Geometry
Problem booklet

Assoc. Prof. Cornel Pintea

E-mail: `cpintea math.ubbcluj.ro`

Contents

Module leader: Assoc. Prof. Cornel Pintea

Department of Mathematics,
"Babeş-Bolyai" University
400084 M. Kogălniceanu 1,
Cluj-Napoca, Romania

1 Week 6: Products of vectors

This section briefly presents the theoretical aspects covered in the tutorial. For more details please check the lecture notes.

1.1 Brief theoretical background. Products of vectors

1.1.1 The vector product

If $[\vec{i}, \vec{j}, \vec{k}]$ is an orthonormal basis, observe that $\vec{i} \times \vec{j} \in \{-\vec{k}, \vec{k}\}$. We say that the orthonormal basis $[\vec{i}, \vec{j}, \vec{k}]$ is *direct* if $\vec{i} \times \vec{j} = \vec{k}$. If, on the contrary, $\vec{i} \times \vec{j} = -\vec{k}$, we say that the orthonormal basis $[\vec{i}, \vec{j}, \vec{k}]$ is *inverse*. Therefore, if $[\vec{i}, \vec{j}, \vec{k}]$ is a direct orthonormal basis, then $\vec{i} \times \vec{j} = \vec{k}$, $\vec{j} \times \vec{k} = \vec{i}$, $\vec{k} \times \vec{i} = \vec{j}$ and obviously $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}$.

Proposition 1.1. *If $[\vec{i}, \vec{j}, \vec{k}]$ is a direct orthonormal basis and*

$$\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}, \ \vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k},$$

then

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2)\vec{i} + (a_3b_1 - a_1b_3)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}, \tag{1.1}$$

or, equivalently,

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k} \tag{1.2}$$

One can rewrite formula (1.1) in the form

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{1.3}$$

the right hand side determinant being understood in the sense of its cofactor expansion along the first line.

1.1.2 Applications of the vector product

• **The area of the triangle ABC.** $S_{ABC} = \frac{1}{2} \| \overrightarrow{AB} \| \cdot \| \overrightarrow{AC} \| \sin \widehat{BAC} = \frac{1}{2} \| \overrightarrow{AB} \times \overrightarrow{AC} \|$. Since the coordinates of the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are

$$(x_B - x_A, y_B - x_A, z_B - z_A) \text{ and } (x_C - x_A, y_C - x_A, z_C - z_A)$$

respectively, we deduce that

$$S_{ABC} = \frac{1}{2} \left\| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_B - x_A & y_B - x_A & z_B - z_A \\ x_C - x_A & y_C - x_A & z_C - z_A \end{vmatrix} \right\|,$$

or, equivalently

$$4S^2_{ABC} = \begin{vmatrix} y_B - y_A & z_B - z_A \\ y_C - y_A & z_C - z_A \end{vmatrix}^2 + \begin{vmatrix} z_B - z_A & x_B - x_A \\ z_C - z_A & x_C - x_A \end{vmatrix}^2 + \begin{vmatrix} x_B - x_A & y_B - y_A \\ x_C - x_A & y_C - y_A \end{vmatrix}^2.$$

• **The distance from one point to a straight line.**

a) *The distance $\delta(A, BC)$ from the point $A(x_A, y_A, z_A)$ to the straight line BC*, where $B(x_B, y_B, z_B)$ şi $C(x_C, y_C, z_C)$. Since

$$S_{ABC} = \frac{\| \overrightarrow{BC} \| \cdot \delta(A, BC)}{2}$$

rezultă că

$$\delta^2(A, BC) = \frac{4S^2_{ABC}}{\| \overrightarrow{BC} \|^2}.$$

Thus, we obtain

$$\delta^2(A, BC) = \frac{\begin{vmatrix} y_B - y_A & z_B - z_A \\ y_C - y_A & z_C - z_A \end{vmatrix}^2 + \begin{vmatrix} z_B - z_A & x_B - x_A \\ z_C - z_A & x_C - x_A \end{vmatrix}^2 + \begin{vmatrix} x_B - x_A & y_B - y_A \\ x_C - x_A & y_C - y_A \end{vmatrix}^2}{(x_C - x_B)^2 + (y_C - y_B)^2 + (z_C - z_B)^2}.$$

(b) The distance from $\delta(A, d)$ from one point $A(A_A, y_A, z_A)$ to the straight line

$$d : \frac{x - x_0}{p} + \frac{y - y_0}{p} + \frac{z - z_0}{p}.$$

$$\delta(A, d) = \frac{\| \vec{d} \times \overrightarrow{A_0A} \|}{\vec{d}}, \quad \text{where } A_0(x_0, y_0, z_0) \in \delta.$$

Since

$$\begin{aligned} \vec{d} \times \overrightarrow{A_0A} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ p & q & r \\ x_A - x_0 & y_A - y_0 & z_A - z_0 \end{vmatrix} \\ &= \begin{vmatrix} q & r \\ y_A - y_0 & z_A - z_0 \end{vmatrix} \vec{i} + \begin{vmatrix} r & p \\ z_A - z_0 & x_A - x_0 \end{vmatrix} \vec{j} + \begin{vmatrix} p & q \\ x_A - x_0 & y_A - y_0 \end{vmatrix} \vec{k} \end{aligned}$$

it follows that

$$\delta(A, d) = \frac{\sqrt{\begin{vmatrix} q & r \\ y_A - y_0 & z_A - z_0 \end{vmatrix}^2 + \begin{vmatrix} r & p \\ z_A - z_0 & x_A - x_0 \end{vmatrix}^2 + \begin{vmatrix} p & q \\ x_A - x_0 & y_A - y_0 \end{vmatrix}^2}}{\sqrt{p^2 + q^2 + r^2}}.$$

1.1.3 The double vector (cross) product

The *double vector (cross) product* of the vectors $\vec{a}, \vec{b}, \vec{c}$ is the vector $\vec{a} \times (\vec{b} \times \vec{c})$

Proposition 1.2 1.2. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\, \vec{b} - (\vec{a} \cdot \vec{b})\, \vec{c} = \begin{vmatrix} \vec{b} & \vec{c} \\ \vec{a} \cdot \vec{b} & \vec{a} \cdot \vec{c} \end{vmatrix}, \ \forall\ \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}.$

Corollary 1.3. *1.* $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\, \vec{b} - (\vec{b} \cdot \vec{c})\, \vec{a} = \begin{vmatrix} \vec{b} & \vec{a} \\ \vec{c} \cdot \vec{b} & \vec{c} \cdot \vec{a} \end{vmatrix}, \forall\, \vec{a}, \vec{b}, \vec{c} \in \mathcal{V};$

2. $\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})=\vec{0}$, $\forall\, \vec{a},\vec{b},\vec{c}\in\mathcal{V}$ *(Jacobi's identity).*

Proof. (1)

$$\begin{aligned}(\vec{a}\times\vec{b})\times\vec{c} &= -\vec{c}\cdot(\vec{a}\times\vec{b}) = -[(\vec{c}\cdot\vec{b})\,\vec{a}-(\vec{c}\cdot\vec{a})\,\vec{b}]\\ &= (\vec{a}\cdot\vec{c})\,\vec{b}-(\vec{b}\cdot\vec{c})\,\vec{a}=\begin{vmatrix}\vec{b} & \vec{a}\\ \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{a}\end{vmatrix}.\end{aligned}$$

(2)

$$\begin{aligned}&\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})\\ &=(\vec{a}\cdot\vec{c})\,\vec{b}-(\vec{a}\cdot\vec{b})\,\vec{c}+(\vec{b}\cdot\vec{a})\,\vec{c}-(\vec{b}\cdot\vec{c})\,\vec{a}+(\vec{c}\cdot\vec{b})\,\vec{a}-(\vec{c}\cdot\vec{a})\,\vec{b}=\vec{0}\,.\end{aligned}$$

□

1.1.4 The triple scalar product

The *triple scalar product* $(\vec{a},\vec{b},\vec{c})$ of the vectors $\vec{a},\vec{b},\vec{c}$ is the real number $(\vec{a}\times\vec{b})\cdot\vec{c}$.

Proposition 1. *If* $[\vec{i},\vec{j},\vec{k}]$ *is a direct orthonormal basis and*

$$\begin{aligned}\vec{a}&=a_1\,\vec{i}+a_2\,\vec{j}+a_3\,\vec{k}\\ \vec{b}&=b_1\,\vec{i}+b_2\,\vec{j}+b_3\,\vec{k}\\ \vec{c}&=c_1\,\vec{i}+c_2\,\vec{j}+c_3\,\vec{k},\end{aligned}$$

then

$$(\vec{a},\vec{b},\vec{c})=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix} \tag{1.4}$$

Proof.

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}=\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}\vec{i}-\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}\vec{j}+\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\vec{k}\,.$$

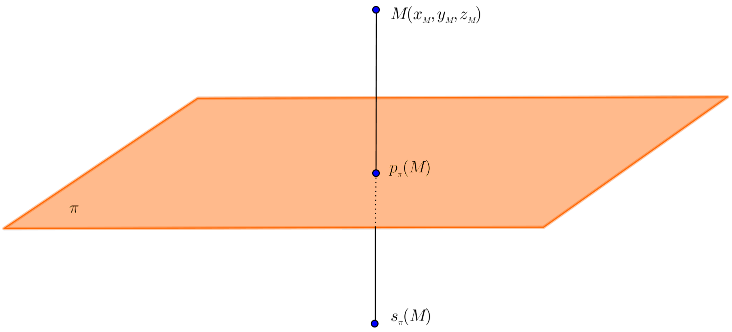
Thus

$$\begin{aligned}(\vec{a},\vec{b},\vec{c}) &= (\vec{a}\times\vec{b})\cdot\vec{c}=\vec{c}\cdot(\vec{a}\times\vec{b})\\ &= c_1\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}-c_2\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}+c_3\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}.\end{aligned}$$

□

1.1.5 Appendix: Orthogonal projections and orthogonal symmetries

• The orthogonal projection on a plane π. For a given plane $\pi : Ax + By + Cz + D = 0$ and a given point $M(x_M, y_M, z_M)$, we shall determine the coordinates of its orthogonal projection on the plane π, as well as the coordinates of its (orthogonal) symmetric with respect to π. The equation of the plane and the coordinates of M are considered with respect to some cartezian coordinate system $R = (O, \overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k})$. In this respect we consider the orthogonal line on π which passes through M.



Its parametric equations are

$$\begin{cases} x = x_M + At \\ y = y_M + Bt \\ z = z_M + Ct \end{cases}, \; t \in \mathbb{R}. \tag{1.5}$$

The orthogonal projection $p_\pi(M)$ of M on the plane π is at its intersection point with the orthogonal line (1.5) and the value of $t \in \mathbb{R}$ for which this orthogonal line (1.5) puncture the plane π can be determined by imposing the condition on the point of coordinates $(x_M + At, y_M + Bt, z_M + Ct)$ to verify the equation of the plane, namely $A(x_M + At) + B(y_M + Bt) + C(z_M + Ct) + D = 0$. Thus

$$t = -\frac{Ax_M + By_M + Cz_M + D}{A^2 + B^2 + C^2} = -\frac{F(x_M, y_M, z_M)}{\| \overrightarrow{n}_\pi \|^2},$$

where $F(x, y, z) = Ax + By + Cz + D$ şi $\overrightarrow{n}_\pi = A \overrightarrow{i} + B \overrightarrow{j} + C \overrightarrow{k}$ is the normal vector of the plasne π.

• The orthogonal projection on the plane π.

The coordinates of the orthogonal projection $p_\pi(M)$ of M on th eplane π are

$$\begin{cases} x_M - A\dfrac{F(x_M, y_M, z_M)}{A^2 + B^2 + C^2} \\ y_M - B\dfrac{F(x_M, y_M, z_M)}{A^2 + B^2 + C^2} \\ z_M - C\dfrac{F(x_M, y_M, z_M)}{A^2 + B^2 + C^2}. \end{cases}$$

Therefore, the position vector of the orthogonal projection $p_\pi(M)$ is

$$\overrightarrow{Op_\pi(M)} = \overrightarrow{OM} - \frac{F(M)}{\| \overrightarrow{n}_\pi \|^2} \overrightarrow{n}_\pi . \tag{1.6}$$

• **The orthogonal symmetry with respect to the plane π.** In order to find the position vector of the orthogonally symmetric point $s_\pi(M)$ of M w.r.t. π, we use the relation

$$\overrightarrow{Op_\pi(M)} = \frac{1}{2}\left(\overrightarrow{OM} + \overrightarrow{Os_\pi(M)}\right),$$

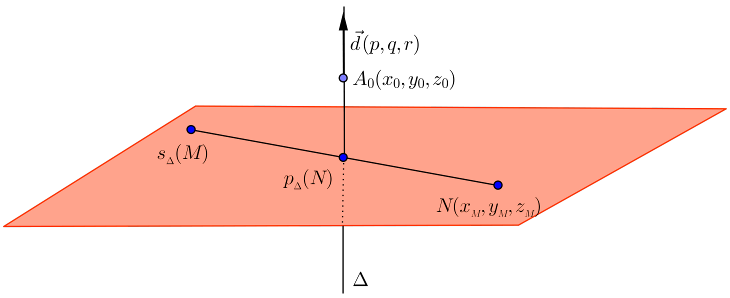
namely

$$\overrightarrow{Os_\pi(M)} = 2\,\overrightarrow{Op_\pi(M)} - \overrightarrow{OM} = \overrightarrow{OM} - 2\frac{F(M)}{\| \overrightarrow{n}_\pi \|^2}\,\overrightarrow{n}_\pi .$$

• **The orthogonal projection on a line Δ.** For a given line

$$\Delta : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

and a point $N(x_N, y_N, z_N)$, we shall find the coordinates of its orthogonal projection on the line Δ, as well as the coordinates of the orthogonally symmetric point M with respect to Δ. The equations of the line and the coordinates of the point N are considered with respect to an orthonormal coordinate system $R = (O, \overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k})$. In this respect we consider the plane $p(x - x_N) + q(y - y_N) + r(z - z_N) = 0$ orthogonal on the line Δ which passes through the point N.



The parametric equations of the line Δ are

$$\begin{cases} x = x_0 + pt \\ y = y_0 + qt \\ z = z_0 + rt \end{cases}, \ t \in \mathbb{R}. \tag{1.7}$$

The orthogonal projection of N on the line Δ is at its intersection point and the plane $p(x - x_N) + q(y - y_N) + r(z - z_N) = 0$, and the value of $t \in \mathbb{R}$ for which the line Δ puncture the orthogonal plane $p(x - x_N) + q(y - y_N) + r(z - z_N) = 0$ can be found by imposing the condition on the point of coordinate $(x_0 + pt, y_0 + qt, z_0 + rt)$ to verify the equation of the plane, namely $p(x_0 + pt - x_N) + q(y_0 + qt - y_N) + r(z_0 + rt - z_N) = 0$. Thus

$$t = -\frac{p(x_0 - x_N) + q(y_0 - y_N) + r(z_0 - z_N)}{p^2 + q^2 + r^2} = -\frac{G(x_0, y_0, z_0)}{\| \overrightarrow{d}_\Delta \|^2},$$

where $G(x, y, z) = p(x - x_N) + q(y - y_N) + r(z - z_N)$ and $\overrightarrow{d}_\pi = p\,\overrightarrow{i} + q\,\overrightarrow{j} + r\,\overrightarrow{k}$ is the director vectoir of the line Δ. Ths coordinates of the orthogonal projection $p_\Delta(N)$ of N on the line Δ are therefore

$$\begin{cases} x_0 - p\dfrac{G(x_0, y_0, z_0)}{p^2 + q^2 + r^2} \\ y_0 - q\dfrac{G(x_0, y_0, z_0)}{p^2 + q^2 + r^2} \\ z_0 - r\dfrac{G(x_0, y_0, z_0)}{p^2 + q^2 + r^2}. \end{cases}$$

Thus, the position vector of the orthogonal projection $p_{\Delta}(N)$ is

$$\overrightarrow{Op_{\Delta}(N)}=\overrightarrow{OA_0} - \frac{G(A_0)}{\| \overrightarrow{d}_{\Delta} \|^2} \overrightarrow{d}_{\Delta}, \tag{1.8}$$

where $A_0(x_0, y_0, z_0) \in \Delta$.

• **The orthogonal symmetry with respect to a line Δ.** In order to find the position vector of the orthogonally symmetric point $s_{\Delta}(N)$ of N with respect to the line Δ we use the relation

$$\overrightarrow{Op_{\Delta}(N)}= \frac{1}{2}\left(\overrightarrow{ON} + \overrightarrow{Os_{\Delta}(N)}\right)$$

i.e.

$$\overrightarrow{Os_{\Delta}(N)}= 2\,\overrightarrow{Op_{\Delta}(N)} - \overrightarrow{ON}= 2\,\overrightarrow{OA_0} -2\frac{G(A_0)}{\| \overrightarrow{d}_{\Delta} \|^2} \overrightarrow{d}_{\Delta} - \overrightarrow{ON}\,.$$

Remark 1.4. *1. The distance from the point $M(x_M, y_M, z_M)$ to the plane $\pi : Ax + By + Cz + D = 0$ can be equally computed by means of* (1.6). *Indeed,*

$$\begin{aligned}\delta(M, \pi) &= \| \overrightarrow{Mp_{\pi}(M)} \| = \| \overrightarrow{Op_{\pi}(M)} - \overrightarrow{OM} \| \\ &= \left| -\tfrac{F(M)}{\|\overrightarrow{n}_{\pi}\|^2}\right| \cdot \| \overrightarrow{n}_{\pi} \| = \frac{|F(M)|}{\| \overrightarrow{n}_{\pi} \|}.\end{aligned}$$

2. *The distance from the point $N(x_N, y_N, z_N)$ to the straight line $\Delta : \dfrac{x - x_0}{p} = \dfrac{y - y_0}{q} = \dfrac{z - z_0}{r}$ can be computed by means of* (1.8). *Indeed,*

$$\begin{aligned}\delta(M, \Delta) &= \| \overrightarrow{Np_{\Delta}(N)} \| = \| \overrightarrow{NO} + \overrightarrow{Op_{\Delta}(N)} \| \\ &= \left\|\overrightarrow{NA_0} - \frac{G(A_0)}{\| \overrightarrow{d}_{\Delta} \|^2} \overrightarrow{d}_{\Delta}\right\| = \left\|\overrightarrow{NA_0} - \frac{\overrightarrow{d}_{\Delta} \cdot \overrightarrow{NA_0}}{\| \overrightarrow{d}_{\Delta} \|^2} \overrightarrow{d}_{\Delta}\right\|.\end{aligned} \tag{1.9}$$

Proposition 1.5. *Taking into account the formula* (1.9) *for the distance $\delta(M, \Delta)$ from the point $N(x_N, y_N, z_N)$ to the straight line $\Delta : \dfrac{x - x_0}{p} = \dfrac{y - y_0}{q} = \dfrac{z - z_0}{r}$ as well as Proposition 1.2 we deduce that*

$$\begin{aligned}\delta(M, \Delta) &= \left\| \overrightarrow{NA_0} - \frac{\overrightarrow{d}_{\Delta} \cdot \overrightarrow{NA_0}}{\| \overrightarrow{d}_{\Delta} \|^2} \overrightarrow{d}_{\Delta} \right\| = \frac{\| \left(\overrightarrow{d}_{\Delta} \cdot \overrightarrow{d}_{\Delta}\right) \overrightarrow{NA_0} - \left(\overrightarrow{d}_{\Delta} \cdot \overrightarrow{NA_0}\right) \overrightarrow{d}_{\Delta} \|}{\| \overrightarrow{d}_{\Delta} \|^2} \\ &= \frac{\| \overrightarrow{d}_{\Delta} \times (\overrightarrow{NA_0} \times \overrightarrow{d}_{\Delta})\|}{\| \overrightarrow{d}_{\Delta} \|^2} = \frac{\| \overrightarrow{NA_0} \times \overrightarrow{d}_{\Delta} \|}{\| \overrightarrow{d}_{\Delta} \|}.\end{aligned}$$

1.2 Problems

1. If two pairs of opposite edges of the tetrahedron $ABCD$ are perpendicular ($AB \perp CD$, $AD \perp BC$), show that

(a) The third pair of opposite edges are perpendicular too ($AC \perp BD$).

(b) $AB^2 + CD^2 = AC^2 + BD^2 = BC^2 + AD^2$.

(c) The heights of the tetrahedron are concurrent.
(Such a tetrahedron is said to be orthocentric)

2. Two triangles ABC şi $A'B'C'$ are said to be *orthologic* if they are in the same plane and the perpendicular lines from the vertices A', B', C' on the sides BC, CA, AB are concurrent. Show that, in this case, the perpendicular lines from the vertices A, B, C on the sides $B'C'$, $C'A'$, $A'B'$ are concurrent too.

3. Show that $\| \vec{a} \times \vec{b} \| \leq \| \vec{a} \| \cdot \| \vec{b} \|, \forall \vec{a}, \vec{b}, \in \mathcal{V}$.

Solution. $\| \vec{a} \times \vec{b} \| = \| \vec{a} \| \cdot \| \vec{b} \| \sin(\widehat{\vec{a}, \vec{b}}) \leq \| \vec{a} \| \cdot \| \vec{b} \|$.

4. Let $\vec{a}, \vec{b}, \vec{c}$ be noncollinear vectors. Show that the necesssary and sufficient condition for the existence of a triangle ABC with the properties $\overrightarrow{BC} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$, $\overrightarrow{AB} = \vec{c}$ is

$$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a} .$$

From the equalities of the norms deduce the low of sines.

5. Show that the sum of some outer-pointing vectors perpendicular on the faces of a tetrahedron which are proportional to the areas of the faces is the zero vector.

6. Find the orthogonal projection

(a) of the point $A(1, 2, 1)$ on the plane $\pi : \ x + y + 3z + 5 = 0$.

(b) of the point $B(5, 0, -2)$ on the straight line (d) $\dfrac{x-2}{3} = \dfrac{y-1}{2} = \dfrac{z-3}{4}$.

7. Compute the distance from the point $A(3, 1, -1)$ to the plane $\pi : 22x + 4y - 20z - 45 = 0$.

8. Find the equations of the bisector planes of the dihedral angles of the planes

$$(\pi_1)\ 2x + y - 3z - 5 = 0,\ (\pi_2)\ x + 3y + 2z + 1 = 0.$$

9. Find the angle between:

(a) the straight lines

$$(d_1) \begin{cases} x + 2y + z - 1 = 0 \\ x - 2y + z + 1 = 0 \end{cases} \quad (d_2) \begin{cases} x - y - z - 1 = 0 \\ x - y + 2z + 1 = 0. \end{cases}$$

(b) the planes

$$\pi_1 : \ x + 3y + 2z + 1 = 0 \text{ and } \pi_2 : 3x + 2y - z = 6.$$

(c) the plane xOy and the straight line M_1M_2, where $M_1(1, 2, 3)$ and $M_2(-2, 1, 4)$.

References

[1] Andrica, D., Ţopan, L., Analytic geometry, Cluj University Press, 2004.

[2] Galbură Gh., Radó, F., Geometrie, Editura didactică şi pedagogică-Bucureşti, 1979.

[3] Pintea, C. Geometrie. Elemente de geometrie analitică. Elemente de geometrie diferenţială a curbelor şi suprafeţelor, Presa Universitară Clujeană, 2001.

[4] Radó, F., Orban, B., Groze, V., Vasiu, A., Culegere de Probleme de Geometrie, Lit. Univ. ”Babeş-Bolyai”, Cluj-Napoca, 1979.

