Lab 8

Quadrature formulas (2)

The rectangle (midpoint) quadrature formula is

$$\int_a^b f(x)dx = (b-a)f\left(\frac{a+b}{2}\right) + R_1(f).$$

The repeated rectangle (midpoint) quadrature formula is

$$\int_a^b f(x)dx = \frac{b-a}{n}\sum_{i=1}^{n} f(x_i) + R_n(f),$$

with $x_1 = a + \frac{b-a}{2n}$, $x_i = x_1 + (i-1)\frac{b-a}{n},\ i = 2, ..., n.$

Problems:

1. a) Use the rectangle formula to evaluate the integral

$$\int_1^{1.5} e^{-x^2} dx.$$

   b) Use the repeated rectangle formula, for $n = 150$ and 500, to evaluate the integral

$$\int_1^{1.5} e^{-x^2} dx.$$

   (*Result:* 0.1094)

2. Use Romberg's algorithm for trapezium and Simpson's formulas to approximate the integral

$$\int_0^1 \frac{2}{1+x^2} dx,$$

   with precision $\varepsilon = 10^{-5}$.

3. Plot the graph of $f : [1, 3] \to \mathbb{R}$, $f(x) = \frac{100}{x^2} \sin \frac{10}{x}$. Use an adaptive quadrature algorithm for Simpson's formula to approximate the integral

$$\int_1^3 f(x)dx,$$

with precision $\varepsilon = 10^{-4}$. Compare the obtained result with the one obtained applying repeated Simpson formula for $n = 50$ and 100. (The exact value is -1.4260247818.)

Facultative problem

Quadrature formula of Gauss type for double integral

Consider the integral $I = \int_a^b \int_c^d f(x, y)dydx$.

We change the variable y from $[c, d]$, in variable t from $[-1, 1]$. The linear transformation gives:

$$f(x, y) = f\left(\frac{(d-c)t + d + c}{2}\right) \quad dy = \frac{d-c}{2}dt.$$

$$\int_c^d f(x, y)dy = \int_{-1}^1 f\left(x, \frac{(d-c)t + d + c}{2}\right) dt.$$

We obtain

$$\int_a^b \int_c^d f(x, y)dx \approx \int_a^b \frac{d-c}{2} \sum_{j=1}^n c_{n,j} f\left(x, \frac{(d-c)r_{n,j} + d + c}{2}\right) dt,$$

with $c_{n,j}$ and $r_{n,j}$ given in tables. Then, it is changed the interval $[a, b]$ in the interval $[-1, 1]$ and it is repeated the same procedure.

Algorithm:

INPUT: a,b,c,d,m,n

the coefficients $c_{i,j}$ and nodes r_{ij} for $i = \max\{m, n\}$ and $1 \le j \le i$

OUTPUT: the approximant J of the integral I

$h_1 = (b-a)/2;$

$h_2 = (b+a)/2;$

$J = 0.$

For $i = 1, 2, ..., m$ do

$JX = 0$

$x = h_1 r_{m,i} + h_2;$

$k_1 = (d-c)/2;$

$k_2 = (d+c)/2.$

For $j = 1, 2, ..., n$ do

$y = k_1 r_{n,j} + k_2;$
$Q = f(x, y);$
$JX = JX + c_{n,j} Q.$
end{for}
Let $J = J + c_{m,i} \cdot k_1 \cdot JX.$
end{for}
$J = h_1 J$

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1. The volume of a solid is given by $\int_{0.1}^{0.5} \int_{0.01}^{0.25} e^{\frac{y}{x}} dydx$. Approximate the volume applying the algorithm for Gauss type quadratures for double integrals for $m = n = 5$. Compare the result with the one obtained applying Simpson's algorithm for double integrals for $m = n = 10$. (Result: 0.178571)

We know the following data:

Nodes $r_{5,i}$	*Coefficients* $c_{5,i}$
0.9062	0.2369
0.5385	0.4786
0	0.5689
−0.5385	0.4786
−0.9062	0.2369