SIT292 LINEAR ALGEBRA 2021
Assignment 3

Total marks: 120

1. State the definition of the row-rank. For the following matrix

$$A = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & 4 & -2 & -2 \\ 3 & 6 & -4 & 10 \end{bmatrix}$$

   (a) determine the row-rank.

   (b) find a set of generators for the row space of A. Explain why these vectors are generators.

   (c) find a basis for the row space of A. Explain why it is a basis.

   Solution:

   (a) We can do Gauss elimination

$$A \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 5 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   And we determine row-rank of A is 3. (4 marks)

   (b) We can simply take the rows of matrix A as by definition the rows of a matrix generate the row space. Or course we could take any other set of rows we obtain during Gauss elimination. (2 marks)

   (c) A basis for the row space of A would be the three rows of the row-echelon form, because clearly these rows are **linearly independent and belong to the row-space of** A. But in this example, because the row-rank is 3 and not smaller, the rows of the original A are also linearly independent, so we could just take the rows of A as the basis. This won't work in other examples when the row-rank is smaller than the number of rows, as they may turn to be linearly dependent. (4 marks)

2. For the following matrix

$$\begin{bmatrix} -2 & 2 & 3 \\ -2 & 3 & 2 \\ -4 & 2 & 5 \end{bmatrix}$$

   (a) find the eigenvalues
   (b) find the eigenvectors corresponding to these eigenvalues
   (c) starting with the eigenvectors you found in (a) construct a set of orthonormal vectors (use the Gram-Schmidt procedure).

Solution:

   (a) Here we follow the standard procedure (see Assignment 2 for explanations, but your detailed solution is needed) and get the eigenvalues as $1, 2, 3$.

   (5 marks)

   (b) The corresponding eigenvectors will be $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

   (10 marks)

   (c) Gram-Schmidt procedure

$$u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

$$y_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 2 \\ -1/2 \end{bmatrix}.$$

   and then

$$u_2 = \frac{1}{\sqrt{18}} \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}.$$

Next,

$$y_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{4}{18}\begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}.$$

And so

$$u_3 = \frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}.$$

At this stage you should check orthogonality of u_1, u_2, u_3, which is very easy to do. (5 marks)

3. The set of ordered 4-tuples $\{(0,2,1,0),(1,-1,0,0),(1,2,0,-1),(1,0,0,1)\}$ presumably forms a basis for $\mathbb{R}^4$. Show that it is a basis indeed. Then, starting with this basis use the Gram-Schmidt procedure to construct an orthonormal basis for $\mathbb{R}^4$. Verify that the resulting vectors are orthonormal.

Solution:

Form a matrix from these vectors as rows (or columns). Calculate the rank by doing Gauss elimination

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 2 & 0 & -1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 3 & 0 & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 3 & 0 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So we see that the four vectors are linearly independent and hence form a basis for $\mathbb{R}^4$.

(3 marks)

Next, Gram-Schmidt procedure, starting from any of those vectors, take vector one, for example.

$$u_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}.$$

$$y_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \frac{2}{5} \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1/5 \\ 2/5 \\ 0 \end{bmatrix}.$$

and then

$$u_2 = \frac{1}{\sqrt{30}} \begin{bmatrix} 5 \\ -1 \\ 2 \\ 0 \end{bmatrix}.$$

You can check orthogonality right away here. Next,

$$y_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} - \frac{4}{5} \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} - \frac{3}{30} \begin{bmatrix} 5 \\ -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \\ -1 \end{bmatrix}.$$

And so

$$u_3 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 1 \\ -2 \\ -2 \end{bmatrix}.$$

Again, easy to check orthogonality here (not to carry any mistakes to the next step). Finally,

$$y_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - 0 \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} - \frac{5}{30} \begin{bmatrix} 5 \\ -1 \\ 2 \\ 0 \end{bmatrix} + \frac{1}{10} \begin{bmatrix} 1 \\ 1 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 8/30 \\ 8/30 \\ -16/30 \\ 8/10 \end{bmatrix}.$$

And so

$$u_4 = \frac{1}{\sqrt{15}} \begin{bmatrix} 1 \\ 1 \\ -2 \\ 3 \end{bmatrix}.$$

Finally check orthogonality here. (7 marks)

4. Denote by $\mathbb{R}^n$ the set of all n-tuples of real numbers. $\mathbb{R}^n$ is called the Euclidean vector space, with equality, addition and multiplication defined in the obvious way. Let V be the set of all vectors in $\mathbb{R}^4$ orthogonal to the vector $(0, -1, 1, 1)$; i.e. all vectors $v \in V$ so that $v^T(0, -1, 1, 1) = 0$.

   (a) Prove that V is a subspace of $\mathbb{R}^4$.

   (b) What is the dimension of V (provide an argument for this), and find a basis of V. (Hint: observe that the vector $(0, -1, 1, 1)$ does not belong to V).

   (c) Let A be a matrix consisting of the basis vectors of V (taken as rows of A). What is the Null-space of A and what is its row-space? Does the rank-nullity theorem hold for this matrix?

Solution:

(a) Prove that V is a subspace of R^4.

As explained on p. 190 of SG, you need to prove that given two vectors u_1, u_2, both orthogonal to v, and a scalar α, the vectors $u = u_1 + u_2$ and αu are both orthogonal to v. Clearly,

$$(\alpha u^T) \cdot v = \alpha(u^T \cdot v) = \alpha \times 0 = 0$$

and

$$v^t \cdot (u_1 + u_2) = v^t \cdot u_1 + v^t \cdot u_2 = 0 + 0 = 0.$$

(5 marks)

(b) What is the dimension of V?

This is at most 4. But not every 4-tuple is in V, for instance $(0, -1, 1, 1) \notin V$ as it cannot be orthogonal to itself. Then it could be 1,2, or 3. Let us guess 3 vectors orthogonal to $v = (0, -1, 1, 1)$, prove that they are linearly independent, and that will prove that they form a basis in V, and the dimension of V is therefore 3.

Take $u_1 = (1,0,0,0)$. Clearly $v^t \cdot u_1 = 0$.

Take $u_2 = (0,1,0,1)$, again $v^t \cdot u_2 = 0$.

take $u_3 = (0,1,1,0)$, again $v^t \cdot u_3 = 0$.

Test for linear independence. Do Gaussian elimination

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The matrix is in the row-echelon form, its rank is 3, hence the three vectors we guessed are linearly independent, the dimension of V is 3.

Another option is to use Rank-Nullity Theorem. Let $v = (a,b,c,d)$. The condition v is orthogonal to the given vector is $-b+c+d = 0$. This is $Av = 0$ in matrix form, and A has just one row $A = [0,-1,1,1]$. So $rank(A) = 1$. Then nullity, which is the dimension of the nullspace, the space of solutions to $Av = 0$ is $4-1 = 3$. So $dim(V) = 3$. Next we pick 3 linearly independent solutions to $Av = 0$, either as before, or differently. Check they are linearly independent. (14 marks)

(c) We take A as in part (b).

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Its row space is the set of all linear combinations of its rows. It has dimension 3, as the rank of A is 3. The nullspace of A is the set of all vectors orthogonal to its rows, and that is precisely the space generated by the given vector v, i.e., all its multiples. This nullspace has dimension 1 and it agrees with the Rank-Nullity Theorem: 3+1=4 = number of columns of A. (5 marks)

5. Determine the dimension of the subspace of $\mathbb{R}^4$ generated by the set of 4-tuples
$$\{(1,2,1,2),(2,4,3,5),(3,6,4,9),(1,2,4,3)\}$$

Solution:

Put all vectors into a matrix (row-wise or column-wise)

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 3 & 5 \\ 3 & 6 & 4 & 9 \\ 1 & 2 & 4 & 3 \end{bmatrix}$$

Did you notice the second column is just a multiple of the first? So we can safely eliminate it in order to find the rank of the matrix (and save on calculations). In fact it makes sense to put the vectors into columns, but i'll proceed with the reduced A. So I take

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \\ 3 & 4 & 9 \\ 1 & 4 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

So the rank of A is 3 here, and therefore there are three linearly independent vectors in the given collection. So the dimension of the generated subspace is 3. (6 marks)

6. The code words

$$u_1 = 1010010, u_2 = 1100001, u_3 = 0101000, u_4 = 0010101$$

form a basis for a $(7,4)$ linear binary code.

(a) Write down a generator matrix for this code.

(b) Construct code words for the messages 1001 and 0101.

(c) Write down the parity check matrix for this code.

(d) Find the syndromes for the received words
$1110011, 1001010, 0001101, 1101010$

(a) Write down a generator matrix for this code.

Since we need the matrix G in the standard form, write the vectors u_i as rows in this order and do Gauss elimination to get I on the right side. remember that we can only apply operation + here and 1+1=0. The first step is rearrangement of the rows here. And then R2+R4.

$$A = \begin{bmatrix} 1010010 \\ 1100001 \\ 0101000 \\ 0010101 \end{bmatrix} \sim \begin{bmatrix} 0101000 \\ 0010101 \\ 1010010 \\ 1100001 \end{bmatrix} \sim \begin{bmatrix} 0101000 \\ 1110100 \\ 1010010 \\ 1100001 \end{bmatrix}$$

$$G = A^T = \begin{bmatrix} 0111 \\ 1101 \\ 0110 \\ \hline 1000 \\ 0100 \\ 0010 \\ 0001 \end{bmatrix} = \begin{bmatrix} P_{3\times4} \\ \hline I_{4\times4} \end{bmatrix}$$

(b) Construct code words for the messages 1001 and 0101.

Doing multiplication Gm we get 1001001 and 0010101

(c) Write down the parity check matrix for this code.

Since G is a 7×4 matrix, the parity check matrix H is a $(7-4) \times 7$ matrix:

$$H := \left[\begin{array}{c|c} 100 & 0111 \\ 010 & 1101 \\ 001 & 0110 \end{array}\right] = \left[\begin{array}{c|c} I_{3\times3} & P_{3\times4} \end{array}\right]$$

(d) Find the syndromes for the received words
$1110011, 1001010, 0001101, 1101010$

Multiply Hw. We get 100, 011, 011, 001.

$(6 + 4 + 4 + 8 = 20$ marks$)$

7. (Challenges for higher marks)

The challenges require students to do independent reading and researching, as while the methods of solution are mentioned in the classes, no solution templates or solutions to similar questions are provided (and will not be provided). Yet all the technical tools required to solve these questions have been studied in detail.

The set of vectors in P_2 is $\{1-x+2x^2, 2x+x^2, 2+5x-x^2\}$. Determine whether this set is linearly independent. Does it form a basis for P_2? Using Gram-Schmidt process starting from the first vector build an orthonormal basis for P_2. Here the Inner product of two polynomials is defined as per

$$\langle p, q\rangle = p_0q_0 + p_1q_1 + p_2q_2$$

where p_i, q_i are the coefficients of p and q.

Solution:

The best approach is to note that there is a mapping between the polynomials and their coefficients. So it is the same question as whether the vectors $(2,-1,1),(1,2,0),(-1,5,2)$ form a basis in $\mathbb{R}^3$. Which we know how to do: Put them into a matrix and do Gauss elimination to find its rank. So we have

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 0 \\ -1 & 5 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & -5 & 1 \\ 0 & 7 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & -5 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

We see that the rank is 3, so there are three linearly independent polynomials in our collection. Hence they form a basis in P_2

We proceed as forming the orthogonal basis in the space of tuples. Gram-Schmidt procedure. The important part here is the definition of the inner product which matches(!) the scalar product of tuples, so whenever we perform scalar products with polynomials, we can do it with their coefficients instead. (This would be different if the inner (scalar) product were defined differently, which is often the case). So it is a carbon copy of the Gram-Schmidt procedure with tuples.

$$u_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

$$y_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - 0 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

and then

$$u_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

Next,

$$y_3 = \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix} + \frac{5}{6} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} - \frac{9}{5} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = 17 \begin{bmatrix} -2/30 \\ 1/30 \\ 5/30 \end{bmatrix}.$$

And so

$$u_3 = \frac{1}{\sqrt{30}} \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}.$$

hence the orthogonal basis for polynomials is $\frac{1}{6}(2x^2 - x + 1)$, $\frac{1}{\sqrt{5}}(x^2 + 2x)$, $\frac{1}{\sqrt{30}}(-2x^2 + x + 5)$.

(7 + 8 marks)

8. (Challenges for higher marks)

The table shows the revenue of a corporation (in billions of dollars) for six different years

Year	2008	2009	2010	2011	2012	2013
Revenue	29.3	32.0	32.5	32.7	31.7	31.2

Let $x = 8$ represents the year 2008 (i.e. $x = year - 2000$). Find the least squares regression quadratic polynomial $y = c_0 + c_1 x + c_2 x^2$ for the data in the table and use your model to estimate the revenue for 2015. You will need to state the system of interpolating conditions and the normal system of equations as part of your solution, and then resolve the normal system for c_i.

Solution.

This is the least squares approximation (or polynomial regression) problem. Our data becomes

x	8	9	10	11	12	13
y	29.3	32.0	32.5	32.7	31.7	31.2

The interpolating conditions are therefore

$$64c_2 + 8c_1 + c_0 \approx 29.3, \quad 9^2c_2 + 9c_1 + c_0 \approx 32, \ldots$$

Or in matrix form the system of interpolating conditions is

$$\begin{bmatrix} 8^2 & 8 & 1 \\ 9^2 & 9 & 1 \\ 10^2 & 10 & 1 \\ 11^2 & 11 & 1 \\ 12^2 & 12 & 1 \\ 13^2 & 13 & 1 \end{bmatrix} \begin{bmatrix} c_2 \\ c_1 \\ c_0 \end{bmatrix} = \begin{bmatrix} 29.3 \\ 32.0 \\ 32.5 \\ 32.7 \\ 31.7 \\ 31.2 \end{bmatrix}$$

This system has no solution because its rank is smaller than the rank of the augmented system. Instead we convert it to the system of normal equations, so that the RHS is satisfied in the least squares sense. Now the system of normal equations is obtained by multiplying $A^tAc = A^ty$

It has the matrix of the form

$$\begin{bmatrix} \sum x_i^4 & \sum x_i^3 & \sum x_i^2 \\ \sum x_i^3 & \sum x_i^2 & \sum x_i^1 \\ \sum x_i^2 & \sum x_i^1 & \sum x_i^0 \end{bmatrix} \begin{bmatrix} c_2 \\ c_1 \\ c_0 \end{bmatrix} = \begin{bmatrix} \sum x_i^2 y_i \\ \sum x_i y_i \\ \sum y_i \end{bmatrix}$$

it gives

$$\begin{bmatrix} 84595 & 7497 & 679 \\ 7497 & 679 & 63 \\ 679 & 63 & 6 \end{bmatrix} \begin{bmatrix} c_2 \\ c_1 \\ c_0 \end{bmatrix} = \begin{bmatrix} 21511.5 \\ 1993.1 \\ 189.4 \end{bmatrix}$$

Solving numerically we get (2 d.p.) $c_2 = -0.39$, $c_1 = 8.50$, $c_0 = -13.24$. Hence the quadratic model becomes $y(x) = -0.39x^2 + 8.5x - 13.24$

For year 2015 we have $x = 15$ and $y(15) = -0.39 \times 225 + 8.5 \times 15 - 13.24 = 26.5$ billion dollars. (The actual values will depend on the accuracy you used for the coefficients, especially c_2, so you may get it as 25.89).

(15 marks)