One Dimensional Discrete Quantum Walk

Kun Wang

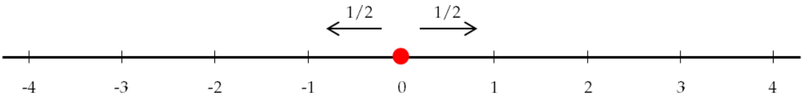
Department of Computer Science and Technology
Nanjing University, China

June 30, 2014

Outline

Outline

One dimensional discrete random walk



- One of the most elementary stochastic processes
- On the integer number line $\mathbb{Z}$
- Starts at 0, at each step moves left or right of unit with equal probability
- S_t gives the distance from origin after walking t steps

$$S_0 = 0, S_t = \sum_{i=1}^{t} Y_i, \quad \mathbb{P}(Y_i = 1) = \mathbb{P}(Y_i = -1) = \frac{1}{2}$$

- Series $S = \{S_t\}$ is a *one dimensional discrete random walk*

- S_t gives the distance from origin after walking t steps

$$\mathbb{P}(S_t = n | S_0 = 0) = \begin{cases} \frac{1}{2^t}\binom{t}{\frac{n+t}{2}}, & \frac{n+t}{2} \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}$$

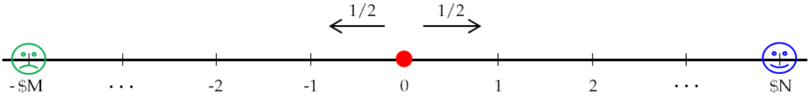
n	-5	-4	-3	-2	-1	0	1	2	3	4	5
$2^0\mathbb{P}(S_0 = n$						1					
$2^1\mathbb{P}(S_1 = n)$					1		1				
$2^2\mathbb{P}(S_2 = n)$				1		2		1			
$2^3\mathbb{P}(S_3 = n)$			1		3		3		1		
$2^4\mathbb{P}(S_4 = n)$		1		4		6		4		1	
$2^5\mathbb{P}(S_5 = n)$	1		5		10		10		5		1

Table: Simple random walk of first 5 steps[1]

Some parameters of random walks

$$
\begin{aligned}
\mathrm{E}\left(Y_i\right) &= 0 \\
\mathrm{E}\left(Y_i^2\right) &= 1 \\
\mathrm{E}\left(S_t\right) &= \sum_{i=1}^{t} \mathrm{E}\left(Y_i\right) = 0 \\
\mathrm{E}\left(S_t^2\right) &= \sum_{i=1}^{t} \mathrm{E}\left(Y_i^2\right) = t \\
\lim_{t\to\infty} \frac{\mathrm{E}\left(\mid S_t \mid\right)}{\sqrt{t}} &= \sqrt{\frac{2}{\pi}}
\end{aligned}
$$

Gambler's ruin



- A gambler starts with an initial fortune of m dollars
- On each successive round, the gambler either wins or loses 1 dollar independent of the past with equal probabilities
- Play the gamble until either his earning reaches n(win) or his wealth drops to 0 (lose)
- *What is the probability that the gambler will win/lose?*

Gambler's ruin (cont.)

- Denote the probability that the gambler will lose as p_n, then the win probability is $1 - p_n$
- Let W^t is the gain of the gambler after t steps, then

$$E[W^t] = 0 = mp_n - n(1 - p_n) \quad \Rightarrow \quad p_n = \frac{n}{m+n}$$

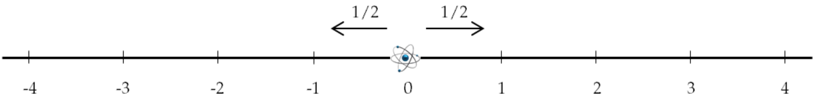
- Probability of losing m dollars is proportional to the amount of money n the gambler is willing to win[2]
- *Really a ruin!*

Outline

What is quantum walk

- A random walk is the simulation of the random movement of a particle around a graph
- A quantum walk is the same - but with a quantum particle
- Random walks are a useful model for developing classical algorithms
- Quantum walks provide a new way of developing quantum algorithms
  - Quantum Fourier transform - Shor's algorithm[3]
  - Amplitude amplification - Grover's algorithm[4]
  - Quantum walks

What we really care



- We only care *one dimensional discrete quantum walk*
- Main components of a discrete quantum walk
  - Walker and coin: actors of the quantum walk
  - Evolution operators for both walker and coin: how to play the wandering game?
  - Projection operators for measurement: where is the walker?
- One dimensional discrete quantum walk has specified restrictions on these components

The walker and the coin

- *The walker* is a quantum system living in a infinite but countable dimension Hilbert space $\mathcal{H}_p$
  - $\{|i\rangle_p, i \in \mathbb{Z}\}$ are computational basis states that span $\mathcal{H}_p$
  - Denote the walker's state as $|position\rangle \in \mathcal{H}_p$
  - Walker can be any valid superposition state of system $\mathcal{H}_p$

$$|position\rangle = \sum_i \alpha_i |i\rangle_p, \ s.t. \ \sum_i |\alpha_i|^2 = 1, \alpha_i \in \mathbb{C},$$

    the walker is at position i with probability $|\alpha_i|^2$
  - The walker is usually initialized at the origin, i.e.

$$|position\rangle_{init} = |0\rangle_p$$

The walker and the coin (cont.)

- *The coin* is a quantum system living in a 2-dimensional Hilbert space $\mathcal{H}_c$
  - $\{|0\rangle, |1\rangle\}$ are computational basis states that span $\mathcal{H}_c$
  - Denote the coin's state as $|coin\rangle \in \mathcal{H}_c$
  - Coin can be any valid superposition state of the system $\mathcal{H}_c$

    $$|coin\rangle = \alpha|0\rangle + \beta|1\rangle,\ s.t.\ |\alpha|^2 + |\beta|^2 = 1, \alpha, \beta \in \mathbb{C},$$

    the coin is at state $0, 1$ with probability $|\alpha|^2, |\beta|^2$
  - $0, 1$ can be used to indicate "move left", "move right"
- State of a quantum walk $|\psi\rangle$ resides in space $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{H}_c$

  $$|\psi\rangle = |position\rangle \otimes |coin\rangle \in \mathcal{H}$$

- Initial state of the quantum walk system is

  $$|\psi\rangle_{init} = |position\rangle_{init} \otimes |coin\rangle_{init}$$

Evolution operators

- Each step of the evolution is a *coin flip transformation* followed by a *shift transformation*
  - *coin flip transformation* $\mathcal{C}$ operates on the coin state $|coin\rangle$

$$\begin{array}{l} \mathcal{C}|0\rangle = a|0\rangle + b|1\rangle \\ \mathcal{C}|1\rangle = c|0\rangle + d|1\rangle \end{array} , \quad \mathcal{C} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

  - Among all possible coin operators, the Hadamard operator $\mathcal{H}$ has been extensively employed

$$\begin{array}{l} \mathcal{H}|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \\ \mathcal{H}|1\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \end{array} , \quad \mathcal{H} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\mathcal{H}|coin\rangle = \mathcal{H}(\alpha|0\rangle + \beta|1\rangle) = \frac{\alpha+\beta}{\sqrt{2}}|0\rangle + \frac{\alpha-\beta}{\sqrt{2}}|1\rangle$$

  - $\mathcal{H}$ is a unbaised coin operator

Evolution operators (cont.)

- Each step of the evolution is a *coin flip transformation* followed by a *shift transformation*
  - *shift transformation* $\mathcal{S}$ on system state ψ

$$\begin{aligned} \mathcal{S}|n\rangle|0\rangle &= |n-1\rangle|0\rangle \\ \mathcal{S}|n\rangle|1\rangle &= |n+1\rangle|1\rangle \end{aligned}$$

- $\mathcal{H}, \mathcal{S}$ are unitary operations, hence the combination is too. The combined operator $\mathcal{U}$ lives on total Hilbert space $\mathcal{H}$

$$\mathcal{U} = \mathcal{S} \cdot (\mathcal{I} \otimes \mathcal{H})$$

- A *discrete quantum walk* after t steps is

$$|\psi\rangle_t = \mathcal{U}^t|\psi\rangle_{init} = \sum_i \underbrace{|i\rangle_p}_{\text{walker position}} \underbrace{(\alpha_i|0\rangle + \beta_i|1\rangle)}_{\text{coin state}}$$

Measurement

- Perform a measurement at some point in order to know the outcome of quantum walk
- A set of projection operators to answer the question "Is the walker at position n?"

$$\prod_{yes}^{n} = |n\rangle\langle n| \otimes \mathcal{I}_c, \quad \prod_{no}^{n} = \mathcal{I} - \prod_{yes}^{n}$$

- Example: "Is the walker at position 0?"

$$|\psi\rangle = \frac{1}{2}|0\rangle|1\rangle - \frac{1}{2}|0\rangle|0\rangle + \frac{1}{2}|2\rangle|1\rangle + \frac{1}{2}|4\rangle|0\rangle$$

$$\left\| \prod_{yes}^{0} |\psi\rangle \right\| = \left\| \frac{1}{2}|0\rangle|1\rangle - \frac{1}{2}|0\rangle|0\rangle \right\| = \frac{1}{2}, \quad |\psi'\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle - |0\rangle|0\rangle)$$

Evolution of discrete quantum walk

Step1. Initialize the system to state $|\psi\rangle_{init} = |0\rangle|0\rangle$

Step2. For any choosen number of steps t, apply $\mathcal{U}$ to the system t times

$$|\psi\rangle_t = \mathcal{U}^t|\psi\rangle_{init}.$$

Step3. Apply the projection operator $\prod_{yes}^{n}$ to $|\psi\rangle_t$.

Example of evolution

$$|\psi\rangle_{init} = |0\rangle|0\rangle$$

$$\xrightarrow{\mathcal{I}\otimes\mathcal{C}} \quad \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |0\rangle|1\rangle) \xrightarrow{\mathcal{S}} \frac{1}{\sqrt{2}}(|-1\rangle|0\rangle + |1\rangle|1\rangle)$$

$$\xrightarrow{\mathcal{I}\otimes\mathcal{C}} \quad \frac{1}{2}(|-1\rangle|0\rangle + |-1\rangle|1\rangle + |1\rangle|0\rangle - |1\rangle|1\rangle)$$

$$\xrightarrow{\mathcal{S}} \quad \frac{1}{2}(|-2\rangle|0\rangle + |0\rangle|1\rangle + |0\rangle|0\rangle - |2\rangle|1\rangle)$$

$$\xrightarrow{\mathcal{I}\otimes\mathcal{C}} \quad \frac{1}{2\sqrt{2}}(|-2\rangle|0\rangle + |-2\rangle|1\rangle + |0\rangle|0\rangle - |0\rangle|1\rangle + |0\rangle|0\rangle + |0\rangle|1\rangle - |2\rangle|0\rangle + |2\rangle|1\rangle)$$

$$\xrightarrow{\mathcal{S}} \quad \frac{1}{2\sqrt{2}}(|-3\rangle|0\rangle + |-1\rangle|1\rangle + 2|-1\rangle|0\rangle - |1\rangle|0\rangle + |3\rangle|1\rangle)$$

$$\xrightarrow{\prod_{yes}^{-1}} \quad \frac{1}{\sqrt{5}}(|-1\rangle|1\rangle + 2|-1\rangle|0\rangle) \quad \text{with probability } 5/8$$

Differences between two walks

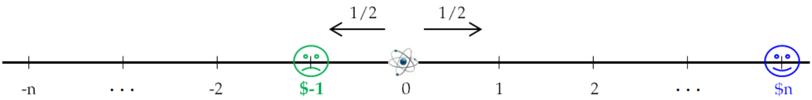
	random walk	quantum walk
expected pos	0	0
max prob pos	0	$\pm\frac{t}{\sqrt{2}}$
hitting time H_{0n}	$\mathrm{O}\left(\sqrt{t}\right), \quad n \ll t$ $\mathrm{O}\left(2^t\right), \quad n \approx t$	$\mathrm{O}\left(\sqrt{t}\right), \quad n \in [\frac{-t}{\sqrt{2}}, \frac{t}{\sqrt{2}}]$ $\mathrm{O}\left(2^t\right), \quad$ otherwise
mixing time	$\mathrm{O}\left(t^2\right)$	$\mathrm{O}\left(t\right)$

Table: Comparison of some parameters after walking t steps[5]

Differences between two walks (cont.)

Gambler's ruin again

What about the Quantum Money?



- Quantum walk with two boundaries
- Assume the gambler has only 1 "quantum money", but wants to earn n "quantum money"
- Probability of winning is $1/(n+1)$ in random walk, nearly impossible as n becomes larger

Gambler's ruin again (cont.)

- Denote the probability that the gambler will lose as p_n, then the win probability is $1 - p_n$

$$\textbf{Theorem} \quad [6]. \ \lim_{n \to \infty} p_n = \frac{1}{\sqrt{2}}.$$

- The "quantum gambler" has a probability of $1 - \frac{1}{\sqrt{2}} \approx 0.3$ to win a big amount of "quantum money" even he has little
- Probability of winning n "quantum money" is proportional to the amount of money n the gambler is willing to win
- *Not a ruin at all!*

Outline

Summary and future work

- Quantum walks are a way to develop quantum algorithms[7]
  - Exponentially fast hitting
  - Quantum walk search: hypercube, grid, $\cdots$
- Reasons for studying one dimensional quantum walk[8]
  - Build quantum walks on more sophisticated structures
  - A simple model to understand properties of quantum walks for the development of quantum algorithms
  - Employed to test the quantumness of physical systems
- Some problems which require further study
  - Algorithms that apply the model of quantum walk
  - Quantum walk with boundaries
  - More general graph structures

Q & A

Thank you !

Any questions ?

References I

Wikipedia.
Random walk.
url="http://en.wikipedia.org/wiki/Random_walk", June 2014.

Michael Mitzenmacher and Eli Upfal.
Probability and computing: Randomized algorithms and probabilistic analysis.
Cambridge University Press, 2005.

Peter W Shor.
Algorithms for quantum computation: discrete logarithms and factoring.
In *Foundations of Computer Science, 1994 Proceedings., 35th Annual Symposium on*, pages 124--134. IEEE, 1994.

References II

Lov K Grover.
A fast quantum mechanical algorithm for database search.
In *Proceedings of the twenty-eighth annual ACM symposium on Theory of computing*, pages 212--219. ACM, 1996.

Takuya Machida.
Quantum walk.
url="http://machida.stat.t.u-tokyo.ac.jp/pukiwiki/index.php?Quantum%20walk",
February 2014.

References III

Andris Ambainis, Eric Bach, Ashwin Nayak, Ashvin Vishwanath, and John Watrous.
One-dimensional quantum walks.
In *Proceedings of the thirty-third annual ACM symposium on Theory of computing*, pages 37--49. ACM, 2001.

Andris Ambainis.
Quantum walks and their algorithmic applications.
International Journal of Quantum Information, 1(04):507--518, 2003.

Salvador Elías Venegas-Andraca.
Quantum walks: a comprehensive review.
Quantum Information Processing, 11(5):1015--1106, 2012.