# Columbia University
**MATH G6071 Spring 2015**
**Numerical Methods in Finance**
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**Topic: PDE (outline)**

The purpose of this document is to provide a brief class note for the topic concerning PDE and its application in quantitative finance.

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# INTRODUCTION

## OUTLINE

PDE has its historical importance in the subject, and it provides an angle to understand derivative pricing without Stochastic Calculus. Main reference: **[Seydel]**

## REVIEW OF THE BLACK-SCHOLES PDE

A reference to arrive at the Black Scholes PDE is [Hull] page 291 section 13.6.

The standard point is the stochastic differential equation that defines the dynamics of the stock:

$\frac{dS}{S} = (r-\delta)\,dt + \sigma dW$ Then one can derive the Black Scholes PDE:

$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta) S \frac{\partial V}{\partial S} - rV = 0$ Note that this PDE holds if we assume that the option will not be early exercised locally. Note also that this PDE is of Euler type.

## REDUCING TO THE HEAT EQUATION

Here is the heat equation that we are interested in: $\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$ for $t > 0$, $-\infty < x < \infty$ with initial data $u(x,0) = u_0(x)$ and $u \to 0$ as $x \to \pm\infty$. The solution of this PDE would depend on boundary conditions. In the class we will detail the transformation required to turn the BS PDE into the heat equation: $S = Ke^x$, $t = T - \frac{2\tau}{\sigma^2}$, $q = \frac{2r}{\sigma^2}$, $q_\delta = \frac{2(r-\delta)}{\sigma^2}$, $y(x,\tau) = \frac{1}{Ke^{\left\{\frac{-1}{2}[q_\delta - 1]x - \left[\frac{1}{4}[q_\delta - 1]^2 + q\right]\tau\right\}}} v(x,\tau)$, we get to the simplified PDE $\frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial x^2}$.

This PDE can have solutions of different form, for example, you can verify that both $u(x,t) = \frac{1}{2\sqrt{\pi t}} e^{\frac{-x^2}{4t}}$ and $u(x,t) = \sin(nx) e^{-n^2 t}$ satisfy the PDE $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$.

Hence it is important to figure out the boundary conditions.

### OBSERVATIONS ABOUT THE CHANGE OF VARIABLES

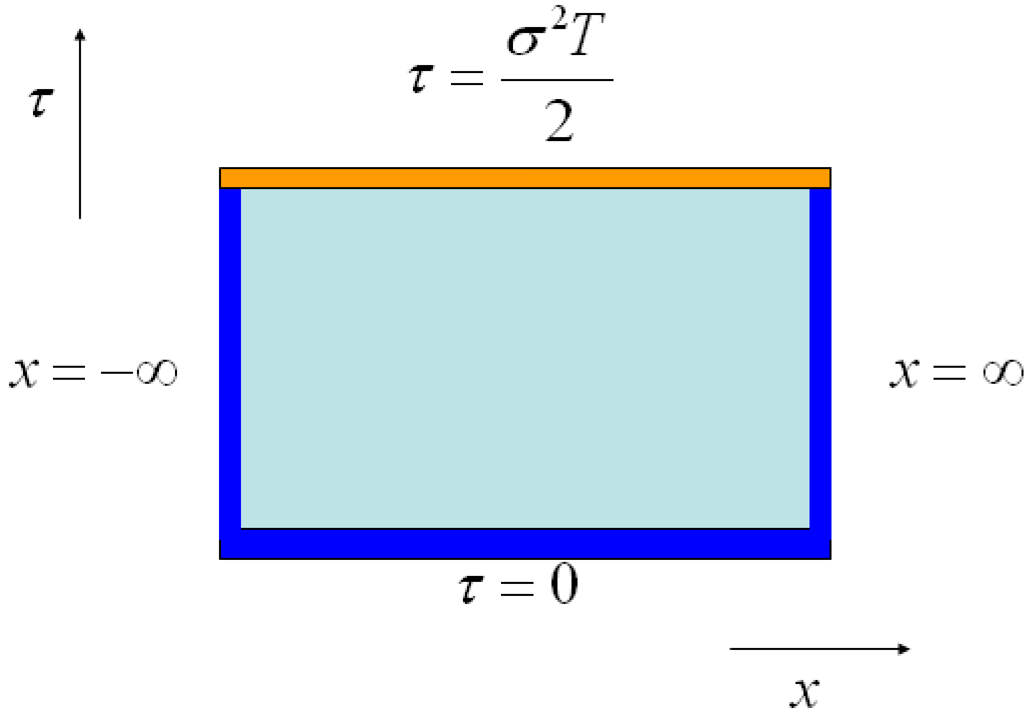
- Time $\tau$ is reversed and, of finite domain and dimensionless
- The space variable $x = \ln\left(\frac{S}{K}\right)$ also because dimensionless

## BOUNDARY CONDITION FOR EUROPEAN OPTIONS

We will discuss in class the transformed boundary conditions for European calls and puts

| | $x \to -\infty$ | $x \to \infty$ | $\tau = 0$ |
|---|---|---|---|
| $y(x,0)$ | $S = 0$ | $S \to \infty$ | $t = T$ |
| Call Value | 0 | $e^{\frac{x}{2}[q_\delta+1] + \frac{\tau}{4}[q_\delta+1]^2}$ | $\max\left(e^{\left\{\frac{x}{2}[q_\delta+1]\right\}} - e^{\left\{\frac{x}{2}[q_\delta-1]\right\}}, 0\right)$ |
| Put Value | $e^{\frac{x}{2}[q_\delta-1] + \frac{\tau}{4}[q_\delta-1]^2}$ | 0 | $\max\left(e^{\left\{\frac{x}{2}[q_\delta-1]\right\}} - e^{\left\{\frac{x}{2}[q_\delta+1]\right\}}, 0\right)$ |

The boundary situation:




- A question to ponder: do we really need the boundary $x \to -\infty$ and $x \to \infty$ if we are interested in only today's value of the option?

# SOLVING PDE WITH FINITE DIFFERENCE METHOD

## THE SET UP AND NOTATIONS

We will consider a rectangular grid defined by a partition on the $x$-axis with a equal spacing $\Delta x$ and a partition on the $\tau$-axis with a equal spacing $\Delta\tau$. So for each index $\gamma$, we have $\tau_{\gamma+1} = \tau_\gamma + \Delta\tau$ and $x_{i+1} = x_i + \Delta x$.

We will find only with a finite grid, so we have

- $x_0 = a$ corresponding to a choice of minimum stock price $S_{\min} = Ke^a$
- $x_m = b$ corresponding to a choice of minimum stock price $S_{\max} = Ke^b$
- $\tau_0 = 0$
- $\tau_{\gamma_{\max}} = \frac{\sigma^2 T}{2}$

The intersection of the $x$ grid line and $\tau$ grid line is called a node. Note that the equal spacing on the $x$ dimension does not translate into an equal spacing in the stock prices (the $S$ dimension). It is denser for small values of $S$ and sparse for large values of $S$.

We want to solve for the function $y(x,\tau)$. Sine we have used a finite grid we can only solve for the function values on the grid. Let the true value be $y_{i\gamma} = y(x_i,\tau_\gamma)$. We will calculate a set of approximation $w_{i\gamma}$ for $y_{i\gamma}$

## Explicit Method

From the practical Greek section we have

$$\frac{\partial y_{i\gamma}}{\partial \tau} = \frac{y_{i,\gamma+1} - y_{i\gamma}}{\Delta\tau} + O(\Delta\tau)$$

And

$$\frac{\partial^2 y_{i\gamma}}{\partial x^2} = \frac{y_{i+1,\gamma} - 2y_{i\gamma} + y_{i-1,\gamma}}{\Delta x^2} + O(\Delta x^2)$$

Put it into the heat equation $\frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial x^2}$, we get (with our approximations ready for iterations)

$$\frac{w_{i,\gamma+1} - w_{i\gamma}}{\Delta\tau} = \frac{w_{i+1,\gamma} - 2w_{i\gamma} + w_{i-1,\gamma}}{\Delta x^2}$$

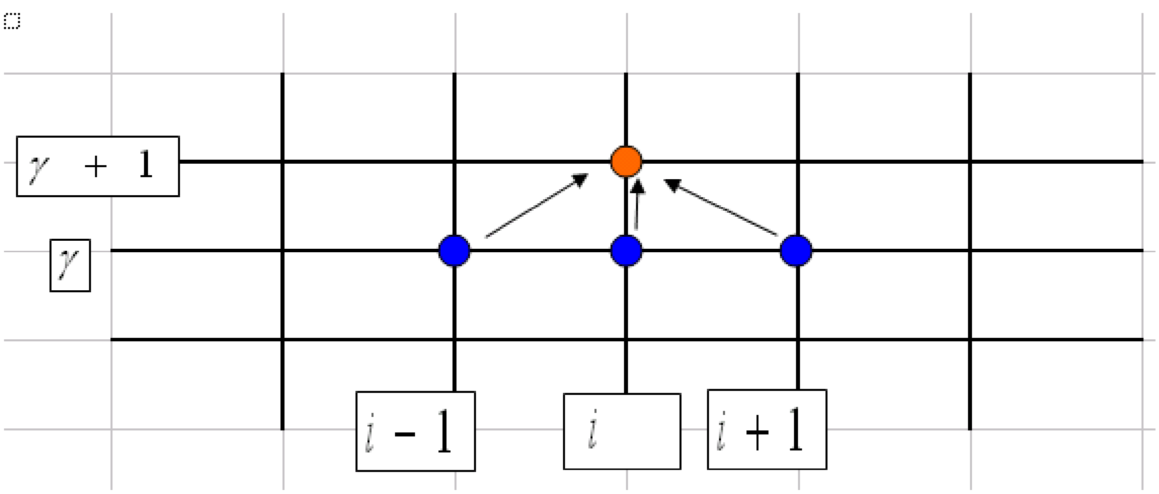
And we note that this method the order of accuracy on the $x$ direction is $O(\Delta x^2)$ and on the $\tau$ direction is $O(\Delta\tau)$. Just like the practical Greek calculations, we will see that they are ways to improve it using similar techniques.

Let $\lambda = \frac{\Delta\tau}{\Delta x^2}$, we get

$$w_{i,\gamma+1} = \lambda w_{i+1,\gamma} + (1-2\lambda)w_{i\gamma} + \lambda w_{i-1,\gamma} \quad \text{for } i = 1,2,\ldots,m-1$$

This is call the "explicit" method because to perform one iteration on the $\tau$ direction, each value can be explicitly calculated.

Picture




It is also known as the forward difference method.

In vector notation, let $\vec{w}^{(\gamma)} = \left(w_{1\gamma}, \ldots, w_{m-1,\gamma}\right)^T$

Let $A = \begin{pmatrix} 1-2\lambda & \lambda & & \\ \lambda & 1-2\lambda & \ddots & \\ & \ddots & \ddots & \lambda \\ & & \lambda & 1-2\lambda \end{pmatrix}$ an $(m-1)$ by $(m-1)$ matrix

Then we have $\vec{w}^{(\gamma+1)} = A\vec{w}^{(\gamma)}$.

## BOUNDARY CONDITION

The boundary conditions are given by the first vector $\vec{w}^{(0)} = \left(w_{10}, \ldots, w_{m-1,0}\right)^T = \left(y_{10}, \ldots, y_{m-1,0}\right)^T$. The assumption about is that are $w_{0\tau}$ and $w_{m\tau}$ are zeros. This is okay if $2m \geq \gamma_{\max}$ and we only want only the currently option value.

The boundary condition table needed is therefore

| | $\tau = 0$ |
|---|---|
| $y(x,0)$ | $t = T$ |
| Call Value | $\max\left(e^{\left\{\frac{x}{2}[q_\delta+1]\right\}} - e^{\left\{\frac{x}{2}[q_\delta-1]\right\}},0\right)$ |
| Put Value | $\max\left(e^{\left\{\frac{x}{2}[q_\delta-1]\right\}} - e^{\left\{\frac{x}{2}[q_\delta+1]\right\}},0\right)$ |

To obtain the entire surface, each $w_{0\tau}$ and $w_{m\tau}$ will need to be considered.

We want to take include the boundary conditions $w_{0\gamma} = r_1(a,\tau)$ and $w_{m\gamma} = r_2(b,\tau)$ for all $\tau$.

Adapting to the boundary conditions for this method:

$$\vec{w}^{(\gamma+1)} = A\vec{w}^{(\gamma)} + \vec{d}^{(\gamma)}$$

With $\vec{d}^{(\gamma)} = \lambda \begin{pmatrix} r_1(a,\tau_\gamma) \\ 0 \\ \vdots \\ 0 \\ r_2(b,\tau_\gamma) \end{pmatrix}$

Note that the above equations concerns $(m-1)$-vectors

Where

| | $x \to -\infty$ | $x \to \infty$ | $\tau = 0$ |
|---|---|---|---|

| $y(x,0)$ | $S=0$ | $S \to \infty$ | $t=T$ |
|---|---|---|---|
| Call Value | $r_1(x,\tau)=0$ | $r_2(x,\tau)= e^{\frac{x}{2}[q_\delta+1]+\frac{\tau}{4}[q_\delta+1]^2}$ | $\max\left(e^{\left\{\frac{x}{2}[q_\delta+1]\right\}}-e^{\left\{\frac{x}{2}[q_\delta-1]\right\}},0\right)$ |
| Put Value | $r_1(x,\tau)=e^{\frac{x}{2}[q_\delta-1]+\frac{\tau}{4}[q_\delta-1]^2}$ | $r_2(x,\tau)=0$ | $\max\left(e^{\left\{\frac{x}{2}[q_\delta-1]\right\}}-e^{\left\{\frac{x}{2}[q_\delta+1]\right\}},0\right)$ |

### Stability of the Method

We want the errors to die down instead of being amplified. Mathematically, we want $\lim_{\gamma\to\infty} A^\gamma \vec{e}^{(0)} = \vec{0}$ for any vector $\vec{e}^{(0)}$. This means that we want $\lim_{\gamma\to\infty} A^\gamma$ to be a zero matrix.

In the class we will arrive at the stability condition for explicit method:

$$\boxed{\frac{\Delta\tau}{\Delta x^2} \le \frac{1}{2}}$$

The consequence: Grid resolution cannot be chosen independent of each other.

Note also that the stability condition implies that there is a probabilistic interpretation. We will discuss this in class.

## Implicit Method

From the practical Greek section we have

$$\frac{\partial y_{i\gamma}}{\partial\tau} = \frac{y_{i,\gamma} - y_{i\gamma-1}}{\Delta\tau} + O(\Delta\tau)$$

And

$$\frac{\partial^2 y_{i\gamma}}{\partial x^2} = \frac{y_{i+1,\gamma} - 2y_{i\gamma} + y_{i-1,\gamma}}{\Delta x^2} + O(\Delta x^2)$$

Put it into the heat equation $\frac{\partial y}{\partial\tau} = \frac{\partial^2 y}{\partial x^2}$, we get (with our approximations ready for iterations)

$$\frac{w_{i,\gamma} - w_{i,\gamma-1}}{\Delta\tau} = \frac{w_{i+1,\gamma} - 2w_{i\gamma} + w_{i-1,\gamma}}{\Delta x^2}$$

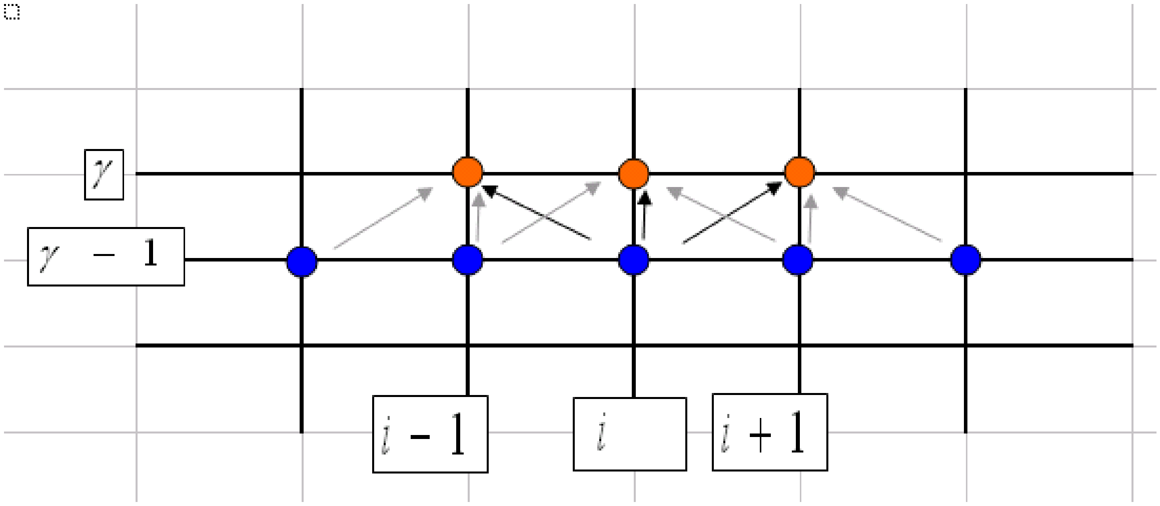
And we note that this method the order of accuracy on the $x$ direction is $O(\Delta x^2)$ and on the $\tau$ direction is $O(\Delta\tau)$. This does not therefore represent an improvement on this front.

Like before we let $\lambda = \frac{\Delta\tau}{\Delta x^2}$, we get

$$-\lambda w_{i+1,\gamma} + (2\lambda+1)w_{i\gamma} - \lambda w_{i-1,\gamma} = w_{i,\gamma-1} \quad \text{for } i = 1,2,\ldots,m-1$$

This is call the “implicit” method because to perform one iteration on the $\tau$ direction, the value can be no longer be explicitly calculated. They are implicitly given only by an equation.

Picture




In vector notation, let $\vec{w}^{(\gamma)} = \left(w_{1\gamma}, \ldots, w_{m-1,\gamma}\right)^T$

Let $A = \begin{pmatrix} 2\lambda+1 & -\lambda & & \\ -\lambda & 2\lambda+1 & \ddots & \\ & \ddots & \ddots & -\lambda \\ & & -\lambda & 2\lambda+1 \end{pmatrix}$ an $(m-1)$ by $(m-1)$ matrix

Then we have $A\vec{w}^{(\gamma)} = \vec{w}^{(\gamma-1)}$. In a form in line with the explicit method, this time we have $\vec{w}^{(\gamma)} = A^{-1}\vec{w}^{(\gamma-1)}$

## BOUNDARY CONDITION

The boundary conditions are given by the first vector $\vec{w}^{(0)} = \left(w_{10}, \ldots, w_{m-1,0}\right)^T = \left(y_{10}, \ldots, y_{m-1,0}\right)^T$.

We want to take include the boundary conditions $w_{0\gamma} = r_1(a,\tau)$ and $w_{m\gamma} = r_2(b,\tau)$ for all $\tau$.

Adapting to the boundary conditions for this method:

$$A\vec{w}^{(\gamma+1)} = \vec{w}^{(\gamma)} + \vec{d}^{(\gamma)}$$

With $\vec{d}^{(\gamma)} = \lambda \begin{pmatrix} r_1(a,\tau_{\gamma+1}) \\ 0 \\ \vdots \\ 0 \\ r_2(b,\tau_{\gamma+1}) \end{pmatrix}$

Where

| | $x \to -\infty$ | $x \to \infty$ | $\tau = 0$ |
|---|---|---|---|

| $y(x,0)$ | $S=0$ | $S \to \infty$ | $t=T$ |
|---|---|---|---|
| Call Value | $r_1(x,\tau)=0$ | $r_2(x,\tau)= e^{\frac{x}{2}[q_\delta+1]+\frac{\tau}{4}[q_\delta+1]^2}$ | $\max\left(e^{\left\{\frac{x}{2}[q_\delta+1]\right\}}-e^{\left\{\frac{x}{2}[q_\delta-1]\right\}},0\right)$ |
| Put Value | $r_1(x,\tau)=e^{\frac{x}{2}[q_\delta-1]+\frac{\tau}{4}[q_\delta-1]^2}$ | $r_2(x,\tau)=0$ | $\max\left(e^{\left\{\frac{x}{2}[q_\delta-1]\right\}}-e^{\left\{\frac{x}{2}[q_\delta+1]\right\}},0\right)$ |

### STABILITY OF THE METHOD

We will discuss in class: The implicit method is unconditionally stable regardless of the choice of $\lambda$.

## CRANK-NICOLSON METHOD

Crank Nicolson is the average of explicit and implicit.

So, putting here (Explicit Method)

$$\frac{w_{i,\gamma+1}-w_{i\gamma}}{\Delta\tau}=\frac{w_{i+1,\gamma}-2w_{i\gamma}+w_{i-1,\gamma}}{\Delta x^2}$$

And (Implicit method)

$$\frac{w_{i,\gamma}-w_{i,\gamma-1}}{\Delta\tau}=\frac{w_{i+1,\gamma}-2w_{i\gamma}+w_{i-1,\gamma}}{\Delta x^2}$$

And we just average this two, we will break the matrix format. So shift the time index up on the implicit method:

$$\frac{w_{i,\gamma+1}-w_{i,\gamma}}{\Delta\tau}=\frac{w_{i+1,\gamma+1}-2w_{i\gamma+1}+w_{i-1,\gamma+1}}{\Delta x^2}$$

And now we average to get:

$$\frac{w_{i,\gamma+1}-w_{i\gamma}}{\Delta\tau}=\frac{1}{2}\left(\frac{w_{i+1,\gamma}-2w_{i\gamma}+w_{i-1,\gamma}}{\Delta x^2}+\frac{w_{i+1,\gamma+1}-2w_{i\gamma+1}+w_{i-1,\gamma+1}}{\Delta x^2}\right)$$

Rewriting we get

$$w_{i,\gamma+1}-w_{i\gamma}=\frac{\lambda}{2}\left(w_{i+1,\gamma}-2w_{i\gamma}+w_{i-1,\gamma}+w_{i+1,\gamma+1}-2w_{i\gamma+1}+w_{i-1,\gamma+1}\right)$$

Rearranging as an iterative form

$$-\frac{\lambda}{2}w_{i-1,\gamma+1}+(1+\lambda)w_{i,\gamma+1}-\frac{\lambda}{2}w_{i+1,\gamma+1}=\frac{\lambda}{2}w_{i-1,\gamma}+(1-\lambda)w_{i\gamma}+\frac{\lambda}{2}w_{i+1,\gamma}$$

Now if we write

$$A = \begin{pmatrix} 1+\lambda & \frac{-\lambda}{2} & & \\ \frac{-\lambda}{2} & 1+\lambda & \ddots & \\ & \ddots & \ddots & \frac{-\lambda}{2} \\ & & \frac{-\lambda}{2} & 1+\lambda \end{pmatrix} \text{ and } B = \begin{pmatrix} 1-\lambda & \frac{\lambda}{2} & & \\ \frac{\lambda}{2} & 1-\lambda & \ddots & \\ & \ddots & \ddots & \frac{\lambda}{2} \\ & & \frac{\lambda}{2} & 1-\lambda \end{pmatrix}$$

It becomes

$$A\vec{w}^{(\gamma+1)} = B\vec{w}^{(\gamma)}$$

In homework we have shown that Show that for $A$ its eigenvalues all lie in $[1,1+2\lambda]$ hence it is non-singular. So we can write

$$\vec{w}^{(\gamma+1)} = A^{-1}B\vec{w}^{(\gamma)}$$

**BOUNDARY CONDITION**

We want to take include the boundary conditions $w_{0\gamma} = r_1(a,\tau)$ and $w_{m\gamma} = r_2(b,\tau)$ for all $\tau$.

Adapting to the boundary conditions for this method:

$$A\vec{w}^{(\gamma+1)} = B\vec{w}^{(\gamma)} + \vec{d}^{(\gamma)}$$

With $\vec{d}^{(\gamma)} = \frac{\lambda}{2}\begin{pmatrix} r_1(a,\tau_{\gamma+1}) + r_1(a,\tau_\gamma) \\ 0 \\ \vdots \\ 0 \\ r_2(b,\tau_{\gamma+1}) + r_2(b,\tau_\gamma) \end{pmatrix}$

Where

| | $x \to -\infty$ | $x \to \infty$ | $\tau = 0$ |
|---|---|---|---|
| $y(x,0)$ | $S = 0$ | $S \to \infty$ | $t = T$ |
| Call Value | $r_1(x,\tau) = 0$ | $r_2(x,\tau) = e^{\frac{x}{2}[q_\delta+1]+\frac{\tau}{4}[q_\delta+1]^2}$ | $\max\left(e^{\left\{\frac{x}{2}[q_\delta+1]\right\}} - e^{\left\{\frac{x}{2}[q_\delta-1]\right\}},0\right)$ |
| Put Value | $r_1(x,\tau) = e^{\frac{x}{2}[q_\delta-1]+\frac{\tau}{4}[q_\delta-1]^2}$ | $r_2(x,\tau) = 0$ | $\max\left(e^{\left\{\frac{x}{2}[q_\delta-1]\right\}} - e^{\left\{\frac{x}{2}[q_\delta+1]\right\}},0\right)$ |

**STABILITY OF THE METHOD**

We discuss in class: The Crank-Nicolson method is unconditionally stable regardless of the choice of $\lambda$.

## ERROR INVESTIGATION

We will discuss in class that this method the order of accuracy on the $x$ direction is $O(\Delta x^2)$ and on the $\tau$ direction is $O(\Delta \tau^2)$.

## $\theta$ METHOD

Crank Nicolson is the $\theta$ weighted average of explicit and implicit.

| Special case | Corresponding method |
|---|---|
| $\theta$ is zero | Explicit |
| $\theta$ is a half | Crank-Nicolson |
| $\theta$ is one | Implicit |

And the iteration formula is

(Explicit Method)

$$\frac{w_{i,\gamma+1} - w_{i\gamma}}{\Delta\tau} = \frac{w_{i+1,\gamma} - 2w_{i\gamma} + w_{i-1,\gamma}}{\Delta x^2}$$

(Implicit method)

$$\frac{w_{i,\gamma} - w_{i,\gamma-1}}{\Delta\tau} = \frac{w_{i+1,\gamma} - 2w_{i\gamma} + w_{i-1,\gamma}}{\Delta x^2}$$

And now we average to get:

$$\frac{w_{i,\gamma+1} - w_{i\gamma}}{\Delta\tau} = (1-\theta)\left(\frac{w_{i+1,\gamma} - 2w_{i\gamma} + w_{i-1,\gamma}}{\Delta x^2}\right) + \theta\left(\frac{w_{i+1,\gamma+1} - 2w_{i\gamma+1} + w_{i-1,\gamma+1}}{\Delta x^2}\right)$$

Rewriting we get

$$w_{i,\gamma+1} - w_{i\gamma} = \lambda\left((1-\theta)w_{i+1,\gamma} - 2(1-\theta)w_{i\gamma} + (1-\theta)w_{i-1,\gamma} + \theta w_{i+1,\gamma+1} - 2\theta w_{i\gamma+1} + \theta w_{i-1,\gamma+1}\right)$$

Rearranging as an iterative form

$$-\lambda\theta w_{i-1,\gamma+1} + (1+2\lambda\theta)w_{i,\gamma+1} - \lambda\theta w_{i+1,\gamma+1} = \lambda(1-\theta)w_{i-1,\gamma} + (1-2\lambda(1-\theta))w_{i\gamma} + \lambda(1-\theta)w_{i+1,\gamma}$$

Notions: $A = \begin{pmatrix} (1+2\lambda\theta) & -\lambda\theta & & \\ -\lambda\theta & (1+2\lambda\theta) & \ddots & \\ & \ddots & \ddots & -\lambda\theta \\ & & -\lambda\theta & (1+2\lambda\theta) \end{pmatrix}$ and

$$B = \begin{pmatrix} (1-2\lambda(1-\theta)) & \lambda(1-\theta) & & \\ \lambda(1-\theta) & (1-2\lambda(1-\theta)) & \ddots & \\ & \ddots & \ddots & \lambda(1-\theta) \\ & & \lambda(1-\theta) & (1-2\lambda(1-\theta)) \end{pmatrix}$$

It becomes

$$A\vec{w}^{(\gamma+1)} = B\vec{w}^{(\gamma)}$$

And things proceed like before.

### IMPLEMENTATION NOTE

When we have $A\vec{w}^{(\gamma+1)} = B\vec{w}^{(\gamma)} + \vec{d}^{(\gamma)}$, sometimes it would be convenient to do it with transposes. Note that $\left(M^{-1}\right)^T = \left(M^T\right)^{-1}$, we can write $\left(\vec{w}^{(\gamma+1)}\right)^T = \left[\left(\vec{w}^{(\gamma)}\right)^T B + \left(\vec{d}^{(\gamma)}\right)^T\right] A^{-1}$

## REFERENCES


**[StBu]** J. Stoer, R. Bulirsch, *Introduction to Numerical Analysis*, Springer,

**[Hull]** John Hull, *Options, Futures and other derivatives*, Prentice Hall,

**[Seydel]** Rüdiger U. Seydel ,*Tools for Computational Finance*, Springer,