1. Consider the following Hasse diagram

   (a) List the pairs of elements that are not related via this partial order.

   (b) Construct, as far as possible, the table of least upper bounds for pairs of elements under this partial order.

   (c) Is this poset a lattice? (provide an argument for your answer).

   (d) Let $\rho = \{(b,b),(a,b),(b,a),(c,c),(c,d),(d,c)\}$ be a relation on the set $\{a,b,c,d\}$. Construct ρ^2, list explicitly all the ordered pairs that form ρ^2.

   (e) Draw a directed graph diagram for ρ and ρ^2. Which ordered pair(s) must be adjoined to ρ^2 to complete it into an equivalence relation on the set $\{a,b,c,d\}$?

$(2+5+3+4+4=18 \text{ marks})$

2. For the following system of linear equations:

$$\begin{array}{rcrcrcrcr} -x & + & 2y & - & z & + & w & = & 4 \\ x & - & y & + & z & - & w & = & -2 \\ 2x & + & 2y & - & z & - & w & = & -1 \\ x & + & y & + & z & + & w & = & 10 \end{array}$$

   (a) Use Gaussian elimination to put the system into row echelon form

   (b) Use Gauss-Jordan elimination to change your row echelon form into reduced row echelon form.

   (c) Solve the system of equations using either your answer to part (a) or your answer to part (b).

   (d) Verify that your solutions satisfy this system of equations.

$(8+5+3+2=18 \text{ marks})$

3. Consider the following matrix

$$A = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

(a) Confirm that the matrix A is invertible.

(b) Find all the cofactors C_{ij} of the matrix A and hence find $\operatorname{adj}(A)$.

(c) Verify that the $\operatorname{adj}(A)$ you obtained is correct by multiplying it with A.

$(2 + 6 + 2 = 10 \text{ marks})$

4. For the following matrix

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

(a) Find the eigenvalues (one eigenvalue is $\lambda_1 = 2$).

(b) For each eigenvalue find the corresponding eigenvector(s).

(c) Determine if possible a matrix P so that $B = P^{-1}AP$ is in diagonal form. Write down B and P.

$(2 + 8 + 6 = 16 \text{ marks})$

5. The vector space of solutions of $\mathbf{Ax} = \mathbf{0}$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 2 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$$

is generated by

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

(a) Verify that each vector is a solution.

(b) Show that any solution can be written as a linear combination

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

for a suitable choice of $c_1,\ c_2,\ c_3$.

(c) What is the dimension of the row-space of A and the dimension of its nullspace? State one basis for the nullspace of A.

$(6 + 8 + 8 = 22 \text{ marks})$

6. For the following matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 1 & 0 & -2 & 1 \\ 0 & 2 & 2 & -2 \end{bmatrix}$$

(a) Determine the row-rank.

(b) Find a set of generators for the row space of A.

(c) State one possible basis for the row space of A and explain why it is a basis.

$(6 + 4 + 6 = 16$ marks$)$

7. The 6-tuples:

$$u_1 = 011100, u_2 = 111010, u_3 = 110011$$

form a basis for a (6,3) linear code.

(a) Write down the generator matrix for this code.

(b) Construct code words for the message blocks: $101, 110, 011, 010$.

(c) Construct the parity check matrix for this code.

(d) Decode if possible

$$010001, 010101, 101101, 000111, 011111, 110011$$

$(6+ 4 + 4 + 6= 20$ marks$)$