Let's start with a very basic mathematical fact :

$$F(x) = e^{\alpha x} \quad : \alpha \equiv \text{any number real, complex or pure imaginary}$$

$$\underbrace{\frac{dF(x)}{dx}}_{\text{rate of change of } F(x)} = \alpha e^{\alpha x} = \underbrace{\alpha}_{\text{proportial constant}} F(x)$$

$$\begin{cases} F_1(x) = \sin kx \Rightarrow \frac{dF_1(x)}{dx} = k\cos kx \\ F_2(x) = \cos kx \Rightarrow \frac{dF_2(x)}{dx} = -k\sin kx \end{cases}$$

When you dfⁿ one then you will get other with proportial const. [in one case with '–']

$$f(x) = \cos kx + i \sin kx \qquad e^{ikx}$$

$$\frac{df(x)}{dx} = -k\sin kx + ik\cos kx$$

$$= -k(\sin kx - i\cos kx)$$

$$= ik\underbrace{(\cos kx + i\sin kx)}$$

Let's consider a wave with frequency, f = freq. c/s

ω = angular frequency

$= 2\pi f$

λ = wave length

In case of light,

$$\underbrace{\lambda}_{\text{distance}} \underbrace{f}_{\text{inverse of time}} = \underbrace{c}_{\text{speed of light}}$$

!! Picture of Single Photon !!

In quantum mechanics, where we can correlate between wave like motion with particle.

One of the important thing in quantum mechanics : h

plank const. from Uncertainty Principle $(\Delta x)(\Delta p) \sim \hbar$

$$\hbar = \frac{h}{2\pi} \simeq 10^{-34}$$

reduced planck const.

☺ (Lazy Physicists!!)

Energy of a single quantum

From quantum mechanics of harmonic oscillator

$$E = \hbar\omega = hf \simeq O(J)$$

very very very small $\sim 10^{-34}$

typically very large $\sim 10^{15}$

momentum = typically in the direction of motion of an object, $\vec{p} = (p_x, p_y, p_z)$

typically consered in nature !!
at least in non-relativistic case

$$\vec{p} = m\vec{v}$$

But what will happen in case of relativistic photon momentum :

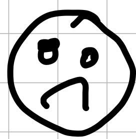
By default relativistic in case of Photon

Is it just $|p| = mc$??

What is this mass :: rest mass

So if we consider in relativity into our picture then m is not just mass but

$$m = m_0 \equiv \text{rest mass}$$

but for photon , $m_0 = 0$ 

So momentum of a single photon (magnitude)

$$\lambda f = c$$
$$\Rightarrow f/c = \frac{1}{\lambda}$$

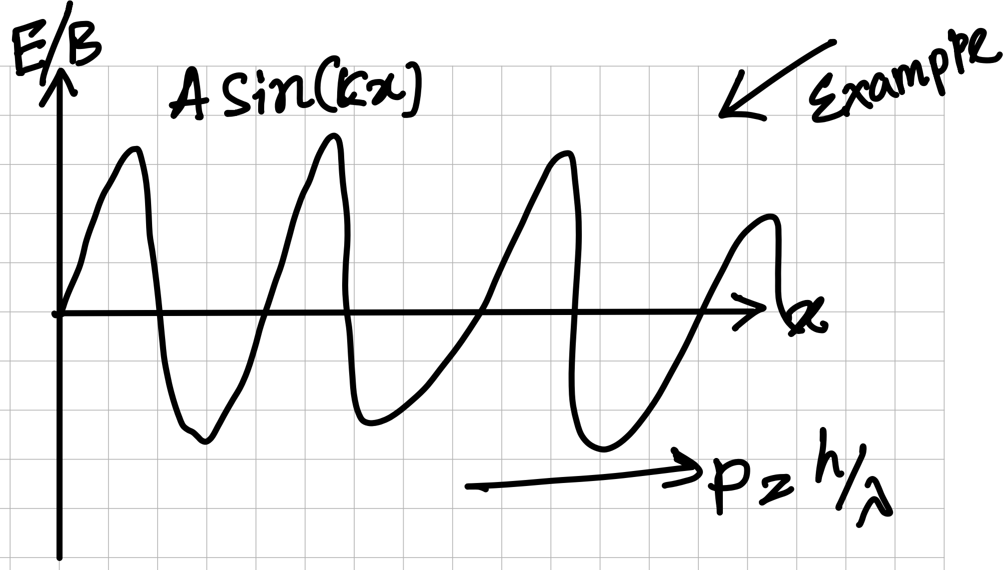
$$p = \frac{hf}{c} = \frac{h}{\lambda}$$

In case of bunch of "n" number of photons

$$E = n\hbar\omega$$

again from Harmonic oscillator of quantum mechanical system

Like typically Electromagnetic wave which carries appricated amount of energy has enormous number of photons and amplitude of the wave $A \sim \sqrt{n}$



So, let's try to write relationship of Energy and Momentum in more general case :

Let's simplify the above problem:

What is relationship between frequency and wave length for a slow moving electron or neucleon ??

Kinetic energy, $E = \frac{1}{2} m v^2$

$= \frac{1}{2} m \cdot (p/m)^2$

$= p^2/2m$

for a single electron, $hf = \frac{1}{2} p^2/m$

$= \frac{1}{2m} \cdot h^2/\lambda^2$

time-dependence of wave

space-dependence of wave

So, $\boxed{f = \frac{h}{2m\lambda^2}}$

Isn't it same like Schrödinger equn

Let ask a very general question:

When we consider a particle kinematics relativistic or non-relativistic !! ?

If momentum is large compare to it's rest mass (in assumption $c=1$). then dynamics of particle will be relativistic kinematics

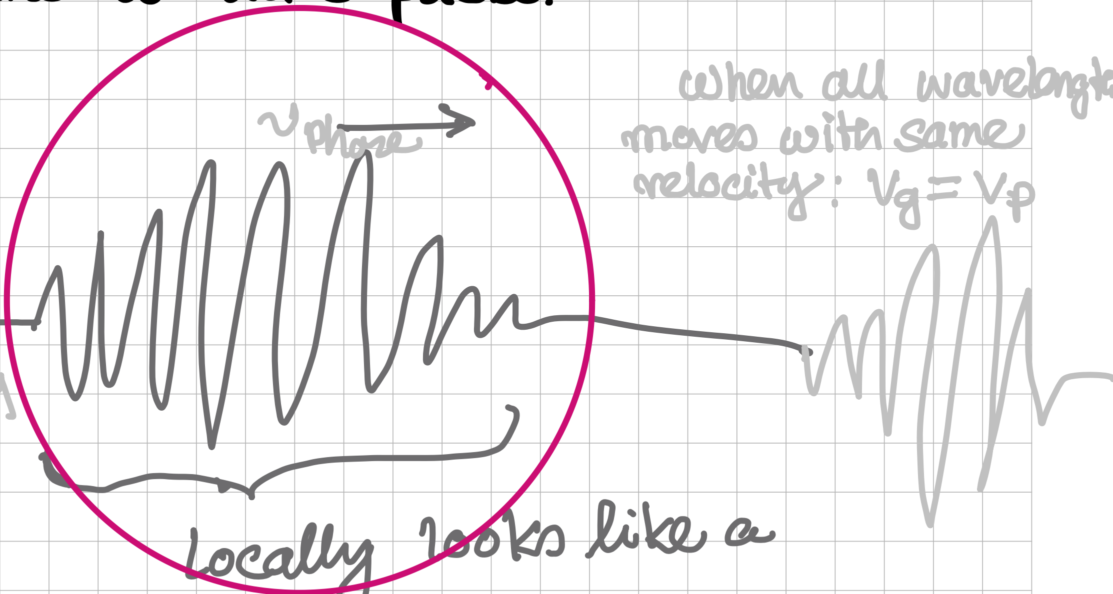
Let's talk a bit about group velocity and Phase velocity:

Let's assume a wave packet

$f = \frac{h}{2m\lambda^2}$

$f\lambda = \frac{h}{2m\lambda}$

So, different λ moves with different velocity



when all wavelength moves with same velocity: $V_g = V_p$

Locally looks like a plane wave confine in a region

V_{group}

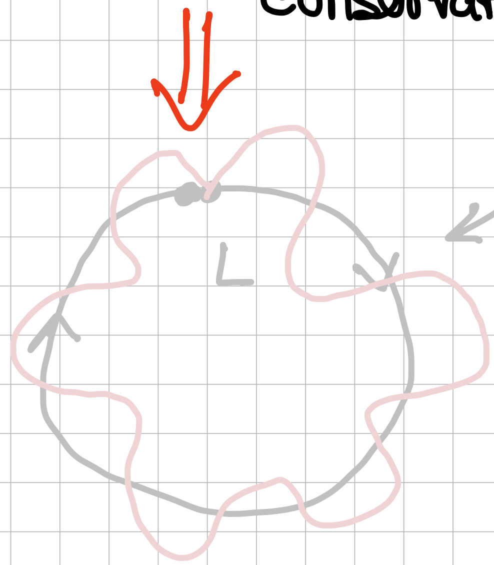
• Guess which one you will identify as non-relativistic velocity of particle: Group velocity

$V_{group} < V_{Phase}$

N.B

In quantum mechanics, QFT, particle physics, we always concentrated our ideas of wave-equation in a finite system first then take a limit to extend the space

What about momentum conservation ??



periodic space

This trick will help you to solve ~~two fundamental~~ problems:

(i) momentum conservation

(ii) Infinite space

But this has draw ~~back~~:

(i) momentum is quantize

• Quantization of momentum

$L = 2\pi r$

$\lambda = \frac{L}{N}$; N = number of wave = integer

wavelength become discrete

$P = h/\lambda = \frac{Nh}{L} = \frac{h}{L} \cdot N$

$$p = \frac{Nh}{2\pi r}$$; Angular momentum,

$$L = \cancel{mv}r = rp \ ??$$

$$= \frac{Nh}{2\pi} = N\hbar$$

Quantize

in unit of $\hbar$

- Now let's talk about relation between particle and field

$\underbrace{\text{particle and field}}_{\text{Real world of Quantum field}}$

⇒ Now consider an example of Quantum Harmonic oscillator:

Why Harmonic oscillator?

- Harmonic oscillator is important in many different reason particular in case of field theory: wave is oscillation where electric field and magnetic field are oscillating as harmonic oscillator → simplest sinusoidal oscillation

classical system with quantum analogue

To understand concept of Quantum field let's go back Harmonic oscillator

- wave superposition of different wave length
- wave can be reconstruct as individual wave with different wave length and quantized the system as harmonic oscillator

• Commutational relation:

$$AB - BA \neq 0$$

$\left.\begin{matrix} A \\ B \end{matrix}\right\} \rightarrow$ operator/ measureable

A and B can't measure simulteneously

$\hat{x}, \hat{p}$

$$[A, B] \neq 0 \quad \{\text{order matter}\}$$

like particle has different component of position and momentum

$$\vec{r} = (x, y, z)$$

$$\vec{p} = (p_x, p_y, p_z)$$

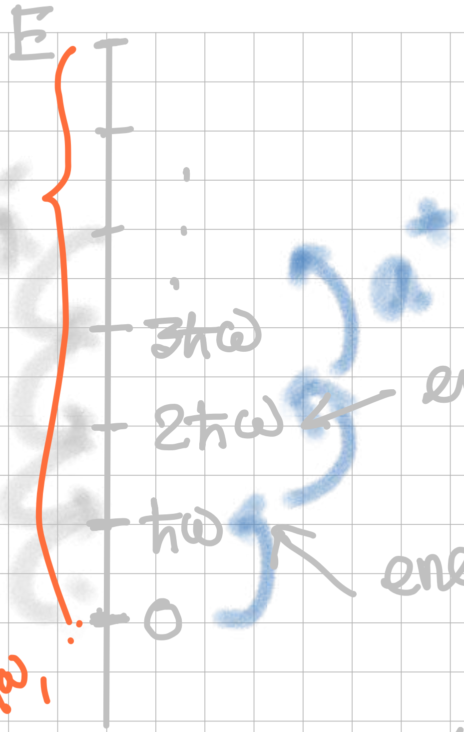
$$[x, p_x] \neq 0 = i\hbar \quad (\text{small but non-zero})$$

$$[y, p_y] = [z, p_z] = i\hbar$$

- $\omega = 2\pi f$ (frequency of harmonic oscillator)

In quantum harmonic oscillator,

Energy is quantize not momentum



energy of two photon

energy of 1 photon

Those are possible energy of harmonic oscillator $0, \hbar\omega, 2\hbar\omega,$

Braket

integer

$$E = n\hbar\omega \quad [\, n = 0, 1, 2, \ldots \,]$$

$E_n = n\hbar\omega$; $|n\rangle$ = State of the System

Just mathematical juggling $\Rightarrow$ not correspond to expt.

$$a^+ |n\rangle = \sqrt{n+1}\, |n+1\rangle$$ ← Abra Ka Dabra !! Of operator mathematics

$$a^- |n\rangle = \sqrt{n}\, |n-1\rangle$$

$$a^+ a^- |n\rangle = \sqrt{n}\, a^+ |n-1\rangle$$
$$= \sqrt{n} \cdot \sqrt{n}\, |n\rangle$$
$$= n\, |n\rangle$$

$$\boxed{a^+ a^- = n}$$

$$h\omega\, a^+ a^- = n h\omega$$

$$a^- a^+ |n\rangle = \sqrt{n+1}\, a^- |n+1\rangle$$

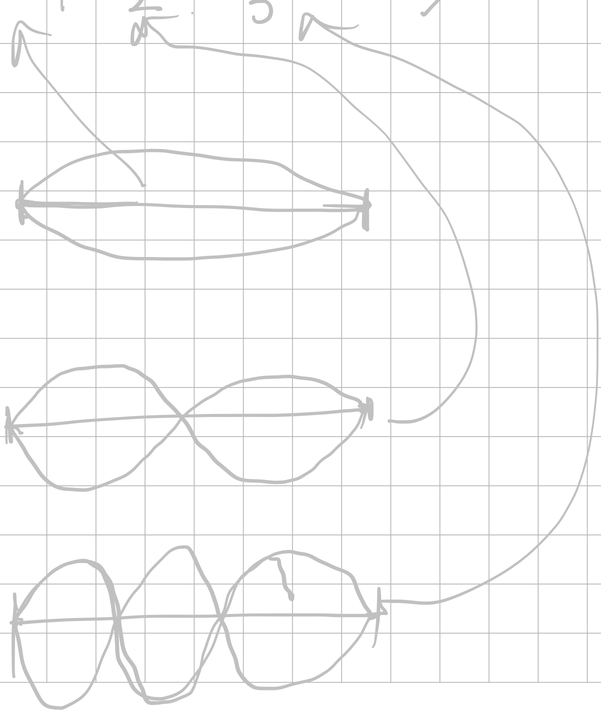
$$= \sqrt{n+1}\sqrt{n+1} \cdot |n\rangle$$

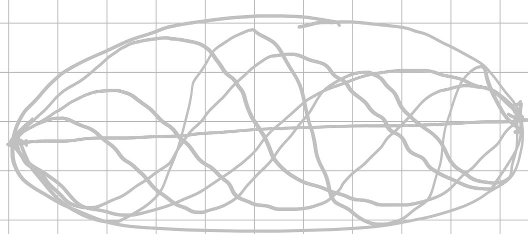
$$= (n+1)|n\rangle$$

So, $[a^+, a^-] \neq 0$ → you can't measure a^+, a^- simultaneously measureable

When a^+, a^- operator acts on any state it create or destroy photon : which represent as annihilation/creation/or ladder operator

$|n_1, n_2, n_3, \dots\rangle$





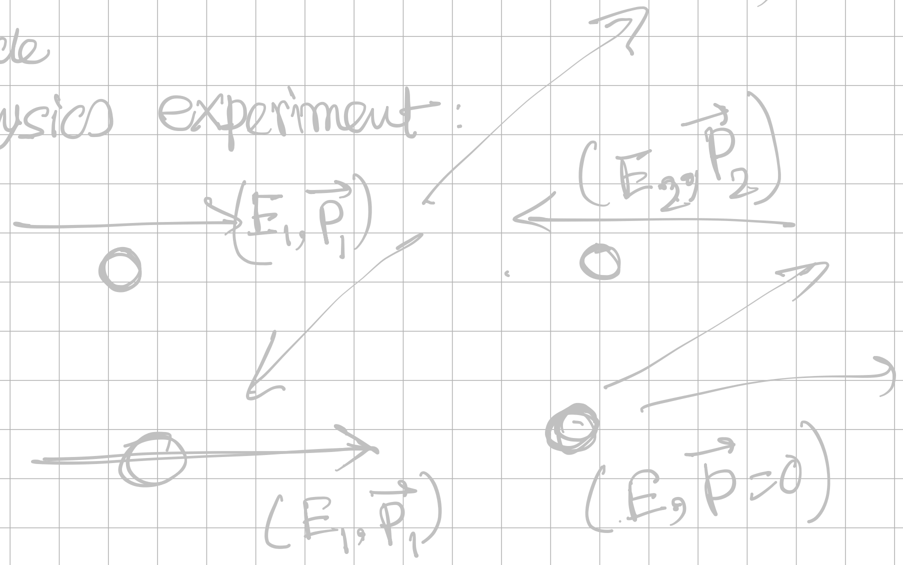
$\lambda = L/N$

ω_N

⇒ when infinite number of harmonic oscillator create and annihilation infinite times mathematically that collection just infinite number of oscillator which create and annihilate photon due to excitation: neively we could called it "Quantum Field".

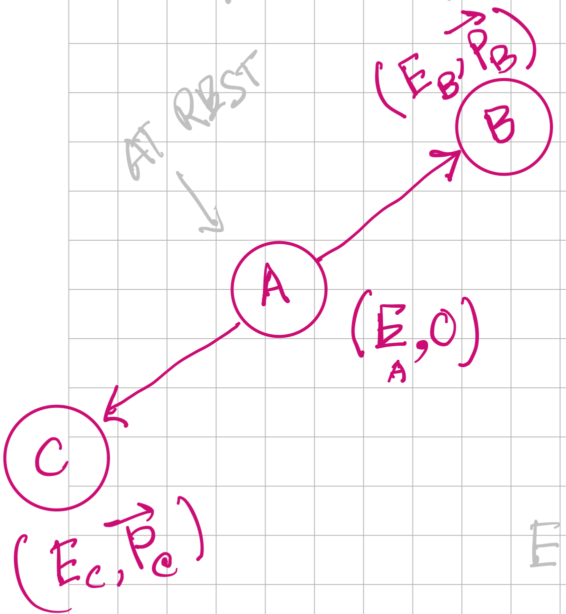
If you are interested in particle physics such notation should be known by heart $\Rightarrow |n_1 n_2 n_3 n_4 \dots\rangle$

In high energy particle physics experiment:



Now play with more example of relativistic kinematics

- TWO BODY DECAY OF AN UNSTABLE PARTICLE



— USING 4-VECTOR MOMENTUM CONSERVATION

$$\underset{m_A}{A} \longrightarrow \underset{m_B}{B} + \underset{m_C}{C}$$

REST MASS

Energy momentum 4-vector,

$$p^\mu = (E/c, p^1, p^2, p^3)$$

$$P_A^\mu = \left(\frac{E_A}{c}, 0, 0, 0\right) \leftarrow \text{FOR PARTICLE 'A'}$$

$$P_B^\mu = \left(\frac{E_B}{c}, p_B^1, p_B^2, p_B^3\right) \leftarrow \text{FOR PARTICLE B}$$

$$P_C^\mu = \left(\frac{E_C}{c}, p_C^1, p_C^2, p_C^3\right) \leftarrow \text{FOR PARTICLE C}$$

NOW LET USE FUNDAMENTEL CONSERVATION OF MOMENTUM

$$P_A^\mu = P_B^\mu + P_C^\mu$$

$\Rightarrow$ AHH !! WE NEED TO LEARN FOUR VECTOR BEFORE PROCEED FURTHER !!

SPACE-TIME WHICH DESCRIBE BY FOUR CO-ORDINATE OF AN EVENT :

$$X = (ct, \vec{r}) = (x_0, x_1, x_2, x_3)$$

$$x_0' = \frac{(x_1 - \beta x_0)}{\sqrt{1 - v^2/c^2}} \quad ; \quad \beta = v/c \; ; \; \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$x_1' = \gamma(x_1 - \beta x_0)$$

$$x_0' = \gamma(x_0 - \beta x_1)$$

$$X \cdot X = x_0^2 - x_1^2 - x_2^2 - x_3^2 = x_0^2 - \vec{r}\cdot\vec{r} = S^2$$

inner product

SPACE-TIME INTERVAL

Similarly, if we have two event seperated by space-time interval,

$$(\Delta S)^2 = (\Delta x_0)^2 - (\Delta r)^2$$

$\simeq$ Lorentz invariant

same in all frame of reference doesn't effect by Lorentz transformation

LET'S USE THE SAME ANALOGY IN MOMENTUM FOUR VECTOR:

$$\vec{p} = m \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t}$$

vector

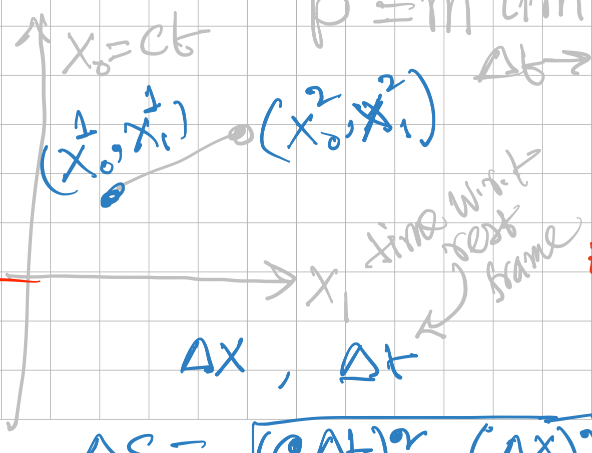
scaler

i.e. Something doesn't affected by rotation

P = FOUR-MOMENTUM

HOW TO Def^n ??

$$P = m \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{??}$$



Δx , Δt

$$\Delta S = \sqrt{(c\Delta t)^2 - (\Delta x)^2} = c\Delta t\sqrt{1 - \left(\frac{\Delta x}{\Delta t}\right)^2 \frac{1}{c^2}}$$

$$\Delta s = c\,d\tau \quad ; \quad d\tau = dt\sqrt{1-\left(\frac{\Delta x}{\Delta t}\right)^2 \frac{1}{c^2}}$$

YOU CAN EVALUATE THIS IN ANY FRAME YOU LIKE

What is $d\tau$??

$\Rightarrow$ TIME IN THE PARTICLE FRAME OF REF^n

$\equiv$ TIME ELAPSED ACCORDING TO THE PARTICLE

$$\frac{d\tau}{\sqrt{1-v^2/c^2}} = dt$$

LOOKS FAMILIER LIKE TIME-DILATION

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1-v^2/c^2}} = \gamma$$

FOUR MOMENTUM:

$$P = \left(m\frac{dx_0}{d\tau},\ m\frac{d\vec{r}}{d\tau}\right)$$

FOUR DIMENSIONAL MOMENTUM

$$= \left(\underbrace{mc\frac{dt}{d\tau}}_{P_0},\ \underbrace{m\frac{d\vec{r}}{d\tau}}_{\vec{P}}\right)$$

$$= \left(\frac{mc}{\sqrt{1-v^2/c^2}},\ \frac{m\vec{v}}{\sqrt{1-v^2/c^2}}\right)$$

$$\frac{d\vec{r}}{d\tau} = \frac{d\vec{r}}{dt}\frac{dt}{d\tau}$$

$$\vec{v}\cdot\frac{dt}{d\tau}$$

$$= \gamma\vec{v}$$

$$p_0' = \gamma(p_0 - \beta p_1)$$

$$p_1' = \gamma(p_1 - \beta p_0)$$

NOW LET'S MORE CLOSER TO P_0 TERM

$$p_0 = \frac{mc}{\sqrt{1 - v^2/c^2}}$$

Taylor expansion

$$= mc\left(1 + \frac{v^2}{2c^2} + \cdots\cdots\right)$$

$$= mc + \frac{1}{2}mv^2 \cdot \frac{1}{c} + \cdots\cdots$$

If $v/c \simeq 0$ then higher order term could be ignore but please think twice before ignore them

$$cp_0 = mc^2 + \frac{1}{2}mv^2 + \cdots\cdots$$

THIS SHOULD BE ENERGY

Kinetic ENERGY

$$E = mc^2 + \frac{1}{2}mv^2 + \cdots\cdots$$

So, $p_0 = E/c$

$$P = \left(E/c, \vec{p}\right)$$

ABOVE MATHEMATICAL JUGGLING IS JUST SOME ABRA KA DABRA !!
BUT ONE SHOULD KNOW THIS

$$X = (ct, \vec{r})$$

Four-vector

scaler

vector

$$P = (E/c, \vec{P})$$

Energy momentum Four vector

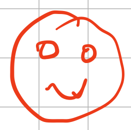
UNDER LORENTZ TRANSFORMATION $E, \vec{P}$ WILL MIXED WITH EACH OTHER SAME LIKE $X = (x_0, \vec{r})$

WHEN WE WILL STUDY PARTICLE DYNAMICS IN PARTICLE PHYSICS, ONE NEED TO KNOW :

$$P = (E/c, \vec{p}); \quad E = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$

$$p = \frac{mv}{\sqrt{1 - v^2/c^2}}$$

So, in old days, when physicists were too cleaver but not well equipped with detector which could play with high energy particle ($v \simeq c$), then life was simple and physicists were busy with low energy particles



But now-a-days, thanks to huge development of technology (why makes physicists life more complicated), we than even think about higher order terms of:

$$c p_0 = mc^2 + \frac{1}{2} m v^2 + \cdots$$

BY KNOWING ENERGY-MOMENTUM IN ONE FRAME THEN YOU COULD KNOW ENERGY-MOMENTUM IN OTHER FRAME OF REF^n

$$A = (A_0, \vec{A}) \qquad A \cdot B = A_0 B_0 - \vec{A} \cdot \vec{B}$$

$$B = (B_0, \vec{B}) \qquad \vec{A} \cdot \vec{A} = A_0^2 - |\vec{A}|^2$$

$$\text{So,}\quad P.P = p_0^2 - p^2 = \left(\frac{mc}{\sqrt{1-v^2/c^2}}\right)^2 - \left(\frac{mv}{\sqrt{1-v^2/c^2}}\right)^2$$

$$= \frac{m^2c^2}{\left(1-v^2/c^2\right)}\left(1-\frac{v^2}{c^2}\right)$$

$$\underbrace{\qquad}_{\text{THIS SHOULD BE INVARINT UNDER L.T.}} = m^2c^2$$

RELATIVISTIC LENGTH SQURE OF FOUR-MOMENTUM

THIS DOESN'T DEPEND HOW FAST PARTICLE MOVING

LETS TRYING TO THINK HOW YOU CAN CALCULATE P.P WITHOUT ABOVE MATHEMATICAL JUGGLING !!

P.P SHOULD BE INVARIANT IN ALL FRAME OF REF^n : SO IF YOU THINK IN PARTICLE FRAME THEN $\vec{p} = 0$ (AS PARTICLE IS MOVING WITH RESPECT TO REST FRAME)

$$\text{SO}\quad P.P = p_0^2 = m^2c^2$$

$$P \cdot P = p_0^2 - p^2 = m^2c^2$$

$$\Rightarrow \frac{E^2}{c^2} - p^2 = m^2c^2$$

$$\Rightarrow \boxed{E^2 = p^2c^2 + m^2c^4}$$

Navie way to remember !! ☺

If particle at rest

$p = 0$

then $E = mc^2$

rest energy of a particle

If you consider photon then

$$\boxed{E^2 = p^2c^2 + \cancel{m^2c^4}}$$

$$E = pc$$

Summary:

$$\mathbb{P} = \left(E/c, \vec{p}\right)$$

$$= \left(\frac{mc}{\sqrt{1 - v^2/c^2}}, \frac{m\vec{v}}{\sqrt{1 - v^2/c^2}}\right)$$

$$P \cdot P = m^2c^2$$

General convention,

momentum of photon, $\vec{K}$

$$\mathbb{K} = \left(\frac{\omega}{c}, \vec{K}\right); \ \omega = K_0 c$$

$$\mathbb{K} \cdot \mathbb{K} = 0$$

Probabily you are thinking why we are discussing so much about four-momentum or in simply momentum !!

$\Longrightarrow$ Momentum Conservation !!

What is big deal of energy-momentum !!??

Like the problem we tried to solve there energy-momentum before and after should be conserved

$\Longrightarrow$ Also energy-momentum should be conserved in all frame of ref^n.

IN RELATIVISTICS EXPERIMENT :

By collision of two particle can create many particles in high energy accelerator

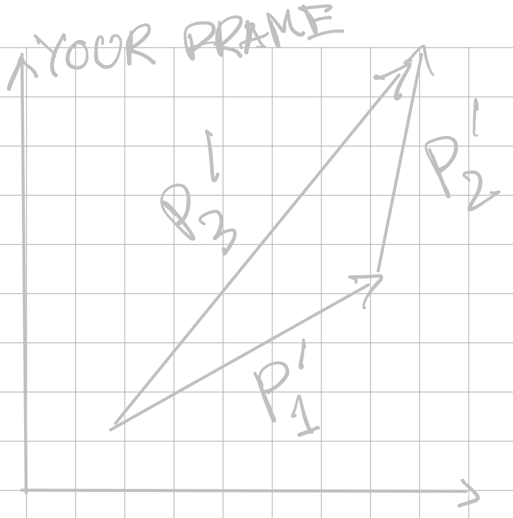
Four energy-momentum vector

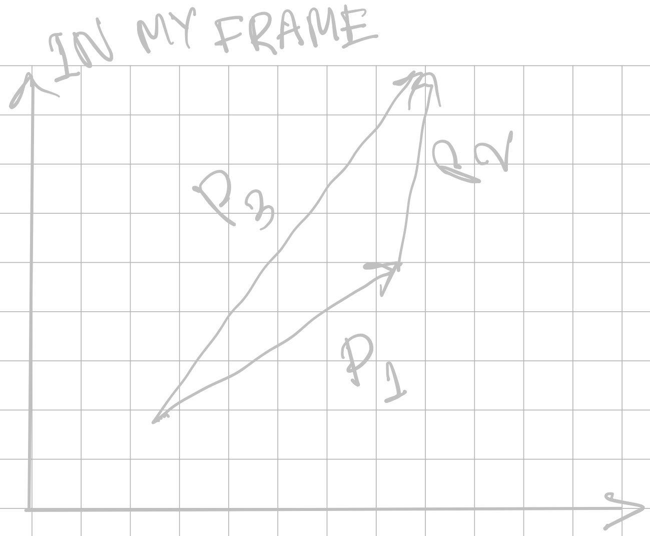
$$\underbrace{P_1 + P_2}_{P_{income}} = \underbrace{P_3 + P_4 + P_5 + \dots}_{P_{out}}$$

NOW SUPPOSE ; YOU AND ME ARE IN TWO DIFFERENT FRAME OF REF^N

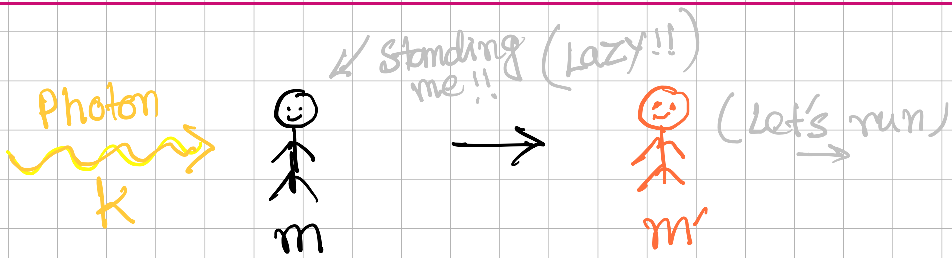
YOU OBSERVED THE WHOLE COLLISION IN YOUR FRAME OF REF^N :

$$\underbrace{P_1' + P_2'}_{P_{in}'} = \underbrace{P_3' + P_4' + P_5' + \dots}_{P_{out}'} \;??$$

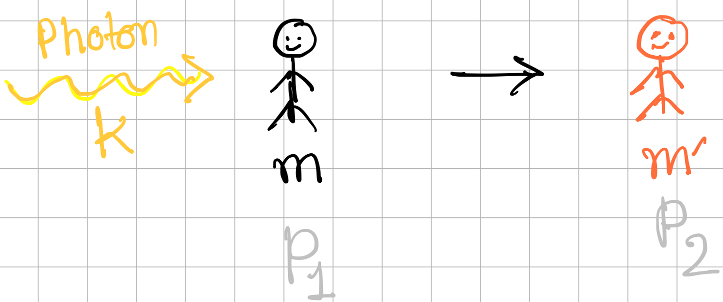




$\boxed{\text{NOW TRYING TO SOLVE SOME PROBLEMS}}$



Question is what is m' $\longrightarrow x$



[Energy of photon, $E = \omega$]

Use energy-momentum conservation

$[k_y = k_z = 0]$

$$k = P_1 + P_2 \longleftrightarrow \text{Everything is four-vector}$$

$\omega = k_x c$ for photon

$$\left(\frac{\omega}{c}, k_x\right) \qquad (mc, 0) \qquad \left(\frac{m'c}{\sqrt{1-v^2/c^2}}, \frac{m'v}{\sqrt{1-v^2/c^2}}\right)$$

when you chose energy then you don't have any option to choose momentum

$$\frac{\omega}{c} + mc = \frac{m'c}{\sqrt{1-v^2/c^2}} \; ; \; k_x = \frac{m'v}{\sqrt{1-v^2/c^2}}$$

What is given to you; $\frac{\omega}{c}$, k_x, m

Your job is to calculate m', v (two equations & two unknown)

Fourvector $\rightarrow P_2 = P_1 + K$

$\Rightarrow$ Remember we already discussed inner product of two four vector, $P_2 \cdot P_2 = m'^2 c^2$

So, $m'^2 c^2 = (P_1 + K)^2 = P_1^2 + K^2 + 2P_1 K$

$= m^2 c^2 + 0 + 2 \cdot m\omega$

$$\boxed{\begin{aligned} P_1 &= (mc, 0) \\ K &= \left(\frac{\omega}{c}, k_x\right); \quad \omega = k_x c \\ P_1 \cdot K &= m\omega - 0 \end{aligned}}$$

for photon $K \cdot K = 0$

So, $m'^2 c^2 = m^2 c^2 + 2m\omega$

$$\Rightarrow m'^2 = m^2 + \frac{2m\omega}{c^2}$$

$$\Rightarrow m' = \sqrt{m^2 + \frac{2m\omega}{c^2}}$$

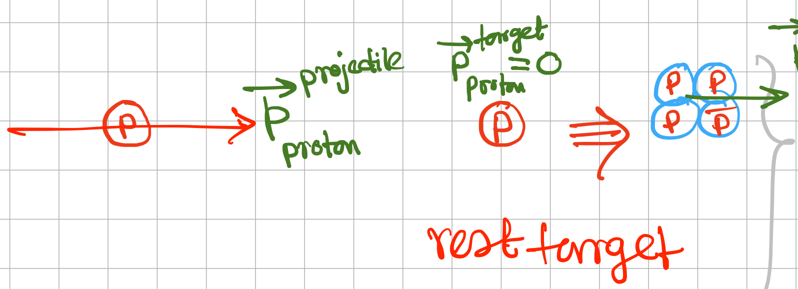
Now from where this extra mass come into picture!!

when I absorbed photon then I had to recoile to conserve momentum ⇒ consequently my final energy should be equal to initial energy (but this energy isn't my rest energy !! we need to think in terms of total energy)

Now let's make our life more simple by choosing unit:

$$\left.\begin{matrix} c = 1 \\ h = 1 \end{matrix}\right\} \text{fundamental}$$

Now let's think one realistic example:



An experiment to create anti-particle

$p + p \rightarrow p p p \bar{p}$

All p stands for proton

Question is what is the energy of incoming proton to create anti-proton ??

→ Before solving the problem let's trying to make a random reasonable guess ☺

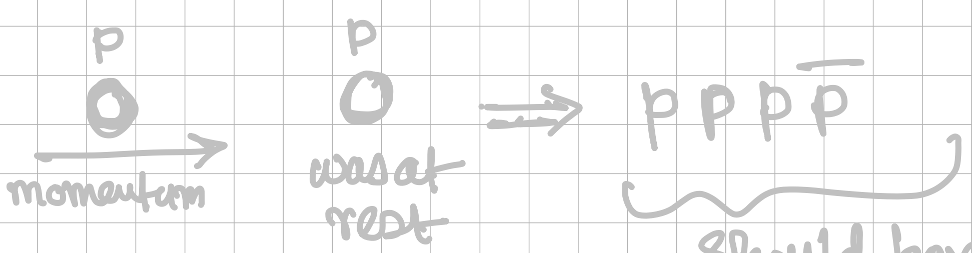
$p + p \longrightarrow p\ p\ p\ \bar{p}$

this proton at rest and all protons have rest mass "m"

One clearer but stupid guess could be just

$E = 3m$ ← Please note $c = 1$

$\Rightarrow$ This guess couldn't lead us towards right answer because of energy-momentum conservation



Should have momentum to conserve E-M

So energy of the incoming proton should be more than $3m$ for sure ☹

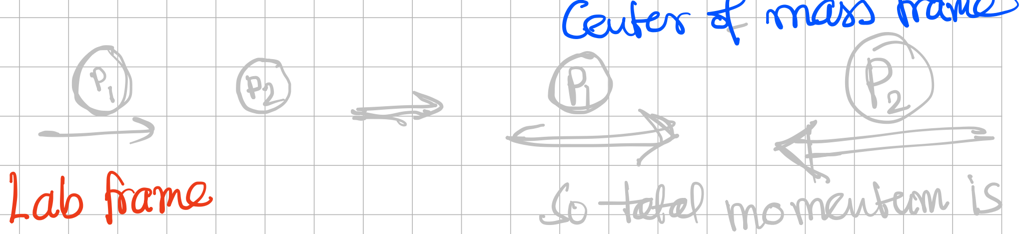
Now let's do the mathematics,

Initial four-momentum,

$$P_{ini} = \left(E+m , p \right) = P_{final}$$

$$P_{ini}^2 = P_{final}^2$$

Now if we think in terms of center of mass frame



So total momentum is zero

So, in center of mass frame, the final particle momentum should be zero

So, $\mathbb{P}^{CMS}_{final} = (4m, 0)$

So, $P^{CM\,2}_{final} = 16m^2 = P_{ini}^2$

$= (E+m)^2 - p^2$

$\Rightarrow 16m^2 = E^2 + 2Em + m^2 - p^2$

$= m^2 + 2Em + m^2$

$= 2m^2 + 2Em$

$\Rightarrow 14m^2 = 2Em$

$\Rightarrow E = 7m$

$E^2 = p^2c^2 + m^2c^4$

$c = 1$

$E^2 = p^2 + m^2$

$\Rightarrow E^2 - p^2 = m^2$

So it is not 3m but 7m