NOTATKI - ANALIZA II, 2020

1. Ćwiczenia 17-03-2020

Zadanie 3/2.

$$\lim_{t\to 0}\frac{1}{t}\int_0^t \cos^2 x dx \overset{\text{H}}{=} \lim_{t\to 0}\frac{\cos^2 t}{1} = 1$$

$$\cos^2 x = (\cos(2x)-1)/2$$

$$\cos^2 x = \frac{\cos(2x)-1}{2}$$

Zadanie 3/3. Niech $F'(x) = f(x)$. Najpierw policzmy:

(1) $$(xF(x))' = xf(x) + F(x)$$

(2) $$xF(x) - 0 = \int_0^x (xF(x))'dx = \int_0^x xf(x)dx + \int_0^x F(x)dx$$

(3) $$x\int_0^x f(x)dx - \int_0^x xf(x)dx = \int_0^x F(x)dx$$

Również $F(x) = \int_0^x f(x)dx + f(0) > 0, dla$ x≥ 0, bo f(x)$>0.Czyli\int_0^x F(x)dx > 0$.
Sprawdzamy czy g(x) rosnąca licząc pochodna:

$$g'(x) = \left(\frac{\int_0^x tf(t)dt}{\int_0^x f(t)dt}\right)' = \frac{xf(x)\int_0^x f(t)dt - f(x)\int_0^x tf(t)dt}{\left(\int_0^x f(t)dt\right)^2} = \frac{f(x)\int_0^x F(x)dx}{\left(\int_0^x f(t)dt\right)^2} > 0$$

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Zadanie 3/5.

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Zadanie 3/6.

$$S_n = n^{-3} \sum_{k=1}^{n} (n+k)\sqrt{n^2 + (2k)^2} = \frac{1}{n} \sum_{k=1}^{n} \left(1 + \frac{k}{n}\right) \sqrt{1 + \frac{(2k)^2}{n^2}} \tag{4}$$

$$\begin{gathered} S_n \to \int_0^1 (1+x)\sqrt{1+4x^2}dx = \\ \int_0^1 \sqrt{1+4x^2}\,dx + \int_0^1 x\sqrt{1+4x^2}\,dx \end{gathered} \tag{5}$$

$$\begin{aligned} \int \sqrt{1+4x^2}dx = [2x = \tan y,\ 2dx = dy/\cos^2 y] = \\ = \int \frac{1}{\cos^3 y}\,dy = ... \end{aligned} \tag{6}$$

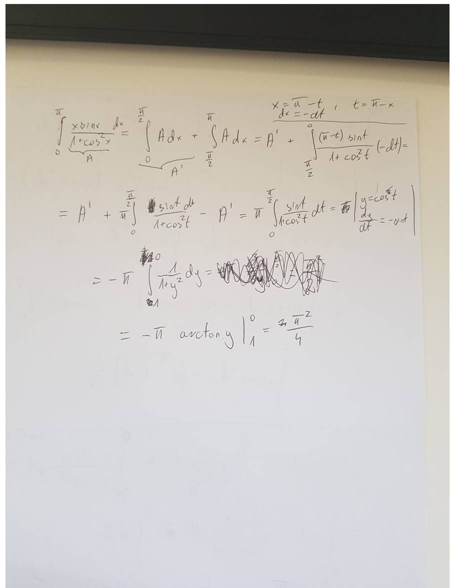
$$\cosh^2 x - \sinh^2 x = 1$$

$$\cosh^2 y = (\cosh(2y) + 1)/2$$

$$\begin{aligned} \int \sqrt{1+4x^2}dx = [2x = \sinh y,\ 2dx = \cosh y dy] = \\ = \int \cosh^2 y\,\frac{dy}{2} = \sinh(2y)/8 + y/4 + C \end{aligned} \tag{7}$$

$$\int_0^1 \sqrt{1+4x^2}dx = (\sinh 2y/8 + y/4)\,\Big|_0^{\mathrm{arsinh} 2} \tag{8}$$

Zadanie 3/7.



Zadanie 3/8.

$$\int_0^{\pi/2} f(\cos x)dx = [x = \arccos t, t = \cos x, dx = -\frac{1}{\sqrt{1-t^2}}dt]$$
$$= -\int_1^0 \frac{f(t)}{\sqrt{1-t^2}}dt = \int_0^1 \frac{f(t)}{\sqrt{1-t^2}}dt$$

$$\int_0^{\pi/2} f(\sin x)dx = [x = \arcsin t, dx = \frac{1}{\sqrt{1-t^2}}dt] =$$
$$\int_0^1 \frac{f(t)}{\sqrt{1-t^2}}dt$$

(9) $$\int_0^{\pi/2} f(\cos x)dx = [y = \pi/2 - x] = -\int_{\pi/2}^0 f\left(\cos(\pi/2 - y)\right)\, dy$$

Zadanie 3/10.

$$\int_a^b \left(\int_a^b (f(x)g(y) - f(y)g(x))^2 dx\right) dy \geq 0 \tag{10}$$

$$\int_a^b (\int_a^b (f^2(x)g^2(y))dx)dy = \int_a^b (f^2(x))dx \int_a^b (g^2(x))dx \tag{11}$$

$$\int_a^b (\int_a^b (g^2(x)f^2(y))dx)dy = \int_a^b (f^2(x))dx \int_a^b (g^2(x))dx \tag{12}$$

(13)

$$\int_a^b (\int_a^b (2g(x)f(y)f(x)g(y)dx))dy = 2\int_a^b (\int_a^b f(x)g(x)dx)f(y)g(y)dy = 2\left(\int_a^b (f(x)g(x))dx\right)^2$$

$$2\int_a^b (f^2(x))dx \int_a^b (g^2(x))dx - 2\left(\int_a^b (f(x)g(x))dx\right) \geq 0 \tag{14}$$

3/10(2). Lemat 1. Dla wszystkich $x \in \mathbb{R}$ zachodzi $x^2 \geq 0$.

Lemat 2. Jeżeli dla wszystkich $x \in \mathbb{R}$ zachodzi $f(x) \geq 0$, to dla wszystkich $a, b \in \mathbb{R}$, $a < b$ zachodzi $\int_a^b f(x)\,\mathrm{d}x \geq 0$.

Lemat 3. Jeżeli dla wszystkich $x \in \mathbb{R}$ zachodzi $ax^2 + bx + c \geq 0$, to $\Delta = b^2 - 4ac \leq 0$.

Niech t będzie dowolną liczbą rzeczywistą. Wówczas dla dowolnej liczby $x \in \mathbb{R}$ z lematu 1. mamy

$$(t \cdot f(x) - g(x))^2 \geq 0 \iff t^2 \cdot f(x)^2 - 2t \cdot f(x)g(x) + g(x)^2 \geq 0.$$

Z lematu 2. zastosowanego dla powyższej funkcji mamy dla dowolnych liczb $a, b \in \mathbb{R}$, $a < b$

$$\int_a^b t^2 \cdot f(x)^2 - 2t \cdot f(x)g(x) + g(x)^2\,\mathrm{d}x \geq 0 \iff t^2 \cdot \int_a^b f(x)^2\,\mathrm{d}x - 2t \cdot \int_a^b f(x)g(x)\,\mathrm{d}x + \int_a^b g(x)^2\,\mathrm{d}x \geq 0.$$

Z lematu 3. gdzie $a := \int_a^b f(x)^2\,\mathrm{d}x$, $b := 2 \cdot \int_a^b f(x)g(x)\,\mathrm{d}x$, $c := \int_a^b g(x)^2\,\mathrm{d}x$ mamy nierówność

$$4\left(\int_a^b f(x)g(x)\,\mathrm{d}x\right)^2 \leq 4\left(\int_a^b f(x)^2\,\mathrm{d}x\right)\left(\int_a^b g(x)^2\,\mathrm{d}x\right) \blacksquare$$