1.

a.

$$A = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & 4 & -2 & -2 \\ 3 & 6 & -4 & 10 \end{bmatrix}$$

$Use\ row -\ echelon\ form$

$R2 = R2 - R1 * 2$

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & -2 \\ 3 & 6 & -4 & 10 \end{bmatrix}$$

$R3 = R3 - R1 * 3$

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 5 & 10 \end{bmatrix}$$

$R3 = R3 - R2 * \frac{5}{4}$

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 0 & \frac{25}{2} \end{bmatrix}$$

$Row - rank = 3$

b.

$v_1 = \{1, 2, -3, 0\}$

$v_2 = \{0, 0, 4, -2\}$

$v_3 = \left\{0, 0, 0, \frac{25}{2}\right\}$

$\boldsymbol{Explain\ why\ these\ vectors\ are\ generators}.$

Because they generate the new row-space of A

c.

$Basis\ for\ Row(A)$

$$= \left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{25}{2} \end{bmatrix} \right\}$$

These are the basis of A because the non-zero rows in the reduced row-echelon form the basis for the row space A.

$\dim(Row(A)) = 3$. The vectors are linear independent.

2.

a.

$$\begin{bmatrix} -2 & 2 & 3 \\ -2 & 3 & 2 \\ -4 & 2 & 5 \end{bmatrix}$$

$$\det\left(\begin{bmatrix} -2 & 2 & 3 \\ -2 & 3 & 2 \\ -4 & 2 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$\det\left(\begin{bmatrix} -2-\lambda & 2 & 3 \\ -2 & 3-\lambda & 2 \\ -4 & 2 & 5-\lambda \end{bmatrix}\right) = 0$$

$$(-2-\ \lambda) * det \begin{bmatrix} 3-\lambda & 2 \\ 2 & 5-\lambda \end{bmatrix} - (2) * det \begin{bmatrix} -2 & 2 \\ -4 & 5-\lambda \end{bmatrix} + (3) * det \begin{bmatrix} -2 & 3-\lambda \\ -4 & 2 \end{bmatrix} = 0$$

$$(-2-x)(x^2-8x+11) - 2(2x-2) + 3(-4x+8) = 0$$

$$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$$

b.

$$\boldsymbol{Find\ eigenvectors\ for\ A}$$

$$\boldsymbol{Find\ eigenvectors\ when\ \lambda_1 = 1}$$

$$(A-\ \lambda \mathrm{I}) = \begin{bmatrix} -3 & 2 & 3 \\ -2 & 2 & 2 \\ -4 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$Using\ gaussian\ elimination$$

$$\begin{bmatrix} -3 & 2 & 3 & | & 0 \\ -2 & 2 & 2 & | & 0 \\ -4 & 0 & 4 & | & 0 \end{bmatrix}$$

$$R2 = R2 - R1 * \frac{2}{3}$$

$$\begin{bmatrix} -3 & 2 & 3 & | & 0 \\ 0 & \frac{2}{3} & 0 & | & 0 \\ -4 & 0 & 4 & | & 0 \end{bmatrix}$$

$$R3 = R3 - R1 * \frac{4}{3}$$

$$\begin{bmatrix} -3 & 2 & 3 & | & 0 \\ 0 & \frac{2}{3} & 0 & | & 0 \\ 0 & -\frac{2}{3} & 0 & | & 0 \end{bmatrix}$$

$R3 = R3 + R2$

$$\begin{bmatrix} -3 & 2 & 3 & | & 0 \\ 0 & \frac{2}{3} & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$x_3 = t$

$x_2 = 0$

$x_1 = x_3$

$$v_1 = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix}, v_1 = t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\boldsymbol{Find\ eigenvectors\ when\ \lambda_2 = 2}$

$$(A - \lambda \mathrm{I}) = \begin{bmatrix} -4 & 2 & 3 \\ -2 & 1 & 2 \\ -4 & 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$Using\ gaussian\ elimination$

$$\begin{bmatrix} -4 & 2 & 3 & | & 0 \\ -2 & 1 & 2 & | & 0 \\ -4 & 0 & 3 & | & 0 \end{bmatrix}$$

$R2 = R2 - R1 * \frac{1}{2}$

$$\begin{bmatrix} -4 & 2 & 3 & | & 0 \\ 0 & 0 & \frac{1}{2} & | & 0 \\ -4 & 0 & 3 & | & 0 \end{bmatrix}$$

$R3 = R3 - R1$

$$\begin{bmatrix} -4 & 2 & 3 & | & 0 \\ 0 & 0 & \frac{1}{2} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$x_3 = 0$

$x_2 = t$

$x_1 = \frac{1}{2}x_2$

$$v_2 = \begin{bmatrix} \frac{1}{2}t \\ t \\ 0 \end{bmatrix}, v_2 = t\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$$

$\boldsymbol{Find\ eigenvectors\ when\ \lambda_3 = 3}$

$$(A - \lambda \mathrm{I}) = \begin{bmatrix} -5 & 2 & 3 \\ -2 & 0 & 2 \\ -4 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using gaussian elimination

$$\begin{bmatrix} -5 & 2 & 3 & | & 0 \\ -2 & 0 & 2 & | & 0 \\ -4 & 2 & 2 & | & 0 \end{bmatrix}$$

$$R2 = R2 - R1 * \frac{2}{5}$$

$$\begin{bmatrix} -5 & 2 & 3 & | & 0 \\ 0 & -\frac{4}{5} & \frac{4}{5} & | & 0 \\ -4 & 2 & 2 & | & 0 \end{bmatrix}$$

$$R3 = R3 - R1 * \frac{4}{5}$$

$$\begin{bmatrix} -5 & 2 & 3 & | & 0 \\ 0 & -\frac{4}{5} & \frac{4}{5} & | & 0 \\ 0 & \frac{2}{5} & -\frac{2}{5} & | & 0 \end{bmatrix}$$

$$R3 = R3 + R2 * \frac{1}{2}$$

$$\begin{bmatrix} -5 & 2 & 3 & | & 0 \\ 0 & -\frac{4}{5} & \frac{4}{5} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$x_3 = t$

$x_2 = x_3$

$x_1 = x_3$

$$v_3 = \begin{bmatrix} t \\ t \\ t \end{bmatrix}, v_3 = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\boldsymbol{Answers}$:

$$v_1 = \begin{bmatrix}1\\0\\1\end{bmatrix},\; v_2 = \begin{bmatrix}\frac{1}{2}\\1\\0\end{bmatrix}, v_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

C.

$$\boldsymbol{Gram - Schmidt}$$

$$v_1 = \begin{bmatrix}1\\0\\1\end{bmatrix},\; v_2 = \begin{bmatrix}\frac{1}{2}\\1\\0\end{bmatrix}, v_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$\boldsymbol{u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}} * \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix}}$$

$$u_2' = v_2 - (u_1 \cdot v_2)u_1 = \begin{bmatrix}\frac{1}{2}\\1\\0\end{bmatrix} - \left(\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix} \cdot \begin{bmatrix}\frac{1}{2}\\1\\0\end{bmatrix}\right) * \begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\1\\0\end{bmatrix} - \begin{bmatrix}\frac{1}{4}\\0\\\frac{1}{4}\end{bmatrix} = \begin{bmatrix}\frac{1}{4}\\1\\-\frac{1}{4}\end{bmatrix}$$

$$\boldsymbol{u_2 = \frac{u_2'}{\|u_2'\|} = \begin{bmatrix}\frac{1}{4}\\1\\-\frac{1}{4}\end{bmatrix} * \frac{4}{\sqrt{18}} = \begin{bmatrix}\frac{1}{\sqrt{18}}\\\frac{4}{\sqrt{18}}\\-\frac{1}{\sqrt{18}}\end{bmatrix} = \begin{bmatrix}\frac{1}{3\sqrt{2}}\\\frac{2\sqrt{2}}{3}\\-\frac{1}{3\sqrt{2}}\end{bmatrix}}$$

$$u_3' = v_3 - (u_2 \cdot v_3)u_2 - (u_1 \cdot v_3)u_1 = \begin{bmatrix}1\\1\\1\end{bmatrix} - \left(\begin{bmatrix}\frac{1}{3\sqrt{2}}\\\frac{2\sqrt{2}}{3}\\-\frac{1}{3\sqrt{2}}\end{bmatrix} \cdot \begin{bmatrix}1\\1\\1\end{bmatrix}\right) * \begin{bmatrix}\frac{1}{3\sqrt{2}}\\\frac{2\sqrt{2}}{3}\\\frac{1}{3\sqrt{2}}\end{bmatrix} - \left(\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix} \cdot \begin{bmatrix}1\\1\\1\end{bmatrix}\right) * \begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix} =$$

$$\begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}\frac{2}{9}\\\frac{8}{9}\\-\frac{2}{9}\end{bmatrix} - \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}-\frac{2}{9}\\\frac{1}{9}\\\frac{2}{9}\end{bmatrix}$$

$$\boldsymbol{u_3} = \frac{\boldsymbol{u_3'}}{\|\boldsymbol{u_3'}\|} = \begin{bmatrix} -\frac{2}{9} \\ \frac{1}{9} \\ \frac{2}{9} \end{bmatrix} * 3 = \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

3.

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 2 & -1 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$Swap\ R1\ and\ R2$

$$\begin{bmatrix} 2 & -1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$R3 = R3 - R1 * \frac{1}{2}$

$$\begin{bmatrix} 2 & -1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & \frac{1}{2} & -1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$R3 = R3 - R2 * \frac{1}{2}$

$$\begin{bmatrix} 2 & -1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$R4 = R4 - R3 * \frac{2}{3}$

$$\begin{bmatrix} 2 & -1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{4}{3} \end{bmatrix}$$

$Rank = 4, number\ of\ vectors\ is\ 4, hence\ linearly\ independent.$

$However, those\ 4$
$-\ tuples\ isn't\ a\ basis\ for\ \mathbb{R}^3 because\ \mathbb{R}^3\ requires\ 3\ linearly\ independent\ vectors.$

$But\ it\ is\ however, a\ basis\ of\ \mathbb{R}^4. 4 > 3$

$$c_1(0,2,1,0) + c_2(1,-1,0,0) + c_3(1,2,0,-1) + c_4(1,0,0,1) = (a,b,c,d)$$

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 2 & -1 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} * \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

The rank = 4, thus there are unique solutions, equal to the number of unknows, hence we can always find $c_1, \dots\dots, c_4$, which means that the set generates the space, hence it is a basis.

$$v_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix},\ v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, v_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\boldsymbol{u_1} = \frac{\boldsymbol{v_1}}{\|\boldsymbol{v_1}\|} = \frac{\boldsymbol{1}}{\sqrt{\boldsymbol{5}}} * \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{2} \\ \boldsymbol{1} \\ \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \frac{\boldsymbol{2}}{\sqrt{\boldsymbol{5}}} \\ \frac{\boldsymbol{1}}{\sqrt{\boldsymbol{5}}} \\ \boldsymbol{0} \end{bmatrix}$$

$$u_2' = v_2 - (u_1 \cdot v_2)u_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right) * \begin{bmatrix} 0 \\ \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -\frac{4}{5} \\ -\frac{2}{5} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{5} \\ \frac{2}{5} \\ 0 \end{bmatrix}$$

$$\boldsymbol{u_2} = \frac{\boldsymbol{u_2'}}{\|\boldsymbol{u_2'}\|} = \frac{\boldsymbol{5}}{\sqrt{\boldsymbol{30}}} * \begin{bmatrix} \boldsymbol{1} \\ -\frac{\boldsymbol{1}}{\boldsymbol{5}} \\ \frac{\boldsymbol{2}}{\boldsymbol{5}} \\ \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{\boldsymbol{5}}{\boldsymbol{6}}} \\ -\frac{\boldsymbol{1}}{\sqrt{\boldsymbol{30}}} \\ \sqrt{\frac{\boldsymbol{2}}{\boldsymbol{15}}} \\ \boldsymbol{0} \end{bmatrix}$$

$$u_3' = v_3 - (u_2 \cdot v_3)u_2 - (u_1 \cdot v_3)u_1$$

$$= \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} - \left(\begin{bmatrix} \sqrt{\frac{5}{6}} \\ -\frac{1}{\sqrt{30}} \\ \sqrt{\frac{2}{15}} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} \right) * \begin{bmatrix} \sqrt{\frac{5}{6}} \\ -\frac{1}{\sqrt{30}} \\ \sqrt{\frac{2}{15}} \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} \right) * \begin{bmatrix} 0 \\ \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{10} \\ \frac{1}{5} \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{8}{5} \\ \frac{4}{5} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \\ -1 \end{bmatrix}$$

$$\boldsymbol{u_3} = \frac{\boldsymbol{u_3'}}{\|\boldsymbol{u_3'}\|} = \frac{\mathbf{2}}{\sqrt{\mathbf{10}}} * \begin{bmatrix} \frac{\mathbf{1}}{\mathbf{2}} \\ \frac{\mathbf{1}}{\mathbf{2}} \\ -\mathbf{1} \\ -\mathbf{1} \end{bmatrix} = \begin{bmatrix} \frac{\mathbf{1}}{\sqrt{\mathbf{10}}} \\ \frac{\mathbf{1}}{\sqrt{\mathbf{10}}} \\ -\frac{\mathbf{2}}{\sqrt{\mathbf{10}}} \\ -\frac{\mathbf{2}}{\sqrt{\mathbf{10}}} \end{bmatrix}$$

$$u_4' = v_4 - (u_3 \cdot v_4)u_3 - (u_2 \cdot v_4)u_2 - (u_1 \cdot v_4)u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \\ -\frac{2}{\sqrt{10}} \\ -\frac{2}{\sqrt{10}} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) * \begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \\ -\frac{2}{\sqrt{10}} \\ -\frac{2}{\sqrt{10}} \end{bmatrix} -$$

$$\left(\begin{bmatrix} \sqrt{\frac{5}{6}} \\ -\frac{1}{\sqrt{30}} \\ \sqrt{\frac{2}{15}} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) * \begin{bmatrix} \sqrt{\frac{5}{6}} \\ -\frac{1}{\sqrt{30}} \\ \sqrt{\frac{2}{15}} \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) * \begin{bmatrix} 0 \\ \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -\frac{1}{10} \\ -\frac{1}{10} \\ \frac{1}{5} \\ \frac{1}{5} \end{bmatrix} - \begin{bmatrix} \frac{5}{6} \\ -\frac{1}{6} \\ \frac{1}{3} \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{4}{15} \\ \frac{4}{15} \\ -\frac{8}{15} \\ \frac{4}{5} \end{bmatrix}$$

$$\boldsymbol{u_4} = \frac{\boldsymbol{u'_4}}{\|\boldsymbol{u'_4}\|} = \begin{bmatrix} \frac{4}{15} \\ \frac{4}{15} \\ -\frac{8}{15} \\ \frac{4}{5} \end{bmatrix} * \frac{\sqrt{15}}{4} = \begin{bmatrix} \frac{1}{\sqrt{15}} \\ \frac{1}{\sqrt{15}} \\ -\frac{2}{\sqrt{15}} \\ \sqrt{\frac{3}{5}} \end{bmatrix}$$

Answers:

$$\boldsymbol{u_1} = \begin{bmatrix} 0 \\ \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix}, \boldsymbol{u_2} = \begin{bmatrix} \sqrt{\frac{5}{6}} \\ -\frac{1}{\sqrt{30}} \\ \sqrt{\frac{2}{15}} \\ 0 \end{bmatrix}, \boldsymbol{u_3} = \begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \\ -\frac{2}{\sqrt{10}} \\ -\frac{2}{\sqrt{10}} \end{bmatrix}, \boldsymbol{u_4} = \begin{bmatrix} \frac{1}{\sqrt{15}} \\ \frac{1}{\sqrt{15}} \\ -\frac{2}{\sqrt{15}} \\ \sqrt{\frac{3}{5}} \end{bmatrix}$$

Verify that the resulting vectors are orthonormal

$$u_1 \cdot u_2 = \begin{bmatrix} 0 \\ \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{\frac{5}{6}} \\ -\frac{1}{\sqrt{30}} \\ \sqrt{\frac{2}{15}} \\ 0 \end{bmatrix} = 0$$

$$u_1 \cdot u_3 = \begin{bmatrix} 0 \\ \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \\ -\frac{2}{\sqrt{10}} \\ -\frac{2}{\sqrt{10}} \end{bmatrix} = 0$$

$$u_1 \cdot u_4 = \begin{bmatrix} 0 \\ \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{15}} \\ \frac{1}{\sqrt{15}} \\ -\frac{2}{\sqrt{15}} \\ \sqrt{\frac{3}{5}} \end{bmatrix} = 0$$

$$u_2 \cdot u_3 = \begin{bmatrix} \sqrt{\frac{5}{6}} \\ -\frac{1}{\sqrt{30}} \\ \sqrt{\frac{2}{15}} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \\ -\frac{2}{\sqrt{10}} \\ -\frac{2}{\sqrt{10}} \end{bmatrix} = 0$$

$$u_2 \cdot u_4 = \begin{bmatrix} \sqrt{\frac{5}{6}} \\ -\frac{1}{\sqrt{30}} \\ \sqrt{\frac{2}{15}} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{15}} \\ \frac{1}{\sqrt{15}} \\ -\frac{2}{\sqrt{15}} \\ \sqrt{\frac{3}{5}} \end{bmatrix} = 0$$

$$u_3 \cdot u_4 = \begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \\ -\frac{2}{\sqrt{10}} \\ -\frac{2}{\sqrt{10}} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{15}} \\ \frac{1}{\sqrt{15}} \\ -\frac{2}{\sqrt{15}} \\ \sqrt{\frac{3}{5}} \end{bmatrix} = 0$$

4.

a.

$$(v_1 + v_2)^T \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} = v_1^T \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} + v_2^T \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} = 0 + 0 + 0$$

$hence\ (v_1 + v_2)^T \in V$

$$(kv)^T\begin{bmatrix}0\\-1\\1\\1\end{bmatrix} = (kv)^T\begin{bmatrix}0\\-1\\1\\1\end{bmatrix} = k \times 0 = 0$$

$hence\ kv \in V$

b.

$-b + c + d = 0$

$b = c + d$

$= (a, c + d, c, d)$

$= a(1,0,0,0) + c(0,1,1,0) + d(0,1,0,1)$

$$= \left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}\right\}$$

$$= \begin{bmatrix}1 & 0 & 0 & | & 0\\0 & 1 & 1 & | & 0\\0 & 1 & 0 & | & 0\\0 & 0 & 1 & | & 0\end{bmatrix}$$

$R3 = R3 - R2$

$$\begin{bmatrix}1 & 0 & 0 & | & 0\\0 & 1 & 1 & | & 0\\0 & 0 & -1 & | & 0\\0 & 0 & 1 & | & 0\end{bmatrix}$$

$R4 = R4 + R3$

$$\begin{bmatrix}1 & 0 & 0 & | & 0\\0 & 1 & 1 & | & 0\\0 & 0 & -1 & | & 0\\0 & 0 & 0 & | & 0\end{bmatrix}$$

$Rank = 3 = \dim(3)$

c.

$nullspace = (0, -1, 1, 1)$

$rowspace = (1, 0, 0, 0), (0, 1, 1, 0), (0, 1, -1, 0)$

$\dim(nullspace) = 1$

$\dim(rowspace) = 3$

$\dim(nullspace) + \dim(rowspace) = 4 = \mathbb{R}^4, therefore\ it\ holds\ for\ rank\ nullity\ theorom.$

5.

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 3 & 5 \\ 3 & 6 & 4 & 9 \\ 1 & 2 & 4 & 3 \end{bmatrix}$$

$Put\ it\ in\ row - reduced\ form$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 3 & 5 \\ 3 & 6 & 4 & 9 \\ 1 & 2 & 4 & 3 \end{bmatrix}$$

$R2 = R2 - R1 * 2$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 3 & 6 & 4 & 9 \\ 1 & 2 & 4 & 3 \end{bmatrix}$$

$R3 = R3 - R1 * 3$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 3 \\ 1 & 2 & 4 & 3 \end{bmatrix}$$

$R4 = R4 - R1$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

$R3 = R3 - R2$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

$R4 = R4 - R2 * 3$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$R4 = R4 + R3$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$Rank(A) = 3 = \dim(rowspace) = \dim\,(V)$

6.

$u_1 = 1010010$

$u_2 = 1100001$

$u_3 = 0101000$

$u_4 = 0010101$

a)

$v_1, v_2, v_3, v_4\ must\ end\ in\ 1000, 0100, 0010, 0001.$

$v_1 = u_3,\ v_2 = u_4, v_3 = u_1, v_4 = u_2$

$\{u_3, u_4, u_1, u_2\}^T$

$$\begin{bmatrix} 0 & 1 & 0 & | & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & | & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & | & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & | & 0 & 0 & 0 & 1 \end{bmatrix}$$

$R2 = R2 + R4$

$$\begin{bmatrix} 0 & 1 & 0 & | & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & | & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & | & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & | & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$G = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b)

$\boldsymbol{Code\ words\ for\ the\ message\ 1001}$

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0*1+1*0+1*0+1*1=\mathbf{1} \\ 1*1+1*0+0*0+1*1=\mathbf{0} \\ 0*1+1*0+1*0+0*1=\mathbf{0} \\ 1*1+0*0+0*0+0*1=\mathbf{1} \\ 0*0+1*0+0*0+0*1=\mathbf{0} \\ 0*0+0*0+1*0+0*1=\mathbf{0} \\ 0*0+0*0+0*0+1*1=\mathbf{1} \end{bmatrix}$$

$\boldsymbol{Code\ words\ for\ the\ message\ 0101}$

$$\begin{bmatrix}0 & 1 & 1 & 1\\1 & 1 & 0 & 1\\0 & 1 & 1 & 0\\1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix} \odot \begin{bmatrix}0\\1\\0\\1\end{bmatrix} = \begin{bmatrix}0*0+1*1+1*0+1*1=\mathbf{0}\\1*0+1*1+0*0+1*1=\mathbf{0}\\0*0+1*1+1*0+0*1=\mathbf{1}\\1*0+0*1+0*0+0*1=\mathbf{0}\\0*0+1*1+0*0+0*1=\mathbf{1}\\0*0+0*1+1*0+0*1=\mathbf{0}\\0*0+0*1+1*0+1*1=\mathbf{1}\end{bmatrix}$$

c)

$$P = \begin{bmatrix}0 & 1 & 1 & 1\\1 & 1 & 0 & 1\\0 & 1 & 1 & 0\end{bmatrix}$$

$$H = \begin{bmatrix}1 & 0 & 0 & 0 & 1 & 1 & 1\\0 & 1 & 0 & 1 & 1 & 0 & 1\\0 & 0 & 1 & 0 & 1 & 1 & 0\end{bmatrix}$$

$$H = [I_3 P]$$

d)

$\boldsymbol{Syndromes\ for\ 1110011}$

$$\begin{bmatrix}1 & 0 & 0 & 0 & 1 & 1 & 1\\0 & 1 & 0 & 1 & 1 & 0 & 1\\0 & 0 & 1 & 0 & 1 & 1 & 0\end{bmatrix} \odot \begin{bmatrix}1\\1\\1\\0\\0\\1\\1\end{bmatrix} = \begin{bmatrix}1+0+0+0+0+1+1=\mathbf{1}\\0+1+0+0+0+0+1=\mathbf{0}\\0+0+1+0+0+1+0=\mathbf{0}\end{bmatrix}$$

$\boldsymbol{Syndromes\ for\ 1001010}$

$$\begin{bmatrix}1 & 0 & 0 & 0 & 1 & 1 & 1\\0 & 1 & 0 & 1 & 1 & 0 & 1\\0 & 0 & 1 & 0 & 1 & 1 & 0\end{bmatrix} \odot \begin{bmatrix}1\\0\\0\\1\\0\\1\\0\end{bmatrix} = \begin{bmatrix}1+0+0+0+0+1+0=\mathbf{0}\\0+0+0+1+0+0+0=\mathbf{1}\\0+0+0+0+0+1+0=\mathbf{1}\end{bmatrix}$$

$\boldsymbol{Syndromes\ for\ 0001101}$

$$\begin{bmatrix}1 & 0 & 0 & 0 & 1 & 1 & 1\\0 & 1 & 0 & 1 & 1 & 0 & 1\\0 & 0 & 1 & 0 & 1 & 1 & 0\end{bmatrix} \odot \begin{bmatrix}0\\0\\0\\1\\1\\0\\1\end{bmatrix} = \begin{bmatrix}0+0+0+0+1+0+1=\mathbf{0}\\0+0+0+1+1+0+1=\mathbf{1}\\0+0+0+0+1+0+0=\mathbf{1}\end{bmatrix}$$

$\boldsymbol{Syndromes\ for\ 1101010}$

$$\begin{bmatrix}1&0&0&0&1&1&1\\0&1&0&1&1&0&1\\0&0&1&0&1&1&0\end{bmatrix}\odot\begin{bmatrix}1\\1\\0\\1\\0\\1\\0\end{bmatrix}=\begin{bmatrix}1+0+0+0+0+1+0=\mathbf{0}\\0+1+0+1+0+0+0=\mathbf{0}\\0+0+0+0+0+1+0=\mathbf{1}\end{bmatrix}$$

7.

$$det\begin{bmatrix}1&-1&2\\0&2&1\\2&5&-1\end{bmatrix}$$

$$1\begin{bmatrix}2&1\\5&-1\end{bmatrix}+(-1)\begin{bmatrix}0&1\\2&-1\end{bmatrix}+2\begin{bmatrix}0&2\\2&5\end{bmatrix}$$

$$=-7-2-8$$

$$=-17\neq 0$$

$Therefore, it\ is\ linearly\ independent$

$$\begin{bmatrix}1&-1&2\\0&2&1\\2&5&-1\end{bmatrix}$$

$$R3=R3-R1*2$$

$$\begin{bmatrix}1&-1&2\\0&2&1\\0&7&-5\end{bmatrix}$$

$$R3=R3-R2*\frac{7}{2}$$

$$\begin{bmatrix}1&-1&2\\0&2&1\\0&0&-\frac{17}{2}\end{bmatrix}$$

$$Rank=3=dim=3$$

$Therefore, P_2\ is\ a\ basis.$

$$v_1=\begin{bmatrix}1\\-1\\2\end{bmatrix},\ v_2=\begin{bmatrix}0\\2\\1\end{bmatrix}, v_3=\begin{bmatrix}2\\5\\-1\end{bmatrix}$$

$$\boldsymbol{u_1} = \frac{\boldsymbol{v_1}}{\|\boldsymbol{v_1}\|} = \frac{\mathbf{1}}{\sqrt{\mathbf{6}}} * \begin{bmatrix} \mathbf{1} \\ \mathbf{-1} \\ \mathbf{2} \end{bmatrix} = \begin{bmatrix} \frac{\mathbf{1}}{\sqrt{\mathbf{6}}} \\ -\frac{\mathbf{1}}{\sqrt{\mathbf{6}}} \\ \frac{\mathbf{2}}{\sqrt{\mathbf{6}}} \end{bmatrix}$$

$$u_2' = v_2 - (u_1 \cdot v_2)u_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right) * \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$\boldsymbol{u_2} = \frac{\boldsymbol{u_2'}}{\|\boldsymbol{u_2'}\|} = \begin{bmatrix} \mathbf{0} \\ \mathbf{2} \\ \mathbf{1} \end{bmatrix} * \frac{\mathbf{1}}{\sqrt{\mathbf{5}}} = \begin{bmatrix} \mathbf{0} \\ \frac{\mathbf{2}}{\sqrt{\mathbf{5}}} \\ \frac{\mathbf{1}}{\sqrt{\mathbf{5}}} \end{bmatrix}$$

$$u_3' = v_3 - (u_2 \cdot v_3)u_2 - (u_1 \cdot v_3)u_1$$

$$= \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} \right) * \begin{bmatrix} 0 \\ \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} - \left(\begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} \right) * \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{18}{5} \\ \frac{9}{5} \end{bmatrix} - \begin{bmatrix} -\frac{5}{6} \\ \frac{5}{6} \\ -\frac{5}{3} \end{bmatrix} = \begin{bmatrix} \frac{17}{6} \\ \frac{17}{30} \\ -\frac{17}{15} \end{bmatrix}$$

$$\boldsymbol{u_3} = \frac{\boldsymbol{u_3'}}{\|\boldsymbol{u_3'}\|} = \frac{\mathbf{1}}{\frac{\mathbf{17}}{\sqrt{\mathbf{30}}}} * \begin{bmatrix} \frac{\mathbf{17}}{\mathbf{6}} \\ \frac{\mathbf{17}}{\mathbf{30}} \\ -\frac{\mathbf{17}}{\mathbf{15}} \end{bmatrix} = \begin{bmatrix} \frac{\mathbf{5}}{\sqrt{\mathbf{30}}} \\ \frac{\mathbf{1}}{\sqrt{\mathbf{30}}} \\ -\frac{\mathbf{2}}{\sqrt{\mathbf{30}}} \end{bmatrix}$$

Answers:

$$\boldsymbol{u_1} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}, \boldsymbol{u_2} = \begin{bmatrix} 0 \\ \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}, \boldsymbol{u_3} = \begin{bmatrix} \frac{5}{\sqrt{30}} \\ \frac{1}{\sqrt{30}} \\ -\frac{2}{\sqrt{30}} \end{bmatrix}$$

8.

$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{x^2}$	$\boldsymbol{x^3}$	$\boldsymbol{x^4}$	$\boldsymbol{xy}$	$\boldsymbol{x^2y}$
8	29.3	64	512	4096	234.4	1875.2
9	32	81	729	6261	288	2592
10	32.5	100	1000	10000	325	3250
11	32.7	121	1331	14641	359.7	3956.7
12	31.7	144	1728	20736	380.4	4564.8
13	31.2	169	2197	28561	405.6	5272.8
Total	189.4	679	7497	84595	1993.1	21511.5

$6c_0 + 63c_1 + 679c_2 = 189.4$

$63c_0 + 679c_1 + 7497c_2 = 1993.1$

$679c_0 + 7497c_1 + 84595c_2 = 21511.5$

$c_0 = -13.24$

$c_1 = 8.5014$

$c_2 = -0.3929$

$y = -13.24 + 8.5014x - 0.3929x^2$

$26.5 = -13.24 + 8.5014(15) - 0.3929(15)^2$

$\boldsymbol{26.5\ billion}$