

1.

i)

$$C_{1,1} = \det \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} = 1 * 2 - 0 * 2 = 2$$

$$C_{1,2} = \det \begin{pmatrix} 3 & 0 \\ -4 & 2 \end{pmatrix} = 3 * 2 - 0 * (-4) = 6$$

$$C_{1,3} = \det \begin{pmatrix} 3 & 1 \\ -4 & 2 \end{pmatrix} = 3 * 2 - 1 * (-4) = 10$$

$$C_{2,1} = \det \begin{pmatrix} 2 & -1 \\ 2 & 2 \end{pmatrix} = 2 * 2 - (-1) * 2 = 6$$

$$C_{2,2} = \det \begin{pmatrix} 1 & -1 \\ -4 & 2 \end{pmatrix} = 1 * 2 - (-1)(-4) = -2$$

$$C_{2,3} = \det \begin{pmatrix} 1 & 2 \\ -4 & 2 \end{pmatrix} = 1 * 2 - 2(-4) = 10$$

$$C_{3,1} = \det \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} = 2 * 0 - (-1) * 1 = 1$$

$$C_{3,2} = \det \begin{pmatrix} 1 & -1 \\ 3 & 0 \end{pmatrix} = 1 * 0 - (-1) * 3 = 3$$

$$C_{3,3} = \det \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = 1 * 1 - 2 * 3 = -5$$

$$\begin{pmatrix} 2 & 6 & 10 \\ 6 & -2 & 10 \\ 1 & 3 & -5 \end{pmatrix}$$

$$\begin{pmatrix} (-1)^{1+1} * 2 & (-1)^{1+2} * 6 & (-1)^{1+3} * 10 \\ (-1)^{2+1} * 6 & (-1)^{2+2} * (-2) & (-1)^{2+3} * 10 \\ (-1)^{3+1} * 1 & (-1)^{3+2} * 3 & (-1)^{3+3} * (-5) \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -6 & 10 \\ -6 & -2 & -10 \\ 1 & -3 & -5 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -6 & 10 \\ -6 & -2 & -10 \\ 1 & -3 & -5 \end{pmatrix}^T = \begin{pmatrix} 2 & -6 & 1 \\ -6 & -2 & -3 \\ 10 & -10 & -5 \end{pmatrix}$$

ii)

$$\begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -4 & 2 & 2 \end{pmatrix} * \begin{pmatrix} 2 & -6 & 1 \\ -6 & -2 & -3 \\ 10 & -10 & -5 \end{pmatrix} = \begin{pmatrix} -20 & 0 & 0 \\ 0 & -20 & 0 \\ 0 & 0 & -20 \end{pmatrix} = -20 * I$$

$$= \det(A) * I$$

2.

$$(5)(\alpha) + (\alpha)(\alpha) + (3\alpha)(3\alpha) + (1)(-1) + (5)(1) = 8\alpha^2 + 5\alpha + 4$$

$$8\alpha^2 + 5\alpha + 4 = 0$$

$$\alpha_{1,2} = \frac{-5 \pm \sqrt{5^2 - 4 * 8 * 4}}{2 * 8}$$

$$\sqrt{5^2 - 128} = i\sqrt{128 - 5^2} = \sqrt{103} i$$

$$\alpha = \frac{-5 + \sqrt{103} i}{2 * 8}, \alpha = \frac{-5 - \sqrt{103} i}{2 * 8}$$

**Answers:**

$$\alpha = -\frac{5}{16} + i\frac{\sqrt{103}}{16}, \quad \alpha = -\frac{5}{16} - i\frac{\sqrt{103}}{16}$$

3.

$$\begin{bmatrix} 2x_1 - 2x_2 + 2x_3 - 4x_4 = 2 \\ x_1 + 4x_2 + 8x_3 + 2x_4 = 5 \\ -x_1 + 9x_2 + 3x_3 - 4x_4 = 5 \end{bmatrix}$$

$$\left[ \begin{array}{cccc|c} 2 & -2 & 2 & -4 & 2 \\ 1 & 4 & 8 & 2 & 5 \\ -1 & 9 & 3 & -5 & 5 \end{array} \right]$$

$$R2 = R2 - R1(\frac{1}{2})$$

$$\left[ \begin{array}{cccc|c} 2 & -2 & 2 & -4 & 2 \\ 0 & 5 & 7 & 4 & 4 \\ -1 & 9 & 3 & -5 & 5 \end{array} \right]$$

$$R3 = R3 + R1\left(\frac{1}{2}\right)$$

$$\left[ \begin{array}{cccc|c} 2 & -2 & 2 & -4 & 2 \\ 0 & 5 & 7 & 4 & 4 \\ 0 & 8 & 4 & -6 & 6 \end{array} \right]$$

$$R3 = R3 - \frac{8R2}{5}$$

$$\left[ \begin{array}{cccc|c} 2 & -2 & 2 & -4 & 2 \\ 0 & 5 & 7 & 4 & 4 \\ 0 & 0 & -\frac{36}{5} & -\frac{62}{5} & -\frac{2}{5} \end{array} \right]$$

$$x_4 = t$$

**Find  $x_3$**

$$-\frac{36}{5}x_3 - \frac{62}{5}t = -\frac{2}{5}$$

$$-\frac{36}{5}x_3 = -\frac{2}{5} - \frac{62}{5}t$$

$$-36x_3 = -2 - 62t$$

$$x_3 = \frac{1}{18} - \frac{31}{18}t$$

**Find  $x_2$**

$$5x_2 + 7\left(\frac{1}{18} - \frac{31}{18}t\right) + 4t = 4$$

$$5x_2 = 4 - \left(7\left(\frac{1}{18} - \frac{31}{18}t\right) + 4t\right)$$

$$5x_2 = \frac{62}{18} + \frac{145}{18}t$$

$$x_2 = \frac{13}{18} + \frac{29}{18}t$$

**Find  $x_1$**

$$2x_1 - 2\left(\frac{13}{18} + \frac{29}{18}t\right) + 2\left(\frac{1}{18} - \frac{31}{18}t\right) - 4t = 2$$

$$2x_1 = 2 - \left(-2\left(\frac{13}{18} + \frac{29}{18}t\right) + 2\left(\frac{1}{18} - \frac{31}{18}t\right) - 4t\right)$$

$$2x_1 = \frac{10}{3} + \frac{32}{3}t$$

$$x_1 = \frac{5}{3} + \frac{16}{3}t$$

**Answers:**

$$x_1 = \frac{5}{3} + \frac{16}{3}t, x_2 = \frac{13}{18} + \frac{29}{18}t, x_3 = \frac{1}{18} - \frac{31}{18}t$$

**Justify the correctness using matrix ranks:**

$$A = \begin{bmatrix} 2 & -2 & 2 & -4 \\ 0 & 5 & 7 & 4 \\ 0 & 0 & -\frac{36}{5} & -\frac{62}{5} \end{bmatrix}$$

$$([Ab]) = \begin{bmatrix} 2 & -2 & 2 & -4 & | & 2 \\ 0 & 5 & 7 & 4 & | & 4 \\ 0 & 0 & -\frac{36}{5} & -\frac{62}{5} & | & -\frac{2}{5} \end{bmatrix}$$

$$R(A) = 3, R([Ab]) = 3$$

There are 4 unknowns and rank of both matrix A and R([Ab]) are 3. So,

$$R(A) = R([Ab]) < 4$$

Meaning that the system is consistent and has infinitely many solutions

**4.**

$$3x_2 + 11x_3 = 6$$

$$x_1 + x_2 + 3x_3 = 2$$

$$3x_1 - 3x_2 - 13x_3 = -6$$

$$-x_1 + 2x_2 + 8x_3 = 4$$

$$\begin{bmatrix} 0 & 3 & 11 & | & 6 \\ 1 & 1 & 3 & | & 2 \\ 3 & -3 & -13 & | & -6 \\ -1 & 2 & 8 & | & 4 \end{bmatrix}$$

Swap R2 with R1

$$\begin{bmatrix} 1 & 1 & 3 & | & 2 \\ 0 & 3 & 11 & | & 6 \\ 3 & -3 & -13 & | & -6 \\ -1 & 2 & 8 & | & 4 \end{bmatrix}$$

$$R3 = R3 - 3R1$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 3 & 11 & 6 \\ 0 & -6 & -22 & -12 \\ -1 & 2 & 8 & 4 \end{array} \right]$$

$$R4 = R4 + R1$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 3 & 11 & 6 \\ 0 & -6 & -22 & -12 \\ 0 & 3 & 11 & 6 \end{array} \right]$$

$$R3 = R3 + 2R2$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 3 & 11 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 11 & 6 \end{array} \right]$$

$$R4 = R4 - R2$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 3 & 11 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_3 = t$$

**Find  $x_2$**

$$3x_2 + 11t = 6$$

$$x_2 = \frac{6 - 11t}{3} = 2 - \frac{11t}{3}$$

**Find  $x_1$**

$$1x_1 + \left(2 - \frac{11}{3}t\right) + 3t = 2$$

$$x_1 = 2 - \left(2 - \frac{11}{3}t\right) + 3t$$

$$x_1 = \frac{2}{3}t$$

**Answers:**

$$x_1 = \frac{2}{3}t, x_2 = \frac{6 - 11t}{3} = 2 - \frac{11t}{3}$$

**Justify the correctness using matrix ranks:**

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 3 & 11 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$([Ab]) = \begin{bmatrix} 1 & 1 & 3 & | & 2 \\ 0 & 3 & 11 & | & 6 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$R(A) = 2, R([Ab]) = 2$$

There are 3 unknowns and rank of both matrix  $A$  and  $([Ab]) =$  are 2. So,

$$R(A) = R([Ab]) < 3$$

Meaning that the system is consistent and has infinitely many solutions

5.

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 & | & 1 & 0 & 0 \\ 1 & 0 & 2 & | & 0 & 1 & 0 \\ 1 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

$$R2 = R2 - R1$$

$$\begin{bmatrix} 1 & 2 & 2 & | & 1 & 0 & 0 \\ 0 & -2 & 0 & | & -1 & 1 & 0 \\ 1 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

$$R3 = R3 - R1$$

$$\begin{bmatrix} 1 & 2 & 2 & | & 1 & 0 & 0 \\ 0 & -2 & 0 & | & -1 & 1 & 0 \\ 0 & -2 & -2 & | & -1 & 0 & 1 \end{bmatrix}$$

$$R2 = R2 \div -2$$

$$\begin{bmatrix} 1 & 2 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & 1/2 & -1/2 & 0 \\ 0 & -2 & -2 & | & -1 & 0 & 1 \end{bmatrix}$$

$$R3 = R3 + 2R2$$

$$\begin{bmatrix} 1 & 2 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & 1/2 & -1/2 & 0 \\ 0 & 0 & -2 & | & 0 & -1 & 1 \end{bmatrix}$$

$$R1 = R1 - 2R2$$

$$\begin{bmatrix} 1 & 0 & 2 & | & 0 & 1 & 0 \\ 0 & 1 & 0 & | & 1/2 & -1/2 & 0 \\ 0 & 0 & -2 & | & 0 & -1 & 1 \end{bmatrix}$$

$$R3 = R3 \div -2$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 1/2 & -1/2 \end{array} \right]$$

$$R1 = R1 - 2R3$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 1/2 & -1/2 \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0.5 & -0.5 & 0 \\ 0 & 0.5 & -0.5 \end{bmatrix}$$

6.

**Find eigenvalues for A**

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\det \left( \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$\det \left( \begin{bmatrix} 0-\lambda & 1 & -1 \\ 1 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{bmatrix} \right) = 0$$

$$1 \begin{vmatrix} 1 & -1 \\ 0 & 1-\lambda \end{vmatrix} - (1-\lambda) \begin{vmatrix} -\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} + 0 \begin{vmatrix} -\lambda & 1 \\ 1 & 0 \end{vmatrix}$$

$$-\lambda + 1 - 1 - \lambda^2 + \lambda^3 + 0 = 0$$

$$\lambda^3 - \lambda^2 = 0$$

$$\lambda^2(\lambda - 1) = 0$$

$$\lambda_1 = 0, \lambda_2 = 1$$

**Find eigenvectors for A**

**Find eigenvectors when  $\lambda_1 = 0$**

$$\lambda_1 = 0$$

$$(A - \lambda I) = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

*Using gaussian elimination*

$$\left[ \begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array} \right]$$

*Swap 1 and 2*

$$\left[ \begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 \end{array} \right]$$

$$R3 = R3 - R1$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array} \right]$$

$$R3 = R3 + R2$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_3 = t$$

$$x_2 = t$$

$$x_1 = -t$$

$$v_1 = \begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

**Find eigenvectors when  $\lambda_2 = 1$**

$$(A - \lambda I) = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

*Using gaussian elimination*

$$\left[ \begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right]$$

$$R2 = R2 + R1$$

$$\left[ \begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right]$$

$$R3 = R3 + R1$$

$$\left[ \begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array} \right]$$

$$R3 = R3 - R2$$



$$\left[ \begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_3 = t$$

$$x_2 = t$$

$$x_1 = 0$$

$$v_2 = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Answers:

$$v_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

### ***Find eigenvalues for B***

$$B = \begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix}$$

$$\det \left( \begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$\det \left( \begin{bmatrix} 5-\lambda & -4 & 0 \\ 1 & 0-\lambda & 2 \\ 0 & 2 & 5-\lambda \end{bmatrix} \right) = 0$$

$$-\lambda^3 + 10\lambda^2 - 25\lambda = 0$$

$$\lambda_1 = 0, \lambda_2 = 5$$

### ***Find eigenvectors for B***

***Find eigenvectors when  $\lambda_1 = 0$***

$$(A - \lambda I) = \begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

*Using gaussian elimination*

$$\left[ \begin{array}{ccc|c} 5 & -4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 5 & 0 \end{array} \right]$$

$$R2 = R2 - R1 \frac{1}{5}$$

$$\begin{bmatrix} 5 & -4 & 0 & | & 0 \\ 0 & \frac{4}{5} & 2 & | & 0 \\ 0 & 2 & 5 & | & 0 \end{bmatrix}$$

$$R3 = R3 - R2 \frac{5}{2}$$

$$\begin{bmatrix} 5 & -4 & 0 & | & 0 \\ 0 & \frac{4}{5} & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$x_3 = t$$

$$x_2 = -\frac{5}{2}t$$

$$x_1 = -2t$$

Let  $x_3$  be 1

$$v_1 = \begin{bmatrix} -2t \\ -\frac{5}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ -\frac{5}{2} \\ 1 \end{bmatrix}$$

**Find eigenvectors when  $\lambda_1 = 5$**

$$(A - \lambda I) = \begin{bmatrix} 0 & -4 & 0 \\ 1 & -5 & 2 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using gauss – jordan elimination

$$\begin{bmatrix} 0 & -4 & 0 & | & 0 \\ 1 & -5 & 2 & | & 0 \\ 0 & 2 & 0 & | & 0 \end{bmatrix}$$

Swap R1 and R2

$$\begin{bmatrix} 1 & -5 & 2 & | & 0 \\ 0 & -4 & 0 & | & 0 \\ 0 & 2 & 0 & | & 0 \end{bmatrix}$$

$$R2 = R2 * -\frac{1}{4}$$

$$\begin{bmatrix} 1 & -5 & 2 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$R1 = R1 + 5R2$$

$$\begin{bmatrix} 1 & 0 & 2 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 2 & 0 & | & 0 \end{bmatrix}$$

$$R3 = R3 - 2R2$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_3 = t$$

$$x_2 = 0$$

$$x_1 = -2t$$

$$v_2 = \begin{bmatrix} -2t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Answers:

$$v_1 = \begin{bmatrix} -2 \\ 5 \\ -2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

7.

**Find eigenvalues for A**

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\det \left( \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$\det \left( \begin{bmatrix} 1-\lambda & 1 & 0 \\ 1 & 0-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix} \right) = 0$$

$$1 - \lambda \begin{bmatrix} 0-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1-\lambda \end{bmatrix} + 0 \begin{bmatrix} 1 & 0-\lambda \\ 0 & 1 \end{bmatrix}$$

$$(1 - \lambda)(-\lambda(1 - \lambda) - 1) - (-\lambda + 1) = 0$$

$$(-\lambda - 1)(\lambda - 1)(\lambda - 2) = 0$$

$$\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 2$$

**Find eigenvectors for A**

**Find eigenvectors when  $\lambda_1 = 1$**

$$(A - \lambda I) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

*Using gaussian elimination*

$$\begin{bmatrix} 0 & 1 & 0 & | & 0 \\ 1 & -1 & 1 & | & 0 \\ 0 & 1 & 0 & | & 0 \end{bmatrix}$$

*Swap R1 and R2*

$$\begin{bmatrix} 1 & -1 & 1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \end{bmatrix}$$

$R3 = R3 - R2$

$$\begin{bmatrix} 1 & -1 & 1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$x_3 = t$$

$$x_2 = 0$$

$$x_1 = -t$$

$$v_1 = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

**Find eigenvectors when  $\lambda_2 = -1$**

$$(A - \lambda I) = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

*Using gaussian elimination*

$$\begin{bmatrix} 2 & 1 & 0 & | & 0 \\ 1 & 1 & 1 & | & 0 \\ 0 & 1 & 2 & | & 0 \end{bmatrix}$$

$$R2 = R2 - R1 * \frac{1}{2}$$

$$\begin{bmatrix} 2 & 1 & 0 & | & 0 \\ 0 & \frac{1}{2} & 1 & | & 0 \\ 0 & 1 & 2 & | & 0 \end{bmatrix}$$

$R3 = R3 - 2R2$

$$\begin{bmatrix} 2 & 1 & 0 & | & 0 \\ 0 & \frac{1}{2} & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$x_3 = t$$

$$x_2 = -2t$$

$$x_1 = t$$

$$v_2 = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

**Find eigenvectors when  $\lambda_2 = 2$**

$$(A - \lambda I) = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

*Using gaussian elimination*

$$\left[ \begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array} \right]$$

$$R2 = R2 + R1$$

$$\left[ \begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array} \right]$$

$$R3 = R3 + R2$$

$$\left[ \begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_3 = t$$

$$x_2 = t$$

$$x_1 = t$$

$$v_3 = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

**Eigenvalues:**

$$\lambda = 1, \lambda = -1, \lambda = 2$$

**Eigenvectors:**

$$v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A = PDP^{-1}$$

$$P = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1}$$

**Find inverse of P**

$$\left[ \begin{array}{ccc|ccc} -1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$R1 = R1 * -1$$

$$\left[ \begin{array}{ccc|ccc} 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$R3 = R3 - R1$$

$$\left[ \begin{array}{ccc|ccc} 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 & 0 & 1 \end{array} \right]$$

$$R2 = R2 * -\frac{1}{2}$$

$$\left[ \begin{array}{ccc|ccc} 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 2 & 2 & 1 & 0 & 1 \end{array} \right]$$

$$R1 = R1 + R2$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & -\frac{3}{2} & -1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 2 & 2 & 1 & 0 & 1 \end{array} \right]$$

$$R3 = R3 - 2R2$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & -\frac{3}{2} & -1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 3 & 1 & 1 & 1 \end{array} \right]$$

$$R3 = R3 * \frac{1}{3}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & -\frac{3}{2} & -1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array} \right]$$

$$R1 = R1 + R3 * \frac{3}{2}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array} \right]$$

$$R2 = R2 + R3 * \frac{1}{2}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array} \right]$$

$$P^{-1} = \left[ \begin{array}{ccc} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array} \right]$$

$$\left[ \begin{array}{ccc} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{array} \right] = \left[ \begin{array}{ccc} -1 & 1 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & 1 \end{array} \right] * \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{array} \right] * \left[ \begin{array}{ccc} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array} \right]$$

$$A = PDP^{-1}$$

Another way to check if A can be diagonalized, is check if  $AD = PD$  instead of figuring out P inverse.

8.

a)

**Find eigenvalues for A**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\det \left( \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 2 \\ 0 & 0 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$\det \left( \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & -2-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{bmatrix} \right)$$

$$0 \begin{bmatrix} 1-\lambda & 2 \\ 0 & -2-\lambda \end{bmatrix} - 0 \begin{bmatrix} 1-\lambda & 3 \\ 0 & 2 \end{bmatrix} + (3-\lambda) \begin{bmatrix} 1-\lambda & 2 \\ 0 & -2-\lambda \end{bmatrix}$$

$$(3-\lambda)(-\lambda+1)(-\lambda-2) = 0$$

$$\lambda_1 = 3, \lambda_2 = 1, \lambda_3 = -2$$

## ***Find eigenvectors for A***

***Find eigenvectors when  $\lambda_1 = 3$***

$$(A - \lambda I) = \begin{bmatrix} -2 & 2 & 3 \\ 0 & -5 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[ \begin{array}{ccc|c} -2 & 2 & 3 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_3 = t$$

$$x_2 = \frac{2}{5}t$$

$$x_1 = \frac{19}{10}t$$

$$v_1 = \begin{bmatrix} \frac{19}{10}t \\ \frac{2}{5}t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{19}{10} \\ \frac{2}{5} \\ 1 \end{bmatrix}$$

***Find eigenvectors when  $\lambda_2 = 1$***

$$(A - \lambda I) = \begin{bmatrix} 0 & 2 & 3 \\ 0 & -3 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[ \begin{array}{ccc|c} 0 & 2 & 3 & 0 \\ 0 & -3 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array} \right]$$

$$R2 = R2 + R1 * \frac{3}{2}$$

$$\left[ \begin{array}{ccc|c} 0 & 2 & 3 & 0 \\ 0 & 0 & \frac{13}{2} & 0 \\ 0 & 0 & 2 & 0 \end{array} \right]$$

$$R3 = R3 - R2 * \frac{4}{13}$$



$$\left[ \begin{array}{ccc|c} 0 & 2 & 3 & 0 \\ 0 & 0 & \frac{13}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_3 = 0$$

$$x_2 = 0$$

$$x_1 = t$$

$$v_2 = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

**Find eigenvectors when  $\lambda_3 = -2$**

$$(A - \lambda I) = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[ \begin{array}{ccc|c} 3 & 2 & 3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 5 & 0 \end{array} \right]$$

$$R3 = R3 - R2 * \frac{5}{2}$$

$$\left[ \begin{array}{ccc|c} 3 & 2 & 3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_3 = 0$$

$$x_2 = t$$

$$x_1 = -\frac{2}{3}t$$

$$v_3 = \begin{bmatrix} -\frac{2}{3}t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix}$$

Answers:

$$v_1 = \begin{bmatrix} \frac{19}{10} \\ 2 \\ \frac{5}{1} \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix}$$

**Find eigenvalues for B**

$$B = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\det \left( \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$\det \left( \begin{bmatrix} 3-\lambda & 1 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{bmatrix} \right) = 0$$

$$0 \begin{bmatrix} 1 & 0 \\ 1-\lambda & 2 \end{bmatrix} - 0 \begin{bmatrix} 3-\lambda & 0 \\ 0 & 2 \end{bmatrix} + (-2-\lambda) \begin{bmatrix} 3-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} = 0$$

$$(-2-\lambda)(-\lambda+3)(-\lambda+1) = 0$$

$$\lambda_1 = -2, \lambda_2 = 3, \lambda_3 = 1$$

## ***Find eigenvectors for B***

***Find eigenvectors when  $\lambda_1 = 3$***

$$(A - \lambda I) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[ \begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & -5 & 0 \end{array} \right]$$

$$R2 = R2 + 2R1$$

$$\left[ \begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -5 & 0 \end{array} \right]$$

$$R3 = R3 + R2 * \frac{5}{2}$$

$$\left[ \begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_3 = 0$$

$$x_2 = 0$$

$$x_1 = t$$

$$v_1 = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

***Find eigenvectors when  $\lambda_2 = 1$***

$$(A - \lambda I) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[ \begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -3 & 0 \end{array} \right]$$

$$R3 = R3 + R2 * \frac{3}{2}$$

$$\left[ \begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_3 = 0$$

$$x_2 = t$$

$$x_1 = -\frac{1}{2}t$$

$$v_2 = \begin{bmatrix} -\frac{1}{2}t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$$

**Find eigenvectors when  $\lambda_3 = -2$**

$$(A - \lambda I) = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[ \begin{array}{ccc|c} 5 & 1 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_3 = t$$

$$x_2 = -\frac{2}{3}t$$

$$x_1 = \frac{2}{15}t$$

$$v_3 = \begin{bmatrix} \frac{2}{15}t \\ -\frac{2}{3}t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{15} \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

Answers:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} \frac{2}{15} \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

b)

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q^{-1}AQ = R^{-1}BR$$

$$(Rx) \quad RQ^{-1}AQ = RR^{-1}BR$$

$$RQ^{-1}AQ = BR$$

$$(xR^{-1}) \quad RQ^{-1}AQR^{-1} = BRR^{-1}$$

$$RQ^{-1}AQR^{-1} = B$$

$$\text{Let } P = QR^{-1}$$

$$P^{-1}AP = B$$

A matrix  $P$  is possible

**Both  $A$  and  $B$  have same eigenvalues**

$$\lambda_1 = 3, \lambda_2 = 1, \lambda_3 = -2$$

**Matrix  $A$ 's eigenvectors:**

$$v_1 = \begin{bmatrix} \frac{19}{10} \\ 2 \\ \frac{5}{1} \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix}$$

**Matrix  $B$ 's eigenvectors:**

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} \frac{2}{15} \\ 2 \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

**Find  $P$**

$$Q^{-1}AQ = D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \text{ where } Q = \begin{bmatrix} \frac{19}{10} & 1 & -\frac{2}{3} \\ 2 & 0 & 1 \\ \frac{5}{1} & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$R^{-1}BR = D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \text{ where } R = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{2}{15} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q^{-1}AQ = R^{-1}BR$$

$$RQ^{-1}AQR^{-1} = B$$

$$P^{-1}AP = B, \text{ where } P = QR^{-1}$$

**Find  $R^{-1}$**

*Using Gauss – Jordan to find inverse*

$$\left[ \begin{array}{ccc|ccc} 1 & -\frac{1}{2} & \frac{2}{15} & 1 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$R1 = R1 + R2 * \frac{1}{2}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & \frac{2}{15} & 1 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3} & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$R1 = R1 + R3 * \frac{1}{5}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{5} \\ 0 & 1 & -\frac{2}{3} & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$R2 = R2 + R3 * \frac{2}{3}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{5} \\ 0 & 1 & 0 & 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$R^{-1} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{5} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

**Calculate  $P$ :**

$$P = QR^{-1} = \begin{bmatrix} \frac{19}{10} & 1 & -\frac{2}{3} \\ \frac{2}{5} & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} * \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{5} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{19}{10} & \frac{39}{20} & \frac{19}{50} \\ \frac{2}{5} & \frac{1}{5} & \frac{27}{25} \\ 1 & \frac{1}{2} & \frac{1}{5} \end{bmatrix}$$

9.

a)

**Find eigenvalues for A**

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$$

$$\det \left( \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$\det \left( \begin{bmatrix} 1-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ -3 & 5 & 2-\lambda \end{bmatrix} \right)$$

$$(1-\lambda) \begin{bmatrix} 2-\lambda & 0 \\ 5 & 2-\lambda \end{bmatrix} - 0 \begin{bmatrix} 2-\lambda & 0 \\ -3 & 2-\lambda \end{bmatrix} + 0 \begin{bmatrix} 1 & 2-\lambda \\ -3 & 5 \end{bmatrix} = 0$$

$$(1-\lambda)(-\lambda+2)^2 = 0$$

$$\lambda_1 = 1, \lambda_2 = 2$$

b)

**Find eigenvectors for A**

**Find eigenvectors when  $\lambda_1 = 1$**

$$(A - \lambda I) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ -3 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[ \begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -3 & 5 & 1 & 0 \end{array} \right]$$

Swap R1 and R2

$$\left[ \begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & 5 & 1 & 0 \end{array} \right]$$

$R3 = R3 + 3R1$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 8 & 1 & 0 \end{array} \right]$$

Swap R2 and R3

$$\left[ \begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 8 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_3 = t$$

$$x_2 = -\frac{1}{8}t$$

$$x_1 = \frac{1}{8}t$$

$$v_1 = \begin{bmatrix} \frac{1}{8}t \\ -\frac{1}{8}t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{8} \\ -\frac{1}{8} \\ 1 \end{bmatrix}$$

**Find eigenvectors when  $\lambda_2 = 2$**

$$(A - \lambda I) = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ -3 & 5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[ \begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -3 & 5 & 0 & 0 \end{array} \right]$$

$$R2 = R2 + R1$$

$$\left[ \begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 5 & 0 & 0 \end{array} \right]$$

$$R3 = R3 - 3R2$$

$$\left[ \begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 5 & 0 & 0 \end{array} \right]$$

$$R3 = R3 - 5R2$$

$$\left[ \begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_3 = t$$

$$x_2 = 0$$

$$x_1 = 0$$

$$v_2 = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

**c)**

*Formula:*

$$B = P^{-1}AP$$

*Eigenvalues Calculated From Before:*

$$(1 - \lambda)(-\lambda + 2)^2 = 0$$

$$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 2$$

*Form  $R = [v_1, v_2, w]$*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$$

*For the last column for  $R$ , it can be anything but its can't equal the first 2 column.*

*Has to be linear indepdent.*

$$R = \begin{bmatrix} \frac{1}{8} & 0 & 1 \\ -\frac{1}{8} & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

*Check if determinant of  $R \neq 0$*

$$\text{Det}(R) = \frac{1}{8}, \text{ which means } R \text{ can be inverted}$$

**Find  $R^{-1}$**

*Using gass – jordan to find inverse*

$$\left[ \begin{array}{ccc|ccc} \frac{1}{8} & 0 & 1 & 1 & 0 & 0 \\ -\frac{1}{8} & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right]$$

$$R1 = R1 * 8$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 8 & 8 & 0 & 0 \\ -\frac{1}{8} & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right]$$

$$R2 = R2 + R1 * \frac{1}{8}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 8 & 8 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right]$$

$$R3 = R3 - R1$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 8 & 8 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & -8 & -8 & 0 & 1 \end{array} \right]$$



Swap R2 and R3

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 8 & 8 & 0 & 0 \\ 0 & 1 & -8 & -8 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array} \right]$$

$$R1 = R1 - R3 * 8$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -8 & 0 \\ 0 & 1 & -8 & -8 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array} \right]$$

$$R2 = R2 + R3 * 8$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -8 & 0 \\ 0 & 1 & 0 & 0 & 8 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array} \right]$$

$$R^{-1} = \begin{bmatrix} 0 & -8 & 0 \\ 0 & 8 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Let  $R = P$

Put it in the formula  $B = P^{-1}AP$

$$\begin{bmatrix} 0 & -8 & 0 \\ 0 & 8 & 1 \\ 1 & 1 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix} * \begin{bmatrix} \frac{1}{8} & 0 & 1 \\ -\frac{1}{8} & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{-8} \\ \mathbf{0} & \mathbf{2} & \mathbf{5} \\ \mathbf{0} & \mathbf{0} & \mathbf{2} \end{bmatrix} = \mathbf{B}$$

As we can see on the diagonal, we have our eigenvalues we found earlier.

**Verify that the determinant of B agrees with what you used in (a):**

We can see that the determinant of B is  $1 * 2 * 2 = 4$ , which was using in (a).