Zadanie 0 EGZAMIN CZ. 2 ~ Piotr Piesiak

$X \sim N(\mu_1, \sigma_1^2)$ , $Y \sim N(\mu_2, \sigma_2^2)$, X i Y są niezależne

Niech:

a) $\begin{cases} Z = X + Y \\ V = Y \end{cases} \xrightarrow{\text{odwrotne}} \begin{cases} X = Z - V \\ Y = V \end{cases}$  b) jakobian: $\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$

zatem $g(z,v) = f(x(z,v), y(z,v)) \cdot 1.$

$$f(x,y) \overset{X,Y \text{ są niezależne}}{=\!=\!=} f_1(x) \cdot f_2(x) = \frac{1}{\sqrt{2\pi}\cdot\sqrt{2\pi}\,\sigma_1\sigma_2} \cdot \exp\left(-\frac{(x-\mu_1)^2\sigma_2^2 + (y-\mu_2)^2\sigma_1^2}{2\sigma_1^2\sigma_2^2}\right)$$

$$g(z,v) = \frac{1}{2\pi\sigma_1\sigma_2} \cdot \exp\left(-\frac{((z-v)^2 - 2(z-v)\mu_1 + \mu_1^2)\sigma_2^2 + (v^2 - 2v\mu_2 + \mu_2^2)\sigma_1^2}{2\sigma_1^2\sigma_2^2}\right)$$

$$= \frac{1}{2\pi\sigma_1\sigma_2} \cdot \exp\left(-\frac{(z^2 + v^2 - 2zv - 2z\mu_1 + 2v\mu_1 + \mu_1^2)\sigma_2^2 + (v^2 - 2v\mu_2 + \mu_2^2)\sigma_1^2}{2\sigma_1^2\sigma_2^2}\right)$$

$$= \frac{1}{2\pi\sigma_1\sigma_2} \cdot \exp\left(-\frac{v^2(\sigma_1^2 + \sigma_2^2) - 2v((z-\mu_1)\sigma_2^2 + \mu_2\sigma_1^2) + \sigma_2^2(z^2 - 2z\mu_1 + \mu_1^2) + \sigma_1^2\mu_2^2}{2\sigma_1^2\sigma_2^2}\right)$$

Niech $\sigma_3 = \sqrt{\sigma_1^2 + \sigma_2^2}$ , wtedy:

chcemy „zwinąć", więc odejmujemy w nast. równości

$$g(z,v) = \frac{1}{2\pi \frac{\sigma_1\sigma_2}{\sigma_3} \cdot \sigma_3} \cdot \exp\left(-\frac{v^2 - \frac{2v((z-\mu_1)\sigma_2^2 + \mu_2\sigma_1^2)}{\sigma_3^2} + \frac{\sigma_2^2(z-\mu_1)^2 + \sigma_1^2\mu_2^2}{\sigma_3^2}}{2\left(\frac{\sigma_1\sigma_2}{\sigma_3}\right)^2}\right)$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_3} \cdot \exp\left(-\frac{\sigma_3^2(\sigma_2^2(z-\mu_1)^2 + \sigma_1^2\mu_2^2) - (\sigma_2^2(z-\mu_1) + \mu_2\sigma_1^2)^2}{2\sigma_1^2\sigma_2^2\sigma_3^2}\right) \cdot$$

$$\cdot \frac{1}{\sqrt{2\pi}\,\frac{\sigma_1\sigma_2}{\sigma_3}} \cdot \exp\left(-\frac{\left(v - \frac{\sigma_2^2(z-\mu_1) + \mu_2\sigma_1^2}{\sigma_3^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sigma_3}\right)^2}\right) =$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_3} \cdot \exp\left(-\frac{\sigma_2^2(z-\mu_1)^2(\sigma_3^2 - \sigma_2^2) + \sigma_1^2\mu_2^2(\sigma_3^2 - \sigma_1^2) - 2\sigma_2^2\sigma_1^2(z-\mu_1)\mu_2}{2\sigma_1^2\sigma_2^2\sigma_3^2}\right)$$

$$\cdot \frac{1}{\sqrt{2\pi} \cdot \frac{\sigma_1\sigma_2}{\sigma_3}} \cdot \exp\left(-\frac{\left(v - \frac{\sigma_2^2(z-\mu_1) + \mu_2\sigma_1^2}{\sigma_3^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sigma_3}\right)^2}\right)$$

czyli:

minus (niewyrażny)

$$g(z,v)=\frac{1}{\sqrt{2\pi}\,\sigma_3}\exp\left(\frac{(z-(\mu_1+\mu_2))^2}{2\sigma_3^2}\right)\cdot\frac{1}{\sqrt{2\pi}\,\frac{\sigma_1\sigma_2}{\sigma_3}}\exp\left(-\frac{\left(v-\frac{\sigma_2^2(z-\mu_1)+\mu_2\sigma_1^2}{\sigma_3^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sigma_3}\right)^2}\right)$$

c) gęstość brzegowa $g_Z(z)$: // $-\infty<Z<+\infty,\ -\infty<V<+\infty$

$$g_Z(z)=\int_{-\infty}^{+\infty}g(z,v)\,dv=\frac{1}{\sqrt{2\pi}\,\sigma_3}\cdot\exp\left(-\frac{(z-(\mu_1+\mu_2))^2}{2\sigma_3^2}\right)\cdot$$

$$\cdot\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}\,\frac{\sigma_1\sigma_2}{\sigma_3}}\cdot\exp\left(-\frac{\left(v-\frac{\sigma_2^2(z-\mu_1)+\mu_2\sigma_1^2}{\sigma_3^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sigma_3}\right)^2}\right)dv$$

Zauważmy, że podkreślona całka jest równa 1, ponieważ funkcja wewnątrz jest f. gęstości zmiennej o rozkładzie normalnym z parametrami: $N\left(\frac{\sigma_2^2(z-\mu_1)+\mu_2\sigma_1^2}{\sigma_3^2},\left(\frac{\sigma_1\sigma_2}{\sigma_3}\right)^2\right)$

Zatem $g_Z(z)=\frac{1}{\sqrt{2\pi}\,\sigma_3}\cdot\exp\left(-\frac{(z-(\mu_1+\mu_2))^2}{2\sigma_3^2}\right)$

czyli zmienna $Z$ ma rozkład normalny $Z\sim N(\mu_1+\mu_2,\ \sigma_1^2+\sigma_2^2)$.