1.

i)

$$C_{1,1} = det\begin{pmatrix}1 & 0\\2 & 2\end{pmatrix} = 1*2-0*2 = 2$$

$$C_{1,2} = det\begin{pmatrix}3 & 0\\-4 & 2\end{pmatrix} = 3*2-0*(-4) = 6$$

$$C_{1,3} = \det\begin{pmatrix}3 & 1\\-4 & 2\end{pmatrix} = 3*2-1*(-4) = 10$$

$$C_{2,1} = det\begin{pmatrix}2 & -1\\2 & 2\end{pmatrix} = 2*2-(-1)*2 = 6$$

$$C_{2,2} = det\begin{pmatrix}1 & -1\\-4 & 2\end{pmatrix} = 1*2-(-1)(-4) = -2$$

$$C_{2,3} = det\begin{pmatrix}1 & 2\\-4 & 2\end{pmatrix} = 1*2-2(-4) = 10$$

$$C_{3,1} = det\begin{pmatrix}2 & -1\\1 & 0\end{pmatrix} = 2*0-(-1)*1 = 1$$

$$C_{3,2} = det\begin{pmatrix}1 & -1\\3 & 0\end{pmatrix} = 1*0-(-1)*3 = 3$$

$$C_{3,3} = det\begin{pmatrix}1 & 2\\3 & 1\end{pmatrix} = 1*1-2*3 = -5$$

$$\begin{pmatrix}2 & 6 & 10\\6 & -2 & 10\\1 & 3 & -5\end{pmatrix}$$

$$\begin{matrix}(-1)^{1+1}*2 & (-1)^{1+2}*6 & (-1)^{1+3}*10\\(-1)^{2+1}*6 & (-1)^{2+2}*(-2) & (-1)^{2+3}*10\\(-1)^{3+1}*1 & (-1)^{3+2}*3 & (-1)^{3+3}*(-5)\end{matrix}$$

$$= \begin{pmatrix}2 & -6 & 10\\-6 & -2 & -10\\1 & -3 & -5\end{pmatrix}$$

$$\begin{pmatrix} 2 & -6 & 10 \\ -6 & -2 & -10 \\ 1 & -3 & -5 \end{pmatrix}^T = \begin{pmatrix} 2 & -6 & 1 \\ -6 & -2 & -3 \\ 10 & -10 & -5 \end{pmatrix}$$

ii)

$$\begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -4 & 2 & 2 \end{pmatrix} * \begin{pmatrix} 2 & -6 & 1 \\ -6 & -2 & -3 \\ 10 & -10 & -5 \end{pmatrix} = \begin{pmatrix} -20 & 0 & 0 \\ 0 & -20 & 0 \\ 0 & 0 & -20 \end{pmatrix} = -20 * I$$
$$= \det(A) * I$$

2.

$(5)(\alpha) + (\alpha)(\alpha) + (3\alpha)(3\alpha) + (1)(-1) + (5)(1) = 8\alpha^2 + 5\alpha + 4$

$8\alpha^2 + 5\alpha + 4 = 0$

$\alpha_{1,2} = \dfrac{-5 \pm \sqrt{5^2 - 4 * 8 * 4}}{2 * 8}$

$\sqrt{5^2 - 128} = i\sqrt{128 - 5^2} = \sqrt{103}\, i$

$\alpha = \dfrac{-5 + \sqrt{103}\, i}{2 * 8}, \alpha = \dfrac{-5 - \sqrt{103}\, i}{2 * 8}$

Answers:

$\alpha = -\dfrac{5}{16} + i\dfrac{\sqrt{103}}{16}, \qquad \alpha = -\dfrac{5}{16} - i\dfrac{\sqrt{103}}{16}$

3.

$$\begin{bmatrix} 2x_1 - 2x_2 + 2x_3 - 4x_4 = 2 \\ x_1 + 4x_2 + 8x_3 + 2x_4 = 5 \\ -x_1 + 9x_2 + 3x_3 - 4x_4 = 5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 & 2 & -4 & | & 2 \\ 1 & 4 & 8 & 2 & | & 5 \\ -1 & 9 & 3 & -4 & | & 5 \end{bmatrix}$$

$R2 = R2 - R1(\frac{1}{2})$

$$\begin{bmatrix} 2 & -2 & 2 & -4 & | & 2 \\ 0 & 5 & 7 & 4 & | & 4 \\ -1 & 9 & 3 & -4 & | & 5 \end{bmatrix}$$

$R3 = R3 + R1(\frac{1}{2})$

$$\begin{bmatrix} 2 & -2 & 2 & -4 & | & 2 \\ 0 & 5 & 7 & 4 & | & 4 \\ 0 & 8 & 4 & -6 & | & 6 \end{bmatrix}$$

$R3 = R3 - \frac{8R2}{5}$

$$\begin{bmatrix} 2 & -2 & 2 & -4 & | & 2 \\ 0 & 5 & 7 & 4 & | & 4 \\ 0 & 0 & -\frac{36}{5} & -\frac{62}{5} & | & -\frac{2}{5} \end{bmatrix}$$

$\boldsymbol{x_4 = t}$

$\boldsymbol{Find\ x_3}$

$$-\frac{36}{5}x_3 - -\frac{62}{5}t = -\frac{2}{5}$$

$$-\frac{36}{5}x_3 = -\frac{2}{5} - \frac{62}{5}t$$

$$-36x_3 = -2 - 62t$$

$$x_3 = \frac{1}{18} - \frac{31}{18}t$$

$\boldsymbol{Find\ x_2}$

$$5x_2 + 7\left(\frac{1}{18} - \frac{31}{18}t\right) + 4t = 4$$

$$5x_2 = 4 - (7\left(\frac{1}{18} - \frac{31}{18}t\right) + 4t)$$

$$5x_2 = \frac{62}{18} + \frac{145}{18}t$$

$$x_2 = \frac{13}{18} + \frac{29}{18}t$$

$\boldsymbol{Find\ x_1}$

$$2x_1 - 2\left(\frac{13}{18} + \frac{29}{18}t\right) + 2\left(\frac{1}{18} - \frac{31}{18}t\right) - 4t = 2$$

$$2x_1 = 2 - \left(-2\left(\frac{13}{18} + \frac{29}{18}t\right) + 2\left(\frac{1}{18} - \frac{31}{18}t\right) - 4t\right)$$

$$2x_1 = \frac{10}{3} + \frac{32}{3}t$$

$x_1 = \frac{5}{3} + \frac{16}{3}t$

Answers:

$$x_1 = \frac{5}{3} + \frac{16}{3}t, x_2 = \frac{13}{18} + \frac{29}{18}t, x_3 = \frac{1}{18} - \frac{31}{18}t$$

Justify the correctness using matrix ranks:

$$A = \begin{bmatrix} 2 & -2 & 2 & -4 \\ 0 & 5 & 7 & 4 \\ 0 & 0 & -\frac{36}{5} & -\frac{62}{5} \end{bmatrix}$$

$$([Ab]) = \begin{bmatrix} 2 & -2 & 2 & -4 & | & 2 \\ 0 & 5 & 7 & 4 & | & 4 \\ 0 & 0 & -\frac{36}{5} & -\frac{62}{5} & | & -\frac{2}{5} \end{bmatrix}$$

$R(A) = 3, R([Ab]) = 3$

$There\ are\ 4\ unknowns\ and\ rank\ of\ both\ matrix\ A\ and\ R([Ab])\ are\ 3. So,$

$R(A) = R([Ab]) < 4$

$Meaning\ that\ the\ system\ is\ consistent\ and\ has\ infinitely\ many\ solutions$

4.

$3x_2 + 11x_3 = 6$
$x_1 + x_2 + 3x_3 = 2$
$3x_1 - 3x_2 - 13x_3 = -6$
$-x_1 + 2x_2 + 8x_3 = 4$

$$\begin{bmatrix} 0 & 3 & 11 & | & 6 \\ 1 & 1 & 3 & | & 2 \\ 3 & -3 & -13 & | & -6 \\ -1 & 2 & 8 & | & 4 \end{bmatrix}$$

$Swap\ R2\ with\ R1$

$$\begin{bmatrix} 1 & 1 & 3 & | & 2 \\ 0 & 3 & 11 & | & 6 \\ 3 & -3 & -13 & | & -6 \\ -1 & 2 & 8 & | & 4 \end{bmatrix}$$

$R3 = R3 - 3R1$

$$\begin{bmatrix} 1 & 1 & 3 & | & 2 \\ 0 & 3 & 11 & | & 6 \\ 0 & -6 & -22 & | & -12 \\ -1 & 2 & 8 & | & 4 \end{bmatrix}$$

$R4 = R4 + R1$

$$\begin{bmatrix} 1 & 1 & 3 & | & 2 \\ 0 & 3 & 11 & | & 6 \\ 0 & -6 & -22 & | & -12 \\ 0 & 3 & 11 & | & 6 \end{bmatrix}$$

$R3 = R3 + 2R2$

$$\begin{bmatrix} 1 & 1 & 3 & | & 2 \\ 0 & 3 & 11 & | & 6 \\ 0 & 0 & 0 & | & 0 \\ 0 & 3 & 11 & | & 6 \end{bmatrix}$$

$R4 = R4 - R2$

$$\begin{bmatrix} 1 & 1 & 3 & | & 2 \\ 0 & 3 & 11 & | & 6 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$\boldsymbol{x_3 = t}$

$\boldsymbol{Find\ x_2}$

$3x_2 + 11t = 6$

$$x_2 = \frac{6 - 11t}{3} = 2 - \frac{11t}{3}$$

$\boldsymbol{Find\ x_1}$

$$1x_1 + \left(2 - \frac{11}{3}t\right) + 3t = 2$$

$$x_1 = 2 - \left(2 - \frac{11}{3}t\right) + 3t$$

$$x_1 = \frac{2}{3}t$$

Answers:

$$x_1 = \frac{2}{3}t, x_2 = \frac{6-11t}{3} = 2 - \frac{11t}{3}$$

Justify the correctness using matrix ranks:

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 3 & 11 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$([Ab]) = \begin{bmatrix} 1 & 1 & 3 & | & 2 \\ 0 & 3 & 11 & | & 6 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$R(A) = 2, R([Ab]) = 2$

$There\ are\ 3\ unknowns\ and\ rank\ of\ both\ matrix\ A\ and\ ([Ab]) =\ are\ 2. So,$

$R(A) = R([Ab]) < 3$

$Meaning\ that\ the\ system\ is\ consistent\ and\ has\ infinitely\ many\ solutions$

5.

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 & | & 1 & 0 & 0 \\ 1 & 0 & 2 & | & 0 & 1 & 0 \\ 1 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

$R2 = R2 - R1$

$$\begin{bmatrix} 1 & 2 & 2 & | & 1 & 0 & 0 \\ 0 & -2 & 0 & | & -1 & 1 & 0 \\ 1 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

$R3 = R3 - R1$

$$\begin{bmatrix} 1 & 2 & 2 & | & 1 & 0 & 0 \\ 0 & -2 & 0 & | & -1 & 1 & 0 \\ 0 & -2 & -2 & | & -1 & 0 & 1 \end{bmatrix}$$

$R2 = R2 * -\frac{1}{2}$

$$\begin{bmatrix} 1 & 2 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & 1/2 & -1/2 & 0 \\ 0 & -2 & -2 & | & -1 & 0 & 1 \end{bmatrix}$$

$R3 = R3 + 2R2$

$$\begin{bmatrix} 1 & 2 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & 1/2 & -1/2 & 0 \\ 0 & 0 & -2 & | & 0 & -1 & 1 \end{bmatrix}$$

$R1 = R1 - 2R2$

$$\begin{bmatrix} 1 & 0 & 2 & | & 0 & 1 & 0 \\ 0 & 1 & 0 & | & 1/2 & -1/2 & 0 \\ 0 & 0 & -2 & | & 0 & -1 & 1 \end{bmatrix}$$

$$R3 = R3 * -\frac{1}{2}$$

$$\begin{bmatrix} 1 & 0 & 2 & | & 0 & 1 & 0 \\ 0 & 1 & 0 & | & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & | & 0 & 1/2 & -1/2 \end{bmatrix}$$

$$R1 = R1 - 2R3$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 & 0 & 1 \\ 0 & 1 & 0 & | & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & | & 0 & 1/2 & -1/2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0.5 & -0.5 & 0 \\ 0 & 0.5 & -0.5 \end{bmatrix}$$

6.

$\boldsymbol{Find\ eigenvalues\ for\ A}$

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$\det\left(\begin{bmatrix} 0 - \lambda & 1 & -1 \\ 1 & 1 - \lambda & 0 \\ 1 & 0 & 1 - \lambda \end{bmatrix}\right) = 0$$

$$1 \begin{bmatrix} 1 & -1 \\ 0 & 1 - \lambda \end{bmatrix} - \left(1 - \lambda \begin{bmatrix} -\lambda & -1 \\ 1 & 1 - \lambda \end{bmatrix}\right) + 0 \begin{bmatrix} -\lambda & 1 \\ 1 & 0 \end{bmatrix}$$

$$-\lambda + 1 - 1 - \lambda^2 + \lambda^3 + 0 = 0$$

$$\lambda^3 - \lambda^2 = 0$$

$$\lambda^2(\lambda - 1) = 0$$

$$\lambda_1 = 0, \lambda_2 = 1$$

$\boldsymbol{Find\ eigenvectors\ for\ A}$

$\boldsymbol{Find\ eigenvectors\ when\ \lambda_1 = 0}$

$\lambda_1 = 0$

$$(A - \lambda \mathrm{I}) = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using gaussian elimination

$$\begin{bmatrix} 0 & 1 & -1 & | & 0 \\ 1 & 1 & 0 & | & 0 \\ 1 & 0 & 1 & | & 0 \end{bmatrix}$$

Swap 1 *and* 2

$$\begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 1 & 0 & 1 & | & 0 \end{bmatrix}$$

$R3 = R3 - R1$

$$\begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & -1 & 1 & | & 0 \end{bmatrix}$$

$R3 = R3 + R2$

$$\begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$x_3 = t$

$x_2 = t$

$x_1 = -t$

$$v_1 = \begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Find eigenvectors when $\boldsymbol{\lambda_2 = 1}$

$$(A - \lambda \mathrm{I}) = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using gaussian elimination

$$\begin{bmatrix} -1 & 1 & -1 & | & 0 \\ 1 & 0 & 0 & | & 0 \\ 1 & 0 & 0 & | & 0 \end{bmatrix}$$

$R2 = R2 + R1$

$$\begin{bmatrix} -1 & 1 & -1 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 1 & 0 & 0 & | & 0 \end{bmatrix}$$

$R3 = R3 + R1$

$$\begin{bmatrix} -1 & 1 & -1 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 1 & -1 & | & 0 \end{bmatrix}$$

$R3 = R3 - R2$

$$\begin{bmatrix} -1 & 1 & -1 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$x_3 = t$

$x_2 = t$

$x_1 = 0$

$$v_2 = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Answers:

$$v_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Find eigenvalues for B

$$B = \begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$\det\left(\begin{bmatrix} 5-\lambda & -4 & 0 \\ 1 & 0-\lambda & 2 \\ 0 & 2 & 5-\lambda \end{bmatrix}\right) = 0$$

$-\lambda^3 + 10\lambda^2 - 25\lambda = 0$

$\lambda_1 = 0, \lambda_2 = 5$

Find eigenvectors for B

Find eigenvectors when $\lambda_1 = 0$

$$(A - \lambda \mathrm{I}) = \begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using gaussian elimination

$$\begin{bmatrix} 5 & -4 & 0 & | & 0 \\ 1 & 0 & 2 & | & 0 \\ 0 & 2 & 5 & | & 0 \end{bmatrix}$$

$$R2 = R2 - R1\frac{1}{5}$$

$$\begin{bmatrix} 5 & -4 & 0 & | & 0 \\ 0 & \frac{4}{5} & 2 & | & 0 \\ 0 & 2 & 5 & | & 0 \end{bmatrix}$$

$$R3 = R3 - R3\frac{5}{2}$$

$$\begin{bmatrix} 5 & -4 & 0 & | & 0 \\ 0 & \frac{4}{5} & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$x_3 = t$$

$$x_2 = -\frac{5}{2}t$$

$$x_1 = -2t$$

$$Let\ x_3\ be\ 1$$

$$v_1 = \begin{bmatrix} -2t \\ -\frac{5}{2}t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ -\frac{5}{2} \\ 1 \end{bmatrix}$$

$$\boldsymbol{Find\ eigenvectors\ when\ \lambda_1 = 5}$$

$$(A - \lambda \mathrm{I}) = \begin{bmatrix} 0 & -4 & 0 \\ 1 & -5 & 2 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$Using\ gauss - jordan\ elimination$$

$$\begin{bmatrix} 0 & -4 & 0 & | & 0 \\ 1 & -5 & 2 & | & 0 \\ 0 & 2 & 0 & | & 0 \end{bmatrix}$$

$$Swap\ R1\ and\ R2$$

$$\begin{bmatrix} 1 & -5 & 2 & | & 0 \\ 0 & -4 & 0 & | & 0 \\ 0 & 2 & 0 & | & 0 \end{bmatrix}$$

$$R2 = R2 * -\frac{1}{4}$$

$$\begin{bmatrix} 1 & -5 & 2 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$R1 = R1 + 5R2$

$$\begin{bmatrix} 1 & 0 & 2 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 2 & 0 & | & 0 \end{bmatrix}$$

$R3 = R3 - 2R2$

$$\begin{bmatrix} 1 & 0 & 2 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$x_3 = t$

$x_2 = 0$

$x_1 = -2t$

$$v_2 = \begin{bmatrix} -2t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Answers:

$$v_1 = \begin{bmatrix} -2 \\ -\frac{5}{2} \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

7.

$\boldsymbol{Find\ eigenvalues\ for\ A}$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$\det\left(\begin{bmatrix} 1-\lambda & 1 & 0 \\ 1 & 0-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix}\right) = 0$$

$$1 - \lambda \begin{bmatrix} 0-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1-\lambda \end{bmatrix} + 0 \begin{bmatrix} 1 & 0-\lambda \\ 0 & 1 \end{bmatrix}$$

$$(1-\lambda)(-\lambda(1-\lambda) - 1) - (-\lambda + 1) = 0$$

$$(-\lambda - 1)(\lambda - 1)(\lambda - 2) = 0$$

$$\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 2$$

$Find\ eigenvectors\ for\ A$

$Find\ eigenvectors\ when\ \lambda_1 = 1$

$$(A - \lambda I) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$Using\ gaussian\ elimination$

$$\begin{bmatrix} 0 & 1 & 0 & | & 0 \\ 1 & -1 & 1 & | & 0 \\ 0 & 1 & 0 & | & 0 \end{bmatrix}$$

$Swap\ R1\ and\ R2$

$$\begin{bmatrix} 1 & -1 & 1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \end{bmatrix}$$

$R3 = R3 - R2$

$$\begin{bmatrix} 1 & -1 & 1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$x_3 = t$

$x_2 = 0$

$x_1 = -t$

$$v_1 = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$Find\ eigenvectors\ when\ \lambda_2 = -1$

$$(A - \lambda I) = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$Using\ gaussian\ elimination$

$$\begin{bmatrix} 2 & 1 & 0 & | & 0 \\ 1 & 1 & 1 & | & 0 \\ 0 & 1 & 2 & | & 0 \end{bmatrix}$$

$$R2 = R2 - R1 * \frac{1}{2}$$

$$\begin{bmatrix} 2 & 1 & 0 & | & 0 \\ 0 & \frac{1}{2} & 1 & | & 0 \\ 0 & 1 & 2 & | & 0 \end{bmatrix}$$

$R3 = R3 - 2R2$

$$\begin{bmatrix} 2 & 1 & 0 & | & 0 \\ 0 & \frac{1}{2} & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$x_3 = t$

$x_2 = -2t$

$x_1 = t$

$$v_2 = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$\boldsymbol{Find\ eigenvectors\ when\ \lambda_2 = 2}$

$$(A - \lambda \mathrm{I}) = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$Using\ gaussian\ elimination$

$$\begin{bmatrix} -1 & 1 & 0 & | & 0 \\ 1 & -2 & 1 & | & 0 \\ 0 & 1 & -1 & | & 0 \end{bmatrix}$$

$R2 = R2 + R1$

$$\begin{bmatrix} -1 & 1 & 0 & | & 0 \\ 0 & -1 & 1 & | & 0 \\ 0 & 1 & -1 & | & 0 \end{bmatrix}$$

$R3 = R3 + R2$

$$\begin{bmatrix} -1 & 1 & 0 & | & 0 \\ 0 & -1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$x_3 = t$

$x_2 = t$

$x_1 = t$

$$v_3 = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\boldsymbol{Eigenvalues}$:

$\lambda = 1, \lambda = -1, \lambda = 2$

$\boldsymbol{Eigenvectors}$:

$$v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\boldsymbol{A = PDP^{-1}}$

$$P = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1}$$

$\boldsymbol{Find\ inverse\ of\ P}$

$$\begin{bmatrix} -1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & -2 & 1 & | & 0 & 1 & 0 \\ 1 & 1 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

$R1 = R1 * -1$

$$\begin{bmatrix} 1 & -1 & -1 & | & -1 & 0 & 0 \\ 0 & -2 & 1 & | & 0 & 1 & 0 \\ 1 & 1 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

$R3 = R3 - R1$

$$\begin{bmatrix} 1 & -1 & -1 & | & -1 & 0 & 0 \\ 0 & -2 & 1 & | & 0 & 1 & 0 \\ 0 & 2 & 2 & | & 1 & 0 & 1 \end{bmatrix}$$

$R2 = R2 * -\frac{1}{2}$

$$\begin{bmatrix} 1 & -1 & -1 & | & -1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & | & 0 & -\frac{1}{2} & 0 \\ 0 & 2 & 2 & | & 1 & 0 & 1 \end{bmatrix}$$

$R1 = R1 + R2$

$$\begin{bmatrix} 1 & 0 & -\frac{3}{2} & | & -1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & | & 0 & -\frac{1}{2} & 0 \\ 0 & 2 & 2 & | & 1 & 0 & 1 \end{bmatrix}$$

$R3 = R3 - 2R2$

$$\begin{bmatrix} 1 & 0 & -\frac{3}{2} & | & -1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & | & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 3 & | & 1 & 1 & 1 \end{bmatrix}$$

$R3 = R3 * \frac{1}{3}$

$$\begin{bmatrix} 1 & 0 & -\frac{3}{2} & | & -1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & | & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & | & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$R1 = R1 + R3 * \frac{3}{2}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & | & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & | & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$R2 = R2 + R3 * \frac{1}{2}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & | & \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ 0 & 0 & 1 & | & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} * \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$\boldsymbol{A = PDP^{-1}}$$

$$Another\ way\ to\ check\ if\ A\ can\ be\ digaonalized, is\ check\ if\ AD = PD\ instead\ of\ figuring\ out\ P\ inverse.$$

8.

a)

$$\boldsymbol{Find\ eigenvalues\ for\ A}$$

$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

$\det\left(\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 2 \\ 0 & 0 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$

$\det\left(\begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & -2-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{bmatrix}\right)$

$0 \begin{bmatrix} 1-\lambda & 2 \\ 0 & -2-\lambda \end{bmatrix} - 0 \begin{bmatrix} 1-\lambda & 3 \\ 0 & 2 \end{bmatrix} + (3-\lambda) \begin{bmatrix} 1-\lambda & 2 \\ 0 & -2-\lambda \end{bmatrix}$

$(3-\lambda)(-\lambda+1)(-\lambda-2) = 0$

$\lambda_1 = 3, \lambda_2 = 1, \lambda_3 = -2$

$\boldsymbol{Find\ eigenvectors\ for\ A}$

$\boldsymbol{Find\ eigenvectors\ when\ \lambda_1 = 3}$

$(A - \lambda \mathrm{I}) = \begin{bmatrix} -2 & 2 & 3 \\ 0 & -5 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -2 & 2 & 3 & | & 0 \\ 0 & -5 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$

$x_3 = t$

$x_2 = \frac{2}{5} t$

$x_1 = \frac{19}{10} t$

$v_1 = \begin{bmatrix} \frac{19}{10} t \\ \frac{2}{5} t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{19}{10} \\ \frac{2}{5} \\ 1 \end{bmatrix}$

$\boldsymbol{Find\ eigenvectors\ when\ \lambda_2 = 1}$

$(A - \lambda \mathrm{I}) = \begin{bmatrix} 0 & 2 & 3 \\ 0 & -3 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 2 & 3 & | & 0 \\ 0 & -3 & 2 & | & 0 \\ 0 & 0 & 2 & | & 0 \end{bmatrix}$

$R2 = R2 + R1 * \frac{3}{2}$

$$\begin{bmatrix} 0 & 2 & 3 & | & 0 \\ 0 & 0 & \frac{13}{2} & | & 0 \\ 0 & 0 & 2 & | & 0 \end{bmatrix}$$

$R3 = R3 - R2 * \frac{4}{13}$

$$\begin{bmatrix} 0 & 2 & 3 & | & 0 \\ 0 & 0 & \frac{13}{2} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$x_3 = 0$

$x_2 = 0$

$x_1 = t$

$$v_2 = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\boldsymbol{Find\ eigenvectors\ when\ \lambda_3 = -2}$

$$(A - \lambda \mathrm{I}) = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 3 & | & 0 \\ 0 & 0 & 2 & | & 0 \\ 0 & 0 & 5 & | & 0 \end{bmatrix}$$

$R3 = R3 - R2 * \frac{5}{2}$

$$\begin{bmatrix} 3 & 2 & 3 & | & 0 \\ 0 & 0 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$x_3 = 0$

$x_2 = t$

$x_1 = -\frac{2}{3} t$

$$v_3 = \begin{bmatrix} -\frac{2}{3} t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix}$$

Answers:

$$v_1 = \begin{bmatrix} \frac{19}{10} \\ \frac{2}{5} \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix}$$

$\boldsymbol{Find\ eigenvalues\ for\ B}$

$$B = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$\det\left(\begin{bmatrix} 3-\lambda & 1 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{bmatrix}\right) = 0$$

$$0\begin{bmatrix} 1 & 0 \\ 1-\lambda & 2 \end{bmatrix} - 0\begin{bmatrix} 3-\lambda & 0 \\ 0 & 2 \end{bmatrix} + (-2-\lambda)\begin{bmatrix} 3-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} = 0$$

$$(-2-x)(-x+3)(-x+1) = 0$$

$$\lambda_1 = -2, \lambda_2 = 3, \lambda_3 = 1$$

$\boldsymbol{Find\ eigenvectors\ for\ B}$

$\boldsymbol{Find\ eigenvectors\ when\ \lambda_1 = 3}$

$$(A - \lambda \mathrm{I}) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & | & 0 \\ 0 & -2 & 2 & | & 0 \\ 0 & 0 & -5 & | & 0 \end{bmatrix}$$

$$R2 = R2 + 2R1$$

$$\begin{bmatrix} 0 & 1 & 0 & | & 0 \\ 0 & 0 & 2 & | & 0 \\ 0 & 0 & -5 & | & 0 \end{bmatrix}$$

$$R3 = R3 + R2 * \frac{5}{2}$$

$$\begin{bmatrix} 0 & 1 & 0 & | & 0 \\ 0 & 0 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$x_3 = 0$$

$$x_2 = 0$$

$$x_1 = t$$

$$v_1 = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Find eigenvectors when $\boldsymbol{\lambda_2 = 1}$

$$(A - \lambda \mathrm{I}) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 & | & 0 \\ 0 & 0 & 2 & | & 0 \\ 0 & 0 & -3 & | & 0 \end{bmatrix}$$

$$R3 = R3 + R2 * \frac{3}{2}$$

$$\begin{bmatrix} 2 & 1 & 0 & | & 0 \\ 0 & 0 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$x_3 = 0$$

$$x_2 = t$$

$$x_1 = -\frac{1}{2} t$$

$$v_2 = \begin{bmatrix} -\frac{1}{2} t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$$

Find eigenvectors when $\boldsymbol{\lambda_3 = -2}$

$$(A - \lambda \mathrm{I}) = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 1 & 0 & | & 0 \\ 0 & 3 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$x_3 = t$$

$$x_2 = -\frac{2}{3} t$$

$$x_1 = \frac{2}{15} t$$

$$v_3 = \begin{bmatrix} \frac{2}{15} t \\ -\frac{2}{3} t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{15} \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

Answers:

$$v_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, v_2 = \begin{bmatrix}-\frac{1}{2}\\1\\0\end{bmatrix}, v_3 = \begin{bmatrix}\frac{2}{15}\\-\frac{2}{3}\\1\end{bmatrix}$$

b)

$$D = \begin{bmatrix}3 & 0 & 0\\0 & 1 & 0\\0 & 0 & -2\end{bmatrix}$$

$$Q^{-1}AQ = R^{-1}BR$$

$$(Rx) \quad RQ^{-1}AQ = RR^{-1}BR$$

$$RQ^{-1}AQ = BR$$

$$(xR^{-1}) \quad RQ^{-1}AQR^{-1} = BRR^{-1}$$

$$RQ^{-1}AQR^{-1} = B$$

$$Let\ P = QR^{-1}$$

$$P^{-1}AP = B$$

$A\ matrix\ P\ is\ possible$

$\boldsymbol{Both\ A\ and\ B\ have\ same\ eigenvalues}$

$\lambda_1 = 3, \lambda_2 = 1, \lambda_3 = -2$

$\boldsymbol{Matrix\ A's\ eigenvectors}$:

$$v_1 = \begin{bmatrix}\frac{19}{10}\\\frac{2}{5}\\1\end{bmatrix}, v_2 = \begin{bmatrix}1\\0\\0\end{bmatrix}, v_3 = \begin{bmatrix}-\frac{2}{3}\\1\\0\end{bmatrix}$$

$\boldsymbol{Matrix\ B's\ eigenvectors}$:

$$v_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, v_2 = \begin{bmatrix}-\frac{1}{2}\\1\\0\end{bmatrix}, v_3 = \begin{bmatrix}\frac{2}{15}\\-\frac{2}{3}\\1\end{bmatrix}$$

$\boldsymbol{Find\ P}$

$$Q^{-1}AQ = D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, where\ Q = \begin{bmatrix} \frac{19}{10} & 1 & -\frac{2}{3} \\ \frac{2}{5} & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$R^{-1}BR = D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, where\ R = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{2}{15} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q^{-1}AQ = R^{-1}BR$$

$$RQ^{-1}AQR^{-1} = B$$

$$P^{-1}AP = B, where\ P = QR^{-1}$$

$\boldsymbol{Find\ R^{-1}}$

$Using\ gass - jordan\ to\ find\ inverse$

$$\begin{bmatrix} 1 & -\frac{1}{2} & \frac{2}{15} & | & 1 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

$$R1 = R1 + R2 * \frac{1}{2}$$

$$\begin{bmatrix} 1 & 0 & \frac{2}{15} & | & 1 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3} & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

$$R1 = R1 + R3 * \frac{1}{5}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 1 & \frac{1}{2} & \frac{1}{5} \\ 0 & 1 & -\frac{2}{3} & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

$$R2 = R2 + R3 * \frac{2}{3}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 1 & \frac{1}{2} & \frac{1}{5} \\ 0 & 1 & 0 & | & 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{R^{-1}} = \begin{bmatrix} \mathbf{1} & \frac{\mathbf{1}}{\mathbf{2}} & \frac{\mathbf{1}}{\mathbf{5}} \\ \mathbf{0} & \mathbf{1} & \frac{\mathbf{2}}{\mathbf{3}} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

$\boldsymbol{Calculate\ P}$:

$$\boldsymbol{P = QR^{-1}} = \begin{bmatrix} \frac{\mathbf{19}}{\mathbf{10}} & \mathbf{1} & -\frac{\mathbf{2}}{\mathbf{3}} \\ \frac{\mathbf{2}}{\mathbf{5}} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \end{bmatrix} * \begin{bmatrix} \mathbf{1} & \frac{\mathbf{1}}{\mathbf{2}} & \frac{\mathbf{1}}{\mathbf{5}} \\ \mathbf{0} & \mathbf{1} & \frac{\mathbf{2}}{\mathbf{3}} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} \frac{\mathbf{19}}{\mathbf{10}} & \frac{\mathbf{39}}{\mathbf{20}} & \frac{\mathbf{19}}{\mathbf{50}} \\ \frac{\mathbf{2}}{\mathbf{5}} & \frac{\mathbf{1}}{\mathbf{5}} & \frac{\mathbf{27}}{\mathbf{25}} \\ \mathbf{1} & \frac{\mathbf{1}}{\mathbf{2}} & \frac{\mathbf{1}}{\mathbf{5}} \end{bmatrix}$$

9.

a)

$\boldsymbol{Find\ eigenvalues\ for\ A}$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 2 \\ -3 & 5 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$\det\left(\begin{bmatrix} 1-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ -3 & 5 & 2-\lambda \end{bmatrix}\right)$$

$$(1-\lambda)\begin{bmatrix} 2-\lambda & 0 \\ 5 & 2-\lambda \end{bmatrix} - 0\begin{bmatrix} 2-\lambda & 0 \\ -3 & 2-\lambda \end{bmatrix} + 0\begin{bmatrix} 1 & 2-\lambda \\ -3 & 5 \end{bmatrix} = 0$$

$$(1-\lambda)(-\lambda+2)^2 = 0$$

$$\lambda_1 = 1, \lambda_2 = 2$$

b)

$\boldsymbol{Find\ eigenvectors\ for\ A}$

$\boldsymbol{Find\ eigenvectors\ when\ \lambda_1 = 1}$

$$(A - \lambda \mathrm{I}) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ -3 & 5 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & | & 0 \\ 1 & 1 & 0 & | & 0 \\ -3 & 5 & 1 & | & 0 \end{bmatrix}$$

$Swap\ R1\ and\ R2$

$$\begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ -3 & 5 & 1 & | & 0 \end{bmatrix}$$

$R3 = R3 + 3R1$

$$\begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 8 & 1 & | & 0 \end{bmatrix}$$

$Swap\ R2\ and\ R3$

$$\begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 8 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$x_3 = t$

$x_2 = -\frac{1}{8}t$

$x_1 = \frac{1}{8}t$

$$v_1 = \begin{bmatrix} \frac{1}{8}t \\ -\frac{1}{8}t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{1}{8} \\ -\frac{1}{8} \\ 1 \end{bmatrix}$$

$\boldsymbol{Find\ eigenvectors\ when\ \lambda_2 = 2}$

$$(A - \lambda \mathrm{I}) = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ -3 & 5 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 & | & 0 \\ 1 & 0 & 0 & | & 0 \\ -3 & 5 & 0 & | & 0 \end{bmatrix}$$

$R2 = R2 + R1$

$$\begin{bmatrix} -1 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ -3 & 5 & 0 & | & 0 \end{bmatrix}$$

$R3 = R3 - R1 * 3$

$$\begin{bmatrix} -1 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 5 & 0 & | & 0 \end{bmatrix}$$

$Swap\ R2\ and\ R3$

$$\begin{bmatrix} -1 & 0 & 0 & | & 0 \\ 0 & 5 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$x_3 = t$

$x_2 = 0$

$x_1 = 0$

$$v_2 = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

c)

$Formula:$

$B = P^{-1}AP$

$Eigenvalues\ Calculated\ From\ Before:$

$(1-\lambda)(-\lambda+2)^2 = 0$

$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 2$

$Form\ R = [v_1, v_2, w]$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ -3 & 5 & 2 \end{bmatrix}$$

$For\ the\ last\ column\ for\ R, it\ can\ be\ anything\ but\ it\ can't\ be\ equal\ to\ the\ first\ 2\ column.$

$Has\ to\ be\ linear\ indepdent$

$$R = \begin{bmatrix} \frac{1}{8} & 0 & 1 \\ -\frac{1}{8} & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$Check\ if\ determinant\ of\ R \neq 0$

$Det(R) = \frac{1}{8}, which\ means\ R\ can\ be\ inverted$

$\boldsymbol{Find\ R^{-1}}$

$Using\ gass - jordan\ to\ find\ inverse$

$$\begin{bmatrix} \frac{1}{8} & 0 & 1 & | & 1 & 0 & 0 \\ -\frac{1}{8} & 0 & 0 & | & 0 & 1 & 0 \\ 1 & 1 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

$R1 = R1 * 8$

$$\begin{bmatrix} 1 & 0 & 8 & | & 8 & 0 & 0 \\ -\frac{1}{8} & 0 & 0 & | & 0 & 1 & 0 \\ 1 & 1 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

$R2 = R2 + R1 * \frac{1}{8}$

$$\begin{bmatrix} 1 & 0 & 8 & | & 8 & 0 & 0 \\ 0 & 0 & 1 & | & 1 & 1 & 0 \\ 1 & 1 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

$R3 = R3 - R1$

$$\begin{bmatrix} 1 & 0 & 8 & | & 8 & 0 & 0 \\ 0 & 0 & 1 & | & 1 & 1 & 0 \\ 0 & 1 & -8 & | & -8 & 0 & 1 \end{bmatrix}$$

$Swap\ R2\ and\ R3$

$$\begin{bmatrix} 1 & 0 & 8 & | & 8 & 0 & 0 \\ 0 & 1 & -8 & | & -8 & 0 & 1 \\ 0 & 0 & 1 & | & 1 & 1 & 0 \end{bmatrix}$$

$R1 = R1 - R3 * 8$

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 & -8 & 0 \\ 0 & 1 & -8 & | & -8 & 0 & 1 \\ 0 & 0 & 1 & | & 1 & 1 & 0 \end{bmatrix}$$

$R2 = R2 + R3 * 8$

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 & -8 & 0 \\ 0 & 1 & 0 & | & 0 & 8 & 1 \\ 0 & 0 & 1 & | & 1 & 1 & 0 \end{bmatrix}$$

$$R^{-1} = \begin{bmatrix} 0 & -8 & 0 \\ 0 & 8 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$Let\ R = P$

$Put\ in\ in\ the\ formula\ B = P^{-1}AP$

$$\begin{bmatrix} 0 & -8 & 0 \\ 0 & 8 & 1 \\ 1 & 1 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix} * \begin{bmatrix} \frac{1}{8} & 0 & 1 \\ -\frac{1}{8} & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{-8} \\ \mathbf{0} & \mathbf{2} & \mathbf{5} \\ \mathbf{0} & \mathbf{0} & \mathbf{2} \end{bmatrix} = \boldsymbol{B}$$

$As\ we\ can\ see\ on\ the\ digonal, we\ have\ our\ eigenvalues\ we\ found\ earlier.$

$\boldsymbol{Verify\ that\ the\ determinant\ of\ B\ agrees\ with\ what\ you\ used\ in}\ (\boldsymbol{a})$:

$We\ can\ see\ that\ the\ determinant\ of\ B\ is\ 1 * 2 * 2 = 4, which\ was\ used\ in\ (a).$