

LinLog und LinDisCats

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1 Introduction

what am I even writing about?

2 Categorical Preliminaries

2.1 Categories

sources for basic definitions. (nlab proly doesn't suffice)

Definition 2.1 (Category) A Category \mathfrak{C} consists of the following data:

- A class $\text{Obj}(\mathfrak{C})$ of objects.
- For every pair of objects $A, B \in \text{Obj}(\mathfrak{C})$ there is a class $\text{Hom}(A, B)$ of morphisms $f : A \rightarrow B$ from A to B .
- Morphisms compose: For $f \in \text{Hom}(A, B)$ and $g \in \text{Hom}(B, C)$ there is a morphism $g \circ f \in \text{Hom}(A, C)$. That composition is associative:

$$h \circ (g \circ f) = (h \circ g) \circ f$$

We will write gf for $g \circ f$ when appropriate.

- For every object A there is an identity morphism $\text{id}_A \in \text{Hom}(A, A)$:

$$f \circ \text{id}_A = f, \quad \text{id}_B \circ f = f$$

for $f \in \text{Hom}(A, B)$

If the hom-classes are sets, we call the category locally small. If the object class is a set, we call the category small. Otherwise, we call the category large.

If we have a category \mathfrak{C} , we call its opposite category $\mathfrak{C}^{\text{opp}}$ the category with the following structure:

- $\text{Obj}(\mathfrak{C}^{\text{opp}}) = \text{Obj}(\mathfrak{C})$
- $\forall A, B \in \text{Obj}(\mathfrak{C}) : \text{Hom}_{\mathfrak{C}^{\text{opp}}}(A, B) = \text{Hom}_{\mathfrak{C}}(B, A)$

Definition 2.2 (Functor) A (covariant) functor $F : \mathfrak{C} \rightarrow \mathfrak{D}$ between two categories \mathfrak{C} and \mathfrak{D} is a function mapping each object $A \in \text{Obj}(\mathfrak{C})$ to an object $F(A) \in \text{Obj}(\mathfrak{D})$ and each morphism $f \in \text{Hom}(A, B)$ to a morphism $F(f) \in \text{Hom}(F(A), F(B))$ such that identity and composition are preserved:

$$F(\text{id}_A) = \text{id}_{F(A)}, \quad F(g \circ f) = F(g) \circ F(f)$$

A functor $F : \mathfrak{C}^{\text{opp}} \rightarrow \mathfrak{D}$ is called *contravariant* on \mathfrak{C} .

We will drop the parentheses when appropriate.

Definition 2.3 (Natural transformation) A natural transformation $\tau : F \rightarrow G$ between two functors $F, G : \mathfrak{C} \rightarrow \mathfrak{D}$ is family of morphisms in \mathfrak{D} :

$$\tau = \{\tau_A : FA \rightarrow GA \mid A \in \text{Obj}(\mathfrak{C})\}$$

such that $\tau_B F(f) = G(f) \tau_A$ for all $f : A \rightarrow B \in \text{Morph}(\mathfrak{C})$, i.e. the following diagram commutes:

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \downarrow \tau_A & & \downarrow \tau_B \\ GA & \xrightarrow{Gf} & GB \end{array}$$

If τ_A is an isomorphism for all $A \in \text{Obj}(\mathfrak{C})$, we call τ a *natural isomorphism*. We will often represent a natural transformation by a single one of its members and also drop the index when appropriate.

definition: equivalence of categories

Definition 2.4 (Adjunction) Let \mathfrak{C} and \mathfrak{D} be categories. We call $F : \mathfrak{C} \rightarrow \mathfrak{D}$ the left adjoint of $G : \mathfrak{D} \rightarrow \mathfrak{C}$ and G the right adjoint of F if there is an isomorphism that is natural in $A \in \mathfrak{C}$ and $B \in \mathfrak{D}$:

$$\forall A \in \mathfrak{C}, \forall B \in \mathfrak{D} : \text{Hom}_{\mathfrak{D}}(FA, B) \cong \text{Hom}_{\mathfrak{C}}(A, GB)$$

Equivalently, we call F the left adjoint of G if there are natural transformations

$$\begin{aligned} \eta : \text{id}_{\mathfrak{C}} &\rightarrow G \circ F \\ \varepsilon : F \circ G &\rightarrow \text{id}_{\mathfrak{D}} \end{aligned}$$

fulfilling the following condition:

$$\begin{aligned} \text{id}_{FA} &= \varepsilon_{FA} \circ F(\eta_A) \\ \text{id}_{GB} &= G(\varepsilon_B) \circ \eta_{GB} \end{aligned}$$

We write $F \dashv G$.

Definition 2.5 (Monoidal category) A monoidal category $(\mathfrak{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ is a category \mathfrak{C} with the following additional data:

source:
Brandenburg

- a functor $\otimes : \mathfrak{C} \times \mathfrak{C} \rightarrow \mathfrak{C}$, called monoidal or tensor product;
- an object $\mathbb{1} \in \text{Obj}(\mathfrak{C})$, called unit object;
- a natural isomorphism $\alpha : A \otimes (B \otimes C) \rightarrow (A \otimes B) \otimes C$, called associator;
- a natural isomorphism $\lambda : \mathbb{1} \otimes A \rightarrow A$, called left unitor;
- a natural isomorphism $\rho : A \otimes \mathbb{1} \rightarrow A$, called right unitor;

such that the following diagrams commute:

- the pentagon diagram:

$$\begin{array}{ccccc}
 & & A \otimes (B \otimes (C \otimes D)) & & \\
 & \swarrow \text{id} \otimes \alpha & & \searrow \alpha & \\
 A \otimes ((B \otimes C) \otimes D) & & & & (A \otimes B) \otimes (C \otimes D) \\
 \downarrow \alpha & & & & \downarrow \alpha \\
 (A \otimes (B \otimes C)) \otimes D & \xrightarrow{\alpha \otimes \text{id}} & & & ((A \otimes B) \otimes C) \otimes D
 \end{array}$$

- the unit diagram:

$$\begin{array}{ccc}
 (A \otimes \mathbb{1}) \otimes B & \xrightarrow{\alpha} & A \otimes (\mathbb{1} \otimes B) \\
 \searrow \rho \otimes \text{id} & & \swarrow \text{id} \otimes \lambda \\
 & A \otimes B &
 \end{array}$$

A symmetric monoidal category has an additional natural isomorphism:

$$s_{A,B} : A \otimes B \rightarrow B \otimes A$$

satisfying the following conditions:

- the hexagon law:

$$\begin{array}{ccccc}
 A \otimes (B \otimes C) & \xrightarrow{\alpha} & (A \otimes B) \otimes C & \xrightarrow{s_{A \otimes B, C}} & C \otimes (A \otimes B) \\
 \downarrow \text{id} \otimes s_{B, C} & & & & \downarrow \alpha \\
 A \otimes (C \otimes B) & \xrightarrow{\alpha} & (A \otimes C) \otimes B & \xrightarrow{s_{A, C \otimes B}} & (C \otimes A) \otimes B
 \end{array}$$

- the inverse law: $s_{A,B}^{-1} = s_{B,A}$
- the unit law: $\lambda = \rho s$

$$\begin{array}{ccc}
 \mathbb{1} \otimes A & \xrightarrow{s_{\mathbb{1}, A}} & A \otimes \mathbb{1} \\
 \searrow \lambda & & \swarrow \rho \\
 & A &
 \end{array}$$

Definition 2.6 (Closed monoidal category) A monoidal category \mathfrak{C} is called right closed, resp. left closed, iff there is a functor $- \multimap - : \mathfrak{C}^{\text{opp}} \times \mathfrak{C} \rightarrow \mathfrak{C}$, resp. $- \multimap - : \mathfrak{C} \times \mathfrak{C}^{\text{opp}} \rightarrow \mathfrak{C}$, such that there is a natural isomorphism $\text{Hom}(A \otimes B, C) \cong \text{Hom}(B, A \multimap C)$, resp. $\text{Hom}(A \otimes B, C) \cong \text{Hom}(A, C \multimap B)$, i.e. iff the functor $- \otimes B$, resp. $A \otimes -$, has a right adjoint. A monoidal \mathfrak{C} is called biclosed iff it is both left and right closed.

lengthy, rambling... maybe cut into three definitions?

definitely mixed up the right and left homs there

If \mathfrak{C} is symmetric, these functors coincide and we call the category closed.

The functors $- \multimap - : \mathfrak{C}^{\text{opp}} \times \mathfrak{C} \rightarrow \mathfrak{C}$ and $- \multimap - : \mathfrak{C} \times \mathfrak{C}^{\text{opp}} \rightarrow \mathfrak{C}$ are called internal hom functors.

2.2 LinDisCats

Definition 2.7 (Linearly distributive category) A linearly distributive category $(\mathfrak{C}, \otimes, 1, \mathfrak{A}, \perp)$ is a category \mathfrak{C} consisting of:

- a monoidal category $(\mathfrak{C}, \otimes, 1, \alpha_{\otimes}, \lambda_{\otimes}, \rho_{\otimes})$, with \otimes called "tensor";
- a monoidal category $(\mathfrak{C}, \mathfrak{A}, \perp, \alpha_{\mathfrak{A}}, \lambda_{\mathfrak{A}}, \rho_{\mathfrak{A}})$, with \mathfrak{A} called "par";
- two natural transformations called left and right linear distributors respectively:

$$\begin{aligned} \partial_L : A \otimes (B \mathfrak{A} C) &\rightarrow (A \otimes B) \mathfrak{A} C \\ \partial_R : (A \mathfrak{A} B) \otimes C &\rightarrow A \mathfrak{A} (B \otimes C) \end{aligned}$$

satisfying the following coherence conditions:

- coherence between the distributors and unitors:

$$\begin{array}{ccc} 1 \otimes (A \mathfrak{A} B) & \xrightarrow{\partial_L} & (1 \otimes A) \mathfrak{A} B \\ & \searrow \lambda_{\otimes} \quad \swarrow \lambda_{\otimes} \mathfrak{A} \text{id} & \\ & A \mathfrak{A} B & \\ (A \mathfrak{A} B) \otimes 1 & \xrightarrow{\partial_R} & A \mathfrak{A} (B \otimes 1) \\ & \searrow \rho_{\otimes} \quad \swarrow \text{id} \otimes \rho_{\otimes} & \\ & A \mathfrak{A} B & \\ A \otimes (B \mathfrak{A} \perp) & \xrightarrow{\partial_L} & (A \otimes B) \mathfrak{A} \perp \\ & \searrow \text{id} \otimes \rho_{\mathfrak{A}} \quad \swarrow \rho_{\mathfrak{A}} & \\ & A \otimes B & \\ (\perp \mathfrak{A} A) \otimes B & \xrightarrow{\partial_R} & \perp \mathfrak{A} (A \otimes B) \\ & \searrow \lambda_{\mathfrak{A}} \otimes \text{id} \quad \swarrow \lambda_{\mathfrak{A}} & \\ & A \otimes B & \end{array}$$

- coherence between distributors and associators:

$$\begin{array}{ccc} & (A \otimes B) \otimes (C \mathfrak{A} D) & \\ & \swarrow \alpha_{\otimes} \quad \searrow \partial_L & \\ A \otimes (B \otimes (C \mathfrak{A} D)) & & ((A \otimes B) \otimes C) \mathfrak{A} D \\ \downarrow \text{id} \otimes \partial_L & & \downarrow \alpha_{\otimes} \mathfrak{A} \text{id} \\ A \otimes ((B \otimes C) \mathfrak{A} D) & \xrightarrow{\partial_L} & (A \otimes (B \otimes C)) \mathfrak{A} D \end{array}$$

$$\begin{array}{ccc}
& A \otimes (B \wp (C \wp D)) & \\
\swarrow \text{id} \otimes \alpha_{\wp} & & \searrow \partial_L \\
A \otimes ((B \wp C) \wp D) & & (A \otimes B) \wp (C \wp D) \\
\downarrow \partial_L & & \downarrow \alpha_{\wp} \\
(A \otimes (B \wp C)) \wp D & \xrightarrow{\partial_L \wp \text{id}} & ((A \otimes B) \wp C) \wp D
\end{array}$$

$$\begin{array}{ccc}
& (A \wp B) \otimes (C \otimes D) & \\
\swarrow \alpha_{\otimes} & & \searrow \partial_R \\
((A \wp B) \otimes C) \otimes D & & A \wp (B \otimes (C \otimes D)) \\
\downarrow \partial_R \otimes \text{id} & & \downarrow \text{id} \wp \alpha_{\otimes} \\
(A \wp (B \otimes C)) \otimes D & \xrightarrow{\partial_R} & A \wp ((B \otimes C) \otimes D)
\end{array}$$

$$\begin{array}{ccc}
& ((A \wp B) \wp C) \otimes D & \\
\swarrow \alpha_{\wp} \otimes \text{id} & & \searrow \partial_R \\
(A \wp (B \wp C)) \otimes D & & (A \wp B) \wp (C \otimes D) \\
\downarrow \partial_R & & \downarrow \alpha_{\wp} \\
A \wp ((B \wp C) \otimes D) & \xrightarrow{\text{id} \wp \partial_R} & A \wp (B \wp (C \otimes D))
\end{array}$$

- coherence between the distributors:

$$\begin{array}{ccc}
& (A \wp B) \otimes (C \wp D) & \\
\swarrow \partial_R & & \searrow \partial_L \\
((A \wp B) \otimes C) \wp D & & A \wp (B \otimes (C \wp D)) \\
\downarrow \partial_R \wp \text{id} & & \downarrow \text{id} \wp \partial_L \\
(A \wp (B \otimes C)) \wp D & \xleftarrow{\alpha_{\wp}} & A \wp ((B \otimes C) \wp D)
\end{array}$$

$$\begin{array}{ccc}
& A \otimes ((B \wp C) \otimes D) & \\
\swarrow \alpha_{\otimes} & & \searrow \text{id} \otimes \partial_R \\
(A \otimes (B \wp C)) \otimes D & & A \otimes (B \wp (C \otimes D)) \\
\downarrow \partial_L \otimes \text{id} & & \downarrow \partial_L \\
((A \otimes B) \wp C) \otimes D & \xrightarrow{\partial_R} & (A \otimes B) \wp (C \otimes D)
\end{array}$$

2.3 *-aut. cats

Definition 2.8 (Dual object) Let \mathfrak{C} be a linearly distributive category and $A, A^* \in \text{Obj}(\mathfrak{C})$, then we call A^* left dual (or left linearly adjoint) to A , if there are morphisms $\tau : 1 \rightarrow A^* \wp A$ and $\gamma : A \otimes A^* \rightarrow \perp$, called unit and counit resp., such that the following diagrams commute:

$$\begin{array}{ccc}
A^* & \xlongequal{\text{id}} & A^* \\
\rho_{\mathfrak{A}} \uparrow & & \downarrow \lambda_{\otimes}^{-1} \\
A^* \mathfrak{A} \perp & & 1 \otimes A^* \\
\text{id} \mathfrak{A} \gamma \uparrow & & \downarrow \tau \otimes \text{id} \\
A^* \mathfrak{A} (A \otimes A^*) & \xleftarrow{\partial_R} & (A^* \mathfrak{A} A) \otimes A^*
\end{array}
\quad
\begin{array}{ccc}
A & \xlongequal{\text{id}} & A \\
\lambda_{\mathfrak{A}} \uparrow & & \downarrow \rho_{\otimes}^{-1} \\
\perp \mathfrak{A} A & & A \otimes 1 \\
\gamma \mathfrak{A} \text{id} \uparrow & & \downarrow \text{id} \otimes \tau \\
(A^* \mathfrak{A} A) \otimes A^* & \xleftarrow{\partial_L} & A^* \mathfrak{A} (A \otimes A^*)
\end{array}$$

We write $(\tau, \gamma) : A^* \dashv A$ and also call A right dual to A^* .

Lemma 2.9 1. In an LDC: if $A^* \dashv A$ and $A' \dashv A$, then A^* and A' are isomorphic. We will from now on only talk about the dual object, when equality up to isomorphism is sufficient.

2. In a symmetric LDC: $(\tau, \gamma) : A^* \dashv A \iff (\tau s_{\mathfrak{A}}, s_{\otimes} \gamma) : A \dashv A^*$

proof

Definition 2.10 (*-autonomous categories [Sri23]) An LDC \mathfrak{C} in which for every Object $A \in \mathfrak{C}$ there exists a left and right dual, resp. $(\tau^*, \gamma^*) : A^* \dashv A$ and $(\tau, \gamma) : A \dashv A^*$, is called *-autonomous category.

cut Barr's following stuff as it doesn't lead to anything right now?

Definition 2.11 (*-autonomous categories ([Bar95], A)) A *-autonomous category is a biclosed monoidal category \mathfrak{C} with a closed functor $(-)^* : \mathfrak{C} \rightarrow \mathfrak{C}^{\text{opp}}$, which is a strong equivalence of categories.

definition closed functor, strong equivalence. (strong closed functor?)

Definition 2.12 (Dualizing object) Let \mathfrak{C} be a biclosed monoidal category. An object \perp is called a dualizing object if for every $A \in \text{Obj}(\mathfrak{C})$ the natural map $A \rightarrow \perp \multimap (A \multimap \perp)$ gotten by transposing $\text{id} : A \multimap \perp \rightarrow A \multimap \perp$ twice is an isomorphism.

Definition 2.13 (*-autonomous categories (Barr, B)) A *-autonomous category is a biclosed category with a dualizing object.

Definition 2.14 (*-autonomous categories (Barr, C)) A *-autonomous category is a monoidal category \mathfrak{C} equipped with an equivalence $(-)^* : \mathfrak{C} \rightarrow \mathfrak{C}^{\text{opp}}$ such that there is a natural isomorphism

$$\text{Hom}(A, B^*) \rightarrow \text{Hom}(1, (A \oplus B)^*)$$

details
(Barr, 99)

3 LinLog Preliminaries

3.1 Intuition

feels kinda rambling... combine intuition with syntax definition below

Classical (and intuitionistic) logic deals with the propagation of stable truth values. If one has a true sentence A and an implication $A \Rightarrow B$, then B follows while A remains true. However, real-life implications are often causal and modify their premises. They cannot therefore be iterated arbitrarily. For example if A describes the ownership of 1€ and B owning a chocolate bar, an implication $A \multimap B$ (to be formally introduced later) would describe the process of buying such a chocolate bar for 1€, losing the 1€ in the process.

While such dealings with resources can of course be modeled in classical logic, it is easier done in the resource-sensitive *linear logic*, first described by Girard in 1987 [Gir87].

Here, we have two conjunctions simultaneously \otimes ("times" or "tensor") and $\&$ ("with"), which describe the availability of resources:

Suppose C is the ownership of a cookie and it costs also 1€ (i.e. we have $A \multimap C$). Then $B \otimes C$ states that one owns both a chocolate bar and a cookie. The implication $A \multimap B \otimes C$ is not possible, as it would mean that you are buying both, cookie and chocolate bar at the same time, for just 1€ total. However, from $A \multimap B$ and $A \multimap C$ we get $A \otimes A \multimap B \otimes C$, i.e. the process of buying both for 2€.

On the other hand, $B \& C$ states that one has a choice between either B or C (imagine a token). From the implications $A \multimap B$ and $A \multimap C$ we get the implication $A \multimap B \& C$, i.e. the process of buying a token to be exchanged for a chocolate bar or a cookie at a later time (with the choice lying with oneself). While this may seem like a disjunction, both implications $B \& C \multimap B$ and $B \& C \multimap C$ (exchanging the token for either product) are provable from $B \& C$, although not simultaneously.

Dually, we have two disjunctions \wp ("par") and \oplus ("plus"):

Suppose now that B and C are the ownership of a figurine of Pikachu or Mew respectively. Then $B \oplus C$ may be the ownership of a Kinder Egg containing either figurine. This means when buying that egg ($A \multimap B \oplus C$) we do not know which one we will get.

Our second disjunction, dual to \otimes , can be understood by linear implication and the linear negation (denoted as $(\cdot)^\perp$): Under the interpretation of ownership the linear negation is not interpreted as the absence of ownership but as negative ownership, i.e. debt. That means the negation of owning 1€, A , is owing someone 1€, A^\perp . With the par operator we can now write the linear implication $A \multimap B$ symmetrically as $A^\perp \wp B$.

(explicit interpretation?! shared pool between people containing Pika and Mew -> you have to get rid of one to use the other?)

In order to regain our stable truths known from classical logic, we need to employ two unitary connectives ! ("of course" or "bang") and ? ("why not"). The bang operator informs us that there is an infinite amount of a resource: The statement $!A$ translates into the ownership of an amount of money that is large enough for us to ignore resource sensitivity. Imagine for example a billionaire buying a Pokemon figurine instead of a social media site: His amount of money will not be noticeably smaller after buying the figurine.

We can informally say $!A = (1 \& A) \otimes (1 \& A) \otimes \dots$ and therefore view classical and intuitionistic logic as some sort of limit of linear logic, just as classical mechanics is a limit of quantum mechanics and the theory of relativity.

interpretation of $?A$ and constants.

3.2 Syntax

Quellen, Quellen, Quellen!!!

main source: IntroLinLog

Definition 3.1 (Formula) *Formulas are defined inductively from atomic formulas and four constants 1 , \perp , \top and 0 :*

- *Every constant is a formula.*
- *Every atomic formula is a formula.*
- *If A is a formula, so are A^\perp , $!A$ and $?A$.*
- *If A and B are formulas, so are $A \otimes B$, $A \wp B$, $A \& B$ and $A \oplus B$.*

Let Γ , Δ etc. be arbitrary, finite lists of formulas (e.g.: $\Gamma = p_1, \dots, p_n$), and A , B etc. formulas.

As we will later consider fragments without negation, we shall define linear logic with a two-sided calculus.

Definition 3.2 (Proof (tree)) *A proof tree (often just proof) is a rooted tree with sequents of the form $\Gamma \vdash \Delta$ as edges and the following sequent rules as vertices (except for the root).*

Structural rules We only have the exchange rule as a structural, missing the (general) weakening and contraction rules known from classical logic:

$$\frac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, B, A, \Gamma' \vdash \Delta} \text{ex.L} \quad \frac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, B, A, \Delta'} \text{ex.R}$$

Identity rules We have the identity and negation rules:

$$\frac{}{A \vdash A} \text{id} \quad \frac{\Gamma_1 \vdash \Delta_1, A, \Delta'_1 \quad \Gamma_2, A, \Gamma'_2 \vdash \Delta_2}{\Gamma_2, \Gamma_1, \Gamma'_2 \vdash \Delta_1, \Delta_2, \Delta'_1} \text{Cut}$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, A^\perp \vdash \Delta} \text{neg.L} \quad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A^\perp, \Delta} \text{neg.R}$$

As already mentioned, the classical conjunction \wedge and disjunction \vee as well as their respective units split into two respectively. These can be classified as multiplicative and additive connectives.

add ex-
ample
with more
graph-like
proof tree

Multiplicatives The calculus rules for the *multiplicative* conjunction \otimes , disjunction \wp ("par") and their units 1 and \perp ("bottom") are as follows:

$$\begin{array}{c}
\frac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, A \otimes B, \Gamma' \vdash \Delta} \otimes_L \quad \frac{\Gamma \vdash \Delta, A \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash \Delta, A \otimes B, \Delta'} \otimes_R \\
\frac{\Gamma, A \vdash \Delta \quad B, \Gamma' \vdash \Delta'}{\Gamma, A \wp B, \Gamma' \vdash \Delta, \Delta'} \wp_L \quad \frac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, A \wp B, \Delta'} \wp_R \\
\frac{\Gamma \vdash \Delta}{\Gamma, 1 \vdash \Delta} 1_L \quad \frac{}{\vdash 1} 1_R \\
\frac{}{\perp \vdash} \perp_L \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \perp, \Delta} \perp_R
\end{array}$$

Remark 3.3 The rules \otimes_L and \wp_R imply, that the commas are to be read as \otimes on the left-hand side and as \wp on the right-hand side. That means $A, B \vdash C, D$ is provable iff $A \otimes B \vdash C \wp D$ is provable.

PROOF: We only have to show that $A, B \vdash C, D$ follows from $A \otimes B \vdash C \wp D$ as the other direction is just our introduction rule:

$$\frac{\frac{\frac{}{A \vdash A} \text{id} \quad \frac{}{B \vdash B} \text{id}}{A, B \vdash A \otimes B} \otimes_R \quad \frac{A \otimes B \vdash C \wp D}{A, B \vdash C \wp D} \text{Cut} \quad \frac{\frac{}{C \vdash C} \quad \frac{}{D \vdash D}}{C \wp D \vdash C, D} \wp_L}{A, B \vdash C, D} \quad \square$$

proofsymbol
:/

Additives The calculus rules for the *additive* conjunction $\&$ and disjunction \oplus ("plus") are as follows:

$$\begin{array}{c}
\frac{\Gamma, A \vdash \Delta}{\Gamma, A \& B \vdash \Delta} \&_{L1} \quad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \& B \vdash \Delta} \&_{L2} \\
\frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \& B} \&_R \\
\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta} \oplus_L \\
\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \oplus B, \Delta} \oplus_{R1} \quad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \oplus B, \Delta} \oplus_{R2} \\
\frac{}{\Gamma, 0 \vdash \Delta} 0_L \quad \frac{}{\Gamma \vdash \top, \Delta} \top_R
\end{array}$$

Notice the difference between $\&_R$ and \otimes_R (and dually between \oplus_L and \wp_L): while for \otimes_R the contexts Γ etc. are arbitrary and get combined in the conclusion, $\&_R$ requires the contexts to be equal. In classical logic these rules can be shown to be equivalent using its additional structural rules.

Furthermore, note that the unit introduction rules also introduce arbitrary contexts, and that each unit only has one introduction rule as opposed to the multiplicative units.

Remark 3.4 Similarly to the multiplicative connectives above, the additive connectives have a invertibility statement: $\Gamma \vdash A \& B$ is provable iff $\Gamma \vdash A$ and $\Gamma \vdash B$ are provable. Dually, $A \oplus B \vdash \Delta$ iff $A \vdash \Delta$ and $B \vdash \Delta$.

remark
unnötig?

PROOF: As with the multiplicative statement, one direction suffices:

$$\frac{\Gamma \vdash A \& B \quad \frac{\overline{A \vdash A}}{A \& B \vdash A}}{\Gamma \vdash A}$$

$\Gamma \vdash B$ follows the same way. □

Definition 3.5 We call two formulas A and B (linearly) equivalent iff $A \vdash B$ and $B \vdash A$ are provable, and write $A \equiv B$.

Remark 3.6 The names "multiplicative" and "additive" are motivated by the following relations:

$$\begin{aligned} A \otimes (B \oplus C) &\equiv (A \otimes B) \oplus (A \otimes C) \\ A \wp (B \& C) &\equiv (A \wp B) \& (A \wp C) \end{aligned}$$

The similarities to basic arithmetic don't end there:

$$A \otimes 0 \equiv 0, \quad A \wp \top \equiv \top$$

PROOF: We only show $A \otimes (B \oplus C) \equiv (A \otimes B) \oplus (A \otimes C)$.

- $A \otimes (B \oplus C) \vdash (A \otimes B) \oplus (A \otimes C)$:

$$\frac{\frac{\frac{\overline{A \vdash A}}{A, B \vdash A \otimes B} \quad \frac{\overline{B \vdash B}}{A, B \vdash A \otimes B}}{A, B \vdash (A \otimes B) \oplus (A \otimes C)} \quad \frac{\frac{\overline{A \vdash A}}{A, C \vdash A \otimes C} \quad \frac{\overline{C \vdash C}}{A, C \vdash A \otimes C}}{A, C \vdash (A \otimes B) \oplus (A \otimes C)}}{\frac{A, B \oplus C \vdash (A \otimes B) \oplus (A \otimes C)}{A \otimes (B \oplus C) \vdash (A \otimes B) \oplus (A \otimes C)}}$$

- $(A \otimes B) \oplus (A \otimes C) \vdash A \otimes (B \oplus C)$:

$$\frac{\frac{\frac{\overline{A \vdash A}}{A, B \vdash A \otimes (B \oplus C)} \quad \frac{\overline{B \vdash B}}{B \vdash B \oplus C}}{A \otimes B \vdash A \otimes (B \oplus C)} \quad \frac{\frac{\overline{A \vdash A}}{A, C \vdash A \otimes (B \oplus C)} \quad \frac{\overline{C \vdash C}}{C \vdash B \oplus C}}{A \otimes C \vdash A \otimes (B \oplus C)}}{(A \otimes B) \oplus (A \otimes C) \vdash A \otimes (B \oplus C)}$$

The proof for $A \wp (B \& C) \equiv (A \wp B) \& (A \wp C)$ is similar, the proof for the equations of the constants a trivial application of their introduction rules. □

Theorem 3.7 We have the following equivalencies for the linear negation:

- For the constants:

$$\begin{aligned} 1^\perp &\equiv \perp & \perp^\perp &\equiv 1 \\ \top^\perp &\equiv 0 & 0^\perp &\equiv \top \end{aligned}$$

- The negation is involutory: $(A^\perp)^\perp \equiv A$
- The De Morgan equations hold:

$$\begin{aligned} (A \otimes B)^\perp &\equiv A^\perp \wp B^\perp & (A \wp B)^\perp &\equiv A^\perp \otimes B^\perp \\ (A \& B)^\perp &\equiv A^\perp \oplus B^\perp & (A \oplus B)^\perp &\equiv A^\perp \& B^\perp \end{aligned}$$

PROOF: We shall only prove parts.

- $1^\perp \vdash \perp$:

$$\frac{\frac{\overline{\vdash 1} \quad 1_R}{\vdash 1, \perp} \quad \perp_R}{1^\perp \vdash \perp} \text{neg.L}$$

- $(A^\perp)^\perp \vdash A$:

$$\frac{\frac{\overline{A \vdash A}}{\vdash A^\perp, A}}{(A^\perp)^\perp \vdash A}$$

- $(A \otimes B)^\perp \vdash A^\perp \wp B^\perp$:

$$\frac{\frac{\frac{\overline{A \vdash A}}{\vdash A^\perp, A} \quad \frac{\overline{B \vdash B}}{\vdash B^\perp, B}}{\vdash A \otimes B, A^\perp, B^\perp}}{(A \otimes B)^\perp \vdash A^\perp, B^\perp} \quad \frac{}{(A \otimes B)^\perp \vdash A^\perp \wp B^\perp}$$

The rest is shown similarly. □

With the negation our calculus rules become quite redundant and we can restrict them on the rules for the right-hand side by translating any two-sided sequent $A_1, \dots, A_n \vdash B_1, \dots, B_n$ into a one-sided sequent $\vdash A_1^\perp, \dots, A_n^\perp, B_1, \dots, B_n$. However, these redundancies are necessary when dealing with a negation-free fragment of LL as we will later do.

Definition 3.8 (Syntactical implication and equivalence) *We define linear implication with the par-operator:*

$$A \multimap B := A^\perp \wp B$$

We further define (syntactical) linear equivalence:

$$A \circ\!\!\multimap B := (A \multimap B) \& (B \multimap A)$$

It is easily seen that $\vdash A \multimap B$ iff $A \vdash B$ and that $\vdash A \circ\!\!\multimap B$ iff $A \equiv B$.

Exponentials The modality connectives reintroduce stable truths and with them the the weakening and contraction rules known from classical logic:

$$\begin{array}{c}
\frac{\Gamma \vdash \Delta}{\Gamma, !A \vdash \Delta} !_W \text{ (weakening)} \quad \frac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta} !_D \text{ (dereliction)} \\
\frac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta} !_C \text{ (contraction)} \quad \frac{! \Gamma \vdash ? \Delta, A}{! \Gamma \vdash ? \Delta, !A} !_R \\
\frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, ?A} ?_W \text{ (weakening)} \quad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, ?A} ?_D \text{ (dereliction)} \\
\frac{\Gamma \vdash \Delta, ?A, ?A}{\Gamma \vdash \Delta, ?A} ?_C \text{ (contraction)} \quad \frac{! \Gamma, A \vdash ? \Delta}{! \Gamma, ?A \vdash ? \Delta} ?_L
\end{array}$$

$$(!A)^\perp \equiv ?(A^\perp) \quad (?A)^\perp \equiv !(A^\perp)$$

Here, the context $! \Gamma$ is given by applying the $!$ -modality to every formula of Γ , i.e. $! \Gamma = !q, !p, \dots$ for $\Gamma = q, p, \dots$; same for $? \Delta$.

Remark 3.9 These modalities are called exponentials because of the following relation:

$$!(A \& B) \equiv !A \otimes !B, \quad ?(A \oplus B) \equiv ?A \wp ?B$$

PROOF: We will only prove the second equivalence.

- $?A \wp ?B \vdash ?(A \oplus B)$:

$$\frac{\frac{\frac{\overline{A \vdash A}}{A \vdash A \oplus B} \oplus_R}{A \vdash ?(A \oplus B)} ?_D \quad \frac{\frac{\frac{\overline{B \vdash B}}{B \vdash A \oplus B} \oplus_R}{B \vdash, ?(A \oplus B)} ?_D}{?A \vdash ?(A \oplus B)} ?_L \quad \frac{\frac{\frac{\overline{B \vdash B}}{B \vdash ?B} \oplus_R}{B \vdash ?B, ?A} ?_W}{B \vdash ?B, ?A} ?_W}{\frac{A \oplus B \vdash ?A, ?B}{?(A \oplus B) \vdash ?A, ?B} ?_L} \oplus_L \quad \frac{?(A \oplus B) \vdash ?A, ?B}{?(A \oplus B) \vdash ?A \wp ?B} \wp_R}{\frac{?(A) \wp (B) \vdash ?(A \oplus B), ?(A \oplus B)}{?A \wp ?B \vdash ?(A \oplus B)} ?_D} \wp$$

- $?(A \oplus B) \vdash ?A \wp ?B$:

$$\frac{\frac{\frac{\overline{A \vdash A}}{A \vdash ?A} ?_D}{A \vdash ?A, ?B} ?_W \quad \frac{\frac{\frac{\overline{B \vdash B}}{B \vdash ?B} ?_D}{B \vdash ?B, ?A} ?_W}{B \vdash ?B, ?A} ?_W}{\frac{A \oplus B \vdash ?A, ?B}{?(A \oplus B) \vdash ?A, ?B} ?_L} \oplus_L \quad \frac{?(A \oplus B) \vdash ?A, ?B}{?(A \oplus B) \vdash ?A \wp ?B} \wp_R$$

$!(A \& B) \equiv !A \otimes !B$ is acquired the same way. □

3.3 Cut elimination

add "spoiler" to categoru theoru

As is the case with most, if not all, logical system, the Cut rule of our calculus is admissible. That means for any proof π of a sequent $\Gamma \vdash \Delta$ we can find a *cut-free* proof π' of that same sequent. We will prove this for the multiplicative fragment by simplifying the proof due to Braüner [Bra96, Appendix B]. Baüner's proof includes the exponentials which require additional care because of the contraction rule. The general strategy consists of either, moving a cut upwards in a proof or replacing it with cuts of simpler formulas. To do this, we need two definitions:

Definition 3.10 (Degree) *The degree $\partial(A)$ of a formula A is defined inductively:*

- $\partial(A) = 1$, for A atomic or constant
- $\partial(A^\perp) = \partial(A)$
- $\partial(A \otimes B) = \partial(A \wp B) = \max\{\partial(A), \partial(B)\} + 1$
- $\partial(!A) = \partial(?A) = \partial(A) + 1$

The degree of a cut is the degree of the cut formula. The cut formula is called principal formula. The degree $\partial(\pi)$ of a proof π is the supremum of the degrees of the cuts in the proof.

Definition 3.11 (Height) *The height $h(\pi)$ of a proof π is just its height as a rooted tree:*

- $h(\pi) = 1$ when π is an instance of a rule with zero premises, i.e. an axiom.
- $h(\pi) = h(\tau) + 1$, for $\pi = \frac{\tau}{\Gamma \vdash \Delta}$
- $h(\pi) = \max\{h(\tau), h(\tau')\} + 1$, for $\pi = \frac{\tau \quad \tau'}{\Gamma \vdash \Delta}$

With these, we can now eliminate cuts in a proof:

Lemma 3.12 *Let τ be a proof consisting of two subproofs τ_1 and τ_2 of strictly lower degree, i.e.:*

$$\tau = \frac{\frac{\tau_1}{\vdots} \quad \frac{\tau_2}{\vdots}}{\frac{\Gamma_1 \vdash \Delta_1, A, \Delta'_1 \quad \Gamma_2, A, \Gamma'_2 \vdash \Delta_2}{\Gamma_2, \Gamma_1, \Gamma'_2 \vdash \Delta_1, \Delta_2, \Delta'_1}}$$

Then, we can find a proof τ' of that same sequent $\Gamma_2, \Gamma_1, \Gamma'_2 \vdash \Delta_1, \Delta_2, \Delta'_1$ with $\partial(\tau') < \partial(\tau)$.

PROOF: Induction on $h(\tau_1) + h(\tau_2)$. The immediate subproofs of τ_i are denoted by π_j . We perform transformations case by case:

- If one of the the subproofs is an axiom, we just take the remaining one, resulting in a proof with lower degree:

$$\begin{array}{ccc}
\frac{\frac{A \vdash A}{\vdots} \quad \frac{A, \Gamma \vdash \Delta}{\vdots}}{A, \Gamma \vdash \Delta} & \rightsquigarrow & \frac{A, \Gamma \vdash \Delta}{\vdots} \\
\frac{\frac{\Gamma \vdash \Delta, A}{\vdots} \quad \frac{A \vdash A}{\vdots}}{\Gamma \vdash \Delta, A} & \rightsquigarrow & \frac{\Gamma \vdash \Delta, A}{\vdots}
\end{array}$$

- τ_1 does not introduce the principal formula in its last rule. Depending on the subcase, we perform one of the following transformations, pushing τ_2 upwards, and use the induction hypothesis on the resulting subproof.

Problem: Resulting proofs for the \mathfrak{A}_L -cases are illegal under current formulation of the \mathfrak{A}_L -rule. (Also: they're kinda the same?) Possible solutions:

- 1: add exchanges to transformation (easy; ugly)
- 2: find generalized \mathfrak{A}_L -rule (more difficult; possibly uglier)

To be added: \otimes_R -cases.

Maybe combine \mathfrak{A}_L - and \otimes_R -cases into one? (Requires generalized formulation of the rules)

$$\begin{array}{ccc}
\frac{\frac{\pi_1}{\Gamma_1, A \vdash \Delta_1, C, \Delta'_1} \quad \frac{\pi_2}{B, \Gamma_2 \vdash \Delta_2} \quad \frac{\tau_2}{\Gamma_3, C, \Gamma'_3 \vdash \Delta_3}}{\Gamma_3, \Gamma_1, A \mathfrak{A} B, \Gamma_2, \Gamma'_3 \vdash \Delta_1, \Delta_3, \Delta'_1 \Delta_2} \mathfrak{A}_L & \rightsquigarrow & \frac{\frac{\pi_1}{\Gamma_1, A \vdash \Delta_1, C, \Delta'_1} \quad \frac{\tau_2}{\Gamma_3, C, \Gamma'_3 \vdash \Delta_3}}{\Gamma_3, \Gamma_1, A, \Gamma'_3 \vdash \Delta_1, \Delta_3, \Delta'_1} \quad \frac{\pi_2}{B, \Gamma_2 \vdash \Delta_2} \\
\frac{\frac{\pi_1}{\Gamma_1, A \vdash \Delta_1} \quad \frac{\pi_2}{B, \Gamma_2 \vdash \Delta_2, C, \Delta'_2} \quad \frac{\tau_2}{\Gamma_3, C, \Gamma'_3 \vdash \Delta_3}}{\Gamma_3, \Gamma_1, A \mathfrak{A} B, \Gamma_2, \Gamma'_3 \vdash \Delta_1, \Delta_2, \Delta_3, \Delta'_2} \mathfrak{A}_L & \rightsquigarrow & \frac{\frac{\pi_1}{\Gamma_1, A \vdash \Delta_1} \quad \frac{\pi_2}{B, \Gamma_2 \vdash \Delta_2, C, \Delta'_2} \quad \frac{\tau_2}{\Gamma_3, C, \Gamma'_3 \vdash \Delta_3}}{\Gamma_3, \Gamma_1, A \mathfrak{A} B, \Gamma_2, \Gamma'_3 \vdash \Delta_1, \Delta_2, \Delta_3, \Delta'_2} \\
\frac{\frac{\pi}{\Psi_1 \vdash \Psi_2, A, \Psi'_2} \quad \frac{\tau_2}{\Gamma, A, \Gamma' \vdash \Delta}}{\Gamma, \Phi_1, \Gamma' \vdash \Phi_2, \Delta, \Phi'_2} r & \rightsquigarrow & \frac{\frac{\pi}{\Psi_1 \vdash \Psi_2, A, \Psi'_2} \quad \frac{\tau_2}{\Gamma, A, \Gamma' \vdash \Delta}}{\Gamma, \Psi_1, \Gamma' \vdash \Psi_2, \Delta, \Psi'_2} r \\
& & \frac{\Gamma, \Psi_1, \Gamma' \vdash \Psi_2, \Delta, \Psi'_2}{\Gamma, \Phi_1, \Gamma' \vdash \Phi_2, \Delta, \Phi'_2} r
\end{array}$$

Here, r is one of the following rules: $\otimes_L, \mathfrak{A}_R, 1_L, \perp_R, \text{ex.R}$ or ex.L . Note that the exchange rule might be performed on A and a neighboring formula B , thus possibly requiring multiple applications of the exchange rule in the transformed proof. This is denoted by the double line.

- τ_2 does not introduce the principal formula in its last rule. Dually to the case above, we perform one of the following transformations, pushing τ_1 upwards, and use the induction hypothesis on the resulting subproof.

To be added: \mathfrak{A}_L - and \otimes_R -cases (same problems as above)

$$\begin{array}{ccc}
\frac{\frac{\tau_1}{\vdots} \quad \frac{\pi}{\Psi_1, A, \Psi'_1 \vdash \Psi_2}}{\Gamma \vdash \Delta, A, \Delta'} & \rightsquigarrow & \frac{\frac{\tau_1}{\vdots} \quad \frac{\pi}{\Psi_1, A, \Psi'_1 \vdash \Psi_2}}{\Gamma \vdash \Delta, A, \Delta'} \\
\frac{\Gamma \vdash \Delta, A, \Delta'}{\Phi_1, \Gamma, \Phi'_1 \vdash \Delta, \Phi_2, \Delta'} & & \frac{\Gamma \vdash \Delta, A, \Delta'}{\Gamma, \Psi_1, \Gamma' \vdash \Psi_2, \Delta, \Psi'_2} r \\
& & \frac{\Gamma, \Psi_1, \Gamma' \vdash \Psi_2, \Delta, \Psi'_2}{\Phi_1, \Gamma, \Phi'_1 \vdash \Delta, \Phi_2, \Delta'} r
\end{array}$$

Again, r is one of the following rules: $\otimes_L, \wp_R, 1_L, \perp_R, \text{ex.R}$ or ex.L .

- Finally, if both τ_1 and τ_2 introduce the principle formula with their last rule, we perform one the following transformations resulting in a proof of lower degree:

$$\begin{array}{c}
\frac{\frac{\pi_1}{\vdots} \quad \frac{\pi_2}{\vdots} \quad \frac{\pi_3}{\vdots}}{\frac{\Gamma_1 \vdash \Delta_1, A \quad \Gamma_2 \vdash B, \Delta_2 \quad \Gamma_3, A, B, \Gamma'_3 \vdash \Delta_3}{\Gamma_1, \Gamma_2 \vdash \Delta_1, A \otimes B, \Delta_2} \quad \frac{\Gamma_3, A \otimes B, \Gamma'_3 \vdash \Delta_3}{\Gamma_3, \Gamma_1, \Gamma_2, \Gamma'_3 \vdash \Delta_1, \Delta_3, \Delta_2}} \rightsquigarrow \frac{\frac{\pi_1}{\vdots} \quad \frac{\pi_2}{\vdots} \quad \frac{\pi_3}{\vdots}}{\frac{\Gamma_1 \vdash \Delta_1, A \quad \Gamma_2 \vdash B, \Delta_2 \quad \Gamma_3, A, B, \Gamma'_3 \vdash \Delta_3}{\Gamma_3, \Gamma_1, \Gamma_2, \Gamma'_3 \vdash \Delta_1, \Delta_3, \Delta_2}} \\
\\
\frac{\frac{\pi_1}{\vdots} \quad \frac{\pi_2}{\vdots} \quad \frac{\pi_3}{\vdots}}{\frac{\Gamma_1 \vdash \Delta_1, A, B, \Delta'_1 \quad \Gamma_2, A \vdash \Delta_2 \quad B, \Gamma_3 \vdash \Delta_3}{\Gamma_1 \vdash \Delta_1, A \wp B, \Delta'_1} \quad \frac{\Gamma_2, A \wp B, \Gamma_3 \vdash \Delta_2, \Delta_3}{\Gamma_2, \Gamma_1, \Gamma_3 \vdash \Delta_1, \Delta_2, \Delta_3, \Delta'_1}} \rightsquigarrow \frac{\frac{\pi_1}{\vdots} \quad \frac{\pi_2}{\vdots} \quad \frac{\pi_3}{\vdots}}{\frac{\Gamma_1 \vdash \Delta_1, A, B, \Delta'_1 \quad \Gamma_2, A \vdash \Delta_2 \quad B, \Gamma_3 \vdash \Delta_3}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, B, \Delta'_1} \quad \frac{B, \Gamma_3 \vdash \Delta_3}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta_1, \Delta_2, \Delta_3, \Delta'_1}} \\
\\
\frac{\frac{\vdash 1 \quad \frac{\pi}{\vdots} \quad \frac{\Gamma \vdash \Delta}{\Gamma, 1 \vdash \Delta}}{\Gamma \vdash \Delta} \quad \frac{\pi}{\vdots}}{\Gamma \vdash \Delta} \rightsquigarrow \frac{\pi}{\vdots} \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta} \\
\\
\frac{\vdash \perp \quad \frac{\pi}{\vdots} \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \perp, \Delta}}{\Gamma \vdash \Delta} \rightsquigarrow \frac{\pi}{\vdots} \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta}
\end{array}$$

proofsymbol

testing for proofsymbol

□

Lemma 3.13 *Let τ be a proof of the sequent $\Gamma \vdash \Delta$ of degree $\partial(\tau) > 0$. Then we can construct a proof τ' with $\partial(\tau') < \partial(\tau)$.*

PROOF: Induction on $h(\tau)$.

If the last rule of τ is not a cut of degree equal $\partial(\tau)$, we gain our proof by applying the induction hypothesis on the immediate subproofs of τ .

If the last rule of τ is a cut of degree equal to $\partial(\tau)$, we apply the induction hypothesis on the immediate subproofs of τ and obtain a proof of the following form

$$\tau = \frac{\frac{\tau_1}{\vdots} \quad \frac{\tau_2}{\vdots}}{\frac{\Gamma_1 \vdash \Delta_1, A, \Delta'_1 \quad \Gamma_2, A, \Gamma'_2 \vdash \Delta_2}{\Gamma_2, \Gamma_1, \Gamma'_2 \vdash \Delta_1, \Delta_2, \Delta'_1}}$$

with $\partial(\tau_1), \partial(\tau_2) < \partial(A)$. Note that $\Gamma = \Gamma_2 \parallel \Gamma_1 \parallel \Gamma'_2$ and $\Delta = \Gamma_2 \parallel \Gamma_1 \parallel \Gamma'_2$, with \parallel being the concatenation operator. We then apply Lemma 3.12 to the proof tree. □

With this we now obtain Gantzen's Hauptsatz:

Theorem 3.14 (Cut elimination) *Given any proof of a sequent we can construct a cut-free proof of that same sequent.*

PROOF: Iteration of Lemma 3.13 on a non-cut-free proof. □

4 CatSem of LinLog

4.1 Proof invariants

text

Definition 4.1 (η -expansion) *We call the following proof transformations η -expansions:*

$$\begin{array}{ccc}
 \frac{}{A \otimes B \vdash A \otimes B} \text{id} & \rightsquigarrow & \frac{\frac{}{A \vdash A} \text{id} \quad \frac{}{B \vdash B} \text{id}}{A, B \vdash A \otimes B} \otimes_R \\
 & & \frac{}{A \otimes B \vdash A \otimes B} \otimes_L \\
 \\
 \frac{}{A \wp B \vdash A \wp B} \text{id} & \rightsquigarrow & \frac{\frac{}{A \vdash A} \text{id} \quad \frac{}{B \vdash B} \text{id}}{A \wp B \vdash A, B} \wp_L \\
 & & \frac{}{A \wp B \vdash A \wp B} \wp_R \\
 \\
 \frac{}{1 \vdash 1} \text{id} & \rightsquigarrow & \frac{\frac{}{1 \vdash 1} 1_R}{1 \vdash 1} 1_L \\
 \\
 \frac{}{\perp \vdash \perp} \text{id} & \rightsquigarrow & \frac{\frac{}{\perp \vdash \perp} \perp_L}{\perp \vdash \perp} \perp_R
 \end{array}$$

Remark 4.2 From now on, when talking about "cut elimination procedure" or "modulo cut elimination", we mean the above mentioned transformations and the proof transformations from Lemma 3.12.

We will now define proof invariants as outlined by Milliès.

Definition 4.3 (Proof invariant) *Let π be a proof tree. We call any function*

$$\pi \mapsto [\pi]$$

that remains constant under cut elimination proof invariant. The entity $[\pi]$ is called denotation of the proof π .

Definition 4.4 (Modularity) *We call a proof invariant modular iff there exists a binary operation \circ such that for any two proofs*

$$\frac{\pi_1}{\vdots} \quad \text{and} \quad \frac{\pi_2}{\vdots}$$

the denotation of the proof

$$\pi = \frac{\frac{\pi_1}{\vdots} \quad \frac{\pi_2}{\vdots}}{A \vdash B \quad B \vdash C} \text{Cut}$$

is given by $[\pi] = [\pi_2] \circ [\pi_1]$. As the symbol suggests, we call this operation composition.

add citation

restrict definition on proofs of singular formula sequents

Proposition 4.5 *A modular invariant of proofs forms a category with the formulae as objects and the proof denotations as morphisms.*

PROOF: Associativity is given by commuting cuts. The identity morphisms are given by the axiom rule. \square

more like polycategory if we allow general sequents...

Definition 4.6 (tensorial) *A proof invariant is called tensorial iff there is a binary operation \otimes such that for any two proofs $\frac{\pi_1}{A \vdash C}$ and $\frac{\pi_2}{B \vdash D}$ the denotation of the proof*

$$\pi = \frac{\frac{\pi_1}{A \vdash C} \quad \frac{\pi_2}{B \vdash D}}{A, B \vdash C \otimes D} \otimes_R \otimes_L$$

is given by $[\pi] = [\pi_1] \otimes [\pi_2]$.

Definition 4.7 (cotensorial) *A proof invariant is called cotensorial iff there is a binary operation \otimes such that for any two proofs $\frac{\pi_1}{A \vdash C}$ and $\frac{\pi_2}{B \vdash D}$ the denotation of the proof*

$$\pi = \frac{\frac{\pi_1}{A \vdash C} \quad \frac{\pi_2}{B \vdash D}}{A \wp B \vdash C, D} \wp_L \wp_R$$

is given by $[\pi] = [\pi_1] \wp [\pi_2]$.

Lemma 4.8 *A (co-)tensorial, modular proof invariant forms a monoidal category.*

PROOF: We focus on tensorial proof invariants as the proof trees for the cotensorial case are geometrically the same. From now on we abuse our notation and write π for the denotation $[\pi]$ of the proof π .

It is easy to see that \otimes forms a bifunctor: By η -expansion the two proofs

$$\frac{}{A \otimes B \vdash A \otimes B} \text{id} \quad \text{and} \quad \frac{\frac{}{A \vdash A} \text{id} \quad \frac{}{B \vdash B} \text{id}}{A, B \vdash A \otimes B} \otimes_R \otimes_L$$

have the same denotation $\text{id}_{A \otimes B} = \text{id}_A \otimes \text{id}_B$.

Given four proofs

$$\frac{\pi_1}{A_1 \vdash A_2}, \quad \frac{\pi_2}{B_1 \vdash B_2}, \quad \frac{\pi_3}{A_2 \vdash A_3} \quad \text{and} \quad \frac{\pi_4}{B_2 \vdash B_3}$$

the proof tree

$$\frac{\frac{\frac{\pi_1}{\vdots}}{A_1 \vdash A_2} \quad \frac{\frac{\pi_2}{\vdots}}{B_1 \vdash B_2}}{A_1, B_1 \vdash A_2, B_2} \otimes_R \quad \frac{\frac{\frac{\pi_3}{\vdots}}{A_2 \vdash A_3} \quad \frac{\frac{\pi_4}{\vdots}}{B_2 \vdash B_3}}{A_2, B_2 \vdash A_3 \otimes B_3} \otimes_R}{\frac{A_1 \otimes B_1 \vdash A_2, B_2}{A_1 \otimes B_1 \vdash A_3 \otimes B_3} \otimes_L} \text{Cut}$$

with denotation $(\pi_3 \otimes \pi_4) \circ (\pi_1 \otimes \pi_2)$ is transformed via cut elimination into the proof

$$\frac{\frac{\frac{\pi_1}{\vdots}}{A_1 \vdash A_2} \quad \frac{\frac{\pi_3}{\vdots}}{A_2 \vdash A_3} \text{Cut} \quad \frac{\frac{\pi_2}{\vdots}}{B_1 \vdash B_2} \quad \frac{\frac{\pi_4}{\vdots}}{B_2 \vdash B_3} \text{Cut}}{\frac{A_1 \vdash A_3 \quad B_1 \vdash B_3}{A_1, B_1 \vdash A_3 \otimes B_3} \otimes_R} \otimes_L \frac{}{A_1 \otimes B_1 \vdash A_3 \otimes B_3}$$

with denotation $(\pi_3 \circ \pi_1) \otimes (\pi_4 \circ \pi_2)$ which gives us the following equation

$$(\pi_3 \otimes \pi_4) \circ (\pi_1 \otimes \pi_2) = (\pi_3 \circ \pi_1) \otimes (\pi_4 \circ \pi_2)$$

The associator α is given by the denotation of the proof

$$\frac{\frac{\frac{\frac{}{A \vdash A} \text{id} \quad \frac{\frac{}{B \vdash B} \text{id} \quad \frac{}{C \vdash C} \text{id}}{B, C \vdash B \otimes C} \otimes_R}{A, B, C \vdash A \otimes (B \otimes C)} \otimes_R}{A \otimes B, C \vdash A \otimes (B \otimes C)} \otimes_L}{(A \otimes B) \otimes C \vdash A \otimes (B \otimes C)} \otimes_L$$

while the unitors λ and ρ are given by

$$\frac{\frac{\overline{A \vdash A}}{1, A \vdash A} \text{id}}{1 \otimes A \vdash A} 1_L \quad \text{and} \quad \frac{\frac{\overline{A \vdash A}}{A, 1 \vdash A} \text{id}}{A \otimes 1 \vdash A} 1_L$$

- requires generalized unit rules

We have to show isomorphy as well as the naturality and coherence conditions. The inverse morphisms are given by

$$\frac{\frac{\frac{}{A \vdash A} \text{id} \quad \frac{}{B \vdash B} \text{id}}{A, B \vdash A \otimes B} \otimes_R \quad \frac{}{C \vdash C} \text{id}}{\frac{A, B, C \vdash (A \otimes B) \otimes C}{A, B \otimes C \vdash (A \otimes B) \otimes C} \otimes_L} \otimes_R \otimes_L$$

as well as

$$\frac{\frac{}{1 \vdash} 1_R \quad \frac{}{A \vdash A} \text{id}}{A \vdash 1 \otimes A} \otimes_R \quad \text{and} \quad \frac{\frac{}{1 \vdash} 1_R \quad \frac{}{A \vdash A} \text{id}}{A \vdash 1 \otimes A} \otimes_R \quad \square$$

finish: naturality example on α , isomorphism with $\text{id}_{A \otimes B} \circ \rho^{-1} \circ \rho = \text{id}_{A \otimes B}$

Theorem 4.9 *A proof invariant that is modular, tensorial and cotensorial forms a linearly distributive category.*

PROOF: The distributors ∂_L and ∂_R are given by:

write ∂ proofs and maybe show coherence.

4.1.1 Commuting cuts

is a separate definition even needed? integrate that stuff into expansions.

$$\frac{\frac{\frac{}{\vdots} \pi_1}{\Gamma_1 \vdash \Delta_1, A, \Delta'_1} \quad \frac{\frac{\frac{}{\vdots} \pi_2}{\Gamma_2, A, \Gamma'_2 \vdash \Delta_2, B, \Delta'_2} \quad \frac{\frac{}{\vdots} \pi_3}{\Gamma_3, B, \Gamma'_3 \vdash \Delta_3}}{\Gamma_3, \Gamma_2, A, \Gamma'_2, \Gamma'_3 \vdash \Delta_2, \Delta_3, \Delta'_2} \text{Cut}}{\Gamma_3, \Gamma_2, \Gamma_1, \Gamma'_2, \Gamma'_3 \vdash \Delta_1, \Delta_2, \Delta_3, \Delta'_2, \Delta'_1} \text{Cut} \quad \longleftrightarrow \quad \frac{\frac{\frac{}{\vdots} \pi_1}{\Gamma_1 \vdash \Delta_1, A, \Delta'_1} \quad \frac{\frac{\frac{}{\vdots} \pi_2}{\Gamma_2, A, \Gamma'_2 \vdash \Delta_2, B, \Delta'_2} \quad \frac{\frac{}{\vdots} \pi_3}{\Gamma_3, B, \Gamma'_3 \vdash \Delta_3}}{\Gamma_2, \Gamma_1, \Gamma'_2 \vdash \Delta_2, B, \Delta'_2} \text{Cut}}{\Gamma_3, \Gamma_2, \Gamma_1, \Gamma'_2, \Gamma'_3 \vdash \Delta_1, \Delta_2, \Delta_3, \Delta'_2, \Delta'_1} \text{Cut}$$

The proof

$$\frac{\frac{\frac{}{\vdots} \pi_1}{\Gamma_1 \vdash \Delta_1, A, \Delta'_1} \quad \frac{\frac{\frac{}{\vdots} \pi_2}{\Gamma_2 \vdash \Delta_2, B, \Delta'_2} \quad \frac{\frac{}{\vdots} \pi_3}{\Gamma_3, A, B, \Gamma'_3 \vdash \Delta_3}}{\Gamma_3, A, \Gamma_2, \Gamma'_3 \vdash \Delta_2, \Delta_3, \Delta'_2} \text{Cut}}{\Gamma_3, \Gamma_1, \Gamma_2, \Gamma'_3 \vdash \Delta_1, \Delta_2, \Delta_3, \Delta'_2, \Delta'_1} \text{Cut}$$

is transformed into

$$\frac{\frac{\frac{}{\vdots} \pi_2}{\Gamma_2 \vdash \Delta_2, B, \Delta'_2} \quad \frac{\frac{\frac{}{\vdots} \pi_1}{\Gamma_1 \vdash \Delta_1, A, \Delta'_1} \quad \frac{\frac{}{\vdots} \pi_3}{\Gamma_3, A, B, \Gamma'_3 \vdash \Delta_3}}{\Gamma_3, \Gamma_1, B, \Gamma'_3 \vdash \Delta_1, \Delta_3, \Delta'_1} \text{Cut}}{\Gamma_3, \Gamma_1, \Gamma_2, \Gamma'_3 \vdash \Delta_2, \Delta_1, \Delta_3, \Delta'_1, \Delta'_2} \text{Cut}$$

and vice versa

think about the deltas...: aufteilen in zwei Regeln mit einseitigen Deltas für A und B oder allgemein halten und viele exchanges drunter klatschen (was es asymmetrisch macht und damit)?

add refs

put that stuff back into a normal center instead of tabularx?

References

- [Bar95] Michael Barr. “Nonsymmetric *-autonomous categories”. In: *Theoretical Computer Science* 139.1 (1995), pp. 115–130. ISSN: 0304-3975. DOI: [https://doi.org/10.1016/0304-3975\(94\)00089-2](https://doi.org/10.1016/0304-3975(94)00089-2). URL: <https://www.sciencedirect.com/science/article/pii/0304397594000892>.
- [Bra96] Torben Braüner. “Introduction to Linear Logic”. In: (1996).
- [Gir87] Jean-Yves Girard. “Linear logic”. In: *Theoretical Computer Science* 50.1 (1987), pp. 1–101. ISSN: 0304-3975. DOI: [https://doi.org/10.1016/0304-3975\(87\)90045-4](https://doi.org/10.1016/0304-3975(87)90045-4). URL: <https://www.sciencedirect.com/science/article/pii/0304397587900454>.
- [Sri23] Priyaa Srinivasan. “Dagger linear logic and categorical quantum mechanics”. PhD thesis. Mar. 2023.

Todo list

■ what am I even writing about?	1
■ sources for basic definitions. (nlab proly doesn't suffice)	1
■ definition: equivalence of categories	2
■ source: Brandenburg	2
■ lengthy, rambling... maybe cut into three definitions?	3
■ definitely mixed up the right and left homs there	3
■ Diagramme besser alignen	4
■ proof	6
■ cut Barr's following stuff as it doesn't lead to anything right now?	6
■ definition closed functor, strong equivalence. (strong closed functor?)	6
■ details (Barr, 99)	6
■ feels kinda rambling... combine intuition with syntax definition below	7
■ (explicit interpretation?! shared pool between people containing Pika and Mew -> you have to get rid of one to use the other?)	7
■ interpretation of ?wn and constants.	8
■ Quellen, Quellen, Quellen!!!	8
■ main source: IntroLinLog	8
■ add example with more graph-like proof tree	8
■ proofsymbols :/	9
■ remark unnötig?	10
■ add "spoiler" to categoru theoru	13
■ Problem: Resulting proofs for the \mathfrak{A}_L -cases are illegal under current formulation of the \mathfrak{A}_L -rule. (Also: they're kinda the same?) Possible solutions: 1: add exchanges to transformation (easy; ugly) 2: find generalized \mathfrak{A}_L -rule (more difficult; possibly uglier)	14
■ To be added: \otimes_R -cases. Maybe combine \mathfrak{A}_L - and \otimes_R -cases into one? (Requires generalized formulation of the rules)	14
■ To be added: \mathfrak{A}_L - and \otimes_R -cases (same problems as above)	14
■ proofsymbols	15
■ text	16
■ add citation	16
■ restrict definition on proofs of singular formula sequents	16

■ more like polycategory if we allow general sequents...	17
■ requires generalized unit rules	18
■ finish: naturality example on α , isomorphy with $\text{id}_{A \otimes B} \circ \rho^{-1} \circ \rho = \text{id}_{A \otimes B}$	19
■ write ∂ proofs and maybe show coherence.	19
■ is a separate definition even needed? integrate that stuff into expansions.	19
■ put that stuff back into a normal center instead of tabularx?	19
■ think about the deltas...: aufteilen in zwei Regeln mit einseitigen Deltas für A und B oder allgemein halten und viele exchanges drunter klatschen (was es asymmetrisch macht und damit)?	19
■ add refs	19