SIT292 LINEAR ALGEBRA 2021
Assignment 1 Solutions

Total marks: 110

1. Define the sets

$$A = \{1, 2, 3\} \qquad B = \{\{1\}, \{2\}, \{3\}\}$$
$$C = \{1, 2, 3, \{2\}, \{3\}, \{1, 2, 3\}\} \qquad D = \{\{3\}, \{2\}, \{1\}, \{1, 2\}, \{1, 2, 3\}\}.$$

Solution: In solving this question it is important to understand that the set A is a set of numbers, while B and D are sets of sets and C contains both numbers and their subsets. For the inclusion like $A \subset C$ we need to check that *every* element of A is also in C. If a is an element of D then we should be able to *replace* the corresponding entry in D with a (exactly as a is written) (so here we can replace $\{1, 2, 3\}$ in D with A).

Therefore

(a) $A = B$: *false*, as A contains numbers while B contains sets.

(b) $A \subseteq B$: *false*, as any subset of B will be a set of sets, but A is a set of numbers (i.e., all the elements of B are sets, so neither $1 \in B$ nor $2 \in B$ nor $3 \in B$).

(c) $A \subset C$: *true*, because $1, 2$ and 3 are also elements of C.

(d) $A \in C$: *true*, the sixth element of C is $\{1, 2, 3\} = A$.

(e) $A \subset D$: *false*, same argument as part (b).

(f) $C \subset D$: *false*, for example, element $1 \in C$ but $1 \notin D$.

(g) $B \subset D$: *true*, the statement "if $b \in B$, then $b \in D$" holds for all elements $b \in B$, which are $\{1\}$, $\{2\}$ and $\{3\}$ (of course the order of appearance in a set does not matter).

(h) $B \in D$: *false*, the *elements* of D are sets of numbers, but B is a set of sets. So B is not an element of D. None of the elements of D can be replaced with B.

(i) $A \in D$: *true*, D contains the set $\{1, 2, 3\} = A$.

$9 \times (1 + 1)$ marks = 1 mark for correct answer, 1 mark for giving a valid reason

2 Determine (and explain why) whether the relation R on the set of all dogs is reflexive, symmetric, antisymmetric and/or transitive, where $(a, b) \in R$ if and only if

a) a runs faster than b;

b) a and b have the same fur colour;

c) a ate from the same bowl as b.

Solution: For each relation, we need to check whether it is reflexive, symmetric and/or transitive.

(a) S is the set of all dogs, $a\rho b$ if a runs faster than b:

i. Reflexive: Is $a\rho a$? Is a runs faster than a: No, therefore not reflexive.

ii. Symmetric: If $a\rho b$, then is $b\rho a$? If a runs faster than b, is b runs faster than a? No, therefore not symmetric.

iii. Antisymmetric: If $a\rho b$, then it is not $b\rho a$. If a runs faster than b, then b does not run faster than a? Yes, therefore antisymmetric.

iv. Transitive: If $a\rho b$ and $b\rho c$, then is $a\rho c$? If a runs faster than b and b runs faster than c, is a runs faster than c? Yes. Therefore transitive.

(b) S is the set of all dogs, $a \sim b$ if a and b have the same fur colour:

i. Reflexive: Is $a \sim a$? Did a has the same fur colour a: Yes, therefore reflexive.

ii. Symmetric: If $a \sim b$, then is $b \sim a$? Yes, therefore symmetric. But it is not antysymmetric.

iii. Transitive: If $a \sim b$ and $b \sim c$, then is $a \sim c$? Yes, c and a have the same fur colour. So the relation is transitive.

Note that since the relation is reflexive, symmetric and transitive, it is an equivalence relation.

(c) S is the set of all dogs, $a \sim b$ if a and b ate from the same bowl:

i. Reflexive: Is $a \sim a$? Did a eat from the same bowl a: Yes, therefore reflexive.

ii. Symmetric: If $a \sim b$, then is $b \sim a$? Yes, therefore symmetric. It is not antisymmetric.

iii. Transitive: If $a \sim b$ and $b \sim c$, then is $a \sim c$? Consider this: a and b ate from the same bowl F, and b ate from the same bowl G as c on another occasion, but a did not eat from the same bowl as c so it did not eat from G. So a and c did not necessarily eat from the same bowl. Therefore this relation is not transitive.

4+4+4=12 marks

3. Sets describing intervals of **real** numbers are expressed with brackets and endpoints: a square bracket [ ] if the endpoint is included, a round bracket ( ) if the endpoint is excluded. Set $A = [0, 2)$. Then A is the set of all **real** numbers from 0 to 2, including 1 but not including 2. Define also the sets $B = (-5, 0)$ and $C = [1, 3]$.

   (a) Write as intervals the 3 possible pairwise intersections and the 3 possible unions of sets A, B, C. Name the resulting sets as $D, E, ...$ Do not use different letters to denote the same set.

   (b) You have several sets now. Define a relation ρ to be "is a subset of" $\subseteq$, on the set consisting of all sets you obtained. Write down the ordered pairs of this relation and draw the Hasse diagram of this partial ordering.

   (c) Does the resulting relation define a lattice? (explain why yes or why no)

   (d) What is the least upper bound and the greatest lower bound of the set $\{A, B, C\}$?

Solution:

We have $A = [0, 2)$, $B = (-5, 0)$ and $C = [1, 3]$. Draw the real line and mark these sets to facilitate your reasoning.

(a) The 3 possible unions and the 3 possible intersections are

$$\begin{aligned} D = A \cup B &= (-5, 2) & G &= A \cap B = \emptyset \\ E = A \cup C &= [0, 3] & H &= A \cap C = [1, 2) \\ F = B \cup C &= (-5, 0) \cup [1, 3] & I &= B \cap C = \emptyset = G \end{aligned}$$

2 marks

Note that we talk about real numbers (in boldface in this problem statement). Some students still confuse real numbers (including rationals like 3.1 and irrationals like π) with integers. The intervals above contain all integers, rationals and irrationals within the respective bounds, not just a few integers). This is assumed knowledge.

(b) We need to write down the elements of the relation $\subseteq$, which is "is a subset of". (Note there is no difference between the symbols $\subset$ and $\subseteq$ here.) To do this we first construct the set of all pairs (the Cartesian product $U \times U$) on which this relation is defined, where $U = \{A, B, C, D, E, F, G, H\}$ (no I, as it is redundant!).

Next we go through each of the 8 sets from U and check them against all of the other 8 sets.

We get

$$\subseteq = \left\{ \begin{matrix} (A, A), & (B, B), & (C, C), & (D, D), \\ (E, E), & (F, F), & (G, G), & (H, H), \\ (G, A) & (G, H), & (G, C), & (G, B), \\ (G, E) & (G, D), & (G, F), & (H, A), \\ (H, C) & (H, E), & (H, D), & (H, F), \\ (A, E), & (A, D), & (C, E), & (C, F), \\ (B, D), & (B, F) & & \end{matrix} \right\}$$

Hasse diagram (Fig.1) is obtained from part (b). We need to draw all possible links between the 8 sets $A, \ldots, H$ and then remove those obtained by using transitivity, e.g. link (H, E), etc. 8 marks

(c) For the lattice we need that every pair of U have the meet and join. Every pair has meet, but not so for join. For instance E and D do not have a join (least upper bound). Hence this is not a lattice. 6 marks

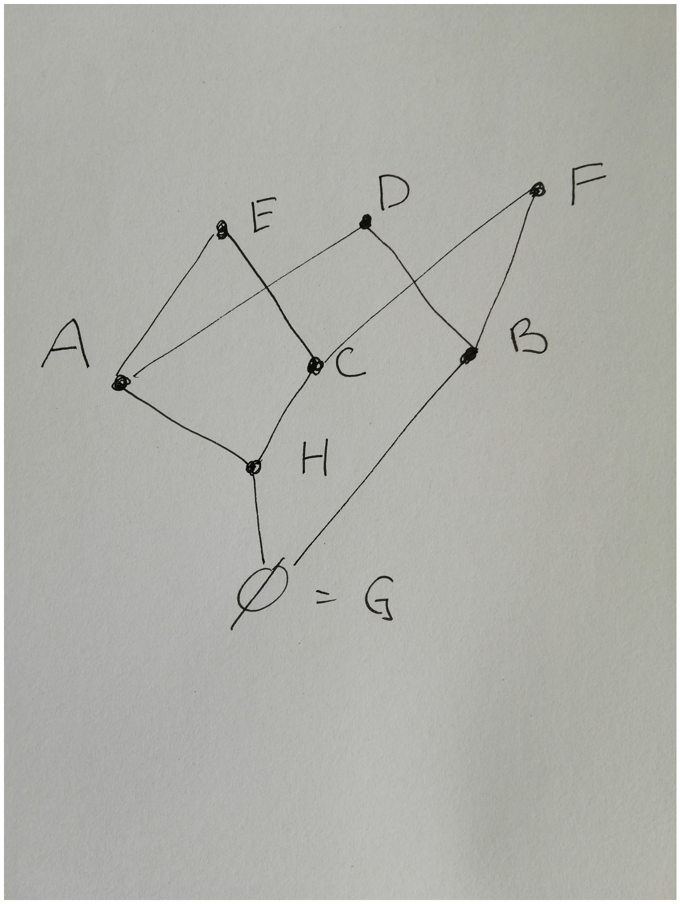


Figure 1: Hasse diagram.

(d) For the subset $\{A, B, C\}$ from the Hasse diagram there is no LUB. The GLB is $\emptyset = G$, because B is not comparable to H. 4 marks

4. Define the relation ρ on the set $S = \{a, b, c, d, e, f\}$ by

$$\rho = \{(a, a), (b, b), (c, c), (d, d), (f, f)(a, b), (a, c), (c, a),$$
$$(b, c), (c, b), (e, d), (d, f), (e, f), (f, e)\}$$

(a) Draw the directed graph of this relation.

(b) Which ordered ordered pairs (if any) must be adjoined to ρ to complete it into an equivalence relation on the set S. Write down the equivalence classes of the new relation on S.

Solution:

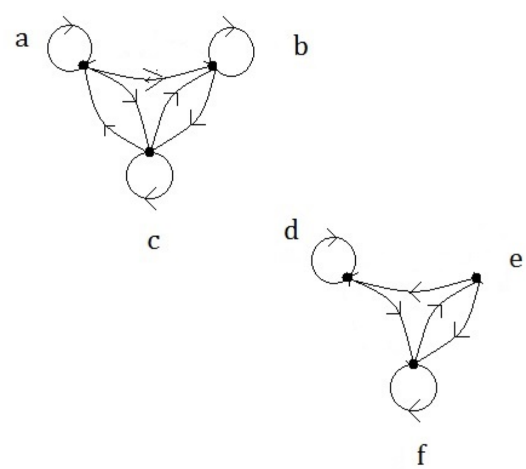


Figure 2: Directed graph of the relation ρ.

To verify that this is an equivalence relation we need to verify Reflexivity, Symmetry and Transitivity. Reflexivity and Symmetry fail (from the graph). So we need to add the pairs (e, e) (reflexivity) and (b, a), (f, d), (d, e) (symmetry). For **transitivity** : on the graph the triangles a, b, c and d, e, f need to be complete. Hence after the addition of the mentioned four pairs the relation becomes transitive, and hence it becomes an equivalence relation.

The equivalence classes from the graph are the clearly separated groups $[a] = \{a, b, c\}$ and $[d] = \{d, e, f\}$.

5+5 marks

5. Given the binary relations on the set $A = \{1, 2, 3, 4\}$ defined by:

$$\rho_1 = \{(1,4),(2,1),(2,2),(3,3),(4,3)\}$$

and

$$\rho_2 = \{(1,2),(1,3),(2,3),(3,3),(4,4)\}$$

determine (construct the ordered pairs) of the composite relations:

(a) ρ_1^2

(b) $\rho_1 \circ \rho_2$

(c) $\rho_2 \circ \rho_1$

(d) $\rho_1 \circ \rho_2 \circ \rho_1$

10 marks

Solution:

An easy way to find the composite relations is to use the mappings, see figures below. Counting the paths joining elements on the left and on the right we get

(a) $\rho_1^2 = \{(1,3),(2,4),(2,1),(2,2),(3,3),(4,3)\}$

(b) $\rho_1 \circ \rho_2 = \{(1,4),(2,2),(2,3),(3,3),(4,3)\}$

(c) $\rho_2 \circ \rho_1 = \{(1,1),(1,2),(1,3),(2,3),(3,3),(4,3)\}$

(d) $\rho_1 \circ \rho_2 \circ \rho_1 = \{(1,3),(2,1),(2,2),(2,3),(3,3),(4,3)\}$

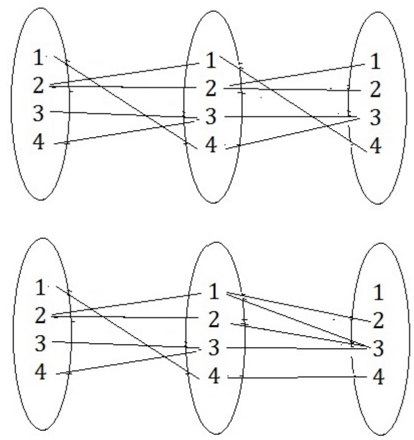


Figure 3: Composition of relations.

6. (i) Use the properties of determinants (page 72 Study Guide (SG)) to evaluate the determinants of A and B

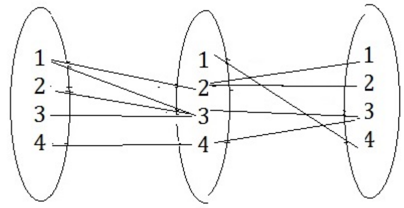
$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ -1 & -3 & -4 & 1 \\ 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 5 \end{bmatrix}, B = \begin{bmatrix} 1 & -3 & -2 & 4 \\ 3 & 2 & 5 & 1 \\ 4 & -1 & 3 & 5 \\ 5 & -4 & 1 & 9 \end{bmatrix}$$

(ii) Using the definition of **rank** of a matrix (3.3.1 P 74 SG), evaluate rank(A) and rank(B).

Solution:

(i) The properties of the determinants are to make our calculations easier. While we can expand the determinants along any row or column (preferably with a zero in it), we can also check if some of the rows or columns are multiples of one another, or are linear combinations of one another.

For matrix A we can check that row 1 minus row 3 equals 1/5th of 4. Immediately we obtain that the determinant of A is zero.



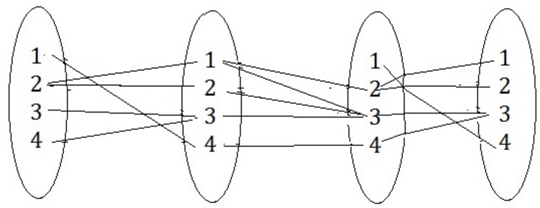


Figure 4: Composition of relations.

For matrix B we see that row 1 plus row 2 equals row 3. Again the rows are linear combinations of one another, and hence the determinant is zero.

This reasoning (or similar reasoning using the properties of the determinants) gives **6 marks**.

While students can evaluate the determinants directly, they can get 3 marks for that if the answer is correct i.e., 0 in both cases).

(ii) Since both determinants are zero, the ranks of both matrices cannot be 4. Let us try to find a 3 by 3 determinant which gives us a non-zero value.

For matrix A it is pointless to use the top left corner, because the rows 1,2 and 4 are dependent and we will get 0. It is worth using many zeros, so take the bottom right 3 by 3 block Expanding along the last row. We get

$$5\begin{vmatrix} -3 & -4 \\ 2 & 3 \end{vmatrix} = 5(-9-(-8)) = -5 \neq 0.$$

We conclude that $rank(A) = 3$.

For matrix B we try all 3 by 3 blocks and always get 0 as their determinants. This is because row 1 plus row 3 give row 4, so not only the first three rows but also rows 1,2,4 are linearly dependent. We find a 2 by 2 determinant which is not zero

$$\begin{vmatrix} 1 & -3 \\ 3 & 2 \end{vmatrix} = 2 + 9 \neq 0$$

We get $rank(B) = 2$. (4 marks)

Note that there are many different correct ways of calculating these determinants using properties of determinants. In all cases the final answer must be the same. The solutions provided show just one possible way of doing so.

7. (Extensions for higher marks) Calculate the determinants of the following matrices, and then solve for x the equations $Det(A) = 0$, $Det(B) = 0$

$$A = \begin{bmatrix} 2 & x & 0 \\ x & 2 & x \\ 0 & x & 2 \end{bmatrix}, B = \begin{bmatrix} x-1 & 0 & 4 \\ 0 & x+1 & 3 \\ 0 & 3 & x+1 \end{bmatrix}.$$

Solution:

Expand $|A|$ along row 1. We get

$$2(4 - x^2) - x(2x) = 8 - 2x^2 - 2x^2 = -4x^2 + 8.$$

Solving $|A| = -4x^2 + 8 = 0$ we have $x^2 = 2$ hence there are two solutions $x = \sqrt{2}, -\sqrt{2}$.

For B, expanding along column 1:

$$|B| = (x-1)((x+1)^2 - 9).$$

Then

$$(x-1)((x+1)^2 - 9) = 0$$

gives $x = 1$ or $(x+1)^2 = 9$, which in turn gives $(x+1) = \pm 3$, and so $x = 2$, $x = -4$.

Each matrix: 10 marks

8. Prove that points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ are collinear if and only if

$$det\left(\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}\right) = 0.$$

Solution:

There are various ways to solve this. The points A, B, C are collinear if (assuming B is in between A and C) $B = \alpha A + (1-\alpha)C$, $\alpha \in [0, 1]$. Which means $x_2 = \alpha x_1 + (1-\alpha)x_3$, $y_2 = \alpha y_1 + (1-\alpha)y_3$.

To prove collinearity $\Rightarrow$ det=0 (necessity): Assume collinearity and hence the above expressions. The matrix becomes

$$\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & \alpha x_1 + (1-\alpha)x_3 & \alpha y_1 + (1-\alpha)y_3 \\ 1 & x_3 & y_3 \end{bmatrix}$$

Let us now combine the rows in this way. Take α row1 + $(1-\alpha)row3$, and take away row 2

$$\begin{bmatrix} 1 & x_1 & y_1 \\ \alpha + (1-\alpha) - 1 & 0 & 0 \\ 1 & x_3 & y_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 0 & 0 & 0 \\ 1 & x_3 & y_3 \end{bmatrix}$$

The determinant of the above matrix is clearly 0 (and it is a multiple of the determinant of the original matrix). So the fact that the point B was a linear combination of the other points implied zero determinant.

Now the other way around. Assume the determinant is 0. Let us simplify calculations by subtracting rows 2 and 3 from row 1. We get

$$det\left(\begin{bmatrix} 1 & x_1 & y_1 \\ 0 & x_1 - x_2 & y_1 - y_2 \\ 0 & x_1 - x_3 & y_1 - y_3 \end{bmatrix}\right) = 1det\left(\begin{bmatrix} x_1 - x_2 & y_1 - y_2 \\ x_1 - x_3 & y_1 - y_3 \end{bmatrix}\right).$$

$$= (x_1 - x_2)(y_1 - y_3) - (y_1 - y_2)(x_1 - x_3) = 0.$$

This is possible in these cases:

1. $x_1 - x_2 = 0$ and $x_1 - x_3 = 0$ (meaning the three points are on a vertical line, hence collinear).

2. $y_1 - y_2 = 0$ and $y_1 - y_3 = 0$ (meaning the three points are on a horisintal line, hence collinear).

3. $x_1 - x_2 = 0$ and $y_1 - y_2 = 0$ (or the same for the other two points, just use symmetry), meaning $A = B$, again the points are collinear.

4. Neither of the above but $\frac{y_1-y_2}{x_1-x_2} = \frac{y_1-y_3}{x_1-x_3} = m$. Recall the equation of a straight line $y = mx + b$ which we can also write as $y - y_1 = m(x - x_1)$. m is the gradient. We note that it translates to $\frac{y-y_1}{x-x_1} = m$ for all points (x, y) on that line. Since both B and C satisfy this equation, they must be on the same straight line given by $y - y_1 = m(x - x_1)$.

So in each of these cases zero determinant implies the points are either coinciding or are on the same line (vertical, horisontal or with gradient m). No other possibilities. Which proves the sufficiency.

20 marks