One-Way Quantum Computation

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Outline

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QCM v.s. 1WQC

- Quantum computation is usually described in terms of quantum circuit model (QCM).
- One-way quantum computation (1WQC) is a very different way of describing quantum computation.
- Different models of quantum computation give us
  - New approaches to quantum computer design.
  - Different ways of understanding quantum computation.

General framework for 1WQC

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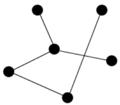
Step1. A special a special entangled state called a *Cluster State* or *Graph State* is prepared.



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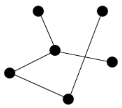

Step2. The qubits are then measured *one* at a time (single-qubit measurements), with local measurements, in a sequence of *different measurement bases*.

Outline

Single qubit rotations

- A single qubit rotation through angle ϕ around the X, Y, or Z axis corresponds to the operator

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$$R_x(\phi) = \exp[-i\frac{\phi}{2}X]$$

- **Euler Theorem:** Any rotation on a sphere can be formed from three successive rotations about mutually orthogonal axes.
- Thus any single qubit rotation can be expressed as

$$— \boxed{R_z(\alpha)} — \boxed{R_x(\beta)} — \boxed{R_z(\gamma)} —$$

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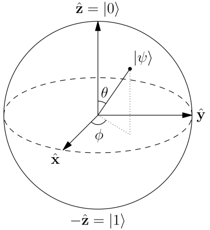

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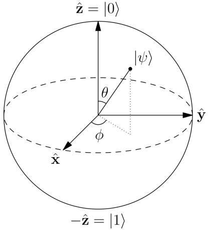
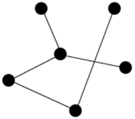


- Single qubit measurements applied in 1WQC:
  - Z measurement projects on $|0\rangle, |1\rangle$.
  - X measurement projects on $|\pm\rangle = [|0\rangle + |1\rangle]/\sqrt{2}$.
  - "X-Y plane" measurement $\cos(\alpha)X + \sin(\alpha)Y$ projects onto $[|0\rangle \pm e^{-i\alpha}|1\rangle]/\sqrt{2}$.

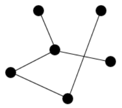
Outline

Graph states



- Graph states are entangled states over many qubits, with a very simple description.
- A collection of vertices, some of whom are linked by an edge. This leads to their name "graph states".
- Each vertex represents a qubit, each edge represents an entangle operation.
- The graph structure gives a method of constructing the graph states.

Constructing the graph states



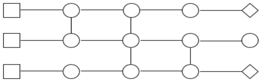
- With every vertex we associate a qubit, initially prepared in state $|+\rangle = [|0\rangle + |1\rangle]/\sqrt{2}$.
- The graph state is obtained by applying a CZ gate (entangle operation) to each edge on the graph.
- A CZ gate has the following truth table and is order-independent:

$$|00\rangle \mapsto |00\rangle,\ |01\rangle \mapsto |01\rangle,\ |10\rangle \mapsto |10\rangle,\ |00\rangle \mapsto -|11\rangle$$

$$\mathrm{CZ}|i\rangle|j\rangle = (-1)^{ij}|i\rangle|j\rangle,\ i,j \in \{0,1\}$$

Cluster state

- The term *cluster state* is reserved for a graph state on a square lattice:



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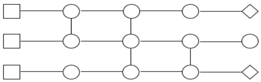

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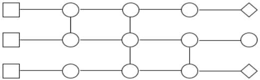

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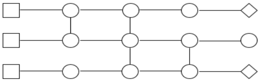
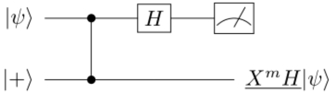


- In one-way quantum computation, by simply making single-qubit measurements on cluster states of sufficient size, any quantum circuit can be simulated.
- Extra flexibility of graph states require far fewer qubits when implementing the computation.
- Regular layout of cluster states means that they can be generated efficiently in physical implementation.

One-qubit teleportation

- The key idea in 1WQC is known as *one-bit teleportation*



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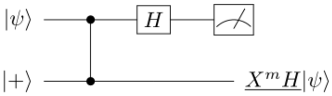

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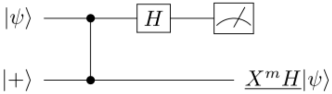
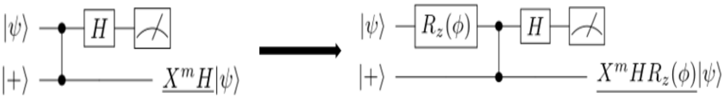


- m is the outcome (0 or 1) of the computational basis measurement on the first qubit.
- Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, then the evolution is

$$\begin{aligned}
|\psi\rangle|+\rangle &\xrightarrow{\text{CZ}} \alpha|0\rangle|+\rangle + \beta|1\rangle|-\rangle \\
&\xrightarrow{\text{H}\otimes\text{I}} \alpha|+\rangle|+\rangle + \beta|-\rangle|-\rangle \\
&= [|0\rangle \otimes \text{H}|\psi\rangle + |1\rangle \otimes \text{XH}|\psi\rangle]/\sqrt{2} \\
&\xrightarrow{\text{M}} \text{X}^m\text{H}|\psi\rangle
\end{aligned}$$

From one-bit teleportation to kernel pattern

- Imagine now that a Z-rotation $R_z(\phi)$ is added at the start of this circuit



1WQC

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Introduction

Preliminaries

One-Way Quantum Computation

Graph States and Cluster States

Kernel Pattern

Connecting Kernel Patterns

Implementation

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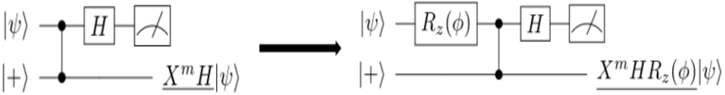
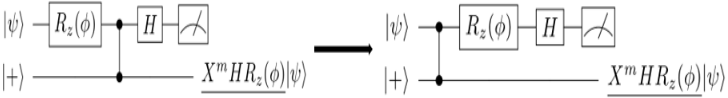


- Note that $[R_z(\phi), \mathrm{CZ}] = 0$, thus we can rearrange the order of the gates



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- Introduce a new symbol which explicitly states the measurement's basis states $\mathrm{X}^m|0\rangle$
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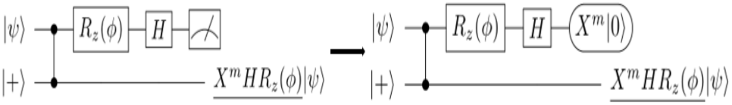

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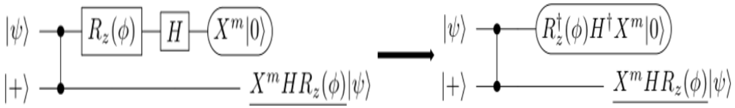
- Unitary operators before a measurement can be "absorbed" into the measurement with result reserved.



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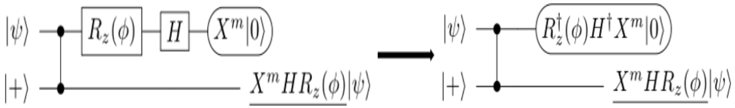
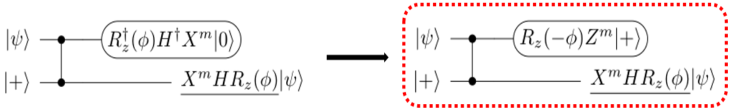
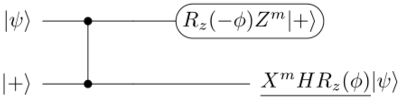


- $H^\dagger X^m|0\rangle = Z^m|+\rangle$, $R_z^\dagger(\phi) = R_z(-\phi)$, thus



Kernel pattern

- From the one-bit teleportation protocol, we derive the kernel pattern in 1WQC:



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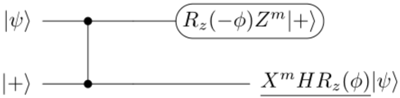

- In the Bloch sphere picture, the measurement with basis $R_z(-\phi)|+\rangle$ and $R_z(-\phi)|-\rangle$ lies on the $X - Y$ plane.

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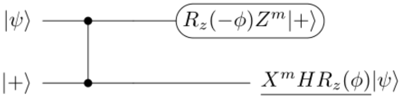
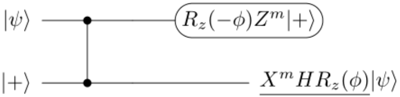


- In the Bloch sphere picture, the measurement with basis $R_z(-\phi)|+\rangle$ and $R_z(-\phi)|-\rangle$ lies on the $X - Y$ plane.
- Actually, the first qubit is measured in the basis

$$\frac{1}{\sqrt{2}}[|0]\rangle \pm e^{i\phi}|1\rangle]$$

Kernel pattern (cont.)



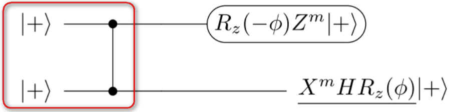
- The kernel pattern is implementing $\mathrm{H}R_z(\phi)$ modulo a Pauli correction X^m

$$\underbrace{X^m}_{\text{Random "by-product"}} \underbrace{\mathrm{H}R_z(\phi)}_{\text{Implemented unitary operator}}$$

- m is the measurement result of first qubit, that's why we say measurement based!

Cluster states in kernel pattern

- Imagine we set the input state to $|+\rangle$



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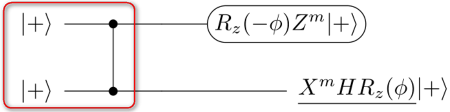

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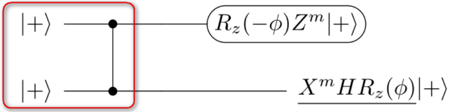

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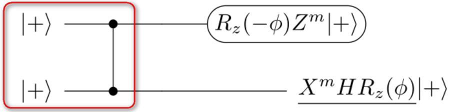

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- The circuit consists of a single qubit measurement on a cluster state.
- How can we connect the kernel patterns to implement more complicated unitary operators?

Simulating arbitrary single-qubit gates

- **Euler's Theorem:** any single-qubit rotation U can be decomposed as a product of three rotations

$$\forall\, U \in SU(2), \exists\, \alpha, \beta, \gamma \text{ such that } U = R_z(\gamma) R_x(\beta) R_z(\alpha).$$

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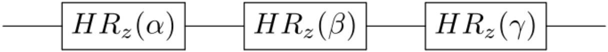

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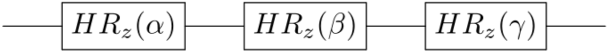
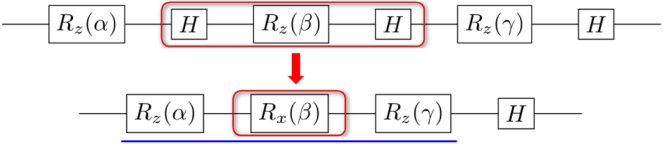
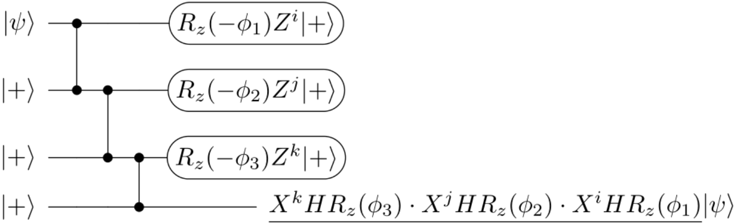


- This gives a circuit for simulating the rotation U

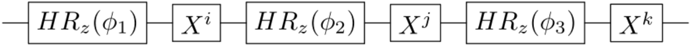


Concatenating the kernel patterns

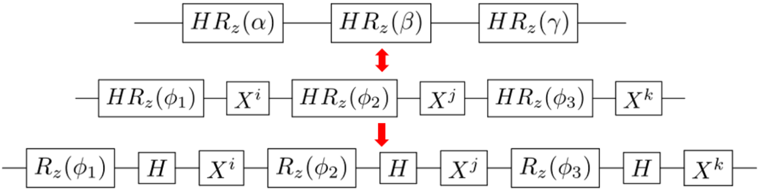
- We can simulate the above circuit by concatenating three copies of kernel pattern



which implements the following unitary operator:



By-product Pauli operators



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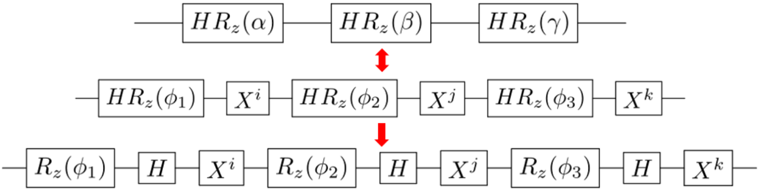

- Measurement-dependent by-product operators X^i are sandwiched among the H and R_z gates.
- We need to change the orders between X^i and H and R_z, so that all H and R_z operations are performed before X^i operations.

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  1. p **and** P **anti-commute**. In this case

$$\begin{aligned} R_p(\alpha)P = \exp[-i\frac{\alpha}{2}p]P &= [\cos(\frac{\alpha}{2}) - i\sin(\frac{\alpha}{2})p]P \\ &= P[\cos(\frac{\alpha}{2}) + i\sin(\frac{\alpha}{2})p] \\ &= PR_p(-\alpha) \end{aligned}$$

After the order has been interchanged, P is unchanged, the rotation remains a rotation but the rotation angle α is reversed!

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     After the order has been interchanged, P is unchanged, the rotation remains a rotation but the rotation angle α is reversed!
  2. p **and** P **commute**. The two can be interchanged without change, i.e. $R_p(\alpha)P = PR_p(\alpha)$.

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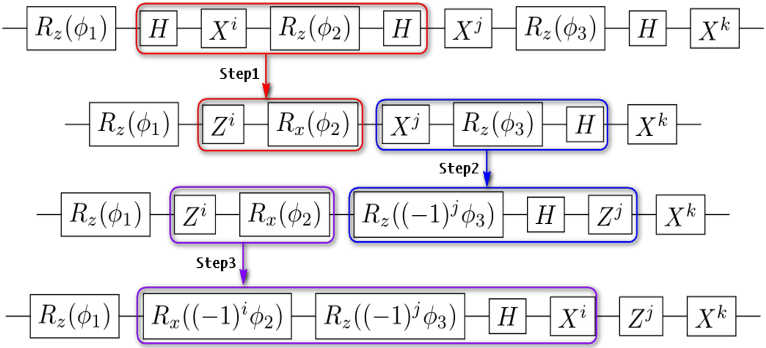
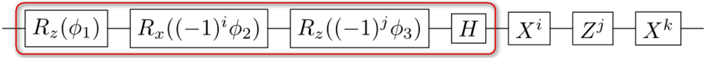
- We see that on interchanging the order:
  1. The Clifford group operator C remains invariant,
  2. The Pauli operator remains a (possibly different) Pauli operator.

Permutation of byproducts

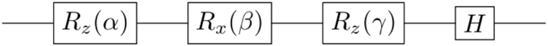
- We can now apply these permutation rules:
  1. **Rotation**: $R_z(\alpha)X = XR_z(-\alpha)$, $R_z(\alpha)Z = ZR_z(\alpha)$;
  2. **Clifford group**: $HX^i = Z^iH$, $HZ^i = X^iH$



Simulating arbitrary single-qubit gates



- Choosing $\phi_1 = \alpha$, $\phi_2 = (-1)^i\beta$, $\phi_3 = (-1)^j\gamma$, we implement the single-qubit gate $U = R_z(\gamma)R_x(\beta)R_z(\alpha)$ (up to the Pauli corrections).



Simulating arbitrary single-qubit gates

$$-\boxed{R_z(\phi_1)}-\boxed{R_x((-1)^i\phi_2)}-\boxed{R_z((-1)^j\phi_3)}-\boxed{H}-\boxed{X^i}-\boxed{Z^j}-\boxed{X^k}-$$

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$$-\boxed{R_z(\alpha)}-\boxed{R_x(\beta)}-\boxed{R_z(\gamma)}-\boxed{H}-$$

- Hence, we apply kernel patterns to successfully simulate arbitrary single-qubit gates.

Graph notations

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- We are going to represent each qubit by a shape:
  - The "input qubit" by a square $\square$
  - The "output qubit" by a diamond $\lozenge$
  - The other qubits by a circle $\bigcirc$

Graph notations

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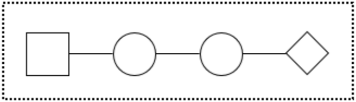
- One-dimensional lattice graphical notation for the clusters states of arbitrary single-qubit gates:



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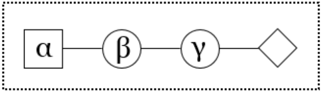

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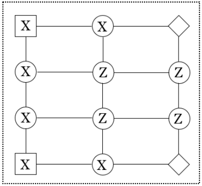

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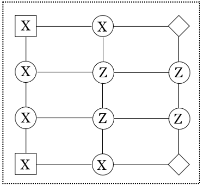

- Measurement patterns are not unique and subject to the construction approaches.

Outline

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- There are systems where the one-way model has some significant practical advantages.

Physical implementation of 1WQC (cont.)

- Optical lattices
  - **Pro:** cluster states can be generated efficiently, already implemented in labs.
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- Cavity QED
  - **Pro:** atoms interact in cavities, implementing CZ gates and cluster state.
  - **Con:** hard to implement single-qubit measurements.

Acknowledgement

Thank you !

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