One Dimensional Quantum Cellular Automata

A Brief Introduction

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Outline

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2. One Dimensional Quantum Cellular Automata
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   - Definition
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3. Summary

History

- von Neumann proposed cellular automata in 1952 as formal models of self-reproducing organisms[Bur70].
- Conway presented the *Game of Life* in 1970, which is a two-dimensional cellular automata[Gar70].
- Fredkin proposed that the world we live in is a huge cellular automata.
- Wolfram entered th field of cellular automata in early 1980s and studied in detail the one-dimensional cellular automata[Wol02].
- *A New Kind of Science*, extensively argues that the discoveries about cellular automata are not isolated facts but are robust and have significance for all disciplines of science.

Definition

- A *one-dimensional cellular automata* (1d-CA), is determined by a quintuple

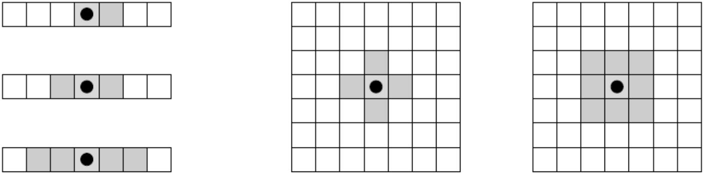
$$\mathcal{M} =< Q, N, \delta, k, A >$$

- Q is a finite set of states;
- N is the neighborhood of given cell, $N(i) = \{n_1, n_2, \cdots, n_r\} \subseteq \mathbb{Z}$. $\mathcal{M}$ is assumed to have a two-way infinite sequence of cells indexed by integers (denoted by $\mathbb{Z}$);
- k is the *accepting cell*, $A \subset Q$ is the *accepting state set*;
- δ is a local transition function

$$\delta : Q^r \longrightarrow Q.$$

Neighbors

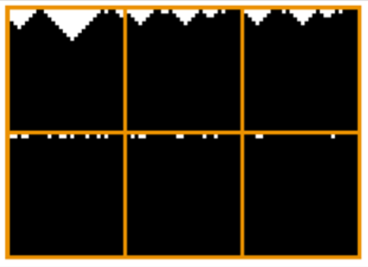
- The neighborhood of a cell c (usually including itself, but not always the case) is the set of cells in the network which will locally determine the evolution of c.
- 1d-CA's neighbors.
- 2d-CA's neighbors: Von Neumann neighborhood.
- 2d-CA's neighbors: Moore neighborhood.

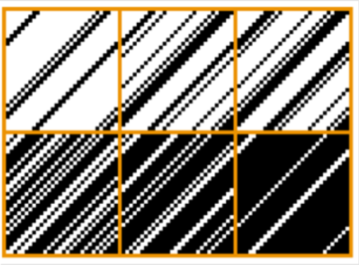


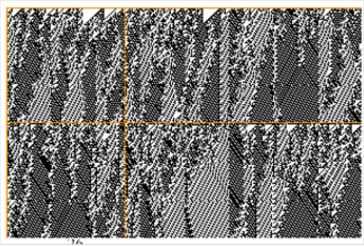
Two-state 1d-CAs

- There are totally $2^{2^3} = 256$ two-state 1d-CAs. Denote $Q = \{0, 1\}$, each two-state 1d-CA can be represented as a 8 bit string.
- Two-state 1d-QCA can be classified into four classes based on their evolution behavior:
  1. **Class 1:** evolution leads to a homogeneous state.
  2. **Class 2:** evolution leads to a set of separated simple stable or periodic structures.
  3. **Class 3:** evolution leads to a chaotic pattern.
  4. **Class 4:** evolution leads to complex localized structures which are sometimes long-lived. It is in-between class 2 and class 3.

Two-state 1d-CAs (Cont.)







Rule 110

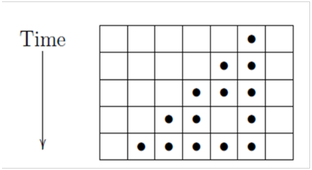
- Rule 110 is a special two-state 1d-CA which is in class 4.

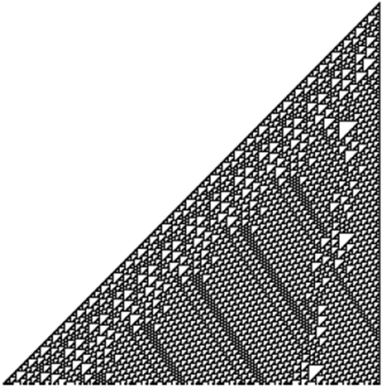
current pattern	111	110	101	100	011	010	001	000
new state	0	1	1	0	1	1	1	0

Table: Transition function for Rule 110

Rule 110 (Cont.)

- Rule 110 is Turing complete and thus computational universal[Coo04].
- Rule 110 is arguably the simplest known Turing complete system.





Outline

History

- Feynman mentioned QCA as possible way of making universal quantum computer in 1982[Fey82].
- Grossing and Zeilinger had tried to introduce the concept of QCA.
- Watrous first gave a mathematical definition of a 1-d QCA[Wat95].
- Since then, several alternative definitions have been given, but Watrous's QCA is still the most straightforward.
- QCAs have an advantage because cells are not required to be able to distinguish one from another.

Quantum Amplitudes

- Basic idea is to replace classical, deterministic transitions with quantum amplitudes.
- Let Q be finite state set, δ be transition function, then

classical case $$\delta : \underbrace{Q}_{\text{left}} \times \underbrace{Q}_{\text{old}} \times \underbrace{Q}_{\text{right}} \longrightarrow \underbrace{Q}_{\text{new}}$$

quantum case $$\delta : \underbrace{Q}_{\text{left}} \times \underbrace{Q}_{\text{old}} \times \underbrace{Q}_{\text{right}} \times \underbrace{Q}_{\text{new}} \longrightarrow \underbrace{\mathbb{C}}_{\text{amplitude}}$$

Definition[Gru99]

- A *one-dimensional quantum cellular automata* (1d-QCA), is determined by a sextuple

$$\mathcal{M} =< Q, \epsilon, N, \delta, k, A >$$

- Q is a finite set of states, $\epsilon \in Q$ is the so-called *quiescent state*;
- N is the neighborhood of given cell, $N(i) = \{n_1, n_2, \cdots, n_r\} \subseteq \mathbb{Z}$. $\mathcal{M}$ is assumed to have a two-way infinite sequence of cells indexed by integers (denoted by $\mathbb{Z}$);
- k is the *accepting cell*, $A \subset Q$ is the *accepting state set*;
- δ is a local transition function

$$\delta : Q^{r+1} \longrightarrow \mathbb{C}_{[0,1]}$$

satisfying certain conditions.

Local Transition Function

A local transition function δ must satisfy following three conditions:

1. **Local probability condition.** For any $(q_1, q_2, \cdots, q_r) \in Q^r$,

$$\sum_{q \in Q} |\delta(q_1, q_2, \cdots, q_r, q)|^2 = 1$$

2. **Stability of the quiescent state condition.** if $q \in Q$, then

$$\delta(\epsilon, \cdots, \epsilon, q) = \begin{cases} 1, & \text{if} \quad q = \epsilon; \\ 0, & \text{otherwise.} \end{cases}$$

3. **Unitary of evolution condition.** In order to define the third condition, several concepts have to be introduced first.

Unitary of Evolution Condition

- An valid *configuration* $c : \mathbb{Z} \to Q$ is a mapping such that $c(i) \neq \epsilon$ only for finitely many cells i;
- Let $C(\mathcal{M})$ denotes the set of all valid configurations. Computation of $\mathcal{M}$ is then done in the inner product space $H_{\mathcal{M}} = l_2(C(\mathcal{M}))$ with the basis $\{|c\rangle | c \in C(\mathcal{M})\}$;
- In one step, $\mathcal{M}$ transfers from one basis state $|c_1\rangle$ to another $|c_2\rangle$. The amplitude $\alpha(c_1, c_2)$ of such a transition is:

$$\alpha(c_1, c_2) = \prod_{i \in \mathbb{Z}} \delta(\underbrace{c_1(i + n_1), c_1(i + n_2), \cdots, c_1(i + n_r)}_{r \text{ neighbors of cell } i}, c_2(i)).$$

Unitary of Evolution Condition (Cont.)

- A state in $H_{\mathcal{M}}$ therefore has form in general:

$$|\phi\rangle = \sum_{c \in C(\mathcal{M})} \alpha_c |c\rangle, \quad \text{where} \quad \sum_{c \in C(\mathcal{M})} |\alpha_c|^2 = 1.$$

- The evolution operator $E_{\mathcal{M}}$ of $\mathcal{M}$ maps any state $|\phi\rangle \in H_{\mathcal{M}}$ into state $|\varphi\rangle = E_{\mathcal{M}}|\phi\rangle$ such that

$$|\varphi\rangle = E_{\mathcal{M}}|\phi\rangle = \sum_{c \in C(\mathcal{M})} \beta_c |c\rangle, \quad \text{where} \quad \beta_c = \sum_{c' \in C(\mathcal{M})} \alpha'_c \alpha(c', c).$$

- Now, we cam formulate the third condition δ to satisfy:

the evolution operator $E_{\mathcal{M}}$ has to be unitary.

Well-formed 1d-QCA

Denote 1d-QCAs which have unitary evolution operator $E_{\mathcal{M}}$ as well-formed 1d-QCAs.

Well-formedness of 1d-QCA

Given an arbitrary 1d-QCA $\mathcal{M}$, it is not a trivial matter to determine $\mathcal{M}$ is well-formed or not.

Define the size of a 1d-QCA to be $n = |Q|^{r+1}$. Solutions:

- Find a sufficiently large and interesting class of 1d-QCAs for which checking well-formedness is easier than in general.
- There exists an algorithm which takes a simple 1d-QCA as input, and decides in time $O(n^{\frac{3r-1}{r+1}}) = O(n^3)$ if it is unitary[DS96].

1d-PQCA

- The *partitioned quantum cellular automata* (1d-PQCA) is a subclass of 1d-QCA, for which the well-fromedness can be easily determined.
- In a 1d-PQCA, the state of each cell is made up of the states of three *subcells*. Here is a single cell with the state (α, β, γ):

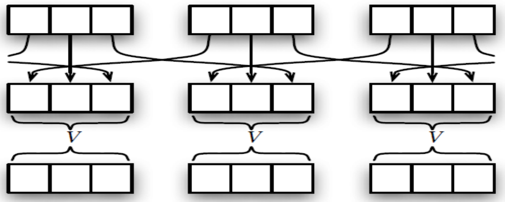
$\in Q$		
α	β	γ
$\in Q_L$	$\in Q_M$	$\in Q_R$

$$Q = Q_L \times Q_M \times Q_R$$

Local Transition Function of 1d-PQCA

- The transition amplitudes depends on:
  1. the right substate of the left neighbor,
  2. the middle substate of itself,
  3. the left substate of the right neighbor.

  A cell can only "see" the right side of left neighbor, the middle of itself, the left side of right neighbor.
- The action of transition function δ is a two-step method: first permutates three subcells, then applies cell-wise transformation V:



Well-formedness of 1d-PQCA

- Each step of the operation consists of:
  1. a permutation of registers;
  2. disjointly-acting three-register mapping.
- If the three-register mapping is unitary, then the global time-step operation will be unitary, and visa versa.
- Checking well-formedness of a 1d-PQCA down to checking the matrix determining the transition amplitudes is unitary[Hor08].

Equivalence of QTM and 1d-PQCA

Theorem

Given any quantum Turing machine $\mathcal{M}_{tm}$, there is an one-dimensional partitioned quantum cellular automata $\mathcal{M}_{ca}$ which simulates $\mathcal{M}_{tm}$ with constant slow down[Wat95].

Theorem

Given any one-dimensional partitioned quantum cellular automata $\mathcal{M}_{ca}$, there is an quantum Turing machine $\mathcal{M}_{tm}$ which simulates $\mathcal{M}_{ca}$ with linear slow down[Wat95].

Implementation of QCA[Wie09]

- Quantum dot.
- Optical lattices.
- All of them are confined to a very finite number of elements and are no way near to a quantum Turing machine.

Outline

Summary

- One dimensional cellular automata
- One dimensional quantum cellular automata
- One dimensional partitioned quantum cellular automata
- Physical implementations of QCA
- Some open problems:
  1. Is there exists a 1d-QCA which would be universal in a sense that it could efficiently simulate any 1d-QCA?
  2. The equivalence was proven between QTMs and restricted 1d-QCAs. Are 1d-QCA any more powerful than QTMs?

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Thank you !