

# LinLog und LinDisCats

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## 1 Introduction

what am I even writing about?

## 2 LinLog Preliminaries

### 2.1 Motivation/Intuition

feels kinda rambling... combine intuition with syntax definition below

Classical (and intuitionistic) logic deals with the propagation of stable truth values. If one has a true sentence  $A$  and an implication  $A \Rightarrow B$ , then  $B$  follows while  $A$  remains true. However, real-life implications are often causal and modify their premises. They cannot therefore be iterated arbitrarily. For example if  $A$  describes the ownership of 1€ and  $B$  owning a chocolate bar, an implication  $A \multimap B$  (to be formally introduced later) would describe the process of buying such a chocolate bar for 1€, losing the 1€ in the process.

While such dealings with resources can of course be modeled in classical logic, it is easier done in the resource-sensitive *linear logic*, first described by Girard in 1987 [Gir87].

Here, we have two conjunctions simultaneously  $\otimes$  ("times" or "tensor") and  $\&$  ("with"), which describe the availability of resources:

Suppose  $C$  is the ownership of a cookie and it costs also 1€ (i.e. we have  $A \multimap C$ ). Then  $B \otimes C$  states that one owns both a chocolate bar and a cookie. The implication  $A \multimap B \otimes C$  is not possible, as it would mean that you are buying both, cookie and chocolate bar at the same time, for just 1€ total. However, from  $A \multimap B$  and  $A \multimap C$  we get  $A \otimes A \multimap B \otimes C$ , i.e. the process of buying both for 2€.

On the other hand,  $B \& C$  states that one has a choice between either  $B$  or  $C$  (imagine a token). From the implications  $A \multimap B$  and  $A \multimap C$  we get the implication  $A \multimap B \& C$ , i.e. the process of buying a token to be exchanged for a chocolate bar or a cookie at a later time (with the choice lying with oneself). While this may seem like a disjunction, both implications  $B \& C \multimap B$  and  $B \& C \multimap C$  (exchanging the token for either product) are provable from  $B \& C$ , although not simultaneously.

Dually, we have two disjunctions  $\wp$  ("par") and  $\oplus$  ("plus"):

Suppose now that  $B$  and  $C$  are the ownership of a figurine of Pikachu or Mew respectively. Then  $B \oplus C$  may be the ownership of a Kinder Egg containing either figurine. This means when buying that egg ( $A \multimap B \oplus C$ ) we do not know which one we will get.

Our second disjunction, dual to  $\otimes$ , can be understood by linear implication and the linear negation (denoted as  $(\cdot)^\perp$ ): Under the interpretation of ownership the linear negation is no interpreted as the absence of ownership but as negative ownership, i.e. debt. That means the negation of owning 1€,  $A$ , is owing someone 1€,  $A^\perp$ . With the par operator we can now write the linear implication  $A \multimap B$  symmetrically as  $A^\perp \wp B$ .

(explicit interpretation?! shared pool between people containing Pika and Mew -> you have to get rid of one to use the other?)

In order to regain our stable truths known from classical logic, we need to employ two unitary connectives ! ("of course" or "bang") and ? ("why not"). The bang operator informs us that there is an infinite amount of a resource: The statement  $!A$  translates into the ownership of an amount of money that is large enough for us to ignore resource sensitivity. Imagine for example a billionaire buying a Pokemon figurine instead of a social media site: His amount of money will not be noticeably smaller after buying the figurine. We can informally say  $!A = (1 \& A) \otimes (1 \& A) \otimes \dots$  and therefore view classical logic as some sort of limit of linear logic, just as classical mechanics is a limit of quantum mechanics and the theory of relativity.

interpretation of ?wn.

interpretation of constants

## 2.2 Syntax

section needs serious reformatting...

main source: IntroLinLog

Formulas are built from atomic formulas  $p, q, \phi, \psi, p^\perp$  etc. and constants  $1, \perp, 0, \top$  with connectives  $\otimes, \wp, \&, \oplus, !, ?$ .

Let  $\Gamma, \Delta$  etc. be arbitrary, finite lists of formulas (e.g.:  $\Gamma = (p_1, \dots, p_n)$ ) and  $A$  and  $B$  formulas. As we will later consider fragments without negation, we shall define linear logic with a two-sided calculus. For structural rules we only have the exchange rule, missing the (general) weakening and contraction rules known from classical logic:

formula?  
term?  
look up  
basic terminology

$$\frac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, B, A, \Gamma' \vdash \Delta} \text{ex.L} \quad \frac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, B, A, \Delta'} \text{ex.R}$$

Obviously, we have an identity and a cut rule:

$$\frac{}{A \vdash A} \text{id} \quad \frac{\Gamma \vdash \Delta, A \quad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{Cut}$$

As already mentioned, the classical conjunction  $\wedge$  and disjunction  $\vee$  split into two respectively. These can be classified as multiplicative and additive connectives. The calculus rules for the *multiplicative* conjunction  $\otimes$  and disjunction  $\wp$  ("par") are as follows:

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma A \otimes B \vdash \Delta} \otimes_L \quad \frac{\Gamma \vdash \Delta, A \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash \Delta, A \otimes B, \Delta'} \otimes_R$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, A \wp B, \Gamma' \vdash \Delta, \Delta'} \wp_L \quad \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \wp B} \wp_R$$

**Remark 2.1** The rules  $\otimes_L$  and  $\wp_R$  imply, that the commas are to be read as  $\otimes$  on the left-hand side and as  $\wp$  on the right-hand side. That means  $A, B \vdash C, D$  is provable iff  $A \otimes B \vdash C \wp D$  is provable.

PROOF: We only have to show that  $A, B \vdash C, D$  follows from  $A \otimes B \vdash C \wp D$  as the other direction is just our introduction rule:

$$\frac{\frac{A \otimes B \vdash C \wp D \quad \frac{\overline{A \vdash A} \text{ id} \quad \overline{B \vdash B} \text{ id}}{A, B \vdash A \otimes B} \otimes_R}{A, B \vdash C \wp D} \text{Cut} \quad \frac{\overline{C \vdash C} \quad \overline{D \vdash D}}{C \wp D \vdash C, D} \wp_L \quad \square$$

$$\frac{A, B \vdash C \wp D}{A, B \vdash C, D}$$

The calculus rules for the *additive* conjunction  $\&$  and disjunction  $\oplus$  ("plus") are as follows:

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \& B \vdash \Delta} \&_{L1} \quad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \& B \vdash \Delta} \&_{L2}$$

$$\frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \& B} \&_R$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta} \oplus_L$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \oplus B, \Delta} \oplus_{R1} \quad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \oplus B, \Delta} \oplus_{R2}$$

Notice the difference between  $\&_R$  and  $\otimes_R$  (and dually between  $\oplus_L$  and  $\wp_L$ ): while for  $\otimes_R$  the contexts  $\Gamma$  etc. are arbitrary and get combined in the conclusion,  $\&_R$  requires the contexts to be equal. In classical logic these rules would be equivalent using its additional structural rules.

**Remark 2.2** Similarly to the multiplicative connectives above, the additive connectives have a invertibility statement:  $\Gamma \vdash A \& B$  is provable iff  $\Gamma \vdash A$  and  $\Gamma \vdash B$  are provable. (The statement for  $\oplus$  is formulated dually)

PROOF: As with the multiplicative statement, one direction suffices:

$$\frac{\Gamma \vdash A \& B \quad \frac{\overline{A \vdash A}}{A \& B \vdash A}}{\Gamma \vdash A}$$

$\Gamma \vdash B$  follows the same way. □

above remark unnötig?

We now define the linear negation  $(\cdot)^\perp$  as follows:

- For the constants:

$$\begin{aligned} 1^\perp &:= \perp & \perp^\perp &:= 1 \\ \top^\perp &:= 0 & 0^\perp &:= \top \end{aligned}$$

- For atomic formulas: the negation of  $p$  is  $p^\perp$ . The negation  $(p^\perp)^\perp$  of  $p^\perp$  is  $p$ .
- For non-atomic formulas we define the negation by the De Morgan equations:

$$\begin{aligned} (A \otimes B)^\perp &:= A^\perp \wp B^\perp & (A \wp B)^\perp &:= A^\perp \otimes B^\perp \\ (A \& B)^\perp &:= A^\perp \oplus B^\perp & (A \oplus B)^\perp &:= A^\perp \& B^\perp \end{aligned}$$

- We define linear implication with the par-operator:

$$A \multimap B := A^\perp \wp B$$

- As with classical logic, we translate a two-sided sequent  $A_1, \dots, A_n \vdash B_1, \dots, B_n$  into a right-sided sequent  $\vdash A_1^\perp, \dots, A_n^\perp, B_1, \dots, B_n$  by negation of the left-hand side and vice versa. With this it is easily seen that  $\vdash A \multimap B$  iff  $A \vdash B$ .

With this our calculus rules above become quite redundant and we can restrict them on the rules for the right-hand side. However, these redundancies become necessary when dealing with a negation-free fragment of LL.

The modality connectives reintroduce stable truths and with them the the weakening and contraction rules known from classical logic:

$$\begin{aligned} \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A} ! \quad & \frac{\vdash \Gamma}{\vdash \Gamma, ?A} ?_W \text{ (weakening)} \\ \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A} ?_D \text{ (dereliction)} \quad & \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A} ?_C \text{ (contraction)} \end{aligned}$$

$$(!A)^\perp = ?(A^\perp) \quad (?A)^\perp = !(A^\perp)$$

Here the Kontext  $?\Gamma$  is given by applying the  $?$ -modality to every formula in the list of  $\Gamma$ , i.e.  $?\Gamma = ?q, ?p, \dots$  for  $\Gamma = q, p, \dots$

**Remark 2.3** These modalities are called exponentials because of the following relation:

$$!(A \& B) \equiv !A \otimes !B, \quad ?(A \oplus B) \equiv ?A \wp ?B$$

PROOF: We will only prove the second equivalence. The first is acquired by duality.

Definition of  $?\Gamma$  korrekt?

definition linear equivalence

$$\begin{array}{c}
\frac{\frac{\frac{\vdash A, A^\perp}{A^\perp, A \oplus B} \oplus}{\vdash A^\perp, ?(A \oplus B)} ?_D \quad \frac{\frac{\frac{\vdash B, B^\perp}{B^\perp, A \oplus B} \oplus}{\vdash B^\perp, ?(A \oplus B)} ?_D}{\vdash !(A^\perp), ?(A \oplus B)} ! \quad \frac{\frac{\frac{\vdash B, B^\perp}{B^\perp, A \oplus B} \oplus}{\vdash B^\perp, ?(A \oplus B)} ?_D \quad \frac{\frac{\vdash A, A^\perp}{A^\perp, A \oplus B} \oplus}{\vdash !(B^\perp), ?(A \oplus B)} !}{\vdash !(A^\perp) \otimes !(B^\perp), ?(A \oplus B), ?(A \oplus B)} \otimes \\
\frac{\vdash !(A^\perp) \otimes !(B^\perp), ?(A \oplus B), ?(A \oplus B)}{\vdash !(A^\perp) \otimes !(B^\perp), ?(A \oplus B)} ?_D \\
\frac{\vdash !(A^\perp) \otimes !(B^\perp), ?(A \oplus B)}{\vdash (?A \wp ?B)^\perp, ?(A \oplus B)} (-)^\perp \\
\frac{\vdash (?A \wp ?B)^\perp, ?(A \oplus B)}{\vdash ?A \wp ?B \multimap ?(A \oplus B)} \\
\\
\frac{\frac{\frac{\vdash A^\perp, A}{\vdash A^\perp, ?A} ?_W}{\vdash A^\perp, ?A, ?B} ?_W \quad \frac{\frac{\frac{\vdash B^\perp, B}{\vdash B^\perp, ?B} ?_W}{\vdash B^\perp, ?B, ?A} ?_W}{\vdash A^\perp \& B^\perp, ?A, ?B} \& \\
\frac{\vdash A^\perp \& B^\perp, ?A, ?B}{\vdash !(A^\perp \& B^\perp), ?A, ?B} \\
\frac{\vdash !(A^\perp \& B^\perp), ?A \wp ?B}{\vdash ?(A \oplus B) \multimap ?A \wp ?B}
\end{array}$$

□

proofsymbol  
なんで?:(

MAYBE to be added: \forall and \exists. Do we want predicate logic?

## 2.3 Blabla proof-structures and proof-nets

We shall now leave behind any capitalistic resource interpretation and concentrate on what linear logic was invented for: analyzing proofs.

to be continued...

how to typeset axiom links? :(

## 3 Categorical Preliminaries

### 3.1 Categories

sources for basic definitions. (nlab proly doesn't suffice)

**Definition 3.1 (Category)** A Category  $\mathfrak{C}$  consists of the following data:

- A class  $\text{Obj}(\mathfrak{C})$  of objects.
- For every pair of objects  $A, B \in \text{Obj}(\mathfrak{C})$  there is a class  $\text{Hom}(A, B)$  of morphisms  $f : A \rightarrow B$  from  $A$  to  $B$ .
- Morphisms compose: For  $f \in \text{Hom}(A, B)$  and  $g \in \text{Hom}(B, C)$  there is a morphism  $g \circ f \in \text{Hom}(A, C)$ . That composition is associative:

$$h \circ (g \circ f) = (h \circ g) \circ f$$

We will write  $gf$  for  $g \circ f$  when appropriate.

- For every object  $A$  there is an identity morphism  $\text{id}_A \in \text{Hom}(A, A)$ :

$$f \circ \text{id}_A = f, \quad \text{id}_B \circ f = f$$

for  $f \in \text{Hom}(A, B)$

If the hom-classes are sets, we call the category locally small. If the object class is a set, we call the category small. Otherwise, we call the category large.

If we have a category  $\mathfrak{C}$ , we call its opposite category  $\mathfrak{C}^{\text{opp}}$  the category with the following structure:

- $\text{Obj}(\mathfrak{C}^{\text{opp}}) = \text{Obj}(\mathfrak{C})$
- $\forall A, B \in \text{Obj}(\mathfrak{C}) : \text{Hom}_{\mathfrak{C}^{\text{opp}}}(A, B) = \text{Hom}_{\mathfrak{C}}(B, A)$

extra definition for opposite structure?

**Definition 3.2 (Functor)** A (covariant) functor  $F : \mathfrak{C} \rightarrow \mathfrak{D}$  between two categories  $\mathfrak{C}$  and  $\mathfrak{D}$  is a function mapping each object  $A \in \text{Obj}(\mathfrak{C})$  to an object  $F(A) \in \text{Obj}(\mathfrak{D})$  and each morphism  $f \in \text{Hom}_{\mathfrak{C}}(A, B)$  to a morphism  $F(f) \in \text{Hom}_{\mathfrak{D}}(F(A), F(B))$  such that identity and composition are preserved:

$$F(\text{id}_A) = \text{id}_{F(A)}, \quad F(g \circ f) = F(g) \circ F(f)$$

A functor  $F : \mathfrak{C}^{\text{opp}} \rightarrow \mathfrak{D}$  is called contravariant on  $\mathfrak{C}$ .

We will drop the parentheses when appropriate.

**Definition 3.3 (Natural transformation)** A natural transformation  $\tau : F \rightarrow G$  between two functors  $F, G : \mathfrak{C} \rightarrow \mathfrak{D}$  is family of morphisms in  $\mathfrak{D}$ :

$$\tau = \{\tau_A : FA \rightarrow GA \mid A \in \text{Obj}(\mathfrak{C})\}$$

such that  $\tau_B F(f) = G(f) \tau_A$  for all  $f : A \rightarrow B \in \text{Morph}(\mathfrak{C})$ , i.e. the following diagram commutes:

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \downarrow \tau_A & & \downarrow \tau_B \\ GA & \xrightarrow{Gf} & GB \end{array}$$

If  $\tau_A$  is an isomorphism for all  $A \in \text{Obj}(\mathfrak{C})$ , we call  $\tau$  a natural isomorphism. We will often represent a natural transformation by a single one of its members and also drop the index when appropriate.

definition: equivalence of categories

**Definition 3.4 (Adjunction)** *Let  $\mathfrak{C}$  and  $\mathfrak{D}$  be categories. We call  $F : \mathfrak{C} \rightarrow \mathfrak{D}$  the left adjoint of  $G : \mathfrak{D} \rightarrow \mathfrak{C}$  and  $G$  the right adjoint of  $F$  if there is an isomorphism that is natural in  $A \in \mathfrak{C}$  and  $B \in \mathfrak{D}$ :*

source:  
Branden-  
burg

$$\text{Hom}_{\mathfrak{D}}(FA, B) \cong \text{Hom}_{\mathfrak{C}}(A, GB)$$

Equivalently, we call  $F$  the left adjoint of  $G$  if there are natural transformations

$$\begin{aligned} \eta : \text{id}_{\mathfrak{C}} &\rightarrow G \circ F \\ \varepsilon : F \circ G &\rightarrow \text{id}_{\mathfrak{D}} \end{aligned}$$

fulfilling the following condition:

$$\begin{aligned} \text{id}_{FA} &= \varepsilon(FA) \circ F\eta(A) \\ \text{id}_{GB} &= G\varepsilon(B) \circ \eta(GB) \end{aligned}$$

i.e. the following diagrams commute:

maybe  
index  
Schreib-  
weise?

$$\begin{array}{ccc} & F \circ G \circ F & \\ \begin{array}{c} \nearrow F \bullet \eta \\ \text{F id}_{\mathfrak{C}} \\ \xlongequal{\quad \text{id} \quad} \\ \text{id}_{\mathfrak{C}} \circ F \end{array} & & \begin{array}{c} \nwarrow \varepsilon \bullet F \\ \text{id}_{\mathfrak{C}} \circ G \\ \xlongequal{\quad \text{id} \quad} \\ G \circ \text{id}_{\mathfrak{D}} \end{array} \end{array}$$

cut triangle diagrams or define •

We write  $F \dashv G$ .

**Definition 3.5 (Monoidal category)** A monoidal category  $(\mathfrak{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$  is a category  $\mathfrak{C}$  with the following additional data:

- a functor  $\otimes : \mathfrak{C} \times \mathfrak{C} \rightarrow \mathfrak{C}$ , called monoidal or tensor product;
- an object  $\mathbb{1} \in \text{Obj}(\mathfrak{C})$ , called unit object;
- a natural isomorphism  $\alpha : A \otimes (B \otimes C) \rightarrow (A \otimes B) \otimes C$ , called associator;
- a natural isomorphism  $\lambda : \mathbb{1} \otimes A \rightarrow A$ , called left unitor;
- a natural isomorphism  $\rho : A \otimes \mathbb{1} \rightarrow A$ , called right unitor;

such that the following diagrams commute:

- the pentagon diagram:

$$\begin{array}{ccc} & A \otimes (B \otimes (C \otimes D)) & \\ \swarrow \text{id} \otimes \alpha & & \searrow \alpha \\ A \otimes ((B \otimes C) \otimes D) & & (A \otimes B) \otimes (C \otimes D) \\ \downarrow \alpha & & \downarrow \alpha \\ (A \otimes (B \otimes C)) \otimes D & \xrightarrow{\alpha \otimes \text{id}} & ((A \otimes B) \otimes C) \otimes D \end{array}$$

- the unit diagram:

$$\begin{array}{ccc}
 (A \otimes \mathbb{1}) \otimes B & \xrightarrow{\alpha} & A \otimes (\mathbb{1} \otimes B) \\
 \searrow \rho \otimes \text{id} & & \swarrow \text{id} \otimes \lambda \\
 & A \otimes B &
 \end{array}$$

A symmetric monoidal category has an additional natural isomorphism:

$$s_{A,B} : A \otimes B \rightarrow B \otimes A$$

satisfying the following conditions:

- the hexagon law:

$$\begin{array}{ccccc}
 A \otimes (B \otimes C) & \xrightarrow{\alpha} & (A \otimes B) \otimes C & \xrightarrow{s_{A \otimes B, C}} & C \otimes (A \otimes B) \\
 \downarrow \text{id} \otimes s_{B, C} & & & & \downarrow \alpha \\
 A \otimes (C \otimes B) & \xrightarrow{\alpha} & (A \otimes C) \otimes B & \xrightarrow{s_{A, C \otimes B}} & (C \otimes A) \otimes B
 \end{array}$$

- the inverse law:  $s_{A,B}^{-1} = s_{B,A}$
- the unit law:  $\lambda = \rho s$

$$\begin{array}{ccc}
 \mathbb{1} \otimes A & \xrightarrow{s_{\mathbb{1}, A}} & A \otimes \mathbb{1} \\
 \searrow \lambda & & \swarrow \rho \\
 & A &
 \end{array}$$

Diagramme verschönern

fix associators throughout the document (direction)

**Definition 3.6 (Closed monoidal category)** A monoidal category  $\mathfrak{C}$  is called right closed, resp. left closed, iff there is a functor  $- \multimap - : \mathfrak{C}^{\text{opp}} \times \mathfrak{C} \rightarrow \mathfrak{C}$ , resp.  $- \multimap - : \mathfrak{C} \times \mathfrak{C}^{\text{opp}} \rightarrow \mathfrak{C}$ , such that there is a natural isomorphism  $\text{Hom}(A \otimes B, C) \cong \text{Hom}(B, A \multimap C)$ , resp.  $\text{Hom}(A \otimes B, C) \cong \text{Hom}(A, C \multimap B)$ , i.e. iff the functor  $- \otimes B$ , resp.  $A \otimes -$ , has a right adjoint. A monoidal  $\mathfrak{C}$  is called biclosed iff it is both left and right closed.

If  $\mathfrak{C}$  is symmetric, these functors coincide we call the category closed.

The functors  $- \multimap - : \mathfrak{C}^{\text{opp}} \times \mathfrak{C} \rightarrow \mathfrak{C}$  and  $- \multimap - : \mathfrak{C} \times \mathfrak{C}^{\text{opp}} \rightarrow \mathfrak{C}$  are called internal hom functors.

**Definition 3.7 (Enriched category)** Let  $(\mathfrak{M}, \mathbb{1}, \alpha, \lambda, \rho)$  be a monoidal category. An  $\mathfrak{M}$ -enriched category  $\mathfrak{C}$  consists of the following data:

- A class  $\text{Obj}(\mathfrak{C})$  of objects;
- For every pair of objects  $(A, B) \in \text{Obj}(\mathfrak{C}) \times \text{Obj}(\mathfrak{C})$  an object  $\mathfrak{C}(A, B) \in \text{Obj}(\mathfrak{M})$  called hom object;
- For every triple of objects  $(A, B, C) \in \text{Obj}(\mathfrak{C}) \times \text{Obj}(\mathfrak{C}) \times \text{Obj}(\mathfrak{C})$  a morphism  $\text{Morph}(\mathfrak{M}) \ni \circ_{A, B, C} : \mathfrak{C}(B, C) \otimes \mathfrak{C}(A, B) \rightarrow \mathfrak{C}(A, C)$  called composition morphism;

lengthy,  
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bling...  
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elegant  
gehen



- For every object  $A \in \mathfrak{C}$  a morphism  $\text{id}_A : \mathbb{1} \rightarrow \mathfrak{C}(A, A)$ ;

such that the following diagrams commute:

$$\begin{array}{ccc}
(\mathfrak{C}(C, D) \otimes \mathfrak{C}(B, C)) \otimes \mathfrak{C}(A, B) & \xrightarrow{\alpha} & \mathfrak{C}(C, D) \otimes (\mathfrak{C}(B, C) \otimes \mathfrak{C}(A, B)) \\
\downarrow \circ_{B, C, D} \otimes \text{id}_{\mathfrak{C}(A, B)} & & \downarrow \text{id}_{\mathfrak{C}(C, D)} \otimes \circ_{A, B, C} \\
\mathfrak{C}(D, B) \otimes \mathfrak{C}(A, B) & & \mathfrak{C}(C, D) \otimes \mathfrak{C}(A, C) \\
& \searrow \circ_{A, B, D} \quad \swarrow \circ_{A, C, D} & \\
& \mathfrak{C}(A, D) &
\end{array}$$
  

$$\begin{array}{ccccc}
\mathbb{1} \otimes \mathfrak{C}(A, B) & \xrightarrow{\lambda} & \mathfrak{C}(A, B) & \xleftarrow{\rho} & \mathfrak{C}(A, B) \otimes \mathbb{1} \\
\text{id}_B \otimes \text{id}_{\mathfrak{C}(A, B)} \downarrow & & \nearrow \circ_{A, B, B} & & \downarrow \text{id}_{\mathfrak{C}(A, B)} \otimes \text{id}_A \\
\mathfrak{C}(B, B) \otimes \mathfrak{C}(A, B) & & \mathfrak{C}(A, B) & & \mathfrak{C}(A, B) \otimes \mathfrak{C}(A, A) \\
& & \nwarrow \circ_{A, A, B} & &
\end{array}$$

**Definition 3.8 (Enriched functors)** Let  $\mathfrak{C}$  and  $\mathfrak{D}$  be  $\mathfrak{M}$ -enriched categories. An enriched functor  $F : \mathfrak{C} \rightarrow \mathfrak{D}$  consists of:

- a function  $F_0 : \text{Obj}(\mathfrak{C}) \rightarrow \text{Obj}(\mathfrak{D})$  between the objects
- a family of morphisms in  $\mathfrak{M}$ :  $F_{A, B} : \mathfrak{C}(A, B) \rightarrow \mathfrak{D}(F_0 A, F_0 B)$

such that the following diagrams commute:

$$\begin{array}{ccc}
\mathfrak{C}(B, C) \otimes \mathfrak{C}(A, B) & \xrightarrow{\circ} & \mathfrak{C}(A, C) \\
\downarrow F_{B, C} \otimes F_{A, B} & & \downarrow F_{A, C} \\
\mathfrak{D}(F_0 B, F_0 C) \otimes \mathfrak{D}(F_0 A, F_0 B) & \xrightarrow{\circ} & \mathfrak{D}(F_0 A, F_0 C)
\end{array}$$
  

$$\begin{array}{ccc}
& \mathbb{1} & \\
\text{id}_A \swarrow & & \searrow \text{id}_{F_0 A} \\
\mathfrak{C}(A, A) & \xrightarrow{F_{A, A}} & \mathfrak{D}(F_0 A, F_0 A)
\end{array}$$

### 3.2 LinDisCats

**Definition 3.9 (Linearly distributive category)** A linearly distributive category  $(\mathfrak{C}, \otimes, 1, \mathfrak{A}, \perp)$  is a category  $\mathfrak{C}$  consisting of:

- a monoidal category  $(\mathfrak{C}, \otimes, 1, \alpha_\otimes, \lambda_\otimes, \rho_\otimes)$ , with  $\otimes$  called "tensor";
- a monoidal category  $(\mathfrak{C}, \mathfrak{A}, \perp, \alpha_\mathfrak{A}, \lambda_\mathfrak{A}, \rho_\mathfrak{A})$ , with  $\mathfrak{A}$  called "par";
- two natural transformations called left and right linear distributors respectively:

$$\begin{aligned}
\partial_L : A \otimes (B \mathfrak{A} C) &\rightarrow (A \otimes B) \mathfrak{A} C \\
\partial_R : (A \mathfrak{A} B) \otimes C &\rightarrow A \mathfrak{A} (B \otimes C)
\end{aligned}$$

satisfying the following coherence conditions:

- coherence between the distributors and unitors:

$$\begin{array}{ccc}
1 \otimes (A \wp B) & \xrightarrow{\partial_L} & (1 \otimes A) \wp B \\
& \searrow \lambda_{\otimes} & \swarrow \lambda_{\otimes} \wp \text{id} \\
& A \wp B & \\
(A \wp B) \otimes 1 & \xrightarrow{\partial_R} & A \wp (B \otimes 1) \\
& \searrow \rho_{\otimes} & \swarrow \text{id} \otimes \rho_{\otimes} \\
& A \wp B & \\
A \otimes (B \wp \perp) & \xrightarrow{\partial_L} & (A \otimes B) \wp \perp \\
& \searrow \text{id} \otimes \rho_{\wp} & \swarrow \rho_{\wp} \\
& A \otimes B & \\
(\perp \wp A) \otimes B & \xrightarrow{\partial_R} & \perp \wp (A \otimes B) \\
& \searrow \lambda_{\wp} \otimes \text{id} & \swarrow \lambda_{\wp} \\
& A \otimes B &
\end{array}$$

- coherence between distributors and associators:

Diagramme  
besser  
alignen

$$\begin{array}{ccc}
& (A \otimes B) \otimes (C \wp D) & \\
& \swarrow \alpha_{\otimes} & \searrow \partial_L \\
A \otimes (B \otimes (C \wp D)) & & ((A \otimes B) \otimes C) \wp D \\
\downarrow \text{id} \otimes \partial_L & & \downarrow \alpha_{\otimes} \wp \text{id} \\
A \otimes ((B \otimes C) \wp D) & \xrightarrow{\partial_L} & (A \otimes (B \otimes C)) \wp D
\end{array}$$

$$\begin{array}{ccc}
& A \otimes (B \wp (C \wp D)) & \\
& \swarrow \text{id} \otimes \alpha_{\wp} & \searrow \partial_L \\
A \otimes ((B \wp C) \wp D) & & (A \otimes B) \wp (C \wp D) \\
\downarrow \partial_L & & \downarrow \alpha_{\wp} \\
(A \otimes (B \wp C)) \wp D & \xrightarrow{\partial_L \wp \text{id}} & ((A \otimes B) \wp C) \wp D
\end{array}$$

$$\begin{array}{ccc}
& (A \wp B) \otimes (C \otimes D) & \\
& \swarrow \alpha_{\otimes} & \searrow \partial_R \\
((A \wp B) \otimes C) \otimes D & & A \wp (B \otimes (C \otimes D)) \\
\downarrow \partial_R \otimes \text{id} & & \downarrow \text{id} \wp \alpha_{\otimes} \\
(A \wp (B \otimes C)) \otimes D & \xrightarrow{\partial_R} & A \wp ((B \otimes C) \otimes D)
\end{array}$$

$$\begin{array}{ccc}
& ((A \wp B) \wp C) \otimes D & \\
& \swarrow \alpha_{\wp} \otimes \text{id} & \searrow \partial_R \\
(A \wp (B \wp C)) \otimes D & & (A \wp B) \wp (C \otimes D) \\
\downarrow \partial_R & & \downarrow \alpha_{\wp} \\
A \wp ((B \wp C) \otimes D) & \xrightarrow{\text{id} \wp \partial_R} & A \wp (B \wp (C \otimes D))
\end{array}$$

- coherence between the distributors:

$$\begin{array}{ccc}
& (A \wp B) \otimes (C \wp D) & \\
\swarrow \partial_R & & \searrow \partial_L \\
((A \wp B) \otimes C) \wp D & & A \wp (B \otimes (C \wp D)) \\
\downarrow \partial_R \wp \text{id} & & \downarrow \text{id} \wp \partial_L \\
(A \wp (B \otimes C)) \wp D & \xleftarrow{\alpha_{\wp}} & A \wp ((B \otimes C) \wp D)
\end{array}$$

obiges Diagramm rumdrehen

\parr \mid id sieht kacke aus...

$$\begin{array}{ccc}
& A \otimes ((B \wp C) \otimes D) & \\
\swarrow \alpha_{\otimes} & & \searrow \text{id} \otimes \partial_R \\
(A \otimes (B \wp C)) \otimes D & & A \otimes (B \wp (C \otimes D)) \\
\downarrow \partial_L \otimes \text{id} & & \downarrow \partial_L \\
((A \otimes B) \wp C) \otimes D & \xrightarrow{\partial_R} & (A \otimes B) \wp (C \otimes D)
\end{array}$$

itemize anpassen?

### 3.3 \*-aut. cats

structure an Mellies, CatSemOfLinLog, anpassen; equivalence Srinivasan and Barr definitions

**Definition 3.10 (Dual object)** Let  $\mathfrak{C}$  be a linearly distributive category and  $A, B \in \text{Obj}(\mathfrak{C})$ , then we call  $B$  left dual (or left linearly adjoint) to  $A$ , if there are morphisms  $\eta : 1 \rightarrow B \wp A$  and  $\varepsilon : A \otimes B \rightarrow \perp$ , called unit and counit resp., such that the following diagrams commute:

$$\begin{array}{ccc}
B & \xlongequal{\text{id}} & B \\
r_{\wp} \uparrow & & \downarrow l_{\otimes}^{-1} \\
B \wp \perp & & 1 \otimes B \\
\text{id} \wp \varepsilon \uparrow & & \downarrow \eta \otimes \text{id} \\
B \wp (A \otimes B) & \xleftarrow{\partial_R} & (B \wp A) \otimes B
\end{array}
\quad
\begin{array}{ccc}
A & \xlongequal{\text{id}} & A \\
l_{\wp} \uparrow & & \downarrow r_{\otimes}^{-1} \\
\perp \wp A & & A \otimes 1 \\
\varepsilon \wp \text{id} \uparrow & & \downarrow \text{id} \otimes \eta \\
B \wp (A \otimes B) & \xleftarrow{\partial_L} & (B \wp A) \otimes B
\end{array}$$

das geht schöner

We write  $(\eta, \varepsilon) : B \dashv A$  and also call  $A$  right dual to  $B$ .

**Lemma 3.11** 1. In an LDC: if  $B \dashv A$  and  $C \dashv A$ , then  $B$  and  $C$  are isomorphic  
2. In a symmetric LDC:  $(\eta, \varepsilon) : B \dashv A \iff (\eta s_{\wp}, s_{\otimes} \varepsilon) : A \dashv B$

proof

Srinivasan Lemma 2.9 (iii)?

definition (iso)mix cat?

**Definition 3.12 (\*-autonomous categories (Srinivasan))** *An LDC  $\mathfrak{C}$  in which for every Object  $A \in \mathfrak{C}$  there exists a left and right dual, resp.  $(\eta^*, \varepsilon^*) : A^* \multimap A$  and  $(\eta, \varepsilon) : A \multimap A^*$ , is called \*-autonomous category.*

is Srinivasan the origin of this version?

**Definition 3.13 (\*-autonomous categories (Barr (95), A))** *A \*-autonomous category is a biclosed monoidal category  $\mathfrak{C}$  with a closed functor  $(-)^* : \mathfrak{C} \rightarrow \mathfrak{C}^{\text{opp}}$ , which is a strong equivalence of categories.*

necessary? meaning of "closed" (functor between C-enriched categories)

**Definition 3.14 (Dualizing object)** *Let  $\mathfrak{C}$  be a biclosed monoidal category. An object  $\perp$  is called a dualizing object if for every  $A \in \text{Obj}(\mathfrak{C})$  the natural map  $A \rightarrow \perp \multimap (A \multimap \perp)$  gotten by transposing  $\text{id} : A \multimap \perp \rightarrow A \multimap \perp$  twice is an isomorphism.*

details (Barr, 99)

**Definition 3.15 (\*-autonomous categories (Barr, B))** *A \*-autonomous category is a biclosed category with a dualizing object.*

**Definition 3.16 (\*-autonomous categories (Barr, C))** *A \*-autonomous category is a monoidal category  $\mathfrak{C}$  equipped with an equivalence  $(-)^* : \mathfrak{C} \rightarrow \mathfrak{C}^{\text{opp}}$  such that there is a natural isomorphism*

main definition, Barr

$$\text{Hom}(A, B^*) \rightarrow \text{Hom}(1, (A \oplus B)^*)$$

**Definition 3.17 (\*-autonomous categories (Barr, D))** *A \*-autonomous category is a closed category  $\mathfrak{C}$  in the sense of Eilenberg and Kelly (1966) together with an equivalence  $(-)^* : \mathfrak{C}^{\text{opp}} \rightarrow \mathfrak{C}$  such that*

nachschauen

$$A \multimap (B \multimap C) \cong (A \multimap B) \multimap C$$

where  $A \multimap B := A^* \multimap B^*$ .

unnötig? C impliziert offensichtlich D, von D nach C sind drei Zeilen.

Äquivalenz von Barrs Definitionen:

**Theorem 3.18** *Barr's definitions are equivalent.*

Folgerung Barr  $A \Rightarrow B$

PROOF: Suppose the category  $\mathfrak{C}$  and the functor  $(-)^{\perp} : \mathfrak{C} \rightarrow \mathfrak{C}^{\text{opp}}$  fulfill the conditions of Definition A □

define useful control sequence

Verlinkungen einfügen

Folgerung Barr  $B \Rightarrow$  Barr  $c$

Folgerung Barr  $C \Rightarrow$  Barr  $A$

what did I need Barr's definition for, again?

Cats as LinLog semantics and LinLog syntax as cat

bibliography

## References

- [Gir87] Jean-Yves Girard. “Linear logic”. In: *Theoretical Computer Science* 50.1 (1987), pp. 1–101. ISSN: 0304-3975. DOI: [https://doi.org/10.1016/0304-3975\(87\)90045-4](https://doi.org/10.1016/0304-3975(87)90045-4). URL: <https://www.sciencedirect.com/science/article/pii/0304397587900454>.

## Todo list

what am I even writing about?	1
feels kinda rambling... combine intuition with syntax definition below	1
(explicit interpretation?! shared pool between people containing Pika and Mew -> you have to get rid of one to use the other?)	2
interpretation of ?wn.	2
interpretation of constants	2
section needs serious reformatting...	2
main source: IntroLinLog	2
formula? term? look up basic terminology	2
permanente Lösung für empty axiom finden	3
proofsymbol :/	3
above remark unnötig?	4
Definition of ? $\Gamma$ korrekt?	4
definition linear equivalence	4
proofsymbol なんて?:(	5
MAYBE to be added: \forall and \exists. Do we want predicate logic?	5
to be continued...	5
how to typeset axiom links? :(	5
sources for basic definitions. (nlab prolly doesn't suffice)	5
extra definition for opposite structure?	6
definition: equivalence of categories	6
source: Brandenburg	7
maybe index Schreibweise?	7
cut triangle diagrams or define •	7
Diagramme verschönern	8
fix associators throughout the document (direction)	8
lengthy, rambling... muss eleganter gehen	8
Diagramme besser alignen	10
obiges Diagramm rumdrehen	11
\parr \id sieht kacke aus...	11
itemize anpassen?	11

■ structure an Mellies, CatSemOfLinLog, anpassen; equivalence Srinivasan and Barr definitions . . . . .	11
■ das geht schöner . . . . .	11
■ proof . . . . .	12
■ Srinivasan Lemma 2.9 (iii)? . . . . .	12
■ definition (iso)mix cat? . . . . .	12
■ is Srinivasan the origin of this version? . . . . .	12
■ necessary? meaning of "closed" (functor between C-enriched categories) . . . . .	12
■ details (Barr, 99) . . . . .	12
■ main definition, Barr . . . . .	12
■ nachschauen . . . . .	12
■ unnötig? C impliziert offensichtlich D, von D nach C sind drei Zeilen. . . . .	12
■ Äquivalenz von Barrs Definitionen: . . . . .	12
■ Folgerung Barr A $\Rightarrow$ Barr B . . . . .	12
■ define useful control sequence . . . . .	12
■ Verlinkungen einfügen . . . . .	12
■ Folgerung Barr B $\Rightarrow$ Barr c . . . . .	13
■ Folgerung Barr C $\Rightarrow$ Barr A . . . . .	13
■ what did I need Barr's definition for, again? . . . . .	13
■ Cats as LinLog semantics and LinLog syntax as cat . . . . .	13
■ bibliography . . . . .	13