LinLog und LinDisCats

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1 Introduction

what am I even writing about?

2 LinLog Preliminaries

2.1 Motivation/Intuition

feels kinda rambling... combine intuition with syntax definition below

Classical (and intuitionistic) logic deals with the propagation of stable truth values. If one has a true sentence A and an an implication $A \Rightarrow B$, then B follows while A remains true. However, real-life implications are often causal and modify their premises. They cannot therefore be iterated arbitrarily. For example if A describes the ownership of 1€ and B owning a chocolate bar, an implication $A \multimap B$ (to be formally introduced later) would describe the process of buying such a chocolate bar for 1€, losing the 1€ in the process.

While such dealings with resources can of course be modeled in classical logic, it is easier done in the resource-sensitive *linear logic*, first described by Girard in 1987 [Gir87].

Here, we have two conjunctions simultaneously $\otimes$ ("times" or "tensor") and & ("with"), which describe the availability of resources:

Suppose C is the ownership of a cookie and it costs also 1€ (i.e. we have $A \multimap C$). Then $B \otimes C$ states that one owns both a chocolate bar and a cookie. The implication $A \multimap B \otimes C$ is not possible, as it would mean that you are buying both, cookie and chocolate bar at the same time, for just 1€ total. However, from $A \multimap B$ and $A \multimap C$ we get $A \otimes A \multimap B \otimes C$, i.e. the process of buying both for 2€.

On the other hand, $B \mathbin{\&} C$ states that one has a choice between either B or C (imagine a token). From the implications $A \multimap B$ and $A \multimap C$ we get the implication $A \multimap B \mathbin{\&} C$, i.e. the process of buying a token to be exchanged for a chocolate bar or a cookie at a later time (with the choice lying with oneself). While this may seem like a disjunction, both implications $B \mathbin{\&} C \multimap B$ and $B \mathbin{\&} C \multimap C$ (exchanging the token for either product) are provable from $B \mathbin{\&} C$, although not simultaneously.

Dually, we have two disjunctions $\parr$ ("par") and $\oplus$ ("plus"):

Suppose now that B and C are the ownership of a figurine of Pikachu or Mew respectively. Then $B \oplus C$ may be the ownership of a Kinder Egg containing either figurine. This means when buying that egg ($A \multimap B \oplus C$) we do not know which one we will get.

Our second disjunction, dual to $\otimes$, can be understood by linear implication and the linear negation (denoted as $(\cdot)^{\perp}$): Under the interpretation of ownership the linear negation is no interpreted as the absence of ownership but as negative ownership, i.e. debt. That means the negation of owning 1€, A, is owing someone 1€, $A^{\perp}$. With the par operator we can now write the linear implication $A \multimap B$ symmetrically as $A^{\perp} \parr B$.

(explicit interpretation?! shared pool between people containing Pika and Mew -> you have to get rid of one to use the other?)

In order to regain our stable truths known from classical logic, we need to employ two unitary connectives ! ("of course" or "bang") and ? ("why not"). The bang operator informs us that there is an infinite amount of a resource: The statement $!A$ translates into the ownership of an amount of money that is large enough for us to ignore resource sensitivity. Imagine for example a billionaire buying a Pokemon figurine instead of a social media site: His amount of money will not be noticeably smaller after buying the figurine. We can informally say $!A = (1 \& A) \otimes (1 \& A) \otimes \cdots$ and therefore view classical logic as some sort of limit of linear logic, just as classical mechanics is a limit of quantum mechanics and the theory of relativity.

interpretation of ?wn.

interpretation of constants

2.2 Syntax

section needs serious reformatting...

main source: IntroLinLog

sequent calculus constants

Formulas are built from atomic formulas $p, q, \phi, \psi, p^{\perp}$ etc. and constants $1, \perp, 0, \top$ with connectives $\otimes, \parr, \&, \oplus, !, ?$.

formula? term? look up basic terminology

Let Γ, Δ etc. be arbitrary, finite lists of formulas (e.g.: $\Gamma = (p_1, ..., p_n)$) and A and B formulas. As we will later consider fragments without negation, we shall define linear logic with a two-sided calculus. For structural rules we only have the exchange rule, missing the (general) weakening and contraction rules known from classical logic:

$$\frac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, B, A, \Gamma' \vdash \Delta}\,\text{ex.L} \qquad \frac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, B, A, \Delta'}\,\text{ex.R}$$

First, we have the identity and negation rules:

$$\frac{}{A \vdash A}\,\mathrm{id} \qquad \frac{\Gamma \vdash \Delta, A \qquad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{Cut}$$

permanente Lösung für empty axiom finden

As already mentioned, the classical conjunction $\wedge$ and disjunction $\vee$ split into two respectively. These can be classified as multiplicative and additive connectives. The calculus rules for the *multiplicative* conjunction $\otimes$ and disjunction $\parr$ ("par") are as follows:

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma A \otimes B \vdash \Delta}\,\otimes_L \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash \Delta, A \otimes B, \Delta'}\,\otimes_R$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, A \parr B, \Gamma' \vdash \Delta, \Delta'}\,\parr_L \qquad \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \parr B}\,\parr_R$$

Remark 2.1 The rules $\otimes_L$ and $\parr_R$ imply, that the commas are to be read as $\otimes$ on the left-hand side and as $\parr$ on the the right-hand side. That means $A, B \vdash C, D$ is provable iff $A \otimes B \vdash C \parr D$ is provable.

Proof: We only have to show that $A, B \vdash C, D$ follows from $A \otimes B \vdash C \parr D$ as the other direction is just our introduction rule:

$$\dfrac{\dfrac{A \otimes B \vdash C \parr D \qquad \dfrac{\dfrac{}{A \vdash A}\,\mathrm{id} \qquad \dfrac{}{B \vdash B}\,\mathrm{id}}{A, B \vdash A \otimes B}\,\otimes_R}{A, B \vdash C \parr D}\,\mathrm{Cut} \qquad \dfrac{\dfrac{}{C \vdash C} \qquad \dfrac{}{D \vdash D}}{C \parr D \vdash C, D}\,\parr_L}{A, B \vdash C, D} \qquad \square$$

proofsymbol :/

The calculus rules for the *additive* conjunction $\&$ and disjunction $\oplus$ ("plus") are as follows:

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \,\&\, B \vdash \Delta}\,\&_{L1} \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \,\&\, B \vdash \Delta}\,\&_{L2}$$

$$\frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \,\&\, B}\,\&_R$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta}\,\oplus_L$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \oplus B, \Delta}\,\oplus_{R1} \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \oplus B, \Delta}\,\oplus_{R2}$$

Notice the difference between $\&_R$ and $\otimes_R$ (and dually between $\oplus_L$ and $\parr_L$): while for $\otimes_R$ the contexts Γ etc. are arbitrary and get combined in the conclusion, $\&_R$ requires the contexts to be equal. In classical logic these rules would be equivalent using its additional structural rules.

Remark 2.2 Similarly to the multiplicative connectives above, the additive connectives have a invertibility statement: $\Gamma \vdash A \,\&\, B$ is provable iff $\Gamma \vdash A$ and $\Gamma \vdash B$ are provable. (The statement for $\oplus$ is formulated dually)

Proof: As with the multiplicative statement, one direction suffices:

$$\dfrac{\Gamma \vdash A \mathbin{\&} B \qquad \dfrac{\dfrac{}{A \vdash A}}{A \mathbin{\&} B \vdash A}}{\Gamma \vdash A}$$

$\Gamma \vdash B$ follows the same way. ◻

above remark unnötig?

Definition 2.3 *We call two formulas A and B* (linearly) equivalent *iff $A \vdash B$ and $A \vdash B$ are provable.*

Theorem 2.4 *We have the following equivalencies for the linear negation:*

- *For the constants:*

$$\begin{aligned} 1^\perp &\equiv \perp & \perp^\perp &\equiv 1 \\ \top^\perp &\equiv & 0^\perp &\equiv \top \end{aligned}$$

- *The negation is involutory: $(A^\perp)^\perp \equiv A$*
- *The De Morgan equations hold:*

$$\begin{aligned} (A \otimes B)^\perp &\equiv A^\perp \parr B^\perp & (A \parr B)^\perp &\equiv A^\perp \otimes B^\perp \\ (A \mathbin{\&} B)^\perp &\equiv A^\perp \oplus B^\perp & (A \oplus B)^\perp &\equiv A^\perp \mathbin{\&} B^\perp \end{aligned}$$

We define linear implication with the par-operator:

$$A \multimap B := A^\perp \parr B$$

As with classical logic, we can now translate a two-sided sequent $A_1, ..., A_n \vdash B_1, ..., B_n$ into a right-sided sequent $\vdash A_1^\perp, ...A_n^\perp, B_1, ...B_n$ by negation of the left-hand side and vice versa. With this it is easily seen that $\vdash A \multimap B$ iff $A \vdash B$.

Proof: We shall only prove parts as the rest is trivial.

- $1^\perp \vdash \perp$:

$$\dfrac{\dfrac{\dfrac{}{\vdash 1}}{\vdash 1, \perp}}{1^\perp \vdash \perp}$$

- $(A^\perp)^\perp \vdash A$:

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}}{\vdash A^\perp, A}}{(A^\perp)^\perp \vdash A}$$

- $(A \otimes B)^\perp \vdash A^\perp \parr B^\perp$:

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$A \vdash A$}
\UnaryInfC{$\vdash A^{\perp}, A$}
\AxiomC{}
\UnaryInfC{$B \vdash B$}
\UnaryInfC{$\vdash B^{\perp}, B$}
\BinaryInfC{$\vdash A \otimes B, A^{\perp}, B^{\perp}$}
\UnaryInfC{$(A \otimes B)^{\perp} \vdash A^{\perp}, B^{\perp}$}
\UnaryInfC{$(A \otimes B)^{\perp} \vdash A^{\perp} \parr B^{\perp}$}
\end{prooftree}
$$

$\square$

proofsymbol

With this our calculus rules above become quite redundant and we can restrict them on the rules for the right-hand side. However, these redundancies are necessary when dealing with a negation-free fragment of LL.

The modality connectives reintroduce stable truths and with them the the weakening and contraction rules known from classical logic:

$$
\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ ! \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ ?_W \ \text{(weakening)}
$$

$$
\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ ?_D \ \text{(dereliction)} \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ ?_C \ \text{(contraction)}
$$

$$
(!A)^{\perp} \equiv ?\big(A^{\perp}\big) \quad (?A)^{\perp} \equiv !\big(A^{\perp}\big)
$$

move in front of negation theorem and include negation equivalencies there?

Here the Kontext $?\Gamma$ is given by applying the ?-modality to every formula in the list of Γ, i.e. $?\Gamma = ?q, ?p, \dots$ for $\Gamma = q, p \dots$

Definition of $?\Gamma$ korrekt?

Remark 2.5 These modalities are called exponentials because of the following relation:

$$
!(A \mathbin{\&} B) \equiv\, !A \otimes !B, \quad ?(A \oplus B) \equiv\, ?A \parr ?B
$$

definition linear equivalence

Proof: We will only prove the second equivalence. The first is acquired by duality.

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\vdash A, A^{\perp}$}
\RightLabel{$\oplus$}
\UnaryInfC{$A^{\perp}, A \oplus B$}
\RightLabel{$?_D$}
\UnaryInfC{$\vdash A^{\perp}, ?(A \oplus B)$}
\RightLabel{$!$}
\UnaryInfC{$\vdash !(A^{\perp}), ?(A \oplus B)$}
\AxiomC{}
\UnaryInfC{$\vdash B, B^{\perp}$}
\RightLabel{$\oplus$}
\UnaryInfC{$\vdash B^{\perp}, A \oplus B$}
\RightLabel{$?_D$}
\UnaryInfC{$\vdash B^{\perp}, ?(A \oplus B)$}
\RightLabel{$!$}
\UnaryInfC{$\vdash !(B^{\perp}), ?(A \oplus B)$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash !(A^{\perp}) \otimes !(B^{\perp}), ?(A \oplus B), ?(A \oplus B)$}
\RightLabel{$?_D$}
\UnaryInfC{$\vdash !(A^{\perp}) \otimes !(B^{\perp}), ?(A \oplus B)$}
\RightLabel{$(-)^{\perp}$}
\UnaryInfC{$\vdash (?A \parr ?B)^{\perp}, ?(A \oplus B)$}
\UnaryInfC{$\vdash ?A \parr ?B \multimap ?(A \oplus B)$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\vdash A^{\perp}, A$}
\UnaryInfC{$\vdash A^{\perp}, ?A$}
\RightLabel{$?_W$}
\UnaryInfC{$\vdash A^{\perp}, ?A, ?B$}
\AxiomC{}
\UnaryInfC{$\vdash B^{\perp}, B$}
\UnaryInfC{$\vdash B^{\perp}, B$}
\RightLabel{$?_W$}
\UnaryInfC{$\vdash B^{\perp}, ?B, ?A$}
\RightLabel{$\&$}
\BinaryInfC{$\vdash A^{\perp} \mathbin{\&} B^{\perp}, ?A, ?B$}
\UnaryInfC{$\vdash !(A^{\perp} \mathbin{\&} B^{\perp}), ?A, ?B$}
\UnaryInfC{$\vdash !(A^{\perp} \mathbin{\&} B^{\perp}), ?A \parr ?B$}
\UnaryInfC{$\vdash ?(A \oplus B) \multimap ?A \parr ?B$}
\end{prooftree}
$$

$\square$

proofsymbol なんで？:(

MAYBE to be added: \forall and \exist. Do we want predicate logic?

2.3 Proof-structures and proof-nets

We shall now leave behind any capitalistic resource interpretation and concentrate on what linear logic was invented for: analyzing proofs.

to be continued...

how to typeset axiom links? :(

3 Categorical Preliminaries

3.1 Categories

sources for basic definitions. (nlab prolly doesn't suffice)

Definition 3.1 (Category) *A Category $\mathfrak{C}$ consists of the following data:*

- *A class* $\mathrm{Obj}(\mathfrak{C})$ *of* objects*.*
- *For every pair of objects $A, B \in \mathrm{Obj}(\mathfrak{C})$ there is a class $\mathrm{Hom}(A, B)$ of* morphisms *$f : A \to B$ from A to B.*
- *Morphisms* compose*: For $f \in \mathrm{Hom}(A, B)$ and $g \in \mathrm{Hom}(B, C)$ there is a morphism $g \circ f \in \mathrm{Hom}(A, C)$. That composition is associative:*

  $$h \circ (g \circ f) = (h \circ g) \circ f$$

  We will write gf for $g \circ f$ when appropriate.
- *For every object A there is an* identity morphism *$\mathrm{id}_A \in \mathrm{Hom}(A, A)$:*

  $$f \circ \mathrm{id}_A = f, \quad \mathrm{id}_B \circ f = f$$

  for $f \in \mathrm{Hom}(A, B)$

If the hom-classes are sets, we call the category locally small*. If the object class is a set, we call the category* small*. Otherwise, we call the category* large*.*

If we have a category $\mathfrak{C}$, we call its opposite *category $\mathfrak{C}^{\mathrm{opp}}$ the category with the following structure:*

- $\mathrm{Obj}\left(\mathfrak{C}^{\mathrm{opp}}\right) = \mathrm{Obj}(\mathfrak{C})$
- $\forall A, B \in \mathrm{Obj}(\mathfrak{C}) : \mathrm{Hom}_{\mathfrak{C}^{\mathrm{opp}}}(A, B) = \mathrm{Hom}_{\mathfrak{C}}(B, A)$

extra definition for opposite structure?

Definition 3.2 (Functor) *A* (covariant) functor $\mathsf{F} : \mathfrak{C} \to \mathfrak{D}$ *between two categories* $\mathfrak{C}$ *and* $\mathfrak{D}$ *is a function mapping each object* $A \in \mathrm{Obj}(\mathfrak{C})$ *to an object* $F(A) \in \mathrm{Obj}(D)$ *and each morphism* $f \in \mathrm{Hom}(A, B)$ *to a morphism* $\mathsf{F}(f) \in \mathrm{Hom}(\mathsf{F}(A), \mathsf{F}(B))$ *such that identity and composition are preserved:*

$$\mathsf{F}(\mathrm{id}_A) = \mathrm{id}_{\mathsf{F}(A)}, \quad \mathsf{F}(g \circ f) = \mathsf{F}(g) \circ \mathsf{F}(f)$$

A functor $\mathsf{F} : \mathfrak{C}^{\mathrm{opp}} \to \mathfrak{D}$ *is called* contravariant *on* $\mathfrak{C}$.

We will drop the parentheses when appropriate.

Definition 3.3 (Natural transformation) *A* natural transformation $\tau : F \to G$ *between two functors* $\mathsf{F}, \mathsf{G} : \mathfrak{C} \to \mathfrak{D}$ *is family of morphisms in* $\mathfrak{D}$*:*

$$\tau = \{\tau_A : \mathsf{F}A \to \mathsf{G}A \,|\, A \in \mathrm{Obj}(\mathfrak{C})\}$$

such that $\tau_B \mathsf{F}(f) = \mathsf{F}(f)\tau_A$ *for all* $f : A \to B \in \mathrm{Morph}(\mathfrak{C})$*, i.e. the following diagram commutes:*

$$\begin{tikzcd}
\mathsf{F}A \arrow[r, "\mathsf{F}f"] \arrow[d, "\tau_A"] & \mathsf{F}B \arrow[d, "\tau_B"] \\
\mathsf{G}A \arrow[r, "\mathsf{G}f"] & \mathsf{G}B
\end{tikzcd}$$

If τ_A *is an isomorphism for all* $A \in \mathrm{Obj}(\mathfrak{C})$*, we call* τ *a* natural isomorphism. *We will often represent a natural transformation by a single one of its members and also drop the index when appropriate.*

definition: equivalence of categories

Definition 3.4 (Adjunction) *Let* $\mathfrak{C}$ *and* $\mathfrak{D}$ *be categories. We call* $\mathsf{F} : \mathfrak{C} \to \mathfrak{D}$ *the* left adjoint *of* $\mathsf{G} : \mathfrak{D} \to \mathfrak{C}$ *and* G *the* right adjoint *of* F *if there is an isomorphism that is natural in* $A \in \mathfrak{C}$ *and* $B \in \mathfrak{D}$*:*

source: Brandenburg

$$\mathrm{Hom}_{\mathfrak{D}}(\mathsf{F}A, B) \cong \mathrm{Hom}_{\mathfrak{C}}(A, \mathsf{G}B)$$

Equivalently, we call F *the left adjoint of* G *if there are natural transformations*

$$\eta : \mathrm{id}_{\mathfrak{C}} \to \mathsf{G} \circ \mathsf{F}$$
$$\varepsilon : \mathsf{F} \circ \mathsf{G} \to \mathrm{id}_{\mathfrak{D}}$$

fulfilling the following condition:

$$\mathrm{id}_{\mathsf{F}A} = \varepsilon(\mathsf{F}A) \circ \mathsf{F}\eta(A)$$
$$\mathrm{id}_{\mathsf{G}B} = \mathsf{G}\varepsilon(B) \circ \eta(\mathsf{G}B)$$

i.e. the following diagrams commute:

maybe index Schreibweise?

$$
\begin{tikzcd}
 & \mathsf{F} \circ \mathsf{G} \circ \mathsf{F} \arrow[dr, "\varepsilon \bullet \mathsf{F}"] & \\
\mathsf{F}\,\mathrm{id}_{\mathfrak{C}} \arrow[ur, "\mathsf{F} \bullet \eta"] \arrow[rr, equal, "\mathrm{id}"] & & \mathrm{id}_{\mathfrak{C}} \circ \mathsf{F}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathsf{G} \circ \mathsf{F} \circ \mathsf{G} \arrow[dr, "\mathsf{G} \bullet \varepsilon"] & \\
\mathrm{id}_{\mathfrak{C}} \circ \mathsf{G} \arrow[ur, "\eta \bullet \mathsf{G}"] \arrow[rr, equal, "\mathrm{id}"] & & \mathsf{G} \circ \mathrm{id}_{\mathfrak{D}}
\end{tikzcd}
$$

cut triangle diagrams or define •

We write $F \dashv G$.

Definition 3.5 (Monoidal category) *A* monoidal category $(\mathfrak{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ *is a category $\mathfrak{C}$ with the following additional data:*

- *a functor $\otimes : \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$, called* monoidal *or* tensor product*;*
- *an object $\mathbb{1} \in \mathrm{Obj}(\mathfrak{C})$, called* unit object*;*
- *a natural isomorphism $\alpha : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$, called* associator*;*
- *a natural isomorphism $\lambda : \mathbb{1} \otimes A \to A$, called left unitor;*
- *a natural isomorphism $\rho : A \otimes \mathbb{1} \to A$, called right unitor;*

such that the following diagrams commute:

- *the pentagon diagram:*

$$
\begin{tikzcd}
 & A \otimes (B \otimes (C \otimes D)) \arrow[dl, "\mathrm{id} \otimes \alpha"'] \arrow[dr, "\alpha"] & \\
A \otimes ((B \otimes C) \otimes D) \arrow[d, "\alpha"] & & (A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha"] \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha \otimes \mathrm{id}"] & & ((A \otimes B) \otimes C) \otimes D
\end{tikzcd}
$$

- *the unit diagram:*

$$
\begin{tikzcd}
(A \otimes \mathbb{1}) \otimes B \arrow[rr, "\alpha"] \arrow[dr, "\rho \otimes \mathrm{id}"'] & & A \otimes (\mathbb{1} \otimes B) \arrow[dl, "\mathrm{id} \otimes \lambda"] \\
 & A \otimes B &
\end{tikzcd}
$$

A symmetric monoidal category *has an additional natural isomorphism:*

$$s_{A,B} : A \otimes B \to B \otimes A$$

satisfying the following conditions:

- *the hexagon law:*

$$
\begin{tikzcd}
A \otimes (B \otimes C) \arrow[r, "\alpha"] \arrow[d, "\mathrm{id} \otimes s_{B,C}"] & (A \otimes B) \otimes C \arrow[r, "s_{A \otimes B, C}"] & C \otimes (A \otimes B) \arrow[d, "\alpha"] \\
A \otimes (C \otimes B) \arrow[r, "\alpha"'] & (A \otimes C) \otimes B \arrow[r, "s_{A,C} \otimes B"'] & (C \otimes A) \otimes B
\end{tikzcd}
$$

- *the inverse law: $s_{A,B}^{-1} = s_{B,A}$*
- *the unit law: $\lambda = \rho s$*

$$\begin{tikzcd}
\mathbb{1} \otimes A \arrow[rr, "s_{\mathbb{1},A}"] \arrow[dr, "\lambda"'] & & A \otimes \mathbb{1} \arrow[dl, "\rho"] \\
& A &
\end{tikzcd}$$

Diagramme verschönern

fix associators throughout the document (direction)

Definition 3.6 (Closed monoidal category) *A monoidal category $\mathfrak{C}$ is called* right closed*, resp.* left closed*, iff there is a functor $- \multimap - : \mathfrak{C}^{\text{opp}} \times \mathfrak{C} \to \mathfrak{C}$, resp. $- \multimapinv - : \mathfrak{C} \times \mathfrak{C}^{\text{opp}} \to \mathfrak{C}$, such that there is a natural isomorphism* $\operatorname{Hom}(A \otimes B, C) \cong \operatorname{Hom}(B, A \multimap C)$*, resp.* $\operatorname{Hom}(A \otimes B, C) \cong \operatorname{Hom}(A, C \multimapinv B)$*, i.e. iff the functor $- \otimes B$, resp. $A \otimes -$, has a right adjoint. A monoidal $\mathfrak{C}$ is called* biclosed *iff it is both left and right closed.*

lengthy, rambling... muss eleganter gehen

definitely mixed up the right and left homs there

If $\mathfrak{C}$ is symmetric, these functors coincide and we call the category closed.

The functors $- \multimap - : \mathfrak{C}^{\text{opp}} \times \mathfrak{C} \to \mathfrak{C}$ and $- \multimapinv - : \mathfrak{C} \times \mathfrak{C}^{\text{opp}} \to \mathfrak{C}$ are called internal hom *functors.*

following definitions prolly unnötig:

Definition 3.7 (Enriched category) *Let $(\mathfrak{M}, \mathbb{1}, \alpha, \lambda, \rho)$ be a monoidal category. An* $\mathfrak{M}$-enriched category $\mathfrak{C}$ *consists of the following data:*

- *A class* $\operatorname{Obj}(\mathfrak{C})$ *of objects;*
- *For every pair of objects $(A, B) \in \operatorname{Obj}(\mathfrak{C}) \times \operatorname{Obj}(\mathfrak{C})$ an object $\mathfrak{C}(A, B) \in \operatorname{Obj}(\mathfrak{M})$ called* hom object*;*
- *For every triple of objects $(A, B, C) \in \operatorname{Obj}(\mathfrak{C}) \times \operatorname{Obj}(\mathfrak{C}) \times \operatorname{Obj}(\mathfrak{C})$ a morphism $\operatorname{Morph}(\mathfrak{M}) \ni \circ_{A,B,C} : \mathfrak{C}(B, C) \otimes \mathfrak{C}(A, B) \to \mathfrak{C}(A, C)$ called* composition morphism*;*
- *For every object $A \in \mathfrak{C}$ a morphism $\mathrm{id}_A : \mathbb{1} \to \mathfrak{C}(A, A)$;*

such that the following diagrams commute:

$$\begin{tikzcd}
(\mathfrak{C}(C,D) \otimes \mathfrak{C}(B,C)) \otimes \mathfrak{C}(A,B) \arrow[rr, "\alpha"] \arrow[d, "\circ_{B,C,D} \otimes \mathrm{id}_{\mathfrak{C}(A,B)}"] & & \mathfrak{C}(C,D) \otimes (\mathfrak{C}(B,C) \otimes \mathfrak{C}(A,B)) \arrow[d, "\mathrm{id}_{\mathfrak{C}(C,D)} \otimes \circ_{A,B,C}"] \\
\mathfrak{C}(D,B) \otimes \mathfrak{C}(A,B) \arrow[dr, "\circ_{A,B,D}"'] & & \mathfrak{C}(C,D) \otimes \mathfrak{C}(A,C) \arrow[dl, "\circ_{A,C,D}"] \\
& \mathfrak{C}(A,D) &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1} \otimes \mathfrak{C}(A,B) \arrow[r, "\lambda"] \arrow[d, "\mathrm{id}_B \otimes \mathrm{id}_{\mathfrak{C}(A,B)}"'] & \mathfrak{C}(A,B) & \mathfrak{C}(A,B) \otimes \mathbb{1} \arrow[l, "\rho"'] \arrow[d, "\mathrm{id}_{\mathfrak{C}(A,B)} \otimes \mathrm{id}_A"] \\
\mathfrak{C}(B,B) \otimes \mathfrak{C}(A,B) \arrow[ur, "\circ_{A,B,B}"'] & & \mathfrak{C}(A,B) \otimes \mathfrak{C}(A,A) \arrow[ul, "\circ_{A,A,B}"]
\end{tikzcd}$$

Definition 3.8 (Enriched functors) *Let $\mathfrak{C}$ and $\mathfrak{D}$ be $\mathfrak{M}$-enriched categories. An* enriched functor $\mathsf{F} : \mathfrak{C} \to \mathfrak{D}$ *consists of:*

- *a function* $\mathsf{F}_0 : \mathrm{Obj}(\mathfrak{C}) \to \mathrm{Obj}(\mathfrak{D})$ *between the objects*
- *a family of morphisms in* $\mathfrak{M}$*:* $\mathsf{F}_{A,B} : \mathfrak{C}(A, B) \to \mathfrak{D}(\mathsf{F}_0 A, \mathsf{F}_0 B)$

such that the following diagrams commute:

$$\begin{tikzcd}
\mathfrak{C}(B,C) \otimes \mathfrak{C}(A,B) \arrow[r, "\circ"] \arrow[d, "\mathsf{F}_{B,C} \otimes \mathsf{F}_{A,B}"] & \mathfrak{C}(A,C) \arrow[d, "\mathsf{F}_{A,C}"] \\
\mathfrak{D}(\mathsf{F}_0 B, \mathsf{F}_0 C) \otimes \mathfrak{D}(\mathsf{F}_0 A, \mathsf{F}_0 B) \arrow[r, "\circ"] & \mathfrak{D}(\mathsf{F}_0 A, \mathsf{F}_0 C)
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathbb{1} \arrow[dl, "\mathrm{id}_A"'] \arrow[dr, "\mathrm{id}_{\mathsf{F}_0 A}"] & \\
\mathfrak{C}(A,A) \arrow[rr, "\mathsf{F}_{A,A}"] & & \mathfrak{D}(\mathsf{F}_0 A, \mathsf{F}_0 A)
\end{tikzcd}$$

3.2 LinDisCats

Definition 3.9 (Linearly distributive category) *A* linearly distributive category $(\mathfrak{C}, \otimes, 1, \parr, \bot)$ *is a category $\mathfrak{C}$ consisting of:*

- *a monoidal category $(\mathfrak{C}, \otimes, 1, \alpha_\otimes, \lambda_\otimes, \rho_\otimes)$, with $\otimes$ called "tensor";*
- *a monoidal category $(\mathfrak{C}, \parr, \bot, \alpha_\parr, \lambda_\parr, \rho_\parr)$, with $\parr$ called "par";*
- *two natural transformations called left and right* linear distributors *respectively:*

$$\begin{aligned}
\partial_L &: A \otimes (B \parr C) \to (A \otimes B) \parr C \\
\partial_R &: (A \parr B) \otimes C \to A \parr (B \otimes C)
\end{aligned}$$

satisfying the following coherence conditions:

- *coherence between the distributors and unitors:*

$$\begin{tikzcd}
1 \otimes (A \parr B) \arrow[rr, "\partial_L"] \arrow[dr, "\lambda_\otimes"'] & & (1 \otimes A) \parr B \arrow[dl, "\lambda_\otimes \parr \mathrm{id}"] \\
& A \parr B &
\end{tikzcd}$$

$$\begin{tikzcd}
(A \parr B) \otimes 1 \arrow[rr, "\partial_R"] \arrow[dr, "\rho_\otimes"'] & & A \parr (B \otimes 1) \arrow[dl, "\mathrm{id} \otimes \rho_\otimes"] \\
& A \parr B &
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes (B \parr \bot) \arrow[rr, "\partial_L"] \arrow[dr, "\mathrm{id} \otimes \rho_\parr"'] & & (A \otimes B) \parr \bot \arrow[dl, "\rho_\parr"] \\
& A \otimes B &
\end{tikzcd}$$

$$\begin{tikzcd}
(\bot \parr A) \otimes B \arrow[rr, "\partial_R"] \arrow[dr, "\lambda_\parr \otimes \mathrm{id}"'] & & \bot \parr (A \otimes B) \arrow[dl, "\lambda_\parr"] \\
& A \otimes B &
\end{tikzcd}$$

Diagramme besser alignen

- *coherence between distributors and associators:*

$$
\begin{tikzcd}
 & (A \otimes B) \otimes (C \parr D) \arrow[dl, "\alpha_\otimes"'] \arrow[dr, "\partial_L"] & \\
A \otimes (B \otimes (C \parr D)) \arrow[d, "\mathrm{id} \otimes \partial_L"] & & ((A \otimes B) \otimes C) \parr D \arrow[d, "\alpha_\otimes \parr \mathrm{id}"] \\
A \otimes ((B \otimes C) \parr D) \arrow[rr, "\partial_L"] & & (A \otimes (B \otimes C)) \parr D
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & A \otimes (B \parr (C \parr D)) \arrow[dl, "\mathrm{id} \otimes \alpha_\parr"'] \arrow[dr, "\partial_L"] & \\
A \otimes ((B \parr C) \parr D) \arrow[d, "\partial_L"] & & (A \otimes B) \parr (C \parr D) \arrow[d, "\alpha_\parr"] \\
(A \otimes (B \parr C)) \parr D \arrow[rr, "\partial_L \parr \mathrm{id}"] & & ((A \otimes B) \parr C) \parr D
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & (A \parr B) \otimes (C \otimes D) \arrow[dl, "\alpha_\otimes"'] \arrow[dr, "\partial_R"] & \\
((A \parr B) \otimes C) \otimes D \arrow[d, "\partial_R \otimes \mathrm{id}"] & & A \parr (B \otimes (C \otimes D)) \arrow[d, "\mathrm{id} \parr \alpha_\otimes"] \\
(A \parr (B \otimes C)) \otimes D \arrow[rr, "\partial_R"] & & A \parr ((B \otimes C) \otimes D)
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & ((A \parr B) \parr C) \otimes D \arrow[dl, "\alpha_\parr \otimes \mathrm{id}"'] \arrow[dr, "\partial_R"] & \\
(A \parr (B \parr C)) \otimes D \arrow[d, "\partial_R"] & & (A \parr B) \parr (C \otimes D) \arrow[d, "\alpha_\parr"] \\
A \parr ((B \parr C) \otimes D) \arrow[rr, "\mathrm{id} \parr \partial_R"] & & A \parr (B \parr (C \otimes D))
\end{tikzcd}
$$

- *coherence between the distributors:*

$$
\begin{tikzcd}
 & (A \parr B) \otimes (C \parr D) \arrow[dl, "\partial_R"'] \arrow[dr, "\partial_L"] & \\
((A \parr B) \otimes C) \parr D \arrow[d, "\partial_R \parr \mathrm{id}"] & & A \parr (B \otimes (C \parr D)) \arrow[d, "\mathrm{id} \parr \partial_L"] \\
(A \parr (B \otimes C)) \parr D & & A \parr ((B \otimes C) \parr D) \arrow[ll, "\alpha_\parr"]
\end{tikzcd}
$$

obiges Diagramm rumdrehen

\parr \id sieht kacke aus...

$$\begin{tikzcd}
 & A \otimes ((B \parr C) \otimes D) \arrow[dl, "\alpha_\otimes"'] \arrow[dr, "\mathrm{id} \otimes \partial_R"] & \\
(A \otimes (B \parr C)) \otimes D \arrow[d, "\partial_L \otimes \mathrm{id}"] & & A \otimes (B \parr (C \otimes D)) \arrow[d, "\partial_L"] \\
((A \otimes B) \parr C) \otimes D \arrow[rr, "\partial_R"] & & (A \otimes B) \parr (C \otimes D)
\end{tikzcd}$$

itemize anpassen?

3.3 *-aut. cats

structure an Mellies, CatSemOfLinLog, anpassen; equivalence Srinivasan and Barr definitions

Definition 3.10 (Dual object) *Let $\mathfrak{C}$ be a linearly distributive category and $A, A^* \in \mathrm{Obj}(\mathfrak{C})$, then we call A^** left dual *(or* left linearly adjoint*) to A, if there are morphisms $\tau : 1 \to A^* \parr A$ and $\gamma : A \otimes A^* \to \bot$, called* unit *and* counit *resp., such that the following diagrams commute:*

$$\begin{tikzcd}
A^* \arrow[r, equal, "\mathrm{id}"] & A^* \arrow[d, "\lambda_\otimes^{-1}"] \\
A^* \parr \bot \arrow[u, "\rho_\parr"] & 1 \otimes A^* \arrow[d, "\tau \otimes \mathrm{id}"] \\
A^* \parr (A \otimes A^*) \arrow[u, "\mathrm{id} \parr \gamma"] & (A^* \parr A) \otimes A^* \arrow[l, "\partial_R"]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, equal, "\mathrm{id}"] & A \arrow[d, "\rho_\otimes^{-1}"] \\
\bot \parr A \arrow[u, "\lambda_\parr"] & A \otimes 1 \arrow[d, "\mathrm{id} \otimes \tau"] \\
(A^* \parr A) \otimes A^* \arrow[u, "\gamma \parr \mathrm{id}"] & A^* \parr (A \otimes A^*) \arrow[l, "\partial_L"]
\end{tikzcd}$$

das geht schöner

We write $(\tau, \gamma) : A^* \dashV A$ and also call A right dual *to A^*.*

mathabx incompatible with amssymb, stix absolutely broken; repalce dashV with dashv + some index?

Lemma 3.11 1. *In an LDC: if $A^* \dashV A$ and $A' \dashV A$, then A^* and A' are isomorphic. We will from now on only talk about* the dual object*, when equality up to isomorphism is sufficient.*

2. *In a symmetric LDC: $(\tau, \gamma) : A^* \dashV A \iff (\tau s_\parr, s_\otimes \gamma) : A \dashV A^*$*

proof

[Sri23] Lemma 2.9 (iii)?

Definition 3.12 (*-autonomous categories (Srinivasan [Sri23])) *An LDC $\mathfrak{C}$ in which for every Object $A \in \mathfrak{C}$ there exists a left and right dual, resp. $(\tau*, \gamma*) : A^* \dashV A$ and $(*\tau, *\gamma) : A \dashV {}^*A$, is called* *-autonomous category*.*

is Srinivasan the origin of this version?

Definition 3.13 (*-autonomous categories (Barr [Bar95], A)) *A *-autonomous category is a biclosed monoidal category $\mathfrak{C}$ with a closed functor $(-)^* : \mathfrak{C} \to \mathfrak{C}^{\mathrm{opp}}$, which is a strong equivalence of categories.*

find a way to make letter part of definition num-

definition closed functor, strong equivalence. (strong closed functor?)

details (Barr, 99)

Definition 3.14 (Dualizing object) *Let $\mathfrak{C}$ be a biclosed monoidal category. An object $\perp$ is called a dualizing object if for every $A \in \mathrm{Obj}(\mathfrak{C})$ the natural map $A \to \perp \multimapinv (A \multimap \perp)$ gotten by transposing $\mathrm{id} : A \multimap \perp \to A \multimap \perp$ twice is an isomorphism.*

Definition 3.15 (*-autonomous categories (Barr, B)) *A *-autonomous category is a biclosed category with a dualizing object.*

main definition, Barr

Definition 3.16 (*-autonomous categories (Barr, C)) *A *-autonomous category is a monoidal category $\mathfrak{C}$ equipped with an equivalence $(-)^* : \mathfrak{C} \to \mathfrak{C}^{\mathrm{opp}}$ such that there is a natural isomorphism*

$$\mathrm{Hom}(A, B^*) \to \mathrm{Hom}(1, (A \oplus B)^*)$$

Definition 3.17 (*-autonomous categories (Barr, D)) *A *-autonomous category is a closed category $\mathfrak{C}$ in the sense of Eilenberg and Kelly (1966) together with an equivalence $(-)^* : \mathfrak{C}^{\mathrm{opp}} \to \mathfrak{C}$ such that*

nachschauen

$$A \multimap (B \multimapinv C) \cong (A \multimap B) \multimapinv C$$

where $A \multimapinv B := A^* \multimap B^*$.

unnötig? C impliziert offensichtlich D, von D nach C sind drei Zeilen.

Theorem 3.18 *Barr's definitions are equivalent.*

Proof:

to be added ([Bar95])

Theorem 3.19 *Srinivasan's definition is equivalent to Barr's definitions.*

Weihnachtsbaustelle:

Proof: Let $\mathfrak{A}$ be a symmetric LDC satisfying definition 3.12.
The mapping $A \mapsto A^*$ is an isomorphism $\mathrm{Obj}(\mathfrak{A}) \to \mathrm{Obj}(\mathfrak{A}^{\mathrm{opp}})$ between objects and defines a Functor by extending it to morphisms with the following diagram:

change definition to diagram?

$$\mathrm{Hom}_{\mathfrak{A}}(A, B) \ni f \mapsto f^* \in \mathrm{Hom}_{\mathfrak{A}^{\mathrm{opp}}}(A^*, B^*) = \mathrm{Hom}_{\mathfrak{A}}(B^*, A^*)$$

$$
\begin{aligned}
f^* : B^* &\xrightarrow{\lambda_\otimes^{-1}} 1 \otimes B^* \xrightarrow{\eta_A \otimes \mathrm{id}} (A^* \parr A) \otimes B^* \xrightarrow{(\mathrm{id} \parr f) \otimes \mathrm{id}} (A^* \parr B) \otimes B^* \\
&\xrightarrow{\partial_R} A^* \parr (B \otimes B^*) \xrightarrow{\mathrm{id} \parr \varepsilon_B} A^* \parr \bot \xrightarrow{\rho_\parr} A^*
\end{aligned}
$$

This mapping of morphisms is also a bijection:

$$
\begin{aligned}
f =& \lambda_{\parr,B} \circ (\varepsilon_A \parr \mathrm{id}_B) \circ ((\mathrm{id}_A \otimes f^*) \parr \mathrm{id}_B) \circ \partial_L \circ (\mathrm{id}_A \otimes \eta_B) \circ \rho_{\otimes,A}^{-1} \\
=& \lambda_{\parr,B} \circ (\varepsilon_A \parr \mathrm{id}_B) \\
&\circ \Big(\Big(\mathrm{id}_A (\otimes \rho_{\parr,A} \circ (\mathrm{id}_{A^*} \parr \varepsilon_B) \circ (\mathrm{id}_{A^*} \parr (f \otimes \mathrm{id}_{B^*})) \circ \partial_R \circ (\eta_A \otimes \mathrm{id}) \circ \lambda_{\otimes,B}^{-1}) \Big) \parr \mathrm{id}_{B^*} \Big) \\
&\circ \partial_L \circ (\mathrm{id}_A \otimes \eta_B) \circ \rho_{\otimes,A}^{-1}
\end{aligned}
$$

(maybe suppress brackets for stuff like $A \otimes 1 \otimes 1$)

$$
\begin{tikzcd}
A \arrow[r, "(\mathrm{id} \otimes \eta_B) \circ \rho_\otimes^{-1}"] \arrow[d, "\rho_\otimes^{-1} \circ \rho_\otimes^{-1}"] \arrow[dddddr, bend left, "f"] & A \otimes (B^* \otimes B) \arrow[r, "\mathrm{id} \otimes ((\eta_A \otimes \mathrm{id}) \parr \mathrm{id})"] & A \otimes (((A^* \parr A) \otimes B^*) \parr B) \arrow[d, "\mathrm{id} \otimes (((\mathrm{id} \parr f) \otimes \mathrm{id}) \parr \mathrm{id})"'] \\
(A \otimes 1) \otimes 1 \arrow[d, "\mathrm{id} \otimes \eta_A \otimes \eta_B"] & & A \otimes (((A^* \parr B) \otimes B^*) \parr B) \arrow[d, "\mathrm{id} \otimes (\partial_R \parr \mathrm{id})"'] \\
(A \otimes (A^* \parr A)) \otimes (B^* \parr B) \arrow[d, "\partial_L \otimes \mathrm{id}"] & & A \otimes ((A^* \parr (B \otimes B^*)) \parr B) \arrow[d, "\mathrm{id} \otimes (\partial_R \parr \mathrm{id})"'] \\
((A \otimes A^*) \parr A) \otimes (B^* \parr B) \arrow[d, "(\lambda_\parr \otimes \mathrm{id}) \circ ((\varepsilon_A \parr f) \otimes \mathrm{id})"] & & A \otimes ((A^* \parr (B \otimes B^*)) \parr B) \arrow[d, "\mathrm{id} \otimes (\rho_\parr \circ (\mathrm{id} \parr \varepsilon_B \parr \mathrm{id}))"'] \\
B \otimes (B^* \parr B) \arrow[r, "\lambda_\parr \circ \varepsilon_B \circ \partial_L"'] & B & A \otimes (A^* \parr B) \arrow[l, "\lambda_\parr \circ \varepsilon_B \circ \partial_L"]
\end{tikzcd}
$$

does this commute?

$$
\begin{aligned}
A &\xrightarrow{\rho_\otimes^{-1}} A \otimes 1 \xrightarrow{\mathrm{id} \otimes \eta_B} A \otimes (B^* \parr B) \xrightarrow{\partial_L} (A \otimes B^*) \parr B \\
&\xrightarrow{(\mathrm{id} \otimes f^*) \parr \mathrm{id}} (A \otimes A^*) \parr B \xrightarrow{\varepsilon_A \parr \mathrm{id}} \bot \parr B \xrightarrow{\lambda_\parr} B
\end{aligned}
$$
□

is this really f?

non-symmetric case to be added (should already be in there, never used symmetry)

what did I need Barr's definition for, again?

Cats as LinLog semantics and LinLog syntax as cat

add refs

References

[Bar95] Michael Barr. “Nonsymmetric $*$-autonomous categories”. In: *Theoretical Computer Science* 139.1 (1995), pp. 115–130. ISSN: 0304-3975. DOI: `https://doi.org/10.1016/0304-3975(94)00089-2`. URL: `https://www.sciencedirect.com/science/article/pii/0304397594000892`.

[Gir87] Jean-Yves Girard. “Linear logic”. In: *Theoretical Computer Science* 50.1 (1987), pp. 1–101. ISSN: 0304-3975. DOI: `https://doi.org/10.1016/0304-3975(87)90045-4`. URL: `https://www.sciencedirect.com/science/article/pii/0304397587900454`.

[Sri23] Priyaa Srinivasan. “Dagger linear logic and categorical quantum mechanics”. PhD thesis. Mar. 2023.

Todo list