SIT292 LINEAR ALGEBRA 2021
Assignment 2 Solutions

Total marks: 120

1. Find all the cofactors C_{ij} of

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -4 & 2 & 2 \end{bmatrix}$$

hence find adj$A$.

(ii) Verify that the adj A you obtained is correct by multiplying it with A.

Solution:

The matrix of cofactors $C_{i,j}$ of A is

$$\begin{bmatrix} 2 & -6 & 10 \\ -6 & -2 & -10 \\ 1 & -3 & -5 \end{bmatrix} \text{ (13.5 marks–1.5 marks/cofactor)}$$

adj(A) is the transpose of this matrix

$$\text{adj}(A) = \begin{bmatrix} 2 & -6 & 1 \\ -6 & -2 & -3 \\ 10 & -10 & -5 \end{bmatrix} \text{ (2.5 marks)}$$

Checking for the correctness of the adjoint:

$$\text{adj}(A)A = \begin{bmatrix} -20 & 0 & 0 \\ 0 & -20 & 0 \\ 0 & 0 & -20 \end{bmatrix} = A \text{ adj}(A) \quad |A| = -20 \text{ (4) marks}$$

2. Find all numbers α such that the vectors

$$\begin{bmatrix} 5 \\ \alpha \\ 3\alpha \\ 1 \\ 5 \end{bmatrix} \text{ and } \begin{bmatrix} \alpha \\ -\alpha \\ 3\alpha \\ -1 \\ 1 \end{bmatrix} \text{ are orthogonal.}$$

Solution:

The orthogonality condition for vectors (P 77 SG) is the matrix product:

$$\mathbf{x}^T\mathbf{y} = 0$$

Applying this condition to the vectors in question gives the equation

$$8\alpha^2 + 5\alpha + 4 \;=\; 0 \;\; (4 \text{ marks})$$

Solving for α gives $\alpha = \frac{-5+i\sqrt{103}}{16}$ or $\alpha = \frac{-5-i\sqrt{103}}{16}$ (2 marks).

3. Use Gaussian elimination to reduce the following system of equations to **row-echelon** form, hence solve for $x_1, \ldots, x_4$. Justify the correctness using matrix ranks correct

$$\begin{aligned} 2x_1 - 2x_2 + 2x_3 - 4x_4 &= 2 \\ x_1 + 4x_2 + 8x_3 + 2x_4 &= 5 \\ -x_1 + 9x_2 + 3x_3 - 4x_4 &= 5 \end{aligned}$$

Solution:

The matrix of the augmented system

$$A = \left[\begin{array}{rrrr|r} 2 & -2 & 2 & -4 & 2 \\ 1 & 4 & 8 & 2 & 5 \\ -1 & 9 & 3 & -4 & 5 \end{array}\right] \quad (2 \text{ marks})$$

Reducing A to the row echelon form by Gaussian elimination gives
divide row 1 by 2
$R_2 \to R_1 - R_2$,$R_3 \to R_3 + R_1$

$$\left[\begin{array}{rrrr|r} 1 & -1 & 1 & -2 & 1 \\ 0 & -5 & -7 & -4 & -4 \\ 0 & 8 & 4 & -6 & 6 \end{array}\right]$$

and then

$$\left[\begin{array}{rrrr|r} 1 & -1 & 1 & -2 & 1 \\ 0 & 5 & 7 & 4 & 4 \\ 0 & 0 & 18 & 31 & 1 \end{array}\right]$$

The last equation tells us $18x_3 + 31x_4 = 1$, taking $x_4 = t$ we get $x_3 = \frac{1}{18}(1 - 31t)$. Then $x_2 = \frac{1}{5}(4 - 4t - \frac{7}{18}(1 - 31t)) = \frac{1}{18}(13 + 29t)$.
$x_1 = 1 + 2t - \frac{1}{18}(1 - 31t) + \frac{1}{18}(13 + 29t) = \frac{1}{3}(5 + 16t)$
We can confirm that the system has infinitely many solutions because a) $rank(A) = 3$ and $rank(A|b) = 3$ (there is a solution), and b) $rank(A) = 3 < 4$ (the solution is not unique).

(8 marks)

4. Use Gaussian elimination to reduce the following system of equations to **row-echelon** form, hence solve for $x_1, \ldots, x_3$. Justify the correctness using matrix ranks correct

$$\begin{aligned} 3x_2 + 11x_3 &= 6 \\ x_1 + x_2 + 3x_3 &= 2 \\ 3x_1 - 3x_2 - 13x_3 &= -6 \\ -x_1 + 2x_2 + 8x_3 &= 4 \end{aligned}$$

Solution:

The matrix of the augmented system

$$A = \left[\begin{array}{rrr|r} 0 & 3 & 11 & 6 \\ 1 & 1 & 3 & 2 \\ 3 & -3 & -13 & -6 \\ -1 & 2 & 8 & 4 \end{array}\right] \quad \text{(2 marks)}$$

Reducing A to the row echelon form by Gaussian elimination gives Swapping row 1 and 2, then traditional row operations

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 3 & 11 & 6 \\ 0 & 3 & 11 & 6 \\ 0 & 3 & 11 & 6 \end{array}\right]$$

and then

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 3 & 11 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So we have $x_3 = t$, $x_2 = \frac{1}{3}(6 - 11t)$, $x_1 = 2 - 3t - \frac{1}{3}(6 - 11t) = 2t/3$

We can confirm that the system has infinitely many solutions because a) $rank(A) = 2$ and $rank(A|b) = 2$(there is a solution), and b) $rank(A) = 2 < 3$ (the solution is not unique).

(8 marks)

5. Use Gauss-Jordan elimination to find the inverse of the matrix

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

Solution: (put row 3 first)

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 2 & 1 & 0 & -1 \\ 0 & 0 & 2 & 0 & 1 & -1 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 & 1 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

Hence the inverse is

$$\begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

(10 marks)

6. Find eigenvalues and eigenvectors of the matrices A and B, where

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix}.$$

Solution:

(a) (1 mark)
The formula on P 102 of the SG gives

$$D_A(\lambda) = \lambda^3 - 2\lambda^2 + \lambda = 0$$

(b) (3 marks)
Solving for λ we get $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 1$. Factoring out λ gives a quadratic equation.

(c) Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be an eigenvector then the eigen equation for $\lambda_1 = 0$ is $A\mathbf{v} = 0\mathbf{v}$. Giving the matrix of the homogeneous system of linear equations

$$\begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Gaussian elimination reduces it to the row-echelon form

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence we have $y = z$ and $x = -z$. Choosing z as the independent variable (or x, y), say $z = 1$, then $y = 1$ and $x = -1$.

The eigenvector has the form of any multiple of

$$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Proceed with the other eigenvalue in the same way, reducing to the row echelon form and choosing one independent variable. We get multiples of $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ (only one eigenvector corresponding to $\lambda = 1$. (6 marks)

Find the roots of the characteristic polynomial of $B = \begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix}$

(a) (1 mark)
The formula on P 102 of the SG gives

$$D_B(\lambda) = \lambda^3 - 10\lambda^2 + 25\lambda = 0$$

(b) (3 marks)
Solving for λ we get $\lambda_1 = 0$, and then $\lambda_2 = \lambda_3 = 5$.

(c) Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be an eigenvector. Then the eigen equation for $\lambda = 0$ is $A\mathbf{v} = 0\mathbf{v}$. Giving the linear equations with the matrix

$$\begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix}$$

Gaussian elimination reduces it to the row-echelon form

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence we have $x + 2z = 0$ and $2y + 5z = 0$. Choosing z as the independent variable (or y, x), say $z = 2$, then $y = -5$ and $x = -4$.

The eigenvector has the form of any multiple of $\begin{bmatrix} -4 \\ -5 \\ 2 \end{bmatrix}$

The other eigenvector has the form of any multiple of

$\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ (6 marks)

Note that there is only one eigenvector that corresponds to this repeated eigenvalue.

7. Diagonalise the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Solution: The characteristic equation is

$$D_B(\lambda) = \lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

Guessing one root (1,-1,2,-2 are the choices), then performing long polynomial division leads to a quadratic equation.

The eigenvalues will be $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 2$.

(3 marks)

Solving for eigenvectors gives us

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(3 marks)

Then in $A = PDP^{-1}$ the matrices are

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ -1 & 1 & 1 \end{bmatrix} \quad P^{-1} = \frac{1}{6}\begin{bmatrix} 3 & 1 & 2 \\ 0 & -2 & 2 \\ -3 & 1 & 2 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(4 marks)

8. For the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 2 \\ 0 & 0 & 3 \end{bmatrix}, \ B = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

(a) find the eigenvalues and eigenvectors

(b) determine, if possible, a matrix P so that $P^{-1}AP = B$. If impossible, provide an argument for that. (Hint: Use the method in the Study Guide p.113).

Both A and B are triangular and their eigenvalues sit on the diagonal, and hence are $1, -2, 3$. (1 mark)

Let us find the eigenvectors. For matrix A, for $\lambda = 1$ we have

$$\begin{bmatrix} 0 & -2 & -3 \\ 0 & 3 & -2 \\ 0 & 0 & -2 \end{bmatrix} x = 0;$$

and hence

$$\begin{bmatrix} 0 & 2 & 3 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} x = 0;$$

Then $x_2 = x_3 = 0$ and x_1 arbitrary, so one eigenvector is

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda = -2$ we have

$$\begin{bmatrix} -3 & -2 & -3 \\ 0 & 0 & -2 \\ 0 & 0 & -5 \end{bmatrix} x = 0;$$

That is $x_3 = 0$, and then $3x_1 = -2x_2$, and x_1 is arbitrary. So we choose

$$v_2 = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$$

For $\lambda = 3$ we have

$$\begin{bmatrix} 2 & -2 & -3 \\ 0 & 5 & -2 \\ 0 & 0 & 0 \end{bmatrix} x = 0;$$

Choose $x_2 = 2$. Then $x_3 = 5$ and then $x_1 = 19/2$. So we have (multiplying by 2 to get integers)

$$v_3 = \begin{bmatrix} 19 \\ 4 \\ 10 \end{bmatrix}$$

For matrix B, for $\lambda = 1$ we have

$$v_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

For $\lambda = -2$ we have

$$v_2 = \begin{bmatrix} 2 \\ -10 \\ 15 \end{bmatrix}$$

For $\lambda = 3$ we have

$$v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(9 marks for all eigenvectors).

(b) Now, as stated in the lecture notes and in the Study Guide p.113, (or tutorial 8): we show that $A \sim D$ and $B \sim D$ then by transitivity $A \sim B$, for some diagonal matrix D.

In order to diagonalise A we use

$$Q = \begin{bmatrix} 1 & 2 & 19 \\ 0 & -3 & 4 \\ 0 & 0 & 10 \end{bmatrix}$$

obtained from its eigenvectors. Then D has the form

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

In order to diagonalise B and have the same D we use

$$R = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -10 & 0 \\ 0 & 15 & 0 \end{bmatrix}$$

(note the order in which we take the eigenvectors, it must be the same as the order of the respective eigenvalues in D). Then $Q^{-1}AQ = D = R^{-1}BR$, and therefore $(QR^{-1})^{-1}A(QR^{-1}) = B$, so the matrix P we need (that is, the matrix satisfying $P^{-1}AP = B$) is $P = QR^{-1}$.

By calculating

$$R^{-1} = \frac{1}{30}\begin{bmatrix} 0 & -15 & -10 \\ 0 & 0 & 2 \\ 30 & 15 & 6 \end{bmatrix}$$

(the method of calculating the inverse is on p. 91 of the SG).

Therefore we obtain

$$P = QR^{-1} = \begin{bmatrix} 19 & 9 & 18/5 \\ 4 & 2 & 3/5 \\ 10 & 5 & 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 95 & 45 & 18 \\ 20 & 10 & 3 \\ 50 & 25 & 10 \end{bmatrix}.$$

It is easy to check that such a matrix P satisfies $AP = PB$. (4 marks)

This problem could have been solved by using Example 4.7 of the SG (but for 9 unknown entries). Also note that students could have used eigenvectors in a different order (like $3, -2, 1$) and the intermediate calculations would be different.

9. For the following matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$$

(a) find the eigenvalues

(b) for each eigenvalue determine the eigenvector(s)

(c) determine a matrix P so that $B = P^{-1}AP$ is in **triangular** form, and verify that the determinant of B agrees with what you used in (a)

(a) For the eigenvalues: they are all on the diagonal, hence $1, 2, 2$.

(5 marks ;-))

(b) the eigenvectors will be $v_1 = \begin{bmatrix} 1 \\ -1 \\ 8 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ or their multiples, and no third eigenvector. Hence this matrix is not diagonalisable.

(5 marks)

(c) For trianglisation we follow the process on p. 124. (OK, Some students could have noticed that the matrix is in the lower triangular form already. I can give 2 marks for that observation, but not 10, as the question asks to follow the trianglisation procedure and arrive to the upper triangular form).

Pick one eigenvector and define matrix

$$S = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 8 & 0 & 1 \end{bmatrix}$$

Note that I picked the first eigenvector. Calculating the inverse of S is not that difficult. Of course the end result would be valid if we pick another eigenvector.

So

$$S^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -8 & 0 & 1 \end{bmatrix}$$

Next we compute

$$S^{-1}AS = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -8 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 8 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 5 & 2 \end{bmatrix}$$

Notice the first column has our eigenvalue 1 and two zeros. Next consider the 2×2 block in the bottom right corner.

$$A_1 = \begin{bmatrix} 2 & 0 \\ 5 & 2 \end{bmatrix}$$

We take the second of our eigenvalues 2 and determine its eigenvector for this matrix A_1, which will be a multiple of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then define the matrix

$$Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

This matrix will trianglise A_1. So in order to apply it to a 3×3 matrix (see p. 127) we use

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Then the desired

$$P = SR = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 8 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 8 & 1 & 0 \end{bmatrix}.$$

Finally check that

$$B = P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

is upper triangular.

Note that the eigenvalues are on the diagonal of B and are exactly $1, 2, 2$ in the same order we took the eigenvectors. The determinant of B is the product of the eigenvalues, which are the same for A, and hence $det(B) = det(A) = 4$.

(10 marks)

Another approach is based on transposition. Note that the matrix A^T is upper triangular as required. Transposition is rearranging rows and columns, in this case first swapping rows 1 and 3, and then swapping columns 1 and 3. Swapping rows is multiplication by a permutation matrix from the left

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Swapping columns is also a permutation matrix but multiplying from the right, and since it is also about columns 1 nd 3, so it will be the same P (notice it is an orthogonal matrix as well). Therefore we have

$$P^{-1}AP = P^TAP = A^T = \begin{bmatrix} 2 & 5 & -3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

So a different but equally valid answer.