Three State Discrete Quantum Walks

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Outline

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The walker and the coin

- *The walker* is a quantum system in infinite-dimensional Hilbert space $\mathcal{H}_p$
- *The coin* is a quantum system in 3-dimensional Hilbert space $\mathcal{H}_c$
    - $\{|L\rangle, |S\rangle, |R\rangle\}$ are computational basis that span $\mathcal{H}_c$
    - $|L\rangle$ indicates *left* move
    - $|S\rangle$ indicates *stay* in current position
    - $|R\rangle$ indicates *right* move
- State of a quantum walk is denoted as $|\Psi\rangle$

$$|\Psi\rangle = |pos\rangle \otimes |coin\rangle \in \mathcal{H}_p \otimes \mathcal{H}_c$$

- Initial state of a quantum walk is

$$|\Psi\rangle_0 = |pos\rangle_0 \otimes |coin\rangle_0$$

Evolution operator

- Each step of the evolution is a *coin flip transformation* followed by a *shift transformation*
- Coin flip operator G (Grover operator)

$$\mathrm{G} = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$$

$$\mathrm{G}|L\rangle = -\frac{1}{3}|L\rangle + \frac{2}{3}|S\rangle + \frac{2}{3}|R\rangle$$

$$\mathrm{G}|S\rangle = \frac{2}{3}|L\rangle - \frac{1}{3}|S\rangle + \frac{2}{3}|R\rangle$$

$$\mathrm{G}|R\rangle = \frac{2}{3}|L\rangle + \frac{2}{3}|S\rangle - \frac{1}{3}|R\rangle$$

Evolution operator (cont.)

- Each step of the evolution is a *coin flip transformation* followed by a *shift transformation*
- Walker shift operator S

$$S(|j\rangle|L\rangle) = |j-1\rangle|L\rangle$$

$$S(|j\rangle|S\rangle) = |j\rangle|S\rangle$$

$$S(|j\rangle|R\rangle) = |j+1\rangle|R\rangle$$

- The combined operator is one-step evolution operator U

$$\mathrm{U} = \mathrm{S} \cdot (\mathrm{I} \otimes \mathrm{C})$$

- A *discrete three state quantum walk* after t steps is

$$|\Psi\rangle_t = \mathrm{U}^t|\Psi\rangle_0$$

Projection measurement

- Perform measurement to get outcome of quantum walk.
- A set of projection operators to answer the question "Is the walker located at position n?"

$$\prod_{yes}^{n} = |n\rangle\langle n| \otimes |L\rangle\langle L| + |n\rangle\langle n| \otimes |S\rangle\langle S| + |n\rangle\langle n| \otimes |R\rangle\langle R|$$

$$\prod_{no}^{n} = I - \prod_{yes}^{n}$$

- Example: "Is the walker at position 0?"

$$|\Psi\rangle = \frac{1}{\sqrt{3}}|0\rangle|L\rangle - \frac{1}{\sqrt{3}}|1\rangle|S\rangle + \frac{1}{\sqrt{3}}|1\rangle|R\rangle$$

$$\left|\prod_{yes}^{0}|\Psi\rangle\right| = \left|\frac{1}{\sqrt{3}}|0\rangle|L\rangle\right| = \frac{1}{3}, \qquad |\Psi'\rangle = |0\rangle|L\rangle$$

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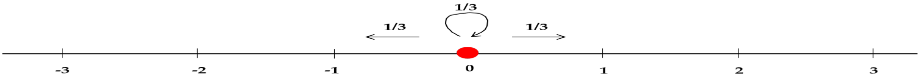
Three state quantum walk



Step1. Initialize the system to state

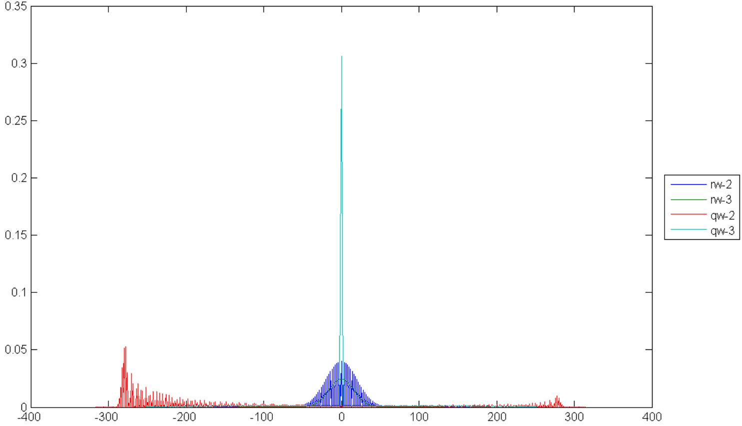
$$|\Psi\rangle_0 = |0\rangle[\frac{1}{\sqrt{3}}(|L\rangle + |S\rangle + |R\rangle)]$$

Step2. For any chosen number of steps t, apply U to the system t times

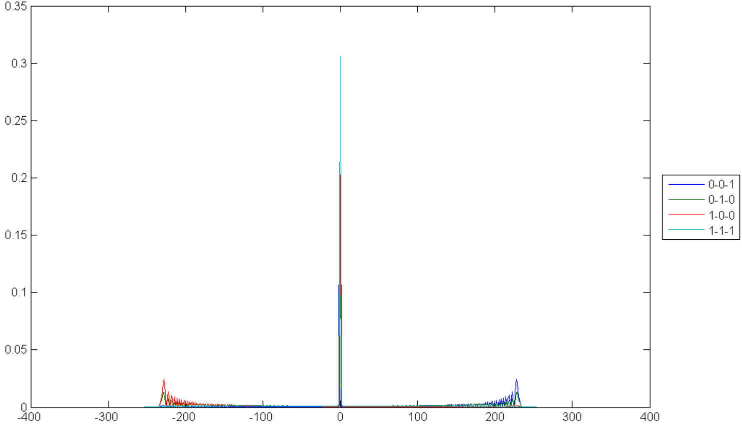
$$|\Psi\rangle_t = \mathrm{U}^t|\Psi\rangle_0$$

Step3. Apply the projection operator $\prod_{yes}^{n}$ to $|\Psi\rangle_t$

Random walks v.s. quantum walks



Different coin initial states



Outline

Description

Step1. Initialize the system to state

$$|\Psi\rangle_0 = |0\rangle|R\rangle$$

Step2. For each step of evolution

- Apply $\mathrm{U} = \mathrm{S} \cdot (\mathrm{I} \otimes \mathrm{G})$ to the system.
- Observe system with projection operators $\{\prod_{yes}^{-1}, \prod_{no}^{-1}\}$.

Step3. If the result of measurement was *"yes"*, terminate; otherwise goto **Step 2**

Exit probability

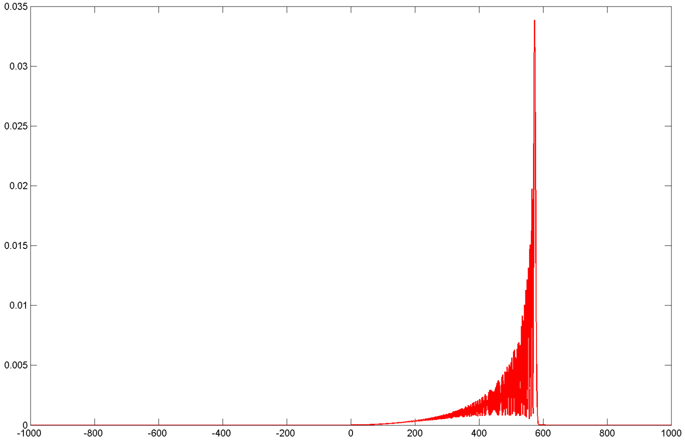
- ***Exit probability:*** the probability that the measurement of whether the walker is at position -1 results in *"yes"*
- Let $P_{\underline{-1,0,\infty}}$ denotes the exit probability. *What is the analytical expression of $P_{\underline{-1,0,\infty}}$?*
- In two state quantum walk case (Proved in 2001, STOC):

$$P_{\underline{-1,0,\infty}} = \frac{2}{\pi}.$$

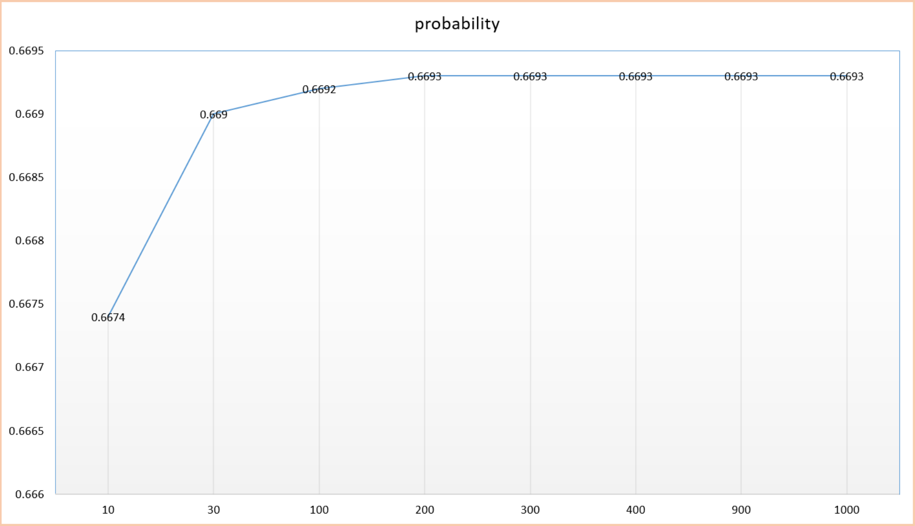
Exit probability after 1000 steps

- quantum walks
- three state
- one boundary

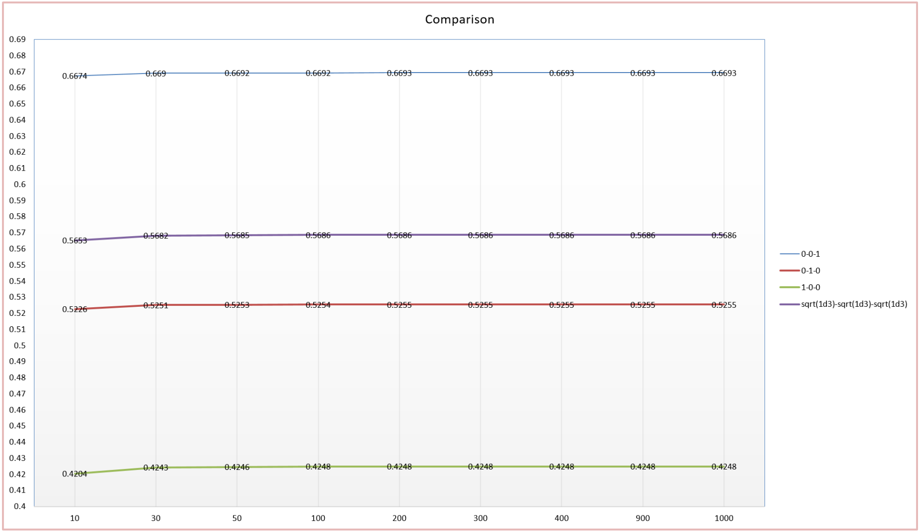
- init coin: [0.000, 0.000, 1.000]
- init pos: 0
- boundary: -1
- steps: 1000
- exit prob: 0.6693
- stay prob: 0.3307
- max prob: 0.0339
- max pos: 573
- mean: 0.0005
- variance: 0.0000



Exit probability convergence



Exit probability in different coin initial states



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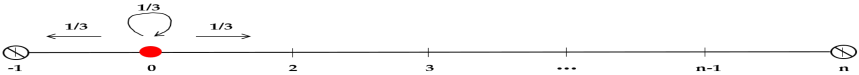


Step1. Initialize the system to state

$$|\Psi\rangle_0 = |-1\rangle|R\rangle$$

Step2. For each step of evolution

- Apply $\mathrm{U} = \mathrm{S} \cdot (\mathrm{I} \otimes \mathrm{G})$ to the system.
- Observe system with projection operators $\{\prod_{yes}^{-1}, \prod_{no}^{-1}\}$.
- Observe system with projection operators $\{\prod_{yes}^{n}, \prod_{no}^{n}\}$.

Step3. If the result of measurement was *"yes"*, terminate; otherwise goto **Step 2**

Exit probability

- ***Exit probability:*** the probability that the measurement of whether the walker is at position -1 results in *"yes"*
- Let $P_{\underline{-1,0,n}}$ denotes the exit probability. *What is the analytical expression of $P_{\underline{-1,0,n}}$?*
- In two state quantum walk case (Conjectured in 2001, STOC. Proved in 2004, JCSS):

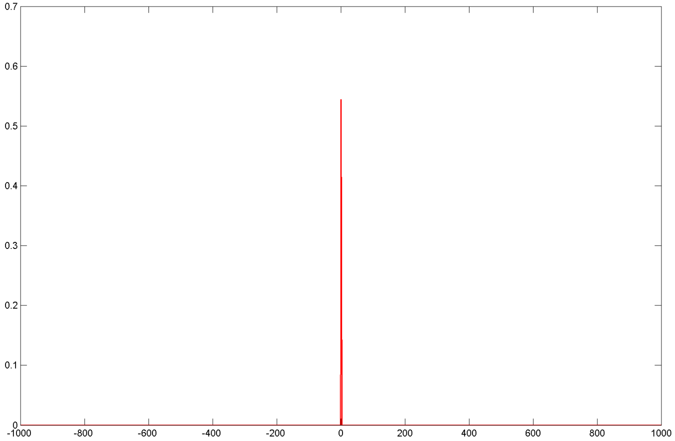
$$\lim_{n\to\infty} P_{\underline{-1,0,n}} = \frac{1}{\sqrt{2}}.$$

$$P_{\underline{-1,0,n+1}} = \frac{1 + 2P_{\underline{-1,0,n}}}{2 + 2P_{\underline{-1,0,n}}}$$

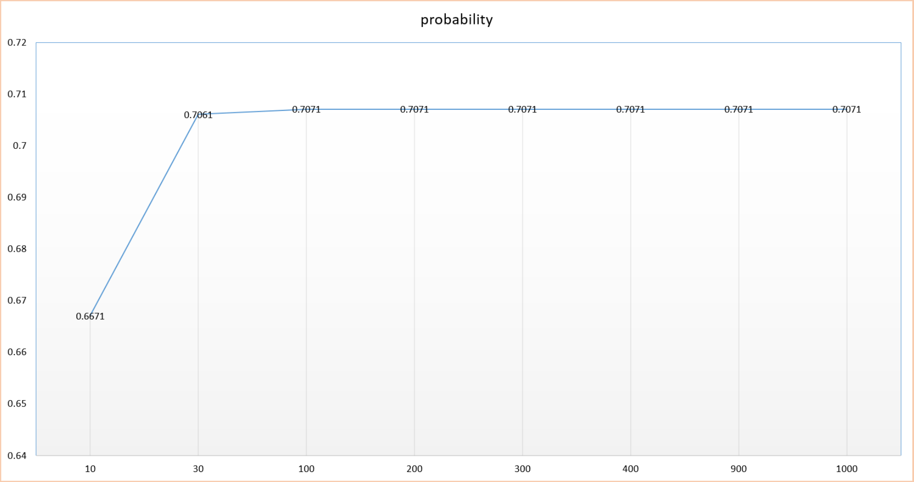
Exit probability after 1000 steps (right boundary in 3)

- quantum walks
- three state
- two boundaries

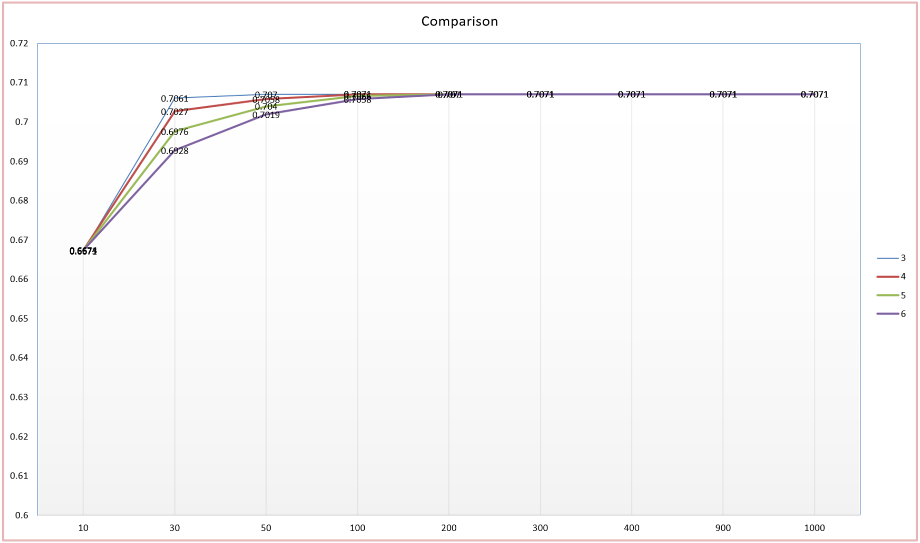
- init coin: [0.000, 0.000, 1.000]
- init pos: 0
- left boundary: -1
- right boundary: 3
- steps: 1000
- left exit prob: 0.7071
- right exit prob: 0.2929
- sum of two: 1.000000
- max prob: 0.5441
- max pos: 1
- mean: 0.0005
- variance: 0.0002



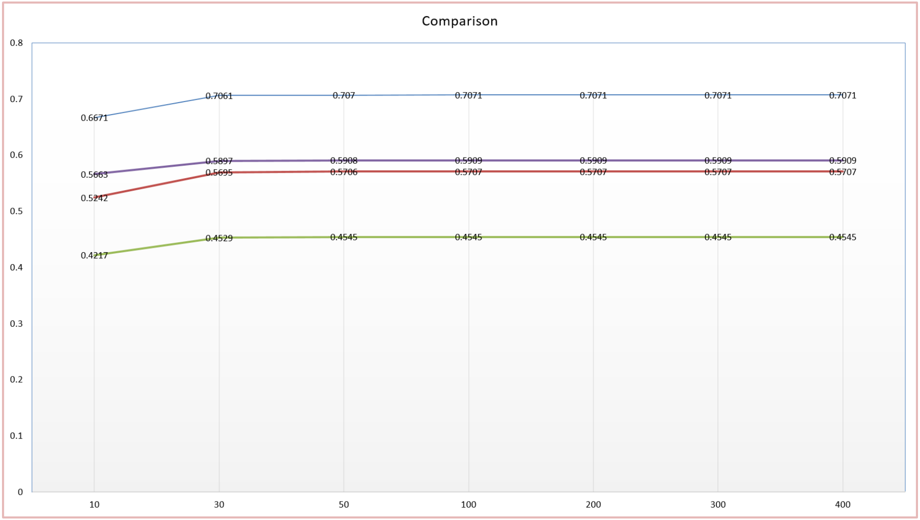
Exit probability convergence (right boundary in 3)



Exit probability in different right boundaries



Exit probability in different coin initial states



Thank you !

Any questions ?