

Lab 7

Quadrature formulas (1)

Repeated trapezium formula:

$$\int_a^b f(x)dx = \frac{b-a}{2n} [f(a) + f(b) + 2 \sum_{k=1}^{n-1} f(x_k)] + R_n(f);$$

with

$$x_k = a + kh, \quad k = 0, 1, \dots, n; \quad h = \frac{b-a}{n}.$$

Repeated Simpson's formula:

$$\int_a^b f(x)dx = \frac{b-a}{6n} [f(a) + f(b) + 4 \sum_{k=1}^n f(\frac{x_{k-1} + x_k}{2}) + 2 \sum_{k=1}^{n-1} f(x_k)] + R_n(f),$$

with

$$x_k = a + kh, \quad k = 0, 1, \dots, n; \quad h = \frac{b-a}{n}.$$

Trapezium formula for double integral

Applying successively trapezium formula with respect to y , and with respect to x , we have:

$$\begin{aligned} \int_a^b \int_c^d f(x, y) dy dx &\approx \int_a^b \left(\frac{d-c}{4} \right) \left[f(x, c) + 2f\left(x, \frac{c+d}{2}\right) + f(x, d) \right] dx \\ &= \frac{b-a}{4} \cdot \frac{d-c}{4} \left[f(a, c) + 2f\left(a, \frac{c+d}{2}\right) + f(a, d) \right] \\ &\quad + \frac{b-a}{4} \cdot 2 \cdot \frac{d-c}{4} \left[f\left(\frac{a+b}{2}, c\right) \right. \\ &\quad \left. + 2f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + f\left(\frac{a+b}{2}, d\right) \right] \\ &\quad + \frac{b-a}{4} \cdot \frac{d-c}{4} \left[f(b, c) + 2f\left(b, \frac{c+d}{2}\right) + f(b, d) \right]. \end{aligned}$$

We get

$$\begin{aligned} \int_a^b \int_c^d f(x, y) dy dx \approx & \frac{(b-a)(d-c)}{16} [f(a, c) + f(a, d) + f(b, c) + f(b, d) \\ & + 2f\left(\frac{a+b}{2}, c\right) + 2f\left(\frac{a+b}{2}, d\right) + 2f\left(a, \frac{c+d}{2}\right) \\ & + 2f\left(b, \frac{c+d}{2}\right) + 4f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)] \end{aligned} \quad (1)$$

Simpson's formula for double integral

Consider the integral $I = \int_a^b \int_c^d f(x, y) dy dx$. Let $m, n \in \mathbb{N}$ and the equidistant points x_0, \dots, x_{2m} in $[a, b]$, with step $h = \frac{b-a}{2m}$, respectively y_0, \dots, y_{2n} in $[c, d]$, with step $k = \frac{d-c}{2n}$.

We apply the repeated Simpson's formula to the integral $\int_c^d f(x, y) dy$ and then to the integral $\int_a^b \int_c^d f(x, y) dy dx$.

Algorithm:

INPUT: a,b,c,d,m,n

OUTPUT: the approximant J of the integral I

$h = (b-a)/(2*n)$;

$j1=0$; $j2=0$; $j3=0$

for $i=0, 1, \dots, 2*n$

 Let $x = a + i*h$;

$h1 = (d-c)/(2*m)$;

$k1 = f(x, c) + f(x, d)$;

$k2=0$;

$k3=0$;

 for $j=1, 2, \dots, 2*m-1$

$y = c + j*h1$;

$z = f(x, y)$;

 if j is even do $k2 = k2 + z$;

 else $k3 = k3 + z$;

 end{if}

 end{for}

$l = (k1 + 2*k2 + 4*k3)*h1/3$;

 if $(i=0) \vee (i=2*n)$ do $j1 = j1 + l$;

 else if i is even do $j2 = j2 + l$;

 else $j3 = j3 + l$;

 end{if}

 end{if}

end

$J = (j1 + 2*j2 + 4*j3)*h/3$

Problems

1. a) Approximate the integral

$$I = \int_0^1 f(x)dx, \quad \text{for } f(x) = \frac{2}{1+x^2},$$

using trapezium formula.

b) Plot the graph of the function f and the graph of the trapezium with vertices $(0, 0)$, $(0, f(0))$, $(1, f(1))$ and $(1, 0)$.

c) Approximate the integral I using Simpson's formula.

2. Approximate the following double integral

$$\int_{1.4}^2 \int_1^{1.5} \ln(x+2y)dydx$$

using trapezium formula for double integrals, given in (1). (*Result:* 0.4295545)

3. Evaluate the integral that arises in electrical field theory:

$$H(p, r) = \frac{60r}{r^2 - p^2} \int_0^{2\pi} \left[1 - \left(\frac{p}{r} \right)^2 \sin x \right]^{1/2} dx,$$

for $r = 110$, $p = 75$, using the repeated trapezium formula for two given values of n . (*Result:* 6.3131)

4. Find the smallest value of n that gives an approximation of the integral $\int_1^2 x \ln(x)dx$ which is correct to three decimals, using the program for the repeated trapezium formula. (*Result:* 0.636294368858383)

5. Evaluate the integral

$$\int_0^\pi \frac{dx}{4 + \sin 20x}$$

using the repeated Simpson's formula for $n = 10$ and 30 . (*Result:* 0.8111579)

6. The error function $\operatorname{erf}(x)$ is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

Use the repeated Simpson's formula to evaluate $\operatorname{erf}(0.5)$ with $n = 4$ and $n = 10$. Estimate the accuracy of your result and compare with the correct value $\operatorname{erf}(0.5) = 0.520499876$.

Facultative:

7. The volume of a solid is given by $\int_{0.1}^{0.5} \int_{0.01}^{0.25} e^{\frac{y}{x}} dydx$. Approximate this volume applying Simpson's algorithm for double integrals considering 21 equidistant points in $[0.1, 0.5]$, respectively in $[0.01, 0.25]$. (*Result:* 0.178571)