LinLog und LinDisCats

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1 Introduction

Classical (and intuitionistic) logic deals with the propagation of stable truth values. If one has a true sentence A and an implication $A \Rightarrow B$, then B follows while A remains true. However, real-life implications are often causal and modify their premises. They cannot therefore be iterated arbitrarily. For example if A describes the ownership of 1€ and B owning a chocolate bar, an implication $A \multimap B$ (to be formally introduced later) would describe the process of buying such a chocolate bar for 1€, losing the 1€ in the process.

While such dealings with resources can of course be modeled in classical logic, it is easier in the resource-sensitive *linear logic*, first described by Girard in 1987 [Gir87].

Here, we have two conjunctions simultaneously $\otimes$ ("times" or "tensor") and & ("with"), which describe the availability of resources:

Suppose C is the ownership of a cookie and it costs also 1€ (i.e. we have $A \multimap C$). Then $B \otimes C$ states that one owns both a chocolate bar and a cookie. The implication $A \multimap B \otimes C$ is not possible, as it would mean that you are buying both, cookie and chocolate bar at the same time, for just 1€ total. However, from $A \multimap B$ and $A \multimap C$ we get $A \otimes A \multimap B \otimes C$, i.e. the process of buying both for 2€.

On the other hand, $B \mathbin{\&} C$ states that one has a choice between either B or C (imagine a token). From the implications $A \multimap B$ and $A \multimap C$ we get the implication $A \multimap B \mathbin{\&} C$, i.e. the process of buying a token to be exchanged for a chocolate bar or a cookie at a later time (with the choice lying with oneself). While this may seem like a disjunction, both implications $B \mathbin{\&} C \multimap B$ and $B \mathbin{\&} C \multimap C$ (exchanging the token for either product) are provable from $B \mathbin{\&} C$, although not simultaneously.

Dually, we have two disjunctions $\parr$ ("par") and $\oplus$ ("plus"):

Suppose now that B and C are the ownership of a figurine of Pikachu or Mew respectively. Then $B \oplus C$ may be the ownership of a Kinder Egg containing either figurine. This means when buying that egg ($A \multimap B \oplus C$) we do not know which one we will get. It mirrors & except, that it is non-deterministic and therefore we cannot

Our second disjunction, dual to $\otimes$, can be understood by linear implication and the linear negation (denoted as $(-)^\perp$): Under the interpretation of ownership the linear negation is no interpreted as the absence of ownership but as negative ownership, i.e. debt. That means the negation of owning 1€, A, is owing someone 1€, $A^\perp$. With the par operator we can now write the linear implication $A \multimap B$ symmetrically as $A^{\perp} \parr B$. It can be viewed as a shared pool of resources, where in order to access one resource, we have to get rid of the other.

In order to regain our stable truths known from classical logic, we need to employ two unitary connectives ! (”of course” or ”bang”) and ? (”why not”). The bang operator informs us that there is an possibly infinite amount of a resource: The statement $!A$ translates into the ownership of an amount of money that is large enough for us to ignore resource sensitivity. Imagine for example a billionaire buying a Pokemon figurine: His amount of money will not be noticeably smaller after buying the figurine. We can informally say $!A = (1 \,\&\, A) \otimes (1 \,\&\, A) \otimes \cdots$ and therefore view classical and intuitionistic logic as some sort of limit of linear logic, just as classical mechanics is a limit of quantum mechanics and the theory of relativity.

Linear logic has applications in various fields from linguistics to proof theory to quantum physics (as a quantum logic), all of which will be of no concern to us [Pai+99; Pra93].

Fragments of linear logic can be modeled by monoidal categories with various additional structures depending on the specific fragment. We will see this with multiplicative linear logic and linearly distributive categories, categories that have two tensor products linked with a ”weak” distributive relation [CS97]. We will start by laying the ground work in category theory in section 2, building toward the introduction of *linearly distributive categories*. Section 3 then presents a sequent calculus of linear logic. Finally, we will see in section 4 how linearly distributive categories form a categorical semantic for multiplicative linear logic.

2 Categorical Preliminaries

Before modeling linear logic with categories, we require a short introduction into category theory. We will first define fundamental concepts such as *categories*, *functors* and *natural transformations*. Using these, we then introduce *monoidal categories*, categories with a tensor product, which form the foundation for any categorical model of linear logic. As we ultimately want to model multiplicative linear logic, we arrive at the definition of *linearly distributive categories*, first introduced by Cocket and Seely [CS97] as *weakly distributive categories*. Finally, we will take a quick look at **-autonomous categories*, which form a model of multiplicative linear logic with linear negation, and their relation to LDCs.

2.1 Categories

Definition 2.1 (Category) *A Category $\mathfrak{C}$ consists of the following data:*

- *A class* $\mathrm{Obj}(\mathfrak{C})$ *of* objects.
- *For every pair of objects $A, B \in \mathrm{Obj}(\mathfrak{C})$ there is a class $\mathrm{Hom}(A, B)$ of* morphisms *$f : A \to B$ from A to B. We denote the class of all morphisms of $\mathfrak{C}$ with* $\mathrm{Morph}(\mathfrak{C})$
- *Morphisms* compose*: For $f \in \mathrm{Hom}(A, B)$ and $g \in \mathrm{Hom}(B, C)$ there is a morphism $g \circ f \in \mathrm{Hom}(A, C)$. That composition is associative:*

$$h \circ (g \circ f) = (h \circ g) \circ f$$

We will write gf for $g \circ f$ when appropriate.

- *For every object A there is an* identity morphism $\mathrm{id}_A \in \mathrm{Hom}(A, A)$*:*

$$f \circ \mathrm{id}_A = f, \quad \mathrm{id}_B \circ f = f$$

for $f \in \mathrm{Hom}(A, B)$

If the hom-classes are sets, we call the category locally small. *If the object class is a set, we call the category* small. *Otherwise, we call the category* large.

If we have a category $\mathfrak{C}$, we call its opposite *category $\mathfrak{C}^{\mathrm{opp}}$ the category with the following structure:*

- $\mathrm{Obj}\left(\mathfrak{C}^{\mathrm{opp}}\right) = \mathrm{Obj}(\mathfrak{C})$
- $\forall A, B \in \mathrm{Obj}(\mathfrak{C}) : \mathrm{Hom}_{\mathfrak{C}^{\mathrm{opp}}}(A, B) = \mathrm{Hom}_{\mathfrak{C}}(B, A)$

We are often interested in the relation between categories and how their structures can be translated into one another. Functors serve this purpose:

Definition 2.2 (Functor) *A* (covariant) functor $\mathsf{F} : \mathfrak{C} \to \mathfrak{D}$ *between two categories $\mathfrak{C}$ and $\mathfrak{D}$ is a function mapping each object $A \in \mathrm{Obj}(\mathfrak{C})$ to an object $F(A) \in \mathrm{Obj}(D)$ and each morphism $f \in \mathrm{Hom}(A, B)$ to a morphism $\mathsf{F}(f) \in \mathrm{Hom}(\mathsf{F}(A), \mathsf{F}(B))$ such that identity and composition are preserved:*

$$\mathsf{F}(\mathrm{id}_A) = \mathrm{id}_{\mathsf{F}(A)}, \quad \mathsf{F}(g \circ f) = \mathsf{F}(g) \circ \mathsf{F}(f)$$

A functor $\mathsf{F} : \mathfrak{C}^{\mathrm{opp}} \to \mathfrak{D}$ is called contravariant *on $\mathfrak{C}$.*

We will drop the parentheses when appropriate.

Just as we are interested in the relation between two categories, we can look into the relation between two functors. These can be described by natural transformations:

Definition 2.3 (Natural transformation) *A* natural transformation $\tau : \mathsf{F} \to \mathsf{G}$ *between two functors $\mathsf{F}, \mathsf{G} : \mathfrak{C} \to \mathfrak{D}$ is family of morphisms in $\mathfrak{D}$:*

$$\tau = \{\tau_A : \mathsf{F}A \to \mathsf{G}A \,|\, A \in \mathrm{Obj}(\mathfrak{C})\}$$

such that $\tau_B \mathsf{F}(f) = \mathsf{F}(f)\tau_A$ for all $f : A \to B \in \mathrm{Morph}(\mathfrak{C})$, i.e. the following diagram commutes:

$$\begin{array}{ccc} \mathsf{F}A & \xrightarrow{\mathsf{F}f} & \mathsf{F}B \\ \downarrow{\scriptstyle \tau_A} & & \downarrow{\scriptstyle \tau_B} \\ \mathsf{G}A & \xrightarrow{\mathsf{G}f} & \mathsf{G}B \end{array}$$

If τ_A is an isomorphism for all $A \in \mathrm{Obj}(\mathfrak{C})$, we call τ a natural isomorphism. *We will often represent a natural transformation by a single one of its members and also drop the index when appropriate, i.e. denoting $\tau : \mathsf{F} \to \mathsf{G}$ by $\tau : \mathsf{F}A \to \mathsf{G}A$.*

Definition 2.4 (Adjunction) *Let $\mathfrak{C}$ and $\mathfrak{D}$ be categories. We call* $\mathsf{F} : \mathfrak{C} \to \mathfrak{D}$ *the* left adjoint *of* $\mathsf{G} : \mathfrak{D} \to \mathfrak{C}$ *and* G *the* right adjoint *of* F *if there is an isomorphism that is natural in $A \in \mathfrak{C}$ and $B \in \mathfrak{D}$:*

$$\forall A \in \mathfrak{C}, \forall B \in \mathfrak{D} : \mathrm{Hom}_{\mathfrak{D}}(\mathsf{F}A, B) \cong \mathrm{Hom}_{\mathfrak{C}}(A, \mathsf{G}B)$$

We write $F \dashv G$.

Proposition 2.5 *The definition above is equivalent to the following: We call* F *the left adjoint of* G *if there are natural transformations*

$$\eta : \mathrm{id}_{\mathfrak{C}} \to \mathsf{G} \circ \mathsf{F}$$
$$\varepsilon : \mathsf{F} \circ \mathsf{G} \to \mathrm{id}_{\mathfrak{D}}$$

fulfilling the following condition:

$$\mathrm{id}_{\mathsf{F}A} = \varepsilon_{\mathsf{F}A} \circ \mathsf{F}(\eta_A)$$
$$\mathrm{id}_{\mathsf{G}B} = \mathsf{G}(\varepsilon_B) \circ \eta_{\mathsf{G}B}$$

PROOF: The proof can be found in most more extensive introductory works. [Bra16] □

Many useful categories have additional structures to their objects and morphisms and allow for the translation of these structures between similar categories. One such example is the category of vector spaces over a fixed field leading to the definition of monoidal or tensorial categories. That definition will form the basis for our categorical model of linear logic.

Definition 2.6 (Monoidal category) *A* monoidal category $(\mathfrak{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ *is a category $\mathfrak{C}$ with the following additional data:*

- *a functor $\otimes : \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$, called* monoidal *or* tensor product*;*
- *an object $\mathbb{1} \in \mathrm{Obj}(\mathfrak{C})$, called* unit object*;*
- *a natural isomorphism $\alpha : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$, called* associator*;*
- *a natural isomorphism $\lambda : \mathbb{1} \otimes A \to A$, called left unitor;*
- *a natural isomorphism $\rho : A \otimes \mathbb{1} \to A$, called right unitor;*

such that the following diagrams commute:

- *the pentagon diagram:*

$$\begin{tikzcd}
 & A \otimes (B \otimes (C \otimes D)) \arrow[dl, "\mathrm{id} \otimes \alpha"'] \arrow[dr, "\alpha"] & \\
A \otimes ((B \otimes C) \otimes D) \arrow[d, "\alpha"] & & (A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha"] \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha \otimes \mathrm{id}"] & & ((A \otimes B) \otimes C) \otimes D
\end{tikzcd}$$

- *the unit diagram:*

$$\begin{tikzcd}
(A \otimes \mathbb{1}) \otimes B \arrow[rr, "\alpha"] \arrow[dr, "\rho \otimes \mathrm{id}"'] & & A \otimes (\mathbb{1} \otimes B) \arrow[dl, "\mathrm{id} \otimes \lambda"] \\
& A \otimes B &
\end{tikzcd}$$

A symmetric monoidal category *has an additional natural isomorphism:*

$$s_{A,B} : A \otimes B \to B \otimes A$$

satisfying the following conditions:

- *the hexagon law:*

$$\begin{tikzcd}
A \otimes (B \otimes C) \arrow[r, "\alpha"] \arrow[d, "\mathrm{id} \otimes s_{B,C}"] & (A \otimes B) \otimes C \arrow[r, "s_{A \otimes B, C}"] & C \otimes (A \otimes B) \arrow[d, "\alpha"] \\
A \otimes (C \otimes B) \arrow[r, "\alpha"'] & (A \otimes C) \otimes B \arrow[r, "s_{A,C} \otimes B"'] & (C \otimes A) \otimes B
\end{tikzcd}$$

- *the inverse law:* $s^{-1}_{A,B} = s_{B,A}$
- *the unit law:* $\lambda = \rho s$

$$\begin{tikzcd}
\mathbb{1} \otimes A \arrow[rr, "s_{\mathbb{1},A}"] \arrow[dr, "\lambda"'] & & A \otimes \mathbb{1} \arrow[dl, "\rho"] \\
& A &
\end{tikzcd}$$

2.2 Linearly distributive categories

We now arrive at the categories that we will use to later model multiplicative linear logic.

Definition 2.7 (Linearly distributive category [Sri23]) *A* linearly distributive category *(LDC)* $(\mathfrak{C}, \otimes, 1, \parr, \bot)$ *is a category* $\mathfrak{C}$ *consisting of:*

- *a monoidal category* $(\mathfrak{C}, \otimes, 1, \alpha_\otimes, \lambda_\otimes, \rho_\otimes)$*, with* $\otimes$ *called "tensor";*
- *a monoidal category* $(\mathfrak{C}, \parr, \bot, \alpha_\parr, \lambda_\parr, \rho_\parr)$*, with* $\parr$ *called "par";*
- *two natural transformations called left and right* linear distributors *respectively:*

$$\begin{aligned} \partial_L &: A \otimes (B \parr C) \to (A \otimes B) \parr C \\ \partial_R &: (A \parr B) \otimes C \to A \parr (B \otimes C) \end{aligned}$$

satisfying the following coherence conditions:

- *coherence between the distributors and unitors:*

$$\begin{tikzcd}
1 \otimes (A \parr B) \arrow[rr, "\partial_L"] \arrow[dr, "\lambda_\otimes"'] & & (1 \otimes A) \parr B \arrow[dl, "\lambda_\otimes \parr \mathrm{id}"] \\
& A \parr B &
\end{tikzcd}$$

$$\begin{tikzcd}
(A \parr B) \otimes 1 \arrow[rr, "\partial_R"] \arrow[dr, "\rho_\otimes"'] & & A \parr (B \otimes 1) \arrow[dl, "\mathrm{id} \parr \rho_\otimes"] \\
& A \parr B &
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes (B \parr \bot) \arrow[rr, "\partial_L"] \arrow[dr, "\mathrm{id} \otimes \rho_\parr"'] & & (A \otimes B) \parr \bot \arrow[dl, "\rho_\parr"] \\
& A \otimes B &
\end{tikzcd}$$

$$\begin{tikzcd}
(\bot \parr A) \otimes B \arrow[rr, "\partial_R"] \arrow[dr, "\lambda_\parr \otimes \mathrm{id}"'] & & \bot \parr (A \otimes B) \arrow[dl, "\lambda_\parr"] \\
& A \otimes B &
\end{tikzcd}$$

- *coherence between distributors and associators:*

$$\begin{tikzcd}
& (A \otimes B) \otimes (C \parr D) \arrow[dl, "\alpha_\otimes"'] \arrow[dr, "\partial_L"] & \\
A \otimes (B \otimes (C \parr D)) \arrow[d, "\mathrm{id} \otimes \partial_L"] & & ((A \otimes B) \otimes C) \parr D \arrow[d, "\alpha_\otimes \parr \mathrm{id}"] \\
A \otimes ((B \otimes C) \parr D) \arrow[rr, "\partial_L"] & & (A \otimes (B \otimes C)) \parr D
\end{tikzcd}$$

$$\begin{tikzcd}
& A \otimes (B \parr (C \parr D)) \arrow[dl, "\mathrm{id} \otimes \alpha_\parr"'] \arrow[dr, "\partial_L"] & \\
A \otimes ((B \parr C) \parr D) \arrow[d, "\partial_L"] & & (A \otimes B) \parr (C \parr D) \arrow[d, "\alpha_\parr"] \\
(A \otimes (B \parr C)) \parr D \arrow[rr, "\partial_L \parr \mathrm{id}"] & & ((A \otimes B) \parr C) \parr D
\end{tikzcd}$$

$$\begin{tikzcd}
& (A \parr B) \otimes (C \otimes D) \arrow[dl, "\alpha_\otimes"'] \arrow[dr, "\partial_R"] & \\
((A \parr B) \otimes C) \otimes D \arrow[d, "\partial_R \otimes \mathrm{id}"] & & A \parr (B \otimes (C \otimes D)) \arrow[d, "\mathrm{id} \parr \alpha_\otimes"] \\
(A \parr (B \otimes C)) \otimes D \arrow[rr, "\partial_R"] & & A \parr ((B \otimes C) \otimes D)
\end{tikzcd}$$

$$\begin{tikzcd}
& ((A \parr B) \parr C) \otimes D \arrow[dl, "\alpha_\parr \otimes \mathrm{id}"'] \arrow[dr, "\partial_R"] & \\
(A \parr (B \parr C)) \otimes D \arrow[d, "\partial_R"] & & (A \parr B) \parr (C \otimes D) \arrow[d, "\alpha_\parr"] \\
A \parr ((B \parr C) \otimes D) \arrow[rr, "\mathrm{id} \parr \partial_R"] & & A \parr (B \parr (C \otimes D))
\end{tikzcd}$$

- *coherence between the distributors:*

$$\begin{tikzcd}
& (A \parr B) \otimes (C \parr D) \arrow[dl, "\partial_R"'] \arrow[dr, "\partial_L"] & \\
((A \parr B) \otimes C) \parr D \arrow[d, "\partial_R \parr \mathrm{id}"] & & A \parr (B \otimes (C \parr D)) \arrow[d, "\mathrm{id} \parr \partial_L"] \\
(A \parr (B \otimes C)) \parr D & & A \parr ((B \otimes C) \parr D) \arrow[ll, "\alpha_\parr"]
\end{tikzcd}$$

$$\begin{tikzcd}
 & A \otimes ((B \parr C) \otimes D) \arrow[dl, "\alpha_\otimes"'] \arrow[dr, "\mathrm{id} \otimes \partial_R"] & \\
(A \otimes (B \parr C)) \otimes D \arrow[d, "\partial_L \otimes \mathrm{id}"] & & A \otimes (B \parr (C \otimes D)) \arrow[d, "\partial_L"] \\
((A \otimes B) \parr C) \otimes D \arrow[rr, "\partial_R"] & & (A \otimes B) \parr (C \otimes D)
\end{tikzcd}$$

A symmetric *LDC is an LDC where both $\otimes$ and $\parr$ are symmetric with symmetry isomorphisms $s_\otimes : A \otimes B \to B \otimes A$ and $s_\parr : A \parr B \to B \parr A$ such that:*

$$\partial_L = s_\parr (s_\otimes \parr \mathrm{id}) \partial_R (\mathrm{id} \otimes s_\parr) s_\otimes$$

Example 2.8 Some examples for LDCs [Sri23, Definition 2.2]:

- Trivially, any monoidal category is a linearly distributive category with $\parr = \otimes$ and $\partial = \alpha$.
- A bounded distributive lattice $(L, \leqslant, \wedge, \top, \vee, \bot)$ viewed as a category with its elements as objects and the pre-order forming the morphisms. The tensor is given by $\wedge$ with its unit being $\top$, the par is given by $\vee$ with its unit being $\bot$.

  The right distributor is given by the following relation:

  $$(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c) \leqslant a \vee (b \wedge c)$$

2.3 *-autonomous Categories

While linearly distributive categories are enough to model multiplicative linear logic, if we want to include negation in our model we need additional structures. Those structures are given by *-autonomous categories, which can defined in various ways.

Definition 2.9 (Dual object) *Let $\mathfrak{C}$ be a linearly distributive category and $A, A^* \in \mathrm{Obj}(\mathfrak{C})$, then we call A^** left dual *(or* left linearly adjoint*) to A, if there are morphisms $\tau : 1 \to A^* \parr A$ and $\gamma : A \otimes A^* \to \bot$, called* unit *and* counit *resp., such that the following diagrams commute:*

$$\begin{tikzcd}
A^* \arrow[r, equal, "\mathrm{id}"] & A^* \arrow[d, "\lambda_\otimes^{-1}"] \\
A^* \parr \bot \arrow[u, "\rho_\parr"] & 1 \otimes A^* \arrow[d, "\tau \otimes \mathrm{id}"] \\
A^* \parr (A \otimes A^*) \arrow[u, "\mathrm{id} \parr \gamma"] & (A^* \parr A) \otimes A^* \arrow[l, "\partial_R"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, equal, "\mathrm{id}"] & A \arrow[d, "\rho_\otimes^{-1}"] \\
\bot \parr A \arrow[u, "\lambda_\parr"] & A \otimes 1 \arrow[d, "\mathrm{id} \otimes \tau"] \\
(A \otimes A^*) \parr A \arrow[u, "\gamma \parr \mathrm{id}"] & A \otimes (A^* \parr A) \arrow[l, "\partial_L"']
\end{tikzcd}$$

We write $(\tau, \gamma) : A^* \dashv\!\!\vert\, A$ and also call A right dual *to A^*.*

Lemma 2.10 *1. In an LDC: if $(\tau^*, \gamma^*) : A^* \dashv\!\!\vert\, A$ and $(\tau', \gamma') : A' \dashv\!\!\vert\, A$, then A^* and A' are uniquely isomorphic. The same is true for $A \dashv\!\!\vert\, {}^*A$ and $A \dashv\!\!\vert\, {}'A$. We will from now on only talk about* the *dual object, when equality up to isomorphism is sufficient.*

2. *In a symmetric LDC:* $(\tau, \gamma) : A^* \dashv\!\!\Vert\, A \Longleftrightarrow (s_{\parr}\tau, \gamma s_{\otimes}) : A \dashv\!\!\Vert\, A^*$

Proof: As the term "linearly adjoint" might suggest, it is straightforward, albeit a bit tedious, to translate a proof for the uniqueness of adjoint functors.

$$\begin{tikzcd}
 & (A^* \parr A) \otimes A^* \arrow[rr, equal] \arrow[dr, "\mathrm{id} \otimes ((\tau' \otimes \mathrm{id})\lambda_{\otimes}^{-1})"] & & (A^* \parr A) \otimes A \arrow[dr, "\rho_{\parr}(\mathrm{id} \parr \gamma^*)\partial_R"] & \\
A^* \arrow[ur, "(\tau^* \otimes \mathrm{id})\lambda_{\otimes}^{-1}"] \arrow[dr, "(\tau' \otimes \mathrm{id})\lambda_{\otimes}^{-1}"'] & & (A^* \parr A) \otimes ((A' \parr A) \otimes A^*) \arrow[ur] \arrow[dr, "\mathrm{id} \otimes (\rho_{\parr}(\mathrm{id} \otimes \gamma^*)\partial_R)"] & & A^* \\
 & (A' \parr A) \otimes A^* \arrow[ur, "(\tau^* \otimes \mathrm{id})\lambda_{\otimes}^{-1}"'] \arrow[dr, "\rho_{\parr}(\mathrm{id} \parr \gamma^*)\partial_R"'] & & (A^* \parr A) \otimes A' \arrow[ur, "\rho_{\parr}(\mathrm{id} \parr \gamma')\partial_R"'] & \\
 & & A' \arrow[ur, "(\tau^* \otimes \mathrm{id})\lambda_{\otimes}^{-1}"'] & &
\end{tikzcd}$$

The upper path is the identity as per definition of γ^* and τ^*, while the lower path gives us our isomorphisms $A^* \xrightarrow{\sim} A'$ and $A' \xrightarrow{\sim} A^*$. Commutativity of the left square follows immediately from the naturality of λ and the functoriality of $\otimes$:

$$\begin{tikzcd}
(A' \parr A) \otimes A^* \arrow[d, "\lambda_{\otimes}^{-1}"] & 1 \otimes A^* \arrow[l, "\tau' \otimes \mathrm{id}"'] \arrow[r, "\tau^* \otimes \mathrm{id}"] \arrow[d, shift right, "\lambda_{\otimes}^{-1}"'] \arrow[d, shift left, "\mathrm{id} \otimes \lambda_{\otimes}^{-1}"] & (A^* \parr A) \otimes A^* \arrow[d, "\mathrm{id} \otimes \lambda_{\otimes}^{-1}"] \\
1 \otimes ((A' \parr A) \otimes A^*) \arrow[dr, "\tau^* \otimes \mathrm{id}"'] & 1 \otimes (1 \otimes A^*) \arrow[l, "\mathrm{id} \otimes (\tau' \otimes \mathrm{id})"'] \arrow[r, "\tau^* \otimes \mathrm{id}"] \arrow[d, "\tau^* \otimes (\tau' \otimes \mathrm{id})"] & (A^* \parr A) \otimes (1 \otimes A) \arrow[dl, "\mathrm{id} \otimes (\tau' \otimes \mathrm{id})"] \\
 & (A^* \parr A) \otimes ((A' \parr A) \otimes A^*) &
\end{tikzcd}$$

The middle square also commutes because of naturality:

$$\begin{tikzcd}
(A' \parr A) \otimes A^* \arrow[r, "\lambda_{\otimes}^{-1}"] \arrow[d, "\partial_R"'] & 1 \otimes ((A' \parr A) \otimes A^*) \arrow[r, "\tau^* \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \partial_R"'] & (A^* \parr A) \otimes ((A' \parr A) \otimes A^*) \arrow[d, "\mathrm{id} \otimes \partial_R"'] \\
A' \parr (A \otimes A^*) \arrow[r, "\lambda_{\otimes}^{-1}"] \arrow[d, "\mathrm{id} \parr \gamma^*"'] & 1 \otimes (A' \parr (A \otimes A^*)) \arrow[r, "\tau^* \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes (\mathrm{id} \parr \gamma^*)"'] & (A^* \parr A) \otimes (A' \parr (A \otimes A^*)) \arrow[d, "\mathrm{id} \otimes (\mathrm{id} \parr \gamma^*)"'] \\
A' \parr \bot \arrow[r, "\lambda_{\otimes}^{-1}"] \arrow[d, "\rho_{\parr}"'] & 1 \otimes (A \parr \bot) \arrow[r, "\tau^* \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \rho_{\parr}"'] & (A^* \parr A) \otimes (A' \parr \bot) \arrow[d, "\mathrm{id} \otimes \rho_{\parr}"'] \\
A' \arrow[r, "\lambda_{\otimes}^{-1}"] & 1 \otimes A' \arrow[r, "\tau^* \otimes \mathrm{id}"] & (A^* \parr A) \otimes A'
\end{tikzcd}$$

For the right square we need the coherence conditions of our transformations:

$$
\begin{tikzcd}
 & (A^* \parr A) \otimes ((A' \parr A) \otimes A^*) \arrow[rr, "\alpha"] \arrow[dl, "\mathrm{id} \otimes \partial_R"'] & & ((A^* \parr A) \otimes (A' \parr A)) \otimes A^* \arrow[dr, "\partial_L \otimes \mathrm{id}"] & \\
(A^* \parr A) \otimes (A' \parr (A \otimes A^*)) \arrow[rr, "\partial_L"] \arrow[d, "\mathrm{id} \otimes (\mathrm{id} \parr \gamma^*)"'] & & ((A^* \parr A) \otimes A') \parr (A \otimes A^*) \arrow[dl, "\mathrm{id} \parr \gamma^*"'] \arrow[d, "\partial_R \parr \mathrm{id}"'] & & (((A^* \parr A) \otimes A') \parr A) \otimes A^* \arrow[ll, "\partial_R"'] \arrow[d, "(\partial_R \parr \mathrm{id}) \otimes \mathrm{id}"] \\
(A^* \parr A) \otimes (A' \parr \bot) \arrow[r, "\partial_L"] \arrow[d, "\mathrm{id} \otimes \rho_\parr"'] & ((A^* \parr A) \otimes A') \parr \bot \arrow[dl, "\rho_\parr"'] \arrow[d, "\partial_R \parr \mathrm{id}"] & (A^* \parr (A \otimes A')) \parr (A \otimes A^*) \arrow[dl, "\mathrm{id} \parr \gamma^*"] \arrow[dr, "(\mathrm{id} \parr \gamma') \parr \mathrm{id}"] & & ((A^* \parr (A \otimes A')) \parr A) \otimes A^* \arrow[ll, "\partial_L"'] \arrow[d, "((\mathrm{id} \parr \gamma') \parr \mathrm{id}) \otimes \mathrm{id}"'] \\
(A^* \parr A) \otimes A' \arrow[d, "\partial_R"] & (A^* \parr (A \otimes A')) \parr \bot \arrow[r, "(\mathrm{id} \parr \gamma') \parr \mathrm{id}"'] \arrow[dl, "\rho_\parr"] & (A^* \parr \bot) \parr \bot \arrow[dl, "\rho_\parr"] & (A^* \parr \bot) \parr (A \otimes A^*) \arrow[l, "\mathrm{id} \otimes \gamma^*"'] \arrow[dl, "\rho_\parr \parr \mathrm{id}"] & ((A^* \parr \bot) \parr A) \otimes A^* \arrow[l, "\partial_R"] \arrow[dl, "(\rho_\parr \parr \mathrm{id}) \otimes \mathrm{id}"] \\
A^* \parr (A \otimes A') \arrow[r, "\mathrm{id} \parr \gamma'"] & A^* \parr \bot & A^* \parr (A \otimes A^*) \arrow[l, "\mathrm{id} \parr \gamma^*"'] & (A^* \parr A) \otimes A^* \arrow[l, "\partial_R"'] &
\end{tikzcd}
$$

Finally, the triangle gives us the identity we want:

$$\begin{tikzcd}
(A^* \parr A) \otimes ((A' \parr A) \otimes A^*) \arrow[d, "\alpha_\otimes"'] & (A^* \parr A) \otimes (1 \otimes A^*) \arrow[l, "\mathrm{id} \otimes (\tau' \otimes \mathrm{id})"'] \arrow[d, "\alpha"'] & (A^* \parr A) \otimes A^* \arrow[l, "\mathrm{id} \otimes \lambda_\otimes^{-1}"'] \arrow[dl, "\rho_\otimes^{-1} \otimes \mathrm{id}"] \arrow[dd, "(\mathrm{id} \parr \rho_\otimes^{-1}) \otimes \mathrm{id}"] \\
((A^* \parr A) \otimes (A' \parr A)) \otimes A^* \arrow[d, "\partial_L \otimes \mathrm{id}"'] \arrow[dr, "\partial_R \otimes \mathrm{id}"] & ((A^* \parr A) \otimes 1) \otimes A^* \arrow[l, "(\mathrm{id} \otimes \tau') \otimes \mathrm{id}"'] \arrow[dr, "\partial_R \otimes \mathrm{id}"] & \\
(((A^* \parr A) \otimes A') \parr A) \otimes A^* \arrow[d, "(\partial_R \parr \mathrm{id}) \otimes \mathrm{id}"'] & (A^* \parr (A \otimes (A' \parr A))) \otimes A^* \arrow[d, "(\mathrm{id} \parr \partial_L) \otimes \mathrm{id}"'] & (A^* \parr (A \otimes 1)) \otimes A^* \arrow[l, "(\mathrm{id} \parr (\mathrm{id} \otimes \tau')) \otimes \mathrm{id}"] \\
((A^* \parr (A \otimes A')) \parr A) \otimes A^* \arrow[r, "\alpha_\parr \otimes \mathrm{id}"] \arrow[d, "((\mathrm{id} \parr \gamma') \parr \mathrm{id}) \otimes \mathrm{id}"'] & (A^* \parr ((A \otimes A') \parr A)) \otimes A^* \arrow[d, "(\mathrm{id} \parr (\gamma' \parr \mathrm{id})) \otimes \mathrm{id}"'] & \\
((A^* \parr \bot) \parr A) \otimes A^* \arrow[r, "\alpha_\parr \otimes \mathrm{id}"] \arrow[dr, "(\rho_\parr \parr \mathrm{id}) \otimes \mathrm{id}"'] & (A^* \parr (\bot \parr A)) \otimes A^* \arrow[d, "(\mathrm{id} \parr \rho_\parr) \otimes \mathrm{id}"] & \\
 & (A^* \parr A) \otimes A^* &
\end{tikzcd}$$

The right hand strand of the above diagram is, again, the identity by definition of the dual object.

Putting it all together we gain the equation:

$$\rho_\parr(\mathrm{id} \parr \gamma')\partial_R(\tau^* \otimes \mathrm{id})\lambda_\otimes^{-1}\rho_\parr(\mathrm{id} \parr \gamma^*)\partial_R(\tau' \otimes \mathrm{id})\lambda_\otimes^{-1} = \mathrm{id}_{A^*}$$

Switching A^* for A' and vice versa in our diagram gives us the analog equation for $\mathrm{id}_{A'}$. Thus, we have $A^* \cong A'$ with the unique isomorphism $d = \rho_\parr(\mathrm{id} \parr \gamma^*)\partial_R(\tau' \otimes \mathrm{id})\lambda_\otimes^{-1}$. For the units and counits we obtain: $\tau' = (d \parr \mathrm{id})\tau^*$ and $\gamma^* = \gamma'(\mathrm{id} \otimes d)$.

An analog proof shows the same for right duals. □

With above lemma we motivated the name ”linearly adjoint”. We will now motivate the name ”dual”:

Lemma 2.11 *For any objects A and B of an LDC: if A^* and B^* exist, the De-Morgan equations hold:*

$$(A \otimes B)^* \cong B^* \parr A^*$$
$$(A \parr B)^* \cong B^* \otimes A^*$$

The same is true for the right dual.

Proof: The unit and counit for $A \otimes B$ are given by

$$\tau_{A \otimes B} = \alpha_\parr\big(\mathrm{id} \parr (\partial_R(\tau_A \otimes \mathrm{id})\lambda_\otimes^{-1})\big)\tau_B : 1 \longrightarrow (B^* \parr A^*) \parr (A \otimes B)$$
$$\gamma_{A \otimes B} = \gamma_A(\mathrm{id} \otimes (\lambda_\parr(\gamma_B \parr \mathrm{id})\partial_L))\alpha_\otimes : (A \otimes B) \otimes (B^* \parr A^*) \longrightarrow \bot$$

A straightforward diagram chase shows that these fulfill the conditions of definition 2.9.

We focus on the second diagram of the definition, i.e. on creating the identity $\mathrm{id}_{A \otimes B}$. The process for $\mathrm{id}_{B^* \parr A^*}$ works the same way. By naturality of $\alpha_\otimes$ (and coherence with $\rho_\otimes$) we first obtain for the unit:

$$\begin{tikzcd}
& A\otimes B & \\
A\otimes(B\otimes 1) \arrow[ur, "\mathrm{id}\otimes\rho_\otimes"] \arrow[rr, "\sim", "\alpha_\otimes"'] \arrow[d, "\mathrm{id}\otimes(\mathrm{id}\otimes\tau_{A\otimes B})"'] & & (A\otimes B)\otimes 1 \arrow[ul, "\rho_\otimes"'] \arrow[d, "\mathrm{id}\otimes\tau_{A\otimes B}"] \\
A\otimes(B\otimes((B^*\parr A^*)\parr(A\otimes B))) \arrow[rr, "\sim", "\alpha_\otimes"'] & & (A\otimes B)\otimes((B^*\parr A^*)\parr(A\otimes B))
\end{tikzcd}$$

On the other hand, we obtain for the co-unit by naturality of ∂_L (and coherence with $\alpha_\otimes$):

$$\begin{tikzcd}
A\otimes(B\otimes((B^*\parr A^*)\parr(A\otimes B))) \arrow[r, "\sim", "\alpha_\otimes"'] \arrow[dd, "\mathrm{id}\otimes\partial_L"'] & (A\otimes B)\otimes((B^*\parr A^*)\parr(A\otimes B)) \arrow[d, "\partial_L"] \\
& ((A\otimes B)\otimes(B^*\parr A^*))\parr(A\otimes B) \arrow[d, "\alpha_\otimes\parr\mathrm{id}"] \\
A\otimes((B\otimes(B^*\parr A^*))\parr(A\otimes B)) \arrow[r, "\partial_L"] \arrow[d, "\mathrm{id}\otimes(\partial_L\parr\mathrm{id})"'] & (A\otimes(B\otimes(B^*\parr A^*)))\parr(A\otimes B) \arrow[d, "(\mathrm{id}\otimes\partial_L)\parr\mathrm{id}"] \\
A\otimes(((B\otimes B^*)\parr A^*)\parr(A\otimes B)) \arrow[r, "\partial_L"] \arrow[d, "\mathrm{id}\otimes((\lambda_\parr(\gamma_B\parr\mathrm{id}))\parr\mathrm{id})"'] & (A\otimes((B\otimes B^*)\parr A^*))\parr(A\otimes B) \arrow[d, "(\mathrm{id}\otimes(\lambda_\parr(\gamma_B\parr\mathrm{id})))\parr\mathrm{id}"] \\
A\otimes(A^*\parr(A\otimes B)) \arrow[r, "\partial_L"] & (A\otimes A^*)\parr(A\otimes B) \arrow[d, "\gamma_A\parr\mathrm{id}"] \\
& \perp\parr(A\otimes B) \arrow[d, "\lambda_\parr"] \\
& A\otimes B
\end{tikzcd}$$

The path on the right is $\lambda_\parr(\gamma_{A\otimes B}\parr\mathrm{id})\partial_L$.

Furthermore, using the coherence between λ and ρ as well as ∂_L and ∂_R, we can rewrite $\mathrm{id}_A\otimes\mathrm{id}_B$:

$$\begin{tikzcd}
& A\otimes B \arrow[dl, "((\mathrm{id}\otimes\tau_A)\rho_\otimes^{-1})\otimes\mathrm{id}"'] \arrow[dr, "\mathrm{id}\otimes((\tau_A\otimes\mathrm{id})\lambda_\otimes^{-1})"] & \\
(A\otimes(A^*\parr A))\otimes B \arrow[rr, "\alpha_\otimes"] \arrow[dd, "\partial_L\otimes\mathrm{id}"'] & & A\otimes((A^*\parr A)\otimes B) \arrow[d, "\mathrm{id}\otimes\partial_R"] \\
& & A\otimes(A^*\parr(A\otimes B)) \arrow[d, "\partial_L"] \\
((A\otimes A^*)\parr A)\otimes B \arrow[rr, "\partial_R"] \arrow[dr, "(\lambda_\parr(\gamma_A\parr\mathrm{id}))\otimes\mathrm{id}"'] & & (A\otimes A^*)\parr(A\otimes B) \arrow[dl, "\lambda_\parr(\gamma_A\parr\mathrm{id})"] \\
& A\otimes B &
\end{tikzcd}$$

Thus, we can draw up the defining diagram (at times suppressing some identities):

$$
\begin{tikzcd}
A \otimes B \arrow[r, "\mathrm{id} \otimes \rho_{\otimes}^{-1}"] \arrow[d, equal] & A \otimes (B \otimes 1) \arrow[r, "\mathrm{id} \otimes (\mathrm{id} \otimes \tau_B)"] \arrow[dr, phantom, "\circlearrowleft : B^* \dashv B"] & A \otimes (B \otimes (B^* \parr B)) \arrow[r, "\mathrm{id} \otimes (\mathrm{id} \parr (\tau_A \otimes \mathrm{id}) \lambda_{\otimes}^{-1})"] \arrow[d, "\mathrm{id} \otimes \partial_L"] & A \otimes (B \otimes (B^* \parr ((A^* \parr A) \otimes B))) \arrow[d, "\mathrm{id} \cdots \partial_R"] \arrow[ddl, "\mathrm{id} \otimes \partial_L"'] \\
A \otimes B \arrow[d, "\mathrm{id} \otimes (\tau_A \otimes \mathrm{id}) \lambda_{\otimes}^{-1}"'] & A \otimes (\bot \parr B) \arrow[l, "\mathrm{id} \otimes \lambda_{\parr}"] \arrow[d, "\mathrm{id} \otimes (\mathrm{id} \parr (\tau_A \otimes \mathrm{id}) \lambda_{\otimes}^{-1})"'] & A \otimes ((B \otimes B^*) \parr B) \arrow[l, "\mathrm{id} \otimes (\gamma_B \parr \mathrm{id})"] \arrow[d, "\mathrm{id} \otimes (\mathrm{id} \parr (\tau_A \otimes \mathrm{id}) \lambda_{\otimes}^{-1})"'] & A \otimes (B \otimes (B^* \parr (A^* \parr (A \otimes B)))) \arrow[d, "\cdots \alpha_{\parr}"] \arrow[ddl, "\mathrm{id} \otimes \partial_L"'] \\
A \otimes ((A^* \parr A) \otimes B) \arrow[d, "\mathrm{id} \otimes \partial_R"'] & A \otimes (\bot \parr ((A^* \parr A) \otimes B)) \arrow[l, "\mathrm{id} \otimes \lambda_{\parr}"] \arrow[d, "\mathrm{id} \otimes (\mathrm{id} \parr \partial_R)"'] & A \otimes ((B \otimes B^*) \parr ((A^* \parr A) \otimes B)) \arrow[l, "\mathrm{id} \otimes (\gamma_B \parr \mathrm{id})"] \arrow[d, "\mathrm{id} \otimes (\mathrm{id} \parr \partial_R)"'] \arrow[r, phantom, "\circlearrowleft : \text{coh. } \partial_L,\ \alpha_{\parr}"] & A \otimes (B \otimes ((B^* \parr A^*) \parr (A \otimes B))) \arrow[d, "\mathrm{id} \otimes \partial_L"] \\
A \otimes (A^* \parr (A \otimes B)) \arrow[d, "\partial_L"'] & A \otimes (\bot \parr (A^* \parr (A \otimes B))) \arrow[l, "\mathrm{id} \otimes \lambda_{\parr}"] \arrow[d, "\mathrm{id} \otimes \alpha_{\parr}"'] & A \otimes ((B \otimes B^*) \parr (A^* \parr (A \otimes B))) \arrow[l, "\mathrm{id} \otimes (\gamma_B \parr \mathrm{id})"] \arrow[d, "\mathrm{id} \otimes \alpha_{\parr}"'] & A \otimes ((B \otimes (B^* \parr A^*)) \parr (A \otimes B)) \arrow[dl, "\mathrm{id} \otimes (\partial_L \parr \mathrm{id})"] \\
(A \otimes A^*) \parr (A \otimes B) \arrow[d, "\gamma_A \parr \mathrm{id}"'] & A \otimes ((\bot \parr A^*) \parr (A \otimes B)) \arrow[ul, "\mathrm{id} \otimes (\lambda_{\parr} \parr \mathrm{id})"] & A \otimes (((B \otimes B^*) \parr A^*) \parr (A \otimes B)) \arrow[l, "\mathrm{id} \otimes ((\gamma_B \parr \mathrm{id}) \parr \mathrm{id})"] & \\
\bot \parr (A \otimes B) \arrow[d, "\lambda_{\parr}"'] & & & \\
A \otimes B & & &
\end{tikzcd}
$$

The left hand path (in orientation of the writing) is the identity on $A \otimes B$, while the top-right-bottom path equals $\lambda_{\parr}(\gamma_{A \otimes B} \parr \mathrm{id})\partial_L(\mathrm{id} \otimes \tau_{A \otimes B})\rho_{\otimes}^{-1}$, as shown above, thus proving that the following diagram commutes:

$$
\begin{tikzcd}
A \otimes B \arrow[r, equal, "\mathrm{id}"] & A \otimes B \arrow[d, "\rho_{\otimes}^{-1}"] \\
\bot \parr (A \otimes B) \arrow[u, "\lambda_{\parr}"] & (A \otimes B) \otimes 1 \arrow[d, "\mathrm{id} \otimes \tau_{A \otimes B}"] \\
((A \otimes B) \otimes (B^* \parr A^*)) \parr (A \otimes B) \arrow[u, "\gamma_{A \otimes B} \parr \mathrm{id}"] & (A \otimes B) \otimes ((B^* \parr A^*) \parr (A \otimes B)) \arrow[l, "\partial_L"']
\end{tikzcd}
$$

A similar argument shows that the following diagram commutes:

$$
\begin{tikzcd}
B^* \parr A^* \arrow[r, equal, "\mathrm{id}"] & B^* \parr A^* \arrow[d, "\lambda_{\otimes}^{-1}"] \\
(B^* \parr A^*) \parr \bot \arrow[u, "\rho_{\parr}"] & 1 \otimes (B^* \parr A^*) \arrow[d, "\tau_{A \otimes B} \otimes \mathrm{id}"] \\
(B^* \parr A^*) \parr ((A \otimes B) \otimes (B^* \parr A^*)) \arrow[u, "\mathrm{id} \parr \gamma_{A \otimes B}"] & ((B^* \parr A^*) \parr (A \otimes B)) \otimes (B^* \parr A^*) \arrow[l, "\partial_R"']
\end{tikzcd}
$$

With definition 2.9 and lemma 2.10 it follows that $(A \otimes B)^* \cong B^* \parr A^*$.

The same is done for the other equations. $\square$

Definition 2.12 (*-autonomous categories [Sri23]) *An LDC $\mathfrak{C}$ in which for every object $A \in \mathfrak{C}$ there exists a left and right dual, resp. $(\tau_A, \gamma_A) : A^* \dashv\!\vdash A$ and $({}_A\tau, {}_A\gamma) : A \dashv\!\vdash {}^*A$, is called* *-autonomous category.

As mentioned, there are various ways to define *-autonomous categories:

Definition 2.13 (*-autonomous categories [Bar95]) *A *-autonomous category is a monoidal category $\mathfrak{C}$ equipped with an equivalence $(-)^* : \mathfrak{C} \to \mathfrak{C}^{\mathrm{opp}}$ such that there is a natural isomorphism*

$$\mathrm{Hom}(A, B^*) \to \mathrm{Hom}(1, (B \otimes A)^*)$$

Barr lists three more ways to define *-autonomous categories, all equivalent to the definition above [Bar95]. We will now show that these are equivalent to Srinivasan's definition:

Theorem 2.14 *Definition 2.12 and definition 2.13 are equivalent.*

Proof: A sketch for the symmetric case was given by Cocket and Seely [CS97, theorem 4.5].

Asume $\mathfrak{C}$ to be a category fulfilling definition 2.12. The object functions $(-)^*$ and ${}^*(-)$ can be extended to contravariant functors by the following maps:

$$
(A \xrightarrow{f} B) \mapsto f^* := \quad
\begin{tikzcd}[column sep=small]
 & B^* \arrow[d, "(\tau_A \otimes \mathrm{id})\lambda_\otimes^{-1}"] & \\
 & (A^* \parr A) \otimes B^* \arrow[dl, "(\mathrm{id} \parr f) \otimes \mathrm{id}"'] \arrow[dr, "\partial_R"] & \\
(A^* \parr B) \otimes B^* \arrow[dr, "\partial_R"] & & A^* \parr (A \otimes B^*) \arrow[dl, "\mathrm{id} \parr (f \otimes \mathrm{id})"] \\
 & A^* \parr (B \otimes B^*) \arrow[d, "\rho_\parr(\mathrm{id} \parr \gamma_B)"] & \\
 & A^* &
\end{tikzcd}
$$

$$
(A \xrightarrow{f} B) \mapsto {}^*f := \quad
\begin{tikzcd}[column sep=small]
 & {}^*B \arrow[d, "(\mathrm{id} \otimes {}_A\tau)\rho_\otimes^{-1}"] & \\
 & {}^*B \otimes (A \parr {}^*A) \arrow[dl, "\mathrm{id} \otimes (f \parr \mathrm{id})"'] \arrow[dr, "\partial_L"] & \\
{}^*B \otimes (B \parr {}^*A) \arrow[dr, "\partial_L"] & & ({}^*B \otimes A) \parr {}^*A \arrow[dl, "(\mathrm{id} \otimes f) \parr \mathrm{id}"] \\
 & ({}^*B \otimes B) \parr A^* \arrow[d, "\lambda_\parr({}_B\gamma \parr \mathrm{id})"] & \\
 & A^* &
\end{tikzcd}
$$

As the squares commute, the order of distributor and f does not matter. For duality to be functor, it needs to preserve composition, leading to the following diagram in the case of $(-)^*$ (surpassing $\alpha_\otimes$ and some identities for better readability):

$$
\begin{tikzcd}[column sep=small]
 & (A^* \parr A) \otimes C^* \arrow[rr, "(\mathrm{id} \parr f) \otimes \mathrm{id}"] \arrow[dr, "\mathrm{id} \otimes (\tau_A \otimes \mathrm{id})\lambda^{-1}"'] & & (A^* \parr B) \otimes C^* \arrow[dr, "(\mathrm{id} \parr \gamma_C)\partial_R((\mathrm{id} \parr g) \otimes \mathrm{id})"] & \\
C^* \arrow[ur, "(\tau_A \otimes \mathrm{id})\lambda^{-1}"] \arrow[dr, "(\tau_B \otimes \mathrm{id})\lambda^{-1}"'] & & (A^* \parr A) \otimes (B^* \parr B) \otimes C^* \arrow[ur, "\cdots\gamma_B\cdots\circ\cdots\partial_R\partial_L\cdots\circ\cdots f\cdots"] \arrow[dr, "(\mathrm{id} \parr \gamma_C)\partial_R((\mathrm{id} \parr g) \otimes \mathrm{id})"'] & & A^* \\
 & (B^* \parr B) \otimes C^* \arrow[ur, "(\tau_A^* \otimes \mathrm{id})\lambda^{-1}"] \arrow[dr, "(\mathrm{id} \parr \gamma_C)\partial_R((\mathrm{id} \parr g) \otimes \mathrm{id})"'] & & (A^* \parr A) \otimes B^* \arrow[ur, "(\mathrm{id} \parr \gamma_B)\partial_R((\mathrm{id} \parr f) \otimes \mathrm{id})"'] & \\
 & & B^* \arrow[ur, "(\tau_A \otimes \mathrm{id})\lambda^{-1}"'] & &
\end{tikzcd}
$$

The path at the top of the diagram is $(gf)^*$ while the path at the bottom is f^*g^*. Commutativity of the diagram is shown the same way as in lemma 2.10. The preservation of identities follows directly from the properties of the (co-)units.

Thus, $(-)^* : \mathfrak{C} \to \mathfrak{C}^{\mathrm{opp}}$ is a functor.

By lemma 2.10 we know ${}^*(A^*) \cong ({}^*A)^* \cong A$ and that the following diagram commutes:

$$
{}^*(f^*) = \begin{tikzcd}
{}^*(A^*) \arrow[rr, "\sim"] \arrow[d, "(\mathrm{id} \otimes_{B^*} \tau)\lambda_{\otimes}^{-1}"'] & & A \arrow[d, "(\mathrm{id} \otimes \tau_B)\lambda_{\otimes}^{-1}"] \\
{}^*(A^*) \otimes (B^* \parr {}^*(B^*)) \arrow[rr, "\sim"] \arrow[d, "\mathrm{id} \otimes (f^* \parr \mathrm{id})"'] & & A \otimes (B^* \parr B) \arrow[d, "\mathrm{id} \otimes (f^* \parr \mathrm{id})"] \\
{}^*(A^*) \otimes (A^* \parr {}^*(B^*)) \arrow[rr, "\sim"] \arrow[d, "\partial_L"'] & & A \otimes (A^* \parr B) \arrow[d, "\partial_L"] \\
({}^*(A^*) \otimes A^*) \parr {}^*(B^*) \arrow[rr, "\sim"] \arrow[d, "\rho_{\parr}({}_{A^*}\gamma \parr \mathrm{id})"'] & & (A \otimes A^*) \parr B \arrow[d, "\rho_{\parr}(\gamma_A \parr \mathrm{id})"] \\
{}^*(B^*) & & B \arrow[ll, "\sim"']
\end{tikzcd}
$$

Thus, in order to check ${}^*(f^*) \cong f$, it is sufficient to check if the following diagram commutes:

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "(\mathrm{id} \otimes \tau_B)\rho_{\otimes}^{-1}"'] & B \arrow[d, "(\mathrm{id} \otimes \tau_B)\rho_{\otimes}^{-1}"] \\
A \otimes (B^* \parr B) \arrow[r, "f \otimes \mathrm{id}"] \arrow[d, "\partial_L"'] & B \otimes (B^* \parr B) \arrow[d, "\partial_L"] \\
(A \otimes B^*) \parr B \arrow[r, "(f \otimes \mathrm{id}) \parr \mathrm{id}"] \arrow[d, "(\mathrm{id} \otimes f^*) \parr \mathrm{id}"'] & (B \otimes B^*) \parr B \arrow[d, "\lambda_{\parr}(\gamma_B \parr \mathrm{id})"] \\
(A \otimes A^*) \parr B \arrow[r, "\lambda_{\parr}(\gamma_A \parr \mathrm{id})"] & B
\end{tikzcd}
$$

The commutativity of the last square can be seen when zooming in:

$$
\begin{tikzcd}
(A \otimes B^*) \parr B \arrow[d, "(\mathrm{id} \otimes (\tau_A \otimes \mathrm{id})\lambda_\otimes^{-1}) \parr \mathrm{id}"'] \arrow[dr, "\mathrm{id} \otimes ((\tau_A \otimes \mathrm{id})\rho_\otimes^{-1} \parr \mathrm{id})"] & & & \\
(A \otimes ((A^* \parr A) \otimes B)) \parr B \arrow[r, "\alpha_\otimes \otimes \mathrm{id}"] \arrow[dd, "(\mathrm{id} \otimes \partial_R) \parr \mathrm{id}"'] \arrow[ddr, phantom, "\circlearrowleft\text{: coh. } \partial_L{,}\partial_R"] & ((A \otimes (A^* \parr A)) \otimes B) \parr B \arrow[d, "(\partial_L \otimes \mathrm{id}) \parr \mathrm{id}"] & & \\
& (((A \otimes A^*) \parr A) \otimes B) \parr B \arrow[r, "\gamma_A \cdots"] \arrow[d, "\partial_R \parr \mathrm{id}"] & ((\bot \parr A)) \otimes B) \parr B \arrow[r, "\lambda_\parr \cdots"] \arrow[d, "\partial_R \parr \mathrm{id}"] & (A \otimes B^*) \parr B \arrow[dd, "(f \otimes \mathrm{id}) \parr \mathrm{id}"] \\
(A \otimes (A^* \parr (A \otimes B))) \parr B \arrow[r, "\partial_L \parr \mathrm{id}"] \arrow[d, "(\mathrm{id} \otimes (\mathrm{id} \parr (f \otimes \mathrm{id}))) \parr \mathrm{id}"'] & ((A \otimes A^*) \parr (A \otimes B^*)) \parr B \arrow[r, "(\gamma_A \parr \mathrm{id}) \otimes \mathrm{id}"] \arrow[d, "(\mathrm{id} \parr (f \otimes \mathrm{id})) \parr \mathrm{id}"] \arrow[dr, phantom, "\circlearrowleft\text{: funct. } \otimes"] & (\bot \parr (A \otimes B^*)) \parr B \arrow[ur, "\lambda_\parr \parr \mathrm{id}"'] \arrow[d, "(\mathrm{id} \parr (f \otimes \mathrm{id})) \parr \mathrm{id}"] & \\
(A \otimes (A^* \parr (B \otimes B^*))) \parr B \arrow[r, "\partial_L \parr \mathrm{id}"] \arrow[d, "(\mathrm{id} \otimes (\mathrm{id} \parr \gamma_B)) \parr \mathrm{id}"'] & ((A \otimes A^*) \parr (B \otimes B^*)) \parr B \arrow[r, "(\gamma_A \parr \mathrm{id}) \parr \mathrm{id}"] \arrow[d, "(\mathrm{id} \parr \gamma_B) \parr \mathrm{id}"] & (\bot \parr (B \otimes B^*)) \parr B \arrow[r, "\lambda_\parr \parr \mathrm{id}"] \arrow[d, "(\mathrm{id} \parr \gamma_B) \parr \mathrm{id}"] & (B \otimes B^*) \parr B \arrow[d, "\gamma_B \parr \mathrm{id}"] \\
(A \otimes (A^* \parr \bot)) \parr B \arrow[r, "(\gamma_A \parr \mathrm{id}) \parr \mathrm{id}"] \arrow[dr, "(\mathrm{id} \otimes \lambda_\parr) \parr \mathrm{id}"'] & ((A \otimes A^*) \parr \bot) \parr B \arrow[r, "(\gamma_A \parr \mathrm{id}) \parr \mathrm{id}"] \arrow[d, "\rho_\parr \parr \mathrm{id}"] & (\bot \parr \bot) \parr B \arrow[r, "\lambda_\parr \parr \mathrm{id}"] \arrow[d, "\rho_\parr \parr \mathrm{id}"] \arrow[dr, phantom, "\circlearrowleft\text{: coh. } \alpha,\ \lambda,\ \rho"] & \bot \parr B \arrow[d, "\lambda_\parr"] \\
& (A \otimes A^*) \parr B \arrow[r, "\gamma_A \parr \mathrm{id}"] & \bot \parr B \arrow[r, "\lambda_\parr"] & B
\end{tikzcd}
$$

The path on the left is $(\mathrm{id} \otimes f^*) \parr \mathrm{id}$ while the path at the top of the diagram is the identity on A per definition of the dual object. With this, we see that ${}^*(f^*) \cong f$. The other direction, $({}^*f)^* \cong f$, is shown similarly. That means $(-)^*$ is indeed an equivalence.

By lemma 2.11 we know $\mathrm{Hom}(1, (B \otimes A)^*) \cong \mathrm{Hom}(1, A^* \parr B^*)$. Therefore, it is enough to check whether $\mathrm{Hom}(A, B^*) \cong \mathrm{Hom}(1, A^* \parr B^*)$. The isomorphism is given by the following map:

$$(f : A \to B^*) \mapsto ((\mathrm{id} \otimes f)\tau_A : 1 \to A^* \parr B^*)$$

and its inverse:

$$(g : 1 \to A^* \parr B^*) \mapsto \big(\lambda_{\parr}(\gamma_A \parr \mathrm{id})\partial_L(\mathrm{id} \otimes g)\rho_{\otimes}^{-1} : A \to B^*\big)$$

Thus, we obtain a monoidal category with an equivalence $(-)^* : \mathfrak{C}^{\mathrm{opp}} \to \mathfrak{C}$ such that $\mathrm{Hom}(A, B^*) \cong \mathrm{Hom}(1, (B \otimes A)^*)$.

Now, assume we have a category $\mathfrak{C}$ fulfilling definition 2.13. As $(-)^*$ is an equivalence, it has an inverse ${}^*(-)$ and we can rewrite the hom-isomorphism as $\mathrm{Hom}(A, B) \cong \mathrm{Hom}(1, ({}^*B \otimes A)^*)$. We can therefore define a second tensor product:

$$A \parr B := ({}^*B \otimes {}^*A)^*$$

This gives us $\mathrm{Hom}(A, B) \cong \mathrm{Hom}(1, A^* \parr B)$.

The unit is defined by $\perp := 1^*$. Using the equivalence, we obtain the following isomorphism:

$$\begin{aligned}
&\mathrm{Hom}(A, \perp) \cong \mathrm{Hom}(1, (1 \otimes A)^*) \overset{{}^*(-)}{\cong} \mathrm{Hom}(1 \otimes A, {}^*1) \cong \mathrm{Hom}(A, {}^*1) \\
&\Rightarrow \perp = 1^* \cong {}^*1
\end{aligned}$$

Similarly, we obtain ${}^*(B^* \otimes A^*) \cong ({}^*B \otimes {}^*A)^* = A \parr B$.

The associator and unitors are inherited from $\otimes$:

$$\begin{tikzcd}
A \parr (B \parr C) \arrow[r, equal] \arrow[d, "\alpha_{\parr}"] & ({}^*(({}^*C \otimes {}^*B)^*) \otimes {}^*A) \arrow[r, "\sim"] & (({}^*C \otimes {}^*B) \otimes {}^*A)^* \arrow[d, "\alpha_{\otimes}^*"] \\
(A \parr B) \parr C \arrow[r, equal] & ({}^*C \otimes {}^*(({}^*B \otimes {}^*A)^*)) & ({}^*C \otimes ({}^*B \otimes {}^*A))^* \arrow[l, "\sim"']
\end{tikzcd}$$

$$\begin{tikzcd}
A \parr \perp \arrow[r, equal] \arrow[d, "\rho_{\parr}"] & ({}^*\perp \otimes {}^*A)^* \arrow[r, "\sim"] & (1 \otimes {}^*A)^* \arrow[d, "(\lambda_{\otimes}^{-1})^*"] \\
A & & ({}^*A)^* \arrow[ll, "\sim"']
\end{tikzcd}$$

$$\begin{tikzcd}
\perp \parr A \arrow[r, equal] \arrow[d, "\lambda_{\parr}"] & ({}^*A \otimes {}^*\perp)^* \arrow[r, "\sim"] & ({}^*A \otimes 1)^* \arrow[d, "(\rho_{\otimes}^{-1})^*"] \\
A & & ({}^*A)^* \arrow[ll, "\sim"']
\end{tikzcd}$$

Accordingly, they fulfill their respective coherence conditions.

Candidates for the unit and co-unit of an object are the identities under the hom-isomorphism:

$$\begin{aligned}\mathrm{Hom}(A, A) &\cong \mathrm{Hom}(1, ({}^*A \otimes A)^*) = \mathrm{Hom}(1, A^* \parr A)\\ \mathrm{id} &\mapsto \tau_A\\ \mathrm{Hom}(A^*, A^*) &\cong \mathrm{Hom}(1, (A \otimes A^*)^*) \cong \mathrm{Hom}(A \otimes A^*, \perp)\\ \mathrm{id} &\mapsto \gamma_A\end{aligned}$$ □

What remains to be done is the construction of the distributors:

joa...

3 Linear Logic Preliminaries

We now delve into propositional linear logic, first defining a Gentzen-style sequent calculus and proving some basic properties. We then show that the Cut-rule is admissible in the multiplicative fragment of linear logic (MLL), meaning that any sequent that can be derived using the Cut-rule can be derived without it. For that proof we develop a cut-elimination procedure that will form the basis for the categorical model of MLL. That procedure can be extended to general linear logic [Bra96] and form the basis for further categorical models of different fragments of linear logic [Mel09].

3.1 Syntax

Definition 3.1 (Atomic formula) *Let $\mathcal{A}$ be a countable set of symbols. We call its elements* atoms *or* atomic formulas.

Definition 3.2 (Formula) *Formulas are defined inductively from atomic formulas and four constants 1, $\perp$, $\top$ and 0 with logical connectives the following way:*

- *Every constant is a formula.*
- *Every atomic formula is a formula.*
- *If A is a formula, so are $A^\perp$, $!A$ and $?A$.*
- *If A and B are formulas, so are $A \otimes B$, $A \parr B$, $A \,\&\, B$ and $A \oplus B$.*

Let Γ, Δ etc. be arbitrary, finite (possibly empty) lists of formulas (e.g.: $\Gamma = p_1, ..., p_n$), and A, B etc. formulas.

As we will later consider fragments of linear logic without negation, we shall define it with a two-sided calculus.

Definition 3.3 (Sequent) *A* sequent $\Gamma \vdash \Delta$ *consists of two Lists Γ and Δ combined by a turnstile $\vdash$. Either Γ or Δ may be empty. While that turnstile serves a purely syntactic purpose, we will later see that it is correctly interpreted as an implication.*

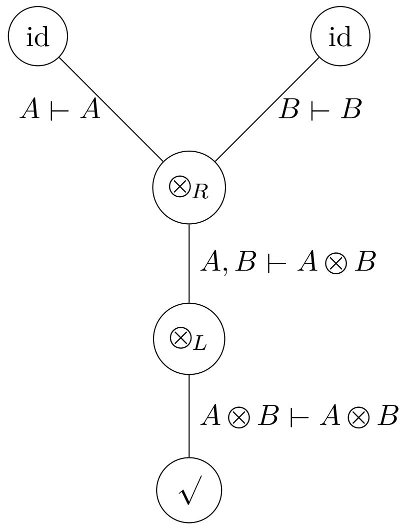
For a sequent $\Gamma_1, A, \Gamma_2 \vdash \Delta_1, B, \Delta_2$ we call Γ_i and Δ_i context *(of A (resp. B)).*

Definition 3.4 (Proof (tree)) *A* proof tree *(often just* proof*) is a rooted tree with sequents as* edges *and the sequent rules defined on the following pages as* vertices *(except for the root).*

Remark 3.5 Normally, proof trees are not written in a very "graph-like" way, which might obfuscate their nature as trees. The proof tree

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\mathrm{id} \qquad \dfrac{}{B \vdash B}\,\mathrm{id}}{A, B \vdash A \otimes B}\,\otimes_R}{A \otimes B \vdash A \otimes B}\,\otimes_L$$

can be displayed more graph-like as



with $\surd$ marking the root. The nodes id, $\otimes_R$ and $\otimes_L$ are sequent rules defined below. As this form is neither efficient nor pretty, we will stick with the traditional display.

Structural rules We only have the exchange rule as a structural rule, missing the (general) weakening and contraction rules known from classical logic:

$$\dfrac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, B, A, \Gamma' \vdash \Delta}\,\text{ex.L} \qquad \dfrac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, B, A, \Delta'}\,\text{ex.R}$$

Identity rules We have the identity and negation rules:

$$\dfrac{}{A \vdash A}\,\mathrm{id} \qquad \dfrac{\Gamma_1 \vdash \Delta_1, A, \Delta_1' \qquad \Gamma_2, A, \Gamma_2' \vdash \Delta_2}{\Gamma_2, \Gamma_1, \Gamma_2' \vdash \Delta_1, \Delta_2, \Delta_1'}\,\text{Cut}$$

$$\dfrac{\Gamma \vdash A, \Delta}{\Gamma, A^\perp \vdash \Delta}\,\text{neg.L} \qquad \dfrac{\Gamma, A \vdash \Delta}{\Gamma \vdash A^\perp, \Delta}\,\text{neg.R}$$

As already mentioned, the classical conjunction $\wedge$ and disjunction $\vee$ as well as their respective units split into two respectively. These can be classified as multiplicative and additive connectives.

Multiplicatives The calculus rules for the *multiplicative* conjunction $\otimes$, disjunction $⅋$ ("par") and their units 1 and $\bot$ ("bottom") are as follows:

$$\frac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, A \otimes B, \Gamma' \vdash \Delta} \otimes_L \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash \Delta, A \otimes B, \Delta'} \otimes_R$$

$$\frac{\Gamma, A \vdash \Delta \qquad B, \Gamma' \vdash \Delta'}{\Gamma, A ⅋ B, \Gamma' \vdash \Delta, \Delta'} ⅋_L \qquad \frac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, A ⅋ B, \Delta'} ⅋_R$$

$$\frac{\Gamma \vdash \Delta}{\Gamma_1, 1, \Gamma_2 \vdash \Delta} 1_L \qquad \frac{}{\vdash 1} 1_R$$

$$\frac{}{\bot \vdash} \bot_L \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta_1, \bot, \Delta_2} \bot_R$$

Note that $\Gamma = \Gamma_1 \| \Gamma_2$ and $\Delta = \Delta_1 \| \Delta_2$ for 1_L and $\bot_R$.

Remark 3.6 The rules $\otimes_L$ and $⅋_R$ imply, that the commas are to be read as $\otimes$ on the left-hand side and as $⅋$ on the the right-hand side. That means $A, B \vdash C, D$ is provable iff $A \otimes B \vdash C ⅋ D$ is provable.

Proof: We only have to show that $A, B \vdash C, D$ follows from $A \otimes B \vdash C ⅋ D$ as the other direction is just our introduction rule:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\text{id} \qquad \dfrac{}{B \vdash B}\text{id}}{A, B \vdash A \otimes B} \otimes_R \qquad A \otimes B \vdash C ⅋ D}{A, B \vdash C ⅋ D} \text{Cut} \qquad \dfrac{\dfrac{}{C \vdash C} \qquad \dfrac{}{D \vdash D}}{C ⅋ D \vdash C, D} ⅋_L}{A, B \vdash C, D}$$

□

Additives The calculus rules for the *additive* conjunction & and disjunction $\oplus$ ("plus") are as follows:

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \,\&\, B \vdash \Delta} \&_{L1} \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \,\&\, B \vdash \Delta} \&_{L2}$$

$$\frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \,\&\, B} \&_R$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta} \oplus_L$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \oplus B, \Delta} \oplus_{R1} \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \oplus B, \Delta} \oplus_{R2}$$

$$\frac{}{\Gamma, 0 \vdash \Delta} 0_L \qquad \frac{}{\Gamma \vdash \top, \Delta} \top_R$$

Notice the difference between $\&_R$ and $\otimes_R$ (and dually between $\oplus_L$ and $\parr_L$): while for $\otimes_R$ the contexts Γ etc. are arbitrary and get combined in the conclusion, $\&_R$ requires the contexts to be equal. In classical logic these rules can be shown to be equivalent using its additional structural rules.

Furthermore, note that the unit introduction rules also introduce arbitrary contexts, and that each unit only has one introduction rule as opposed to the multiplicative units.

Remark 3.7 Similarly to the multiplicative connectives above, the additive connectives have a invertibility statement: $\Gamma \vdash A \& B$ is provable iff $\Gamma \vdash A$ and $\Gamma \vdash B$ are provable. Dually, $A \oplus B \vdash \Delta$ iff $A \vdash \Delta$ and $B \vdash \Delta$.

Proof: As with the multiplicative statement, one direction suffices:

$$\frac{\Gamma \vdash A \& B \qquad \dfrac{\dfrac{}{A \vdash A}}{A \& B \vdash A}}{\Gamma \vdash A}$$

$\Gamma \vdash B$ follows the same way. □

Definition 3.8 *We call two formulas A and B* (linearly) equivalent *iff $A \vdash B$ and $B \vdash A$ are provable, and write $A \equiv B$.*

Remark 3.9 The names ”multiplicative” and ”additive” are motivated by the following relations:

$$\begin{aligned} A \otimes (B \oplus C) &\equiv (A \otimes B) \oplus (A \otimes C) \\ A \parr (B \& C) &\equiv (A \parr B) \& (A \parr C) \end{aligned}$$

The similarities to basic arithmetic don't end there:

$$A \otimes 0 \equiv 0, \quad A \parr \top \equiv \top$$

Proof: We only show $A \otimes (B \oplus C) \equiv (A \otimes B) \oplus (A \otimes C)$.

- $A \otimes (B \oplus C) \vdash (A \otimes B) \oplus (A \otimes C)$:

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A} \quad \dfrac{}{B \vdash B}}{A, B \vdash A \otimes B}}{A, B \vdash (A \otimes B) \oplus (A \otimes C)} \qquad \dfrac{\dfrac{\dfrac{}{A \vdash A} \quad \dfrac{}{C \vdash C}}{A, C \vdash A \otimes C}}{A, C \vdash (A \otimes B) \oplus (A \otimes C)}}{A, B \oplus C \vdash (A \otimes B) \oplus (A \otimes C)}}{A \otimes (B \oplus C) \vdash (A \otimes B) \oplus (A \otimes C)}$$

- $(A \otimes B) \oplus (A \otimes C) \vdash A \otimes (B \oplus C)$:

$$\frac{\dfrac{\dfrac{\dfrac{}{A \vdash A} \quad \dfrac{\dfrac{}{B \vdash B}}{B \vdash B \oplus C}}{A, B \vdash A \otimes (B \oplus C)}}{A \otimes B \vdash A \otimes (B \oplus C)} \qquad \dfrac{\dfrac{\dfrac{}{A \vdash A} \quad \dfrac{\dfrac{}{C \vdash C}}{C \vdash B \oplus C}}{A, C \vdash A \otimes (B \oplus C)}}{A \otimes C \vdash A \otimes (B \oplus C)}}{(A \otimes B) \oplus (A \otimes C) \vdash A \otimes (B \oplus C)}$$

The proof for $A \parr (B \& C) \equiv (A \parr B) \& (A \parr C)$ is similar, the proof for the equations of the constants a trivial application of their introduction rules. $\square$

Theorem 3.10 *We have the following equivalencies for the linear negation:*

- *For the constants:*

$$\begin{array}{ll} 1^\perp \equiv \perp & \perp^\perp \equiv 1 \\ \top^\perp \equiv 0 & 0^\perp \equiv \top \end{array}$$

- *The negation is involutory:* $(A^\perp)^\perp \equiv A$
- *The De Morgan equations hold:*

$$\begin{array}{ll} (A \otimes B)^\perp \equiv A^\perp \parr B^\perp & (A \parr B)^\perp \equiv A^\perp \otimes B^\perp \\ (A \& B)^\perp \equiv A^\perp \oplus B^\perp & (A \oplus B)^\perp \equiv A^\perp \& B^\perp \end{array}$$

Proof: We shall only prove parts.

- $1^\perp \vdash \perp$:

$$\dfrac{\dfrac{\dfrac{}{\vdash 1}\,1_R}{\vdash 1, \perp}\,\perp_R}{1^\perp \vdash \perp}\,\text{neg.L}$$

- $(A^\perp)^\perp \vdash A$:

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}}{\vdash A^\perp, A}}{(A^\perp)^\perp \vdash A}$$

- $(A \otimes B)^\perp \vdash A^\perp \parr B^\perp$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{\vdash A^\perp, A} \quad \dfrac{\dfrac{}{B \vdash B}}{\vdash B^\perp, B}}{\vdash A \otimes B, A^\perp, B^\perp}}{(A \otimes B)^\perp \vdash A^\perp, B^\perp}}{(A \otimes B)^\perp \vdash A^\perp \parr B^\perp}$$

The rest is shown similarly. $\square$

With the negation our calculus rules become quite redundant and we can restrict them on the rules for the right-hand side by translating any two-sided sequent $A_1, ..., A_n \vdash B_1, ..., B_n$ into a one-sided sequent $\vdash A_1^\perp, ...A_n^\perp, B_1, ...B_n$. However, these redundancies are necessary when dealing with a negation-free fragment of LL as we will later do.

Definition 3.11 (Syntactical implication and equivalence) *We define linear implication with the par-operator:*

$$A \multimap B := A^{\perp} \parr B$$

We further define (syntactical) linear equivalence:

$$A \multimap\!\!\!\circ\, B := (A \multimap B) \,\&\, (B \multimap A)$$

It is easily seen that $\vdash A \multimap B$ *iff* $A \vdash B$ *and that* $\vdash A \multimap\!\!\!\circ\, B$ *iff* $A \equiv B$*.*

Exponentials The modality connectives reintroduce stable truths and with them the the weakening and contraction rules known from classical logic:

$$\dfrac{\Gamma \vdash \Delta}{\Gamma, !A \vdash \Delta}\,!_W \text{ (weakening)} \qquad \dfrac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta}\,!_D \text{ (dereliction)}$$

$$\dfrac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta}\,!_C \text{ (contraction)} \qquad \dfrac{!\Gamma \vdash ?\Delta, A}{!\Gamma \vdash ?\Delta, !A}\,!_R$$

$$\dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, ?A}\,?_W \text{ (weakening)} \qquad \dfrac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, ?A}\,?_D \text{ (dereliction)}$$

$$\dfrac{\Gamma \vdash \Delta, ?A, ?A}{\Gamma \vdash \Delta, ?A}\,?_C \text{ (contraction)} \qquad \dfrac{!\Gamma, A \vdash ?\Delta}{!\Gamma, ?A \vdash ?\Delta}\,?_L$$

$$(!A)^{\perp} \equiv ?\big(A^{\perp}\big) \quad (?A)^{\perp} \equiv !\big(A^{\perp}\big)$$

Here, the context $!\Gamma$ is given by applying the !-modality to every formula of Γ, i.e. $!\Gamma = !q, !p, \ldots$ for $\Gamma = q, p \ldots$; same for $?\Delta$.

Remark 3.12 These modalities are called exponentials because of the following relation:

$$!(A \,\&\, B) \equiv !A \otimes !B, \quad ?(A \oplus B) \equiv ?A \parr ?B$$

Proof: We will only prove the second equivalence.

- $?A \parr ?B \vdash ?(A \oplus B)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{A \vdash A \oplus B}\oplus_R}{A \vdash ?(A \oplus B)}\,?_D}{?A \vdash ?(A \oplus B)}\,?_L \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash B}}{B \vdash A \oplus B}\oplus_R}{B \vdash, ?(A \oplus B)}\,?_D}{?B \vdash ?(A \oplus B)}\,?_L}{(?A) \parr (?B) \vdash ?(A \oplus B), ?(A \oplus B)}\parr}{?A \parr ?B \vdash ?(A \oplus B)}\,?_D$$

- $?(A \oplus B) \vdash ?A \parr ?B$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{A \vdash ?A}\,?_D}{A \vdash ?A, ?B}\,?_W \qquad \dfrac{\dfrac{\dfrac{}{B \vdash B}}{B \vdash ?B}}{B \vdash ?B, ?A}\,?_W}{A \oplus B \vdash ?A, ?B}\oplus_L}{?(A \oplus B) \vdash ?A, ?B}\,?_L}{?(A \oplus B) \vdash ?A \parr ?B}\parr_R$$

$!(A \,\&\, B) \equiv !A \otimes !B$ is acquired the same way. $\square$

3.2 Cut elimination

As is the case with most, if not all, logical system, the Cut rule of our calculus is admissible. That means for any proof π of a sequent $\Gamma \vdash \Delta$ we can construct a *cut-free* proof π' of that same sequent. We will prove this for the multiplicative fragment by simplifying the proof due to Braüner [Bra96, Appendix B]. The transformations used in this construction will later form the basis for the categorical semantics of this fragment. Braüner's proof includes the exponentials which require additional care because of the contraction rule. As we are strictly restricted to multiplicative linear logic, the procedure becomes streamlined.

The general strategy consists of either, moving a cut upwards in a proof or replacing it with cuts of simpler formulas. To do this, we need two definitions:

Definition 3.13 (Degree) *The* degree $\partial(A)$ *of a formula A is defined inductively:*

- $\partial(A) = 1$, *for A atomic or constant*
- $\partial(A^{\perp}) = \partial(A)$
- $\partial(A \otimes B) = \partial(A \parr B) = \max\{\partial(A), \partial(B)\} + 1$
- $\partial(!A) = \partial(?A) = \partial(A) + 1$

The degree of a cut is the degree of the cut formula. The cut formula is called principal formula. *The degree $\partial(\pi)$ of a proof π is the supremum of the degrees of the cuts in the proof.*

Remark 3.14 We write $\dfrac{\pi}{\Gamma \vdash \Delta}\, r$ and $\dfrac{\pi \qquad \pi'}{\Gamma \vdash \Delta}\, r$ for the proof that we get when applying the rule r on the last sequent(s) of the proof(s) π (and π'). Furthermore, for the proof τ that ends in the sequent $\Gamma \vdash \Delta$, we write $\dfrac{\begin{array}{c}\tau\\ \vdots\end{array}}{\Gamma \vdash \Delta}$.

Definition 3.15 (Height) *The* height $h(\pi)$ *of a proof π is just its height as a rooted tree:*

- $h(\pi) = 1$ *when π is an instance of a rule with zero premises, i.e. an axiom.*
- $h(\pi) = h(\tau) + 1$, *for* $\pi = \dfrac{\tau}{\Gamma \vdash \Delta}$
- $h(\pi) = \max\{h(\tau), h(\tau')\} + 1$, *for* $\pi = \dfrac{\tau \qquad \tau'}{\Gamma \vdash \Delta}$

With these, we can now eliminate cuts in a proof:

Lemma 3.16 *Let τ be a proof consisting of two subproofs τ_1 and τ_2 of strictly lower degree, i.e.:*

$$\tau = \dfrac{\dfrac{\begin{array}{c}\tau_1\\ \vdots\end{array}}{\Gamma_1 \vdash \Delta_1, A, \Delta_1'} \qquad \dfrac{\begin{array}{c}\tau_2\\ \vdots\end{array}}{\Gamma_2, A, \Gamma_2' \vdash \Delta_2}}{\Gamma_2, \Gamma_1, \Gamma_2' \vdash \Delta_1, \Delta_2, \Delta_1'}$$

Then, we can find a proof τ' of that same sequent $\Gamma_2, \Gamma_1, \Gamma_2' \vdash \Delta_1, \Delta_2, \Delta_1'$ with $\partial(\tau') < \partial(\tau)$.

Proof: Induction on $h(\tau_1) + h(\tau_2)$. The immediate subproofs of τ_i are denoted by π_j. We perform transformations case by case:

- If one of the the subproofs is an axiom, we just take the remaining one, resulting in a proof with lower degree:

\begin{prooftree}
\AxiomC{}
\UnaryInfC{$A \vdash A$}
\AxiomC{τ_2}
\noLine
\UnaryInfC{$\vdots$}
\UnaryInfC{$A, \Gamma \vdash \Delta$}
\BinaryInfC{$A, \Gamma \vdash \Delta$}
\end{prooftree}

$$\overset{(\mathrm{id},-)}{\rightsquigarrow}$$

\begin{prooftree}
\AxiomC{τ_2}
\noLine
\UnaryInfC{$\vdots$}
\UnaryInfC{$A, \Gamma \vdash \Delta$}
\end{prooftree}

\begin{prooftree}
\AxiomC{τ_1}
\noLine
\UnaryInfC{$\vdots$}
\UnaryInfC{$\Gamma \vdash \Delta, A$}
\AxiomC{}
\UnaryInfC{$A \vdash A$}
\BinaryInfC{$\Gamma \vdash \Delta, A$}
\end{prooftree}

$$\overset{(-,\mathrm{id})}{\rightsquigarrow}$$

\begin{prooftree}
\AxiomC{τ_1}
\noLine
\UnaryInfC{$\vdots$}
\UnaryInfC{$\Gamma \vdash \Delta, A$}
\end{prooftree}

- τ_1 does not introduce the principal formula in its last rule. Depending on the subcase, we perform one of the following transformations, pushing τ_2 upwards, and use the induction hypothesis on the resulting subproof.

\begin{prooftree}
\AxiomC{π_1}
\UnaryInfC{$\Gamma_1 \vdash \Delta_1, C, \Delta_1', A$}
\AxiomC{π_2}
\UnaryInfC{$\Gamma_2 \vdash B, \Delta_2$}
\RightLabel{$\otimes_R$}
\BinaryInfC{$\Gamma_1, \Gamma_2 \vdash \Delta_1, C, \Delta_1', A \otimes B, \Delta_2$}
\AxiomC{τ_2}
\noLine
\UnaryInfC{$\vdots$}
\UnaryInfC{$\Gamma_3, C, \Gamma_3' \vdash \Delta_3$}
\RightLabel{Cut}
\BinaryInfC{$\Gamma_3, \Gamma_1, \Gamma_2, \Gamma_3' \vdash \Delta_1, \Delta_3, \Delta_1', A \otimes B, \Delta_2$}
\end{prooftree}

$$\overset{(\downarrow\otimes_R,-)}{\rightsquigarrow}$$

\begin{prooftree}
\AxiomC{π_1}
\UnaryInfC{$\Gamma_1 \vdash \Delta_1, C, \Delta_1', A$}
\AxiomC{τ_2}
\noLine
\UnaryInfC{$\vdots$}
\UnaryInfC{$\Gamma_3, C, \Gamma_3' \vdash \Delta_3$}
\RightLabel{Cut}
\BinaryInfC{$\Gamma_3, \Gamma_1, \Gamma_3' \vdash \Delta_1, \Delta_3, \Delta_1', A$}
\AxiomC{π_2}
\UnaryInfC{$\Gamma_2 \vdash B, \Delta_2$}
\RightLabel{$\otimes_R$}
\BinaryInfC{$\Gamma_3, \Gamma_1, \Gamma_3', \Gamma_2 \vdash \Delta_1, \Delta_3, \Delta_1', A \otimes B, \Delta_2$}
\doubleLine
\RightLabel{ex.}
\UnaryInfC{$\Gamma_3, \Gamma_1, \Gamma_2, \Gamma_3' \vdash \Delta_1, \Delta_3, \Delta_1', A \otimes B, \Delta_2$}
\end{prooftree}

\begin{prooftree}
\AxiomC{π_1}
\UnaryInfC{$\Gamma_1 \vdash \Delta_1, A$}
\AxiomC{π_2}
\UnaryInfC{$\Gamma_2 \vdash B, \Delta_2, C, \Delta_2',$}
\RightLabel{$\otimes_R$}
\BinaryInfC{$\Gamma_1, \Gamma_2 \vdash \Delta_1, A \otimes B, \Delta_2, C, \Delta_2'$}
\AxiomC{τ_2}
\noLine
\UnaryInfC{$\vdots$}
\UnaryInfC{$\Gamma_3, C, \Gamma_3' \vdash \Delta_3$}
\RightLabel{Cut}
\BinaryInfC{$\Gamma_3, \Gamma_1, \Gamma_2, \Gamma_3' \vdash \Delta_1, A \otimes B, \Delta_2, \Delta_3, \Delta_2'$}
\end{prooftree}

$$\overset{(\downarrow\otimes_R,-)}{\rightsquigarrow}$$

\begin{prooftree}
\AxiomC{π_1}
\UnaryInfC{$\Gamma_1 \vdash \Delta_1, A$}
\AxiomC{π_2}
\UnaryInfC{$\Gamma_2 \vdash B, \Delta_2, C, \Delta_2'$}
\AxiomC{τ_2}
\noLine
\UnaryInfC{$\vdots$}
\UnaryInfC{$\Gamma_3, C, \Gamma_3' \vdash \Delta_3$}
\RightLabel{Cut}
\BinaryInfC{$\Gamma_3, \Gamma_2, \Gamma_3' \vdash B, \Delta_2, \Delta_3, \Delta_2'$}
\RightLabel{$\otimes_R$}
\BinaryInfC{$\Gamma_1, \Gamma_3, \Gamma_2, \Gamma_3' \vdash \Delta_1, A \otimes B, \Delta_2, \Delta_3, \Delta_2'$}
\doubleLine
\RightLabel{ex.}
\UnaryInfC{$\Gamma_3, \Gamma_1, \Gamma_2, \Gamma_3' \vdash \Delta_1, A \otimes B, \Delta_2, \Delta_3, \Delta_2'$}
\end{prooftree}

\begin{prooftree}
\AxiomC{π_1}
\UnaryInfC{$\Gamma_1, A \vdash \Delta_1, C, \Delta_1'$}
\AxiomC{π_2}
\UnaryInfC{$B, \Gamma_2 \vdash \Delta_2$}
\RightLabel{$\parr_L$}
\BinaryInfC{$\Gamma_1, A \parr B, \Gamma_2 \vdash \Delta_1, C, \Delta_1', \Delta_2$}
\AxiomC{τ_2}
\noLine
\UnaryInfC{$\vdots$}
\UnaryInfC{$\Gamma_3, C, \Gamma_3' \vdash \Delta_3$}
\RightLabel{Cut}
\BinaryInfC{$\Gamma_3, \Gamma_1, A \parr B, \Gamma_2, \Gamma_3' \vdash \Delta_1, \Delta_3, \Delta_1' \Delta_2$}
\end{prooftree}

$$\overset{(\downarrow\parr_L,-)}{\rightsquigarrow}$$

\begin{prooftree}
\AxiomC{π_1}
\UnaryInfC{$\Gamma_1, A \vdash \Delta_1, C, \Delta_1'$}
\AxiomC{τ_2}
\noLine
\UnaryInfC{$\vdots$}
\UnaryInfC{$\Gamma_3, C, \Gamma_3' \vdash \Delta_3$}
\RightLabel{Cut}
\BinaryInfC{$\Gamma_3, \Gamma_1, A, \Gamma_3' \vdash \Delta_1, \Delta_3, \Delta_1'$}
\doubleLine
\RightLabel{ex.}
\UnaryInfC{$\Gamma_3, \Gamma_1, \Gamma_3', A \vdash \Delta_1, \Delta_3, \Delta_1'$}
\AxiomC{π_2}
\UnaryInfC{$B, \Gamma_2 \vdash \Delta_2$}
\BinaryInfC{$\Gamma_3, \Gamma_1, \Gamma_3', A \parr B, \Gamma_2 \vdash \Delta_1, \Delta_3, \Delta_1' \Delta_2$}
\doubleLine
\RightLabel{ex.}
\UnaryInfC{$\Gamma_3, \Gamma_1, A \parr B, \Gamma_2, \Gamma_3' \vdash \Delta_1, \Delta_3, \Delta_1' \Delta_2$}
\end{prooftree}

\begin{prooftree}
\AxiomC{π_1}
\UnaryInfC{$\Gamma_1, A \vdash \Delta_1$}
\AxiomC{π_2}
\UnaryInfC{$B, \Gamma_2 \vdash \Delta_2, C, \Delta_2'$}
\RightLabel{$\parr_L$}
\BinaryInfC{$\Gamma_1, A \parr B, \Gamma_2 \vdash \Delta_1, \Delta_2, C, \Delta_2'$}
\AxiomC{τ_2}
\noLine
\UnaryInfC{$\vdots$}
\UnaryInfC{$\Gamma_3, C, \Gamma_3' \vdash \Delta_3$}
\RightLabel{Cut}
\BinaryInfC{$\Gamma_3, \Gamma_1, A \parr B, \Gamma_2, \Gamma_3' \vdash \Delta_1, \Delta_2, \Delta_3, \Delta_2'$}
\end{prooftree}

$$\overset{(\downarrow\parr_L,-)}{\rightsquigarrow}$$

\begin{prooftree}
\AxiomC{π_1}
\UnaryInfC{$\Gamma_1, A \vdash \Delta_1$}
\AxiomC{π_2}
\UnaryInfC{$B, \Gamma_2 \vdash \Delta_2, C, \Delta_2'$}
\AxiomC{τ_2}
\noLine
\UnaryInfC{$\vdots$}
\UnaryInfC{$\Gamma_3, C, \Gamma_3' \vdash \Delta_3$}
\RightLabel{Cut}
\BinaryInfC{$\Gamma_3, B, \Gamma_2, \Gamma_3' \vdash \Delta_2, \Delta_3, \Delta_2'$}
\doubleLine
\RightLabel{ex.}
\UnaryInfC{$B, \Gamma_3, \Gamma_2, \Gamma_3' \vdash \Delta_2, \Delta_3, \Delta_2'$}
\BinaryInfC{$\Gamma_1, A \parr B, \Gamma_3, \Gamma_2, \Gamma_3' \vdash \Delta_1, \Delta_2, \Delta_3, \Delta_2'$}
\doubleLine
\RightLabel{ex.}
\UnaryInfC{$\Gamma_3, \Gamma_1, A \parr B, \Gamma_2, \Gamma_3' \vdash \Delta_1, \Delta_2, \Delta_3, \Delta_2'$}
\end{prooftree}

\begin{prooftree}
\AxiomC{π}
\UnaryInfC{$\Psi_1 \vdash \Psi_2, A, \Psi_2'$}
\RightLabel{r}
\UnaryInfC{$\Phi_1 \vdash \Phi_2, A, \Phi_2'$}
\AxiomC{τ_2}
\noLine
\UnaryInfC{$\vdots$}
\UnaryInfC{$\Gamma, A, \Gamma' \vdash \Delta$}
\BinaryInfC{$\Gamma, \Phi_1, \Gamma' \vdash \Phi_2, \Delta, \Phi_2'$}
\end{prooftree}

$$\overset{(\downarrow r,-)}{\rightsquigarrow}$$

\begin{prooftree}
\AxiomC{π}
\UnaryInfC{$\Psi_1 \vdash \Psi_2, A, \Psi_2'$}
\AxiomC{τ_2}
\noLine
\UnaryInfC{$\vdots$}
\UnaryInfC{$\Gamma, A, \Gamma' \vdash \Delta$}
\BinaryInfC{$\Gamma, \Psi_1, \Gamma' \vdash \Psi_2, \Delta, \Psi_2'$}
\doubleLine
\RightLabel{r}
\UnaryInfC{$\Gamma, \Phi_1, \Gamma' \vdash \Phi_2, \Delta, \Phi_2'$}
\end{prooftree}

  Here, r is one of the following rules: $\otimes_L, \parr_R, 1_L, \perp_R$, ex.R or ex.L. Note that the exchange rule might be performed on A and a neighboring formula B, thus possibly requiring multiple applications of the exchange rule in the transformed proof. This is denoted by the double line.

- τ_2 does not introduce the principal formula in its last rule. Dually to the case above, we perform one of the following transformations, pushing τ_1 upwards, and use the induction hypothesis on the resulting subproof.

$$\dfrac{\dfrac{\overset{\tau_1}{\vdots}}{\Gamma_1, \vdash \Delta_1, C, \Delta_1'} \quad \dfrac{\dfrac{\pi_1}{\Gamma_2, C, \Gamma_2', A \vdash \Delta_2} \quad \dfrac{\pi_2}{B, \Gamma_3 \vdash \Delta_3}}{\Gamma_2, C, \Gamma_2', A \parr B, \Gamma_3 \vdash \Delta_2, \Delta_3}\,\parr_L}{\Gamma_2, \Gamma_1, \Gamma_2', A \parr B, \Gamma_3 \vdash \Delta_1, \Delta_2, \Delta_3, \Delta_1'}\,\text{Cut} \quad \underset{\rightsquigarrow}{(-,\downarrow\parr_L)} \quad \dfrac{\dfrac{\dfrac{\dfrac{\overset{\tau_1}{\vdots}}{\Gamma_1, \vdash \Delta_1, C, \Delta_1'} \quad \dfrac{\pi_1}{\Gamma_2, C, \Gamma_2', A \vdash \Delta_2}}{\Gamma_2, \Gamma_1, \Gamma_2', A \vdash \Delta_1, \Delta_2, \Delta_1'}\,\text{Cut} \quad \dfrac{\pi_2}{B, \Gamma_3 \vdash \Delta_3}}{\Gamma_2, \Gamma_1, \Gamma_2', A \parr B, \Gamma_3 \vdash \Delta_1, \Delta_2, \Delta_1', \Delta_3}\,\parr_L}{\Gamma_2, \Gamma_1, \Gamma_2', A \parr B, \Gamma_3 \vdash \Delta_1, \Delta_2, \Delta_3, \Delta_1'}\,\text{ex.}$$

$$\dfrac{\dfrac{\overset{\tau_1}{\vdots}}{\Gamma_1, \vdash \Delta_1, C, \Delta_1'} \quad \dfrac{\dfrac{\pi_1}{\Gamma_2, A \vdash \Delta_2} \quad \dfrac{\pi_2}{B, \Gamma_3, C, \Gamma_3' \vdash \Delta_3}}{\Gamma_2, A \parr B, \Gamma_3, C, \Gamma_3' \vdash \Delta_2, \Delta_3}\,\parr_L}{\Gamma_2, A \parr B, \Gamma_3, \Gamma_1, \Gamma_3' \vdash \Delta_1, \Delta_2, \Delta_3, \Delta_1'}\,\text{Cut} \quad \underset{\rightsquigarrow}{(-,\downarrow\parr_L)} \quad \dfrac{\dfrac{\dfrac{\pi_1}{\Gamma_2, A \vdash \Delta_2} \quad \dfrac{\dfrac{\overset{\tau_1}{\vdots}}{\Gamma_1, \vdash \Delta_1, C, \Delta_1'} \quad \dfrac{\pi_2}{B, \Gamma_3, C, \Gamma_3' \vdash \Delta_3}}{B, \Gamma_3, \Gamma_1, \Gamma_3' \vdash \Delta_1, \Delta_3, \Delta_1'}\,\text{Cut}}{\Gamma_2, A \parr B, \Gamma_3, \Gamma_1, \Gamma_3' \vdash \Delta_1, \Delta_2, \Delta_1', \Delta_3}\,\parr_L}{\Gamma_2, A \parr B, \Gamma_3, \Gamma_1, \Gamma_3' \vdash \Delta_1, \Delta_2, \Delta_3, \Delta_1'}\,\text{ex.}$$

$$\dfrac{\dfrac{\overset{\tau_1}{\vdots}}{\Gamma_1, \vdash \Delta_1, C, \Delta_1'} \quad \dfrac{\dfrac{\pi_1}{\Gamma_2, C, \Gamma_2' \vdash \Delta_2, A} \quad \dfrac{\pi_2}{\Gamma_3 \vdash B, \Delta_3}}{\Gamma_2, C, \Gamma_2', \Gamma_3 \vdash \Delta_2, A \otimes B, \Delta_3}\,\otimes_R}{\Gamma_2, \Gamma_1, \Gamma_2', \Gamma_3 \vdash \Delta_1, \Delta_2, A \otimes B, \Delta_3, \Delta_1'}\,\text{Cut} \quad \underset{\rightsquigarrow}{(-,\downarrow\otimes_R)} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\tau_1}{\vdots}}{\Gamma_1, \vdash \Delta_1, C, \Delta_1'} \quad \dfrac{\pi_1}{\Gamma_2, C, \Gamma_2' \vdash \Delta_2, A}}{\Gamma_2, \Gamma_1, \Gamma_2' \vdash \Delta_1, \Delta_2, A, \Delta_1'}\,\text{Cut}}{\Gamma_2, \Gamma_1, \Gamma_2' \vdash \Delta_1, \Delta_2, \Delta_1', A}\,\text{ex.} \quad \dfrac{\pi_2}{\Gamma_3 \vdash B, \Delta_3}}{\Gamma_2, \Gamma_1, \Gamma_2', \Gamma_3 \vdash \Delta_1, \Delta_2, \Delta_1', A \otimes B, \Delta_3}\,\otimes_R}{\Gamma_2, \Gamma_1, \Gamma_2', \Gamma_3 \vdash \Delta_1, \Delta_2, A \otimes B, \Delta_3, \Delta_1'}\,\text{ex.}$$

$$\dfrac{\dfrac{\overset{\tau_1}{\vdots}}{\Gamma_1, \vdash \Delta_1, C, \Delta_1'} \quad \dfrac{\dfrac{\pi_1}{\Gamma_2 \vdash \Delta_2, A} \quad \dfrac{\pi_2}{\Gamma_3, C, \Gamma_3' \vdash B, \Delta_3}}{\Gamma_2, \Gamma_3, C, \Gamma_3' \vdash \Delta_2, A \otimes B, \Delta_3}\,\otimes_R}{\Gamma_2, \Gamma_3, \Gamma_1, \Gamma_3' \vdash \Delta_1, \Delta_2, A \otimes B, \Delta_3, \Delta_1'}\,\text{Cut} \quad \underset{\rightsquigarrow}{(-,\downarrow\otimes_R)} \quad \dfrac{\dfrac{\dfrac{\pi_1}{\Gamma_2, \vdash \Delta_2, A} \quad \dfrac{\dfrac{\dfrac{\overset{\tau_1}{\vdots}}{\Gamma_1, \vdash \Delta_1, C, \Delta_1'} \quad \dfrac{\pi_2}{\Gamma_3, C, \Gamma_3' \vdash B, \Delta_3}}{\Gamma_3, \Gamma_1, \Gamma_3' \vdash \Delta_1, B, \Delta_3, \Delta_1'}\,\text{Cut}}{\Gamma_3, \Gamma_1, \Gamma_3' \vdash B, \Delta_1, \Delta_3, \Delta_1'}\,\text{ex.}}{\Gamma_2, \Gamma_3, \Gamma_1, \Gamma_3' \vdash \Delta_2, A \otimes B, \Delta_1, \Delta_3, \Delta_1'}\,\otimes_R}{\Gamma_2, \Gamma_3, \Gamma_1, \Gamma_3' \vdash \Delta_1, \Delta_2, A \otimes B, \Delta_3, \Delta_1'}\,\text{ex.}$$

$$\dfrac{\dfrac{\overset{\tau_1}{\vdots}}{\Gamma \vdash \Delta, A, \Delta'} \quad \dfrac{\dfrac{\pi}{\Psi_1, A, \Psi_1' \vdash \Psi_2}}{\Phi_1, A, \Phi_1' \vdash \Phi_2}\,r}{\Phi_1, \Gamma, \Phi_1' \vdash \Delta, \Phi_2, \Delta'} \quad \underset{\rightsquigarrow}{(-,\downarrow r)} \quad \dfrac{\dfrac{\dfrac{\overset{\tau_1}{\vdots}}{\Gamma \vdash \Delta, A, \Delta'} \quad \dfrac{\pi}{\Psi_1, A, \Psi_1' \vdash \Psi_2}}{\Gamma, \Psi_1, \Gamma' \vdash \Psi_2, \Delta, \Psi_2'}}{\Phi_1, \Gamma, \Phi_1' \vdash \Delta, \Phi_2, \Delta'}\,r$$

Again, r is one of the following rules: $\otimes_L, \parr_R, 1_L, \bot_R$, ex.R or ex.L.

- Finally, if both τ_1 and τ_2 introduce the principle formula with their last rule, we perform one the following transformations resulting in a proof of lower degree:

$$\dfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdots}}{\Gamma_1 \vdash \Delta_1, A} \quad \dfrac{\overset{\pi_2}{\vdots}}{\Gamma_2 \vdash B, \Delta_2}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, A \otimes B, \Delta_2}\,\otimes_R \quad \dfrac{\dfrac{\overset{\pi_3}{\vdots}}{\Gamma_3, A, B, \Gamma_3' \vdash \Delta_3}}{\Gamma_3, A \otimes B, \Gamma_3' \vdash \Delta_3}\,\otimes_L}{\Gamma_3, \Gamma_1, \Gamma_2, \Gamma_3' \vdash \Delta_1, \Delta_3, \Delta_2}\,\text{Cut} \quad \underset{\rightsquigarrow}{(\otimes_L,\otimes_R)} \quad \dfrac{\dfrac{\overset{\pi_1}{\vdots}}{\Gamma_1 \vdash \Delta_1, A} \quad \dfrac{\dfrac{\overset{\pi_2}{\vdots}}{\Gamma_2 \vdash B, \Delta_2} \quad \dfrac{\overset{\pi_3}{\vdots}}{\Gamma_3, A, B, \Gamma_3' \vdash \Delta_3}}{\Gamma_3, A, \Gamma_2, \Gamma_3' \vdash \Delta_2, \Delta_3}\,\text{Cut}}{\Gamma_3, \Gamma_1, \Gamma_2, \Gamma_3' \vdash \Delta_1, \Delta_3, \Delta_2}\,\text{Cut}$$

$$\dfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdots}}{\Gamma_1 \vdash \Delta_1, A, B, \Delta_1'}}{\Gamma_1 \vdash \Delta_1, A \parr B, \Delta_1'}\,\parr_R \quad \dfrac{\dfrac{\overset{\pi_2}{\vdots}}{\Gamma_2, A \vdash \Delta_2} \quad \dfrac{\overset{\pi_3}{\vdots}}{B, \Gamma_3 \vdash \Delta_3}}{\Gamma_2, A \parr B, \Gamma_3 \vdash \Delta_2, \Delta_3}\,\parr_L}{\Gamma_2, \Gamma_1, \Gamma_3 \vdash \Delta_1, \Delta_2, \Delta_3, \Delta_1'}\,\text{Cut} \quad \underset{\rightsquigarrow}{(\parr_R,\parr_L)} \quad \dfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdots}}{\Gamma_1 \vdash \Delta_1, A, B, \Delta_1'} \quad \dfrac{\overset{\pi_2}{\vdots}}{\Gamma_2, A \vdash \Delta_2}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, B, \Delta_1'}\,\text{Cut} \quad \dfrac{\overset{\pi_3}{\vdots}}{B, \Gamma_3 \vdash \Delta_3}}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta_1, \Delta_2, \Delta_3, \Delta_1'}\,\text{Cut}$$

$$\dfrac{\dfrac{}{\vdash 1} \quad \dfrac{\dfrac{\overset{\pi}{\vdots}}{\Gamma \vdash \Delta}}{\Gamma, 1 \vdash \Delta}}{\Gamma \vdash \Delta} \quad \underset{\rightsquigarrow}{(1_R,1_L)} \quad \dfrac{\overset{\pi}{\vdots}}{\Gamma \vdash \Delta}$$

$$\dfrac{\dfrac{}{\bot \vdash} \quad \dfrac{\dfrac{\overset{\pi}{\vdots}}{\Gamma \vdash \Delta}}{\Gamma \vdash \bot, \Delta}}{\Gamma \vdash \Delta} \quad \underset{\rightsquigarrow}{(\bot_L,\bot_R)} \quad \dfrac{\overset{\pi}{\vdots}}{\Gamma \vdash \Delta}$$

$\square$

Lemma 3.17 *Let τ be a proof of the sequent $\Gamma \vdash \Delta$ of degree $\partial(\pi) > 0$. Then we can construct a proof τ' with $\partial(\tau') < \partial(\tau)$.*

Proof: Induction on $h(\tau)$.
If the last rule of τ is not a cut of degree equal $\partial(\tau)$, we gain our proof by applying the induction hypothesis on the immediate subproofs of τ.
If the last rule of τ is a cut of degree equal to $\partial(\tau)$, we apply the induction hypothesis on the immediate subproofs of τ and obtain a proof of the following form

$$\tau = \dfrac{\dfrac{\begin{matrix}\tau_1\\ \vdots\end{matrix}}{\Gamma_1 \vdash \Delta_1, A, \Delta_1'} \qquad \dfrac{\begin{matrix}\tau_2\\ \vdots\end{matrix}}{\Gamma_2, A, \Gamma_2' \vdash \Delta_2}}{\Gamma_2, \Gamma_1, \Gamma_2' \vdash \Delta_1, \Delta_2, \Delta_1'}$$

with $\partial(\tau_1), \partial(\tau_2) < \partial(A)$. Note that $\Gamma = \Gamma_2 \| \Gamma_1 \| \Gamma_2'$ and $\Delta = \Gamma_2 \| \Gamma_1 \| \Gamma_2'$, with $\|$ being the concatenation operator. We then apply lemma 3.16 to the proof tree. $\square$

With this we now obtain Gentzen's Hauptsatz:

Theorem 3.18 (Cut elimination) *Given any proof of a sequent we can construct a cut-free proof of that same sequent.*

Proof: For any proof π with $\partial(\pi) > 0$ we can construct a proof π' with $\partial(\pi') = 0$ by a finitely iterated application of lemma 3.17. For $\partial(\pi) = 0$ there is nothing to show. $\square$

4 Categorical Semantics of Linear Logic

Definition 4.1 (η-expansion) *We call the following proof transformations* η-expansions*:*

$$\dfrac{}{A \otimes B \vdash A \otimes B}\,\mathrm{id} \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{}{A \vdash A}\,\mathrm{id} \qquad \dfrac{}{B \vdash B}\,\mathrm{id}}{A, B \vdash A \otimes B}\,\otimes_R}{A \otimes B \vdash A \otimes B}\,\otimes_L$$

$$\dfrac{}{A \parr B \vdash A \parr B}\,\mathrm{id} \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{}{A \vdash A}\,\mathrm{id} \qquad \dfrac{}{B \vdash B}\,\mathrm{id}}{A \parr B \vdash A, B}\,\parr_L}{A \parr B \vdash A \parr B}\,\parr_R$$

$$\dfrac{}{1 \vdash 1}\,\mathrm{id} \quad\rightsquigarrow\quad \dfrac{\dfrac{}{\vdash 1}\,1_R}{1 \vdash 1}\,1_L$$

$$\dfrac{}{\bot \vdash \bot}\,\mathrm{id} \quad\rightsquigarrow\quad \dfrac{\dfrac{}{\bot \vdash}\,\bot_L}{\bot \vdash \bot}\,\bot_R$$

Definition 4.2 *From now on, when talking about "cut elimination procedure" or "modulo cut elimination", we mean the above mentioned transformations, the proof transformations from lemma 3.16 as well as the following commuting cuts:*

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma_1 \vdash \Delta_1, A, \Delta_1'} \qquad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma_2, A, \Gamma_2' \vdash \Delta_2, B, \Delta_2'} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Gamma_3, B, \Gamma_3' \vdash \Delta_3}}{\Gamma_3, \Gamma_2, A, \Gamma_2', \Gamma_3' \vdash \Delta_2, \Delta_3, \Delta_2'}\,\mathit{Cut}}{\Gamma_3, \Gamma_2, \Gamma_1, \Gamma_2', \Gamma_3' \vdash \Delta_1, \Delta_2, \Delta_3, \Delta_2', \Delta_1'}\,\mathit{Cut} \quad\leftrightsquigarrow\quad \dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma_1 \vdash \Delta_1, A, \Delta_1'} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma_2, A, \Gamma_2' \vdash \Delta_2, B, \Delta_2'}}{\Gamma_2, \Gamma_1, \Gamma_2' \vdash \Delta_2, B, \Delta_2'}\,\mathit{Cut} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Gamma_3, B, \Gamma_3' \vdash \Delta_3}}{\Gamma_3, \Gamma_2, \Gamma_1, \Gamma_2', \Gamma_3' \vdash \Delta_1, \Delta_2, \Delta_3, \Delta_2', \Delta_1'}\,\mathit{Cut}$$

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma_1 \vdash \Delta_1, A, \Delta_1'} \qquad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma_2 \vdash \Delta_2, B, \Delta_2'} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Gamma_3, A, B, \Gamma_3' \vdash \Delta_3}}{\Gamma_3, A, \Gamma_2, \Gamma_3' \vdash \Delta_2, \Delta_3, \Delta_2'}\,\mathit{Cut}}{\Gamma_3, \Gamma_1, \Gamma_2, \Gamma_3' \vdash \Delta_1, \Delta_2, \Delta_3, \Delta_2', \Delta_1'}\,\mathit{Cut} \quad\leftrightsquigarrow\quad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma_2 \vdash \Delta_2, B, \Delta_2'} \qquad \dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma_1 \vdash \Delta_1, A, \Delta_1'} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Gamma_3, A, B, \Gamma_3' \vdash \Delta_3}}{\Gamma_3, \Gamma_1, B, \Gamma_3' \vdash \Delta_1, \Delta_3, \Delta_1'}\,\mathit{Cut}}{\Gamma_3, \Gamma_1, \Gamma_2, \Gamma_3' \vdash \Delta_2, \Delta_1, \Delta_3, \Delta_1', \Delta_2'}\,\mathit{Cut}$$

We will now define proof invariants as outlined by Melliès [Mel09].

Definition 4.3 (Proof invariant) *Let π be a proof tree ending in a sequent of the form $A \vdash B$, i.e. with a single formula on either side of the turnstile. We call any function*

$$\pi \mapsto [\pi]$$

that remains constant under cut elimination proof invariant*. The entity $[\pi]$ is called* denotation *of the proof π.*

Definition 4.4 (Modularity) *We call a proof invariant* modular *iff there exists a binary operation $\circ$ such that for any two proofs*

$$\cfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{A \vdash B} \quad \textit{and} \quad \cfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{B \vdash C}$$

the denotation of the proof

$$\pi = \cfrac{\cfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{A \vdash B} \qquad \cfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{B \vdash C}}{A \vdash C}\, Cut$$

is given by $[\pi] = [\pi_2] \circ [\pi_1]$. As the symbol suggests, we call this operation composition*.*

Proposition 4.5 *A modular invariant of proofs forms a category with the formulae as objects and the proof denotations as morphisms.*

Proof: Associativity is given by commuting cuts. The identity morphisms are given by the axiom rule. □

Definition 4.6 (tensoriality) *A proof invariant is called* tensorial *iff there is a binary operation $\otimes$ such that for any two proofs* $\cfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{A \vdash C}$ *and* $\cfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{B \vdash D}$ *the denotation of the proof*

$$\pi = \cfrac{\cfrac{\cfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{A \vdash C} \qquad \cfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{B \vdash D}}{A, B \vdash C \otimes D}\, \otimes_R}{A \otimes B \vdash C \otimes D}\, \otimes_L$$

is given by $[\pi] = [\pi_1] \otimes [\pi_2]$.

Definition 4.7 (cotensorial) *A proof invariant is called* cotensorial *iff there is a binary operation $\otimes$ such that for any two proofs* $\cfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{A \vdash C}$ *and* $\cfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{B \vdash D}$ *the denotation of the proof*

$$\pi \;=\; \dfrac{\dfrac{\dfrac{\pi_1}{\vdots}}{A \vdash C} \qquad \dfrac{\dfrac{\pi_2}{\vdots}}{B \vdash D}}{\dfrac{A \parr B \vdash C, D}{A \parr B \vdash C \parr D}\,\parr_R}\,\parr_L$$

is given by $[\pi] = [\pi_1] \parr [\pi_2]$.

Lemma 4.8 *A (co-)tensorial, modular proof invariant forms a symmetric monoidal category.*

Proof: We focus on tensorial proof invariants as the proof trees for the cotensorial case are geometrically the same. From now on we abuse our notation and write π for the denotation $[\pi]$ of the proof π.

It is easy to see that $\otimes$ forms a bifunctor: By η-expansion the two proofs

$$\dfrac{}{A \otimes B \vdash A \otimes B}\,\mathrm{id} \qquad \text{and} \qquad \dfrac{\dfrac{\dfrac{}{A \vdash A}\,\mathrm{id} \qquad \dfrac{}{B \vdash B}\,\mathrm{id}}{A, B \vdash A \otimes B}\,\otimes_R}{A \otimes B \vdash A \otimes B}\,\otimes_L$$

have the same denotation $\mathrm{id}_{A\otimes B} = \mathrm{id}_A \otimes \mathrm{id}_B$.

Given four proofs

$$\dfrac{\dfrac{\pi_1}{\vdots}}{A_1 \vdash A_2}\,, \qquad \dfrac{\dfrac{\pi_2}{\vdots}}{B_1 \vdash B_2}\,, \qquad \dfrac{\dfrac{\pi_3}{\vdots}}{A_2 \vdash A_3} \qquad \text{and} \qquad \dfrac{\dfrac{\pi_4}{\vdots}}{B_2 \vdash B_3}$$

the proof tree

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\pi_1}{\vdots}}{A_1 \vdash A_2} \quad \dfrac{\dfrac{\pi_2}{\vdots}}{B_1 \vdash B_2}}{A_1, B_1 \vdash A_2, B_2}\,\otimes_R}{A_1 \otimes B_1 \vdash A_2, B_2}\,\otimes_L \qquad \dfrac{\dfrac{\dfrac{\dfrac{\pi_3}{\vdots}}{A_2 \vdash A_3} \quad \dfrac{\dfrac{\pi_4}{\vdots}}{B_2 \vdash B_3}}{A_2, B_2 \vdash A_3 \otimes B_3}\,\otimes_R}{A_2 \otimes B_2 \vdash A_3 \otimes B_3}\,\otimes_L}{A_1 \otimes B_1 \vdash A_3 \otimes B_3}\,\mathrm{Cut}$$

with denotation $(\pi_3 \otimes \pi_4) \circ (\pi_1 \otimes \pi_2)$ is transformed via cut elimination into the proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\pi_1}{\vdots}}{A_1 \vdash A_2} \quad \dfrac{\dfrac{\pi_3}{\vdots}}{A_2 \vdash A_3}}{A_1 \vdash A_3}\,\mathrm{Cut} \qquad \dfrac{\dfrac{\dfrac{\pi_2}{\vdots}}{B_1 \vdash B_2} \quad \dfrac{\dfrac{\pi_4}{\vdots}}{B_2 \vdash B_3}}{B_1 \vdash B_3}\,\mathrm{Cut}}{A_1, B_1 \vdash A_3 \otimes B_3}\,\otimes_R}{A_1 \otimes B_1 \vdash A_3 \otimes B_3}\,\otimes_L$$

with denotation $(\pi_3 \circ \pi_1) \otimes (\pi_4 \circ \pi_2)$ which gives us the following equation

$$(\pi_3 \otimes \pi_4) \circ (\pi_1 \otimes \pi_2) = (\pi_3 \circ \pi_1) \otimes (\pi_4 \circ \pi_2)$$

The associator α is given by the denotation of the proof

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A\vdash A}\,\mathrm{id}\qquad \dfrac{\dfrac{}{B\vdash B}\,\mathrm{id}\quad\dfrac{}{C\vdash C}\,\mathrm{id}}{B,C\vdash B\otimes C}\,\otimes_R}{A,B,C\vdash A\otimes(B\otimes C)}\,\otimes_R}{A\otimes B,C\vdash A\otimes(B\otimes C)}\,\otimes_L}{(A\otimes B)\otimes C\vdash A\otimes(B\otimes C)}\,\otimes_L$$

while the unitors λ and ρ are given by

$$\dfrac{\dfrac{\dfrac{}{A\vdash A}\,\mathrm{id}}{1,A\vdash A}\,1_L}{1\otimes A\vdash A}\qquad\text{and}\qquad\dfrac{\dfrac{\dfrac{}{A\vdash A}\,\mathrm{id}}{A,1\vdash A}\,1_L}{A\otimes 1\vdash A}$$

We have to show isomorphy as well as the naturality and coherence conditions. The inverse morphisms are given by

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A\vdash A}\,\mathrm{id}\quad\dfrac{}{B\vdash B}\,\mathrm{id}}{A,B\vdash A\otimes B}\,\otimes_R\qquad\dfrac{}{C\vdash C}\,\mathrm{id}}{A,B,C\vdash (A\otimes B)\otimes C}\,\otimes_R}{A,B\otimes C\vdash (A\otimes B)\otimes C}\,\otimes_L}{A\otimes(B\otimes C)\vdash (A\otimes B)\otimes C}\,\otimes_L$$

as well as

$$\dfrac{\dfrac{}{\vdash 1}\,1_R\qquad\dfrac{}{A\vdash A}\,\mathrm{id}}{A\vdash 1\otimes A}\,\otimes_R\qquad\text{and}\qquad\dfrac{\dfrac{}{A\vdash A}\,\mathrm{id}\qquad\dfrac{}{\vdash 1}\,1_R}{A\vdash A\otimes 1}\,\otimes_R$$

The equation $\lambda\circ\lambda^{-1}=\mathrm{id}_A$ is easily shown:

$$\dfrac{\dfrac{\dfrac{}{\vdash 1}\,1_R\quad\dfrac{}{A\vdash A}\,\mathrm{id}}{A\vdash 1\otimes A}\,\otimes_R\qquad\dfrac{\dfrac{\dfrac{}{A\vdash A}\,\mathrm{id}}{1,A\vdash A}\,1_L}{1\otimes A\vdash A}\,\otimes_L}{A\vdash A}\,\mathrm{Cut}\quad\overset{(\otimes_R,\otimes_L)}{\rightsquigarrow}\quad\dfrac{\dfrac{}{\vdash 1}\,1_R\qquad\dfrac{\dfrac{}{A\vdash A}\,\mathrm{id}\quad\dfrac{\dfrac{}{A\vdash A}\,\mathrm{id}}{1,A\vdash A}\,1_L}{1,A\vdash A}\,\mathrm{Cut}}{A\vdash A}\,\mathrm{Cut}$$

$$\overset{(-,\mathrm{id})}{\rightsquigarrow}\quad\dfrac{\dfrac{}{\vdash 1}\,1_R\qquad\dfrac{\dfrac{}{A\vdash A}\,\mathrm{id}}{1,A\vdash A}\,1_L}{A\vdash A}\,\mathrm{Cut}\quad\overset{(1_R,1_L)}{\rightsquigarrow}\quad\dfrac{}{A\vdash A}\,\mathrm{id}$$

The equation $\lambda^{-1}\circ\lambda=\mathrm{id}_{1\otimes A}$ requires additional steps. First, we eliminate the cut in $\lambda^{-1}\circ\lambda$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A\vdash A}\,\mathrm{id}}{1,A\vdash A}\,1_L}{1\otimes A\vdash A}\,\otimes_L\qquad\dfrac{\dfrac{}{\vdash 1}\,1_R\quad\dfrac{}{A\vdash A}\,\mathrm{id}}{A\vdash 1\otimes A}\,\otimes_R}{1\otimes A\vdash 1\otimes A}\,\mathrm{Cut}\quad\overset{\downarrow\otimes_L,\downarrow\otimes_R,(-,\mathrm{id})}{\rightsquigarrow}\quad\dfrac{\dfrac{\dfrac{}{\vdash 1}\,1_R\qquad\dfrac{\dfrac{}{A\vdash A}\,\mathrm{id}}{1,A\vdash A}\,1_L}{1,A\vdash 1\otimes A}\,\otimes_R}{1\otimes A\vdash A}\,\otimes_L$$

Then, cut $\lambda^{-1}\circ\lambda$ against the η-expansion of $\mathrm{id}_{1\otimes A}$:

$$\dfrac{\dfrac{\dfrac{}{\vdash 1}\,1_R \quad \dfrac{\dfrac{}{A \vdash A}}{1, A \vdash A}\,1_L}{1, A \vdash 1 \otimes A}\,\otimes_R}{1 \otimes A \vdash A}\,\otimes_L \quad \dfrac{\dfrac{\dfrac{}{1 \vdash 1} \quad \dfrac{}{A \vdash A}}{1, A \vdash 1 \otimes A}}{1 \otimes A \vdash 1 \otimes A}\,\otimes_L \atop \overline{\qquad 1 \otimes A \vdash 1 \otimes A \qquad}\,\mathrm{Cut} \quad \overset{(\downarrow\otimes_L,-),(\otimes_R,\otimes_L)}{\rightsquigarrow} \quad \dfrac{\dfrac{\dfrac{}{\vdash 1}\,1_R \quad \dfrac{\dfrac{\dfrac{}{A \vdash A}}{1, A \vdash A}\,1_L \quad \dfrac{\dfrac{}{1 \vdash 1} \quad \dfrac{}{A \vdash A}}{1, A \vdash 1 \otimes A}\,\otimes_R}{1, 1, A \vdash 1 \otimes A}\,\mathrm{Cut}}{1, A \vdash 1 \otimes A}\,\mathrm{Cut}}{1 \otimes A \vdash 1 \otimes A}\,\otimes_L$$

$$\overset{(-,\downarrow\otimes_R),(-,\mathrm{id})}{\rightsquigarrow} \quad \dfrac{\dfrac{\dfrac{}{\vdash 1}\,1_R \quad \dfrac{\dfrac{}{1 \vdash 1} \quad \dfrac{\dfrac{}{A \vdash A}}{1, A \vdash A}\,1_L}{1, 1, A \vdash 1 \otimes A}\,\otimes_R}{1, A \vdash A}\,\mathrm{Cut}}{1 \otimes A \vdash 1 \otimes A}\,\otimes_L \quad \overset{(-,\downarrow\otimes_R),(1_R,1_L)}{\rightsquigarrow} \quad \dfrac{\dfrac{\dfrac{}{1 \vdash 1} \quad \dfrac{}{A \vdash A}}{1, 1, A \vdash 1 \otimes A}\,\otimes_R}{1 \otimes A \vdash 1 \otimes A}\,\otimes_L$$

Thus, we have $\mathrm{id}_{1 \otimes A} \circ (\lambda^{-1} \circ \lambda) = \mathrm{id}_{1 \otimes A}$ and by modularity $\lambda^{-1} \circ \lambda = \mathrm{id}_{1 \otimes A}$.

Naturality for λ means that for any $\pi : A \to B$ the equation $\lambda_B \circ (\mathrm{id}_1 \otimes \pi) = \pi \circ \lambda_A$ holds. Cut elimination on $\lambda_B \circ (\mathrm{id}_1 \otimes \pi)$ gives us:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{1 \vdash 1} \quad \dfrac{\pi \atop \vdots}{A \vdash B}}{1, A \vdash 1 \otimes B}\,\otimes_R}{1 \otimes A \vdash 1 \otimes B}\,\otimes_L \quad \dfrac{\dfrac{\dfrac{}{B \vdash B}\,\mathrm{id}}{1, B \vdash B}\,1_L}{1 \otimes B \vdash B}\,\otimes_L}{1 \otimes A \vdash B}\,\mathrm{Cut} \quad \overset{(\downarrow\otimes_L,-),(\otimes_R,\otimes_L)}{\rightsquigarrow} \quad \dfrac{\dfrac{\dfrac{}{1 \vdash 1} \quad \dfrac{\dfrac{\pi \atop \vdots}{A \vdash B} \quad \dfrac{\dfrac{}{B \vdash B}\,\mathrm{id}}{1, B \vdash B}\,1_L}{1, A \vdash B}\,\mathrm{Cut}}{1, A \vdash B}\,\mathrm{Cut}}{1 \otimes A \vdash B}\,\otimes_L$$

$$\overset{(-,\downarrow 1_L)(\mathrm{id},-)}{\underset{(-,\mathrm{id})}{\rightsquigarrow}} \quad \dfrac{\dfrac{\dfrac{\pi \atop \vdots}{A \vdash B}}{1, A \vdash B}\,1_L}{1 \otimes A \vdash B}\,\otimes_L$$

Which is the same result as cut elimination on $\pi \circ \lambda_A$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\mathrm{id}}{1, A \vdash A}\,1_L}{1 \otimes A \vdash A}\,\otimes_L \quad \dfrac{\pi \atop \vdots}{A \vdash B}}{1 \otimes A \vdash B} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\pi \atop \vdots}{A \vdash B}}{1, A \vdash B}\,1_L}{1 \otimes A \vdash B}\,\otimes_L$$

Naturality and isomorphy of α and ρ as well as the coherence conditions are shown the same way. $\square$

Theorem 4.9 *A proof invariant that is modular, tensorial and cotensorial forms a linearly distributive category.*

Proof: The distributors ∂_L and ∂_R are given by:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A} \quad \dfrac{\dfrac{}{B \vdash B} \quad \dfrac{}{C \vdash C}}{B \parr C \vdash B, C}\,\parr_L}{A, B \parr C \vdash A \otimes B, C}\,\otimes_R}{A \otimes (B \parr C) \vdash A \otimes B, C}\,\otimes_L}{A \otimes (B \parr C) \vdash (A \otimes B) \parr C}\,\parr_R \quad \text{and} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A} \quad \dfrac{}{B \vdash B}}{A \parr B \vdash A, B}\,\parr_L \quad \dfrac{}{C \vdash C}}{A \parr B, C \vdash A, B \otimes C}\,\otimes_R}{(A \parr B) \parr C \vdash A, B \otimes C}\,\otimes_L}{(A \parr B) \parr C \vdash A \parr (B \otimes C)}\,\parr_R$$

Coherence and naturality are shown the same way as for the associator and unitors above. $\square$

Remark 4.10 As noted in definition 4.3, we have been only dealing with sequents of the form $A \vdash B$. For arbitrary sequents of the form $\Gamma \vdash \Delta$ with $\Gamma = A_1, ..., A_n$ and $\Delta = B_1, ..., B_m$, a natural model are two-tensor-poly-categories as defined by Cockett and Seely [CS97]. These are (2-categorically) equivalent to linearly distributive categories by identifying Γ with $\bigotimes \Gamma = A_1 \otimes \cdots \otimes A_n$ and Δ with $\bigparr \Delta = B_1 \parr \cdots \parr B_m$ [CS97, Theorem 2.1].

Remark 4.11 Modifying the cut elimination in such a way that it can handle the negation rule, we obtain a symmetrical *-autonomous category as in definition 2.12. On the other hand, when starting with intuitionistic multiplicative linear logic (see Melliès for a short list of its calculus rules [Mel09, p43, fig. 2]), which forms a (bi-)closed monoidal category, we obtain a *-autonomous category as defined by Barr [Bar95; Bar91] when we adopt a classical negation, thus giving additional motivation to theorem 2.14.

Accordingly, when we give up on the exchange rule, we need to introduce a second negation ${}^{\perp}A$ corresponding to the non-symmetric defintions in section 2.

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