Chapter 2

A Multiresolution Formulation of Wavelet Systems

Both the mathematics and the practical interpretations of wavelets seem to be best served by using the concept of resolution [Mey93, Mal89b, Mal89c, Dau92] to define the effects of changing scale. To do this, we will start with a *scaling function* $\varphi(t)$ rather than directly with the wavelet $\psi(t)$. After the scaling function is defined from the concept of resolution, the wavelet functions will be derived from it. This chapter will give a rather intuitive development of these ideas, which will be followed by more rigorous arguments in Chapter 5.

This multiresolution formulation is obviously designed to represent signals where a single event is decomposed into finer and finer detail, but it turns out also to be valuable in representing signals where a time-frequency or time-scale description is desired even if no concept of resolution is needed. However, there are other cases where multiresolution is not appropriate, such as for the short-time Fourier transform or Gabor transform or for local sine or cosine bases or lapped orthogonal transforms, which are all discussed briefly later in this book.

2.1 Signal Spaces

In order to talk about the collection of functions or signals that can be represented by a sum of scaling functions and/or wavelets, we need some ideas and terminology from functional analysis. If these concepts are not familiar to you or the information in this section is not sufficient, you may want to skip ahead and read Chapter 5 or [VD95].

A *function space* is a linear vector space (finite or infinite dimensional) where the vectors are functions, the scalars are real numbers (sometime complex numbers), and scalar multiplication and vector addition are similar to that done in (1.1). The *inner product* is a scalar a obtained from two vectors, $f(t)$ and $g(t)$, by an integral. It is denoted

$$a = \langle f(t), g(t) \rangle = \int f^*(t)\, g(t)\, dt \tag{2.1}$$

with the range of integration depending on the signal class being considered. This inner product defines a *norm* or "length" of a vector which is denoted and defined by

$$\|f\| = \sqrt{|\langle f, f \rangle|} \tag{2.2}$$

which is a simple generalization of the geometric operations and definitions in three-dimensional Euclidean space. Two signals (vectors) with non-zero norms are called *orthogonal* if their inner product is zero. For example, with the Fourier series, we see that $\sin(t)$ is orthogonal to $\sin(2t)$.

A space that is particularly important in signal processing is call $L^2(\mathbf{R})$. This is the space of all functions $f(t)$ with a well defined integral of the square of the modulus of the function. The "L" signifies a Lebesque integral, the "2" denotes the integral of the square of the modulus of the function, and $\mathbf{R}$ states that the independent variable of integration t is a number over the whole real line. For a function $g(t)$ to be a member of that space is denoted: $g \in L^2(\mathbf{R})$ or simply $g \in L^2$.

Although most of the definitions and derivations are in terms of signals that are in L^2, many of the results hold for larger classes of signals. For example, polynomials are not in L^2 but can be expanded over any finite domain by most wavelet systems.

In order to develop the wavelet expansion described in (1.5), we will need the idea of an expansion set or a basis set. If we start with the vector space of signals, $\mathcal{S}$, then if any $f(t) \in \mathcal{S}$ can be expressed as $f(t) = \sum_k a_k \, \varphi_k(t)$, the set of functions $\varphi_k(t)$ are called an expansion set for the space $\mathcal{S}$. If the representation is unique, the set is a basis. Alternatively, one could start with the expansion set or basis set and define the space $\mathcal{S}$ as the set of all functions that can be expressed by $f(t) = \sum_k a_k \, \varphi_k(t)$. This is called the *span* of the basis set. In several cases, the signal spaces that we will need are actually the *closure* of the space spanned by the basis set. That means the space contains not only all signals that can be expressed by a linear combination of the basis functions, but also the signals which are the limit of these infinite expansions. The closure of a space is usually denoted by an over-line.

2.2 The Scaling Function

In order to use the idea of multiresolution, we will start by defining the scaling function and then define the wavelet in terms of it. As described for the wavelet in the previous chapter, we define a set of scaling functions in terms of integer translates of the basic scaling function by

$$\varphi_k(t) = \varphi(t-k) \qquad k \in \mathbf{Z} \qquad \varphi \in L^2. \tag{2.3}$$

The subspace of $L^2(\mathbf{R})$ spanned by these functions is defined as

$$\mathcal{V}_0 = \overline{\operatorname*{Span}_k\{\varphi_k(t)\}} \tag{2.4}$$

for all integers k from minus infinity to infinity. The over-bar denotes closure. This means that

$$f(t) = \sum_k a_k \, \varphi_k(t) \quad \text{for any} \quad f(t) \in \mathcal{V}_0. \tag{2.5}$$

One can generally increase the size of the subspace spanned by changing the time scale of the scaling functions. A two-dimensional family of functions is generated from the basic scaling function by scaling and translation by

$$\varphi_{j,k}(t) = 2^{j/2} \, \varphi(2^j t - k) \tag{2.6}$$

whose span over k is

$$\mathcal{V}_j = \overline{\operatorname*{Span}_k\{\varphi_k(2^j t)\}} = \overline{\operatorname*{Span}_k\{\varphi_{j,k}(t)\}} \tag{2.7}$$

for all integers $k \in \mathbf{Z}$. This means that if $f(t) \in \mathcal{V}_j$, then it can be expressed as

$$f(t) = \sum_k a_k \, \varphi(2^j t + k). \tag{2.8}$$

For $j > 0$, the span can be larger since $\varphi_{j,k}(t)$ is narrower and is translated in smaller steps. It, therefore, can represent finer detail. For $j < 0$, $\varphi_{j,k}(t)$ is wider and is translated in larger steps. So these wider scaling functions can represent only coarse information, and the space they span is smaller. Another way to think about the effects of a change of scale is in terms of resolution. If one talks about photographic or optical resolution, then this idea of scale is the same as resolving power.

Multiresolution Analysis

In order to agree with our intuitive ideas of scale or resolution, we formulate the basic requirement of multiresolution analysis (MRA) [Mal89c] by requiring a nesting of the spanned spaces as

$$\cdots \subset \mathcal{V}_{-2} \subset \mathcal{V}_{-1} \subset \mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots \subset L^2 \tag{2.9}$$

or

$$\mathcal{V}_j \subset \mathcal{V}_{j+1} \quad \text{for all} \quad j \in \mathbf{Z} \tag{2.10}$$

with

$$\mathcal{V}_{-\infty} = \{0\}, \qquad \mathcal{V}_{\infty} = L^2. \tag{2.11}$$

The space that contains high resolution signals will contain those of lower resolution also.

Because of the definition of $\mathcal{V}_j$, the spaces have to satisfy a natural scaling condition

$$f(t) \in \mathcal{V}_j \quad \Leftrightarrow \quad f(2t) \in \mathcal{V}_{j+1} \tag{2.12}$$

which insures elements in a space are simply scaled versions of the elements in the next space. The relationship of the spanned spaces is illustrated in Figure 2.1.

The nesting of the spans of $\varphi(2^j t - k)$, denoted by $\mathcal{V}_j$ and shown in (2.9) and (2.12) and graphically illustrated in Figure 2.1, is achieved by requiring that $\varphi(t) \in \mathcal{V}_1$, which means that if $\varphi(t)$ is in $\mathcal{V}_0$, it is also in $\mathcal{V}_1$, the space spanned by $\varphi(2t)$. This means $\varphi(t)$ can be expressed in terms of a weighted sum of shifted $\varphi(2t)$ as

$$\boxed{\varphi(t) = \sum_n h(n) \sqrt{2}\, \varphi(2t - n), \qquad n \in \mathbf{Z}} \tag{2.13}$$

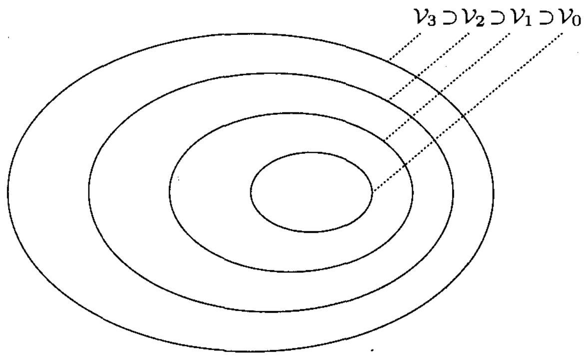


Figure 2.1. Nested Vector Spaces Spanned by the Scaling Functions

where the coefficients $h(n)$ are a sequence of real or perhaps complex numbers called the scaling function coefficients (or the scaling filter or the scaling vector) and the $\sqrt{2}$ maintains the norm of the scaling function with the scale of two.

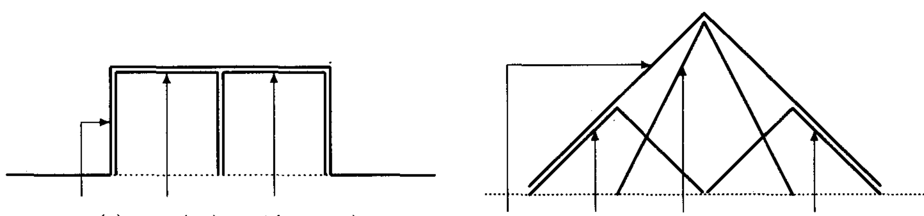
This recursive equation is fundamental to the theory of the scaling functions and is, in some ways, analogous to a differential equation with coefficients $h(n)$ and solution $\varphi(t)$ that may or may not exist or be unique. The equation is referred to by different names to describe different interpretations or points of view. It is called the refinement equation, the multiresolution analysis (MRA) equation, or the dilation equation.

The Haar scaling function is the simple unit-width, unit-height pulse function $\varphi(t)$ shown in Figure 2.2, and it is obvious that $\varphi(2t)$ can be used to construct $\varphi(t)$ by

$$\varphi(t) = \varphi(2t) + \varphi(2t-1) \tag{2.14}$$

which means (2.13) is satisfied for coefficients $h(0) = 1/\sqrt{2}, \quad h(1) = 1/\sqrt{2}$.

The triangle scaling function (also a first order spline) in Figure 2.2 satisfies (2.13) for $h(0) = \frac{1}{2\sqrt{2}}, \quad h(1) = \frac{1}{\sqrt{2}}, \quad h(2) = \frac{1}{2\sqrt{2}}$, and the Daubechies scaling function shown in the first part of



$\phi(t) = \phi(2t) + \phi(2t-1)$

(a) Haar (same as φ_{D2})

$\phi(t) = \frac{1}{2}\phi(2t) + \phi(2t-1) + \frac{1}{2}\phi(2t-2)$

(b) Triangle (same as φ_{S1})

Figure 2.2. Haar and Triangle Scaling Functions

Figure 6.1 satisfies (2.13) for $h = \{0.483, 0.8365, 0.2241, -0.1294\}$ as do all scaling functions for their corresponding scaling coefficients. Indeed, the design of wavelet systems is the choosing of the coefficients $h(n)$ and that is developed later.

2.3 The Wavelet Functions

The important features of a signal can better be described or parameterized, not by using $\varphi_{j,k}(t)$ and increasing j to increase the size of the subspace spanned by the scaling functions, but by defining a slightly different set of functions $\psi_{j,k}(t)$ that span the *differences* between the spaces spanned by the various scales of the scaling function. These functions are the wavelets discussed in the introduction of this book.

There are several advantages to requiring that the scaling functions and wavelets be orthogonal. Orthogonal basis functions allow simple calculation of expansion coefficients and have a Parseval's theorem that allows a partitioning of the signal energy in the wavelet transform domain. The orthogonal complement of $\mathcal{V}_j$ in $\mathcal{V}_{j+1}$ is defined as $\mathcal{W}_j$. This means that all members of $\mathcal{V}_j$ are orthogonal to all members of $\mathcal{W}_j$. We require

$$\langle \varphi_{j,k}(t), \psi_{j,\ell}(t) \rangle \;=\; \int \varphi_{j,k}(t)\; \psi_{j,\ell}(t)\; dt \;=\; 0 \tag{2.15}$$

for all appropriate $j, k, \ell \in \mathbf{Z}$.

The relationship of the various subspaces can be seen from the following expressions. From (2.9) we see that we may start at any $\mathcal{V}_j$, say at $j = 0$, and write

$$\mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots \subset L^2. \tag{2.16}$$

We now define the wavelet spanned subspace $\mathcal{W}_0$ such that

$$\mathcal{V}_1 \;=\; \mathcal{V}_0 \oplus \mathcal{W}_0 \tag{2.17}$$

which extends to

$$\mathcal{V}_2 \;=\; \mathcal{V}_0 \oplus \mathcal{W}_0 \oplus \mathcal{W}_1. \tag{2.18}$$

In general this gives

$$L^2 = \mathcal{V}_0 \oplus \mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \cdots \tag{2.19}$$

when $\mathcal{V}_0$ is the initial space spanned by the scaling function $\varphi(t-k)$. Figure 2.3 pictorially shows the nesting of the scaling function spaces $\mathcal{V}_j$ for different scales j and how the wavelet spaces are the disjoint differences (except for the zero element) or, the orthogonal complements.

The scale of the initial space is arbitrary and could be chosen at a higher resolution of, say, $j = 10$ to give

$$L^2 = \mathcal{V}_{10} \oplus \mathcal{W}_{10} \oplus \mathcal{W}_{11} \oplus \cdots \tag{2.20}$$

or at a lower resolution such as $j = -5$ to give

$$L^2 = \mathcal{V}_{-5} \oplus \mathcal{W}_{-5} \oplus \mathcal{W}_{-4} \oplus \cdots \tag{2.21}$$

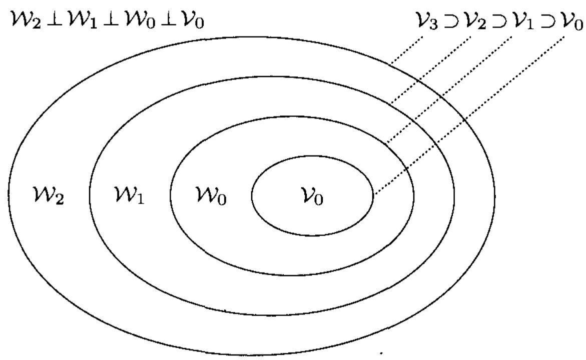


Figure 2.3. Scaling Function and Wavelet Vector Spaces

or at even $j = -\infty$ where (2.19) becomes

$$L^2 = \cdots \oplus \mathcal{W}_{-2} \oplus \mathcal{W}_{-1} \oplus \mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \cdots \tag{2.22}$$

eliminating the scaling space altogether and allowing an expansion of the form in (1.9).

Another way to describe the relation of $\mathcal{V}_0$ to the wavelet spaces is noting

$$\mathcal{W}_{-\infty} \oplus \cdots \oplus \mathcal{W}_{-1} = \mathcal{V}_0 \tag{2.23}$$

which again shows that the scale of the scaling space can be chosen arbitrarily. In practice, it is usually chosen to represent the coarsest detail of interest in a signal.

Since these wavelets reside in the space spanned by the next narrower scaling function, $\mathcal{W}_0 \subset \mathcal{V}_1$, they can be represented by a weighted sum of shifted scaling function $\varphi(2t)$ defined in (2.13) by

$$\boxed{\psi(t) \;=\; \sum_n h_1(n)\, \sqrt{2}\, \varphi(2t-n), \qquad n \in \mathbf{Z}} \tag{2.24}$$

for some set of coefficients $h_1(n)$. From the requirement that the wavelets span the "difference" or orthogonal complement spaces, and the orthogonality of integer translates of the wavelet (or scaling function), it is shown in the Appendix in (12.48) that the wavelet coefficients (modulo translations by integer multiples of two) are required by orthogonality to be related to the scaling function coefficients by

$$h_1(n) = (-1)^n h(1-n). \tag{2.25}$$

One example for a finite even length-N $h(n)$ could be

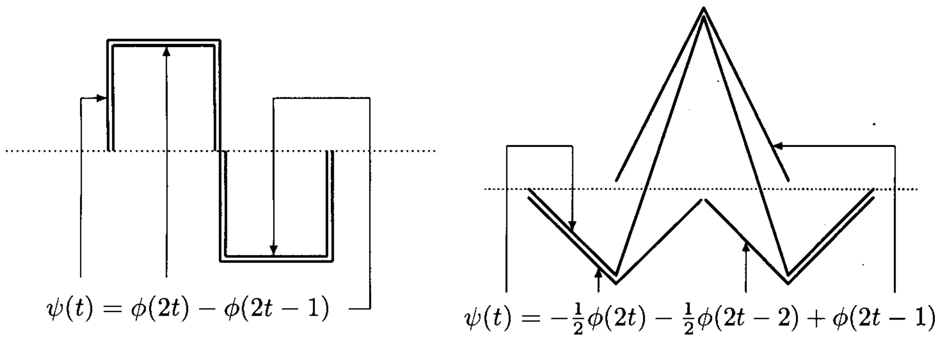
$$h_1(n) = (-1)^n h(N-1-n). \tag{2.26}$$

The function generated by (2.24) gives the prototype or mother wavelet $\psi(t)$ for a class of expansion functions of the form

$$\psi_{j,k}(t) = 2^{j/2}\, \psi(2^j t - k) \tag{2.27}$$

where 2^j is the scaling of t (j is the $\log_2$ of the scale), $2^{-j}k$ is the translation in t, and $2^{j/2}$ maintains the (perhaps unity) L^2 norm of the wavelet at different scales.

The Haar and triangle wavelets that are associated with the scaling functions in Figure 2.2 are shown in Figure 2.4. For the Haar wavelet, the coefficients in (2.24) are $h_1(0) = 1/\sqrt{2}$, $h_1(1) = -1/\sqrt{2}$ which satisfy (2.25). The Daubechies wavelets associated with the scaling functions in Figure 6.1 are shown in Figure 6.2 with corresponding coefficients given later in the book in Tables 6.1 and 6.2.



(a) Haar (same as ψ_{D2}) (b) Triangle (same as ψ_{S1})

Figure 2.4. Haar and Triangle Wavelets

We have now constructed a set of functions $\varphi_k(t)$ and $\psi_{j,k}(t)$ that could span all of $L^2(\mathbf{R})$. According to (2.19), any function $g(t) \in L^2(\mathbf{R})$ could be written

$$g(t) = \sum_{k=-\infty}^{\infty} c(k)\,\varphi_k(t) + \sum_{j=0}^{\infty}\sum_{k=-\infty}^{\infty} d(j,k)\,\psi_{j,k}(t) \tag{2.28}$$

as a series expansion in terms of the scaling function and wavelets.

In this expansion, the first summation in (2.28) gives a function that is a low resolution or coarse approximation of $g(t)$. For each increasing index j in the second summation, a higher or finer resolution function is added, which adds increasing detail. This is somewhat analogous to a Fourier series where the higher frequency terms contain the detail of the signal.

Later in this book, we will develop the property of having these expansion functions form an orthonormal basis or a tight frame, which allows the coefficients to be calculated by inner products as

$$c(k) = c_0(k) = \langle g(t), \varphi_k(t)\rangle = \int g(t)\,\varphi_k(t)\,dt \tag{2.29}$$

and

$$d_j(k) = d(j,k) = \langle g(t), \psi_{j,k}(t)\rangle = \int g(t)\,\psi_{j,k}(t)\,dt. \tag{2.30}$$

The coefficient $d(j,k)$ is sometimes written as $d_j(k)$ to emphasize the difference between the time translation index k and the scale parameter j. The coefficient $c(k)$ is also sometimes written as $c_j(k)$ or $c(j,k)$ if a more general "starting scale" other than $j = 0$ for the lower limit on the sum in (2.28) is used.

It is important at this point to recognize the relationship of the scaling function part of the expansion (2.28) to the wavelet part of the expansion. From the representation of the nested spaces in (2.19) we see that the scaling function can be defined at any scale j. Equation (2.28) uses $j = 0$ to denote the family of scaling functions.

You may want to examine the Haar system example at the end of this chapter just now to see these features illustrated.

2.4 The Discrete Wavelet Transform

Since

$$L^2 = \mathcal{V}_{j_0} \oplus \mathcal{W}_{j_0} \oplus \mathcal{W}_{j_0+1} \oplus \cdots \tag{2.31}$$

using (2.6) and (2.27), a more general statement of the expansion (2.28) can be given by

$$g(t) = \sum_k c_{j_0}(k)\, 2^{j_0/2}\, \varphi(2^{j_0}t - k) + \sum_k \sum_{j=j_0}^{\infty} d_j(k)\, 2^{j/2}\, \psi(2^j t - k) \tag{2.32}$$

or

$$g(t) = \sum_k c_{j_0}(k)\, \varphi_{j_0,k}(t) + \sum_k \sum_{j=j_0}^{\infty} d_j(k)\, \psi_{j,k}(t) \tag{2.33}$$

where j_0 could be zero as in (2.19) and (2.28), it could be ten as in (2.20), or it could be negative infinity as in (1.8) and (2.22) where no scaling functions are used. The choice of j_0 sets the coarsest scale whose space is spanned by $\varphi_{j_0,k}(t)$. The rest of $L^2(\mathbf{R})$ is spanned by the wavelets which provide the high resolution details of the signal. In practice where one is given only the samples of a signal, not the signal itself, there is a highest resolution when the finest scale is the sample level.

The coefficients in this wavelet expansion are called the *discrete wavelet transform* (DWT) of the signal $g(t)$. If certain conditions described later are satisfied, these wavelet coefficients completely describe the original signal and can be used in a way similar to Fourier series coefficients for analysis, description, approximation, and filtering. If the wavelet system is orthogonal, these coefficients can be calculated by inner products

$$c_j(k) = \langle g(t),\, \varphi_{j,k}(t) = \int g(t)\, \varphi_{j,k}(t)\, dt \tag{2.34}$$

and

$$d_j(k) \;=\; \langle g(t),\, \psi_{j,k}(t) = \int g(t)\, \psi_{j,k}(t)\, dt. \tag{2.35}$$

If the scaling function is well-behaved, then at a high scale, the scaling is similar to a Dirac delta function and the inner product simply samples the function. In other words, at high enough resolution, samples of the signal are very close to the scaling coefficients. More is said about this later. It has been shown [Don93b] that wavelet systems form an unconditional basis for a large class of signals. That is discussed in Chapter 5 but means that even for the worst case signal in the class, the wavelet expansion coefficients drop off rapidly as j and k increase. This is why the DWT is efficient for signal and image compression.

The DWT is similar to a Fourier series but, in many ways, is much more flexible and informative. It can be made periodic like a Fourier series to represent periodic signals efficiently.

However, unlike a Fourier series, it can be used directly on non-periodic transient signals with excellent results. An example of the DWT of a pulse was illustrated in Figure 3.3. Other examples are illustrated just after the next section.

2.5 A Parseval's Theorem

If the scaling functions and wavelets form an orthonormal basis[1], there is a Parseval's theorem that relates the energy of the signal $g(t)$ to the energy in each of the components and their wavelet coefficients. That is one reason why orthonormality is important.

For the general wavelet expansion of (2.28) or (2.33), Parseval's theorem is

$$\int |g(t)|^2 \, dt = \sum_{l=-\infty}^{\infty} |c(l)|^2 + \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} |d_j(k)|^2 \tag{2.36}$$

with the energy in the expansion domain partitioned in time by k and in scale by j. Indeed, it is this partitioning of the time-scale parameter plane that describes the DWT. If the expansion system is a tight frame, there is a constant multiplier in (2.36) caused by the redundancy.

Daubechies [Dau88a, Dau92] showed that it is possible for the scaling function and the wavelets to have compact support (i.e., be nonzero only over a finite region) and to be orthonormal. This makes possible the time localization that we desire. We now have a framework for describing signals that has features of short-time Fourier analysis and of Gabor-based analysis but using a new variable, scale. For the short-time Fourier transform, orthogonality and good time-frequency resolution are incompatible according to the Balian-Low-Coifman-Semmes theorem [Dau90, Sie86]. More precisely, if the short-time Fourier transform is orthogonal, either the time or the frequency resolution is poor and the trade-off is inflexible. This is not the case for the wavelet transform. Also, note that there is a variety of scaling functions and wavelets that can be obtained by choosing different coefficients $h(n)$ in (2.13).

Donoho [Don93b] has noted that wavelets are an unconditional basis for a very wide class of signals. This means wavelet expansions of signals have coefficients that drop off rapidly and therefore the signal can be efficiently represented by a small number of them.

We have first developed the basic ideas of the discrete wavelet system using a scaling multiplier of 2 in the defining equation (2.13). This is called a *two-band wavelet system* because of the two channels or bands in the related filter banks discussed in Chapters 3 and 8. It is also possible to define a more general discrete wavelet system using $\varphi(t) = \sum_n h(n) \sqrt{M}\, \varphi(Mt - n)$ where M is an integer [SHGB93]. This is discussed in Section 7.2. The details of numerically calculating the DWT are discussed in Chapter 9 where special forms for periodic signals are used.

2.6 Display of the Discrete Wavelet Transform and the Wavelet Expansion

It is important to have an informative way of displaying or visualizing the wavelet expansion and transform. This is complicated in that the DWT is a real-valued function of two integer indices and, therefore, needs a two-dimensional display or plot. This problem is somewhat analogous to plotting the Fourier transform, which is a complex-valued function.

There seem to be five displays that show the various characteristics of the DWT well:

[1] or a tight frame defined in Chapter 4

1. The most basic time-domain description of a signal is the signal itself (or, for most cases, samples of the signal) but it gives no frequency or scale information. A very interesting property of the DWT (and one different from the Fourier series) is for a high starting scale j_0 in (2.33), samples of the signal are the DWT at that scale. This is an extreme case, but it shows the flexibility of the DWT and will be explained later.

2. The most basic wavelet-domain description is a three-dimensional plot of the expansion coefficients or DWT values $c(k)$ and $d_j(k)$ over the j, k plane. This is difficult to do on a two-dimensional page or display screen, but we show a form of that in Figures 2.5 and 2.8.

3. A very informative picture of the effects of scale can be shown by generating time functions $f_j(t)$ at each scale by summing (2.28) over k so that

$$f(t) = f_{j_0} + \sum_j f_j(t) \tag{2.37}$$

where

$$f_{j_0} = \sum_k c(k)\,\varphi(t-k) \tag{2.38}$$

and

$$f_j(t) = \sum_k d_j(k)\,2^{j/2}\,\psi(2^j t - k). \tag{2.39}$$

This illustrates the components of the signal at each scale and is shown in Figures 2.7 and 2.10.

4. Another illustration that shows the time localization of the wavelet expansion is obtained by generating time functions $f_k(t)$ at each translation by summing (2.28) over k so that

$$f(t) = \sum_k f_k(t) \tag{2.40}$$

where

$$f_k(t) = c(k)\,\varphi(t-k) + \sum_j d_j(k)\,2^{j/2}\,\psi(2^j t - k). \tag{2.41}$$

This illustrates the components of the signal at each integer translation.

5. There is another rather different display based on a partitioning of the time-scale plane as if the time translation index and scale index were continuous variables. This display is called "tiling the time-frequency plane." Because it is a different type of display and is developed and illustrated in Chapter 9, it will not be illustrated here.

Experimentation with these displays can be very informative in terms of the properties and capabilities of the wavelet transform, the effects of particular wavelet systems, and the way a wavelet expansion displays the various attributes or characteristics of a signal.

2.7 Examples of Wavelet Expansions

In this section, we will try to show the way a wavelet expansion decomposes a signal and what the components look like at different scales. These expansions use what is called a length-8 Daubechies basic wavelet (developed in Chapter 6), but that is not the main point here. The local nature of the wavelet decomposition is the topic of this section.

These examples are rather standard ones, some taken from David Donoho's papers and web page. The first is a decomposition of a piecewise linear function to show how edges and constants are handled. A characteristic of Daubechies systems is that low order polynomials are completely contained in the scaling function spaces $\mathcal{V}_j$ and need no wavelets. This means that when a section of a signal is a section of a polynomial (such as a straight line), there are no wavelet expansion coefficients $d_j(k)$, but when the calculation of the expansion coefficients overlaps an edge, there is a wavelet component. This is illustrated well in Figure 2.6 where the high resolution scales gives a very accurate location of the edges and this spreads out over k at the lower scales. This gives a hint of how the DWT could be used for edge detection and how the large number of small or zero expansion coefficients could be used for compression.

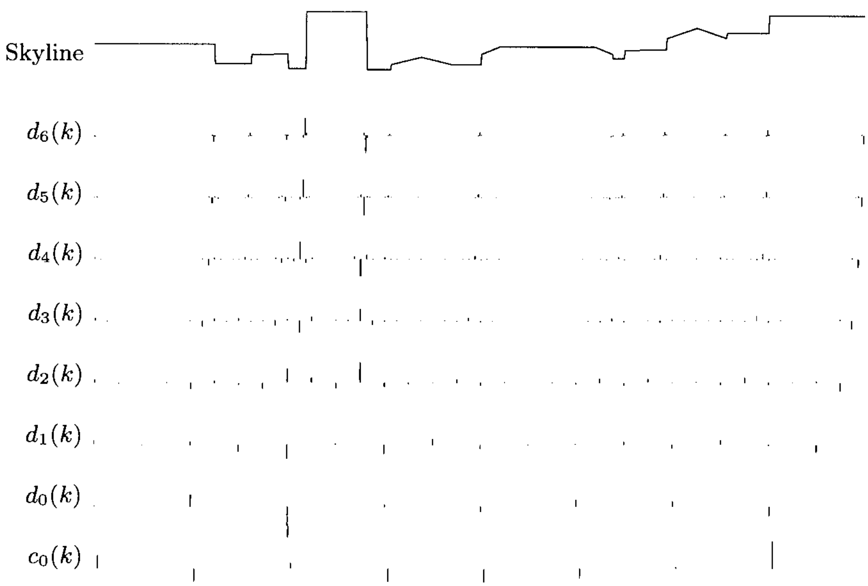
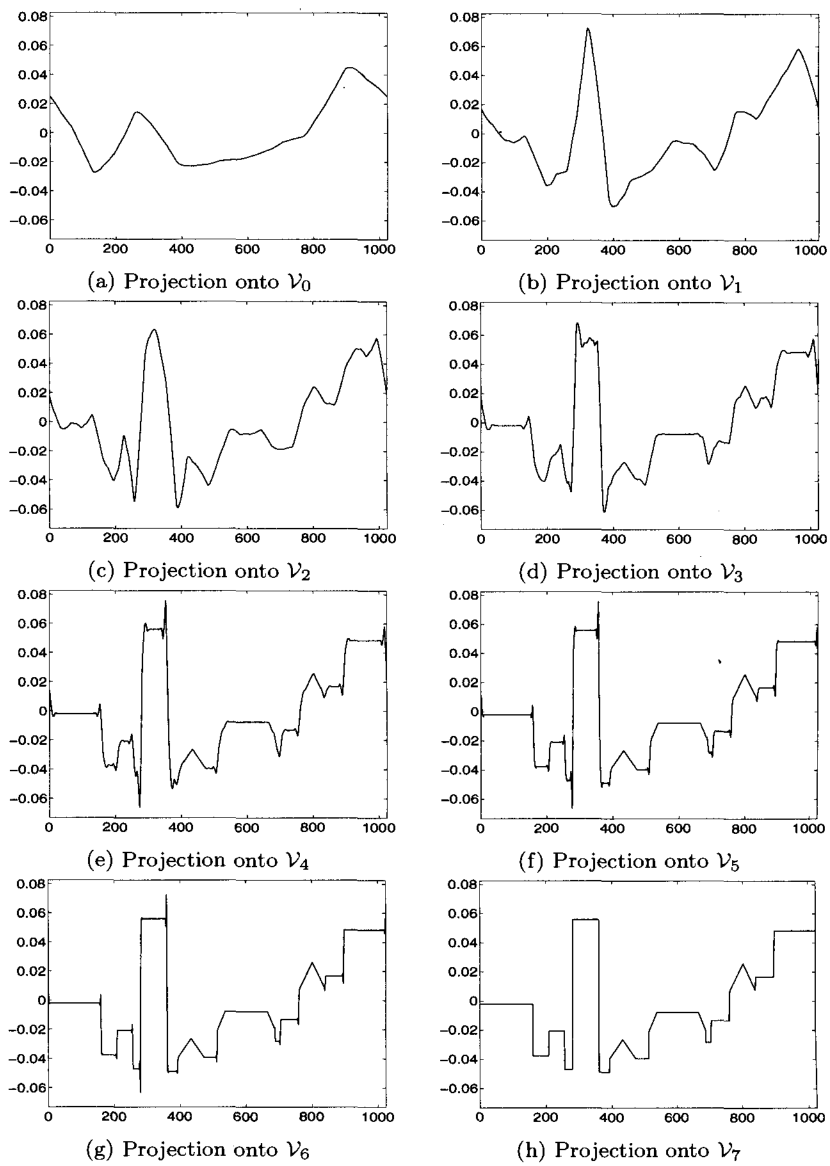


Figure 2.5. Discrete Wavelet Transform of the Houston Skyline, using $\psi_{D8'}$ with a Gain of $\sqrt{2}$ for Each Higher Scale

Figure 2.6 shows the approximations of the skyline signal in the various scaling function spaces $\mathcal{V}_j$. This illustrates just how the approximations progress, giving more and more resolution at higher scales. The fact that the higher scales give more detail is similar to Fourier methods, but the localization is new. Figure 2.7 illustrates the individual wavelet decomposition by showing the components of the signal that exist in the wavelet spaces $\mathcal{W}_j$ at different scales j. This shows the same expansion as Figure 2.6, but with the wavelet components given separately rather than being cumulatively added to the scaling function. Notice how the large objects show up at the lower resolution. Groups of buildings and individual buildings are resolved according to their width. The edges, however, are located at the higher resolutions and are located very accurately.



(a) Projection onto $\mathcal{V}_0$

(b) Projection onto $\mathcal{V}_1$

(c) Projection onto $\mathcal{V}_2$

(d) Projection onto $\mathcal{V}_3$

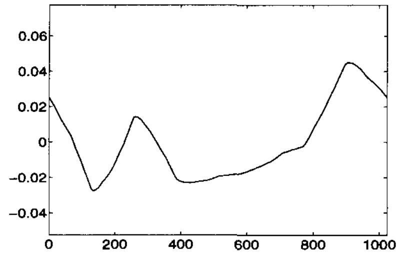
(e) Projection onto $\mathcal{V}_4$

(f) Projection onto $\mathcal{V}_5$

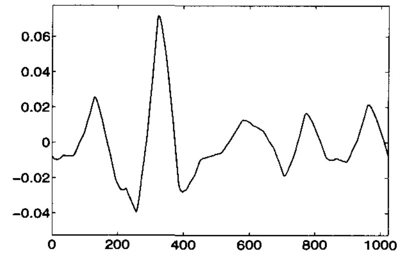
(g) Projection onto $\mathcal{V}_6$

(h) Projection onto $\mathcal{V}_7$

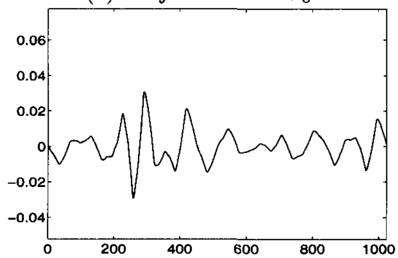
Figure 2.6. Projection of the Houston Skyline Signal onto $\mathcal{V}$ Spaces using $\phi_{D8'}$



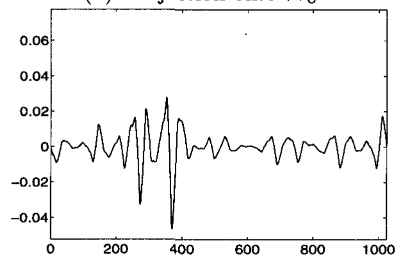
(a) Projection onto $\mathcal{V}_0$



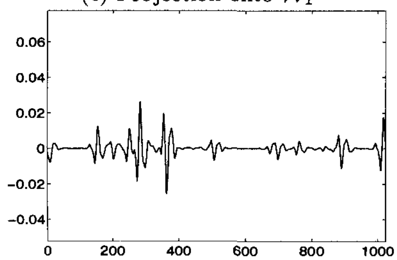
(b) Projection onto $\mathcal{W}_0$



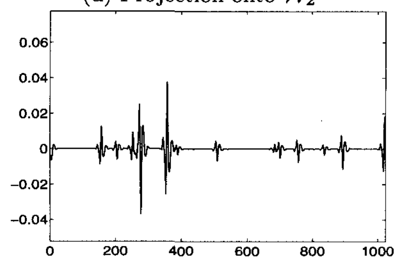
(c) Projection onto $\mathcal{W}_1$



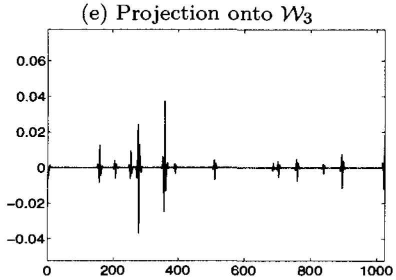
(d) Projection onto $\mathcal{W}_2$



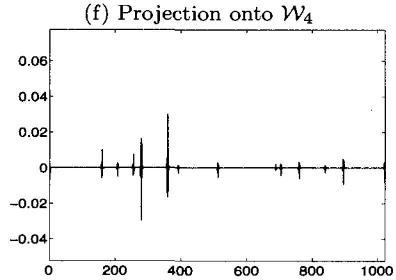
(e) Projection onto $\mathcal{W}_3$



(f) Projection onto $\mathcal{W}_4$



(g) Projection onto $\mathcal{W}_5$



(h) Projection onto $\mathcal{W}_6$

Figure 2.7. Projection of the Houston Skyline Signal onto $\mathcal{W}$ Spaces using $\psi_{D8'}$

The second example uses a chirp or doppler signal to illustrate how a time-varying frequency is described by the scale decomposition. Figure 2.8 gives the coefficients of the DWT directly as a function of j and k. Notice how the location in k tracks the frequencies in the signal in a way the Fourier transform cannot. Figures 2.9 and 2.10 show the scaling function approximations and the wavelet decomposition of this chirp signal. Again, notice in this type of display how the "location" of the frequencies are shown.

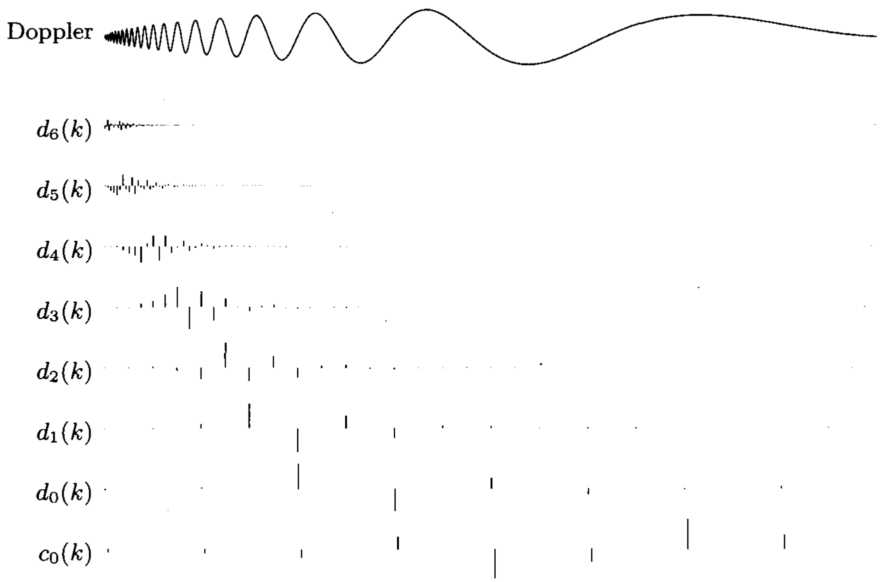


Figure 2.8. Discrete Wavelet Transform of a Doppler, using $\psi_{D8'}$ with a gain of $\sqrt{2}$ for each higher scale.

2.8 An Example of the Haar Wavelet System

In this section, we can illustrate our mathematical discussion with a more complete example. In 1910, Haar [Haa10] showed that certain square wave functions could be translated and scaled to create a basis set that spans L^2. This is illustrated in Figure 2.11. Years later, it was seen that Haar's system is a particular wavelet system.

If we choose our scaling function to have compact support over $0 \le t \le 1$, then a solution to (2.13) is a scaling function that is a simple rectangle function

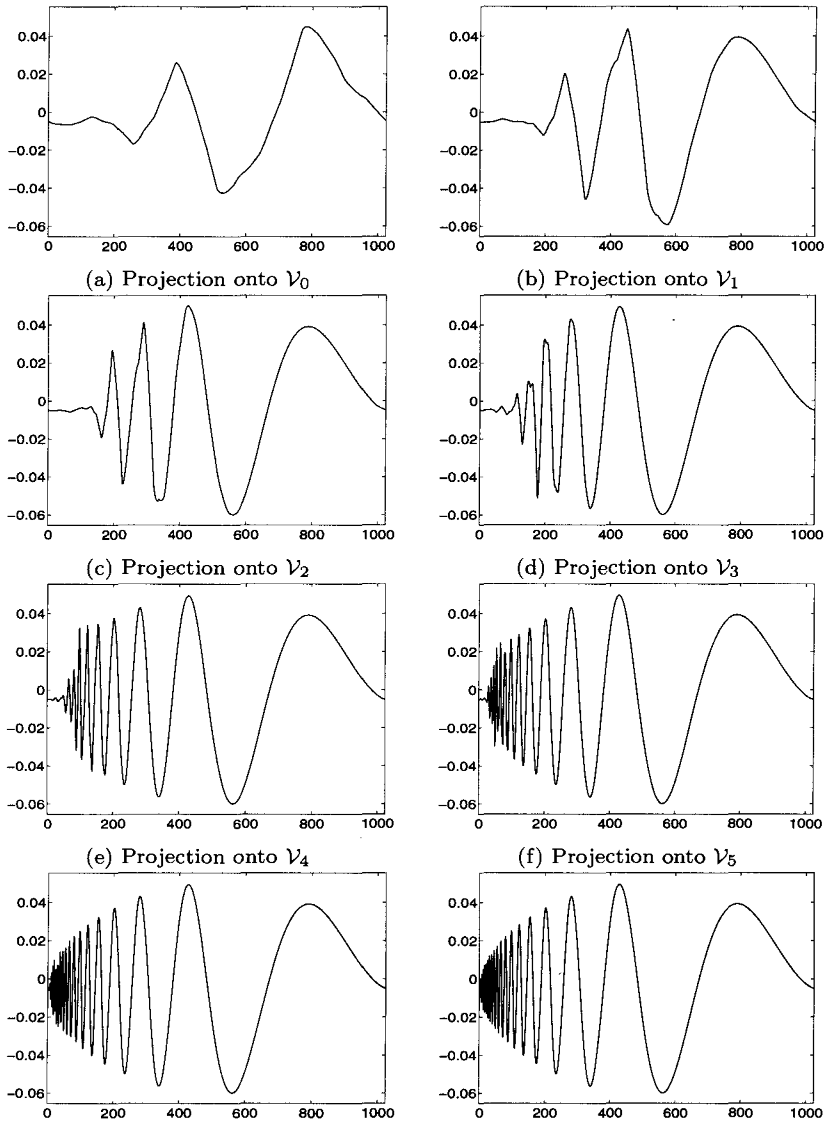
$$\varphi(t) = \begin{cases} 1 & \text{if } 0 < t < 1 \\ 0 & \text{otherwise} \end{cases} \tag{2.42}$$

with only two nonzero coefficients $h(0) = h(1) = 1/\sqrt{2}$ and (2.24) and (2.25) require the wavelet to be

$$\psi(t) = \begin{cases} 1 & \text{for } 0 < t < 0.5 \\ -1 & \text{for } 0.5 < t < 1 \\ 0 & \text{otherwise} \end{cases} \tag{2.43}$$

with only two nonzero coefficients $h_1(0) = 1/\sqrt{2}$ and $h_1(1) = -1/\sqrt{2}$.

$\mathcal{V}_0$ is the space spanned by $\varphi(t-k)$ which is the space of piecewise constant functions over integers, a rather limited space, but nontrivial. The next higher resolution space $\mathcal{V}_1$ is spanned by $\varphi(2t-k)$ which allows a somewhat more interesting class of signals which does include $\mathcal{V}_0$. As we consider higher values of scale j, the space $\mathcal{V}_j$ spanned by $\varphi(2^j t - k)$ becomes better able to approximate arbitrary functions or signals by finer and finer piecewise constant functions.



(a) Projection onto $\mathcal{V}_0$

(b) Projection onto $\mathcal{V}_1$

(c) Projection onto $\mathcal{V}_2$

(d) Projection onto $\mathcal{V}_3$

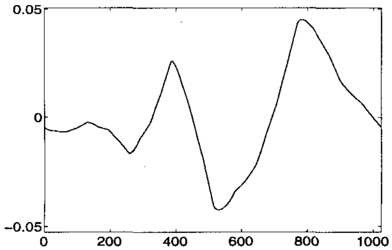
(e) Projection onto $\mathcal{V}_4$

(f) Projection onto $\mathcal{V}_5$

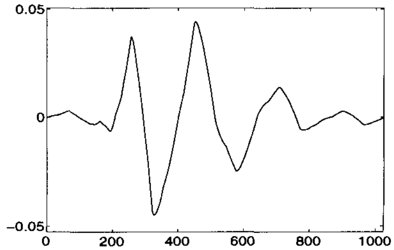
(g) Projection onto $\mathcal{V}_6$

(h) Projection onto $\mathcal{V}_7$

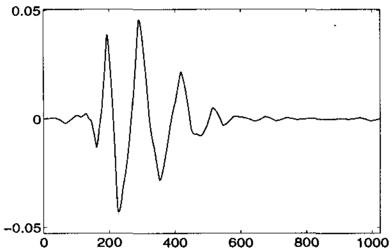
Figure 2.9. Projection of the Doppler Signal onto $\mathcal{V}$ Spaces using $\phi_{D8'}$



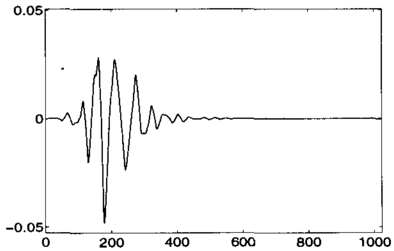
(a) Projection onto $\mathcal{V}_0$



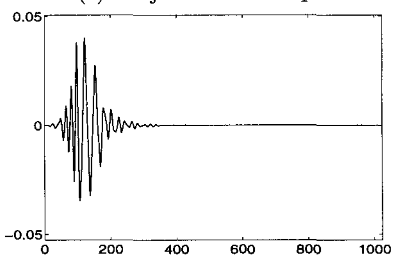
(b) Projection onto $\mathcal{W}_0$



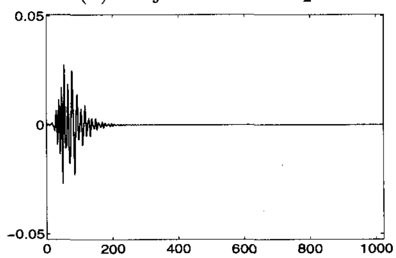
(c) Projection onto $\mathcal{W}_1$



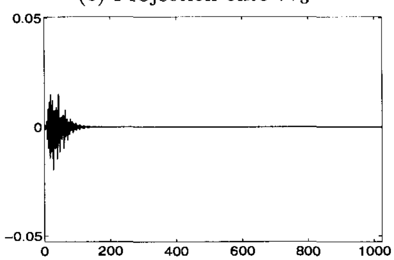
(d) Projection onto $\mathcal{W}_2$



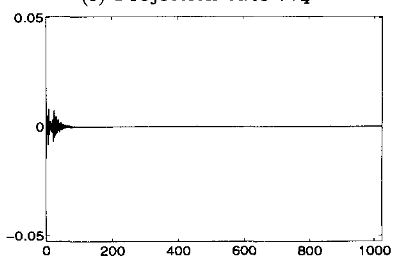
(e) Projection onto $\mathcal{W}_3$



(f) Projection onto $\mathcal{W}_4$



(g) Projection onto $\mathcal{W}_5$



(h) Projection onto $\mathcal{W}_6$

Figure 2.10. Projection of the Doppler Signal onto $\mathcal{W}$ Spaces using $\psi_{D8'}$

Haar showed that as $j \to \infty$, $\mathcal{V}_j \to L^2$. We have an approximation made up of step functions approaching any square integrable function.

The Haar functions are illustrated in Figure 2.11 where the first column contains the simple constant basis function that spans $\mathcal{V}_0$, the second column contains the unit pulse of width one half and the one translate necessary to span $\mathcal{V}_1$. The third column contains four translations of a pulse of width one fourth and the fourth contains eight translations of a pulse of width one eighth. This shows clearly how increasing the scale allows greater and greater detail to be realized. However, using only the scaling function does not allow the decomposition described in the introduction. For that we need the wavelet. Rather than use the scaling functions $\varphi(8t-k)$ in $\mathcal{V}_3$, we will use the orthogonal decomposition

$$\mathcal{V}_3 \quad = \quad \mathcal{V}_2 \oplus \mathcal{W}_2 \tag{2.44}$$

which is the same as

$$\overline{\underset{k}{\mathrm{Span}}\{\varphi(8t-k)\}} \quad = \quad \overline{\underset{k}{\mathrm{Span}}\{\varphi(4t-k)\}} \oplus \overline{\underset{k}{\mathrm{Span}}\{\psi(4t-k)\}} \tag{2.45}$$

which means there are two sets of orthogonal basis functions that span $\mathcal{V}_3$, one in terms of $j=3$ scaling functions, and the other in terms of half as many coarser $j=2$ scaling functions plus the details contained in the $j=2$ wavelets. This is illustrated in Figure 2.12.

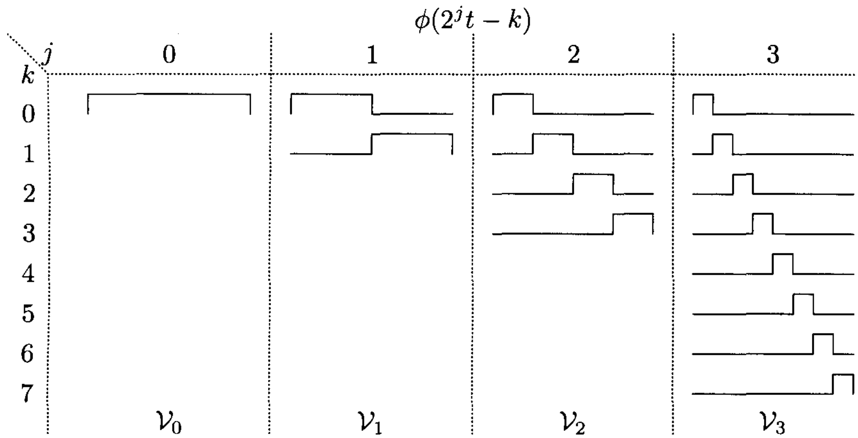


Figure 2.11. Haar Scaling Functions that Span $\mathcal{V}_j$

The $\mathcal{V}_2$ can be further decomposed into

$$\mathcal{V}_2 \quad = \quad \mathcal{V}_1 \oplus \mathcal{W}_1 \tag{2.46}$$

which is the same as

$$\overline{\underset{k}{\mathrm{Span}}\{\varphi(4t-k)\}} \quad = \quad \overline{\underset{k}{\mathrm{Span}}\{\varphi(2t-k)\}} \oplus \overline{\underset{k}{\mathrm{Span}}\{\psi(2t-k)\}} \tag{2.47}$$

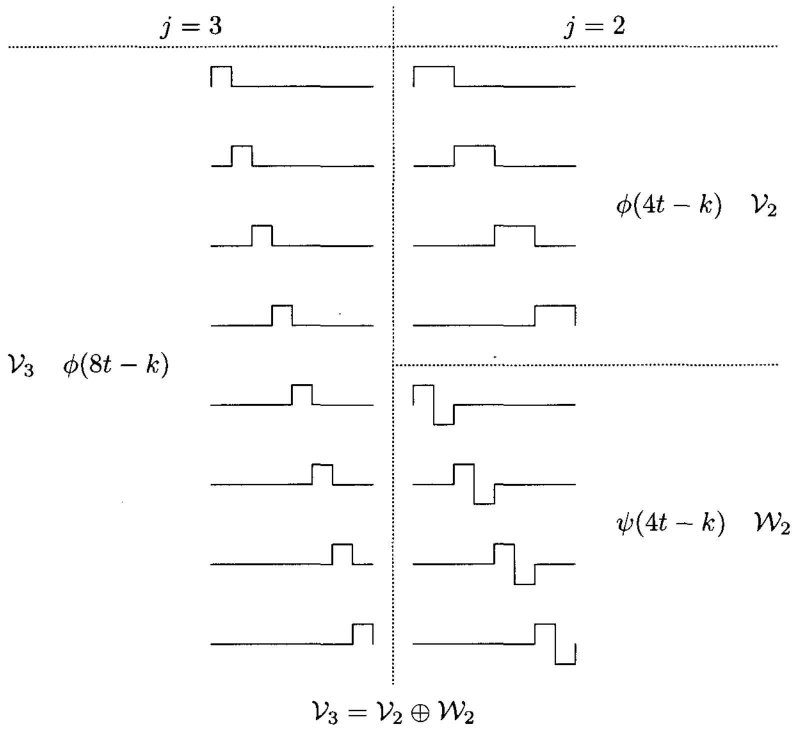


Figure 2.12. Haar Scaling Functions and Wavelets Decomposition of $\mathcal{V}_3$

and this is illustrated in Figure 2.14. This gives $\mathcal{V}_1$ also to be decomposed as

$$\mathcal{V}_1 = \mathcal{V}_0 \oplus \mathcal{W}_0 \tag{2.48}$$

which is shown in Figure 2.13. By continuing to decompose the space spanned by the scaling function until the space is one constant, the complete decomposition of $\mathcal{V}_3$ is obtained. This is symbolically shown in Figure 2.16.

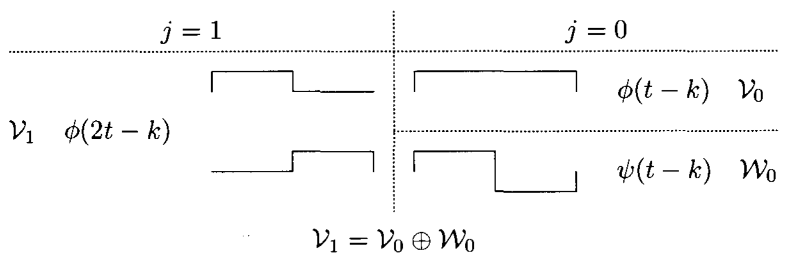


Figure 2.13. Haar Scaling Functions and Wavelets Decomposition of $\mathcal{V}_1$

Finally we look at an approximation to a smooth function constructed from the basis elements in $\mathcal{V}_3 = \mathcal{V}_0 \oplus \mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \mathcal{W}_2$. Because the Haar functions form an orthogonal basis in each subspace, they can produce an optimal least squared error approximation to the smooth function. One can easily imagine the effects of adding a higher resolution "layer" of functions to $\mathcal{W}_3$ giving an approximation residing in $\mathcal{V}_4$. Notice that these functions satisfy all of the conditions that we have considered for scaling functions and wavelets. The basic wavelet is indeed an oscillating function which, in fact, has an average of zero and which will produce finer and finer detail as it is scaled and translated.

The multiresolution character of the scaling function and wavelet system is easily seen from Figure 2.12 where a signal residing in $\mathcal{V}_3$ can be expressed in terms of a sum of eight shifted scaling functions at scale $j = 3$ or a sum of four shifted scaling functions and four shifted wavelets at a scale of $j = 2$. In the second case, the sum of four scaling functions gives a low resolution approximation to the signal with the four wavelets giving the higher resolution "detail". The four shifted scaling functions could be further decomposed into coarser scaling functions and wavelets as illustrated in Figure 2.14 and still further decomposed as shown in Figure 2.13.

Figure 2.15 shows the Haar approximations of a test function in various resolutions. The signal is an example of a mixture of a pure sine wave which would have a perfectly localized Fourier domain representation and a two discontinuities which are completely localized in time domain. The component at the coarsest scale is simply the average of the signal. As we include more and more wavelet scales, the approximation becomes close to the original signal.

This chapter has skipped over some details in an attempt to communicate the general idea of the method. The conditions that can or must be satisfied and the resulting properties, together with examples, are discussed in the following chapters and/or in the references.

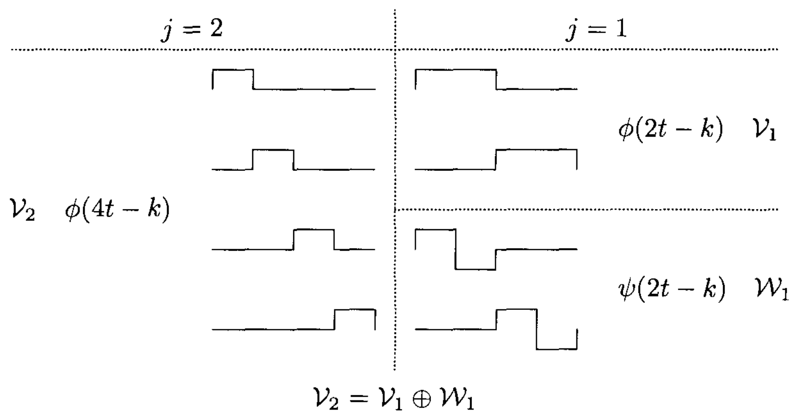


Figure 2.14. Haar Scaling Functions and Wavelets Decomposition of $\mathcal{V}_2$

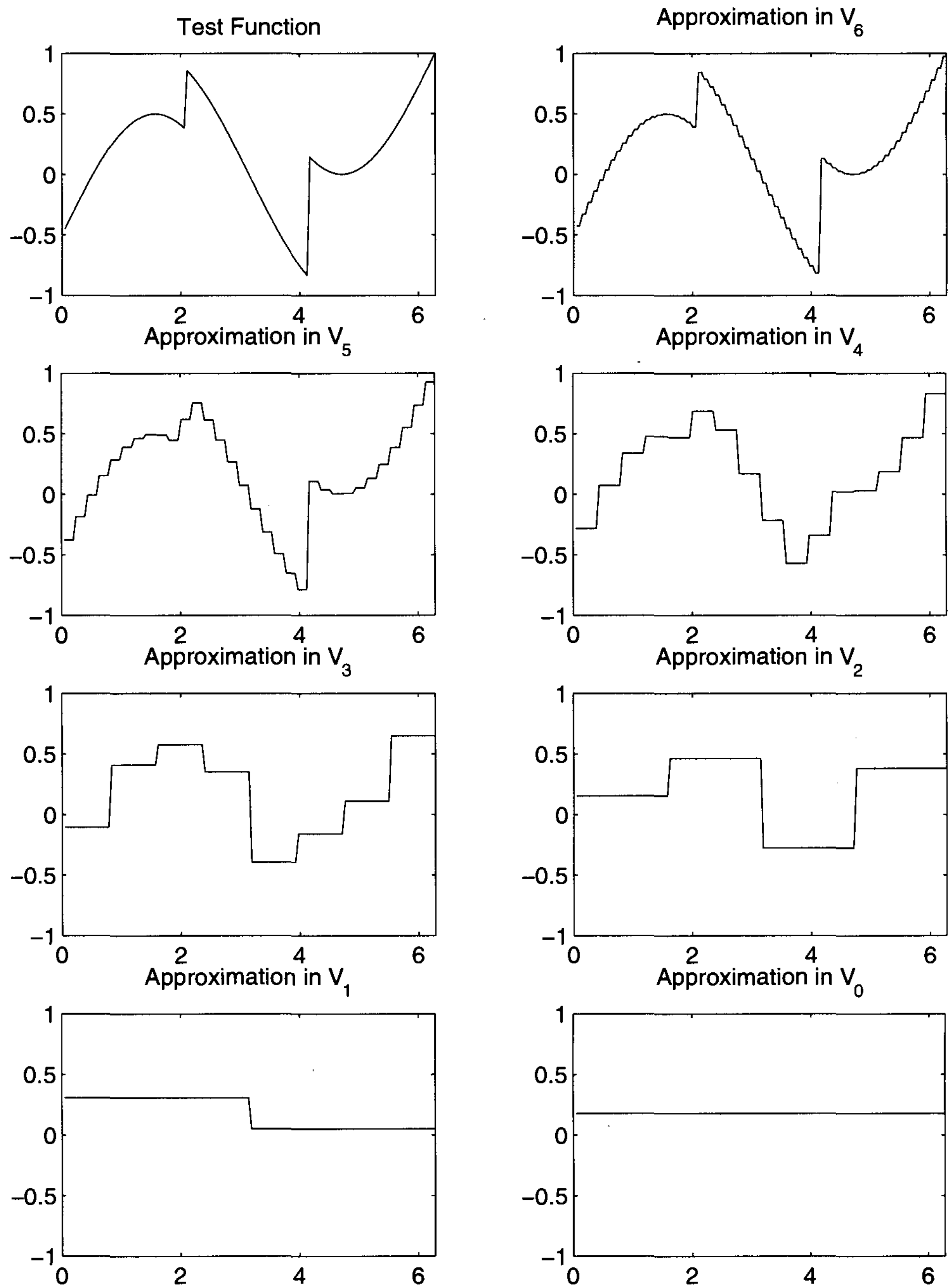
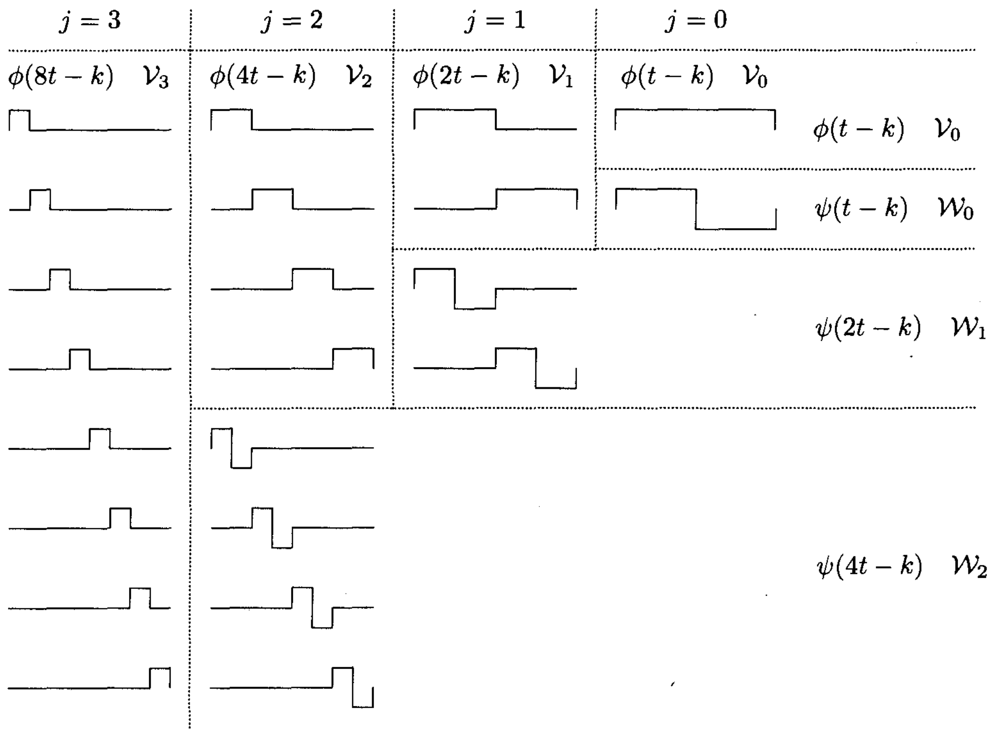


Figure 2.15. Haar Function Approximation in $\mathcal{V}_j$



$$\mathcal{V}_3 = \mathcal{V}_0 \oplus \mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \mathcal{W}_2$$

Figure 2.16. Haar Scaling Functions and Wavelets Decomposition of $\mathcal{V}_3$