Columbia University

MATH G6071 Spring 2015
Numerical Methods in Finance
Professor Tat Sang Fung
Homework 2: Interpolation

	Distribution date	Due date
Homework	Feb 26, 2015 (Thur)	Mar 11, 2015 (Wed) 7:30pm

Note: please send all homework solution to TA before the due date and time. Spreadsheet should be uploaded from CourseWorks. For theory part, please deposit it into the homework box in MATH building 4/F. If you suspect there are typos in this homework, or some questions are wrong, please feel free to email the instructor.

PRACTICE

QUESTION 1

Suppose you are given the market data of the volatility smile by delta for a one year[1] option:

Delta	86.03543	81.03543	76.03543	61.03543	50	25	10	5	0
Vol	**20.14%**	**18.64%**	**17.14%**	**15.64%**	**14.28%**	**15.04%**	**15.74%**	**16.44%**	**17.14%**

Make an Excel spreadsheet that calculates

1. the linear interpolated vol
2. the polynomial interpolated vol using Neville's algorithm[2]
3. the cubic spline interpolated vol [3]

for any given delta that lies between the smallest and largest delta

In particular, calculate the value of the vol that corresponds to the Delta of 45 and save the spreadsheet showing the value and calculation (it is preferable to show some intermediate calculations in some cells)

4. Produce a graph of the interpolated smile

When you plot the graph, plot it in such a way that the values of Delta decreases when you look at the x-axis from left to right.

The spreadsheet should have a toggle (or some control) to let user choose which interpolation he would like to use. And the graph shows the entire interpolated smile of his choice. The user should in general be able to change the values of the input Deltas, and the input vols (then spreadsheet would recalculated the interpolated vol upon request)

For the purpose of this homework, you can assume that the user always gives you nine points, and will not change the number of given points.

[1] Let's assume one year means 365 calendar days
[2] Although there are other methods to find this value, you have to use this method for the purpose of this home work to show the understanding of this particular method. You should have the "tree like" picture displayed somewhere in the spreadsheet
[3] During the computation, you will need to invert a "tri-diagonal" matrix. Please do so without using Excel's matrix inversion routine (you can do it within Excel with row reduction etc)

THEORY

QUESTION 2

The Vandermonde Matrix is one that has the form

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & \vdots & \vdots & & \\ \vdots & \vdots & \vdots & & \\ 1 & x_n & x_n^2 & & x_n^n \end{bmatrix}$$

1. How is Vandermonde Matrix related to the existence and uniqueness of polynomial interpolation[4]?
2. Find the polynomial expression for the determinant of this matrix for $n = 1,2$
3. Show that the determinant of this matrix is not zero if $x_0, x_1, \cdots, x_n$ are distinct for $n = 1,2$
4. Find the polynomial expression for the determinant of this matrix for general n. Show that the determinant of this matrix is not zero if $x_0, x_1, \cdots, x_n$ are distinct.

QUESTION 3

The purpose of this question is to illustrate that in real life we may need to require the interpolated smile be once-differentiable.

1. A call spread is a package where you buy a call and sell a call of different strikes simultaneously. Let $c(K)$ be the European Call with strike K. Consider the payoff of a call spread, explain why a digital call[5] option with strike K is the limiting portfolio $\lim_{K' \to K} \frac{1}{K' - K}\left(c(K) - c(K')\right)$. Let $digital(K)$ be the price of a digital call option with strike K. We therefore have:
$$digital(K) = -\frac{\partial c(K)}{\partial K}$$
2. Hence, show that in the presence of a volatility smile, the value of a digital call option is given by
$$e^{-rT} N(d_2) - Ke^{-rT} N'(d_2)\sqrt{T}\sigma'(K)$$
Where $d_2 = \dfrac{\ln\left(\frac{S}{K}\right) + \left(r - q - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$ and symbols have their usual standard meaning (e.g see [Hull])

QUESTION 4

The purpose of this question is to illustrate that for linear interpolation, the two points (that determines the line) can be chosen strategically to give the smallest theoretical upper bound for the interpolation error.

[4] Hint: we discussed that in class…

[5] In [Hull] a digital call is called a "cash-or-nothing" call option

This question is taken from [StBu] page 134 question 3

1. Consider a function that is twice continuously differentiable on the interval $I = [-1,1]$. Interpolate the function by a linear polynomial through that given points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ where $x_0, x_1 \in I$. Prove that

$$\alpha = \frac{1}{2} \max_{\xi \in I} \left| f''(\xi) \right| \cdot \max_{x \in I} \left| (x - x_0)(x - x_1) \right|$$

   is an upper bound for the maximal absolute interpolation error on the interval I.

2. Which values $x_0, x_1 \in I$ minimize α ?

   - Imagine answering if a trader asks you the question and is only interested in the answer
   - Now, imagine answering the above if a risk manager asks you this and is interested to know the explanation (perhaps some intuition, but not necessarily a proof)
   - Now, imagine a desk Quant or a mathematician asks you this and is interested in a proof when you are to answer

QUESTION 5

1. Let $r(T)$ the continuous compounding zero rate for maturity T. Let $T_1 < T_2$ and $r_F(T_1, T_2)$ be the forward zero rate for the period $[T_1, T_2]$. Prove that[6] $r_F(T_1, T_2) = \dfrac{r(T_2)T_2 - r(T_1)T_1}{T_2 - T_1}$.

2. Let $f(T)$ be the instantaneous forward rate[7] for maturity T. Suppose $r(T)$ is once differentiable. Prove that $f(T) = r(T) + T\dfrac{\partial r(T)}{\partial T}$.

3. Consider a set of zero rates are given: $(t_0, r(t_0)), \ldots, (t_n, r(t_n))$ and piecewise linear interpolation is used on $r(t)$. Discuss under what condition the instantaneous forward rate $f(T)$ would be defined everywhere and is continuous between t_0 and t_n. Show that if all given $r(t_i)$ are non-negative, then $r(t)$ is also non-negative for all $t_0 \le t \le t_n$

4. Consider a set of zero rates are given: $(t_0, r(t_0)), \ldots, (t_n, r(t_n))$ and piecewise linear interpolation is used on $r(t) \cdot t$. Derive an expression[8] for $r(t)$. Show that if all given $r(t_i)$ are non-negative, then $r(t)$ is also non-negative for all $t_0 \le t \le t_n$

QUESTION 6

In [PG Conv] we found without proof formula (2.14)

[6] See [Hull] formula 4.5 on page 85

[7] See [Hull] page 85 for a definition. Here we take the limit as $T_2 \to T_1^+$.

[8] This is sometimes known as the "RT" interpolation among the practitioners…

We can carry out the second step by *replicating* the payoff in 2.12 in terms of payer swaptions. For any smooth function $f(R_s)$ with $f(K) = 0$, we can write

$$f'(K)[R_s - K]^+ + \int_K^\infty [R_s - x]^+ f''(x)dx = \begin{cases} f(R_s) & \text{for } R_s > K \\ 0 & \text{for } R_s < K \end{cases}. \qquad (2.14)$$

1. Please prove (2.14)
2. Hence prove the replication valuation formula (2.17) a and b as below

Together with the first term, this yields

$$V_{cap}^{CMS}(0) = \frac{D(t_p)}{L_0}\left\{\left[1 + f'(K)\right]C(K) + \int_K^\infty C(x)f''(x)dx\right\}, \qquad (2.17a)$$

as the value of the CMS caplet, where

$$C(x) = L_0 \mathrm{E}\left\{[R_s(\tau) - x]^+ \,\middle|\, \mathcal{F}_0\right\} \qquad (2.17b)$$

is the value of an ordinary payer swaption with strike x.

[Hint to 1 and 2: we discussed in class how to do this already. Please write out in details]

REFERENCES

[PG Conv] Patrick Hagan, *Convexity Conundrums: Pricing CMS Swaps, Caps, and Floors* , Wilmott article, available at *http://www.wilmott.com/pdfs/050118_hagan.pdf*

[StBu] J. Stoer, R. Bulirsch, *Introduction to Numerical Analysis*, Springer

[Hull] John Hull, *Options, Futures and other derivatives*, Prentice Hall, 8th edition (you will find similar discussion in the latest 9th edition also)