Optimal Smoothing 2

2. Fixed Lag Smoothing

Fixed Lag Smoothing의 목적은 주어진 N에 대해 $\hat{x}_{k-N,k} = E(x_{k-N}|y_1, \dots, y_k)$을 구하는 것이다.

이를 위해 1에서 했던 것처럼 다음과 같은 system을 정의한다.

$$\begin{bmatrix} x_{k+1} \\ x_{k+1,1} \\ \cdots \\ x_{k+1,N+1} \end{bmatrix} = \begin{bmatrix} F_k & 0 & \cdots & . \\ I & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & I & 0 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k,1} \\ \cdots \\ x_{k,N+1} \end{bmatrix} + \begin{bmatrix} I \\ 0 \\ \cdots \\ 0 \end{bmatrix} w_k$$

$$y_k = \begin{bmatrix} H_k \, 0 \cdots 0 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k,1} \\ \cdots \\ x_{k,N+1} \end{bmatrix} + v_k$$

여기서 x는 다음과 같이 정의된다.

$$\begin{aligned} x_{k+1,1} &= x_k \\ x_{k+1,2} &= x_{k-1} \\ &= x_{k,1} \\ x_{k+1,3} &= x_{k-2} \\ &= x_{k,2} \\ &\cdots \end{aligned}$$

이에 따라 구하고자 하는 x의 각 원소별 추정값은 다음과 같다.

$$\begin{aligned} E(x_{k+1}|y_1, \dots, y_k) &= \hat{x}_{k+1}^{-} \\ &= \hat{x}_{k+1,k} \\ E(x_{k+1,1}|y_1, \dots, y_k) &= E(x_k|y_1, \dots, y_k) \\ &= \hat{x}_k^{+} \\ &= \hat{x}_{k,k} \\ E(x_{k+1,2}|y_1, \dots, y_k) &= E(x_{k-1}|y_1, \dots, y_k) \\ &= \hat{x}_{k-1,k} \\ &\cdots \\ E(x_{k+1,N+1}|y_1, \dots, y_k) &= \hat{x}_{k-N,k} \end{aligned}$$

앞서 1에서 했던 것과 동일하게 해당 system을 One-Step Kalman Filter를 이용하여 전개한다.

Alternate One-Step *a priori* Kalman Filter

$$\begin{aligned} L_k &= F_k P_k^- H_k^T (H_k P_k^- H_k^T + R_k)^{-1} \\ P_{k+1}^- &= F_k P_k^- (F_k - L_k H_k)^T + Q_k \\ \hat{x}_{k+1}^- &= F_k \hat{x}_k^- + L_k (y_k - H_k \hat{x}_k^-) \end{aligned}$$

$$
\begin{bmatrix} \hat{x}_{k+1}^- \\ \hat{x}_{k,k} \\ \cdots \\ \hat{x}_{k-N,k} \end{bmatrix} = \begin{bmatrix} F_k & 0 & \dots & 0 \\ I & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & I & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_k^- \\ \hat{x}_{k-1,k-1} \\ \cdots \\ \hat{x}_{k-1-N,k-1} \end{bmatrix} + \begin{bmatrix} L_{k,0} \\ L_{k,1} \\ \cdots \\ L_{k,N+1} \end{bmatrix} \left(y_k - \left[H_k \, 0 \dots 0 \right] \begin{bmatrix} \hat{x}_k^- \\ \hat{x}_{k-1,k-1} \\ \cdots \\ \hat{x}_{k-1-N,k-1} \end{bmatrix} \right)
$$

$$
\begin{aligned}
L_k = \begin{bmatrix} L_{k,0} \\ L_{k,1} \\ \cdots \\ L_{k,N+1} \end{bmatrix} &= \begin{bmatrix} F_k & 0 & \dots & 0 \\ I & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & I & 0 \end{bmatrix} \begin{bmatrix} P_k^{0,0} & \dots & \left(P_k^{0,N+1}\right)^T \\ \dots & \dots & \dots \\ P_k^{0,N+1} & \dots & P_k^{N+1,N+1} \end{bmatrix} \begin{bmatrix} H_k^T \\ 0 \\ \cdots \\ 0 \end{bmatrix} \\
&\times \left(\left[H_k \, 0 \dots 0 \right] \begin{bmatrix} P_k^{0,0} & \dots & \left(P_k^{0,N+1}\right)^T \\ \dots & \dots & \dots \\ P_k^{0,N+1} & \dots & P_k^{N+1,N+1} \end{bmatrix} \begin{bmatrix} H_k^T \\ 0 \\ \cdots \\ 0 \end{bmatrix} + R_k \right)^{-1} \\
&= \begin{bmatrix} F_k P_k^{0,0} H_k^T \\ P_k^{0,0} H_k^T \\ \cdots \\ P_k^{0,N} H_k^T \end{bmatrix} \left(H_k P_k^{0,0} H_k^T + R_k \right)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
\begin{bmatrix} P_{k+1}^{0,0} & \dots & \left(P_{k+1}^{0,N+1}\right)^T \\ \dots & \dots & \dots \\ P_{k+1}^{0,N+1} & \dots & P_{k+1}^{N+1,N+1} \end{bmatrix} &= \begin{bmatrix} F_k & 0 & \dots & 0 \\ I & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & I & 0 \end{bmatrix} \begin{bmatrix} P_k^{0,0} & \dots & \left(P_k^{0,N+1}\right)^T \\ \dots & \dots & \dots \\ P_k^{0,N+1} & \dots & P_k^{N+1,N+1} \end{bmatrix} \\
&\times \left(\begin{bmatrix} F_k & I & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & I \\ 0 & \dots & 0 & 0 \end{bmatrix} - \begin{bmatrix} H_k^T \\ 0 \\ \cdots \\ 0 \end{bmatrix} L_k^T \right) + \begin{bmatrix} Q_k & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 \end{bmatrix}
\end{aligned}
$$

그에 따라 covariance Matrix의 각 요소들은 다음과 같이 계산된다.

$$
\begin{aligned}
P_{k+1}^{0,0} &= F_k P_k^{0,0} \left[F_k^T - H_k^T (H_k P_k^{0,0} H_k^T + R_k)^{-1} H_k P_k^{0,0} F_k^T \right] + Q_k \\
&= F_k P_k^{0,0} (F_k - L_{k,0} H_k)^T + Q_k \\
P_{k+1}^{0,1} &= P_k^{0,0} (F_k - L_{k,0} H_k)^T \\
&\vdots \\
P_{k+1}^{0,N+1} &= P_k^{0,N} (F_k - L_{k,0} H_k)^T
\end{aligned} \tag{9.50}
$$

$$
\begin{aligned}
P_{k+1}^{1,1} &= P_k^{0,0} \left[I - H_k^T (H_k P_k^{0,0} H_k^T + R_k)^{-1} H_k P_k^{0,0} F_k^T \right] \\
&= P_k^{0,0} - P_k^{0,0} H_k^T L_{k,1}^T F_k^T \\
P_{k+1}^{2,2} &= P_k^{0,1} [-H_k^T L_{k,2}^T F_k^T] + P_k^{1,1} \\
&= P_k^{1,1} - P_k^{0,1} H_k^T L_{k,2}^T F_k^T \\
&\vdots \\
P_{k+1}^{i,i} &= P_k^{i-1,i-1} - P_k^{0,i-1} H_k^T L_{k,i}^T F_k^T
\end{aligned} \tag{9.51}
$$

정리하면 다음과 같다.

The fixed-lag smoother

1. Run the standard Kalman filter of Equation (9.10) to obtain $\hat{x}_{k+1}^-$, L_k, and P_k^-.

2. Initialize the fixed-lag smoother as follows:

$$
\begin{aligned}
\hat{x}_{k+1,k} &= \hat{x}_{k+1}^- \\
L_{k,0} &= L_k \\
P_k^{0,0} &= P_k^-
\end{aligned} \tag{9.52}
$$

3. For $i = 1, \cdots, N+1$, perform the following:

$$
\begin{aligned}
L_{k,i} &= P_k^{0,i-1} H_k^T (H_k P_k^{0,0} H_k^T + R_k)^{-1} \\
P_{k+1}^{i,i} &= P_k^{i-1,i-1} - P_k^{0,i-1} H_k^T L_{k,i}^T F_k^T \\
P_{k+1}^{0,i} &= P_k^{0,i-1} (F_k - L_{k,0} H_k)^T \\
\hat{x}_{k+1-i,k} &= \hat{x}_{k+2-i,k} + L_{k,i} (y_k - H_k \hat{x}_k^-)
\end{aligned} \tag{9.53}
$$

3. Fixed Interval Smoothing

Fixed Interval Smoothing의 목표는 k=1, ..., N에서의 measurement를 이용하여 x_m을 추정하는 것이다. 여기서 m은 1~N 사이의 수다.

여기서는 Standard Kalman Filter를 k=1부터 k=m까지 작동하여 추정한 forward estimation $\hat{x}_f$와, k=N부터 k=m까지 barward로 Kalman Filter를 작동시켜 추정한 $\hat{x}_b$를 gain 합하여 값을 추정한다.

$$\hat{x} = K_f \hat{x}_f + K_b \hat{x}_b$$

여기서 추정값은 unbiased되어야 하므로 $K_f + K_b = I$이다.

$$\hat{x} = K_f \hat{x}_f + (I - K_f)\hat{x}_b$$

해당 추정값의 covariance은 다음과 같다.

$$\begin{aligned} P &= E[(x-\hat{x})(x-\hat{x})^T] \\ &= E[(x - K_f\hat{x}_f - (I-K_f)\hat{x}_b)(x - K_f\hat{x}_f - (I-K_f)\hat{x}_b)^T] \\ &= E[(K_f(e_f - e_b) + e_b)(K_f(e_f - e_b) + e_b)^T] \\ &= E[K_f(e_f e_f^T + e_b e_b^T)K_f^T + e_b e_b^T - K_f e_b e_b^T - e_b e_b^T K_f^T] \end{aligned}$$

각 K들은 covariance을 최소화해야 하기 때문에 미분하여 0으로 두고 풀면 다음과 같다.

$$\begin{aligned} \frac{\partial Tr(P)}{\partial K_f} &= 2E[K_f(e_f e_f^T + e_b e_b^T) - e_b e_b^T] \\ &= 2[K_f(P_f + P_b) - P_b] \end{aligned}$$

따라서 각 gain은 다음과 같다.

$$K_f = P_b(P_f + P_b)^{-1}$$
$$K_b = P_f(P_f + P_b)^{-1}$$

위 gain 값들을 covariance 식에 대입하면 다음과 같다.

위 식을 전개하는 과정에서 다음과 같은 Matrix Lemma가 사용된다.

Matrix Inversion Lemma

$$(A + BC^{-1}D)^{-1} = A^{-1} - A^{-1}B(C + DA^{-1}B)^{-1}DA^{-1}$$
$$(A + BC^{-1}D)^{-1}BC^{-1} = A^{-1}B(C + DA^{-1}B)^{-1}$$

1) $(A + BC^{-1}D)^{-1}BC^{-1} = A^{-1}B(C + DA^{-1}B)^{-1}$

$\rightarrow (A+B)^{-1} = B^{-1}(AB^{-1} + I)^{-1}$

$\rightarrow P_b(P_f + P_b)^{-1} = (P_f P_b^{-1} + I)^{-1}$

$(P_f P_b^{-1} + I)^{-1} P_f = (P_b^{-1} + P_f^{-1})^{-1}$

$(P_f P_b^{-1} + I)^{-1} P_b = (P_b^{-1} P_f P_b^{-1} + P_b^{-1})^{-1}$

2) $(A + BC^{-1}D)^{-1} = A^{-1} - A^{-1}B(C + DA^{-1}B)^{-1}DA^{-1}$

$\rightarrow (P_b^{-1} P_f P_b^{-1} + P_b^{-1})^{-1} = P_b - (P_f^{-1} + P_b^{-1})^{-1}$

위 식들을 이용해 전개하면 다음과 같다.

$$\begin{aligned}
P = & \ E[K_f(e_f e_f^T + e_b e_b^T)K_f^T + e_b e_b^T - K_f e_b e_b^T - e_b e_b^T K_f^T] \\
= & \ P_b(P_f + P_b)^{-1}(P_f + P_b)(P_f + P_b)^{-1}P_b + P_b - P_b(P_f + P_b)^{-1}P_b - P_b(P_f + P_b)^{-1}P_b \\
= & \ (P_f P_b^{-1} + I)^{-1}(P_f + P_b)(P_b^{-1}P_f + I)^{-1} + P_b - 2(P_f P_b^{-1} + I)^{-1}P_b \\
= & \ \left((P_b^{-1} + P_f^{-1})^{-1} + (P_b^{-1}P_f P_b^{-1} + P_b^{-1})^{-1}\right)(P_b^{-1}P_f + I)^{-1} + P_b - 2(P_b^{-1}P_f P_b^{-1} + P_b^{-1})^{-1} \\
= & \ P_b(P_b^{-1}P_f + I)^{-1} + P_b - 2P_b + 2(P_f^{-1} + P_b^{-1})^{-1} \\
= & \ (P_b^{-1}P_f P_b^{-1} + P_b^{-1})^{-1} - P_b + 2(P_f^{-1} + P_b^{-1})^{-1} \\
= & \ P_b - (P_f^{-1} + P_b^{-1})^{-1} - P_b + 2(P_f^{-1} + P_b^{-1})^{-1} \\
= & \ (P_f^{-1} + P_b^{-1})^{-1}
\end{aligned}$$

따라서 위와 같이 forward process와 backward process를 통해 원하는 시점의 추정값을 계산할 수 있다.

1) forward process

forward process의 경우에는 standard Kalman Filter를 이용해 그대로 진행하면 된다. 그 과정은 다음과 같다.

$$\begin{aligned}
x_k &= F_{k-1}x_{k-1} + w_{k-1} \\
y_k &= H_k x_k + v_k \\
w_k &\sim (0, Q_k) \\
v_k &\sim (0, R_k)
\end{aligned}$$

1. Initialize the forward filter as follows:

$$\begin{aligned}
\hat{x}_{f0}^+ &= E(x_0) \\
P_{f0}^+ &= E\left[(x_0 - \hat{x}_{f0}^+)(x_0 - \hat{x}_{f0}^+)^T\right]
\end{aligned} \tag{9.66}$$

2. For $k = 1, \cdots, m$, perform the following:

$$\begin{aligned}
P_{fk}^- &= F_{k-1}P_{f,k-1}^+F_{k-1}^T + Q_{k-1} \\
K_{fk} &= P_{fk}^- H_k^T(H_k P_{fk}^- H_k^T + R_k)^{-1} \\
&= P_{fk}^+ H_k^T R_k^{-1} \\
\hat{x}_{fk}^- &= F_{k-1}\hat{x}_{f,k-1}^+ \\
\hat{x}_{fk}^+ &= \hat{x}_{fk}^- + K_{fk}(y_k - H_k\hat{x}_{fk}^-) \\
P_{fk}^+ &= (I - K_{fk}H_k)P_{fk}^-
\end{aligned} \tag{9.67}$$

2) Backward Process

Backward process의 경우에는 forward process와 독립적으로 진행된다. 따라서 forward filter에서 사용된 정보는 전혀 사용되지 않으며, 그에 따라 covariance $P_{bN}^- = \infty$가 된다. 그렇기 때문에 이 경우에는 Kalman Filter를 사용하지 못하며 Information Filter를 사용해야 한다. 그 과정은 다음과 같다. 다음의 Information Filter 과정을 참고한다.

$$\begin{aligned}
x_{k-1} &= F_{k-1}^{-1}x_k + F_{k-1}^{-1}w_{k-1} \\
&= F_{k-1}^{-1}x_k + w_{b,k-1} \\
y_k &= H_k x_k + v_k \\
w_{bk} &\sim (0, F_k^{-1}Q_k F_k^{-T}) \\
v_k &\sim (0, R_k)
\end{aligned}$$

Information Filter (2) $\mathcal{I} = P^{-1}$, $\hat{z} = P^{-1}\hat{x}$, and $G_k = 0$

$$
\begin{aligned}
\hat{z}_k^- &= Q_{k-1}^{-1}F_{k-1}(\mathcal{I}_{k-1}^+ + F_{k-1}^T Q_{k-1}^{-1} F_{k-1})^{-1}\hat{z}_{k-1}^+ \\
\mathcal{I}_k^- &= Q_{k-1}^{-1} - Q_{k-1}^{-1}F_{k-1}(\mathcal{I}_{k-1}^+ + F_{k-1}^T Q_{k-1}^{-1} F_{k-1})^{-1} F_{k-1}^T Q_{k-1}^{-1} \\
\hat{z}_k^+ &= \hat{z}_k^- + H_k^T R_k^{-1} y_k \\
\mathcal{I}_k^+ &= \mathcal{I}_k^- + H_k^T R_k^{-1} H_k
\end{aligned}
$$

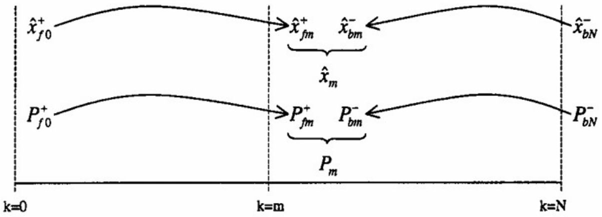
Initialize the information filter at k=N

$$
\begin{aligned}
s_N^- &= 0 \qquad\qquad (s_k = \mathcal{I}_{bk}\hat{x}_{bk}) \\
\mathcal{I}_{bN}^- &= 0
\end{aligned}
$$

Run the information filter backwards

$$
\begin{aligned}
\mathcal{I}_{bk}^+ &= \mathcal{I}_{bk}^- + H_k^T R_k^{-1} H_k \\
s_k^+ &= s_k^- + H_k^T R_k^{-1} y_k \\
\mathcal{I}_{b,k-1}^- &= \left[F_{k-1}^{-1}(\mathcal{I}_{bk}^+)^{-1}F_{k-1}^{-T} + F_{k-1}^{-1}Q_{k-1}F_{k-1}^{-T}\right]^{-1} \\
&= F_{k-1}^T\left[(\mathcal{I}_{bk}^+)^{-1} + Q_{k-1}\right]^{-1}F_{k-1} \\
&= F_{k-1}^T\left[Q_{k-1}^{-1} - Q_{k-1}^{-1}(\mathcal{I}_{bk}^+ + Q_{k-1}^{-1})^{-1}Q_{k-1}^{-1}\right]F_{k-1} \\
s_{k-1}^- &= \mathcal{I}_{b,k-1}^- F_{k-1}^{-1}(\mathcal{I}_{bk}^+)^{-1}s_k^+
\end{aligned}
$$

그렇게 각각의 과정에 따라 m 시점의 각각의 추정값과 covariance, gain을 구하게 되면 앞서 구했던 식에 따라 다음과 같이 추정값을 계산할 수 있다.



$$
\begin{aligned}
K_f &= P_{bm}^-\left(P_{fm}^+ + P_{bm}^-\right)^{-1} \\
P_m &= \left(\left(P_{fm}^+\right)^{-1} + \left(P_{bm}^-\right)^{-1}\right)^{-1} \\
\hat{x}_m &= K_f\hat{x}_{fm}^+ + (I - K_f)\hat{x}_{bm}^- \\
\hat{x}_m &= P_m I_{fm}^+\hat{x}_{fm}^+ + P_m I_{bm}^-\hat{x}_{bm}^- \\
&= P_m\left(I_{fm}^+\hat{x}_{fm}^+ + I_{bm}^-\hat{x}_{bm}^-\right)
\end{aligned}
$$

● Rauch-Tung-Striebel Smoothing (RTS Smoothing)

backward estimate를 진행하지 않아 computational effort를 줄여주는 smoothing 방법이다.

우선 기존의 covariance 식을 Matrix Lemma를 사용해서 바꾸면 다음과 같다.

$$
\begin{aligned}
P_m &= \left(\left(P_{fm}^+\right)^{-1} + \left(P_{bm}^-\right)^{-1}\right)^{-1} \\
&= P_{fm}^+ - P_{fm}^+\left(P_{fm}^+ + P_{bm}^-\right)^{-1}P_{fm}^+
\end{aligned}
$$

여기서 $\left(P_{fm}^+ + P_{bm}^-\right)^{-1}$식에서 P_{bm}^- 를

$$
P_{bm}^- = \left(I_{b,m}^-\right)^{-1} = \left(F_m^T\left(\left(I_{b,m+1}^+\right)^{-1} + Q_m\right)^{-1}F_m\right)^{-1} = F_m^{-1}\left(P_{b,m+1}^+ + Q_m\right)^{-1}F_m^{-T}
$$

를 이용해서 치환하면 다음과 같다.

$$
\begin{aligned}
\left(P_{fm}^{+}+P_{bm}^{-}\right)^{-1} &= \left(P_{fm}^{+}+F_{m}^{-1}\left(P_{b,m+1}^{+}+Q_{m}\right)F_{m}^{-T}\right)^{-1} \\
&= \left(F_{m}^{-1}F_{m}P_{fm}^{+}F_{m}^{T}F_{m}^{-T}+F_{m}^{-1}\left(P_{b,m+1}^{+}+Q_{m}\right)F_{m}^{-T}\right)^{-1} \\
&= \left(F_{m}^{-1}\left(F_{m}P_{fm}^{+}F_{m}^{T}+P_{b,m+1}^{+}+Q_{m}\right)F_{m}^{-T}\right)^{-1} \\
&= F_{m}^{T}\left(F_{m}P_{fm}^{+}F_{m}^{T}+P_{b,m+1}^{+}+Q_{m}\right)^{-1}F_{m} \\
&= F_{m}^{T}\left(P_{f,m+1}^{-}+P_{b,m+1}^{+}\right)^{-1}F_{m}
\end{aligned}
$$

그리고 여기서 $P_{b,m+1}^{+}$은 다음 과정에 따라 치환 가능하다.

$$
\begin{aligned}
I_{fm}^{+} &= I_{fm}^{-}+H_{m}^{T}R_{m}^{-1}H_{m} \\
I_{bm}^{+} &= I_{bm}^{-}+H_{m}^{T}R_{m}^{-1}H_{m} \\
I_{b,m+1}^{+}+I_{f,m+1}^{-} &= I_{f,m+1}^{+}+I_{b,m+1}^{-} \\
P_{m+1} &= \left(\left(P_{f,m+1}^{+}\right)^{-1}+\left(P_{b,m+1}^{-}\right)^{-1}\right)^{-1} \\
&= \left(I_{f,m+1}^{+}+I_{b,m+1}^{-}\right)^{-1} \\
&= \left(I_{b,m+1}^{+}+I_{f,m+1}^{-}\right)^{-1} \\
P_{m+1}^{-1} &= I_{b,m+1}^{+}+I_{f,m+1}^{-} \\
P_{b,m+1}^{+} &= \left(P_{m+1}^{-1}-I_{f,m+1}^{-}\right)^{-1}
\end{aligned}
$$

위 식에 따라 치환하면 다음과 다시 이어서 전개가 가능하다.

$$
\begin{aligned}
\left(P_{fm}^{+}+P_{bm}^{-}\right)^{-1} &= F_{m}^{T}\left(P_{f,m+1}^{-}+P_{b,m+1}^{+}\right)^{-1}F_{m} \\
&= F_{m}^{T}\left(P_{f,m+1}^{-}+\left(P_{m+1}^{-1}-I_{f,m+1}^{-}\right)^{-1}\right)^{-1}F_{m} \\
&= F_{m}^{T}I_{f,m+1}^{-}\left(I_{f,m+1}^{-}+I_{f,m+1}^{-}\left(P_{m+1}^{-1}-I_{f,m+1}^{-}\right)^{-1}I_{f,m+1}^{-}\right)^{-1}I_{f,m+1}^{-}F_{m} \\
&= F_{m}^{T}I_{f,m+1}^{-}\left(P_{f,m+1}^{-}-P_{m+1}\right)I_{f,m+1}^{-}F_{m}
\end{aligned}
$$

따라서 이와 같이 backward covariance 없이 계산 가능하다.

$$
\begin{aligned}
P_{m} &= P_{fm}^{+}-P_{fm}^{+}\left(P_{fm}^{+}+P_{bm}^{-}\right)^{-1}P_{fm}^{+} \\
&= P_{fm}^{+}-P_{fm}^{+}F_{m}^{T}I_{f,m+1}^{-}\left(P_{f,m+1}^{-}-P_{m+1}\right)I_{f,m+1}^{-}F_{m}P_{fm}^{+} \\
&= P_{fm}^{+}-K_{m}\left(P_{f,m+1}^{-}-P_{m+1}\right)K_{m}^{T} \\
K_{m} &= P_{fm}^{+}F_{m}^{T}I_{f,m+1}^{-}
\end{aligned}
$$

정리하면 RTS Smoother의 과정은 다음과 같다.

The RTS smoother

1. The system model is given as follows:

$$
\begin{aligned}
x_k &= F_{k-1}x_{k-1}+G_{k-1}u_{k-1}+w_{k-1} \\
y_k &= H_kx_k+v_k \\
w_k &\sim (0, Q_k) \\
v_k &\sim (0, R_k)
\end{aligned}
\tag{9.134}
$$

2. Initialize the forward filter as follows:

$$
\begin{aligned}
\hat{x}_{f0} &= E(x_0) \\
P_{f0}^{+} &= E\left[(x_0-\hat{x}_{f0})(x_0-\hat{x}_{f0})^T\right]
\end{aligned}
\tag{9.135}
$$

3. For $k = 1, \cdots, N$ (where N is the final time), execute the standard forward Kalman filter:

$$
\begin{aligned}
P_{fk}^- &= F_{k-1} P_{f,k-1}^+ F_{k-1}^T + Q_{k-1} \\
K_{fk} &= P_{fk}^- H_k^T (H_k P_{fk}^- H_k^T + R_k)^{-1} \\
&= P_{fk}^+ H_k^T R_k^{-1} \\
\hat{x}_{fk}^- &= F_{k-1} \hat{x}_{f,k-1}^+ + G_{k-1} u_{k-1} \\
\hat{x}_{fk}^+ &= \hat{x}_{fk}^- + K_{fk}(y_k - H_k \hat{x}_{fk}^-) \\
P_{fk}^+ &= (I - K_{fk} H_k) P_{fk}^- (I - K_{fk} H_k)^T + K_{fk} R_k K_{fk}^T \\
&= \left[(P_{fk}^-)^{-1} + H_k^T R_k^{-1} H_k \right]^{-1} \\
&= (I - K_{fk} H_k) P_{fk}^-
\end{aligned} \tag{9.136}
$$

4. Initialize the RTS smoother as follows:

$$
\begin{aligned}
\hat{x}_N &= \hat{x}_{fN}^+ \\
P_N &= P_{fN}^+
\end{aligned} \tag{9.137}
$$

5. For $k = N-1, \cdots, 1, 0$, execute the following RTS smoother equations:

$$
\begin{aligned}
\mathcal{I}_{f,k+1}^- &= \left(P_{f,k+1}^- \right)^{-1} \\
K_k &= P_{fk}^+ F_k^T \mathcal{I}_{f,k+1}^- \\
P_k &= P_{fk}^+ - K_k (P_{f,k+1}^- - P_{k+1}) K_k^T \\
\hat{x}_k &= \hat{x}_{fk}^+ + K_k (\hat{x}_{k+1} - \hat{x}_{f,k+1}^-)
\end{aligned} \tag{9.138}
$$