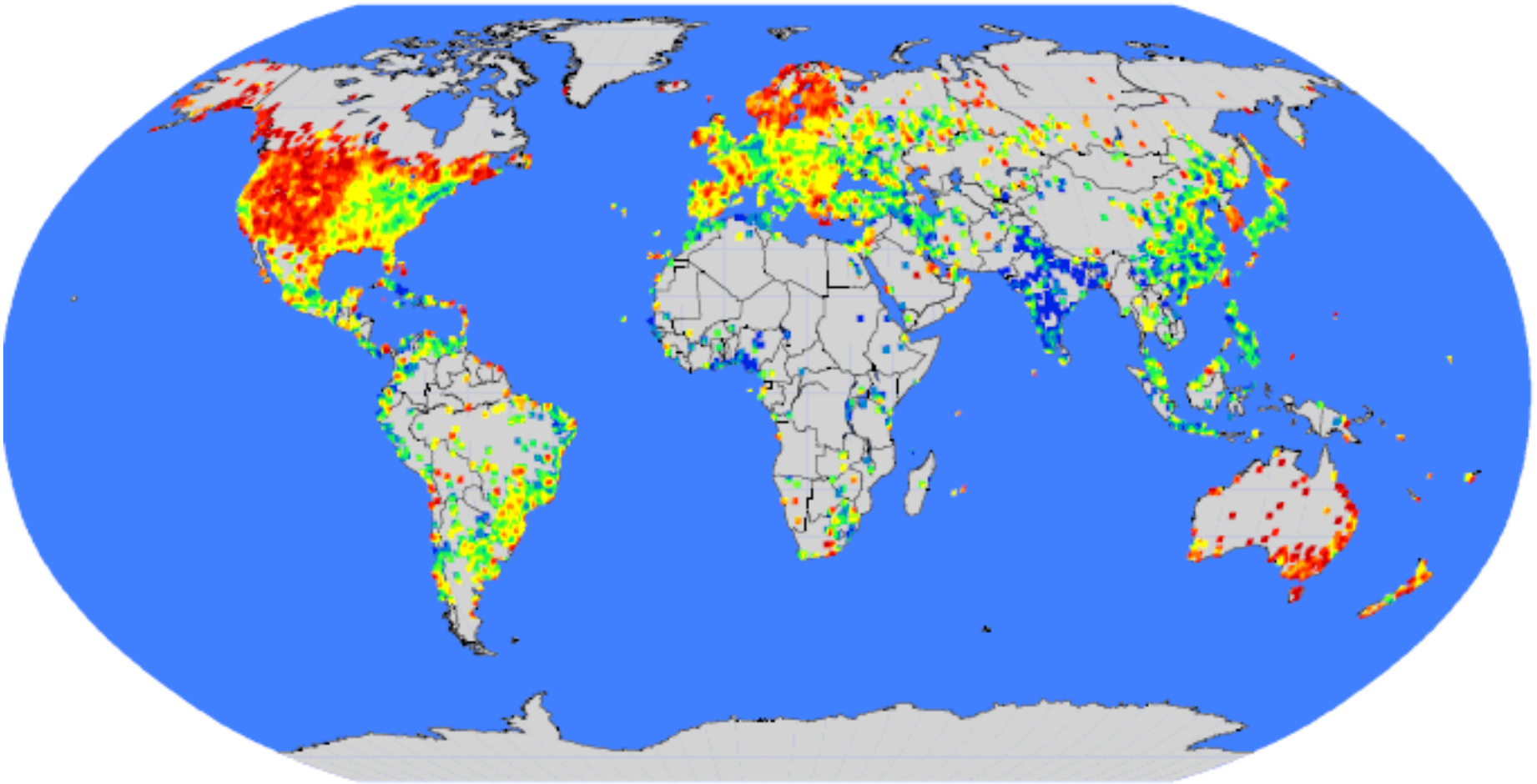
Let's measure these properties on real-world networks!

MSN Messenger

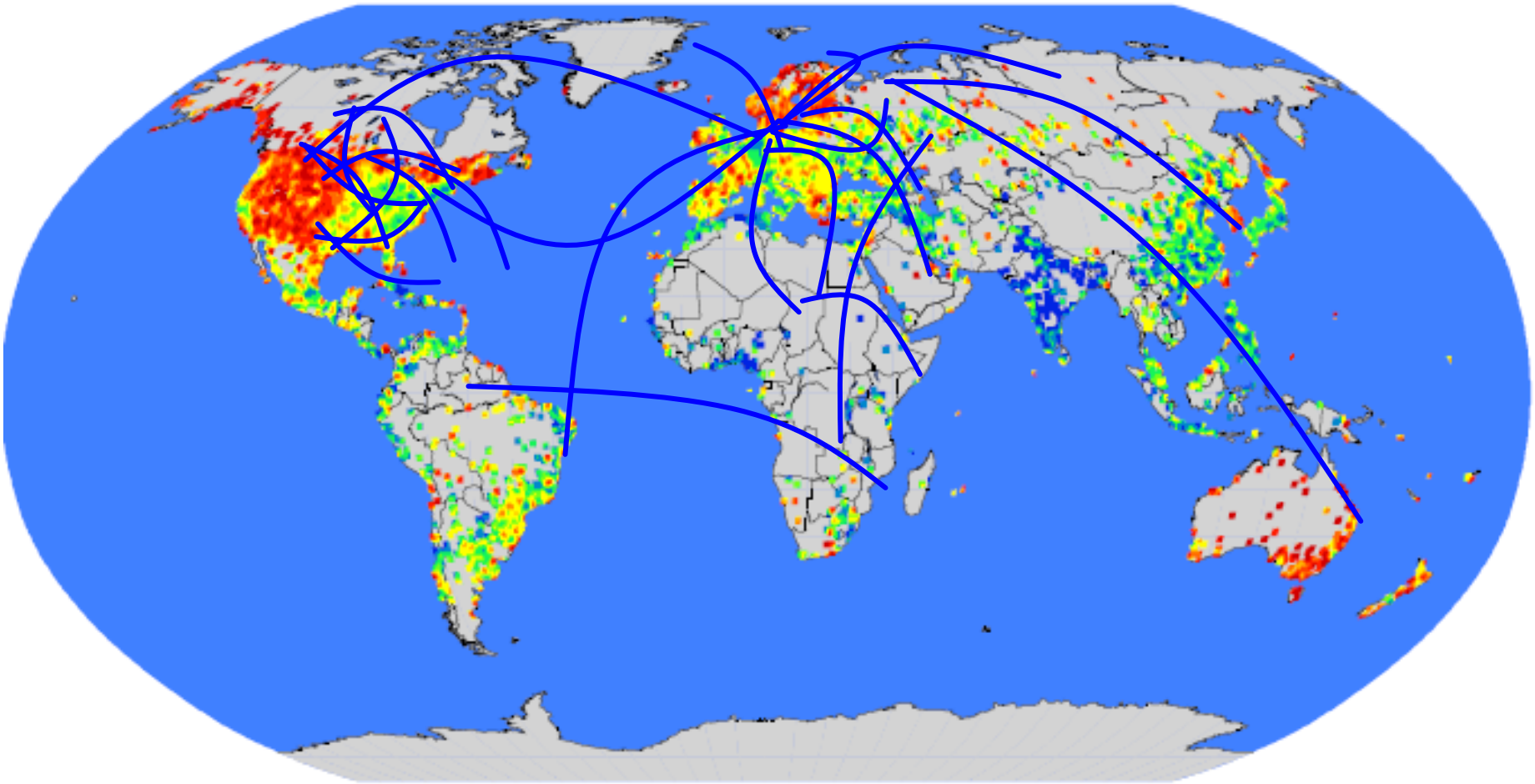
MSN Messenger:

- 1 month of activity
  - 245 million users logged in
  - 180 million users engaged in conversations
  - More than 30 billion conversations
  - More than 255 billion exchanged messages

Geography of Communication

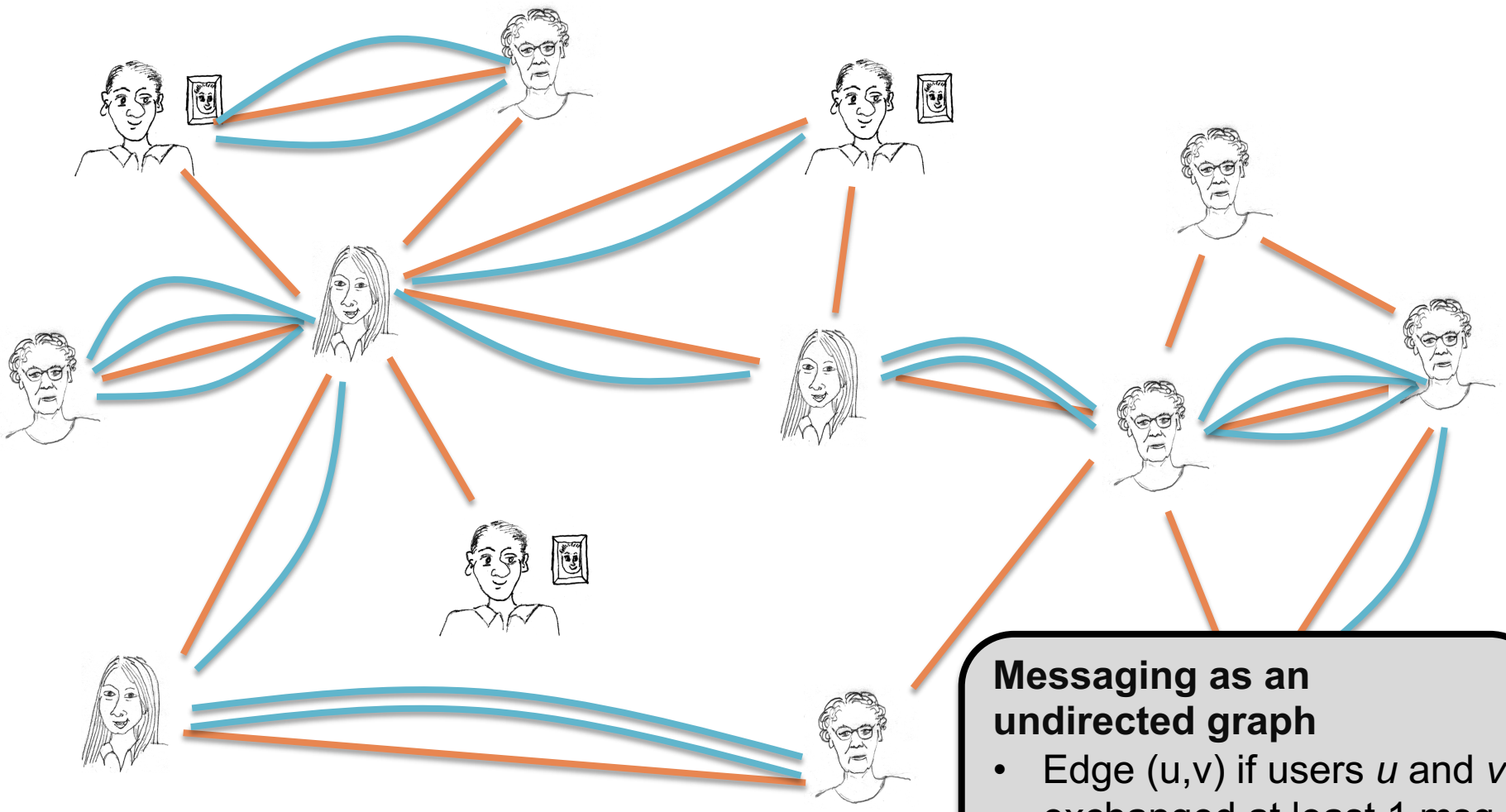


Communication Network



Network: 180M people, 1.3B edges

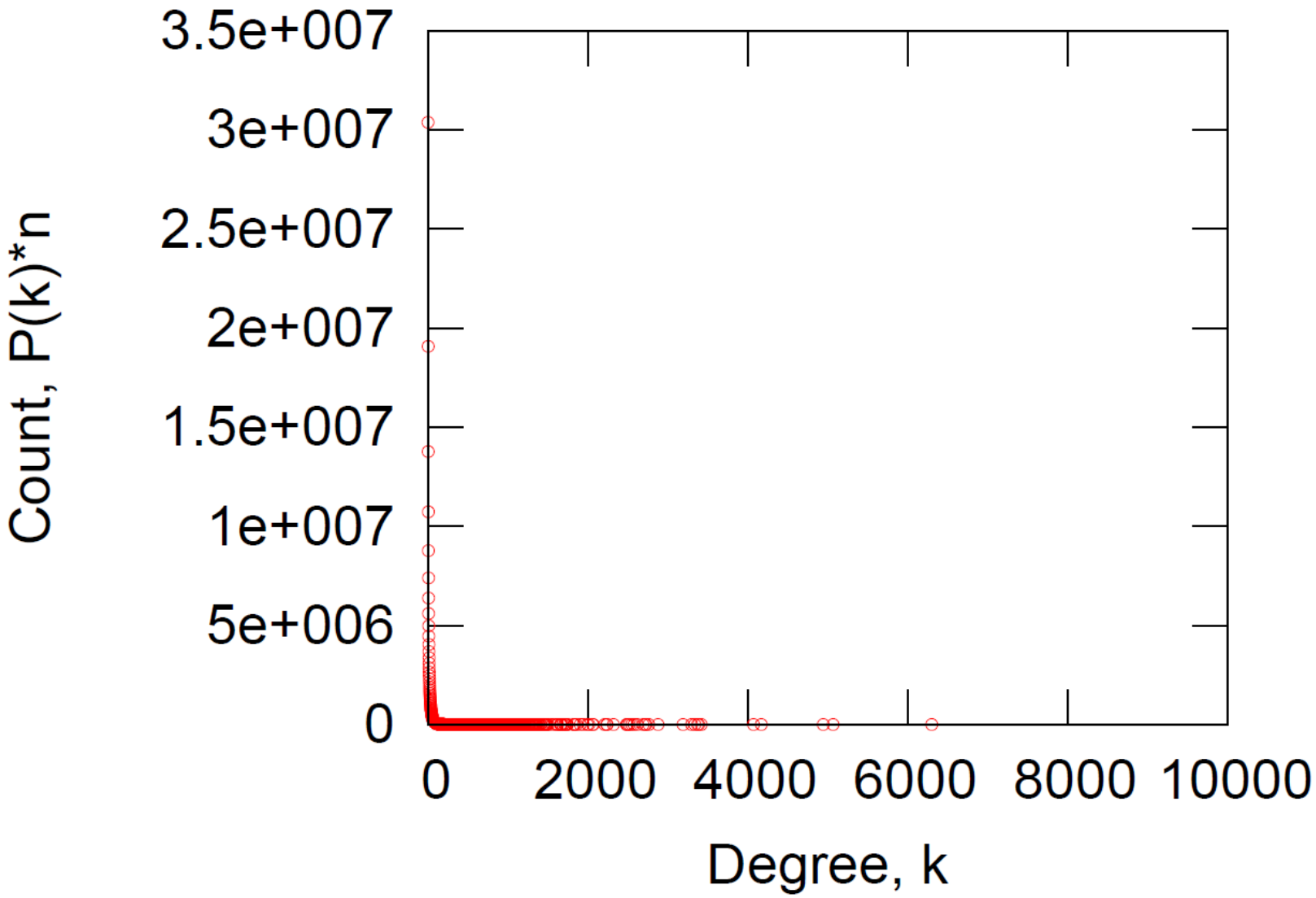
Messaging as a Multigraph



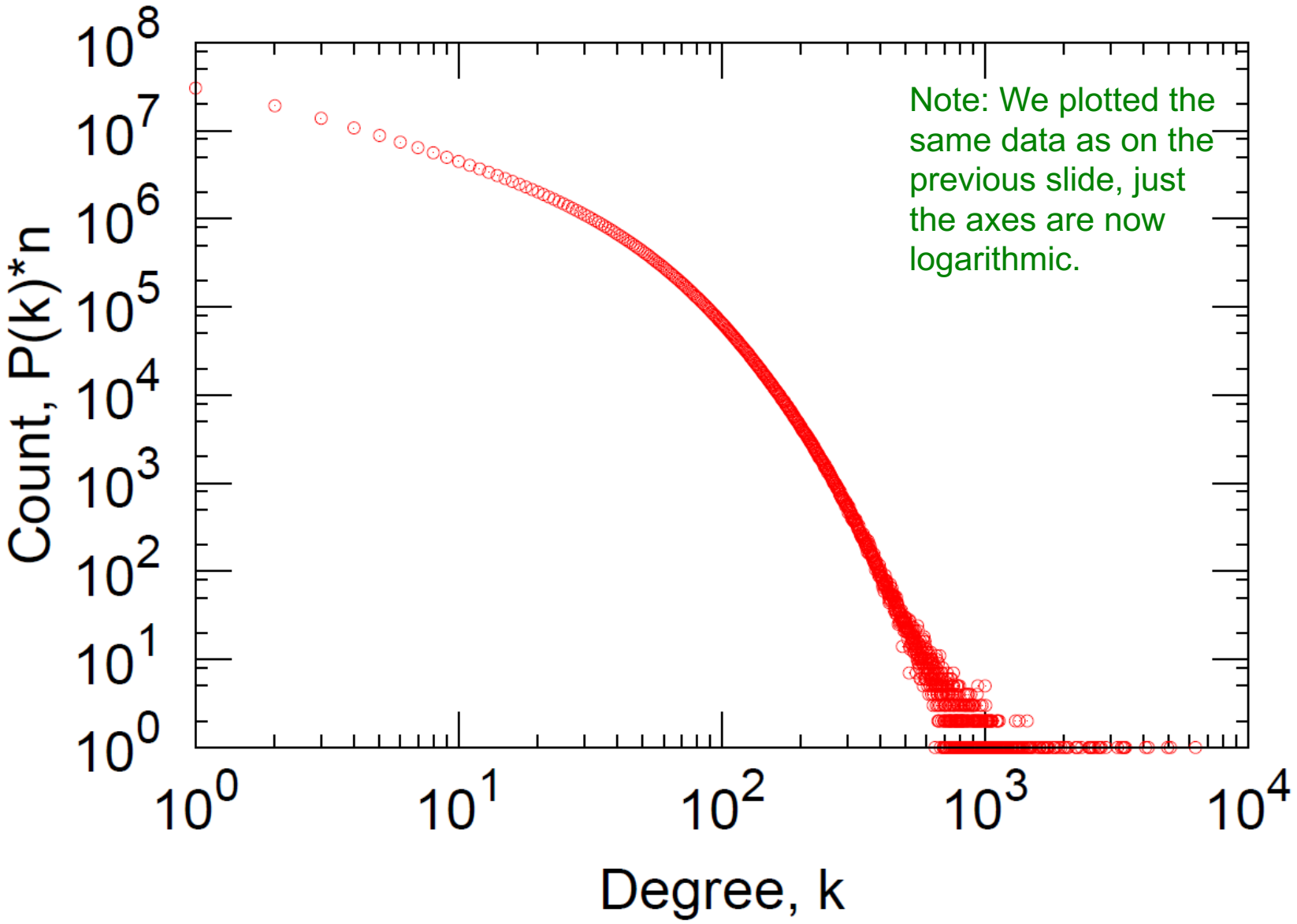
Messaging as an undirected graph

- Edge (u,v) if users *u* and *v* exchanged at least 1 msg
- N=180 million people
- E=1.3 billion edges

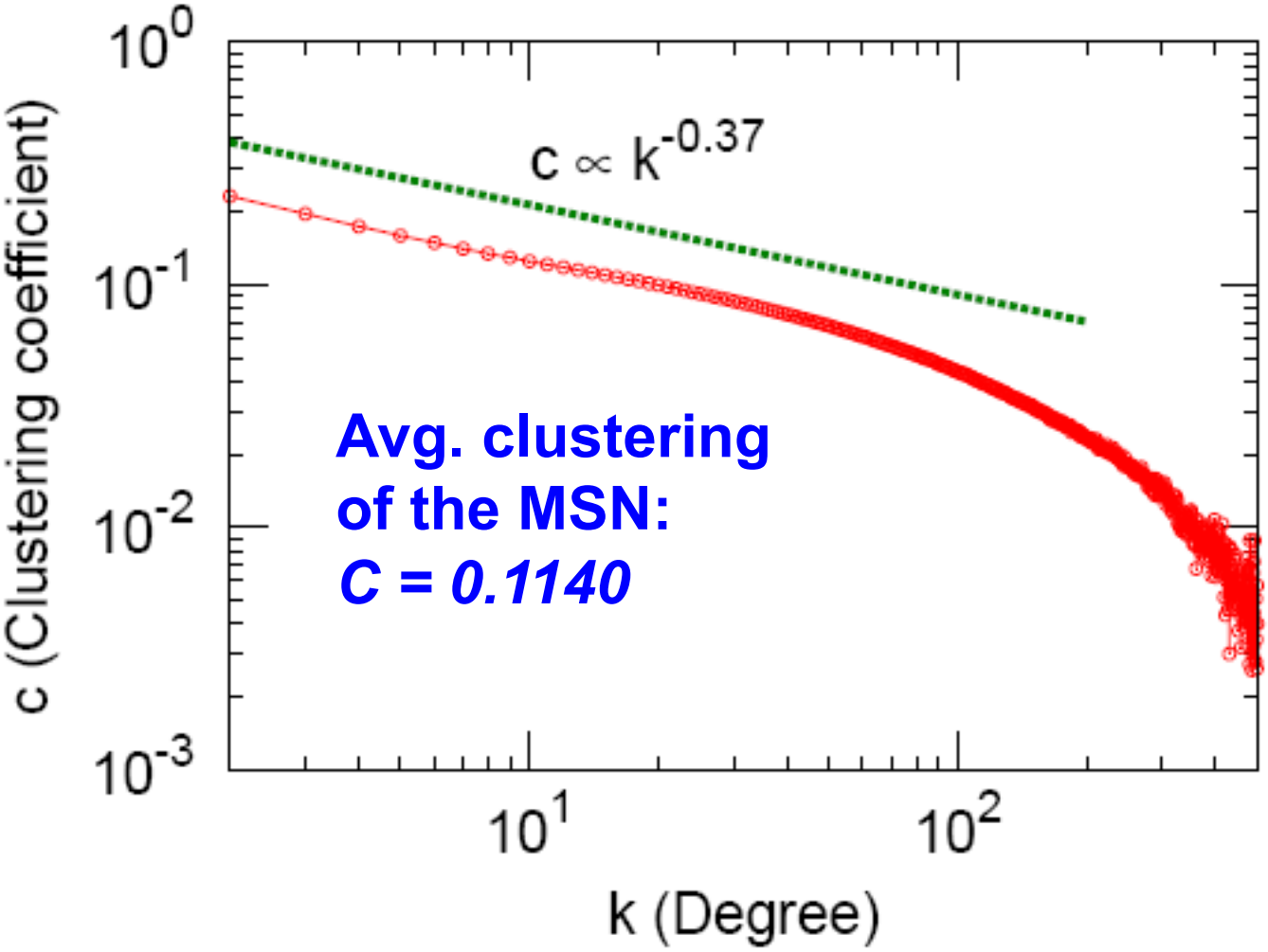
MSN: (1) Degree Distribution



MSN: Log-Log Degree Distribution

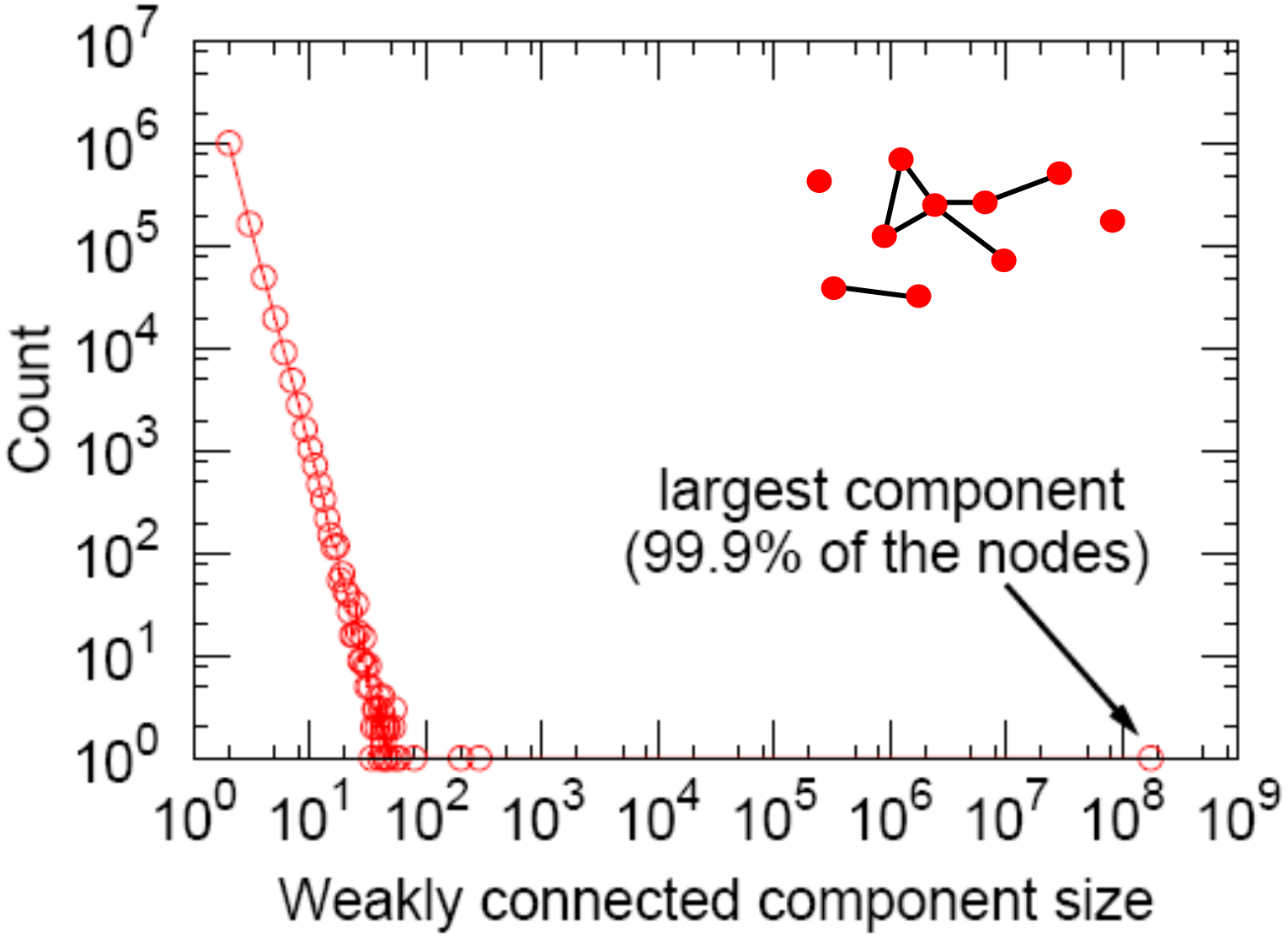


MSN: (2) Clustering

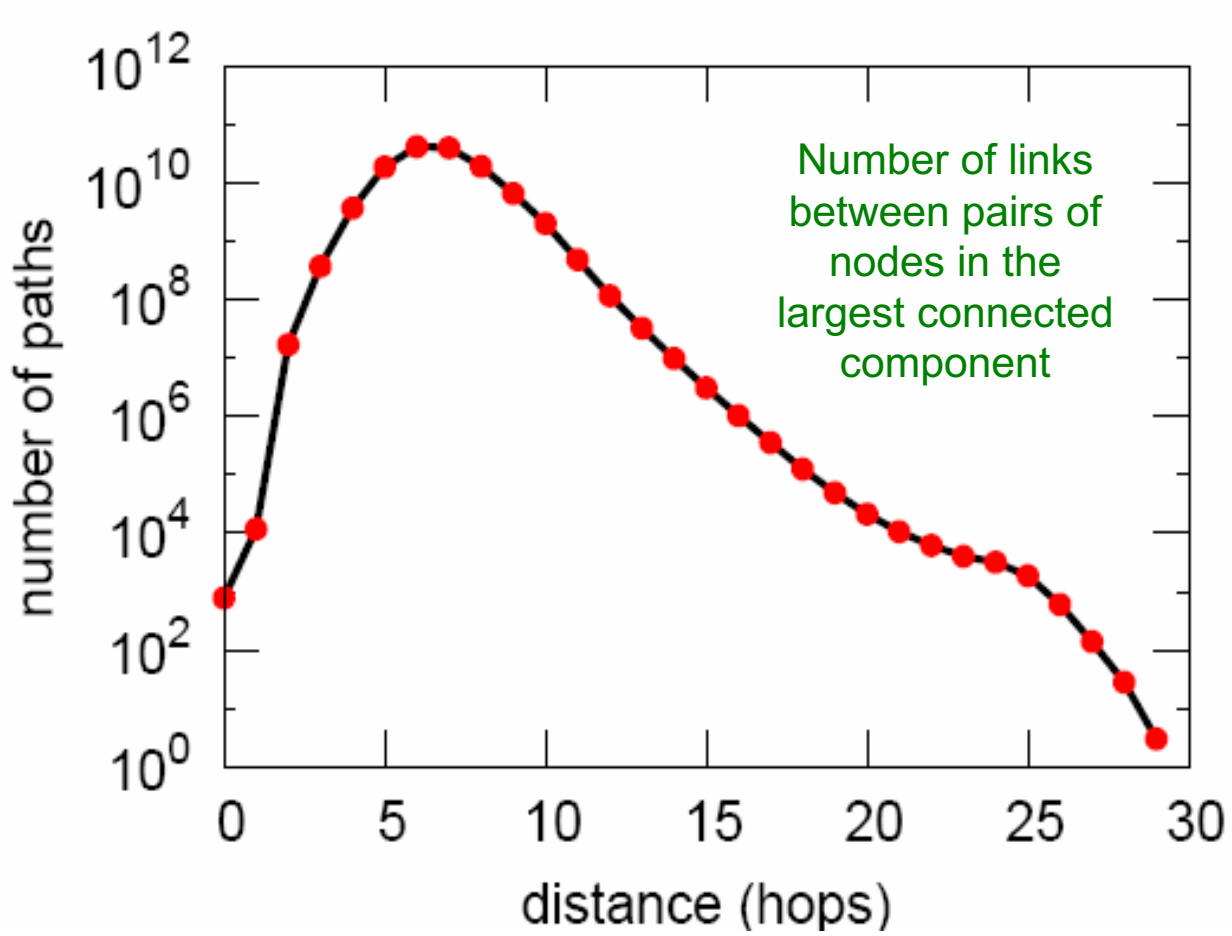


C_k: average C_i of nodes i of degree k: $C_k = \frac{1}{N_k} \sum_{i:k_i=k} C_i$

MSN: (3) Connected Components



MSN: (4) Diameter of WCC



Avg. path length **6.6**
90% of the nodes can be reached in < 8 hops

nodes as we do BFS out of a random node

Steps	#Nodes
0	1
1	10
2	78
3	3,96
4	8,648
5	3,299,252
6	28,395,849
7	79,059,497
8	52,995,778
9	10,321,008
10	1,955,007
11	518,410
12	149,945
13	44,616
14	13,740
15	4,476
16	1,542
17	536
18	167
19	71
20	29
21	16
22	10
23	3
24	2
25	3

MSN: Key Network Properties

Degree distribution: *Heavily skewed; avg. degree = 14.4*

Path length: *6.6*

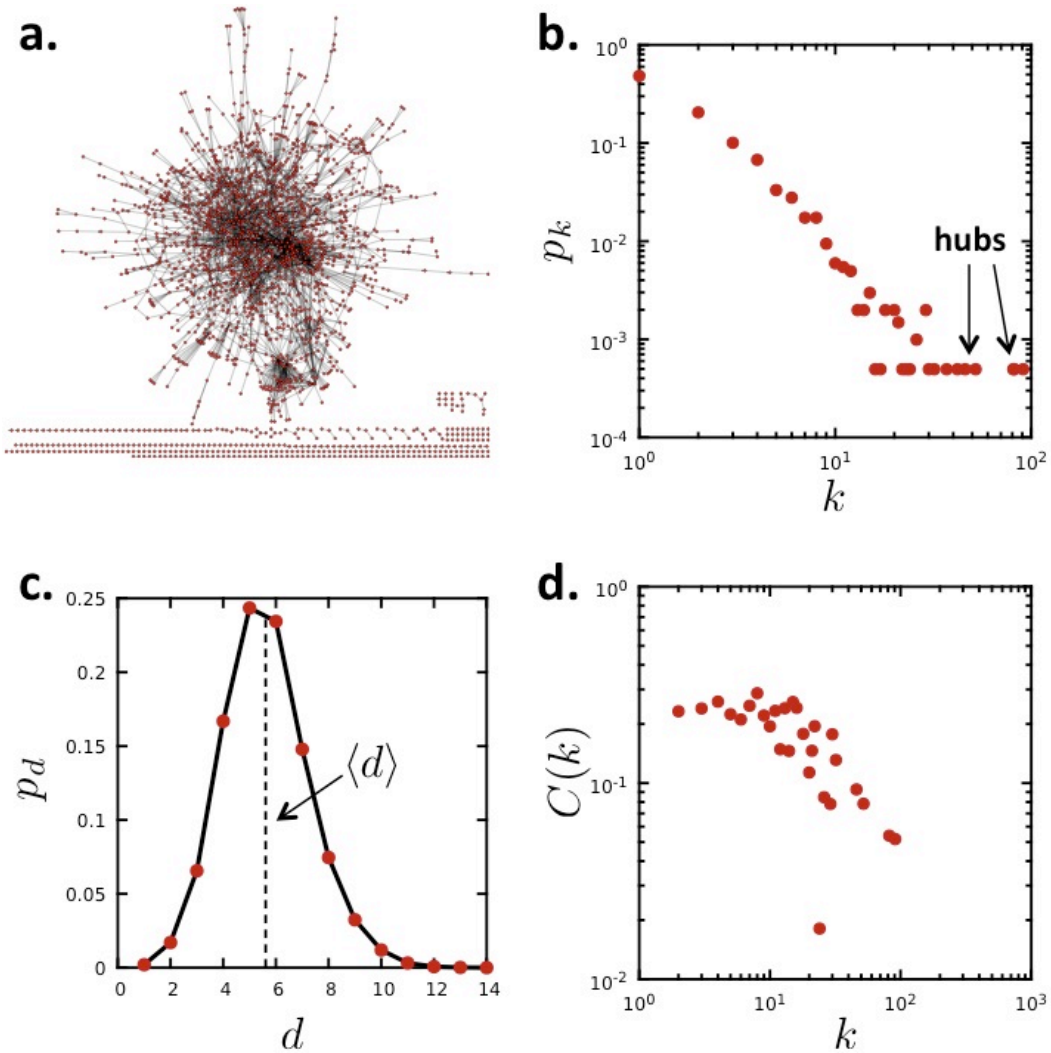
Clustering coefficient: *0.11*

Connectivity: *giant component*

Are these values "expected"?
Are they "surprising"?

To answer this we need a model!

Another example: PPI Network



a. Undirected network

N=2,018 proteins as nodes

E=2,930 binding interactions as links.

b. Degree distribution:

Skewed. Average degree <k>=2.90

c. Diameter:

Avg. path length = 5.8

d. Clustering:

Avg. clustering = 0.12

Connectivity: 185 components

the largest component has

1,647 nodes (81% of nodes)

Erdös-Rényi Random Graph Model

Simplest Model of Graphs

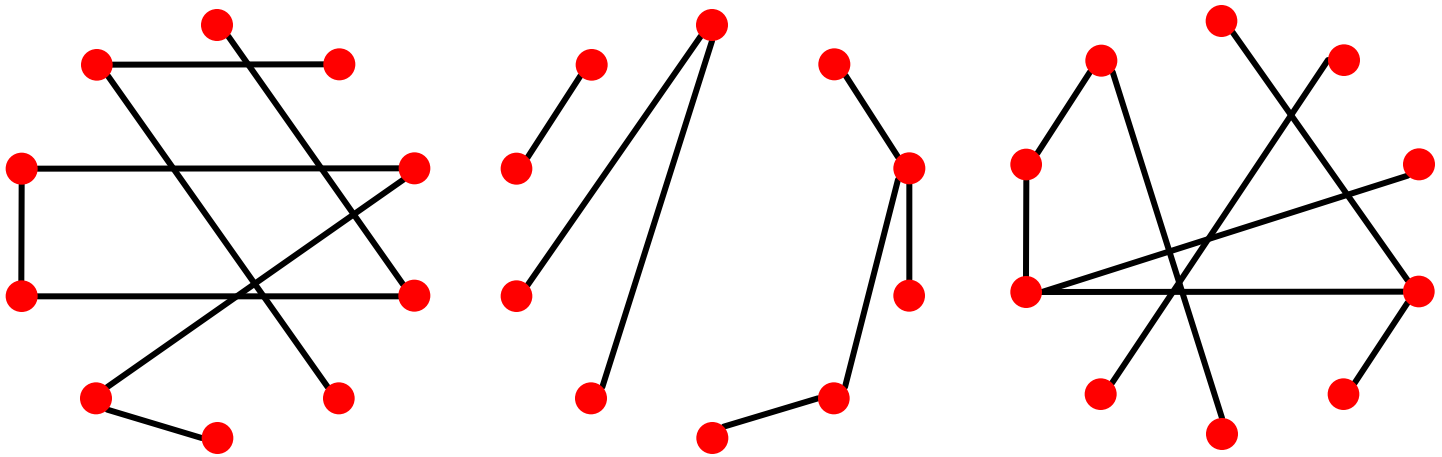
- **Erdös-Renyi Random Graphs** [Erdös-Renyi, '60]
- **Two variants:**
  - G_{np}: undirected graph on n nodes where each edge (u,v) appears i.i.d. with probability p

  - G_{nm} : undirected graph with n nodes, and m edges picked uniformly at random

What kind of networks do such models produce?

Random Graph Model

- ***n* and *p* do not uniquely determine the graph!**
  - The graph is a result of a random process
- We can have many different realizations given the same ***n*** and ***p***



n = 10
p= 1/6

Properties of G_{np}

Degree distribution: $P(k)$

Path length: h

Clustering coefficient: C

What are the values of these properties for G_{np}?

Degree Distribution

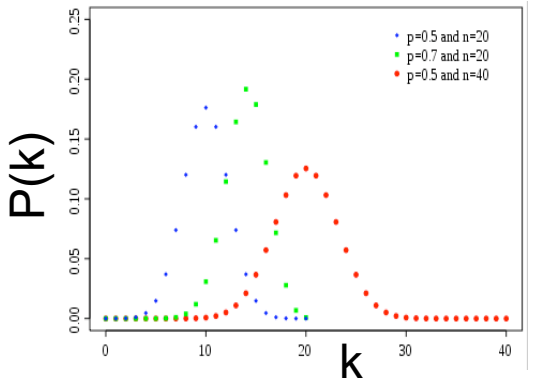
- **Fact:** Degree distribution of G_{np} is binomial.
- Let $P(k)$ denote the fraction of nodes with degree k:

$$P(k) = \binom{n-1}{k} p^k (1-p)^{n-1-k}$$

Select k nodes out of $n-1$

Probability of having k edges

Probability of missing the rest of the $n-1-k$ edges



Mean, variance of a binomial distribution

$$\bar{k} = p(n-1)$$

$$\sigma^2 = p(1-p)(n-1)$$

$$\frac{\sigma}{\bar{k}} = \left[\frac{1-p}{p}\frac{1}{(n-1)}\right]^{1/2} \approx \frac{1}{(n-1)^{1/2}}$$

By the law of large numbers, as the network size increases, the distribution becomes increasingly narrow—we are increasingly confident that the degree of a node is in the vicinity of k.

Clustering Coefficient of G_{np}

- **Remember:** $C_i = \frac{2e_i}{k_i(k_i-1)}$

  Where e_i is the number of edges between i's neighbors

- Edges in G_{np} appear i.i.d. with prob. p

- **So, expected E[e_i] is:** $= p\frac{k_i(k_i-1)}{2}$

  Each pair is connected with prob. p

  Number of distinct pairs of neighbors of node i of degree k_i

- **Then $E[C_i]$:** $= \frac{p \cdot k_i(k_i-1)}{k_i(k_i-1)} = p = \frac{\bar{k}}{n-1} \approx \frac{\bar{k}}{n}$

Clustering coefficient of a random graph is small.
If we generate bigger and bigger graphs with fixed avg. degree k (that is we set $p = k \cdot 1/n$), then C decreases with the graph size n.

Network Properties of G_{np}

Degree distribution: $P(k) = \binom{n-1}{k} p^k (1-p)^{n-1-k}$

Clustering coefficient: $C = p = \bar{k}/n$

Path length: ***next!***

Connectivity:

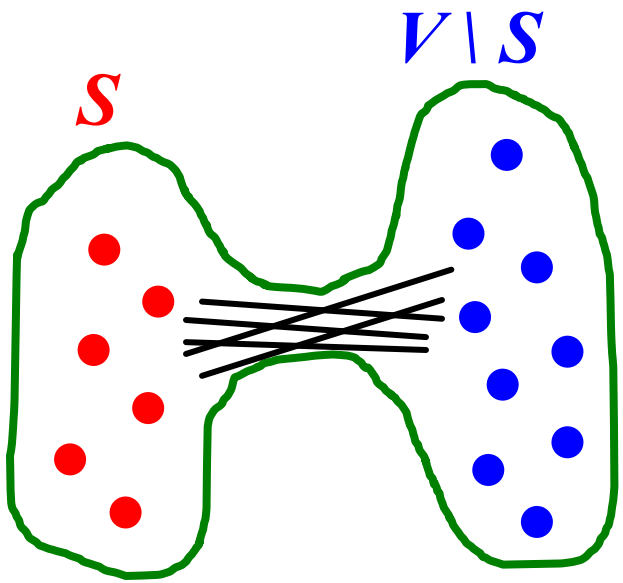
Def: Expansion

- Graph $G(V, E)$ has **expansion α**: if $\forall S \subseteq V$:

  $\#$ of edges leaving $S \geq \alpha \cdot \min(|S|, |V \setminus S|)$

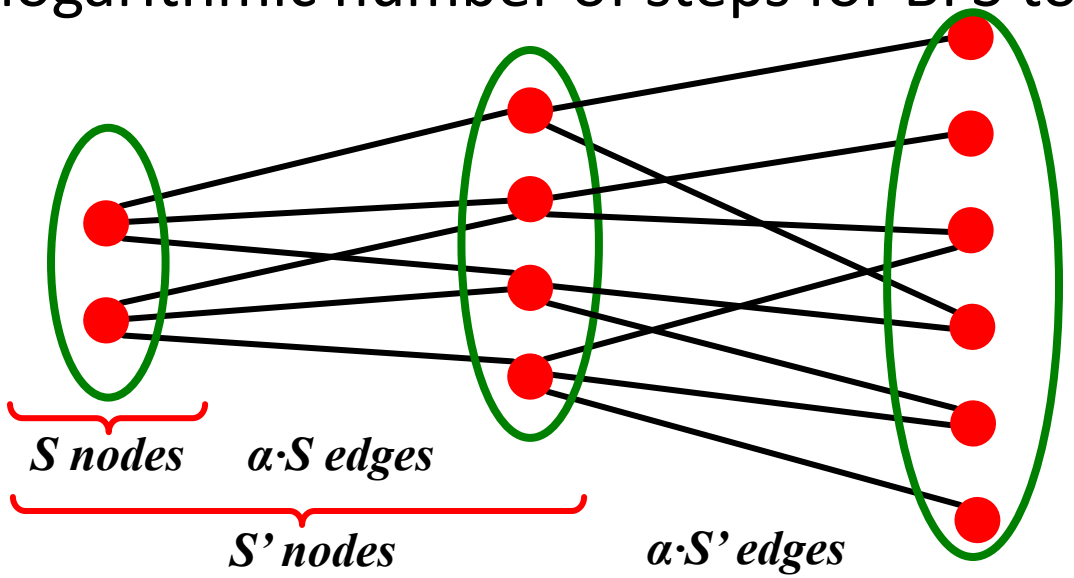
- **Or equivalently:**

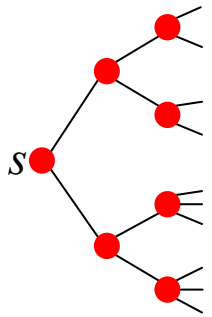
$$\alpha = \min_{S \subseteq V} \frac{\#edges\ leaving\ S}{\min(|S|, |V \setminus S|)}$$



Expansion: Random Graphs

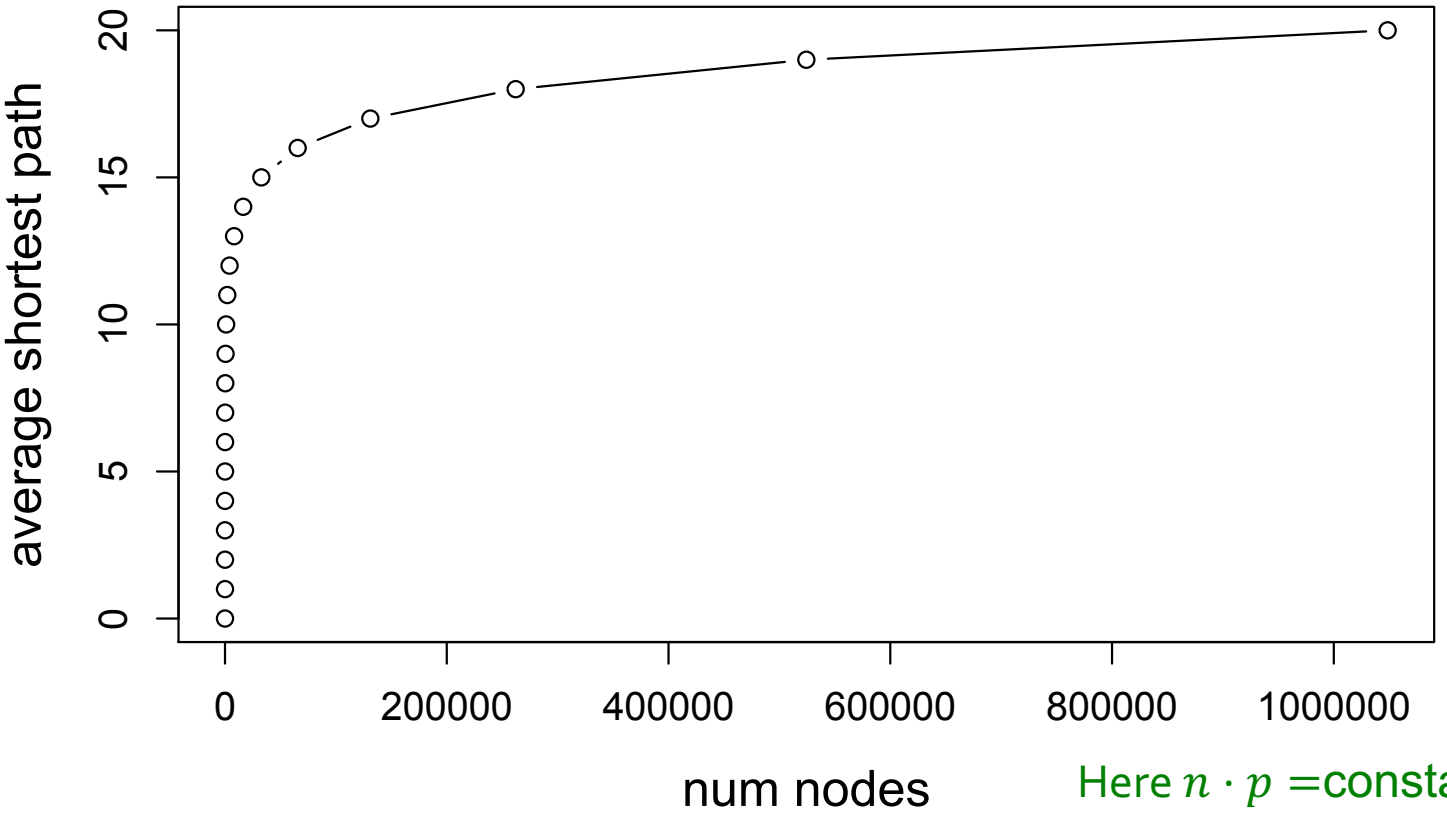
- **Fact:** In a graph on n nodes with expansion α, for all pairs of nodes, there is a path of length $O((\log n)/\alpha)$.
- **Random graph G_{np}:**
  For $\log n > np > c$, diam(G_{np}) = $O(\log n / \log (np))$
  - Random graphs have good expansion so it takes a logarithmic number of steps for BFS to visit all nodes





Erdös-Renyi avg. shortest path

Erdös-Renyi Random Graph can grow very large but nodes will be just a few hops apart



Here $n \cdot p$ =constant
That is, avg deg k is const

Network Properties of G_{np}

Degree distribution: $P(k) = \binom{n-1}{k} p^k (1-p)^{n-1-k}$

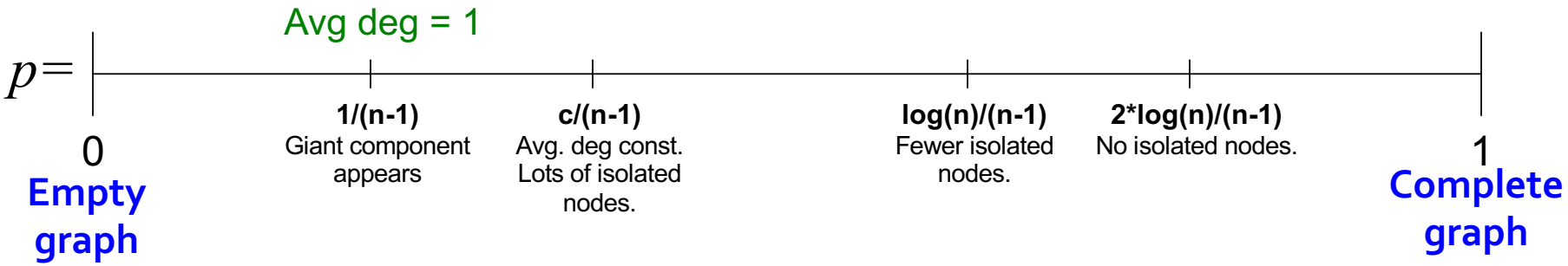
Path length: $O(\log n)$

Clustering coefficient: $C = p = \bar{k} / n$

Connected components: ***next!***

"Evolution" of a Random Graph

- **Graph structure of G_{np} as p changes:**

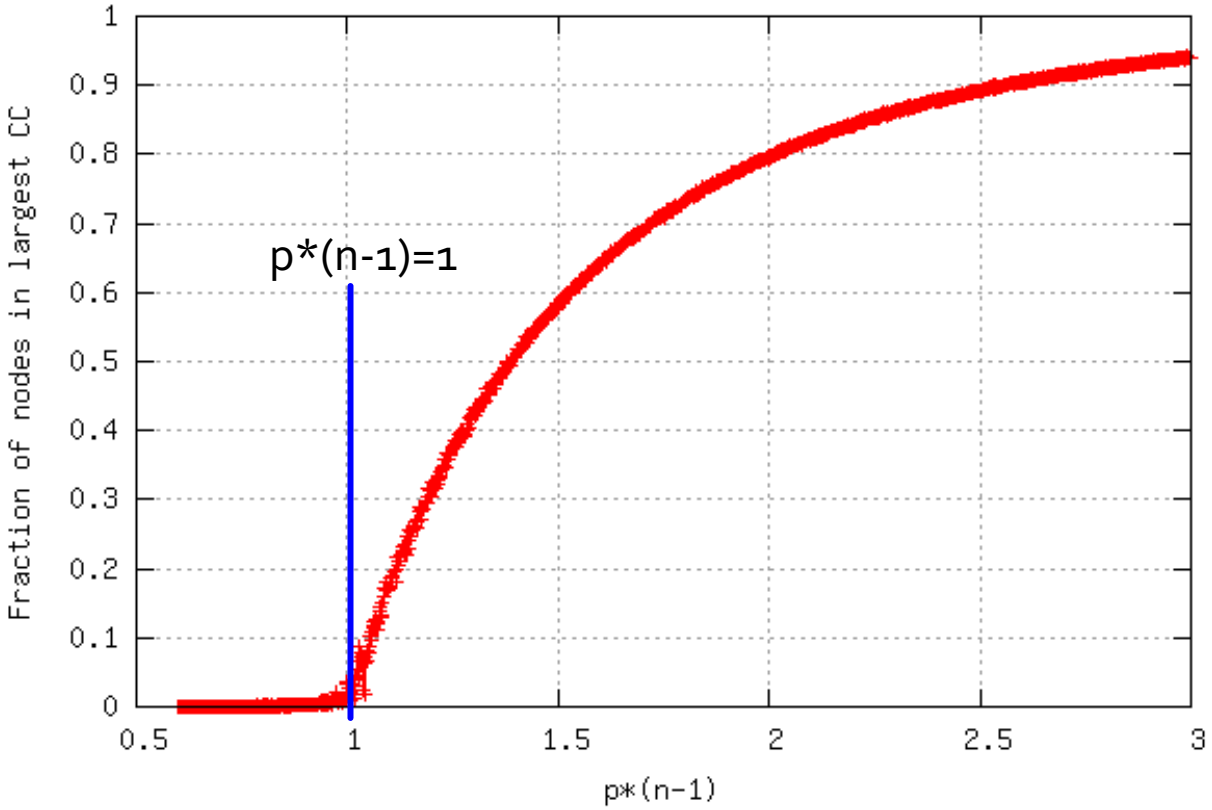


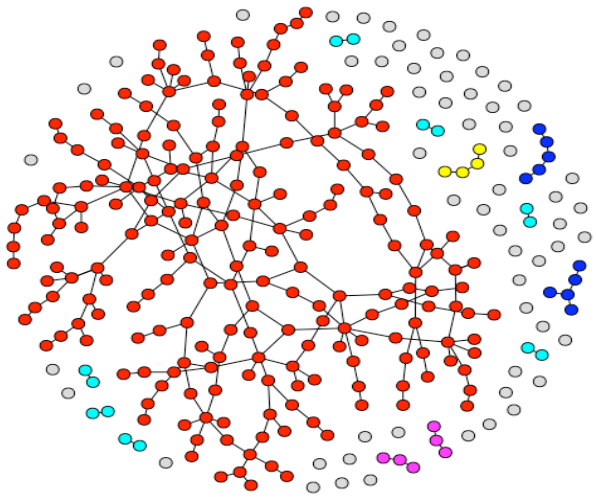
- **Emergence of a giant component:**

  avg. degree $k=2E/n$ or $p=k/(n-1)$

  - $k=1-\varepsilon$: all components are of size $\Omega(\log n)$
  - $k=1+\varepsilon$: 1 component of size $\Omega(n)$, others have size $\Omega(\log n)$
    - Each node has at least one edge in expectation

G_{np} Simulation Experiment

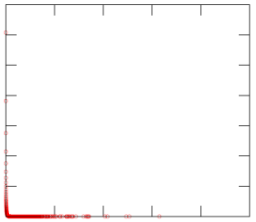
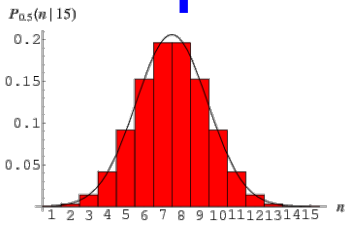




Fraction of nodes in the largest component

- G_{np}, n=100,000, $k=p(n-1)$ = 0.5 … 3

Back to MSN vs. G_{np}

	MSN	G_{np} n=18oM	
Degree distribution:			☒
Avg. path length:	**6.6**	$O(\log n)$ *h ≈ 8.2*	☑
Avg. clustering coef.:	*0.11*	$\bar{k} / n$ *C ≈ 8·10^{-8}*	☒
Largest Conn. Comp.:	*99%*	GCC exists when $\bar{k}>1$. $\bar{k} \approx 14$.	☑

Real Networks vs. G_{np}

- **Are real networks like random graphs?**
  - Giant connected component: ☺
  - Average path length: ☺
  - Clustering Coefficient: ☹
  - Degree Distribution: ☹
- **Problems with the random networks model:**
  - Degree distribution differs from that of real networks
  - Giant component in most real networks does NOT emerge through a phase transition
  - No local structure – clustering coefficient is too low
- **Most important: Are real networks random?**
  - The answer is simply: **NO!**

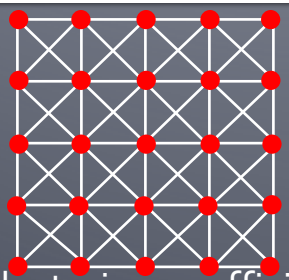
Real Networks vs. G_{np}

- **If G_{np} is wrong, why did we spend time on it?**
  - It is the reference model for the rest of the class
  - It will help us calculate many quantities, that can then be compared to the real data
  - It will help us understand to what degree a particular property is the result of some random process

So, while G_{np} is WRONG, it will turn out to be extremly USEFUL!

The Small-World Model

Can we have high clustering while also having short paths?



High clustering coefficient,
High diameter

Vs.

Low clustering coefficient
Low diameter

Clustering Implies Edge Locality

- **MSN network has 7 orders of magnitude larger clustering than the corresponding G_{np}!**
- **Other examples:**

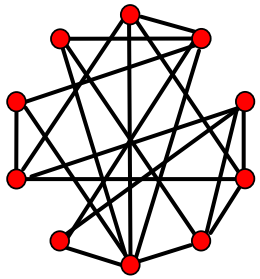
Actor Collaborations (IMDB): N = 225,226 nodes, avg. degree $\overline{k}$ = 61
Electrical power grid: N = 4,941 nodes, $\overline{k}$ = 2.67
Network of neurons: N = 282 nodes, $\overline{k}$ = 14

Network	h_{actual}	h_{random}	C_{actual}	C_{random}
Film actors	3.65	2.99	0.79	0.00027
Power Grid	18.70	12.40	0.080	0.005
C. elegans	2.65	2.25	0.28	0.05

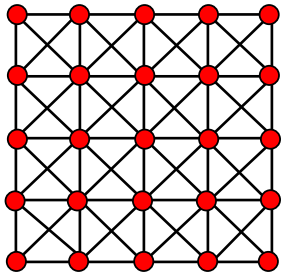
h ... Average shortest path length
C ... Average clustering coefficient
"actual" … real network
"random" … random graph with same avg. degree

The "Controversy"

- **Consequence of expansion:**
  - **Short paths:** $O(\log n)$
    - This is the smallest diameter we can get if we keep the degree constant.
  - But clustering is low!
- **But networks have "local" structure:**
  - **Triadic closure:**
    Friend of a friend is my friend
  - High clustering but diameter is also high
- **How can we have both?**



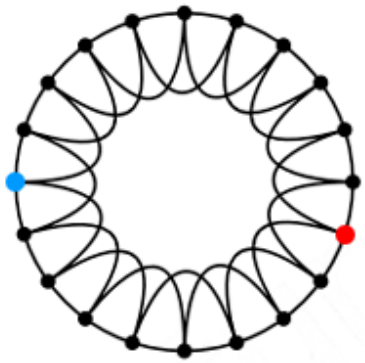
Low diameter
Low clustering coefficient



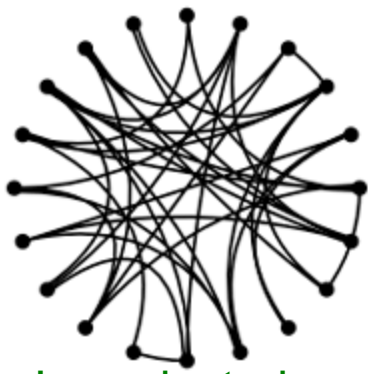
High clustering coefficient
High diameter

Small-World: How?

- **Could a network with high clustering also be small world (have $\log n$ diameter)?**
  - How can we at the same time have **high clustering** and **small diameter?**



High clustering
High diameter



Low clustering
Low diameter

  - Clustering implies edge "locality"
  - Randomness enables "shortcuts"

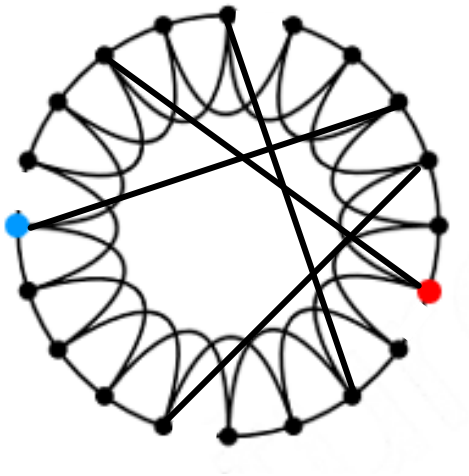
[Watts-Strogatz, '98]

Solution: The Small-World Model

Small-World Model [Watts-Strogatz '98]

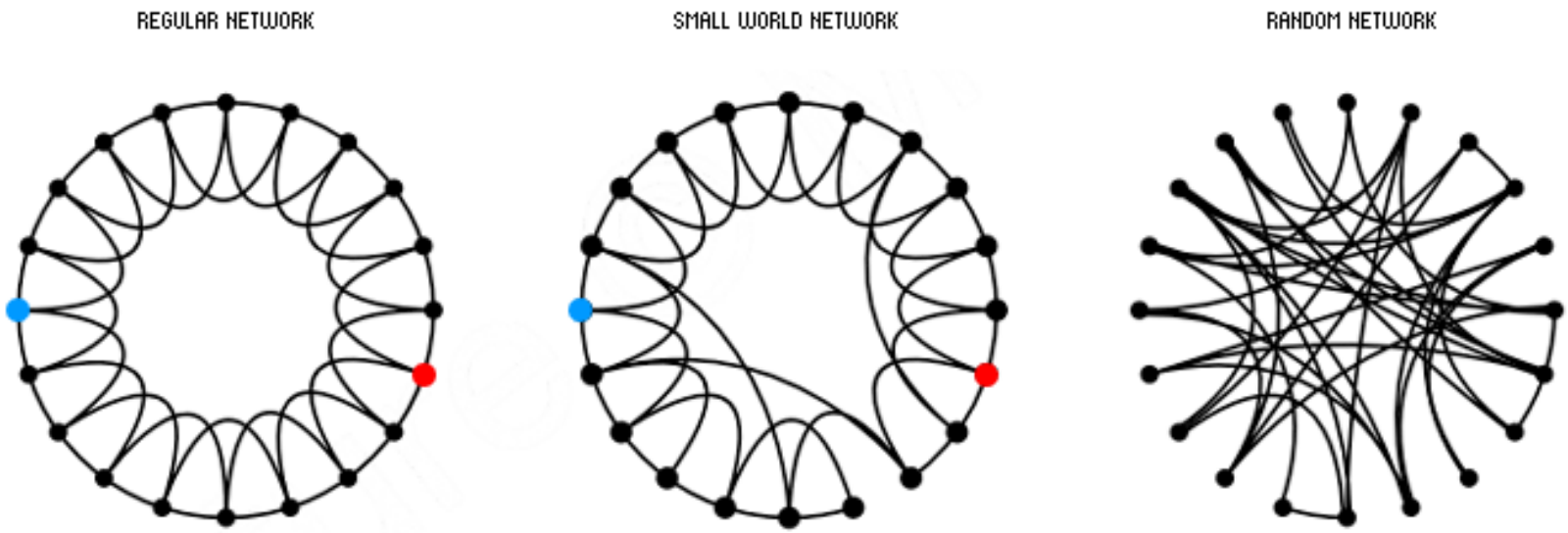
Two components to the model:

- **(1) Start with a low-dimensional regular lattice**
  - (In our case we are using a ring as a lattice)
  - Has high clustering coefficient
- **(2) Rewire: Introduce randomness ("shortcuts")**
  - Add/remove edges to create shortcuts to join remote parts of the lattice
  - For each edge, with prob. p, move the other endpoint to a random node



[Watts-Strogatz, '98]

The Small-World Model



High clustering
High diameter

$h = \frac{N}{2\bar{k}}$ $C = \frac{1}{2}$

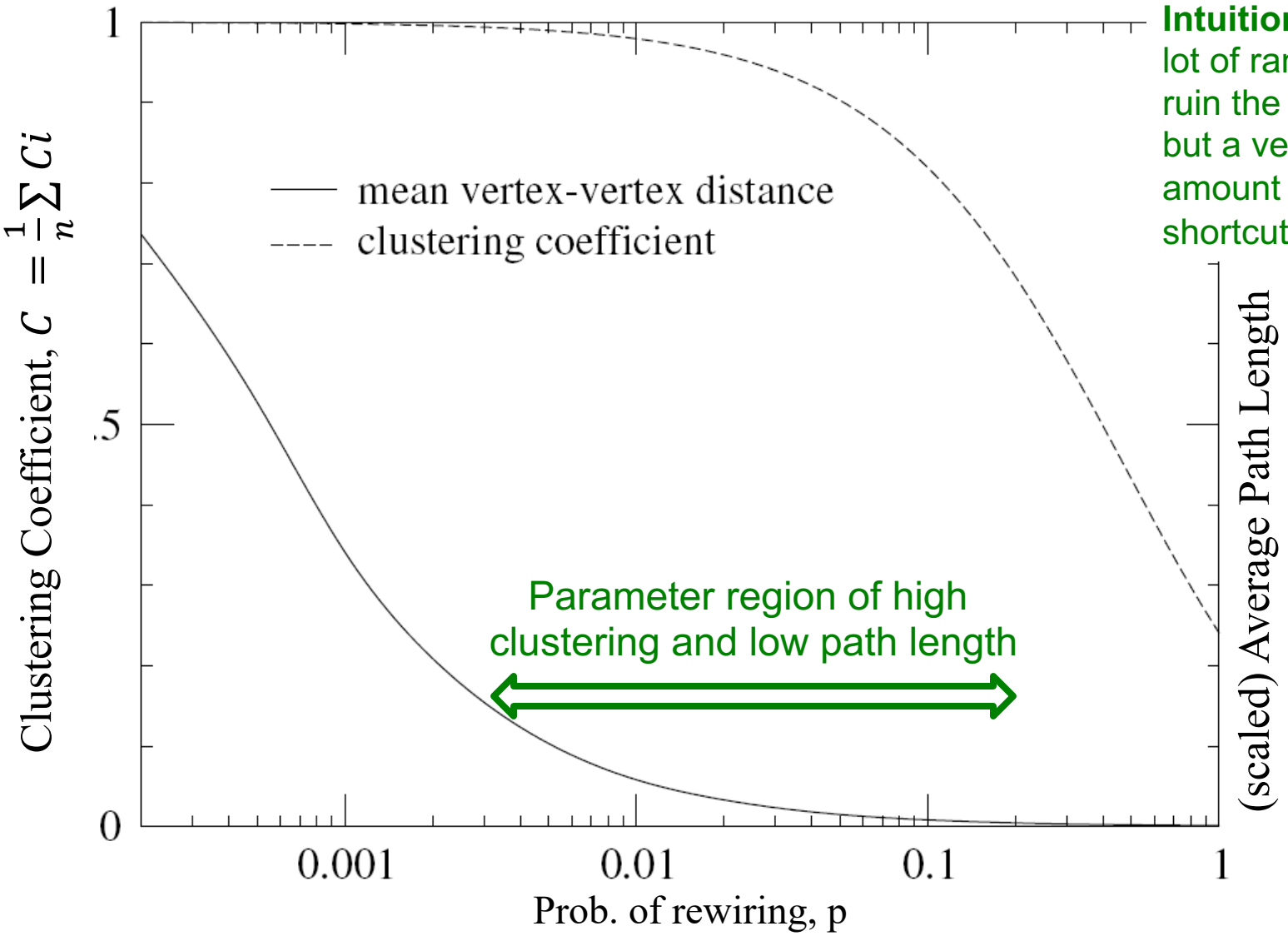
High clustering
Low diameter

Low clustering
Low diameter

$h = \frac{\log N}{\log \alpha}$ $C = \frac{\bar{k}}{N}$

Rewiring allows us to "interpolate" between a regular lattice and a random graph

The Small-World Model



Intuition: It takes a lot of randomness to ruin the clustering, but a very small amount to create shortcuts.

Small-World: Summary

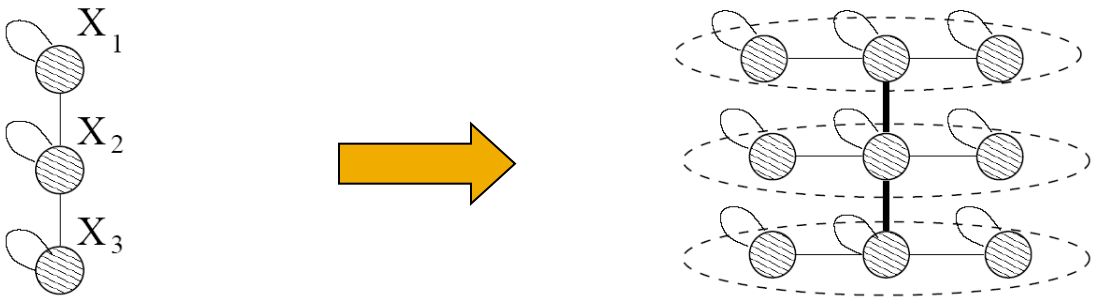
- **Could a network with high clustering be at the same time a small world?**
  - Yes! You don't need more than a few random links
- **The Watts Strogatz Model:**
  - Provides insight on the interplay between clustering and the small-world
  - Captures the structure of many realistic networks
  - Accounts for the high clustering of real networks
  - Does not lead to the correct degree distribution

Kronecker Graph Model

Generating large realistic graphs

Idea: Recursive Graph Generation

- **How can we think of network structure recursively? Intuition:** Self-similarity
  - **Object is similar to a part of itself**: the whole has the same shape as one or more of the parts
- **Mimic recursive graph/community growth:**



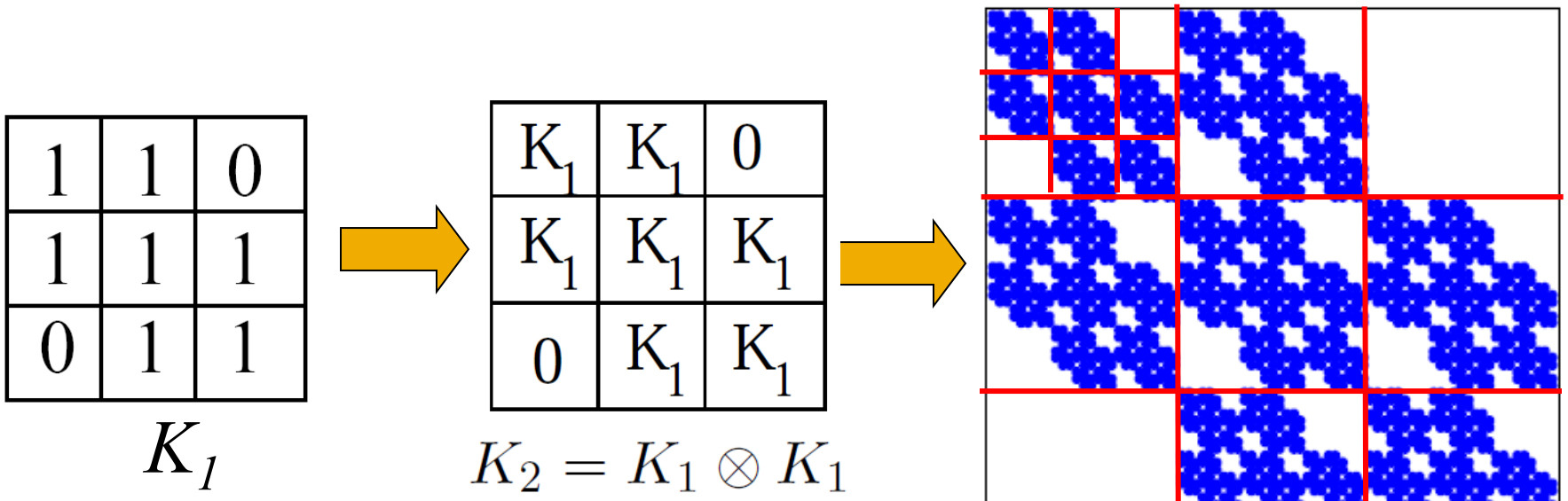
Initial graph | Recursive expansion

- **Kronecker product** is a way of generating self-similar matrices

[PKDD '05]

Kronecker Graph

- **Kronecker graphs:**
  - **A recursive model of network structure**



Kronecker Product: Definition

- **Kronecker product** of matrices A and B is given by

$$\mathbf{C} = \mathbf{A} \otimes \mathbf{B} \doteq \begin{pmatrix} a_{1,1}\mathbf{B} & a_{1,2}\mathbf{B} & \dots & a_{1,m}\mathbf{B} \\ a_{2,1}\mathbf{B} & a_{2,2}\mathbf{B} & \dots & a_{2,m}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}\mathbf{B} & a_{n,2}\mathbf{B} & \dots & a_{n,m}\mathbf{B} \end{pmatrix}$$

N x M *K x L*

*N*K x M*L*

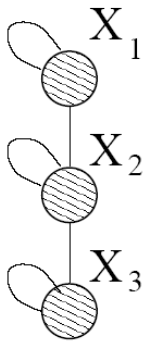
- Define a Kronecker product of two **graphs** as a Kronecker product of their **adjacency matrices**

[PKDD '05]

Kronecker Graphs

- **Kronecker graph** is obtained by growing sequence of graphs by iterating the **Kronecker product over the initiator matrix K_1:**

1	1	0
1	1	1
0	1	1



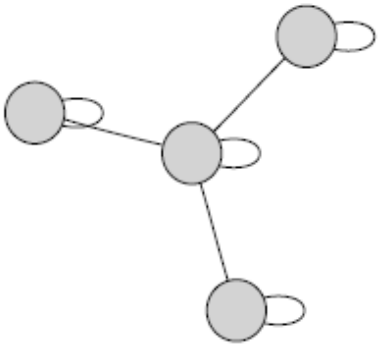
K_1

$$K_1^{[m]} = K_m = \underbrace{K_1 \otimes K_1 \otimes \ldots K_1}_{m \text{ times}} = K_{m-1} \otimes K_1$$

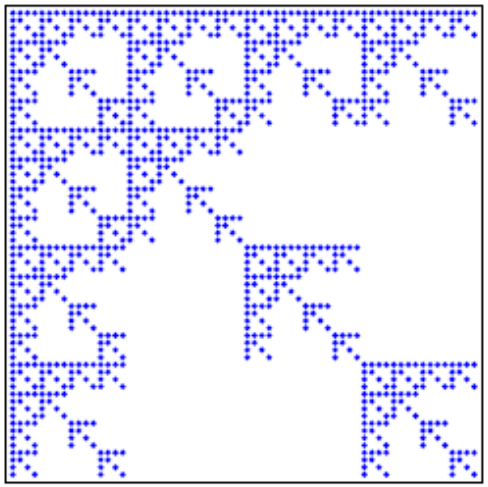
- **Note:** One can easily use multiple initiator matrices (K_1', K_1'', K_1''') (even of different sizes)

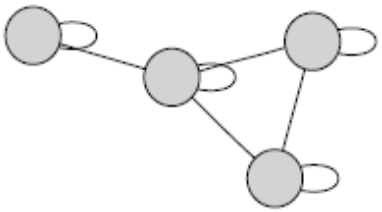
[PKDD '05]

Kronecker Initiator Matrices

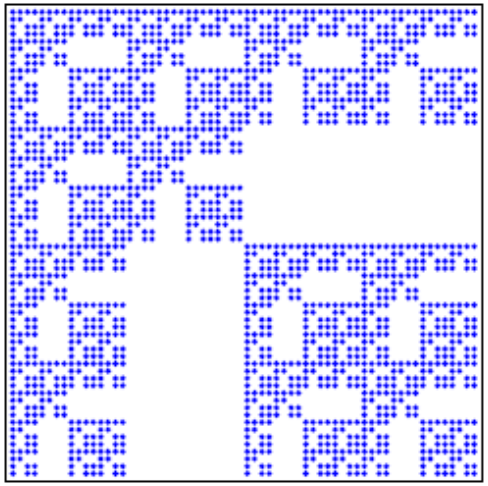


1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	1





1	1	1	1
1	1	0	0
1	0	1	1
1	0	1	1

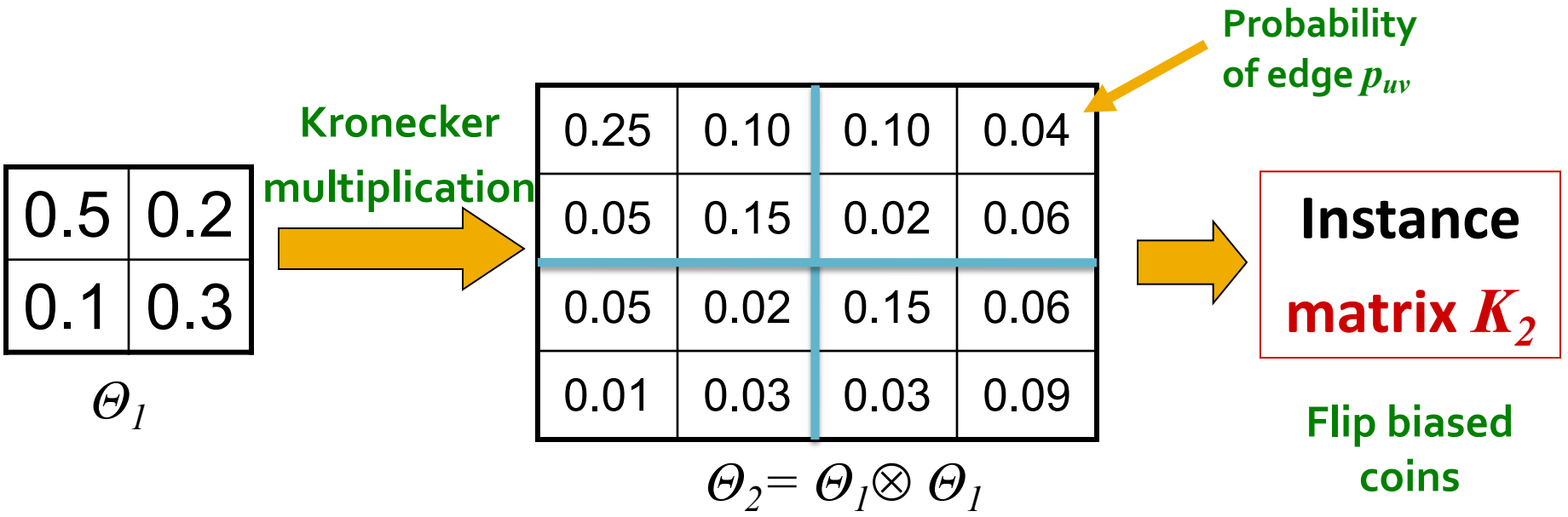


Initiator K_1 | K_1 adjacency matrix | K_3 adjacency matrix

[PKDD '05]

Stochastic Kronecker Graphs

- Create $N_1 \times N_1$ **probability matrix** $\boldsymbol{\Theta_1}$
- Compute the k^{th} Kronecker power $\boldsymbol{\Theta_k}$
- For each entry p_{uv} of $\boldsymbol{\Theta_k}$ include an edge (u, v) in K_k with probability p_{uv}



Generation of Kronecker Graphs

- **How do we generate an instance of a (Directed) stochastic Kronecker graph?**

Probability of edge p_{uv}

0.25	0.10	0.10	0.04
0.05	0.15	0.02	0.06
0.05	0.02	0.15	0.06
0.01	0.03	0.03	0.09

Flip biased coins

1	1	0	0
0	1	0	1
1	0	1	1
0	1	0	1

Need to flip n^2 coins!! Way too slow!!

- **Is there a faster way? YES!**
- **Idea:** Exploit the recursive structure of Kronecker graphs
  - "Drop" edges onto the graph one by one

Generation of Kronecker Graphs

- **A faster way to generate Kronecker graphs**

$\Theta =$

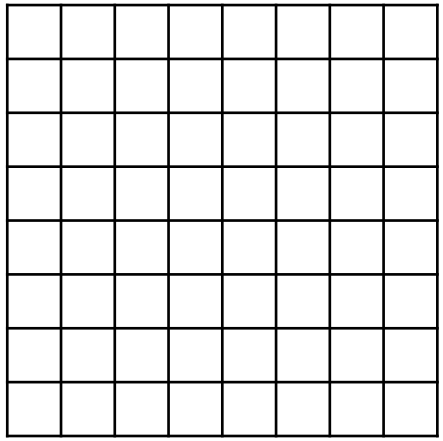
a	b
c	d

	v_1	v_2	v_3	v_4
v_1	a·a	a·b	b·a	b·b
v_2	a·c	a·d	b·c	b·d
v_3	c·a	c·b	d·a	d·b
v_4	c·c	c·d	d·c	d·d

$\Theta \otimes \Theta$ =

	v_1	v_2	v_3	v_4
v_1	a	b	a	b
v_2	c	d	c	d
v_3	a	b	a	b
v_4	c	d	c	d

- **How to "drop" an edge into a graph G on $n = 2^m$ nodes**



Adjacency matrix **G**

Generation of Kronecker Graphs

- **A faster way to generate Kronecker graphs**

$\Theta =$

a	b
c	d

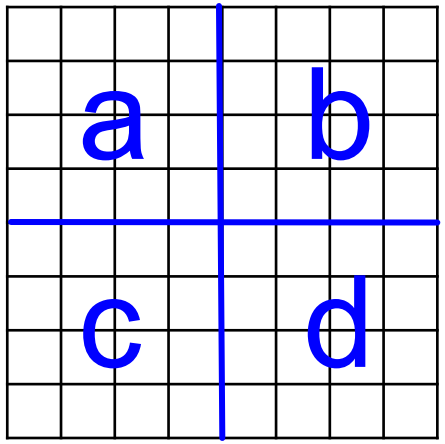
	v_1	v_2	v_3	v_4
v_1	a·a	a·b	b·a	b·b
v_2	a·c	a·d	b·c	b·d
v_3	c·a	c·b	d·a	d·b
v_4	c·c	c·d	d·c	d·d

$\Theta \otimes \Theta$ =

	v_1	v_2	v_3	v_4
v_1	a	b	a	b
v_2	c	d	c	d
v_3	a	b	a	b
v_4	c	d	c	d

(a, b, c, d)

- **How to "drop" an edge into a graph G on $n = 2^m$ nodes**



Adjacency matrix **G**

Generation of Kronecker Graphs

- **A faster way to generate Kronecker graphs**

$\Theta =$

a	b
c	d

	v_1	v_2	v_3	v_4
v_1	a·a	a·b	b·a	b·b
v_2	a·c	a·d	b·c	b·d
v_3	c·a	c·b	d·a	d·b
v_4	c·c	c·d	d·c	d·d

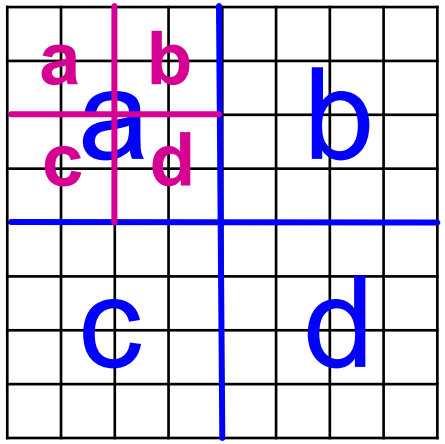
$\Theta \otimes \Theta$

=

	v_1	v_2	v_3	v_4
v_1	a	b	a	b
v_2	c	d	c	d
v_3	a	b	a	b
v_4	c	d	c	d

a b a b c d c d

- **How to "drop" an edge into a graph G on $n = 2^m$ nodes**



Adjacency matrix **G**

Generation of Kronecker Graphs

- **A faster way to generate Kronecker graphs**

$\Theta =$

a	b
c	d

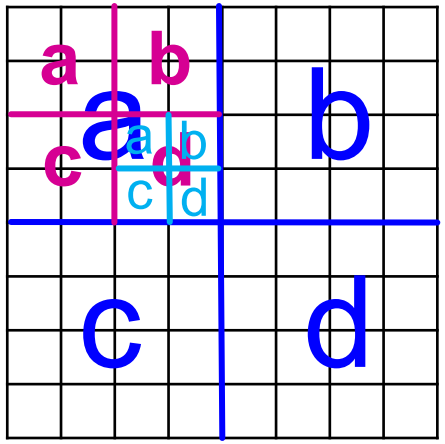
	v_1	v_2	v_3	v_4
v_1	a·a	a·b	b·a	b·b
v_2	a·c	a·d	b·c	b·d
v_3	c·a	c·b	d·a	d·b
v_4	c·c	c·d	d·c	d·d

$\Theta \otimes \Theta$

=

	v_1	v_2	v_3	v_4
v_1	a	b	a	b
v_2	c	d	c	d
v_3	a	b	a	b
v_4	c	d	c	d

- **How to "drop" an edge into a graph G on $n = 2^m$ nodes:**
  - We may get a few edges colliding. We simply reinsert them.

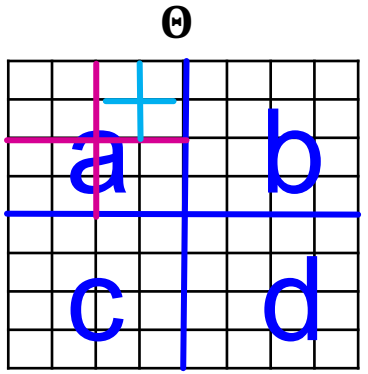


Adjacency matrix **G**

Generation of Kronecker Graphs

Fast Kronecker generator algorithm:

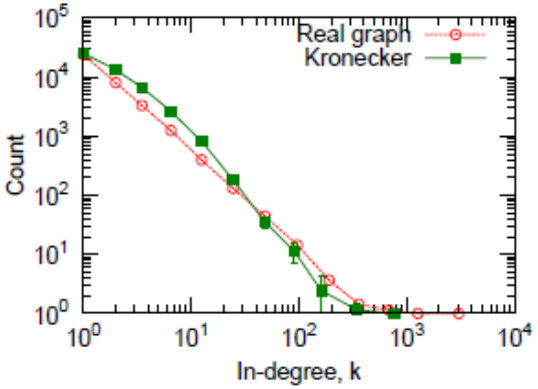
- For generating **directed graphs**
- **Insert 1 edge on graph G on $n = 2^m$ nodes:**
  - Create normalized matrix $L_{uv} = \Theta_{uv}/(\sum_{op} \Theta_{op})$
  - **For $i = 1 \ldots m$**
    - Start with $x = 0, y = 0$
    - Pick a row/column (u, v) with prob. L_{uv}
    - Descend into quadrant (u, v) at level i of G
      - **This means:** $x \mathrel{+}= u \cdot 2^{m-i}$, $y \mathrel{+}= v \cdot 2^{m-i}$
  - Add an edge (x, y) **to** G



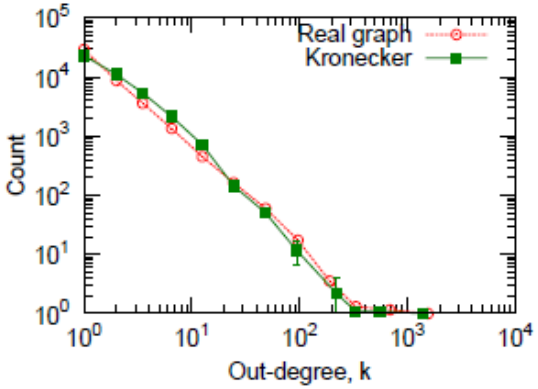
Estimation: Epinions (n=76k, m=510k)

- **Real** and **Kronecker** are very close:

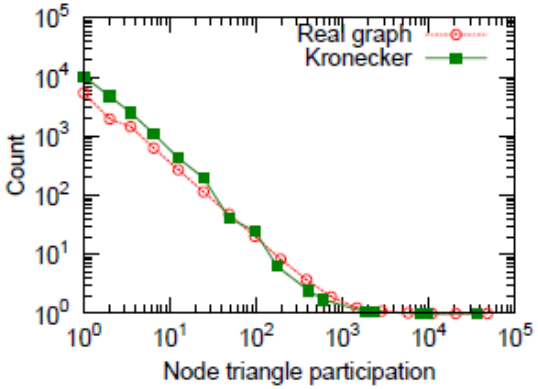
$$\Theta_1 = \begin{bmatrix} 0.99 & 0.54 \\ 0.49 & 0.13 \end{bmatrix}$$



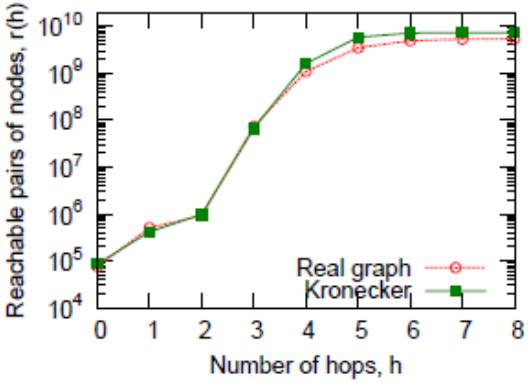
(a) In-Degree



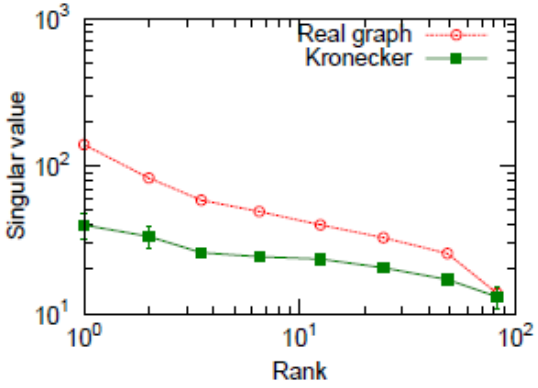
(b) Out-degree



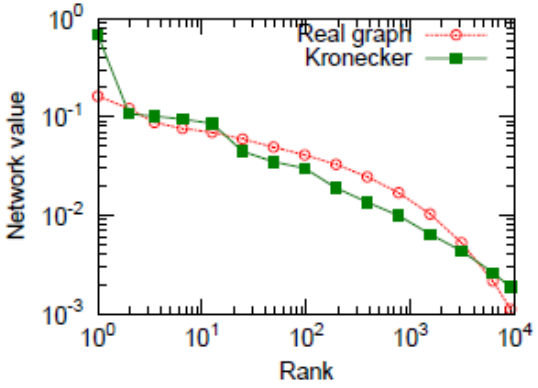
(c) Triangle participation



(d) Hop plot



(e) Scree plot



(f) "Network" value