Nonlinear Kalman Filtering

1. Extended Kalman Filter (EKF)

다음과 같은 nonlinear system을 생각해보자.

$$x_k = f_{k-1}(x_{k-1}, u_{k-1}, w_{k-1})$$
$$y_k = h_k(x_k, v_k)$$
$$w_k \sim (0, Q_k)$$
$$v_k \sim (0, R_k)$$

여기서 Taylor 전개를 통해 state와 measurement를 linearization할 수 있다.

$$\begin{aligned} x_k &= f_{k-1}(\hat{x}_{k-1}^{+}, u_{k-1}, 0) + \frac{\partial f_{k-1}}{\partial x}\Big|_{\hat{x}_{k-1}^{+}}(x_{k-1} - \hat{x}_{k-1}^{+}) + \frac{\partial f_{k-1}}{\partial w}\Big|_{\hat{x}_{k-1}^{+}} w_{k-1} \\ &= f_{k-1}(\hat{x}_{k-1}^{+}, u_{k-1}, 0) + F_{k-1}(x_{k-1} - \hat{x}_{k-1}^{+}) + L_{k-1}w_{k-1} \\ &= F_{k-1}x_{k-1} + [f_{k-1}(\hat{x}_{k-1}^{+}, u_{k-1}, 0) - F_{k-1}\hat{x}_{k-1}^{+}] + L_{k-1}w_{k-1} \\ &= F_{k-1}x_{k-1} + \tilde{u}_{k-1} + \tilde{w}_{k-1} \end{aligned}$$

$$\begin{aligned} y_k &= h_k(\hat{x}_k^{-}, 0) + \frac{\partial h_k}{\partial x}\Big|_{\hat{x}_k^{-}}(x_k - \hat{x}_k^{-}) + \frac{\partial h_k}{\partial v}\Big|_{\hat{x}_k^{-}} v_k \\ &= h_k(\hat{x}_k^{-}, 0) + H_k(x_k - \hat{x}_k^{-}) + M_k v_k \\ &= H_k x_k + [h_k(\hat{x}_k^{-}, 0) - H_k\hat{x}_k^{-}] + M_k v_k \\ &= H_k x_k + z_k + \tilde{v}_k \end{aligned}$$

여기서 치환되는 noise들과 행렬들은 다음과 같다. 여기서 미분값은 자코비안 행렬을 의미한다. 이후엔 치환된 값들을 그대로 Kalman Filter에 적용해주면 된다.

$$\tilde{u}_k = f_k(\hat{x}_k^{+}, u_k, 0) - F_k\hat{x}_k^{+}$$
$$\tilde{w}_k \sim (0, L_k Q_k L_k^T)$$
$$z_k = h_k(\hat{x}_k^{-}, 0) - H_k\hat{x}_k^{-}$$
$$\tilde{v}_k \sim (0, M_k R_k M_k^T)$$

$$F_{k-1} = \frac{\partial f_{k-1}}{\partial x}\Big|_{\hat{x}_{k-1}^{+}} \qquad H_k = \frac{\partial h_k}{\partial x}\Big|_{\hat{x}_k^{-}}$$
$$L_{k-1} = \frac{\partial f_{k-1}}{\partial w}\Big|_{\hat{x}_{k-1}^{+}} \qquad M_k = \frac{\partial h_k}{\partial v}\Big|_{\hat{x}_k^{-}}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Extended Kalman Filter 과정을 정리하면 다음과 같다.

Extended Kalman Filter (EKF)

$$\begin{aligned} P_k^{-} &= F_{k-1}P_{k-1}^{+}F_{k-1}^T + L_{k-1}Q_{k-1}L_{k-1}^T \\ K_k &= P_k^{-}H_k^T(H_kP_k^{-}H_k^T + M_kR_kM_k^T)^{-1} \\ \hat{x}_k^{-} &= f_{k-1}(\hat{x}_{k-1}^{+}, u_{k-1}, 0) \\ z_k &= h_k(\hat{x}_k^{-}, 0) - H_k\hat{x}_k^{-} \\ \hat{x}_k^{+} &= \hat{x}_k^{-} + K_k(y_k - H_k\hat{x}_k^{-} - z_k) \\ &= \hat{x}_k^{-} + K_k[y_k - h_k(\hat{x}_k^{-}, 0)] \\ P_k^{+} &= (I - K_kH_k)P_k^{-} \end{aligned}$$

2. Iterated Extended Kalman Filter

Extended Kalman Filter를 적용하면 nonlinear system을 linearization하는 과정에서 오차가 발생한다. 그래서 IEKF에서는 Measurement update 과정을 여러 번 반복함으로써 발생하는 오차를 최소화한다.

원래의 measurement update식은 다음과 같다.

$$\begin{aligned}\hat{x}_k^+ &= \hat{x}_k^- + K_k\left(y_k - H_k\hat{x}_k^- - z_k\right) \\ &= \hat{x}_k^- + K_k\left(y_k - h_k\left(\hat{x}_k^-, 0\right) - H_k\left(\hat{x}_k^- - \hat{x}_k^-\right)\right) \\ &= \hat{x}_k^- + K_k\left(y_k - h_k\left(\hat{x}_k^-, 0\right)\right)\end{aligned}$$

그런데 여기서 measurement update를 다음과 같이 N개의 단계로 진행하여 각각 i번째 measurement 값을 추정하게 된다. 그렇게 되면 $\hat{x}_{k,i}^+$가 이전 정보로 들어가기 때문에 다음과 같이 식이 달라진다.

$$\begin{aligned}\hat{x}_{k,i}^+ &= \hat{x}_k^- + K_{k,i}\left(y_k - H_{k,i}\hat{x}_k^- - z_{k,i}\right) \\ &= \hat{x}_k^- + K_{k,i}\left(y_k - h\left(\hat{x}_{k,i}^+\right) - H_{k,i}\left(\hat{x}_k^- - \hat{x}_{k,i}^+\right)\right)\end{aligned}$$

이렇게 되면 iteration을 반복할수록 update되는 값들이 점점 작아지면서 오차가 줄어든다.

정리하면 다음과 같다.

The iterated extended Kalman filter

1. The nonlinear system and measurement equations are given as follows:

$$\begin{aligned} x_k &= f_{k-1}(x_{k-1}, u_{k-1}, w_{k-1}) \\ y_k &= h_k(x_k, v_k) \\ w_k &\sim (0, Q_k) \\ v_k &\sim (0, R_k) \end{aligned} \qquad (13.59)$$

2. Initialize the filter as follows.

$$\begin{aligned} \hat{x}_0^+ &= E(x_0) \\ P_0^+ &= E\left[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T\right] \end{aligned} \qquad (13.60)$$

3. For $k = 1, 2, \cdots$, do the following.

   (a) Perform the following time-update equations:

$$\begin{aligned} P_k^- &= F_{k-1}P_{k-1}^+F_{k-1}^T + L_{k-1}Q_{k-1}L_{k-1}^T \\ \hat{x}_k^- &= f_{k-1}(\hat{x}_{k-1}^+, u_{k-1}, 0) \end{aligned} \qquad (13.61)$$

   where the partial derivative matrices F_{k-1} and L_{k-1} are defined as follows:

$$\begin{aligned} F_{k-1} &= \left.\frac{\partial f_{k-1}}{\partial x}\right|_{\hat{x}_{k-1}^+} \\ L_{k-1} &= \left.\frac{\partial f_{k-1}}{\partial w}\right|_{\hat{x}_{k-1}^+} \end{aligned} \qquad (13.62)$$

   Up to this point the iterated EKF is the same as the standard discrete-time EKF.

(b) Perform the measurement update by initializing the iterated EKF estimate to the standard EKF estimate:

$$\begin{aligned} \hat{x}_{k,0}^+ &= \hat{x}_k^- \\ P_{k,0}^+ &= P_k^- \end{aligned} \tag{13.63}$$

For $i = 0, 1, \cdots, N$, evaluate the following equations (where N is the desired number of measurement-update iterations):

$$\begin{aligned} H_{k,i} &= \left.\frac{\partial h}{\partial x}\right|_{\hat{x}_{k,i}^+} \\ M_{k,i} &= \left.\frac{\partial h}{\partial v}\right|_{\hat{x}_{k,i}^+} \\ K_{k,i} &= P_k^- H_{k,i}^T \left(H_{k,i} P_k^- H_{k,i}^T + M_{k,i} R_k M_{k,i}^T\right)^{-1} \\ P_{k,i+1}^+ &= (I - K_{k,i} H_{k,i}) P_k^- \\ \hat{x}_{k,i+1}^+ &= \hat{x}_k^- + K_{k,i}\left[y_k - h(\hat{x}_{k,i}^+) - H_{k,i}(\hat{x}_k^- - \hat{x}_{k,i}^+)\right] \end{aligned} \tag{13.64}$$

(c) The final *a posteriori* state estimate and estimation-error covariance are given as follows:

$$\begin{aligned} \hat{x}_k^+ &= \hat{x}_{k,N+1}^+ \\ P_k^+ &= P_{k,N+1}^+ \end{aligned} \tag{13.65}$$

3. Second Order Extended Kalman Filter

Second Order Extended Kalman Filter는 Taylor 전개를 2차까지 하여 linearization을 진행한다. 이번에는 continuous-time system을 살펴보자.

$$\begin{aligned} \dot{x} &= f(x, u, w, t) \\ y_k &= h(x_k, t_k) + v_k \\ w(t) &\sim (0, Q) \\ v_k &\sim (0, R_k) \end{aligned}$$

여기서 state에 대한 식을 다음과 같이 Taylor 전개를 통해 표현 가능하다.

$$f(x, u, w, t) = f(\hat{x}, u_0, w_0, t) + \frac{\partial f}{\partial x}|_{\hat{x}}(x - \hat{x}) + \frac{1}{2}\sum_{i=1}^{n}\phi_i (x - \hat{x})^T \frac{\partial^2 f}{\partial x^2}|_{\hat{x}}(x - \hat{x})$$

여기서 ϕ_i는 n x 1 벡터로 i 번째 항만 1이고 나머지는 모두 0인 벡터이다. 위 식에서 quadratic 항은 다음과 같이 근사 가능하다.

$$(x - \hat{x})^T \frac{\partial^2 f}{\partial x^2}|_{\hat{x}}(x - \hat{x}) = Tr\left(\frac{\partial^2 f}{\partial x^2}|_{\hat{x}}(x - \hat{x})(x - \hat{x})^T\right) \approx Tr\left(\frac{\partial^2 f}{\partial x^2}|_{\hat{x}} P\right)$$

따라서 구하고자 하는 time-update equation은 다음과 같다.

$$\dot{\hat{x}} = f(\hat{x}, u_0, w_0, t) + \frac{1}{2}\sum_{i=1}^{n}\phi_i Tr\left(\frac{\partial^2 f}{\partial x^2}|_{\hat{x}} P\right)$$

또한 covariance 식에 대한 update도 다음과 같이 주어져 있다.

$$\dot{P} = FP + PF^T + LQL^T$$

다음으로 measurement-update를 구해볼 텐데, 그에 대한 식은 다음과 같다.

$$\hat{x}_k^+ = \hat{x}_k^- + K_k\left(y_k - h(\hat{x}_k^-, t_k)\right) - \pi_k$$

여기서 π는 추정값이 unbiased되기 위한 correction term이다. 위 식을 error항의 식으로 바꿔보면 다음과 같다.

$e_k^- = x_k - \hat{x}_k^-$
$e_k^+ = x_k - \hat{x}_k^+$
$e_k^+ = e_k^- - K_k\left(h(x_k, t_k) - h\left(\hat{x}_k^-, t_k\right)\right) - K_k v_k + \pi_k$

여기서 measurement에 대한 Taylor 전개를 통해 다음 항을 얻을 수 있다.

$$\begin{aligned} h(x_k, t_k) &= h\left(\hat{x}_k^-, t_k\right) + \frac{\partial h}{\partial x}\Big|_{\hat{x}_k^-}\left(x_k - \hat{x}_k^-\right) + \frac{1}{2}\sum_{i=1}^{m}\phi_i\left(x_k - \hat{x}_k^-\right)^T\frac{\partial^2 h(i)}{\partial x^2}\Big|_{\hat{x}_k^-}\left(x_k - \hat{x}_k^-\right) \\ &= h\left(\hat{x}_k^-, t_k\right) + H_k\left(x_k - \hat{x}_k^-\right) + \frac{1}{2}\sum_{i=1}^{m}\phi_i\left(x_k - \hat{x}_k^-\right)^T\frac{\partial^2 h(i)}{\partial x^2}\Big|_{\hat{x}_k^-}\left(x_k - \hat{x}_k^-\right) \end{aligned}$$

$$h(x_k, t_k) - h\left(\hat{x}_k^-, t_k\right) = H_k e_k^- + \frac{1}{2}\sum_{i=1}^{m}\phi_i\left(e_k^-\right)^T D_{k,i} e_k^-$$

이 식을 이용해 위의 measurement update식을 치환하면 다음과 같다.

$$e_k^+ = e_k^- - K_k H_k e_k^- - \frac{1}{2}K_k\sum_{i=1}^{m}\phi_i\left(e_k^-\right)^T D_{k,i} e_k^- - K_k v_k + \pi_k$$

따라서 위 식의 양변에 기댓값을 취하면 기댓값이 0이 되기 위해서 correction term π_k는 다음과 같다.

$$\pi_k = \frac{1}{2}K_k\sum_{i=1}^{m}\phi_i Tr\left(D_{k,i}P_k^-\right)$$

마지막으로 measurement-update에서 Kalman Gain은 covariance matrix를 최소화시키는 값이 된다. 이를 위해 covariance matrix를 살펴보면 다음과 같다.

$$P_k^+ = E[e_k^+\left(e_k^+\right)^T] = (I - K_k H_k)P_k^-(I - K_k H_k)^T + K_k(R_k + \Lambda_k)K_k^T$$

여기서 matrix Λ_k는 다음과 같이 정의된다.

$$\Lambda_k = \frac{1}{4}E\left(\left(\sum_{i=1}^{m}\phi_i Tr\left(D_{k,i}\left(e_k^-\left(e_k^-\right)^T - P_k^-\right)\right)\right)(\ldots)^T\right)$$

계산을 위해 다음과 같은 cost function을 정의하자.

$$J_k = E[\left(e_k^+\right)^T S_k e_k^+] = Tr[S_k P_k^+]$$

위 cost function을 최소화하는 K_k는 다음과 같이 찾을 수 있다.

$$K_k = P_k^- H_k^T\left(H_k P_k^- H_k^T + R_k + \Lambda_k\right)^{-1}$$

이를 이용하여 P_k^+도 계산하면 다음과 같다.

$$P_k^+ = P_k^- - P_k^- H_k^T\left(H_k P_k^- H_k^T + R_k + \Lambda_k\right)^{-1} H_k P_k^-$$

또한 Λ_k의 각 항들은 다음과 같이 구할 수 있다.

$$\Lambda_k(i, j) = \frac{1}{2}Tr\left(D_{k,i}P_k^- D_{k,j}P_k^-\right)$$

최종적으로 second-order extended kalman filter를 정리하면 다음과 같다.

The second-order hybrid extended Kalman filter

1. The system equations are given as follows:

$$\begin{aligned} \dot{x} &= f(x, u, w, t) \\ y_k &= h(x_k, t_k) + v_k \\ w(t) &\sim (0, Q) \\ v_k &\sim (0, R_k) \end{aligned} \tag{13.91}$$

2. The estimator is initialized as follows:

$$\begin{aligned} \hat{x}_0^+ &= E(x_0) \\ P_0^+ &= E\left[(x_0 - \hat{x}_0^+)(x_0 - \hat{x}_0^+)^T\right] \end{aligned} \tag{13.92}$$

3. The time-update equations are given as

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x}, u, 0, t) + \frac{1}{2}\sum_{i=1}^{n} \phi_i \mathrm{Tr}\left[\left.\frac{\partial^2 f_i}{\partial x^2}\right|_{\hat{x}} P\right] \\ \dot{P} &= FP + PF^T + LQL^T \\ \phi_i &= \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{th element} \\ F &= \left.\frac{\partial f}{\partial x}\right|_{\hat{x}} \\ L &= \left.\frac{\partial f}{\partial w}\right|_{\hat{x}} \end{aligned} \tag{13.93}$$

4. The measurement update equations are given as

$$\begin{aligned} \hat{x}_k^+ &= \hat{x}_k^- + K_k\left[y_k - h(\hat{x}_k^-)\right] - \pi_k \\ \pi_k &= \frac{1}{2} K_k \sum_{i=1}^{m} \phi_i \mathrm{Tr}\left[D_{k,i} P_k^-\right] \\ D_{k,i} &= \left.\frac{\partial^2 h_i(x_k, t_k)}{\partial x^2}\right|_{\hat{x}_k^-} \\ K_k &= P_k^- H_k^T \left(H_k P_k^- H_k^T + R_k + \Lambda_k\right)^{-1} \\ H_k &= \left.\frac{\partial h(x_k, t_k)}{\partial x}\right|_{\hat{x}_k^-} \\ \Lambda_k(i,j) &= \frac{1}{2}\mathrm{Tr}(D_{k,i} P_k^- D_{k,j} P_k^-) \\ P_k^+ &= P_k^- - P_k^- H_k^T \left(H_k P_k^- H_k^T + R_k + \Lambda_k\right)^{-1} H_k P_k^- \end{aligned} \tag{13.94}$$

또한, continuous-time system이 아닌 discrete-time system에 대한 SOEKF는 다음과 같이 정리된다. 앞서 했던 과정을 discrete-time에 대해 동일하게 진행하면 된다.

The second-order discrete-time extended Kalman filter

1. The system equations are given as follows:

$$\begin{aligned} x_{k+1} &= f(x_k, u_k, k) + w_k \\ y_k &= h(x_k, k) + v_k \\ w_k &\sim (0, Q_k) \\ v_k &\sim (0, R_k) \end{aligned} \tag{13.98}$$

2. The estimator is initialized as follows:

$$\begin{aligned} \hat{x}_0^+ &= E(x_0) \\ P_0^+ &= E\left[(x_0 - \hat{x}_0^+)(x_0 - \hat{x}_0^+)^T\right] \end{aligned} \tag{13.99}$$

3. The time update equations are given as follows:

$$\begin{aligned} \hat{x}_{k+1}^- &= f(\hat{x}_k^+, u_k, k) + \frac{1}{2}\sum_{i=1}^{n} \phi_i \mathrm{Tr}\left[\left.\frac{\partial^2 f_i}{\partial x}\right|_{\hat{x}_k^+} P_k^+\right] \\ P_{k+1}^- &= F P_k^+ F^T + Q_k \end{aligned}$$

$$
\begin{aligned}
\phi_i &= \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{th element} \\
F &= \left.\frac{\partial f}{\partial x}\right|_{\hat{x}_k^+}
\end{aligned} \tag{13.100}
$$

4. The measurement update equations are given as follows:

$$
\begin{aligned}
\hat{x}_k^+ &= \hat{x}_k^- + K_k[y_k - h(\hat{x}_k^-, k)] - \pi_k \\
\pi_k &= \frac{1}{2} K_k \sum_{i=1}^{m} \phi_i \mathrm{Tr}\left[D_{k,i} P_k^-\right] \\
D_{k,i} &= \left.\frac{\partial^2 h_i(x_k, k)}{\partial x^2}\right|_{\hat{x}_k^-} \\
K_k &= P_k^- H_k^T \left(H_k P_k^- H_k^T + R_k\right)^{-1} \\
H_k &= \left.\frac{\partial h(x_k, k)}{\partial x}\right|_{\hat{x}_k^-} \\
P_k^+ &= (I - K_k H_k) P_k^-
\end{aligned} \tag{13.101}
$$

Other Approach로는 Gaussian sum filter가 있다. 정리하면 다음과 같다.

The Gaussian sum filter

1. The discrete-time n-state system and measurement equations are given as follows:

$$
\begin{aligned}
x_k &= f_{k-1}(x_{k-1}, u_{k-1}, w_{k-1}) \\
y_k &= h_k(x_k, v_k) \\
w_k &\sim (0, Q_k) \\
v_k &\sim (0, R_k)
\end{aligned} \tag{13.102}
$$

2. Initialize the filter by approximating the pdf of the initial state as follows:

$$
\mathrm{pdf}(\hat{x}_0^+) = \sum_{i=1}^{M} a_{0i} N(\hat{x}_{0i}^+, P_{0i}^+) \tag{13.103}
$$

The a_{0i} coefficients (which are positive and add up to 1), the $\hat{x}_{0i}^+$ means, and the P_{0i}^+ covariances, are chosen by the user to provide a good approximation to the pdf of the initial state.

3. For $k = 1, 2, \cdots$, do the following.

   (a) The *a priori* state estimate is obtained by first executing the following time-update equations for $i = 1, \cdots, M$:

$$
\begin{aligned}
\hat{x}_{ki}^- &= f_{k-1}(\hat{x}_{k-1,i}^+, u_{k-1}, 0) \\
F_{k-1,i} &= \left.\frac{\partial f_{k-1}}{\partial x_{k-1}}\right|_{\hat{x}_{k-1,i}^+} \\
P_{ki}^- &= F_{k-1,i} P_{k-1,i}^+ F_{k-1,i}^T + Q_{k-1} \\
a_{ki} &= a_{k-1,i}
\end{aligned} \tag{13.104}
$$

   The pdf of the *a priori* state estimate is obtained by the following sum:

$$
\mathrm{pdf}(\hat{x}_k^-) = \sum_{i=1}^{M} a_{ki} N(\hat{x}_{ki}^-, P_{ki}^-) \tag{13.105}
$$

   (b) The *a posteriori* state estimate is obtained by first executing the following measurement update equations for $i = 1, \cdots, M$:

$$
\begin{aligned}
H_{ki} &= \left.\frac{\partial h_k}{\partial x_k}\right|_{\hat{x}_{ki}^-} \\
K_{ki} &= P_{ki}^- H_{ki}^T (H_{ki} P_{ki}^- H_{ki}^T + R_k)^{-1} \\
P_{ki}^+ &= P_{ki}^- - K_{ki} H_{ki} P_{ki}^- \\
\hat{x}_{ki}^+ &= \hat{x}_{ki}^- + K_{ki}\left[y_k - h_k(\hat{x}_{ki}^-, 0)\right]
\end{aligned} \tag{13.106}
$$

   The weighting coefficients a_{ki} for the individual estimates are obtained as follows:

$$
\begin{aligned}
r_{ki} &= y_k - h_k(\hat{x}_{ki}^-, 0) \\
S_{ki} &= H_{ki} P_{ki}^- H_{ki}^T + R_k \\
\beta_{ki} &= \frac{\exp\left[-r_{ki}^T S_{ki}^{-1} r_{ki}/2\right]}{(2\pi)^{n/2} |S_{ki}|^{1/2}} \\
a_{ki} &= \frac{a_{k-1,i}\beta_{ki}}{\sum_{j=1}^{M} a_{k-1,j}\beta_{kj}}
\end{aligned}
\tag{13.107}
$$

Note that the weighting coefficient a_{ki} is computed by using the measurement y_k to obtain the relative confidence β_{ki} of the estimate $\hat{x}_{ki}^-$. The pdf of the *a posteriori* state estimate is obtained by the following sum:

$$
\mathrm{pdf}(\hat{x}_k^+) = \sum_{i=1}^{M} a_{ki} N(\hat{x}_{ki}^+, P_{ki}^+) \tag{13.108}
$$