

Scientific Computing (M3SC) Project 1

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1 MAIN SOLUTION

The code in *solution.py* contains the main program which carries out the process outlined by the question, i.e modelling the process of the cars moving across the city of Rome using the rules described. I have added scripts *to be added* and *to be added* to help answer the related questions at the end of the project.

Below are the imports used by the main program.

```
1 # Imports
2 import numpy as np
3 import csv
4 import sys
5 import math as ma
6
7 # This import is needed for the last question
8 from solution_accident_occurs import max_index_tracker_no30
```

the variable *max_index_tracker_no30* is imported from another python script, *solution_accident_occurs.py* in order to answer one of the questions. This will be discussed in detail later.

Below are the functions required by the program. The docstring's explain their use.

```

1 # -----
2 # -----          FUNCTIONS USED          -----
3 # -----
4
5 def calcWei(RX, RY, RA, RB, RV):
6     """
7     This function is taken from Tutorials. It calculates the weight matrix
8     given information about each node in the system.
9     :param RX: The x coordinates of each node in the system
10    :param RY: The y coordinates of each node in the system
11    :param RA: the connectivity of each node in the system
12    :param RB: the connectivity of each node in the system
13    :param RV: the speed limits across each edge in the system
14    :return: usable weight matrix
15    """
16
17    n = len(RX)
18    wei = np.zeros((n, n), dtype=float)
19    m = len(RA)
20    for i in range(m):
21        xa = RX[RA[i] - 1]
22        ya = RY[RA[i] - 1]
23        xb = RX[RB[i] - 1]
24        yb = RY[RB[i] - 1]
25        dd = ma.sqrt((xb - xa) ** 2 + (yb - ya) ** 2)
26        tt = dd / RV[i]
27        wei[RA[i] - 1, RB[i] - 1] = tt
28    return wei
29
30 def Dijkst(ist, isp, wei):
31     """
32     This Dijkstra's algorithm implementation is taken from tutorials.
33
34     :param ist: the index of the starting node
35     :param isp: the index of the node to reach
36     :param wei: the associated weight matrix
37     :return:
38     """
39
40     # exception handling (start = stop)
41     if ist == isp:
42         shpath = [ist]
43         return shpath

```

```

44
45     # initialization
46     N = len(wei)
47     Inf = sys.maxint
48     UnVisited = np.ones(N, int)
49     cost = np.ones(N) * 1.e6
50     par = -np.ones(N, int) * Inf
51
52     # set the source point and get its (unvisited) neighbors
53     jj = ist
54     cost[jj] = 0
55     UnVisited[jj] = 0
56     tmp = UnVisited * wei[jj, :]
57     ineigh = np.array(tmp.nonzero()).flatten()
58     L = np.array(UnVisited.nonzero()).flatten().size
59
60     # start Dijkstra algorithm
61     while (L != 0):
62         # step 1: update cost of unvisited neighbors,
63         #         compare and (maybe) update
64         for k in ineigh:
65             newcost = cost[jj] + wei[jj, k]
66             if (newcost < cost[k]):
67                 cost[k] = newcost
68                 par[k] = jj
69
70         # step 2: determine minimum-cost point among UnVisited
71         #         vertices and make this point the new point
72         icnsdr = np.array(UnVisited.nonzero()).flatten()
73         cmin, icmin = cost[icnsdr].min(0), cost[icnsdr].argmin(0)
74         jj = icnsdr[icmin]
75
76         # step 3: update "visited"-status and determine neighbors of new point
77         UnVisited[jj] = 0
78         tmp = UnVisited * wei[jj, :]
79         ineigh = np.array(tmp.nonzero()).flatten()
80         L = np.array(UnVisited.nonzero()).flatten().size
81
82     # determine the shortest path
83     shpath = [isp]
84     while par[isp] != ist:
85         shpath.append(par[isp])
86         isp = par[isp]
87     shpath.append(ist)

```

```

88
89     return shpath[::-1]
90
91 def next_node(path):
92     """ Returns the next index (after the node itself) in the path.
93         If the path contains only one node, returns the node itself.
94     """
95     if len(path) == 1:
96         return path[0]
97     else:
98         return path[1]
99
100
101 def update_weight_matrix(epsilon, c, original_weight_matrix, noNodes=58):
102     """
103     This function updates the weight matrix according to step 5 of the
104     Project. Note the added fix – the weight matrix is not changed if
105     the original entry was 0.
106
107
108     :param epsilon: given in question
109     :param c: the vector containing number of cars at each node
110     :param original_weight_matrix: the weight matrix given by RomeEdges
111     :param noNodes: number of nodes in the system
112     :return: the updated weight matrix
113     """
114
115     new_weight_matrix = np.zeros((noNodes, noNodes))
116     for i in range(noNodes):
117         for j in range(noNodes):
118             if original_weight_matrix[i, j] != float(0):
119                 new_weight_matrix[i, j] = original_weight_matrix[i, j] + \
120                                         (epsilon * (float(c[i]) +
121                                                         float(c[j]))) / float(2)
122     return new_weight_matrix
123
124
125 def extract_data():
126     """
127     This function opens the RomeVertices and RomeEdges files, and creates
128     global variables RomeX, RomeY, RomeA, RomeB and RomeV. These are variables
129     used to create the original weight matrix.
130
131     """

```

```

132 global RomeX, RomeY, RomeA, RomeB, RomeV
133 RomeX = np.empty(0, dtype=float)
134 RomeY = np.empty(0, dtype=float)
135 with open('./data/RomeVertices', 'r') as file:
136     AAA = csv.reader(file)
137     for row in AAA:
138         RomeX = np.concatenate((RomeX, [float(row[1])]))
139         RomeY = np.concatenate((RomeY, [float(row[2])]))
140 file.close()
141 RomeA = np.empty(0, dtype=int)
142 RomeB = np.empty(0, dtype=int)
143 RomeV = np.empty(0, dtype=float)
144 with open('./data/RomeEdges2', 'r') as file:
145     AAA = csv.reader(file)
146     for row in AAA:
147         RomeA = np.concatenate((RomeA, [int(row[0])]))
148         RomeB = np.concatenate((RomeB, [int(row[1])]))
149         RomeV = np.concatenate((RomeV, [float(row[2])]))
150 file.close()

```

Now using these functions, we can execute the main program.

```

1  # -----
2  # ----- Main program -----
3  # -----
4
5
6 if __name__ == '__main__':
7
8     # Import the rome edges file
9     extract_data()
10
11     # Use the calcWei function from tutorials, along with the data set given
12     # to calculate the weight matrix. Also create a copy which is the
13     # temporary weight matrix.
14     weight_matrix = misc.calcWei(RomeX, RomeY, RomeA, RomeB, RomeV)
15     temp_wei = weight_matrix.copy()
16
17     # Initialise minutes and number of nodes
18     minutes = 200
19     total_nodes = weight_matrix.shape[0]
20
21     # Need a vector carNumbers which stores the number of cars at each vertex
22     # in the graph.
23     cars_at_node = np.zeros(total_nodes, dtype=int)

```

```

24 cars_at_node_updated = cars_at_node.copy() # cars_at_node updated is simil
25 max_cars_at_node = cars_at_node.copy() # max_cars_at_node is similar
26
27 # To find the edges utilised, we need a 58x58 matrix of
28 # False's. We will set each element to True if we move
29 # cars from node i to node j.
30 edge_utilised = np.zeros((total_nodes, total_nodes), dtype=bool)
31
32 # Iterate through the 200 minutes
33 for i in range(minutes):
34
35     # Apply Dijkstra's algorithm to find the fastest path to node 52 in
36     # the system. Then use next_node to find the next node in the given
37     # path. (step 1)
38     next_nodes = [next_node(Dijkst(node, 51, temp_wei))
39                   for node in range(total_nodes)]
40
41     # Move all cars as in steps 2,3. Iterate through every node in the
42     # system to do this.
43     for j_node in range(total_nodes):
44
45         if j_node == 51:
46             # We remove 40% of cars from node 52.
47             cars_at_node_updated[51] += int(round(cars_at_node[51] * 0.6))
48         else:
49
50             # Initialise the number of cars at node j_node.
51             number_of_cars = cars_at_node[j_node]
52
53             # Initialise the next node to move to.
54             node_to_move_to = next_nodes[j_node]
55
56             # 70% of cars will move. to keep the total conserved,
57             # the amount staying is just
58             # number_of_cars - amount_moving
59             amount_moving = int(round(0.7 * number_of_cars))
60             amount_staying = number_of_cars - amount_moving
61
62             # We now update cars_at_node.
63             cars_at_node_updated[j_node] += amount_staying
64             cars_at_node_updated[node_to_move_to] += amount_moving
65
66             if amount_moving > 0:
67                 # Update edges_utilised matrix

```

```

68         edge_utilised[j_node, node_to_move_to] = True
69
70     # Now all cars have moved where they need to, we set cars_at_node
71     # to this updated vector, and empty the updated vector for the next
72     # iteration.
73     cars_at_node = cars_at_node_updated.copy()
74     cars_at_node_updated = np.zeros(total_nodes, dtype=int)
75
76     # For the first 180 minutes, 20 cars are injected into node 13.
77     if i <= 179:
78         cars_at_node[12] += 20
79
80     # The temporary weight matrix is updated.
81     temp_weig = update_weight_matrix(0.01, cars_at_node, weight_matrix)
82
83     # We have finished an iteration.
84
85     # Now we calculate the maximum number of cars at each node in the system
86     max_cars_at_node = [max(cars_at_node[node], max_cars_at_node[node])
87                        for node in range(total_nodes)]

```

2 QUESTIONS

Determine for each node the maximum load (maximum number of cars) over the 200 iterations.

As we have already incorporated the calculation of maximums in the loop, this is easy:

```

1 max_index_tracker = [[node+1, max_cars_at_node[node]]
2                       for node in range(total_nodes)]

```

Thus the i'th element of this array gives [node i, maximum number of cars of node i over the 200 iterations]. The code to print is below along with the output.

```

1 print('max_index_tracker is')
2 print(max_index_tracker[0:10])
3 print(max_index_tracker[10:20])
4 print(max_index_tracker[20:30])
5 print(max_index_tracker[30:40])
6 print(max_index_tracker[40:50])
7 print(max_index_tracker[50:(len(max_index_tracker)+1)])

```

```
max_index_tracker is
[[1, 1], [2, 2], [3, 0], [4, 7], [5, 0], [6, 8], [7, 8], [8, 0], [9, 16], [10, 14]]
[[11, 0], [12, 8], [13, 28], [14, 0], [15, 28], [16, 22], [17, 7], [18, 28], [19, 11], [20, 28]]
[[21, 38], [22, 11], [23, 7], [24, 24], [25, 40], [26, 23], [27, 6], [28, 10], [29, 13], [30, 32]]
[[31, 12], [32, 19], [33, 15], [34, 15], [35, 23], [36, 9], [37, 3], [38, 14], [39, 19], [40, 30]]
[[41, 30], [42, 11], [43, 31], [44, 28], [45, 4], [46, 0], [47, 0], [48, 11], [49, 0], [50, 23]]
[[51, 18], [52, 63], [53, 17], [54, 15], [55, 12], [56, 14], [57, 11], [58, 11]]
```

Which are the five most congested nodes?

Given this array *max_index_tracker*, we can simply sort by the second argument to find the top five most congested nodes

```
1 top_five = sorted(max_index_tracker,
2                   key=lambda node_and_max: -1 * node_and_max[1])[:5]
3 print('the five most congested nodes are')
4 print(top_five)
```

```
the five most congested nodes are
[[52, 63], [25, 40], [21, 38], [30, 32], [43, 31]]
```

In other words, the top five most congested nodes (from highest to lowest) is node 52 with 63 cars, node 25 with 40 cars, node 21 with 38 cars, node 30 with 32 cars and node 43 with 31 cars.

Which edges are not utilized at all? Why?

In the main program we defined a 58x58 boolean matrix of False's, where throughout the process if cars moved from index *i* to index *j*, we made the [*i*,*j*] element of the matrix True.

Thus, we have a matrix where indices (corresponding to edges) are *True* if they are traversed by some amount of cars (> 0) in the 200 minute process.

All that is left to do is count the number of False's, making sure that we don't count a False if the edge couldn't be traversed to begin with (in the original weight matrix). This corresponds to an element of the original weight matrix being 0.

```
1 # create a boolean matrix that corresponds to the condition
2 # described above
3 non_utilised_edges_matrix = (weight_matrix != float(0)) \
4                             & (np.logical_not(edge_utilised))
5
6 non_utilised_edges = [[i+1, j+1] for i in range(total_nodes)
7                          for j in range(total_nodes)
8                          if non_utilised_edges_matrix[i, j]]
```

So an element [*l*, *m*] belonging to this vector *non_utilised_edges* implies that the edge *l* → *m* is not utilised.

The code to print this and the output is below.

```
1 print('the non utilised edges are')
2 print(non_utilised_edges[0:10])
3 print(non_utilised_edges[10:20])
4 print(non_utilised_edges[20:30])
5 print(non_utilised_edges[30:40])
6 print(non_utilised_edges[40:50])
7 print(non_utilised_edges[50:60])
8 print(non_utilised_edges[60:(len(non_utilised_edges)+1)])
```

```
the non utilised edges are
[[1, 2], [1, 7], [2, 3], [3, 2], [3, 5], [4, 1], [4, 7], [5, 8], [7, 1], [7, 6]]
[[8, 9], [8, 11], [9, 8], [10, 9], [11, 14], [11, 16], [12, 4], [14, 15], [14, 18], [15, 13]]
[[16, 11], [17, 19], [18, 15], [19, 10], [21, 18], [23, 12], [27, 23], [28, 17], [28, 29], [29, 19]]
[[29, 34], [30, 21], [31, 27], [31, 36], [32, 22], [32, 26], [33, 22], [33, 32], [34, 33], [36, 28]]
[[37, 24], [38, 35], [39, 38], [41, 29], [41, 39], [41, 43], [42, 31], [42, 44], [43, 30], [43, 48]]
[[43, 49], [44, 36], [44, 41], [44, 42], [45, 30], [45, 46], [46, 37], [46, 45], [47, 48], [48, 45]]
[[48, 46], [49, 43], [49, 47], [52, 42], [52, 44], [52, 58], [54, 49], [55, 56], [56, 54], [57, 55], [58, 57]]
```

Now I answer *why* these edges aren't utilised.

Firstly we note that from the first question where we found the maximum load for all cars over the 200 iterations, it is clear that nodes with maximum load 0 (i.e. node 3, 5, 8 etc) will always appear on this list since they were never visited. However, the real question is *why* they were never visited.

If we perform Dijkstra's algorithm on the original weight matrix and find the fastest route from node 13 to 52, we find this:

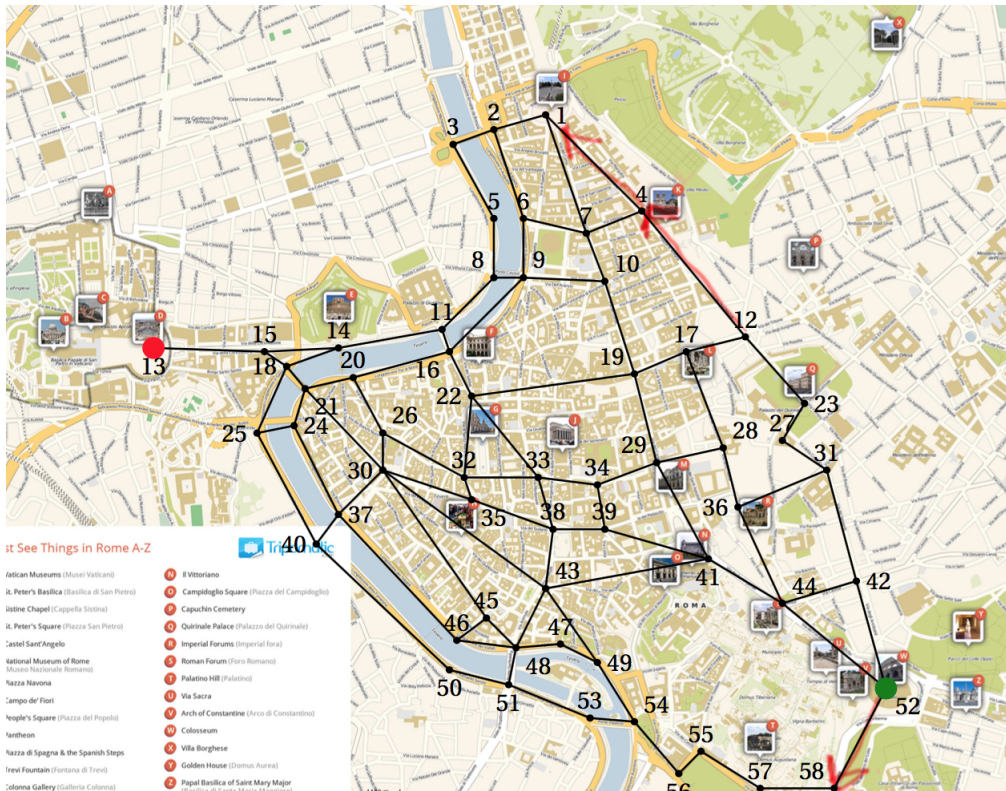
```
1 print(Dijkst(12,51,weight_matrix))
```

```
[12, 14, 17, 24, 39, 49, 50, 52, 53, 55, 54, 56, 57, 51]
```

note that these are the python indices and not the actual node numbers.

If we look at the diagram, this means without taking the effect of congestion, the fastest path throughout the city is almost completely at the bottom of the map, by the river. Such as $40 \rightarrow 50 \rightarrow 51$. This means that it is unlikely for a car to need to travel through the upper west side of the city, such as through nodes 1,3,5 and 2. The only way the shortest path for a car could include such nodes is if the rest of the city is so congested that the shortest path will indeed require these upper left nodes. We can see that edges like [3,2], [8,9] are not optimal for this reason, and are thus unused.

However for the vast majority of unused edges, it is intuitively clear that the reason they are included is because they are pointing completely away from the direction towards node 52. For example, I have highlighted edges $4 \rightarrow 1$, $12 \rightarrow 4$ and $52 \rightarrow 58$ in the diagram below.



These edges in particular are not only in the wrong in the x direction, but wrong in the y direction too. I have created the function below to count how many of these unused edges are pointing *completely* in the wrong direction

```

1 def away_from_52(edge):
2     """
3     Tells you whether a given edge is pointing completely away from
4     node 52, in both the x and y directions.
5     :param edge: an edge of the form [a,b]
6     :return: boolean whether or not this points to or away from 52
7     """
8
9     # extract data for access to global variables
10    extract_data()
11
12    # the edge is of the form [a,b]
13    a = edge[0]
14    b = edge[1]
15
16    # use RomeX and RomeY to find the coordinates for a,b and node 52.
17    a_coord = [RomeX[a - 1], RomeY[a - 1]]
18    b_coord = [RomeX[b - 1], RomeY[b - 1]]
19    coord52 = [RomeX[51], RomeY[51]]
20

```

```

21 # find the change in x/y from a -> b
22 x_change = b_coord[0] - a_coord[0]
23 y_change = b_coord[1] - a_coord[1]
24
25 # find the change in x/y from a -> 52
26 x_changeTo52 = coord52[0] - a_coord[0]
27 y_changeto52 = coord52[1] - a_coord[1]
28
29 # if we're at 52 we're moving away from it
30 if a == 52:
31     return True
32
33 # if both point in same direction, false.
34 if (x_changeTo52 > 0) and (x_change > 0):
35     return False
36 elif (x_changeTo52 < 0) and (x_change < 0):
37     return False
38
39 # if both point in same direction, false.
40 if (y_changeto52 > 0) and (y_change > 0):
41     return False
42 elif (y_changeto52 < 0) and (y_change < 0):
43     return False
44
45 # all other tests have passed, so must be True.
46 return True

```

We then run this to find the number of edges facing completely away from node 52.

```

1 new_unused = list(non_utilised_edges)
2
3 for _, edge in enumerate(non_utilised_edges):
4     if away_from_52(edge):
5         new_unused.remove(edge)
6
7 print('length of new_unused is %i' % len(new_unused))

```

```
length of new_unused is 30
```

So out of 71 unused edges, 41 of them were facing completely away from node 52. In a similar way we can remove nodes that are facing away from node 52 in the x direction only. This, coupled with the discussion above about unused nodes describes why these edges are not utilised by the cars.

What flow pattern do we observe for parameter $\epsilon = 0$?

See the *solution_epsilon0.py* file. The main program is exactly the same, but for one change.

I have replaced $\epsilon = 0.01$ with $\epsilon = \text{float}(0)$ where we update the weight matrix in the main for loop.

```
1 temp_weigh = update_weight_matrix(float(0), cars_at_node, weight_matrix)
```

Now, if we consider the formula for the adjusted weight matrix at each iteration, where $w_{ij} = w_{ij}^{(0)} + \epsilon \frac{c_i + c_j}{2}$, we see that the weight matrix remains unchanged.

As discussed in lectures, Dijkstra's algorithm is an example of a greedy algorithm. Since the time discrete process we are modelling uses Dijkstra's algorithm one node at a time (with the same weight matrix), we know that the shortest path from node 13 to node 52 will remain constant throughout the process.

Furthermore, let P be a node on the shortest path between node 13 and 52,

$$13 \rightarrow \dots \rightarrow P \rightarrow p_1 \rightarrow \dots \rightarrow p_{n-1} \rightarrow 52$$

Then the shortest path from node P to node 52 is

$$P \rightarrow p_1 \rightarrow \dots \rightarrow p_{n-1} \rightarrow 52$$

Since the weight matrix is unchanged. If the shortest path between node P and node 52 were anything else then the shortest path between node 13 and node 52 would not be $13 \rightarrow \dots \rightarrow P \rightarrow p_1 \rightarrow \dots \rightarrow p_{n-1} \rightarrow 52$.

In other words, finding the shortest path from node to node with a constant weight matrix is the same as finding the overall shortest path using the same weight matrix. We can simply use the utilised edges to see that the flow of cars in the system follows this path. Printing the `cars_at_node` vector throughout the 200 minutes verifies this.

```
1 # regular dijkstra's path
2 print('the Dijkstra's path is ')
3 print(dijk.Dijkst(12, 51, weight_matrix))
4
5 utilised_edges = [[i, j] for i in range(noNodes)
6                     for j in range(noNodes)
7                     if edge_utilised[i, j]] # this matrix is defined in the main
8                                             #code
9 print('the utilised edges are')
10 print(utilised_edges)
```

```
the Dijkstra's path is
[12, 14, 17, 24, 39, 49, 50, 52, 53, 55, 54, 56, 57, 51]
the utilised edges are
[[12, 14], [14, 17], [17, 24], [24, 39], [39, 49], [49, 50], [50, 52], [52, 53], [53, 55], [54, 56], [55,
```

Since all of these edges have a distinct head to tail connection, I can be sure the cars follow the path in the order we expect.

An accident occurs at node 30 (python-index 29) which blocks any route to or from node 30. Which nodes are now the most congested and what is their maximum load? Which

nodes (besides node 30) decrease the most in peak value, which nodes increase the most in peak value?

See *solution_accident_occurs.py*. The main code is exactly the same as before, but for these changes:

Since no car can reach node 30, we need to make the 30th row and 30th column of the weight matrix equal to 0s.

```
1 # Use the calcWei function from tutorials, along with the data set given
2 # to calculate the weight matrix. Also create a copy which is the
3 # temporary weight matrix.
4 weight_matrix = calcWei(RomeX, RomeY, RomeA, RomeB, RomeV)
5
6 # The accident at node 30 means that the 30th row and 30th column is all 0
7 weight_matrix[29, :] = np.zeros(58, dtype=float)
8 weight_matrix[:, 29] = np.zeros(58, dtype=float)
9
10 temp_wei = weight_matrix.copy()
```

Once we begin the for loop iterating through the 200 minutes, we compute the next nodes as before. But there is no path between node 30 and 52, so we add this "if node!=29" statement to the next_nodes calculation.

We also insert a 0 at index 29, so the nodes align properly again.

```
1 # Iterate through the 200 minutes
2 for i in range(minutes):
3
4     # Apply Dijkstra's algorithm to find the fastest path to node 52 in
5     # the system. Then use next_node to find the next node in the given
6     # path. (step 1)
7     next_nodes = [next_node(Dijkst(node, 51, temp_wei))
8                   for node in range(total_nodes) if node != 29]
9
10    next_nodes.insert(29, 29) # send the 0 cars from 29 to itself
```

We can ignore node 30 when moving the cars through the system, so we add this else if statement.

```
1 for j_node in range(total_nodes):
2
3     if j_node == 51:
4         # We remove 40% of cars from node 52.
5         cars_at_node_updated[51] += int(round(cars_at_node[51] * 0.6))
6         # really, we can just ignore node 30.
7     elif j_node != 29:
```

Now we create this variable `max_index_tracker_no30`, which is simply the array of all the nodes with the maximum load they carry over the 200 iterations. This is then imported into the *`solution_final.py`* file as outlined before.

```
1 # Find the top 5 most congested nodes.  
2 max_index_tracker_no30 = [[node+1, max_cars_at_node[node]]  
3                             for node in range(total_nodes)]
```