Quantum Walk with Absorbing Boundaries

Wang Kun

Department of Computer Science and Technology
Nanjing University, China

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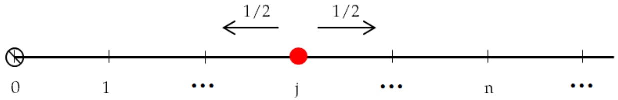
Outline

Quantum walk with one boundary[ABN+01]



Step1. Initialize the system to state $|\psi_{init}\rangle = |j\rangle|R\rangle$

Step2. For each step of evolution

- Apply $\mathcal{U} = \mathcal{S} \cdot (\mathcal{I} \otimes \mathcal{H})$ to the system
- Observe system with projection operator $\{\prod_{yes}^{0}, \prod_{no}^{0}\}$

Step3. If the result of measurement was *"yes"*, terminate; otherwise goto **Step 2**

Exit probability

- ***Exit probability:*** the probability that the measurement of whether the walker is at position 0 eventually results in *"yes"*
- Let $P_{0,j,\infty}$ denotes the exit probability
- Calculating $P_{0,1,\infty}$. $|\psi_{init}\rangle = |1\rangle|R\rangle$

$$\xrightarrow{\mathcal{I}\otimes\mathcal{H}} \quad \frac{1}{\sqrt{2}}(|1\rangle|L\rangle + |1\rangle|R\rangle) \xrightarrow{\mathcal{S}} \frac{1}{\sqrt{2}}(|0\rangle|L\rangle + |2\rangle|R\rangle)$$

$$\xrightarrow{\Pi^0_{yes}} \quad \begin{cases} |0\rangle|L\rangle & \text{w.p. } 1/2\text{, terminates} \\ |2\rangle|R\rangle & \text{w.p. } 1/2\text{, continues} \end{cases}$$

$$\xrightarrow{\mathcal{I}\otimes\mathcal{H}} \quad \frac{1}{\sqrt{2}}(|2\rangle|L\rangle + |2\rangle|R\rangle) \xrightarrow{\mathcal{S}} \frac{1}{\sqrt{2}}(|1\rangle|L\rangle + |3\rangle|R\rangle)$$

$$\xrightarrow{\Pi^0_{yes}} \quad \frac{1}{\sqrt{2}}(|1\rangle|L\rangle + |3\rangle|R\rangle) \quad \text{w.p. } 1/2\text{, continues}$$

The state-of-art

- Mathematical expression of $P_{0,j,\infty}$[BCG+04]

$$P_{0,j,\infty} = \frac{1}{2\pi} \int_0^{2\pi} |F(\theta)|^2 |G(\theta)|^{2j-2} \mathrm{d}\theta$$

  where $F(\theta), G(\theta)$ are pre-defined functions
- Special cases

1. $$P_{0,1,\infty} = \frac{2}{\pi}.$$

   Sharp contrast with the random walk. In random walk, the probability of eventually reaching position 0 is 1.
2. $$\lim_{j \to \infty} P_{0,j,\infty} = \frac{2}{\pi} - \frac{1}{2}.$$

   Sharp contrast with the random walk. In random walk, the probability of eventually reaching position 0 is 1.

Arbitrary coin state

- For an arbitrary coin, the system state may be written as

$$|\psi\rangle_{init} = |j\rangle(\alpha|L\rangle + \beta|R\rangle), \alpha, \beta \in \mathbb{C}.$$

- Mathematical expression of $P_{0,j,\infty}$ [BCG+04]

$$P_{0,j,\infty} = \frac{1}{2\pi}\int_0^{2\pi} |\alpha F(\theta) + \beta G(\theta)|^2 |G(\theta)|^{2i-2} \mathrm{d}\theta$$

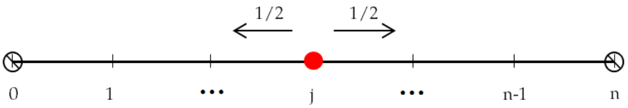
Arbitrary coin transformation

- For an arbitrary unitary operator $\mathcal{C}$, the transformation may be written as

$$\begin{aligned} \mathcal{C}|L\rangle &= a|L\rangle + b|R\rangle \\ \mathcal{C}|R\rangle &= c|L\rangle + d|R\rangle \end{aligned}$$

- Mathematical expression of $P_{0,j,\infty}$[BCG+04]

Quantum walk with two boundaries[ABN+01]



Step1. Initialize the system to state $|\psi_{init}\rangle = |j\rangle|R\rangle$

Step2. For each step of evolution

- Apply $\mathcal{U} = \mathcal{S} \cdot (\mathcal{I} \otimes \mathcal{H})$ to the system
- Observe system with projection operator $\{\prod_{yes}^{0}, \prod_{no}^{0}\}$
- Observe system with projection operator $\{\prod_{yes}^{n}, \prod_{no}^{n}\}$

Step3. If the result of either measurements was *"yes"*, terminate; otherwise goto **Step 2**

Exit probabilities

- ***Exit probability:*** the probability that the measurement of whether the walker is at position 0 eventually results in *"yes"*
- Let $P_{0,j,n}$ denotes the exit probability
- Calculating $P_{0,1,3}$. $|\psi_{init}\rangle = |1\rangle|R\rangle$

$$\xrightarrow{\mathcal{U}} \frac{1}{\sqrt{2}}(|0\rangle|L\rangle + |2\rangle|R\rangle) = \begin{cases} |0\rangle|L\rangle, & \text{w.p. } 1/2\text{, terminates} \\ |2\rangle|R\rangle, & \text{w.p. } 1/2\text{, continues} \end{cases}$$

$$\xrightarrow{\mathcal{U}} \frac{1}{\sqrt{2}}(|1\rangle|L\rangle + |3\rangle|R\rangle) = \begin{cases} |3\rangle|R\rangle, & \text{w.p. } 1/4\text{, terminates} \\ |1\rangle|L\rangle, & \text{w.p. } 1/4\text{, continues} \end{cases}$$

$$\xrightarrow{\mathcal{U}} \frac{1}{\sqrt{2}}(|0\rangle|L\rangle - |2\rangle|R\rangle) = \begin{cases} |0\rangle|L\rangle, & \text{w.p. } 1/8\text{, terminates} \\ |2\rangle|R\rangle, & \text{w.p. } 1/8\text{, continues} \end{cases}$$

$$\xrightarrow{\mathcal{U}} \cdots\cdots$$

- $P_{0,1,3} = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{4} + \cdots = \frac{2}{3}$

The state-of-art

- Mathematical expression of $P_{0,j,n}$[BB09]

$$P_{0,j,n} = \frac{\sqrt{2}}{4} \cdot \frac{(A^{n-j} - B^{n-j})(A^{j-1}B^{j-1})}{A^{n-1} + B^{n-1}}$$

where $A = 2 + \sqrt{2}$, $B = 2 - \sqrt{2}$.

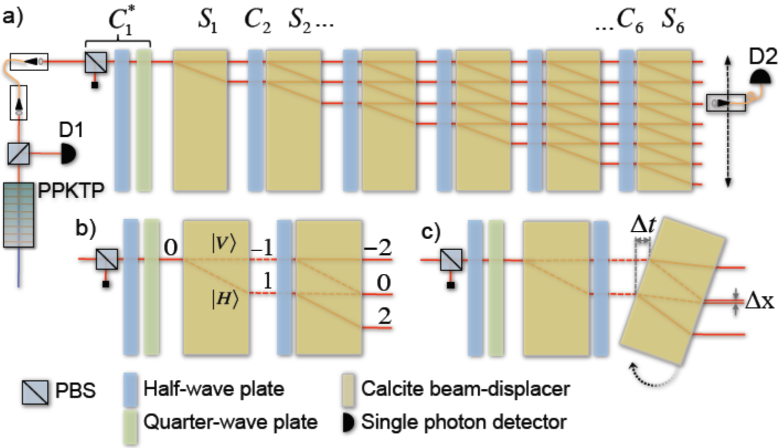
- Special cases

1. $$\lim_{n\to\infty} P_{0,1,n} = \frac{1}{\sqrt{2}}.$$

   Sharp contrast with the random walk. In random walk, the probability of eventually exiting to the left is $\lim_{n\to\infty} P_{0,1,n} = 1$.

2. $$P_{0,1,n+1} = \frac{1 + 2P_{0,1,n}}{2 + 2P_{0,1,n}},\ \forall n \geq 0.$$

Single-photon quantum walk[BFL+10]



Single-photon quantum walk (cont.)

- Tunable decoherence by pure dephasing

$$\rho_{N+1} = (1-q)\mathcal{U}\rho_N\mathcal{U}^\dagger + q\sum_i K_i\mathcal{U}\rho_N\mathcal{U}^\dagger K_i^\dagger$$

  The parameter q is the probability of a dephasing event occur at each step
- A difference between the quantum walk exit probability and classical walk exit probability first occurs after 5 steps
- Absorbing boundaries implemented using *beam blocks* in every spatial mode -1

What we can do

- What are other questions which could be asked about the quantum walk with boundaries?
- In the two boundaries case, arbitrary coin state and arbitrary coin transformation have not been analyzed
- Applications of quantum walk with boundaries
  - *The ruined gambler*
- Demonstrate the differences using other physical schemes
  - How to implement *projection measurement* $\{\prod_{yes}^{0}, \prod_{no}^{0}\}$
  - Walking more steps to show the difference

Summary

- Exit probabilities of quantum walk with boundaries have been intensively studied
- Parameters in quantum walk
    - Discrete or continuous
    - Arbitrary coin initial state
    - Arbitrary coin transformation
    - *Arbitrary walker initial state?*

Acknowledgement

Thank you !

References I

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- Matthew A Broome, Alessandro Fedrizzi, Benjimain P Lanyon, Ivan Kassal, Alan Aspuru-Guzik, and Andrew G White, *Discrete single-photon quantum walks with tunable decoherence*, Physical review letters **104** (2010), no. 15, 153602.